

CALCULATION OF FRICTION FACTORS

by

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(Signature)✓

Acceptance:

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May 19, 1936

Riverbank Court Hotel
Cambridge, Mass.
May 19, 1936

Professor George W. Swett
Secretary of the Faculty
Massachusetts Institute of Technology
Cambridge, Mass.

Dear Sir:

I respectfully submit this thesis entitled
"Calculation of Friction Factors" for consideration
as partial fulfillment of the requirements for the
degree of Bachelor of Science in General Engineering.
I hope that it satisfies the requirements in this
regard.

Yours respectfully,

✓ ✓ ✓
✓
Donald C. Spencer.

Acknowledgement is gratefully made to Dr. H. Peters for his supervision of this thesis and for his many suggestions, and to Dr. R. H. Cameron for suggesting a method of evaluating the definite integral occurring in the solution of the problem of the elliptical pipe.

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TABULATION OF SYMBOLS

c = temporal mean velocity at any point in a circular pipe.

\bar{c} = mean velocity over the pipe section.

δ_1 = thickness of the laminar sublayer.

e = effectivity ($e = \frac{k}{y_0}$).

f = friction factor for pipes ($f = \frac{dp}{dx} \frac{D}{\rho/2 \bar{u}^2}$)

h = distance between the parallel plates.

K = constant of the turbulent exchange ($K = 0.40$ approx.).

k = absolute roughness height.

λ = friction factor for parallel plates ($\lambda = \frac{dp}{dx} \left(\frac{h}{2}\right) \frac{1}{\rho/2 \bar{u}^2}$).

l = mixing length.

μ = dynamic coefficient of viscosity.

ν = kinematic coefficient of viscosity.

r = distance to any point from the center of the pipe.

R = pipe radius.

ρ = density.

Rey = Reynolds Number.

τ = shearing stress.

τ_0 = shearing stress at the wall.

u = temporal mean velocity in direction of flow.

u' = component in the direction of flow of the difference between u and the velocity at any instant of time.

v' = component normal to the flow of the difference between u and the velocity at any instant of time.

y_0 = distance from the wall of a rough pipe to points at which the mean velocity is zero.

Note - Figures in parentheses designate references listed in the appendix.

The purpose of this investigation is to illustrate a method for the calculation of friction which is thought to be simpler and more widely applicable than the existing methods. Von Kármán's similarity concept of the turbulent flow pattern (1) has produced many satisfactory results. However, the logarithmic velocity distribution developed from this concept is not valid, for example, in the center of a pipe line or at the outer part of the boundary layer of a flat plate. Here the correlation between the u' and v' velocity components is not perfect. This equation would require that l become equal to zero since

$$\tau = \rho l^2 \left(\frac{du}{dy} \right)^2$$

and the slope does not vanish for y finite. The mixing length is zero, of course, only at the wall.

It has been pointed out by H. Peters and C. G. Rossby (2) that the solution of many problems involving boundary layer theory might be simplified by the assumption of a maximum value for the mixing length. Nikuradse's measurements in pipe lines indicate a maximum value for l of 0.07 of the diameter, a value varying slightly with the Reynold's Number. Tollmien obtained the same constant in the mixing zone of a two dimensional jet with $l = 0.068b$, b the width of the mixing zone. The value was $0.073 \frac{d}{2}$ for the circular jet with d the diameter. Rossby obtained the result $l = 0.065 \frac{H}{\sqrt{2}}$ for the atmosphere, where H is the

thickness of the layer of variable stress. This constant therefore appears to have considerable significance.

A few simple problems have been solved by making use of this constant in an attempt to establish the validity of this assumption. Near the wall the stress was assumed constant with linear increase of the mixing length with normal distance from the wall. Further out, the mixing length was given the constant value $l = cd$, where $c = 0.07$ and d is some length dimension of the flow; the stress was assumed to diminish rapidly in this region. At the boundary of separation between these two layers continuity in velocity, stress, and mixing length was satisfied. The velocity gradient is zero at the center line.

Smooth Circular Pipes.- In the pure laminar sub-layer next to the pipe wall, the velocity distribution can be obtained from the equation of Navier-Stokes in cylindrical coordinates:

$$\mu \left(\frac{\partial^2 c_z}{\partial r^2} + \frac{1}{r} \frac{\partial c_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 c_z}{\partial \varphi^2} + \frac{\partial^2 c_z}{\partial z^2} \right) - \frac{\partial p}{\partial z} = \rho \frac{Dc_z}{dt}$$

The flow is assumed to be radially symmetrical, hence,

$$\frac{\partial^2 c_z}{\partial \varphi^2} = 0$$

Also,

$$\frac{\partial c_r}{\partial r} + \frac{c_r}{r} + \frac{\partial c_\varphi}{r \partial \varphi} + \frac{\partial c_z}{\partial z} = 0 \quad \text{Equation of Continuity}$$

Since $c_\varphi = 0$ and $c_r = 0$,

$$\frac{\partial c_z}{\partial z} = 0$$

Therefore, for steady flow,

$$\rho \frac{Dc_z}{dt} = \rho \left(\frac{\partial c_z}{\partial t} + c_r \frac{\partial c_z}{\partial r} + c_\varphi \frac{\partial c_z}{r \partial \varphi} + c_z \frac{\partial c_z}{\partial z} \right) = 0$$

