October 1985

LIDS-P-1504

BEYOND SINGULAR VALUES AND LOOP SHAPES

Gunter Stein

Honeywell Systems and Research Center and Massachusetts Institute of Technology

ABSTRACT

This paper reviews the status of singular value loop-shaping as a design paradigm for multivariable feedback systems. It shows that this paradigm is an effective design tool whenever the problem specifications are 'spacially round'. The tool can be arbitrarily conservative, however, when they are not. This happens because singular value conditions for robust performance are not tight (necessary and sufficient) and can severely overstate actual requirements. An alternate paradigm is discussed which overcomes these limitations. The alternative includes a more general problem formulation, a new matrix function μ, and tight conditions for both robust stability and robust performance. The state of the art currently permits analysis of feedback systems within this new paradigm. Synthesis remains a subject of research.

August 30, 1985

BEYOND SINGULAR VALUES AND LOOP SHAPES

Gunter Stein

Honeywell Systems and Research Center and Massachusetts Institute of Technology

1. INTRODUCTION

Ever since the basic work of Nyquist [1], Bode [2] and others, the classical approach to feedback design has followed a frequency domain perspective. We are given a plant described by a rational transfer function, $G(s)$, and wish to design a rational compensator, $K(s)$, such that the closed loop feedback system is stable and meets certain performance and robustness requirements.

As is well known, the stability requirement imposes structural constraints on certain transfer functions of the closed loop system, e.g. a Nyquist encirclement count for the function $det(I+GK)$ [3]. Likewise, the performance requirement imposes magnitude constraints on certain other transfer functions. In particular, for disturbances and commands reflected to the output loop-breaking point, the (output) sensitivity function

$$S(s) \triangleq [I + G(s)K(s)]^{-1} \tag{1}$$

must be small for all frequencies, $s=j\omega$, where the disturbances and/or reference commands are large.

The third feedback design requirement -- tolerance for uncertainty -- also imposes magnitude constraints on transfer functions. A common requirement in this case is that the "complementary (output) sensitivity function"

$$T(s) \triangleq G(s)K(s)[I+G(s)K(s)]^{-1} \tag{2}$$

must be small for all frequencies where so-called unstructured multiplicative model uncertainties are large [4].

For classical single-input single-output (SISO) systems, the meanings of "small" and "large" in these statements are, of course, understood in terms of the absolute values of the respective complex-valued functions at each frequency. Hence, SISO designers working in the frequency domain have viewed the design problem as one of shaping the (Bode) magnitude plots of sensitivity and complementary sensitivity functions to be small enough to meet design specifications. Indeed, because $|S(s)| \approx 1/|GK(s)|$ whenever $|GK(s)|$ is large, and $|T(s)| \approx |GK(s)|$ whenever $|GK(s)|$ is small, the shapes of these magnitude functions are intimately tied to the shape of the loop transfer function $GK(s)$, and the entire design process is often referred to simply as "loop-shaping".

Over the last few years, the loop-shaping process has been successfully generalized to multi-input multi-output (MIMO) design problems [4]. Key ingredients of the generalization include -- 1) the use of singular values as appropriate measures of magnitude for matrix-valued transfer functions, 2) the

This work was supported in part by Honeywell Internal IR&D, and by the NASA Ames and Langley Research Centers under Grant NAG-2-297.

development of formal mathematical conditions which guarantee stability robustness and performance robustness of MIMO feedback systems in terms of these magnitude measures, and 3) the development of certain modifications of existing design procedures (e.g. LQG/LTR) which help to synthesize desired multivariable loop shapes.

Design experience with these new results shows that loop-shaping is an effective MIMO design paradigm for problems whose specifications can be reduced to "spacially round" requirements on $S(s)$ and $T(s)$ alone. Unfortunately, many design issues which arise in MIMO problems cannot be usefully expressed in this form. This paper describes some of these latter design issues and develops a design framework and certain recently developed tools which promise to deal with them more effectively.

The paper begins in Section 2 with a brief review of the MIMO loop-shaping process. It then discusses some of the issues which are not easily handled by this process in Section 3. Finally, a more general design framework and associated research results are presented in Section 4 which promise to address these issues more effectively. Section 5 provides concluding comments. The paper presents no basic new theoretical results and should be viewed only as a brief look at the current status of this branch of frequency domain MIMO design.

2. THE MIMO LOOP-SHAPING GENERALIZATION

Our generic multivariable feedback design problem is illustrated in Figure 1. The loop consists of a plant and a compensator in a unity feedback arrangement. The plant can be any element from a set of plants characterized by a nominal operator, **G**, and a perturbation operator, **δG**. These operators are modelled by rational transfer function matrices $G(s)$ and $\delta G(s)$, respectively. Similarly, the compensator, **K**, is modelled by $K(s)$. The design problem is to find a $K(s)$ which makes the feedback loop internally stable for all possible plants $G(s)+\delta G(s)$ and causes it to respond well to external signals such as commands, $r(s)$, disturbances, $d(s)$, and sensor noise, $n(s)$.*

2.1. A Formal Design Problem

This MIMO design problem can be formalized by specifying precise mathematical statements for the qualitative performance objectives above, and by specifying a set of external signals, $r(s)$, $d(s)$ and $n(s)$, and a set of plants, $G(s)+\delta G(s)$, over which these objectives must be achieved. We will start with some very simple specifications for these three elements of the design problem. More complex situations are treated later.

