October 1986 LIDS-P-1613

CONTROLLED DIFFUSIONS WITH BOUNDARY-CROSSING COSTS

by
Vivek S. Borkar*

Tata Inst. of Fundamental Research
Bangalore Center
P.O. Box 1234
Bangalore 560012, INDIA

ABSTRACT

This paper considers control of nondegenerate diffusions in a bounded domain with a cost associated with the boundary-crossings of a subdomain. Existence of optimal Markov controls and a verification theorem are established.

KEY WORDS

Controlled diffusions, optimal control, Markov controls, boundary-crossing cost, verification theorem

Research supported by ARO contract No. DAAG29-84-K-0005 and AFOSR 85-0227.

*Current address: Laboratory for Information and Decision Systems, Bldg. 35, M.I.T., Cambridge, MA 02139

I. INTRODUCTION

In the classical treatment of controlled diffusion processes, one typically considers a cost which is the expected value of a 'nice' functional of the trajectory of the controlled process. This functional is often the time integral up to a stopping time of a 'running cost' function on the state space [1], [4]. This paper considers a situation where, loosely speaking, the running cost is a Schwartz distribution rather than a function. The specific case we consider has a natural interpretation as the cost ('toll') associated with the boundary crossings of a prescribed region.

The precise formulation of the problem is as follows: Let U be compact metric space and $X(\cdot)$ an R^n-valued controlled diffusion on some probability space described by

$$X(t) = x + \int_0^t m(X(s), u(s))ds + \int_0^t \sigma(X(s))dW(s) \qquad (1.1)$$

for $t \geq 0$, where

(i) $m(\cdot,\cdot) = [m_1(\cdot,\cdot),\ldots,m_n(\cdot,\cdot)]^T : R^n x U \to R^n$ is bounded continuous and Lipschitz in its first argument uniformly with respect to the second,

(ii) $\sigma(\cdot) = [[\sigma_{ij}(\cdot)]] : R^n \to R^{nxn}$ is bounded Lipschitz and satisfies the uniform ellipticity condition

$||\sigma(z)y||^2 \geq \lambda ||y||^2$ for all $z, y \in R^n$

for some $\lambda > 0$,

(iii) $W(\cdot) = [W_1(\cdot),\ldots,W_n(\cdot)]^T$ is an R^n-valued standard Wiener process,

(iv) u(•) is a U-valued process with measurable sample paths and satisfies the nonanticipativity condition: for $t \geq s \geq y$, W(t)-W(s) is independent of u(y).

Call such a u(•) an admissible control. Call it a Markov control if u(•) = v(X(•)) for some measurable $v: R^n \to U$. In this case, it is well-known that (1.1) has a strong solution which is a Markov process. In particular, this implies that Markov controls are admissible. We shall also refer to the map v itself as a Markov control by abuse of terminology.

Let B,D be bounded open sets in R^n with C^2 boundaries δB, δD resp., such that $\bar{B} \subset D$ and $x \varepsilon D \backslash \delta B$. Let $\tau = \inf\{t \geq 0 | X(t) \notin \bar{D}\}$. Let $M(\bar{D})$ denote the space of finite nonnegative measures on $\bar{D}$ with the weakest topology needed to make the maps $\eta \to \int f d\eta$ continuous for $f \varepsilon C(\bar{D})$. For $x \varepsilon \bar{D}$, define $\nu_x \varepsilon M(\bar{D})$ by

$$\int f d\nu_x = E[\int_0^\tau f(X(t))dt], \quad f \varepsilon C(\bar{D}).$$

Note that ν_x depends on u(•). From Krylov inequality ([4], Section 2.2), it follows that ν_x is absolutely continuous with respect to the Lebesgue measure on $\bar{D}$ and thus has a density g(x,•), defined a.e. with respect to the Lebesgue measure. Following standard p.d.e. terminology, we shall call ν_x, g(x,•) the Green measure and the Green function resp. Later on, we shall show that g(x,•) is continuous on $D \backslash \{x\}$. Let h be a finite signed measure on δB, the latter being endowed with the Borel σ-field corresponding to its relative topology. Define the cost associated with control u(•) as

$$J_x(u(\cdot)) = \int_{\delta B} g(x,y)h(dy). \tag{1.2}$$

The control problem is to minimize this over all admissible $u(\cdot)$. For nonnegative h, (1.2) has the heuristic interpretation of being the total toll paid whenever $X(\cdot)$ hits δB, before it exits from $\bar{D}$.

Remark. The restriction $x \varepsilon \delta B$ simplifies the presentation considerably and is therefore retained. It could be relaxed by imposing suitable conditions on h, the nature of which will become apparent as we proceed.

The main results of this paper are as follows:

(i) There exists an optimal Markov control v which is optimal for all initial $x \varepsilon \bar{D} \backslash \delta B$.

(ii) This v is a.e. characterized by a verification theorem involving the value function $V: \bar{D} \backslash \delta B \rightarrow R$ mapping x into $\inf J_x(u(\cdot))$, in analogy with the classical situation.

For technical reasons, we use the relaxed control framework, i.e., we assume that U is the space of probability measures on a compact metric space S with the Prohorov topology and m is of the form

$$m(y,u) = \int_S b(y,s)u(ds) \qquad \text{(termwise integration)}$$

for some $b(\cdot,\cdot) = [b_1(\cdot,\cdot),\ldots,b_n(\cdot,\cdot)]^T : R^n x S \rightarrow R^n$ which is bounded continuous and Lipschitz in its first argument uniformly with respect to the second. This restriction will be dropped eventually.

