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AN APPLICATION OF THE BAKRY-EMERY CRITERION TO INFINITE DIMENSIONAL DIFFUSIONS

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An Application of the Bakry–Emery Criterion to Infinite Dimensional Diffusions

by

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The note [1] by Bakry and Emery contains an important criterion with which to check whether a diffusion semigroup is hypercontractive. Although Bakry and Emery's interest in their criterion stems from its remarkable ability to give "best constants" in certain finite dimensional examples, what will concern us here is its equally remarkable ability to handle some infinite dimensional situations.

We begin by recalling their criterion in the setting with which we will be dealing. Let M be a connected, compact, N–dimensional smooth manifold with Riemannian metric g. Let Φ be a smooth function on M and define the differential operator L by

$$Lf = 1/2\exp(\Phi)\mathrm{div}(\exp(-\Phi)\mathrm{grad}(f)),\ f \in C^{\infty}(M),$$

and the probability measure m by

$$m(dx) = \exp(-\Phi(x))\lambda(dx)/\int\exp(-\Phi(y))\lambda(dy),$$

where λ denotes the Riemann measure on M associated with the metric g. Next, use $\{P_t: t > 0\}$ to denote the diffusion semigroup determined by L. The following facts about $\{P_t: t > 0\}$ are easy to check:

i) $\{P_t: t > 0\}$ on C(M) is a strongly continuous, conservative Markov semigroup under which $C^{\infty}(M)$ is invariant.

ii) $\{P_t: t > 0\}$ is m–reversible (i.e. P_t is symmetric in $L^2(m)$ for each $t > 0$) and $\| P_t f - \int f dm \|_{C(M)} \longrightarrow 0$ as $t \longrightarrow \infty$ for each $f \in C(M)$. In particular, for all $t > 0$ and $p \in [0,\infty)$,

$$\| P_t \|_{L^p(m) \longrightarrow L^p(m)} = 1$$

and there is a unique strongly continuous semigroup $\{\overline{P}_t: t > 0\}$ on $L^2(m)$ such that $\overline{P}_t f = P_t f$ for all $t > 0$ and $f \in C(M)$.

As a consequence, note that, for each $f \in L^2(m)$, $t \longrightarrow (f - P_t f, f)_{L^2(m)}/t$ is a non–negative, non–increasing function and that, therefore, the <u>Dirichlet form</u> given by

$$\mathcal{E}(f,f) = \lim_{t\searrow 0} (f - P_t f, f)_{L^2(m)}/t$$

exists (as an element of $[0,\infty]$) for each $f \in L^2(m)$.

Theorem (Bakry and Emery): Denote by H_Φ the (covariant) Hessian tensor of Φ (i.e. $H_\Phi(X,Y) = X\cdot Y\Phi - \nabla_X Y\Phi$ for $X,Y \in \Gamma(T(M))$) and let Ric be the Ricci curvature on (M,g). If, as quadratic forms, $\mathrm{Ric} + H_\Phi \geq \alpha g$ for some $\alpha > 0$, then the logarithmic Sobolev inequality :

$$\text{(L.S.)} \qquad \int f^2 \log f^2 dm \leq 4/\alpha\, \mathcal{E}(f,f) + \| f \|^2_{L^2(m)} \log \| f \|^2_{L^2(m)}, \quad f \in L^2(m)$$

and, therefore, the hypercontractive estimate :

$$\text{(H.C.)} \qquad \| P_t \|_{L^p(m) \longrightarrow L^q(m)} = 1, \quad 1 < p < q < \infty \text{ and } t > 0 \text{ with } e^{\alpha t} \geq (q-1)/(p-1)$$

hold.

Remark : Actually, Bakry and Emery's result is somewhat more refined than the one just stated. However, the refinement seems to become less and less significant as N becomes large. Since we are interested here in what happens as $N \rightarrow \infty$, the stated result will suffice.