2.1.1. Performance Objectives

As formal performance objectives, we will require that the error signals from the feedback loop, $e(s) = y(s)-r(s)$, be sufficiently small in the $\mathbf{L}_2$-norm sense. That is, we require

$$\| e \|_2 \triangleq \left[\int_0^\infty e(t)^T e(t)\, dt \right]^{1/2} < \delta \tag{3}$$

for all external signals and all plants in sets yet to be defined. The scalar δ sets the desired level of performance.

2.1.2. External Signals

We also use the $\mathbf{L}_2$-norm to characterize the set of external signals. For the time being, this set will consist of a single vector-valued signal, $r(s)$, with $d(s) \equiv 0$, and $n(s) \equiv 0$. The signal $r(s)$ will consists of all time functions which can be generated by passing the functions

$$\eta = \{\eta(t);\, t \geq 0\} \quad \text{with} \quad \|\eta\|_2 \leq 1 \tag{4}$$

through linear systems with specified frequency responses, i.e.

$$r(s) = w_r(s)\eta(s) \tag{5}$$

Note that these signals belong to a unit ball in $\mathbf{L}_2$ which has been distorted in frequency content by the system $\mathbf{w}_r\mathbf{I}$, with transfer function $w_r(s)I$, to represent the spectral content of commands. Without loss of generality, $\mathbf{w}_r\mathbf{I}$ is assumed to be stable and to have a stable inverse.

2.1.3. Magnitudes of Transfer Matrices

The above choices of performance objectives and signal sets imply that singular values are the appropriate measure of magnitude for matrix transfer functions. This follows from the operator norm induced by $\mathbf{L}_2$-functions [5]. To

* Throughout the paper, x(t) and x(s) will be used to designate time functions and their Laplace transforms, respectively.

illustrate, suppose that $\delta\mathbf{G}$ is identically zero. Then

$$e(s) = S(s)r(s) \tag{6}$$

where $S(s)$ is the feedback system's nominal sensitivity matrix defined in equation (1). The performance objective for this case requires that

$$\begin{aligned}\sup_{\eta} \| e \|_2 &= \sup_{\eta} \| \mathbf{S}\mathbf{w}_r\mathbf{I}\,\eta \|_2 \\ &\triangleq \| \mathbf{S}\mathbf{w}_r\mathbf{I} \|_{2\to2} \\ &= \sup_{\omega} \bar{\sigma}[\, S(j\omega) w_r(j\omega)\,] \;<\; \delta \,. \end{aligned} \tag{7}$$

Here the function $\bar{\sigma}[]$ denotes the largest singular value of its matrix-valued argument. It follows that the performance objective is achieved (nominally) if and only if the Bode plot of this singular value for $S(j\omega)$ lies entirely below the Bode plot of $\delta / |w_r(j\omega)|$ over the entire frequency range $0 \le \omega < \infty$.

2.1.4. Set of Plants

The operator norm induced by external $\mathbf{L}_2$-signals also provides a convenient (and reasonable [6]) way to express the set of plants. As a starting point, we will consider $\mathbf{G}+\delta\mathbf{G}$ to be generated by unstructured multiplicative perturbations at the output, i.e.

$$[\, G(s)+\delta G(s)\,] \;=\; [\, I + L(s)\,]\, G(s) \quad , \tag{8}$$

where $L(s)$ is a perturbation matrix which satisfies

$$L(s) \;=\; w_L(s)\, \Delta(s) \tag{9}$$

for a specified stable invertible $\mathbf{w}_L\mathbf{I}$ and some arbitrary stable* operator Δ with induced norm less than or equal to unity, i.e.

$$\| \Delta \|_{2\to2} \;=\; \sup_{\omega} \bar{\sigma}[\, \Delta(j\omega)\,] \;\le\; 1 \tag{10}$$

Note that this characterization makes $\mathbf{L}$ norm-bounded in the sense that $\mathbf{L}_2$-signals in the unit ball, as shaped in frequency content by $\mathbf{w}_L^{-1}\mathbf{I}$, produce outputs whose $\mathbf{L}_2$-norms are less than unity.

2.2. Robust Stability and Robust Performance

Given these specifications for performance objectives, external signal sets, and plant sets, we now seek a compensator $K(s)$ which satisfies two requirements. First, it must achieve stability for all elements in the defined set of plants. This property will be called "robust stability". Second, it must satisfy the stated performance objective for all signals in the defined signal set and for all plants in the plant set. This latter property will be called "robust performance". The process of finding such a compensator is greatly facilitated by two important analysis results. The first of these is the following theorem, proven in [4], which provides necessary and sufficient conditions for robust stability.

Theorem 1

Suppose that the nominal feedback system in Figure 1 is stable (i.e. it is stable when $\delta G(s) \equiv 0$). Then the perturbed system is stable for all $\delta G(s)$ defined by equations (8)-(10) if and only if

* The stability requirement on Δ can be relaxed to the condition that $\mathbf{G}$ and $(\mathbf{I}+\mathbf{L})\mathbf{G}$ have the same number of unstable modes.

$$\bar{\sigma}[\,w_L(j\omega)\,T(j\omega)\,] < 1 \quad \text{for all } \omega, \tag{11}$$

where $T(s)$ is the complementary sensitivity function defined by equation (2).