In the next section, we establish a compactness result for Green measures. Section III derives a corresponding result for Green functions and deduces the existence of an optimal Markov control v for a given initial condition x. Section IV studies the basic properties of the value function. Section V uses these to prove a verification theorem for v which shows among other things that v is optimal for any x.

II. THE GREEN MEASURES

The results of this section allow us to restrict our attention to the class of Markov controls and establish a key compactness result for the set of attainable ν_x. We start with some technical preliminaries.

For the purposes of the following two lemmas, we allow the initial condition of (1.1) to be a random variable X_0 (i.e., $X(0) = X_0$ a.s.) independent of $W(\cdot)$.

Lemma 2.1. For any $T>0$, there exists a $\delta \varepsilon (0,1)$ such that

$$I\{X_0 \varepsilon \bar{D}\} I\{\tau > s\} P(\tau > s+T / X(y), u(y),\ y \leq s) \leq \delta \quad \text{a.s.}$$

under any choice of X_0, $u(\cdot)$, s.

Proof. We need consider only the case $P(X_0 \varepsilon \bar{D},\ \tau > s) > 0$. Let (Ω, F, P) be the underlying probability space. Let $\bar{\Omega} = \Omega \cap \{X_0 \varepsilon \bar{D}\} \cap \{\tau > s\}$, $\bar{F}=F$ relativized to $\bar{\Omega}$, $\bar{X}_0 = X(s)$, $\bar{X}(\cdot) = X(s+\cdot)$ and $\bar{u}(\cdot) = u(s+\cdot)$. Instead of the control system described by $(X(\cdot), X_0, u(\cdot))$ on (Ω, F, P), we could look at $(\bar{X}(\cdot), \bar{X}_0, \bar{u}(\cdot))$ on $(\bar{\Omega}, \bar{F}, \bar{P})$ where $\bar{P}(A) = P(A)/P(\bar{\Omega})$ for $A \varepsilon \bar{F}$. Thus we may take s=0. By a simple conditioning argument, it also suffices to consider $X_0 = x_0$ for some $x_0 \varepsilon \bar{D}$. If the claim is false, we can find a sequence of processes $X^n(\cdot)$, n=1,2,..., satisfying (1.1) on some probability space, with x, $u(\cdot)$ replaced by some x_n, $u_n(\cdot)$ resp. such that: if $\tau^n = \inf\{t \geq 0 \mid X^n(t) \varepsilon \bar{D}\}$, then $P(\tau^n > T) \uparrow 1$. Using the arguments of [5], we may pick a subsequence of {n}, denoted {n} again, so that $x_n \to x_\infty$ for some $x_\infty \varepsilon \bar{D}$ and there exists a process $X^\infty(\cdot)$ satisfying (1.1) on some probability space with $x = x_\infty$ and $u(\cdot)$ = some admissible control $u_\infty(\cdot)$, such that $X^n(\cdot) \to X^\infty(\cdot)$ in law as $C([0,\infty);R^n)$-valued random variables. By Skorohod's theorem, we may assume that this convergence is a.s. on some common probability space. Let $\tau^\infty = \inf\{t \geq 0 \mid X^\infty(t) \notin \bar{D}\}$ and $\sigma^\infty = \inf\{t \geq 0 \mid X^\infty(t) \notin D\}$. From simple geometric considerations, one can see that for each sample point, any limit point of $\{\tau^n\}$ in $[0,\infty]$ must lie between σ^∞ and τ^∞. Under our hypotheses on δD and σ, $\sigma^\infty = \tau^\infty$ a.s. Hence $\tau^n \to \tau^\infty$ a.s. Thus

$$1 = \limsup P(\tau^n \geq T) \leq P(\tau^\infty \geq T),$$

implying $P(\tau^\infty \geq T) = 1$. Thus $X^\infty(T/2) \in \bar{D}$ a.s., which we know to be false under our conditions on m,σ. The claim follows by contradiction. Q.E.D.

Lemma 2.2. There exists a constant $K \in (0,\infty)$ such that

$$E[\tau^2] < K$$

for any $x,u(\cdot)$.

Proof. Let $T>0$. Then for $n=1,2,\ldots,$

$$\begin{aligned} P(\tau>nT) &= E[E[I\{\tau>nT\}/X(y),u(y),y\leq(n-1)T]I\{\tau>(n-1)T\}] \\ &\leq \delta P(\tau>(n-1)T) \end{aligned}$$

by the above lemma. Iterating the argument,

$$P(\tau > nT) \leq \delta^n$$

The rest is easy.

Q.E.D.

We now state and prove the first main result of this section, which is in the spirit of [2]. Let μ_X denote the probability measure on δD defined by

$$\int f d\mu_X = E[f(X(\tau))], \quad f \varepsilon C(\delta D).$$

Theorem 2.1. For each admissible control $u(\cdot)$, there exists a Markov control which yields the same ν_X and μ_X.

Proof. By Lemma 2.2, $E[\tau] < \infty$. Define a probability measure η on $\bar{D}xS$ by

$$\int f(y,s)\eta(dy,ds) = E[\int_0^\tau \int_S f(X(t),s)u(t)(ds)dt]/E[\tau], \quad f \varepsilon C(\bar{D}xS).$$

Disintegrate η as

$$\eta(dy,ds) = \eta_1(dy)\eta_2(y)(ds)$$

where η_1 is the image of η under the projection $\bar{D}xS \to \bar{D}$ and $\eta_2 : \bar{D} \to U$ is the regular conditional law, defined η_1 - a.s. Pick any representative of η_2. Then $u'(\cdot) = \eta_2(X'(\cdot))$ defines a Markov control, $X'(\cdot)$ being the solution to

(1.1) under $u'(\cdot)$. We shall show that $u(\cdot)$, $u'(\cdot)$ lead to the same ν_x, μ_x.