Remark : Several authors (e.g. O. Rathaus [4]) have observed that a logarithmic Sobolev inequality implies a gap in the spectrum of L . To be precise, (L.S.) implies that:

$$\text{(S.G.)} \qquad \| f - \int f dm \|_{L^2(m)} \leq 2/\alpha\, \mathcal{E}(f,f), \qquad f \in L^2(m),$$

or, equivalently,

$$\text{(S.G.')} \qquad \| \overline{P}_t f - \int f dm \|_{L^2(m)} \leq \exp(-2\alpha/t) \| f \|_{L^2(m)}, \quad f \in L^2(m).$$

We now turn to the application of the Bakry-Emery result to infinite dimensional diffusions. For the sake of definiteness, let $d \geq 2$ and $\nu \geq 1$ be given, and, for $n \geq 1$, set

$$M_n = (S^d)^{\Lambda_n},$$

where $\Lambda_n = \{k \in Z^\nu : |k| \equiv \max_{1 \leq i \leq \nu} |k_i| \leq n\}$, and give M_n the product Riemannian structure which it inherits from the standard structure on S^d. Let π_k be the natural projection map from M_n onto the k^{th} sphere S^d, and, for $X \in \Gamma(T(M_n))$, set $X^{(k)} = (\pi_k)_* X$. Noting, as was done in [1], that on S^d the Ricci curvature is equal to $(d-1)$ times the metric, we see that the

Ricci curvature Ric_n and the metric g_n on M_n satisfy the same relationship. Finally, let $\Phi_n \in C_\infty(M_n)$ be given and define the operator L_n, the measure m_n, the semigroup $\{P^n_t : t > 0\}$, and the Dirichlet form $\mathcal{E}_n$ accordingly. As an essentially immediate consequence the the Bakry–Emery theorem, we have the following.

<u>Theorem</u> : Assume that for all X $(T(M_n))$:

$$|H_{\Phi_n}(X,X)| \leq \sum_{k,\ell\in\Lambda_n} \gamma(k - \ )\,\|X^{(k)}\|\ \|X^{(\ell)}\|$$

where $\gamma : Z^\nu \longrightarrow [0,\infty)$ satisfies

$$\sum_{k\in Z^\nu} \gamma(k) \leq (1 - \varepsilon)(d - 1)$$

for some $0 < \varepsilon < 1$. Set $\alpha = \varepsilon(d - 1)$. Then:

$(L.S.)_n$ $\quad \int f^2 \log f^2 dm_n \leq (4/\alpha)\,\mathcal{E}_n(f,f) + \|f\|^2_{L^2(m_n)} \log\|f\|^2_{L^2(m_n)}$,

for $f \in L^2(m_n)$ and

$(H.C.)_n$
$$\|P^n_t\|_{L^p(m_n)\longrightarrow L^q(m_n)} = 1,$$
$1 < p < q < \infty$ and $t > 0$ with $\exp(\alpha t) \geq (q - 1)/(p - 1)$.

In particular,

$(S.G.)_n$ $\quad \|f - \int f dm_n\|_{L^2(m_n)} \leq (2/\alpha)\,\mathcal{E}_n(f,f), \qquad f \in L^2(m_n)$,

and

$(S.G.')_n$ $\quad \|P^n_t f - \int f dm_n\|_{L^2(m_n)} \leq \exp(-\alpha t/2)\|f\|_{L^2(m_n)}, \quad f \in L^2(m_n)$.

<u>Proof</u> : Simply observe that, by Young's inequality, the bound on H_{Φ_n} (as a quadratic form) in terms of g can be dominated by $\|\gamma\|_{\ell^1(Z^\nu)}$.

To complete our program, set $M_\infty = (S^d)^{Z^\nu}$, $\mathcal{F} = \{F \subseteq Z^\nu : \text{card}(F) < \infty\}$, and, for $F \in \mathcal{F}$, denote be π_F the natural projection of M_∞ onto $(S^d)^F$. (Thus, in the notation used before, $\pi_k = \pi_{\{k\}}$ and $M_n = (S^d)^{\Lambda_n}$.) Next, set $\mathfrak{D}_F = \{f \circ \pi_F : f \in C^\infty((S^d)^F)\}$, $\mathfrak{D} = \bigcup\{\mathfrak{D}_F : F \in \mathcal{F}\}$, and let $\Gamma(T(M_\infty))$ be the set of derivations from $\mathfrak{D}$ into itself. We now suppose that we are given a <u>potential</u> $\mathcal{J} = \{J_F : F \in \mathcal{F}\}$, where:

i) for each $F \in \mathcal{F}$, $J_F \in \mathcal{D}_F$, and for each $k \in Z^\nu$ there are only a finite number of $F \ni k$ for which J_F is not identically zero.