Hence,

$$\frac{dp}{dz} = \mu \left(\frac{d^2 c_z}{dr^2} + \frac{1}{r} \frac{dc_z}{dr} \right)$$

or,

$$\begin{aligned} \frac{dp}{dz} \frac{r}{\mu} &= r \frac{d^2 c_z}{dr^2} + \frac{dc_z}{dr} \\ &= \frac{d}{dr} \left(r \frac{dc_z}{dr} \right) \end{aligned}$$

$$\frac{dp}{dz} \frac{rdr}{\mu} = d\left(r \frac{dc}{dr}\right)$$

Since

$$\frac{\tau}{\mu} = \frac{dc}{dr}$$

$$\frac{dp}{dz} \frac{rdr}{\mu} = d\left(\frac{r\tau}{\mu}\right)$$

Hence, integrating from 0 to R,

$$\tau_0 = \frac{dp}{dz} \frac{R}{2}$$

Also, the velocity is given by

$$c = \frac{dp}{dz} \frac{r^2}{4\mu} + K_1 \ln r + K_2$$

Since the fluid clings to the pipe wall, the velocity at the wall must be zero. Also, from symmetry considerations, the velocity is a maximum at the center of the pipe. Therefore, putting

$$c = 0 \text{ when } r = R$$

$$\frac{dc}{dr} = 0 \text{ when } r = 0$$

the velocity equation becomes

$$c = - \frac{dp}{dz} \frac{1}{4\mu} (R^2 - r^2)$$

Letting $R - y = r$

$$c = - \frac{1}{4\mu} \frac{dp}{dz} (y^2 - 2Ry)$$

Neglecting the squared term and substituting for $\frac{dp}{dz}$ in terms of τ_0 ,

$$(I) \frac{c}{\sqrt{\frac{\tau_0}{\rho}}} = \frac{\sqrt{\frac{\tau_0}{\rho}}}{f} y$$

The value $y = \delta_1 = \frac{11.5}{\left(\frac{\tau_0}{\rho}\right)^{1/2}}$ has been fixed by

experiment by Nikuradse as a reasonable limit for the region influenced by laminar friction.

There is theoretical justification, as is shown by a simple derivation, that the stress varies as given by the equation

$$\tau = \tau_0 \frac{r}{R} \quad \text{See Appendix}$$

Taking this variation into account and assuming linear increase of the mixing length with normal distance from the pipe wall, the velocity equation in the region ($\delta_1 \leq y \leq 0.35 R$) takes the form

$$\frac{c_{\max.} - c}{\sqrt{\frac{\tau_0}{\rho}}} = C - \frac{1}{K} \left\{ 2\sqrt{\frac{r}{R}} - \ln \frac{1 + \sqrt{\frac{r}{R}}}{1 - \sqrt{\frac{r}{R}}} \right\}$$

where C is a constant.

However, the complication of this equation as compared with that of the well-known simpler form derived on the assumption of constant stress did not seem to justify its use in this case. Therefore, assuming again that

$$l = ky$$

but that

$$\tau = \tau_0 \quad \text{See Appendix}$$

the velocity equation becomes

$$\tau_o = \rho k^2 y^2 \left(\frac{dc}{dy}\right)^2$$

since

$$\tau = \rho l^2 \left(\frac{dc}{dy}\right)^2$$

Integrating this equation between the limits δ_1 and y ,

$$\frac{c - c_{\delta_1}}{\sqrt{\frac{\tau_o}{\rho}}} = \frac{1}{k} \ln \frac{y}{\delta_1} = \frac{1}{k} \ln \frac{y \sqrt{\frac{\tau_o}{\rho}}}{11.5 \nu}$$

Hence

$$(II) \quad \frac{c}{\sqrt{\frac{\tau_o}{\rho}}} = 5.4 + 5.75 \log_{10} \frac{y \sqrt{\frac{\tau_o}{\rho}}}{\nu}$$

Let l be assumed constant over the center section of the pipe and equal to $0.14 R$, and let the stress be given by

$$\tau = \tau_o \frac{R-y}{0.65R} \quad \text{See Appendix}$$

Then, since $\tau = \rho l^2 \left(\frac{dc}{dy}\right)^2$, the velocity equation is

$$\frac{dc}{dy} \frac{1}{\sqrt{\frac{\tau_o}{\rho}}} = \frac{1}{0.14R^{1/2}} \frac{\sqrt{R-y}}{0.806R}$$

Integrating between the limits $0.35 R$ and y ,

$$\frac{c - c_{0.35R}}{\sqrt{\frac{\tau_o}{\rho}}} = 3.00 - 5.88 \left(\frac{R-y}{R}\right)^{3/2}$$

From (II), at $y = 0.35 R$,

$$\frac{c_{0.35R}}{\sqrt{\frac{\tau_0}{\rho}}} = 2.8 + 5.75 \log_{10} \frac{\sqrt{\frac{\tau_0}{\rho}}}{v} R$$

Hence,

$$(III) \frac{c}{\sqrt{\frac{\tau_0}{\rho}}} = 5.8 + 5.75 \log_{10} \frac{\sqrt{\frac{\tau_0}{\rho}}}{v} R - 5.88 \left(\frac{R-y}{R}\right)^{5/2}$$

$$0.35R \leq y \leq R$$

The mean velocity is given by the integration of equations (I), (II), and (III) over the respective portions of the cross section.