For $y = [y_1,\dots,y_n]^T \varepsilon D$, $u \varepsilon U$, $f \varepsilon H^2_{loc}(D)$, define

$$(Lf)\ (y,u) = \sum_{i=1}^{n} m_i(y,u) \frac{\partial f}{\partial y_i}(y) + \frac{1}{2} \sum_{i,j,k=1}^{n} \sigma_{ik}(y)\sigma_{jk}(y) \frac{\partial^2 f}{\partial y_i \partial y_j}(y).$$

Let $\varphi:\bar{D} \to R$ be smooth and $\phi:\bar{D} \to R$ the map that maps x into

$$E[\int_0^{\tau'} \varphi(X'(t))dt]$$

where $\tau' = \inf\{t \geq 0 | X'(t) \notin \bar{D}\}$. (Recall that $X'(0) = x$.) Then ϕ is the unique solution in $C(\bar{D}) \cap H^2_{loc}(D)$ to

$$-(L\phi)(y,\eta_2(y)) = \varphi(y) \text{ in } D,\ \phi=0 \text{ on } \delta D. \qquad (2.1)$$

(That (2.1) has a unique solution in the given class of functions follows from Theorem 8.30, pp. 196, [3]. That this solution coincides with our definition of ϕ is an easy consequence of Krylov's extension of the Ito formula as in [4], Section 2.10.) Consider the process

$$Y(t) = \phi(X(t)) + \int_0^t \varphi(X(s))ds,\ t \geq 0.$$

Another straightforward application of Krylov's extension of the Ito formula yields (see, e.g., [4], pp. 122)

$$E[Y(\tau)] - E[Y(0)] = E[\int_0^{\tau} (L\phi(X(t),u(t)) + \varphi(X(t)))dt]. \qquad (2.2)$$

Note that the first equality in (2.1) holds a.e. with respect to the Lebesque measure. Since ν_x is absolutely continuous with respect to the Lebesque measure, it holds ν_x-a.s. Hence the right hand side of (2.2) equals

$$E[\int_0^{\tau} \nabla\phi(X(t))\cdot(m(X(t),u(t))-m(X(t),u'(t)))dt],$$

which is zero by our definition of $u'(\cdot)$. Thus $E[Y(\tau)] = E[Y(0)]$, i.e.,

$$E[\int_0^{\tau} \varphi(X(t))dt] = E[\int_0^{\tau'} \varphi(X'(t))dt].$$

Since the choice of φ was arbitrary, it follows that $u(\cdot)$, $u'(\cdot)$ yield the same ν_x. The corresponding claim for μ_x is proved in [2], Theorem 1.2.

Q.E.D.

The second main result of this section combines the foregoing ideas with those of [5].

<u>Theorem 2.2</u>. The set of the pairs (ν_x, μ_x) as x varies over $\bar{D}$ and $u(\cdot)$ varies over all Markov controls (equivalently, all admissible controls) is sequentially compact.

<u>Proof</u>. In view of the preceding theorem, it suffices to consider the case of arbitrary admissible controls. Let $X^n(\cdot)$ be a sequence of processes

satisfying (1.1) on some probability space with $X^n(0) = x_n$, $u(\cdot) = u_n(\cdot)$ for some $x_n \varepsilon \bar{D}$ and admissible controls $u_n(\cdot)$, n=1,2,... As in the proof of Lemma 2.1, we can arrange to have these defined on a common probability space such that $x_n \to x_\infty \varepsilon \bar{D}$ and $X^n(\cdot) \to X^\infty(\cdot)$ a.s. in $C([0,\infty); R^n)$ where $X^\infty(\cdot)$ satisfies (1.1) with x replaced by x_∞ and $u(\cdot)$ by some admissible control $u_\infty(\cdot)$. Defining τ^n, $n=1,2,\dots,\infty$, as in Lemma 2.1, we have $\tau^n \to \tau^\infty$ a.s. Thus for $f \varepsilon C(\bar{D})$,

$$\int_0^{\tau^n} f(X^n(t))dt \to \int_0^{\tau^\infty} f(X^\infty(t))dt \qquad \text{a.s}$$

$$f(X^n(\tau^n)) \to f(X^\infty(\tau^\infty)) \qquad \text{a.s.}$$

By Lemma 2.2, we can take expectations in the above to conclude. Q.E.D.

III. EXISTENCE OF OPTIMAL MARKOV CONTROLS

This section establishes a compactness result for the Green functions which immediately leads to the existence of an optimal Markov control. We start with several preliminary lemmas.

Let $x \in \bar{D}$ and v a Markov control. As in [4], Section 2.6, we construct a family of R^n-valued diffusions X^ε, $0<\varepsilon\leq 1$, with $X^\varepsilon(0) = x$ for all ε, having drift coefficients $m^\varepsilon: R^n \to R^n$ and diffusion coefficients $\sigma^\varepsilon: R^n \to R^{n\times n}$ resp. such that

(i) m^ε, σ^ε are smooth and bounded with the same bounds as m,σ resp.,

(ii) $||\sigma^\varepsilon(z)y||^2 \geq \lambda||y||^2$ for all $y,z \in R^n$ with the same λ as in Section I,

(iii) $X^\varepsilon(\cdot) \to X(\cdot)$ in law as $\varepsilon\downarrow 0$, $X(\cdot)$ being the solution to (1.1) under the Markov control v.