This result follows from the fact that the feedback loop remains stable if and only if the function $det[I+(I+L)GK] = det[I+GK]det[I+\Delta w_L T]$ remains non-zero along the $j\omega$-axis (and therefore in the right half plane) for all Δ.

The second important analysis result provides a sufficient condition and a separate necessary condition for robust performance:

Theorem 2

Suppose that the feedback system in Figure 1 is robustly stable. Then the perturbed system satisfies performance objective (3) for all signals defined by (4)-(5) and all $\delta G(s)$ defined by (8)-(10) if

$$\bar{\sigma}[\,S(j\omega)w_r(j\omega)\,] < \delta\left[1-\bar{\sigma}[\,w_L(j\omega)\,T(j\omega)\,]\right] \quad \text{for all } \omega\,, \tag{12}$$

and it satisfies the objective for all signals and all plants only if

$$\bar{\sigma}[\,S(j\omega)w_r(j\omega)\,] < \delta\left[1-\underline{\sigma}[\,w_L(j\omega)\,T(j\omega)\,]\right] \quad \text{for all } \omega\,, \tag{13}$$

where $\underline{\sigma}[]$ denotes the smallest singular value of its argument.

These conditions follow directly from the perturbed sensitivity function

$$\tilde{S}(s) \triangleq \left\{I+[I+L(s)]\,G(s)K(s)\right\}^{-1}$$

$$= S(s)[\,I+L(s)\,T(s)\,]^{-1} \tag{14}$$

which shows that performance is maintained in the face of $L(s)$ whenever the nominal performance requirement, $\bar{\sigma}[\,S\,w_r\,] \le \delta$ from equation (7), is tightened sufficiently to offset the amplification of the factor $[\,I+LT\,]^{-1}$. Equations (12) and (13) simply reflect the worst and best case values this amplification can take. Note that these equations reduce to a single necessary and sufficient condition whenever the complementary sensitivity function is "spacially round", i.e. when its condition number satisfies

$$\kappa[\,T\,] \triangleq \bar{\sigma}[\,T\,]\,/\,\underline{\sigma}[\,T\,] \approx 1\,. \tag{15}$$

Even without this property, however, the sufficient condition (12) alone is not unduely conservative. This is so because the factor $1-\bar{\sigma}[w_L T]$ must be positive for robust stability and is typically designed to be 0.5 or greater to provide some design margin. On the other hand, the factor $1-\underline{\sigma}[w_L T]$ is never greater than unity. Hence, the true robust performance requirement is typically overstated by less than a factor of two.

The significance of Theorems 1 and 2 is that they define robust stability and robust performance solely in terms of acceptable magnitudes for the nominal functions $\bar{\sigma}[\,T(j\omega)\,]$ and $\bar{\sigma}[\,S(j\omega)\,]$. In particular, we saw from equation (11) that robust stability is achieved if and only if

$$\bar{\sigma}[\,T(j\omega)\,] < \frac{1}{|w_L(j\omega)|} \quad \text{for all } \omega\,. \tag{16}$$

Similarly, equation (12) shows that a linear combination of $\bar{\sigma}[\,T\,]$ and $\bar{\sigma}[\,S\,]$ must be small enough to assure robust performance, namely

$$\bar{\sigma}[\,S(j\omega)\,] + \frac{\delta\,|w_L(j\omega)|}{|w_r(j\omega)|}\,\bar{\sigma}[\,T(j\omega)\,] \;<\; \frac{\delta}{|w_r(j\omega)|} \quad \text{for all } \omega . \tag{17}$$

As in the SISO case, therefore, the design problem defined by the specified performance objectives, external signal sets, and plant sets in Sections 1.1.1-1.1.4 reduces to one of shaping Bode plots of sensitivity and complementary sensitivity magnitudes such that they lie below specified bounds over the entire frequency range.

2.3. Synthesis Methods

While our simple MIMO design problem is thus fundamentally the same as a SISO problem, the actual process of shaping MIMO functions is more difficult than shaping SISO ones. This difficulty has been demonstated over the years by a variety of attempts to generalize SISO synthesis concepts (e.g. inverse Nyquist methods applied to diagonally dominant systems [7], direct-Nyquist and Bode methods applied to characteristic loci [8], root-locus methods applied to multivariable functions [9], etc).

It turns out that some of the most effective methods of shaping MIMO loops use modern optimization-based synthesis tools. For example, modified versions of the LQG problem can be used effectively to achieve trade-offs between singular values of $S(s)$ and $T(s)$ across frequency [4,10]. More recent H_∞-methods have also been developed to synthesize compensators which directly minimize $\sup_\omega \bar{\sigma}[S(j\omega)]$, or $\sup_\omega \bar{\sigma}[T(j\omega)]$, or singular values of weighted augmented combinations $\sup_\omega \bar{\sigma}[S(j\omega) \mid w(j\omega)T(j\omega)]$ [11,12,13]. However, no methods have been developed to date which directly optimize sums of weighted singular values, as might be suggested by equation (17).