$$\begin{aligned} \frac{\bar{c}}{\sqrt{\frac{\tau_0}{\rho}}} &= \frac{2}{R^2} \int_0^R \frac{c}{\sqrt{\frac{\tau_0}{\rho}}} r dr \\ &= \frac{2}{R^2} \left\{ \frac{R}{2} \frac{(11.5)^2 v}{\sqrt{\frac{\tau_0}{\rho}}} - \frac{1}{3} \frac{(11.5)^3 v^2}{\left(\frac{\tau_0}{\rho}\right)} + 0.14 R^2 \right. \\ &\quad - 9.0 R \frac{11.5}{\left(\frac{\tau_0}{\rho}\right)^{1/2}} + 5.1 \frac{(11.5)^2 v^2}{\left(\frac{\tau_0}{\rho}\right)} \\ &\quad - 1.21 R^2 \log_{10} \frac{\sqrt{\frac{\tau_0}{\rho}}}{v} R + 2.88 R^2 \log_{10} \frac{\sqrt{\frac{\tau_0}{\rho}}}{v} R \\ &\quad \left. + 0.85 R^2 + 1.21 R^2 \log_{10} \frac{\sqrt{\frac{\tau_0}{\rho}}}{v} R \right\} \\ &= \frac{2}{R^2} \left\{ R^2 + \left[5.1 - \frac{11.5}{3} \right] \frac{(11.5)^2 v^2}{\left(\frac{\tau_0}{\rho}\right)} \right. \\ &\quad \left. + \left[\frac{11.5R}{2} - 9.0R \right] \frac{11.5 v}{\left(\frac{\tau_0}{\rho}\right)^{1/2}} + 2.88 R^2 \log_{10} \frac{\sqrt{\frac{\tau_0}{\rho}}}{v} R \right\} \end{aligned}$$

Substitution for \bar{c} by means of the formula

$$\frac{\bar{c}}{\sqrt{\frac{\tau_0}{\rho}}} = \frac{\sqrt{g}}{\sqrt{f}}$$

gives

$$\frac{1}{\sqrt{f}} = -0.80 + \frac{5810}{f(\text{Rey})^2} - \frac{150}{\sqrt{f} \text{Rey}} + 2 \log_{10} \sqrt{f} \text{Rey}$$

The equation of von Kármán-Nikuradse is

$$\frac{1}{\sqrt{f}} = -0.80 + 2 \log_{10} \sqrt{f} \text{Rey} \quad \text{See Appendix}$$

Rough Circular Pipes - In this case there is no pure laminar sublayer. Assuming, as for smooth pipes, that

$$\tau = \tau_0, \quad l = Ky \quad \text{for } y \leq 0.35R,$$

$$\frac{dc}{dy} \frac{1}{\sqrt{\frac{\tau_0}{\rho}}} = \frac{1}{Ky}$$

$$c = \frac{1}{K} \ln y + c$$

Since $c = 0$ when $y = y_0$,

$$\frac{c}{\sqrt{\frac{\tau_0}{\rho}}} = 5.75 \log_{10} \frac{y}{y_0}$$

which is customarily written as,

$$(I) \quad \frac{c}{\sqrt{\frac{\tau_0}{\rho}}} = 5.75 \log_{10} \frac{y+y_0}{y_0}$$

in order that $c = 0$ when $y = 0$.

As for smooth pipes ($y \geq 0.35 R$, $l = 0.14R$)

$$\frac{c-c_{0.35}}{\sqrt{\frac{\tau_0}{\rho}}} = 3.00 - 5.88 \left(\frac{R-y}{R}\right)^{3/2}$$

From (I),

$$\frac{c_{0.35R}}{\sqrt{\frac{\tau_0}{\rho}}} = 5.75 \log_{10} \frac{0.35R}{y_0}$$

substituting,

$$(II) \quad \frac{c}{\sqrt{\frac{\tau_0}{\rho}}} = 0.37 + 5.75 \log_{10} \frac{R}{y_0} - 5.88 \left(\frac{R-y}{R}\right)^{3/2}$$

The mean velocity is obtained by the integration of equations (I) and (II) over the cross section.

$$\frac{\bar{c}}{\sqrt{\tau_0 / \rho}} = 5.75 \log_{10} \frac{R}{y_0} - 3.56$$

$$\frac{1}{\sqrt{f}} = 2 \log_{10} \frac{R}{k} - 1.26 + 2 \log_{10} e$$

where $k = ey_0$

The best value of the effectivity for sanded surfaces, as determined from measured values of the friction factor substituted in the above equation, is 30. Hence, the equation is

$$\frac{1}{\sqrt{f}} = 2 \log_{10} \frac{R}{k} + 1.74 \quad \text{For sanded surfaces}$$

See Appendix

Infinite Parallel Smooth Plates.- The velocity equations are the same as those for the circular pipe, and are obtained in the same manner by making similar assumptions concerning the variation of the mixing length and the stress. In the pure laminar sublayer next to the plates, the velocity distribution can be obtained from the equation of Navier-Stokes in rectangular coordinates.

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = - \frac{\partial p}{\partial x} - \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Also,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{Equation of Continuity}$$

Let $u = f(y)$, where y is measured normal to a plane midway between the two plates. Let x be in the direction of flow.

Let $v = 0$

$w = 0$

Then $\frac{\partial u}{\partial x} = 0$ and the continuity is satisfied.