Let ν_x^ε, $g^\varepsilon(x,\cdot)$ denote the Green measure and the Green function resp. corresponding to $X^\varepsilon(\cdot)$ and ν_x, $g(x,\cdot)$ those for $X(\cdot)$. Then the arguments of the preceding section can be used to show that

$$\nu_x^\varepsilon \to \nu_x \text{ in } M(\bar{D}) \text{ as } \varepsilon \to 0 \tag{3.1}$$

Lemma 3.1. Given any open set A such that $\bar{A} \subset D\backslash\{x\}$, there exists an $\alpha>0$ and a $K\in(0,\infty)$ such that

$$|g(x,y) - g(x,z)| < K||y-z||^{\alpha}, \quad y,z \in A,$$

under any choice of v.

Proof. Consider a fixed v to start with. Let L_ε denote the extended generator of $X^\varepsilon(\cdot)$ and L_ε^* its formal adjoint. Then

$$L_\varepsilon^* g^\varepsilon(x,\cdot) = \delta_x(\cdot)$$

in the sense of distributions, where $\delta_x(\cdot)$ is the Dirac measure at x. Hypoellipticity of L_ε^* implies that $g^\varepsilon(x,\cdot)$ is smooth on $D\backslash\{x\}$. By Theorem 8.24, pp. 192, [3], it follows that there exist $\alpha>0$, $0<K<\infty$, such that

$$|g^\varepsilon(x,z) - g^\varepsilon(x,y)| \leq K||z-y||^{\alpha}, \; y,z \in A \tag{3.2}$$

and these α, K depend only on A, the bounds on m,σ and the constant λ. Fix $z \in A$. If $\{g^\varepsilon(x,z),\ 0<\varepsilon\leq 1\}$ is unbounded, there exists a sequence $\{\varepsilon(n)\}$ in (0,1] such that

$$g^{\varepsilon(n)}(x,z) \uparrow \infty \; .$$

By (3.2), it follows that

$$g^{\varepsilon(n)}(x,\cdot) \uparrow \infty$$

uniformly on A. Letting $\tau(\varepsilon) = \inf\{t\geq 0 \,|\, X^\varepsilon(t) \notin \bar{D}\}$, we have

$$E[\tau(\varepsilon(n))] = \int_D g^{\varepsilon(n)}(x,y)dy \uparrow \infty.$$

Recalling the definition of $X^{\varepsilon}(\cdot)$ from [4], Section 2.6, and using an argument similar to that of Lemma 2.2, one can show that $E[\tau(\varepsilon)]$ is bounded uniformly in ε, giving a contradiction. Hence $\{g^{\varepsilon}(x,z),\ 0<\varepsilon\leq 1\}$ is bounded. By Arzela-Ascoli theorem, $g^{\varepsilon}(x,\cdot)$, $0<\varepsilon\leq 1$, is relatively compact in $C(D\backslash\{x\})$ with the topology of uniform convergence on compacts. Pick a sequence $\{\varepsilon(n)\}$ in $(0,1]$ such that $\varepsilon(n) \downarrow 0$ and let $\bar{g}(x,\cdot)$ be a limit point in $C(D\backslash\{x\})$ of $\{g^{\varepsilon(n)}(x,\cdot)\}$. Then for any $f\varepsilon C(\bar{D})$ with support in $D\backslash\{x\}$,

$$\int f(y)g^{\varepsilon(n)}(x,y)dy \to \int f(y)\bar{g}(x,y)dy.$$

From (3.1), it follows that $\bar{g}(x,\cdot) = g(x,\cdot)$. Letting $\varepsilon \to 0$ in (3.2), the claim follows for given v. That it holds uniformly for all v is clear from the fact that α, K depend only on λ,A and the bounds on m,σ. Q.E.D.

Corollary 3.1. The set of $g(x,\cdot)$ as $u(\cdot)$ varies over all Markov (equivalently, all admissible) controls is compact in $C(D\backslash\{x\})$.

Proof. Relative compactness of this set follows as above. That any limit point of it is also a Green function for some Markov control can be proved by using the argument of the last part of the proof of Lemma 3.1 in conjunction with Theorems 2.1 and 2.2. Q.E.D.

Theorem 3.1. An optimal Markov control exists.

Proof. Let $\{u_n(\cdot)\}$ be Markov controls such that

$$J_x(u_n(\cdot)) \downarrow \inf J_x(u(\cdot))$$

where the infimum is over all Markov (equivalently, all admissible) controls. Let $\{g_n(x,.)\}$ be the corresponding Green functions. Let $u(\cdot)$ be a Markov control with $g(x,\cdot)$ the corresponding Green function, such that $g^n(x,\cdot) \rightarrow g(x,\cdot)$ in $C(D\backslash\{x\})$ along a subsequence. Thus $g^n(x,\cdot) \rightarrow g(x,\cdot)$ uniformly on δB along this subsequence. The optimality of $u(\cdot)$ follows easily from this.

Q.E.D.

Let $u(\cdot)$ above be of the form $v(X(\cdot))$. The above theorem does not tell us whether the same v would be optimal for any choice of x. This issue is settled in Section V using the verification theorem, which also allows us to drop the relaxed control framework. As a preparation for that, we derive some regularity properties of the value function V in the next section.

IV. REGULARITY OF THE VALUE FUNCTION

Recall the definition of the value function V from Section I.

Lemma 4.1. V is continuous on $\bar{D}\backslash\delta B$.