These various synthesis methods will not be discussed further here. Rather, it is our objective to identify design issues which do not lend themselves readily to the MIMO loop-shaping generalization even if the associated synthesis problems could be solved effectively.

3. MORE COMPLEX PROBLEMS

The design problem described above can be made more general and more useful for design purposes by including more complex performance requirements, signal sets, and/or plant sets. An obvious addition, for example, is to include disturbances $d(s)$ in the external signals. If these are generated as in equation (5) by an $\mathbf{L}_2$-ball shaped in frequency content by $\mathbf{w}_d\mathbf{I}$, then a robust performance result analogous to equation (17) applies, with $|w_d(s)|$ replacing $|w_r(s)|$. If both $r(s)$ and $d(s)$ exist simultaneously, then $|w_d|+|w_r|$ replaces $|w_r|$ alone.

Another obvious addition is to include sensor noise, $n(s)$. Let this signal again be generated by an $\mathbf{L}_2$-ball shaped by $\mathbf{w}_n\mathbf{I}$. With other signals zero, this input drives the error $e(s)$ via

$$\begin{aligned} e(s) &= (I+L)\,GK[I+(I+L)\,GK]^{-1}w_n\,\eta(s) \\ &= (I+L)\,T[I+L\,T]^{-1}w_n\,\eta(s)\,. \end{aligned} \tag{18}$$

which (after some algebra) leads to the following sufficient condition for performance over all $n(s)$ and all $L(s)$:

$$\bar{\sigma}[\,T(j\omega)\,] < \frac{\delta}{\delta\,|w_L(j\omega)| + |w_n(j\omega)|\,(1+|w_L(j\omega)|)} \quad \text{for all } \omega\,. \tag{19}$$

Note that this constraint on $\bar{\sigma}[T]$ is tighter than the stability robustness constraint (16). Hence, it can be used in place of (16) to cover both design requirements.

3.1. Difficulties

While the above two signal additions can be accomodated quite easily in the MIMO loop-shaping setup, there are other generalizations which cannot. In fact, any performance requirement, signal set, or plant set which is not spacially round as seen at the feedback system's output can cause difficulties.* To illustrate this, we will consider the following more general specifications:

Performance Requirements

Filtered versions of $e(s)$ must be $\mathbf{L}_2$-norm bounded for all external inputs and all plants, i.e.

$$\|\,\varepsilon\,\|_2 < \delta \quad \text{with} \quad \varepsilon(s) \triangleq W_e(s)\,e(s)\,, \tag{20}$$

where $W_e(s)$ is the transfer matrix of a stable invertible operator, $\mathbf{W}_e$, which shapes the spacial directions as well as the frequency content of $e(s)$.

External Signals

$r(s)$ generated by

$$r(s) = W_r(s)\,\eta(s)\,, \tag{21}$$

where matrix $W_r(s)$ shapes the spacial directions and frequency content of the commands.

* Technically, any specification of performance requirements, signal sets, and plant sets which are either all spacially round at the output or all spacially round at the input can be accomodated. In the latter case, the design would be done with S and T defined at the input.

Set of Plants

Unstructured multiplicative perturbations at the output defined by

$$L(s) = W_{Lo}(s)\,\Delta(s)\,W_{Li}(s) \;, \tag{22}$$

where $W_{Li}(s)$ and $W_{Lo}(s)$ shape the spacial directions and frequency content of inputs and outputs of the unit-norm perturbation Δ, respectively.

MIMO loop-shaping results for these specifications can be developed in a manner completely analogous to Section 2. Under the assumption that the nominal feedback system is stable, the principal results are the following:

(1) Nominal performance is satisfied if and only if

$$\bar{\sigma}[\,W_e(j\omega)\,S(j\omega)\,W_r(j\omega)\,] < \delta \quad \text{for all } \omega \tag{23}$$

(2) Stability is robust if and only if

$$\bar{\sigma}[\,W_{Li}(j\omega)\,T(j\omega)\,W_{Lo}(j\omega)\,] < 1 \quad \text{for all } \omega \tag{24}$$

(3) Performance is robust if (not only if)

$$\bar{\sigma}[\,W_e S W_r\,] < \frac{\delta}{\kappa[\,W_{Lo}^{-1} W_r\,]}\left[1-\bar{\sigma}[\,W_{Li}\,T W_{Lo}\,]\right] \quad \text{for all } \omega. \tag{25}$$

The result which causes difficulties here is (25). It is derived via the following manipulations:

$$\begin{aligned}
\varepsilon(s) &= W_e(s)\,\tilde{S}(s)\,W_r(s)\,\eta(s) \\
&= W_e\,S\,[\,I+L\,T\,]^{-1}\,W_r\,\eta(s) \\
&= W_e\,S W_r\,W_r^{-1}[\,I+LT\,]^{-1} W_r\,\eta(s) \\
&= (W_e\,S W_r)\,(W_{Lo}^{-1} W_r)^{-1}[\,I+\Delta(W_{Li}\,T W_{Lo})\,]^{-1}\,(W_{Lo}^{-1} W_r)\,\eta(s)
\end{aligned} \tag{26}$$

The last of these equations shows that the tightest singular value bound which we can place on the transfer matrix from $\eta(s)$ to $\varepsilon(s)$ for all $L(s)$ is $\bar{\sigma}[W_e S W_r]\,\kappa[W_{Lo}^{-1} W_r] \,/\, (1-\bar{\sigma}[W_{Li}\,T W_{Lo}])$. This bound leads directly to Condition (25), and thus, (25) is the weakest sufficient condition for robust stability which can be established via singular values.