ii) there is a constant $B < \infty$ such that

$$\sum_{F\ k} |X^{(k)} J_F| \leq B \, \| X^{(k)} \|, \quad k \in Z^\nu \text{ and } X \in \Gamma(T(M_\infty)),$$

iii) there is a $\gamma : Z^\nu \to [0,\infty)$ such that

$$\sum_{k \in Z^\nu} \gamma(k) < \infty$$

and

$$\sum_{F \supseteq \{k,\ell\}} |H_{J_F}(X^{(k)}, X^{(\ell)})| \leq \gamma(k-\ell) \, \| X^{(k)} \| \| X^{(\ell)} \|$$

for all $k,\ell \in Z^\nu$ and $X \in \Gamma(T(M_\infty))$.

Set $H_k = \sum_{F \ni k} J_F$ and define L_∞ on $\mathcal{D}$ by

$$L_\infty f = 1/2 \sum_{k \in Z^\nu} \exp(H_k) \mathrm{div}_k (\exp(-H_k) \mathrm{grad}_k f)$$

where "div_k" and "grad_k" refer to the corresponding operations in the directions of the k^{th} sphere.

In order to describe the measure m_∞, we will need to introduce the concept of a Gibbs state and this, in turn, requires us to develop a little more notation. For $n \geq 1$ and $x,y \in M_\infty$, define $Q_n(x|y) \in M_\infty$ by

$$Q_n(x|y)_k = \begin{cases} x_k & \text{if } k \in \Lambda_n \\ y_k & \text{if } k \notin \Lambda_n. \end{cases}$$

(It will be convenient, and should cause no confusion, for us to sometimes consider $x \to Q_n(x|y)$, for fixed $y \in M_\infty$, as a function on M_n and $y \to Q_n(x|y)$, for fixed $x \in M_\infty$, as a function on $(S^d)^{\Lambda_n^c}$.) Define

$$\Phi_n(x|y) = \sum_{F \cap \Lambda_n \neq \emptyset} J_F \circ Q_n(x|y)$$

and let $m_n(\cdot|y)$ denote the probability measure on M_n associated with $\Phi_n(\cdot|y)$.

We will say that a probability measure m_∞ on M_∞ is a <u>Gibbs state with potential</u> Υ and will write $m_\infty \in \mathcal{G}(\Upsilon)$ if, for each $n \geq 1$, $y \to m_n(\cdot|y)$ is a regular conditional probability distribution of m_∞ given $\sigma(x_k: k \in \Lambda^c_n)$.

The following lemma summarizes some of the reasonably familiar facts about the sort of situation described above (cf. [2] and [3]).

<u>Lemma</u> : There is exactly one conservative Markov semigroup $\{P^\infty_t: t > 0\}$ on $C(M_\infty)$ such that

$$P^\infty_T f - f = \int_0^T P^\infty_t L_\infty f dt, \quad f \in \mathcal{D}.$$

Moreover, if, for each $n \geq 1$, $\Phi_n \in C^\infty(M_n)$ and the associated operator L_n are given, and if $[L_n(f \circ Q_n(\cdot|y))](x) \to L_\infty f(x)$ uniformly in $x, y \in M_\infty$ for every $f \in \mathcal{D}$, then the associated semigroups $\{P^n_t: t > 0\}$ have the property that

$$[P^n_t f \circ Q_n(\cdot|y)](x) \to P^\infty_t f(x)$$

uniformly in $(t,x,y) \in [0,T] x M_\infty x M_\infty$ for every $T > 0$ and $f \in C(M_\infty)$. Finally: $\mathcal{G}(\Upsilon)$ is a non-empty, compact, convex set; $m_\infty \in \mathcal{G}(\Upsilon)$ if and only if it is a $\{P^\infty_t: t > 0\}$-reversible measure; for each $m_\infty \in \mathcal{G}(\Upsilon)$ there is a $y \in M_\infty$ such that $m_n(\cdot|y) \to m_\infty$; and m_∞ is an extreme point of $\mathcal{G}(\Upsilon)$ if and only if $P^\infty_t f \to \int f dm_\infty$ in $L^2(m_\infty)$ for each $f \in C(M_\infty)$.