Hence,

$$\mu \frac{d^2 u}{dy^2} = \frac{dp}{dx}$$

But

$$\frac{d^2 p}{dx^2} = 0, \text{ since } u = f(y)$$

Hence $\frac{dp}{dx} = \text{a constant.}$

$$\frac{du}{dy} = \frac{dp}{dx} \frac{y}{2\mu} + c_1$$

$$u = \frac{dp}{dx} \frac{y^2}{2\mu} + e_1 y + e_2$$

The velocity is zero at the plate, and from symmetry considerations, the velocity is a maximum midway between the two plates. Putting

$$u = 0 \quad \text{when} \quad y = \pm \frac{h}{2}$$

$$\frac{du}{dy} = 0 \quad \text{when} \quad y = 0$$

the velocity equation becomes

$$u = \frac{dp}{dx} \frac{1}{2\mu} \left[y^2 - \left(\frac{h}{2} \right)^2 \right]$$

Therefore,

$$\frac{du}{dy} = \frac{dp}{dx} \frac{1}{\mu} y$$

$$\tau = \mu \frac{du}{dy} = \frac{dp}{dx} y$$

Assuming constant stress near the wall, $\tau = \tau_0$,

$$\tau_0 = \frac{dp}{dx} y$$

Hence,

$$u = \frac{\tau_0}{\mu} y + C$$

Putting $u = 0$ when $y = \pm \frac{h}{2}$

$$u = \frac{\tau_0}{\mu} \left(y - \frac{h}{2} \right) \quad \text{for } y \text{ positive}$$

$$u = \frac{\tau_0}{\mu} \left(y + \frac{h}{2} \right) \quad \text{for } y \text{ negative.}$$

Since τ_0 is negative for positive y , and positive for negative y ,

$$(I) \begin{cases} \frac{u}{\sqrt{\frac{\tau_0}{\rho}}} = \frac{\sqrt{\frac{\tau_0}{\rho}}}{\nu} \left(\frac{h}{2} - y\right) & \frac{h}{2} - y \leq \delta_1 \\ \frac{u}{\sqrt{\frac{\tau_0}{\rho}}} = \frac{\sqrt{\frac{\tau_0}{\rho}}}{\nu} \left(\frac{h}{2} + y\right) & \frac{h}{2} + y \leq \delta_1 \end{cases}$$

Theoretically, the stress varies as given by

$$\tau = -\tau_0 \frac{2y}{h}$$

However, assuming, as for circular pipes, that

$$\tau = \tau_0$$

$$1 = K \left(\frac{h}{2} - y\right) \text{ for } y \text{ positive} \quad \delta_1 \leq \frac{h}{2} - y \leq 0.175h$$

$$1 = K \left(\frac{h}{2} + y\right) \text{ for } y \text{ negative} \quad \delta_1 \leq \frac{h}{2} + y \leq 0.175h$$

In the range $(0 \leq y \leq \frac{h}{2})$ let $t = \frac{h}{2} - y$, and in the range

$(-\frac{h}{2} \leq y \leq 0)$ let $t = \frac{h}{2} + y$.

$$(II) \frac{u}{\sqrt{\frac{\tau_0}{\rho}}} = 5.4 + 5.75 \log_{10} \frac{\sqrt{\frac{\tau_0}{\rho}} t}{\nu}$$

Assuming $1 = 0.07 h$ for the range

$(-0.175h \leq y \leq 0.175h)$, and let the stress be assumed to vary as

$$\tau = \tau_0 \frac{2y}{0.65h}$$

Then,

$$(III) \frac{u}{\sqrt{\frac{\tau_0}{\rho}}} = 4.00 + 5.75 \log_{10} \frac{\sqrt{\frac{\tau_0}{\rho}}}{\nu} h - 16.65 \left(\frac{y}{h}\right)^{3/2}$$

The above development is almost unnecessary since the velocity is a function of only one variable as for the circular pipe. Therefore, it is obvious the velocity equations must be the same with y replacing r and $\frac{h}{2}$ replacing R .

The mean velocity is given by

$$\begin{aligned} \frac{\bar{u}}{\sqrt{\frac{\tau_0}{\rho}}} &= \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{u}{\sqrt{\frac{\tau_0}{\rho}}} dy = \frac{2}{h} \int_0^{\frac{h}{2}} \frac{u}{\sqrt{\frac{\tau_0}{\rho}}} dy \\ &= -0.74 + 5.74 \log_{10} \frac{\sqrt{\frac{\tau_0}{\rho}}}{\nu} h - 74.6 \frac{\nu}{\sqrt{\frac{\tau_0}{\rho}}} \end{aligned}$$

Let $\lambda = \frac{\tau_0}{\rho/2 \bar{u}^2} = \frac{dp}{dx} \left(\frac{h}{2}\right) \frac{1}{\rho/2 \bar{u}^2}$

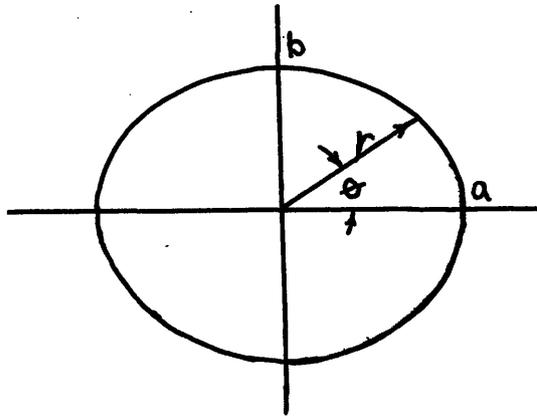
Then,

$$\frac{1}{\sqrt{\lambda}} = -1.13 + 4 \log_{10} \text{Rey} \sqrt{\lambda} - \frac{74.6}{\text{Rey} \sqrt{\lambda}}$$

Elliptical Pipes of Small Eccentricity.- The equation of the ellipse in polar coordinates with the pole at the center is

$$r^2 = \frac{b^2}{1 - e^2 \cos^2 \theta}$$

where e is the eccentricity.