Proof. Let $x(n) \to x(\infty)$ in $\bar{D}\backslash\delta B$. For $n=1,2,\ldots$, let $u_n(\cdot)$ be the optimal Markov control when the initial condition is $x(n)$ and $g_n(x(n), \cdot)$ the corresponding Green function. By arguments similar to those of the preceding section, we can arrange that (by dropping to a subsequence if necessary) $g_n(x(n),\cdot) \to g_\infty(x(\infty), \cdot)$ uniformly on compact subsets of D that are disjoint from $\{x(n), n=1,2,\ldots, \infty\}$ (in fact, disjoint from $x(\infty)$ will do), where $g_\infty(x(\infty),\cdot)$ is the Green function for some Markov control $u_\infty(\cdot)$ when the initial condition is $x(\infty)$. It follows that

$$J_{x(n)}(u_n(\cdot)) \to J_{x(\infty)}(u_\infty(\cdot)). \tag{4.1}$$

Let v be any Markov control. Then

$$J_{x(n)}(v) \to J_{x(\infty)}(v)$$

by Feller property. Since

$$J_{x(n)}(v) \geq J_{x(n)}(u_n(\cdot)),$$

we have

$$J_{x(\infty)}(v) \geq J_{x(\infty)}(u_\infty(\cdot)).$$

Hence $u_\infty(\cdot)$ is optimal for the initial condition $x(\infty)$. Then (4.1) becomes

$$V(x(n)) \to V(x(\infty)). \qquad \text{Q.E.D.}$$

Let a_ε, $0<\varepsilon\leq 1$, be a family of compactly supported mollifiers and define $h_\varepsilon : R^n \to R$ by $h_\varepsilon(y) = \int a_\varepsilon(y-z)h(dz)$. Then $\{h_\varepsilon\}$ are smooth with compact supports decreasing to δB as $\varepsilon\downarrow 0$ (and hence can be assumed to be contained in D for all ε). Also, $h_\varepsilon \to h$ as $\varepsilon \to 0$ in the sense of distributions. Thus

$$h_\varepsilon(y)dy \to h(dy) \text{ as } \varepsilon \to 0$$

as measures in $M(\bar{D})$. Pick $\varepsilon(n) \downarrow 0$ in (0,1) and denote $h_{\varepsilon(n)}$ by h_n by abuse of notation. Define

$$V_n(x) = \inf E[\int_0^\tau h_n(X(t))dt], \quad n=1,2,\ldots,$$

the infimum being over all admissible controls. By the results of [1], Ch. IV, Section 3, this infimum is attained by some Markov control $u_n(\cdot) = v_n(X(\cdot))$. Letting $\{g_n(x,\cdot)\}$ be the corresponding Green functions, we have

$$V_n(x) = \int g_n(x,y)h_n(y)dy, \quad n=1,2,\ldots .$$

Lemma 4.2. $V_n \to V$ uniformly on compact subsets of $D\backslash\delta B$.

Proof. Let $K \subset D\backslash\delta B$ be compact and $Q \subset D$ an open neighbourhood of δB such that $\bar{Q} \cap K = \phi$. By familiar arguments, we conclude that $g_n(x,\cdot)$, $n=1,2,\ldots,x\varepsilon K$, is equicontinuous and pointwise bounded on Q. Fix $x\varepsilon K$.

Again familiar arguments show that any subsequence of {n} has a further subsequence, say {n(k)}, along which

$$g_n(x,\cdot) \to g_\infty(x,\cdot)$$

uniformly on Q, where $g_\infty(x,\cdot)$ is the Green function corresponding to some Markov control $u_\infty(\cdot)$. Without any loss of generality, we may assume that the supports of $\{h_n\}$ are contained in Q. Then for $m = n(k)$, $k=1,2,\ldots,$

$$|V_m(x) - \int g_\infty(x,y)h(dy)| \leq |\int g_\infty(x,y)h_m(y)dy - \int g_\infty(x,y)h(dy)| + \sup_{y\varepsilon Q}|g_m(x,y) - g_\infty(x,y)|\,|h|(\delta B)$$

Hence $V_{n(k)}(x) \to \int g_\infty(x,y)h(dy)$. Let $u(\cdot)$ be an arbitrary Markov control and $g(x,\cdot)$ the corresponding Green function. Then

$$\int g(x,y)h_n(y)dy \to \int g(x,y)h(dy).$$

Since

$$\int g(x,y)h_n(y)dy \geq V_n(x),$$

we have

$$\int g(x,y)h(dy) \geq \int g_\infty(x,y)h(dy).$$

Hence $\int g_\infty(x,y)h(dy) = V(x)$ and $V_n(x) \to V(x)$. Note that for each n, V_n satisfies $L_nV_n = 0$ on $D\backslash supp(h_n)$ where L_n is the extended generator of the Markov process corresponding to $u_n(\cdot)$ [1]. Arguments similar to those of Lemma 3.1 can now be employed to show that $V_n(\cdot)$ are equicontinuous in a neighbourhood of K. It follows that the convergence of V_n to V is uniform on K. Q.E.D.

Let $A \subset D\backslash\delta B$ be open with a C^2 boundary δA that does not intersect δB and define $\xi = \inf\{t \geq 0 | X(t) \notin \bar{A}\}$ for $X(\cdot)$ as in (1.1) with $x \in A$. Define a meaure η_x on $\bar{A}$ by

$$\int f d\eta_x = E[\int_0^\xi f(X(t))dt], \quad f \in C(\bar{A}).$$

We shall briefly digress to insert a technical lemma whose full import is needed only in the next section.