Unfortunately, (25) can be arbitrarily conservative -- requiring much tighter nominal performance than is actually necessary to assure robust performance. Examples which illustrate this difficulty abound [6]. Indeed, (25) is often found to be impossibly tight in light of stability robustness constraints. The potential for this conservatism exists whenever the condition number $\kappa[W_{Lo}^{-1} W_r]$ is large.* This occurs when the external signals and/or the plant perturbations are not spacially round.

3.2. Examples

To conclude this section, we briefly consider two examples which give rise to problem specifications which are not spacially round.

Example 1: Perturbations at the Input

Suppose that the set of plants, **G+δG**, is described not by (8) in Section 2 but rather by

* To a lesser degree, the potential also exists when $\kappa[\,W_{Li}\ T\ W_{Lo}\,])$ is large. As in Section 2, however, the factor $1-\bar{\sigma}[\,W_{Li}\,T W_{Lo}\,]$ is typically designed to be 0.5 or greater, and thus, its contribution to conservatism is not excessive.

$$[G(s)+\delta G(s)] = G(s)[I+\tilde{L}(s)] \tag{27}$$

with $\tilde{L}(s)$ satisfying assumptions (9)-(10). Then a few manipulations show that the corresponding multiplicative perturbation at the output is given by

$$L(s) = G(s)\tilde{L}(s)G(s)^{-1} \tag{28}$$

Thus, $L(s)$ satisfies Specification (22) with

$$W_{Li}(s) = f(s)G_{*}(s)^{-1} \quad \text{and} \quad W_{Lo}(s) = G_{*}(s)w_L(s)/f(s) \tag{29}$$

where $G_{*}(s)$ is a minimum phase stable version of $G(s)$ (with all unstable poles and transmission zeros reflected about the $j\omega$-axis), and $f(s)$ is a low pass filter which makes G_{*}^{-1} proper at high frequencies. It follows that this specification of plant perturbations, which is spacially round at the input, will not be round at the output if the condition number of $G(j\omega)$ is large.

Example 2: Disturbances at the Input

Suppose that the external signals consist of disturbances $d_u(s)$ entering at the input and generated by an $\mathbf{L}_2$-ball shaped with $\mathbf{w}_u\mathbf{I}$. These disturbances can be represented by the following equivalent* disturbances reflected to the output:

$$d(s) = G_{*}(s)d_u(s) \tag{30}$$

This disturbance corresponds to Specification (21) with $W_r(s) = w_u(s)G_{*}(s)$. Again, this external signal is not round whenever the condition number of $G(j\omega)$ is large.

Both of these examples are, of course, common occurances in MIMO feedback design, and there are many others.

*Equivalent in the sense that closed loop responses $y(s)$ will have the same magnitudes for all $s = j\omega$.

4. AN ALTERNATE MIMO DESIGN FRAMEWORK

The loop-shaping limitations described above can be overcome with an alternate design framework which has been developed in the last few years [6,14,15]. This alternate framework consists of a new problem description, a new measure of magnitude for matrix transfer functions, and certain key analysis and synthesis results. These various elements are described briefly in this Section.

4.1. Problem Description

The new problem description is illustrated in Figure 2. It consists of a very general "plant", **P**, whose outputs and inputs comprise three pairs of vector variables. The first pair of variables consists of measured outputs, $y(s)$, to be used for feedback and control inputs, $u(s)$, to be commanded by the compensator, **K**. The second pair consists of performance variables, $\varepsilon(s)$, and external input signals, $\eta(s)$. Finally, the third pair of variables consists of input signals, $\zeta_i(s)$, to the perturbation operator, Δ, and outputs from this operator, $\zeta_o(s)$, which feed back into the plant. The design problem is to find a compensator which keeps the signals $\varepsilon(s)$ in the unit L_2-ball for all signals $\eta(s)$ in the unit-ball and all stable unit-norm operators Δ.

This problem description is very general because the internal structure of **P** can be chosen to represent many different problem specifications. One example of this internal structure is shown in Figure 3. This figure corresponds to the problem specifications in Section 3.1. **P** is seen to include the usual input-output description of the real plant **G**, but it also includes the weighting operators $\mathbf{W}_e$ and $\mathbf{W}_r$, which shape performance variables and external signals, as well as $\mathbf{W}_{Lo}$ and $\mathbf{W}_{Li}$ which shape the plant perturbation. The types of external signals (whether $r(s)$, $d(s)$, $n(s)$ and/or others) are also defined by the internal structure, as are the locations of perturbations (whether at outputs, inputs, and/or elsewhere). Various examples of internal structures of **P** for other problem specifications can be found in [6].