For small values of the eccentricity, the velocity can be defined in terms of distance out from the wall along the radius vector. This is a good approximation for small eccentricity only. With this approximation the velocity equations will be of the same form as for the circular pipe, and continuity in velocity, stress, and mixing length will be satisfied at all points of the elliptical section.

The mean velocity is obtained by integration over the ellipse. Hence

$$\sqrt{\frac{\bar{c}}{\rho}} = \frac{1}{\pi ab} \int_0^{2\pi} \int_0^r \sqrt{\frac{c}{\rho}} r dr d\theta$$

The integration with respect to r does not differ from that over the circular pipe section. Therefore,

$$\begin{aligned} \frac{\bar{c}}{\sqrt{\frac{\tau_0}{\rho}}} &= \frac{1}{\pi ab} \int_0^{2\pi} \left\{ r^2 + 168 \frac{v^2}{\left(\frac{\tau_0}{\rho}\right)} - 37.4 \frac{v}{\left(\frac{\tau_0}{\rho}\right)^{1/2}} r^2 + \right. \\ &\quad \left. 2.88 r^2 \log_{10} \frac{\sqrt{\frac{\tau_0}{\rho}}}{v} r \right\} d\theta \\ &= \frac{2(168)}{ab} \frac{v^2}{\left(\frac{\tau_0}{\rho}\right)} - \frac{37.4}{\pi ab} \frac{v}{\left(\frac{\tau_0}{\rho}\right)^{1/2}} \int_0^{2\pi} r d\theta \\ &\quad + \frac{1}{\pi ab} \int_0^{2\pi} r^2 d\theta + \frac{1.25}{\pi ab} \int_0^{2\pi} r^2 \ln r d\theta \\ &\quad + \frac{2.88}{\pi ab} \log_{10} \frac{\sqrt{\frac{\tau_0}{\rho}}}{v} \int_0^{2\pi} r^2 d\theta \end{aligned}$$

where

$$\int_0^{2\pi} r d\theta = 4b \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - e^2 \sin^2 \theta}} = 4b F(e, \frac{\pi}{2})$$

and

$$\int_0^{2\pi} r^2 d\theta = b^2 \int_0^{2\pi} \frac{d\theta}{1 - e^2 \cos^2 \theta} = \frac{2\pi b^2}{\sqrt{1 - e^2}}$$

Also,

$$\begin{aligned} \int_0^{2\pi} r^2 \ln r d\theta &= \frac{b^2}{2} \int_0^{2\pi} \frac{1}{1 - e^2 \cos^2 \theta} \ln \frac{b^2}{1 - e^2 \cos^2 \theta} d\theta \\ &= \frac{\pi b^2 \ln b^2}{\sqrt{1 - e^2}} - \frac{b^2}{2} \int_0^{2\pi} \frac{\ln(1 - e^2 \cos^2 \theta)}{1 - e^2 \cos^2 \theta} d\theta \end{aligned}$$

But

$$\int_0^{2\pi} \frac{\ln(1-e^2 \cos^2 \theta)}{1-e^2 \cos^2 \theta} d\theta = 4 \int_0^{\pi/2} \frac{\ln(1-e^2 \sin^2 \theta)}{1-e^2 \sin^2 \theta} d\theta$$

This integral can be expressed in terms of $\frac{\partial}{\partial n} p_n(z)$ by transforming Laplace's first integral for the Legendre function $p_n(z)$.

$$p_n(z) = \frac{1}{\pi} \int_0^{\pi} (z + \sqrt{z^2 - 1} \cos \theta)^n d\theta$$

Differentiating with respect to n ,

$$\frac{\partial}{\partial n} p_n(z) = \frac{1}{\pi} \int_0^{\pi} (z + \sqrt{z^2 - 1} \cos \theta)^n \ln(z + \sqrt{z^2 - 1} \cos \theta) d\theta$$

since the resulting integrand is continuous in both θ and n .

Then letting $\theta = 2\theta'$

$$\frac{\partial}{\partial n} p_n(z) = \frac{2}{\pi} \int_0^{\pi/2} (z + \sqrt{z^2 - 1} \cos 2\theta')^n \ln(z + \sqrt{z^2 - 1} \cos 2\theta') d\theta'$$

Substituting $\cos 2\theta = 1 - 2 \sin^2 \theta$

$$\begin{aligned} \frac{\partial}{\partial n} p_n(z) &= \frac{2}{\pi} (z + \sqrt{z^2 - 1}) \int_0^{\pi/2} \frac{(1 - 2\sqrt{z^2 - 1} \sin^2 \theta)^n}{z + \sqrt{z^2 - 1}} \ln(1 - \frac{2\sqrt{z^2 - 1}}{z + \sqrt{z^2 - 1}} \sin^2 \theta) d\theta \\ &\quad + \frac{2}{\pi} (z + \sqrt{z^2 - 1}) \int_0^{\pi/2} (1 - \frac{2\sqrt{z^2 - 1}}{z + \sqrt{z^2 - 1}} \sin^2 \theta)^n \ln(z + \sqrt{z^2 - 1}) d\theta \end{aligned}$$