<u>Lemma 4.3</u>. η_x is mutually absolutely continuous with respect to the Lebesgue measure on $\bar{A}$.

<u>Proof</u>. Absolute continuity of η_x with respect to the Lebesgue measure follows from the Krylov inequality ([4], Section 4.6). To show the converse, first note that by Theorem 3.1 with A replacing D, it suffices to consider $u(\cdot)$ Markov. Let $q(x,\cdot)$ denote the density of η_x with respect to the Lebesgue measure. Then $q(x,\cdot) \geq 0$ on $A\backslash\{x\}$. Our claim follows if we show that the strict inequality holds. Suppose that for some $y \in A\backslash\{x\}$,

$q(x,y) = 0$. Let Q be an open ball in A containing x and disjoint from some open neighbourhood of y. In Q, $q(\cdot,y)$ satisfies $L'q(\cdot,y) = 0$ where L' is the extended generator of the Markov process under consideration. By the maximum principle for elliptic operators, $q(\cdot,y) = 0$ on A. It is easy to see that this leads to $q(\cdot,y) = 0$ on $A\backslash\{y\}$. By Fubini's theorem,

$$\int_A [\int_A q(x,y)dy]dx = \int_A [\int_A q(x,y)dx]dy = 0,$$

implying

$$\int_A q(x,y)dy = E[\xi|X(0) = x] = 0 \quad \text{a.e.},$$

a contradiction. The claim follows. Q.E.D.

Let A,x be as above and $u_n(\cdot)$, $g_n(x,\cdot)$, $n=1,2,\ldots,u(\cdot)$, $g(x,\cdot)$ as in Lemma 4.2. Define $q_n(x,\cdot)$, $n=1,2,\ldots$, and $q(x,\cdot)$ correspondingly.

Lemma 4.4.

$$V(x) = \inf[\int_A q(x,y)h(dy) + E[V(X(\xi))]] \tag{4.2}$$

where the infimum is over all admissible controls. In particular, if $\delta B \subset D\backslash\bar{A}$, this reduces to

$$V(x) = \inf E[V(X(\xi))]. \tag{4.3}$$

Proof. Without any loss of generality, we may assume that the supports of

$\{h_n\}$ are contained in the same connected component of $D\backslash\delta A$ as δB. Let $X^n(\cdot)$, $n=1,2,\ldots$, be the solutions to (1.1) under $u_n(\cdot)$, $n=1,2,\ldots$, resp. and $\xi^n = \inf\{t\geq 0 \,|\, X^n(t) \notin \bar{A}\}$. By the results of [1], Ch. IV, Section 4.3,

$$V_n(x) = \int_A q_n(x,y)h_n(y)dy + E[V_n(X^n(\xi^n))].$$

As in Theorem 2.2, we can have a process $X^\infty(\cdot)$ starting at x and controlled by some Markov control $u_\infty(\cdot)$ such that for $\xi^\infty = \inf\{t\geq 0 \,|\, X^\infty(t) \notin \bar{A}\}$ and $f_1 \in C(\bar{A})$, $f_2 \in C(\delta A)$,

$$E[\int_0^{\xi^n} f_1(X^n(t))dt] \to E[\int_0^{\xi^\infty} f_1(X^\infty(t))dt],\ E[f_2(X^n(\xi^n))] \to E[f_2(X^\infty(\xi^\infty))]$$

Define $q_\infty(x,\cdot)$ correspondingly.

Arguments similar to Lemma 3.1 show that $q_n(x,\cdot) \to q_\infty(x,\cdot)$ uniformly on compact subsets of $A\backslash\{x\}$. Thus

$$\int_A q_n(x,y)h_n(y)dy \to \int_A q_\infty(x,y)h(dy).$$

By the conclusion concerning $\{\mu_x\}$ in Theorem 2.2 (with A replacing D) and Lemma 4.2 above,

$$E[V_n(X^n(\xi^n))] \to E[V(X^\infty(\xi^\infty))].$$

Thus

$$V(x) = \int_A q_\infty(x,y)h(dy) + E[V(X^\infty(\xi^\infty))]$$

The results of [4], Ch. IV, Section 4.3, also imply that if $X(\cdot)$ is the solution to (1.1) under $u(\cdot)$, then

$$V_n(x) \leq \int_A q(x,y)h_n(y)dy + E[V_n(X(\xi))]$$

Taking limits,

$$V(x) \leq \int_A q(x,y)h(dy) + E[V(X(\xi))]$$

The claim follows. Q.E.D.

Theorem 4.1. $V \in H^2_{loc}(D\backslash\delta B)$ and satisfies

$$\inf_u (LV)(x,u) = 0 \quad \text{a.e. on } D\backslash\delta B \tag{4.4}$$

Proof. (4.3) above implies that V restricted to any A in $D\backslash\delta B$ satisfying $\delta B \subset D\backslash\bar{A}$ is the value function for the control problem on A with $E[V(X(\xi))]$ as the cost. The claim follows from [4], Ch. IV, Section 2.2. Q.E.D.

V. A VERIFICATION THEOREM

We shall now derive an analog of the classical vertification theorem that allows us to improve on Theorem 3.1. Let $u(\cdot) = v(X(\cdot))$ be a Markov control which is optimal for the initial condition x, $X(\cdot)$ being the corresponding solution to (1.1).