4.2. Analysis Results

Beyond mere generality, Figure 2 is important because it comes equipped with a non-conservative necessary and sufficient condition for robust performance. In order to describe this new condition, we first close the compensator feedback loop in Figure 2 to get the closed loop system in Figure 4. The operator **F(P,K)** in this figure has a 2x2 block-structured transfer function matrix $F(s)$ whose blocks are defined in terms of the original 3x3 partition of $P(s)$ as follows:

$$F_{ij}(s) = P_{ij}(s) + P_{i3}(s)[I - K(s)P_{33}(s)]^{-1}K(s)P_{3j}(s) \qquad i,j = 1,2 \qquad (31)$$

Suppose that this system is stable. Then the following results apply:

(1) Nominal performance is satisfied if and only if

$$\bar{\sigma}[F_{22}(j\omega)] < 1 \quad \text{for all } \omega \qquad (32)$$

(2) Stability is robust if and only if

$$\bar{\sigma}[F_{11}(j\omega)] < 1 \quad \text{for all } \omega \qquad (33)$$

(3) Performance and stability are robust if **and only if**

$$\mu[F(j\omega)] < 1 \quad \text{for all } \omega \qquad (34)$$

where $\mu[]$ is a function to be defined shortly.

The first two of these results are self-evident. Result(32) follows by definition -- performance is satisfied for $\Delta(s) \equiv 0$ iff the induced norm $\|F_{22}\|_{2\to2}$ is less than or equal to unity. Result(33) follows from the stability condition with the Δ-loop closed -- namely $det[I-\Delta F_{11}]$ must remain non-zero on the $j\omega$-axis for all Δ.

The more significant result is (34). This follows again from the definition that performance is robust if and only if the ε/η transfer matrix with the Δ-loop closed remains stable and norm-bounded by unity, i.e. iff (33) is satisfied and

$$\bar{\sigma}[F_{22}+F_{21}(I-\Delta F_{11})^{-1}\Delta F_{12}] < 1 \quad \text{for all } \omega \text{ and all } \Delta \tag{35}$$

Notice that this last norm-bound is also a necessary and sufficient condition for continued stability if we chose to connect a second norm-bounded perturbation, say $\Delta_0(s)$, across the ε and η terminals of Figure 4 (to see this, compare the form of (35) with our other stability conditions (11), (24), and (33)). It follows, therefore, that robust performance is equivalent to robust stability in the face of two perturbations, Δ and Δ_0, connected around the system $F(s)$ in the diagonally structured arrangement shown in Figure 5.

These observations bring us to the function $\mu[]$. This function was defined in [16] to provide a magnitude measure for the smallest block-structured perturbation which will make a system unstable. The full definition of μ for complex matrices is the following:

$$\mu[M] \triangleq \left[\min\left\{ \delta \;\middle|\; \begin{array}{l} det[I-\delta XM]=0 \\ \text{for some } X = diag[\Delta_1, \Delta_2, \cdots, \Delta_m] \\ \text{with } \bar{\sigma}[\Delta_i] \le 1 \text{ for all } i \end{array} \right\} \right]^{-1} . \tag{36}$$

In words, this equation defines $\mu[]$ to be the reciprocal of the smallest value of scalar δ which makes the matrix $I-\delta XM$ singular for some X in a block-diagonal perturbation set. Notice that this definition reduces to the conventional singular value in the absence of structure (i.e. when the number of blocks, m, in X is one). For this reason, μ has been called the "structured singular value". Note also that the value of μ depends on the number of blocks in the structure as well as on the dimensions of these blocks. Technically, therefore, μ's arguments should include not only matrix M but also a multi-index which describes the structure. By convention and for sake of notational simplicity, this latter dependence is suppressed.

It is clear from this definition that μ can be applied to the transfer matrices of Figure 5 to test whether $det[I-diag(\Delta,\Delta_0)F]$ remains non-zero along the $j\omega$-axis. This establishes tight conditions for robust stability with respect to the two perturbation blocks, and equivalently, tight conditions for robust stability and performance (Condition (34)). Formal details of this argument are given in [14]. Notice, however, that the definition of μ is not limited to the 2x2 diagonal perturbation block structure in Figure 5. It can be used to test stability with respect to any number of diagonal blocks. This makes it possible to establish robust stability with respect to plant sets which are characterized by several unstructured perturbations, and simultaneously, to establish robust performance with respect to several performance requirements*. Indeed, the only limitation on perturbation structures testable via μ appears to be that each perturbation block must be allowed to be complex-valued. Research to remove this remaining potential source of conservatism is under way [17].

* The meaning of several 'performance blocks' is not entirely clear. For such cases, $\mu < 1$, implies that the i-th performance requirement is robust with respect to the real perturba-

4.3. Numerics for μ

Like singular values, μ is useful for practical numerical analyses as well as for theoretical ones. First-generation computer algorithms have been developed to evaluate the function for fixed complex matrices. When used repeatedly, these algorithms can generate Bode plots of μ over frequency for matrix transfer functions such as $F(s)$. This provides a practical Bode-like analysis test of stability/performance robustness for any given candidate design.