Let

$$\frac{2\sqrt{z^2-1}}{z + \sqrt{z^2-1}} = e^2$$

Then,

$$z = \frac{1}{2} \frac{2-e^2}{\sqrt{1-e^2}}$$

$$z + \sqrt{z^2-1} = \frac{1}{\sqrt{1-e^2}}$$

Therefore,

$$\frac{d}{dn} p_n \left(\frac{1}{2} \frac{2-e^2}{\sqrt{1-e^2}} \right) = \frac{2}{\pi\sqrt{1-e^2}} \int_0^{\pi/2} (1-e^2 \sin^2 \theta)^n \ln(1-e^2 \sin^2 \theta) d\theta$$

$$- \frac{2 \ln \sqrt{1-e^2}}{\pi\sqrt{1-e^2}} \int_0^{\pi/2} (1-e^2 \sin^2 \theta)^n d\theta$$

or

$$\int_0^{\pi/2} (1-e^2 \sin^2 \theta)^n \ln(1-e^2 \sin^2 \theta) d\theta =$$

$$\frac{\pi\sqrt{1-e^2}}{2} \frac{d}{dn} p_n \left(\frac{1}{2} \frac{2-e^2}{\sqrt{1-e^2}} \right)$$

$$+ \ln \sqrt{1-e^2} \int_0^{\pi/2} (1-e^2 \sin^2 \theta)^n d\theta$$

Putting $n = -1$

$$\int_0^{\pi/2} \frac{\ln(1-e^2 \sin^2 \theta) d\theta}{(1-e^2 \sin^2 \theta)} = \frac{\pi\sqrt{1-e^2}}{2} \left[\frac{d}{dn} p_n \left(\frac{1}{2} \frac{2-e^2}{\sqrt{1-e^2}} \right) \right]_{n=-1}$$

$$+ \ln \sqrt{1-e^2} \int_0^{\pi/2} \frac{d\theta}{1-e^2 \sin^2 \theta}$$

In Murphy's expression of $p_n(z)$ as a hypergeometric function,

$$p_n(z) = \sum_{r=0}^{\infty} \frac{(n+1)(n+2)\dots(n+r) \cdot (-n)(1-n)\dots(r-1-n)}{(r!)^2} \left(\frac{1}{2} - \frac{1}{2}z\right)^r$$

$$= F(n+1, -n; 1; \frac{1}{2} - \frac{1}{2}z)$$

$$\text{when } |1-z| \leq 2(1-\delta) < 2, \quad 0 < \delta < 1$$

$$\text{let } w = \left(\frac{1}{2} - \frac{1}{2}z\right)$$

Then

$$P_n(z) = \sum_{r=0}^{\infty} \frac{(n+1)(n+2)\dots(n+r) \cdot (-n)(1-n)\dots(r-1-n)}{(r!)^2} w^r$$

$$\left[\frac{\partial}{\partial n} p_n(z) \right]_{n=-1} = \sum_{r=0}^{\infty} \frac{(n+2)\dots(n+r) \cdot (-n)(1-n)\dots(r-1-n)}{(r!)^2} w^r \Big|_{n=-1}$$

since the terms containing the factor $(n+1)$ vanish.

$$\left[\frac{\partial}{\partial n} p_n(z) \right]_{n=-1} = \sum_{r=0}^{\infty} \frac{1 \cdot 2 \dots (r-1) \cdot (1)(2) \dots (r)}{r!r!} w^r$$

$$= \sum_{r=0}^{\infty} \frac{1}{r} w^r$$

$$= -\ln(1-w) = -\ln \frac{1}{2} (1+z)$$

$$= -\ln \frac{1}{2} \left\{ 1 + \frac{1}{2} \frac{2-e^2}{\sqrt{1-e^2}} \right\}$$

Hence,

$$\int_0^{\pi/2} \frac{\ln(1-e^2 \sin^2 \theta)}{(1-e^2 \sin^2 \theta)} d\theta = \frac{\pi}{2} \frac{\ln \sqrt{1-e^2}}{\sqrt{1-e^2}} - \frac{\pi}{2} \sqrt{1-e^2} \ln \frac{1}{2} \left\{ 1 + \frac{1}{2} \frac{2-e^2}{\sqrt{1-e^2}} \right\}$$

Therefore, the expression for the mean velocity

is

$$\frac{\bar{c}}{\sqrt{\frac{\tau_0}{\rho}}} = \frac{336}{ab} \frac{v^2}{\left(\frac{\tau_0}{\rho}\right)} - \frac{150}{\pi a} \frac{v}{\left(\frac{\tau_0}{\rho}\right)^{1/2}} F\left(e, \frac{\pi}{2}\right) + \frac{2b}{a} \frac{1}{\sqrt{1-e^2}} + \frac{2.88b \left\{ (1-e^2) \ln \frac{1}{2} \left(1 + \frac{1}{2} \frac{2-e^2}{\sqrt{1-e^2}}\right) - \ln \sqrt{1-e^2} \right\}}{a \sqrt{1-e^2}} + \frac{5.75 b}{a \sqrt{1-e^2}} \log_{10} \frac{\sqrt{\frac{\tau_0}{\rho}}}{v} b$$