Lemma 5.1. $(LV)(x,v(x)) = 0$ a.e. in $D\backslash\delta B$. (5.1)

Proof. Let A be as in the proof of Theorem 4.1. Let $x \varepsilon A$ and define $\gamma_x \varepsilon M(\bar{D})$ by

$$\int f d\gamma_x = E[\int_\xi^\tau f(X(t))dt] = \int_D f d\nu_x - \int_D f d\eta_x, \tag{5.2}$$

for $f \varepsilon C(\bar{D})$. Then γ_x has a density $p(x,\cdot)$ with respect to the Lebesgue measure which coincides with $g(x,\cdot)$ on $D\backslash\bar{A}$. For any bounded continuous f supported in $D\backslash\bar{A}$,

$$\int g(x,y)f(y)dy = \int p(x,y)f(y)dy = E[\int g(X(\xi),y)f(y)dy]$$

by virtue of (5.2) and the strong Markov property. Letting $f=h_n$, $n=1,2,\ldots$, successively in the above and taking limits,

$$V(x) = E[\int g(X(\xi),y)h(dy)].$$

Thus $V(x) \geq E[V(X(\xi))]$. By (4.3), $V(x) = E[V(X(\xi))]$. By Krylov's extension of the Ito formula ([4], Section 2.10), it follows that

$$\int_A (LV)(y,v(y))\eta_x(dy) = 0.$$

By Theorem 4.1 and Lemma 4.3,

$$(LV)(y,v(y)) \geq 0 \ \eta_x \text{ - a.e. on } A.$$

Hence

$$(LV)(y,v(y)) = 0 \quad \eta_x \text{ - a.e.}$$

on A and hence Lebesgue - a.e. by Lemma 4.3. Q.E.D.

A variation on the above theme yields the following.

Lemma 5.2. If a Markov control v is optimal for some initial condition $x \varepsilon D \backslash \delta B$, it is also optimal for any other initial condition in $D \backslash \delta B$.

Proof. Let A,x be as in Lemma 4.4 with A connected and v an optimal Markov control for $X(0) = x$. Define $g(x,\cdot)$, $q(x,\cdot)$ correspondingly. Then

$$\int g(x,y)h_n(y)dy = E[\int_0^\tau h_n(X(t))dt]$$

$$= E[\int_0^\xi h_n(X(t))dt] + E[\int_\xi^\tau h_n(X(t))dt]$$

$$= \int_A q(x,y)h_n(y)dy + E[\int g(X(\xi),y)h_n(y)dy]$$

by the strong Markov property. Letting $n \to \infty$,

$$V(x) = \int_A q(x,y)h(dy) + E[\int g(X(\xi), y)h(dy)$$

$$\geq \int_A q(x,y)h(dy) + E[V(X(\xi))].$$

By (4.2), equality must hold. Hence

$$\int g(X(\xi), y)h(dy) = V(X(\xi)) \quad \text{a.s.} \tag{5.3}$$

Note that the maps $z \to V(z)$ and $z \to \int g(z,y)h(dy)$ for $z \varepsilon \delta A$ are continuous. Since the support of $X(\xi)$ is the whole of δA (this would follow, e.g., from the Stroock-Varadhan support theorem), this along with (5.3) implies that

$$V(z) = \int g(z,y)h(dy) \text{ for } z \varepsilon \delta A,$$

i.e., v is also optimal for the initial conditions $z \varepsilon \delta A$. Since A can be chosen so as to contain any prescribed point of $D \backslash \delta B$, the claim follows.

Q.E.D.

This allows us to prove the following converse to Lemma 5.1.

<u>Lemma 5.3</u>. A Markov control v is optimal if (5.1) holds.

<u>Proof</u>. Fix $x \varepsilon D \backslash \delta B$. Let A_1, A_2 be open sets in $D \backslash \delta B$ with C^2 boundaries δA_1, δA_2 resp. such that $x \varepsilon A_1 \subset \bar{A}_1 \subset A_2$ and $\delta B \subset D \backslash \bar{A}_2$. Let v_1 be an optimal Markov control. Let $v_2(\cdot) = v(\cdot)$ on A_1 and $= v_1(\cdot)$ elsewhere. Let $X(\cdot)$ be the process starting at x and controlled by v_2. Define the stopping times

$$\tau_0 = 0$$

$$\tau_1 = \inf\{t \geq 0 \,|\, X(t) \notin \bar{A}_2\}$$

$$\tau_{2n} = (\inf\{t \geq \tau_{2n-1} \,|\, X(t) \in A_1\}) \wedge \tau$$

$$\tau_{2n+1} = (\inf\{t \geq \tau_{2n} \,|\, X(t) \notin \bar{A}_2\}) \wedge \tau$$

for n=1,2,... Then $\tau_n \uparrow \tau$ a.s. Define measures β_{2n} on $\bar{A}_2$ and β_{2n+1} on $\bar{D} \backslash A_1$, n=0,1,2,..., by

$$\int f d\beta_{2n} = E[\int_{\tau_{2n}}^{\tau_{2n+1}} f(X(t))dt] \quad \text{for } f \in C(\bar{A}_2)$$

$$\int f d\beta_{2n+1} = E[\int_{\tau_{2n+1}}^{\tau_{2n+2}} f(X(t))dt] \text{ for } f \in C(\bar{D} \backslash A_1)$$