To date, μ-algorithms are based on the the following inequalities, proven in [16]:

$$\max_U |\lambda[UM]| \leq \mu[M] \leq \min_D \bar{\sigma}[DMD^{-1}] \qquad (37)$$

where

$$U \triangleq diag[U_1, U_2, \cdots, U_m] \qquad U_i \text{ unitary}$$
$$D \triangleq diag[d_1 I_1, d_2 I_2, \cdots, d_m I_m] \qquad d_i \text{ scalar}$$

Reference [16] shows that the left hand side of inequality (37) is tight and thus provides a potential way to compute μ. Unfortunately, the implied maximization over the block-structured unitary matrices U has many local maxima. The right hand side of (37) is also tight, at least for structures with three or fewer blocks. Its implied minimization over the block-structured scaling matrices D is convex and thus provides a much nicer problem for numerial search solutions. For this reason, current μ-algorithms are based on the right had side of (37). The issues posed by four or more blocks in the structure remain under study, and further improvements in algorithms are forthcoming [18].

4.4. μ-Synthesis

Progress has also been made in the development of formal synthesis methods for the design framework in Figure 2. These methods seek to design compensators, **K**, to stabilize the nominal system, **P**, and to minimize $\mu[F]$. While complete solutions of this problem are not yet available, an iterative scheme has been invented which yields useful answers [19]. The iterative scheme exploits the fact exhibited in (37) that μ is a scaled version of $\bar{\sigma}$ with block-structured scaling matrix D. This fact suggests the following iteration:

Step(1)

Fix an initial estimate for $D(j\omega)$

Step(2)

Solve a $\bar{\sigma}$-synthesis problem to find a stabilizing $K(s)$ which minimizes $\sup_\omega \bar{\sigma}[D(j\omega)F(j\omega)D(j\omega)^{-1}]$

Step(3)

Evaluate the μ-properties of this solution against Condition (34), and

Step(4)

Stop if Condition (34) is satisfied. Else replace the $D(j\omega)$ estimate with the $D(j\omega)$ values computed as part of the μ-calculation in Step(3) and return to Step(2)

---

tions Δ as well as the This is a stronger requirement than actually necessary.

These iterations are practical primarily because the $\bar{\sigma}$-problem in Step(2) has a numerically tractable solution. This $\bar{\sigma}$-solution has been completed only recently [15], and is itself a significant step forward in MIMO design. It encompasses the various special H_∞-problems from Section 2 as special cases and provides state-space-based computational algorithms which can handle design problems of significant engineering size. Detailed descriptions of the solution are left to [15].

5. SUMMARY AND CONCLUSIONS

This paper has provided a brief status review of the singular value loop-shaping design paradigm for multivarible feedback control systems. It has shown that this paradigm is useful for design problems whose specifications of external signal sets, plant sets, and performance requirements are spacially round at the plant output or, by duality, spacially round at the plant input. For such problems, it is possible to write tight necessary and sufficient analysis conditions for robust stability and also reasonably tight conditions for robust performance. Both conditions take the form of Bode-like magnitude bounds on singular values of sensititivity and complementary sensitivity matrices of the feedback system.

Unfortunately, when problem specifications are not spacially round, the singular value conditions for robust performance can be arbitrarily conservative, leading to highly overspecified design requirements. Design problems in this category abound. An alternate design paradigm is discussed which overcomes these limitations. The alternate paradigm includes of a more general problem formulation, a new matrix function μ, and tight necessary and sufficient conditions for both robust stability and robust performance in terms of this new function. Under current state of the art, μ can be calculated numerically and thus provides an effective analysis tool for existing candidate designs. However, direct synthesis techniques to design compensators which stabilize and also minimize μ are not yet available. Research to provide this capability is underway.

References

1. Nyquist, H., "Regeneration Theory", *Bell System Tech. Journal*, January, 1932.
2. Bode, H.W., *Network Analysis and Feedback Amplifier Design*, D. VanNostrand, Princeton, NJ, 1945.
3. Rosenbrock, H.H., *State Space Methods and Multivariable Theory*, Nelson, London, 1970.
4. Doyle, J.C. and G. Stein, "Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis", *IEEE Trans. Auto. Control*, February, 1981.
5. Desoer, C.A. and M. Vidyasagar, *Feedback Systems: Input-Output Properties*, New York: Academic Press, 1975.
6. Doyle, J.C. and G. Stein (in prep)
7. Rosenbrock, H.H., *Computer-Aided Control System Design*, New York: Academic Press, 1974.
8. MacFarlane, A.G.J, and B. Kouvaritakis, "A Design Technique for Linear Multivariable Feedback Systems", *Int. J. Contr.*, vol 25, 1977.
9. Postlethwaite, I. and A.G.J. MacFarlane, *A Complex Variable Approach to the Analysis of Linear Multivariable Feedback Systems*, Berlin: Springer-Verlag, 1979.
10. Stein, G. and M. Athans, "The LQG/LTR Procedure for Multivariable Feedback Control Design", submitted to *IEEE Trans. Auto. Control*, June, 1984.
11. Zames, G., "Feedback and Optimal Sensitivity: Model Reference Transformations, Multiplicative Seminorms, and Approximate Inverses," *IEEE Trans. Auto. Control*, April 1981
12. Francis, B.A. and G. Zames, "Feedback, Minimax Sensitivity, and Optimal Robustness", *IEEE Trans. Auto. Control*, May, 1983.
13. Kwakernaak, H., "Robustness Optimization of Linear Feedback Systems", *Proc. Conf. on Decision and Control*, San Antonio, TX, December 1983.
14. Doyle, J.C., J.E. Wall, and G. Stein, "Performance and Robustness Analysis for Structured Uncertainty, " *Proc. Conf. on Decision and Control*, 1982.
15. Doyle, J.C and B. Francis, Lecture Notes, *ONR/Honeywell Workshop on Advances in Multivariable Control*, Minneapolis, October 1984.
16. Doyle, J.C., "Analysis of Feedback Systems with Structured Uncertainty," *Proc. IEE*, Nov. 1982.
17. Morton, B.G. and R.M. McAfoos, "A Mu-Test for Robustness Analysis of a Real-Parameter Variation Problem", *Proc. ACC*, Boston, 1985.
18. Fan, M.K.H. and A.L.Tits, "A New Formula for the Structured Singular Value", submitted to 24-th CDC, 1985.
19. Doyle, J.C, "Synthesis of Robust Controllers and Filters", *Proc. CDC* San Antonio, 1983.