Letting

$$\frac{\bar{c}}{\sqrt{\frac{\tau_0}{\rho}}} = \frac{\sqrt{8}}{\sqrt{f}}, \text{ then}$$

$$\frac{1}{\sqrt{f}} = -0.80 \frac{b}{a \sqrt{1-e^2}} + \frac{3810}{f(\text{Rey})^2} \frac{b}{a} - \frac{150}{\sqrt{f} \text{Rey}} \left(\frac{b}{a}\right) \left(\frac{2}{\pi}\right) F\left(e, \frac{\pi}{2}\right) + 2 \frac{b}{a \sqrt{1-e^2}} \log_{10} \sqrt{f} \text{Rey} + \frac{b}{a \sqrt{1-e^2}} \left\{ (1-e^2) \ln \frac{1}{2} \left(1 + \frac{1}{2} \frac{2-e^2}{\sqrt{1-e^2}}\right) - \ln \sqrt{1-e^2} \right\}$$

$$\text{where } \text{Rey} = \frac{2 b \bar{u}}{v}$$

$$\text{Since } e^2 = \frac{a^2 - b^2}{a^2}$$

$$\frac{1}{\sqrt{f}} = 1 - 0.80 + \frac{3810}{f(\text{Rey})^2} \frac{b}{a} - \frac{150}{\sqrt{f} \text{Rey}} \left(\frac{b}{a}\right) \left(\frac{2}{\pi}\right) F\left(e, \frac{\pi}{2}\right)$$

(over)

$$+ 2 \log_{10} \sqrt{f} \operatorname{Re} y$$

$$+ 2.3 \left(\frac{b}{a}\right)^2 \log_{10} \frac{1}{2} \left\{ 1 + \frac{1}{2} \left(\frac{a}{b} + \frac{b}{a} \right) \right\} - 2.3 \log_{10} \frac{b}{a}$$

For these cases the above method seems to give satisfactory results. The velocity gradient is zero at the center line, as it obviously should be from symmetry considerations. In particular, the equation defining the friction factor for the circular pipe obtained by this method agrees very closely with the experimental data and with the equation of von Kármán-Nikuradse. Except for small values of the Reynolds Number where the flow is laminar, the difference between the two equations is negligible (see graph in Appendix).

The friction factor for the elliptical pipe is smaller than that for the circular pipe at the same Reynolds Number as is seen from a comparison of the two equations. This is to be expected since the minor axis $2b$ of the ellipse was used in defining the Reynolds Number. The equation for the elliptical pipe becomes the equation for the circular pipe, as it obviously must, as the eccentricity approaches zero since $F(e, \frac{\pi}{2})$ approaches $\frac{\pi}{2}$.

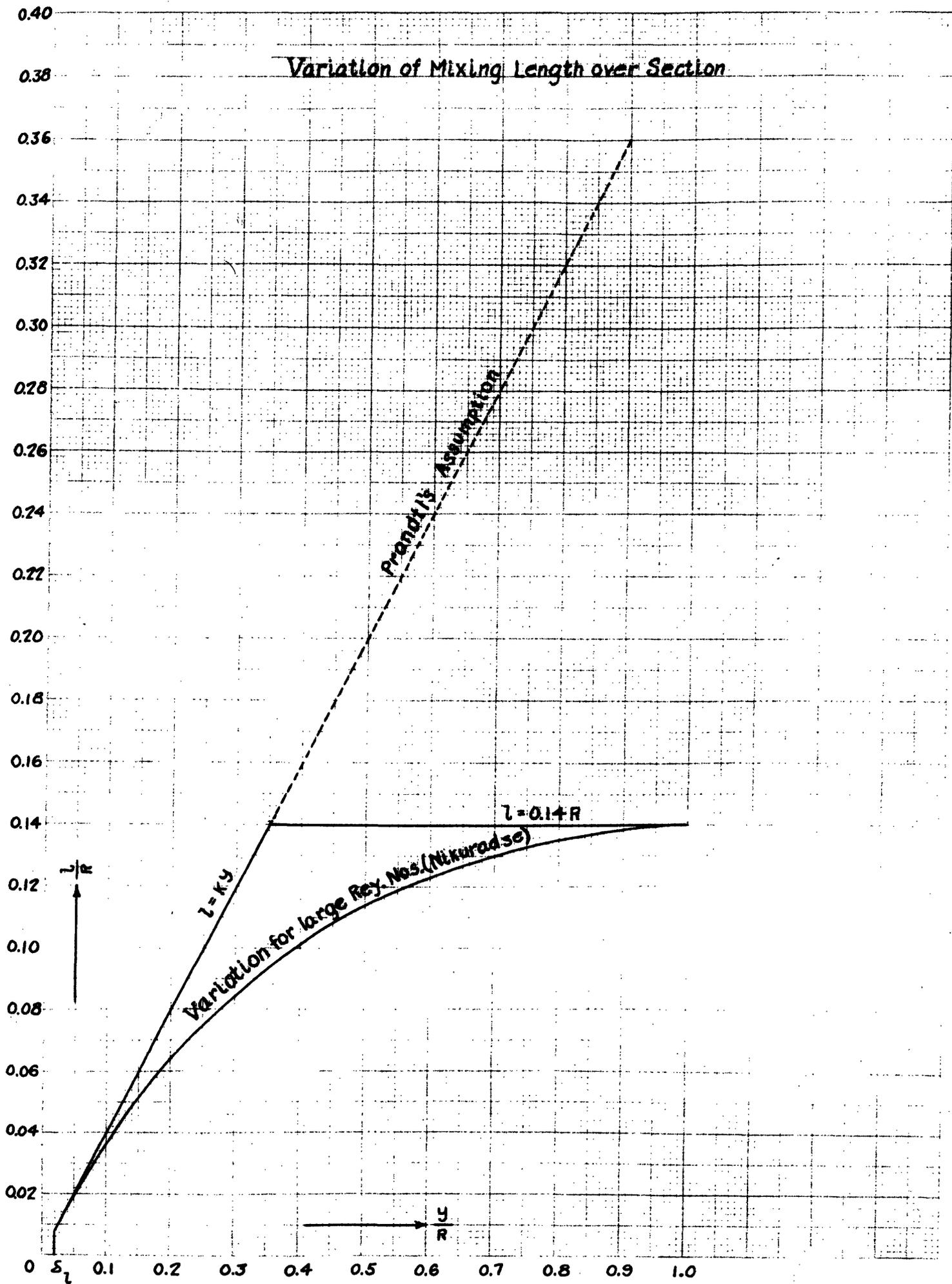
It may be concluded from the good agreement of the equation for the circular pipe with experimental data and with the equation of von Kármán-Nikuradse that Peters and Rossby's assumption of a maximum value for the mixing length is justified. Further, it constitutes a definite improvement in the method of calculating friction factors. The velocity gradient is now zero and the mixing length finite at the center line, thus bringing the equations into better