Since $\{\beta_n\}$ are dominated by ν_x, they have densities with respect to the Lebesgue measure (on $\bar{A}_2$ or $\bar{D} \backslash A_1$ as the case may be). Denote these by $p_n(x,\cdot)$, n = 0,1,2,... By familiar arguments, it can be shown that these are continuous on a neighborhood Q of δB in $D \backslash \bar{A}_2$. Letting $g(x,\cdot)$ be the Green function under v_2, it is clear that

$$\int g(x,y)h(dy) = \sum_{n=0}^{\infty} \int p_{2n+1}(x,y)h(dy) \tag{5.4}$$

where we have used the fact that the supports of β_n for even n are disjoint from δB. Since v, v_1 satisfy (5.1), so does v_2. Hence an application of Krylov's extension of the Ito formula ([4], Section 2.10) yields

$$E[V(X(\tau_{2n}))] = E[V(X(\tau_{2n+1}))], \; n=0,1,2,...$$

On the other hand, in view of the optimality of v_1, arguments similar to those in Lemma 4.4 show that

$$E[V(X(\tau_{2n+1}))] = \int p_{2n+1}(x,y)h(dy) + E[V(X(\tau_{2n+2}))], \; n=0,1,2,...$$

It follows that $V(x) = V(X(\tau_0))$ equals the right hand side of (5.4). Hence v_2 is optimal. Iterating the argument, we construct a sequence of open sets $B_2 \subset B_3 \subset B_4 \subset ...$ in $D\backslash\delta B$ increasing to $D\backslash\delta B$ and optimal Markov controls v_i, $i=2,3,...,$ such that $v_i(\cdot) = v(\cdot)$ on B_i and $= v_1(\cdot)$ elsewhere. Let $\{g_i(\cdot,\cdot)\}$ denote the corresponding Green functions. Fix $x \varepsilon D\backslash\delta B$. By familiar arguments, we have (on dropping to a subsequence if necessary): $g_i(x,\cdot) \to \bar{g}(x,\cdot)$ uniformly on compact subsets of $D\backslash\{x\}$, where $\bar{g}(x,\cdot)$ is the Green function for some optimal Markov control. Now $v_i(x) \to v(x)$ a.e., implying $m_j(x,v_i(x)) \to m_j(x,v(x))$ a.e. as $i \to \infty$, for $1 \leq j \leq n$. For smooth $f:\bar{D} \to R$ with a compact support in $D\backslash\{x\}$,

$$\int g_i(x,y)(Lf)(y,v_i(y))dy = f(x).$$

In view of the foregoing, we can let $i \to \infty$ to obtain

$$\int \bar{g}(x,y)(Lf)(y,v(y))dy = f(x).$$

It follows that $\bar{g}(x,\cdot)$ is the Green function under v. The claim follows.

Q.E.D.

The following theorem recapitulates the above results.

<u>Theorem 5.1</u>. (i) There exists a Markov control v which is optimal under any initial condition in $D\backslash\delta B$.

(ii) A Markov control v is optimal if and only if (5.1) holds.

(iii) An optimal Markov control v may be chosen so that the range of v lies in the set of Dirac measures on S.

<u>Remark</u>. A U-valued control taking values in $U' = \{\text{Dirac measures on } S\} \subset U$ can be associated with an S-valued (i.e. 'ordinary' or 'pre-relaxation') control in an obvious manner. Thus (iii) above allows us to drop the relaxed control framework and replace (4.4) by

$$\inf_{s\varepsilon S}\Big(\sum_{i=1}^{n} b_i(x,s)\frac{\partial V}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j,k=1}^{m} \sigma_{ik}(x)\sigma_{jk}(x)\frac{\partial^2 V}{\partial x_i \partial x_j}(x)\Big) = 0 \quad \text{a.e.}$$

on $D\backslash\delta B$

<u>Proof</u>. Only (iii) needs to be proved. Note that the minimum of

$\nabla V(x) \cdot m(x,\cdot)$, $x \in D\backslash\delta B$, over U will always be attained by an element of U'. Since U' is a compact subset of U, a standard selection theorem (see, e.g., [2], Lemma 1.1) allows us to pick a measurable $v: D\backslash\delta B \to U'$ such that

$$\nabla V(x) \cdot m(x,v(x)) = \min_u \nabla V(x) \cdot m(x,u), \; x \in D\backslash\delta B.$$

Set v(x) = an arbitrary fixed element of U' for $x \in \delta B$. Then v is an optimal Markov control by (ii). Q.E.D.

The above theorem gives a vertification theorem and a recipe for constructing an optimal v, in terms of the function V. Thus it is desirable to have a good characterization of V. Formal dynamic programming considerations and experience in the classical case leads one to expect that V should be characterized as the unique solution to the Hamilton-Jacobi-Bellman equation

$$\inf_u (LV)(x,u) = -h \text{ in } D, \; V=0 \text{ on } \delta D \tag{5.5}$$

in some appropriate sense, where by abuse of terminology, we have let h denote the Schwartz distribution corresponding to the measure h. It is an interesting open problem to make sense of (5.5), thereby obtaining the said characterization of V.

ACKNOWLEDGEMENTS

The work was done while the author was visiting the Laboratory for

Information and Decision Systems, Massachusetts Institute of Technology, during which he was supported by ARO contract DAAG29-84-K-0005 and AFOSR 85-0227.

References

[1] A. Bensoussan, Stochastic control by functional analysis methods, North-Holland, 1982.

[2] V.S. Borkar, A remark on the attainable distributions of controlled diffusions, Stochastic 18(1) (1986), 17-23.

[3] D. Gilbarg, N.S. Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, 1977.

[4] N.V. Krylov, Controlled diffusion processes, Springer-Verlag, 1980.

[5] H.J. Kushner, Existence results for optimal stochastic control, J. Opt. Theory and Appl. 15(4) (1975), 347-359.