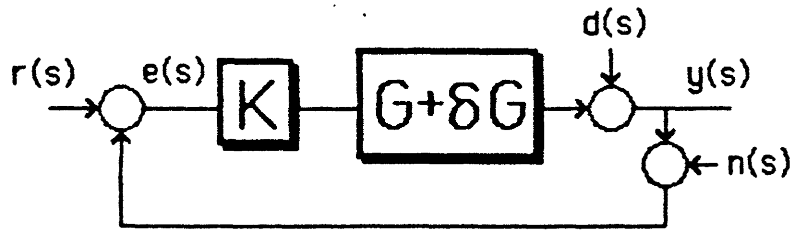

Figure 1: Generic MIMO Feedback Loop

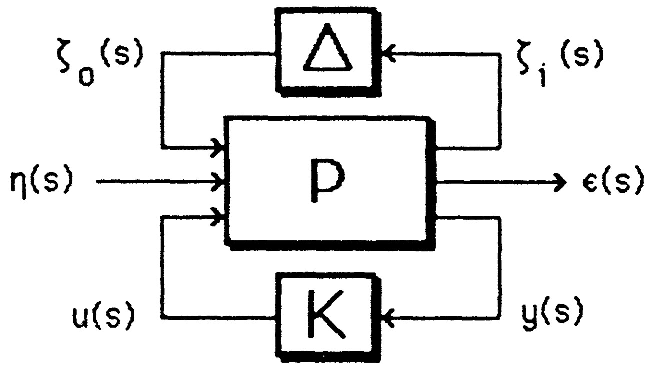

Figure 2: General Problem Description

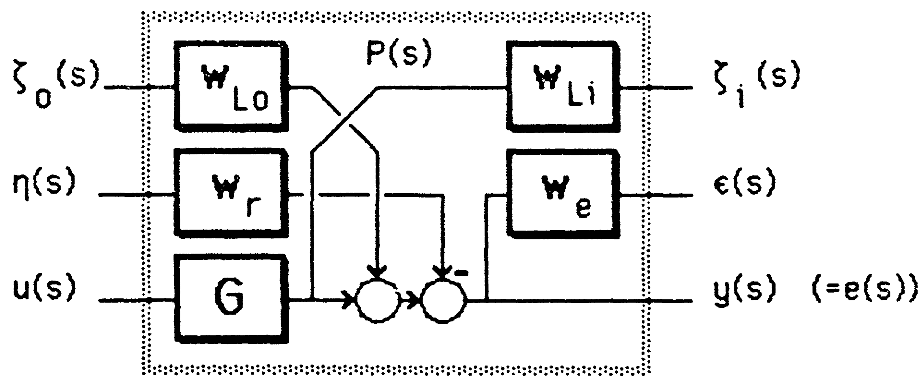

Figure 3: Internal Structure of P(s)

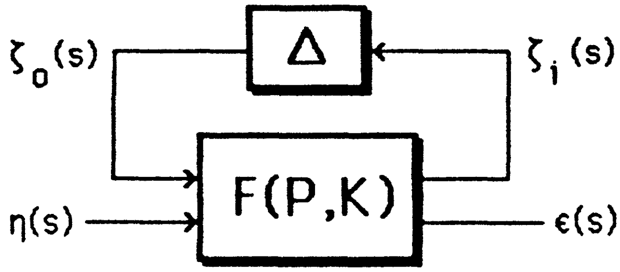

Figure 4: Feedback Loop Closed

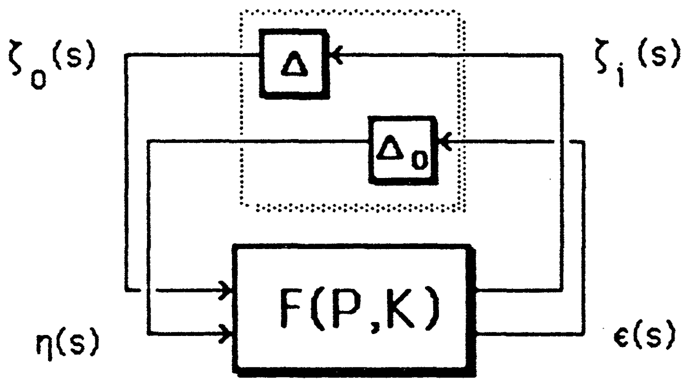

Figure 5: Equivalence of Robust Performance and Robust Stability with Structured Perturbations