agreement with the physical facts. The consideration of the parallel flat plates and of the elliptical pipe indicates that the method might be extended rather generally to problems in which the velocity may be defined as above and in which the stress is unidirectional. The existence of secondary flows, as, for example, in the corners of the rectangular duct, would require a more general method of solution, but would in no way affect the validity of the original assumption.

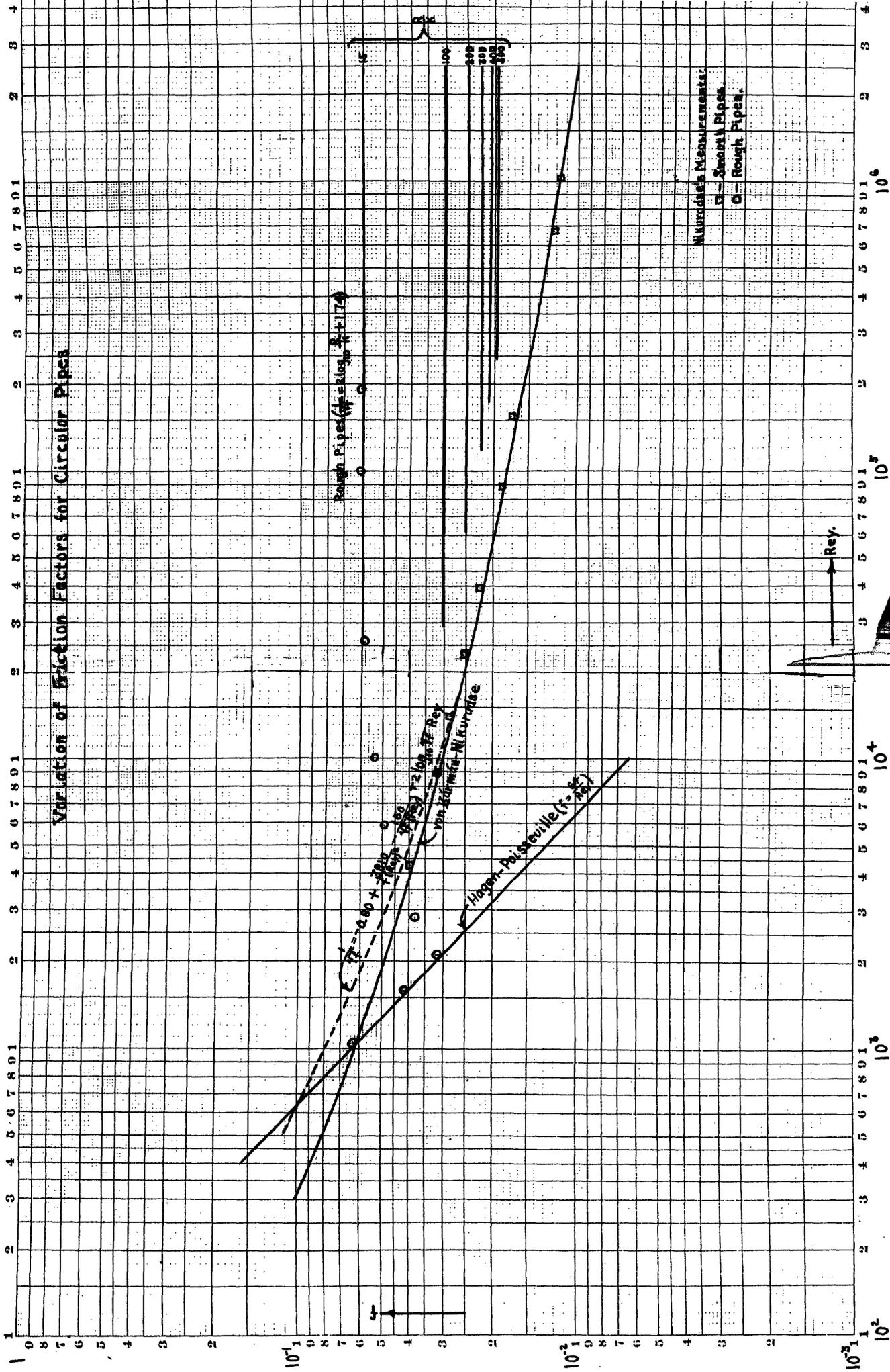
This method might be extended further to the flat plate. But here, although much is known concerning the frictional drag, little is known concerning the velocity distribution. It is therefore improbable that much additional information concerning the validity of this assumption would be obtained from a consideration of this problem.

APPENDIX

Variation of Mixing Length over Section



Variation of Friction Factors for Circular Pipes



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