<u>Theorem</u> : Referring to the situation described above, assume that

$$\sum_{k \in Z^\nu} \gamma(k) \leq (1 - \varepsilon)(d - 1)$$

for some $0 < \varepsilon < 1$. Then $\mathcal{G}(\Upsilon)$ contains precisely one element m_∞, and if $\mathcal{E}_\infty$ is the Dirichlet form determined by $\{P^\infty_t: t > 0\}$ on $L^2(m_\infty)$, then, for $f \in L^2(m_\infty)$:

$$(\text{L.S.})_\infty \quad \int f^2 \log f^2 dm_\infty \leq 4/\alpha\, \mathcal{E}_\infty(f,f) + \| f \|^2_{L^2(m_\infty)} \log \| f \|^2_{L^2(m_\infty)},$$

where $\alpha = \varepsilon(d - 1)$. In particular,

$$(\text{H.C.})_\infty \quad \| P^\infty_t \|_{L^p(m_\infty) \to L^q(m_\infty)} = 1,$$

$$1 < p < q < \infty \text{ and } t \geq 0 \text{ with } e^{\alpha t} \geq (q - 1)/(p - 1),$$

$$(S.G.)_\infty \qquad \| f - \int f\,dm_\infty \|_{L^2(m_\infty)} \leqslant 2/\alpha\ \mathcal{E}_\infty(f,f), \quad f \in L^2(m_\infty),$$

and

$$(S.G.')_\infty \qquad \| \overline{P^\infty_t} f - \int f\,dm_\infty \|_{L^2(m_\infty)} \leqslant \exp(-\alpha t/2) \| f \|_{L^2(m_\infty)}, \quad f \in L^2(m_\infty).$$

<u>Proof</u> : Let $m_\infty \in \mathfrak{G}(\mathcal{T})$ be given and choose and fix $y \in M_\infty$ so that $m_n \equiv m_n(\cdot|y) \to m_\infty$. Set $\Phi_n = \Phi_n(\cdot|y)$ and define L_n and $\{P^n_t : t > 0\}$ accordingly. It is easy to check that the hypotheses of the previous theorem are satisfied for each n. In particular, $(H.C.)_n$ holds for all $n \geqslant 1$. Moreover, the preceding lemma allows us to conclude that

$$\| P^\infty_t \|_{L^p(m_\infty) \to L^q(m_\infty)}$$

$$\leqslant \varliminf_{n\to\infty} \| P^n_t \|_{L^p(m_n) \to L^q(m_n)}$$

for all $1 < p < q < \infty$ and $t > 0$. Hence, we now know that $(H.C.)_\infty$ holds. Since $(L.S.)_\infty$, $(S.G.)_\infty$, and $(S.G.')_\infty$ all follow from $(H.C.)_\infty$, it remains only to check the uniqueness of m_∞. But, by the lemma, if there were two or more elements of $\mathfrak{G}(\mathcal{T})$, then there would be one for which $(S.G.')_\infty$ would not hold. Since we have just seen that $(S.G.')_\infty$ holds for every element of $\mathfrak{G}(\mathcal{T})$, the proof is complete.

<u>Remark</u> : For those who are uninitiated into the mysteries of Gibbs states and related infinite dimensional analysis, it may not be immediately apparent just what the preceding theorem accomplishes. The point is that, as opposed to the finite dimensional counterparts, not even qualitative versions of the assertions made in it are obvious when one is dealing with infinite dimensional situations. In particular, when $\nu \neq 1$, it will not be true, for general potentials $\mathcal{T}$, that $\mathfrak{G}(\mathcal{T})$ contains only one element or that a gap in the spectrum of L_∞ will exist. Indeed, these properties are reasonably hard to prove by any procedure, and so it is pleasing that they come, in the situation just treated, as a dividend of the Bakry–Emery criterion.

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