Parameter Estimation and Control of Nonlinearly Parameterized Systems

by

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Abstract

Parameter estimation in nonlinear systems is an important issue in measurement, diagnosis and modeling. The goal is to find a differentiator free on-line adaptive estimation algorithm which can estimate the internal unknown parameters of dynamic systems using its inputs and outputs. This thesis provides new algorithms for adaptive estimation and control of nonlinearly parameterized (NLP) systems. First, a Hierarchical Min-max algorithm is invented to estimate unknown parameters in NLP systems. To relax the strong condition needed for the convergence in Hierarchical Min-max algorithm, a new Polynomial Adaptive Estimator (PAE) is invented and the Nonlinearly Persistent Excitation Condition for NLP systems, which is no more restrictive than LPE for linear systems, is established for the first time. To reduce computation complexity of PAE, a Hierarchical PAE is proposed. Its performance in the presence of noise is evaluated and is shown to lead to bounded errors. A dead-zone based adaptive filter is also proposed and is shown to accurately estimate the unknown parameters under some conditions.

Based on the adaptive estimation algorithms above, a Continuous Polynomial Adaptive Controller (CPAC) is developed and is shown to control systems with nonlinearities that have piece-wise linear parameterizations. Since large classes of nonlinear systems can be approximated by piece-wise linear functions through local linearization, it opens the door for adaptive control of general NLP systems. The robustness of CPAC under bounded output noise and disturbances is also established.

Thesis Supervisor: A.M. Annaswamy
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Chapter 1

Introduction

The importance of mathematical models in every aspect of the physical, biological, and social sciences is well known. Starting with a phenomenological model structure that characterizes the cause and effect links of the observed phenomenon in these areas, the parameters of the model are tuned so that the behavior of the model approximates the observed behavior. Alternately, a general mathematical model such as a differential equation or a difference equation can be used to represent the input-output behavior of the given process and model outputs in some sense. In the model identification problem, parameter estimation using observed input output signals in a given model structure is inevitable.

Besides the modeling problem, on-line estimation and diagnoses is another important application area of adaptive estimators. Many systems contains uncertain information which are always represented by unknown parameters especially for some systems which work in changing outer-environments like aircraft or robots. In many cases, there is no way or it is very difficult to measure these unknown or time changing parameters directly, like the shifting of internal parameters or changing outer environments. In some situations the unknown parameter can even be a virtual one if the mathematical model is not a physical one and just predicts the input output relationship well. In those situations, for either estimation, diagnosis or control purposes, it is important to have some knowledge of the unknown or changing internal parameters from measurable input output signals. Adaptive estimators serve as a perfect tool in these applications since they are fast on-line recursive algorithms and the estimation is updated adaptively with the changing parameters.

In addition to the pure estimation applications, adaptive estimators are always related with adaptive control of partial known plants. Many adaptive controllers use adaptive estimators to construct the control law. This thesis concerns the parameter estimation and adaptive control in a special class of dynamic systems where parameters occur nonlinearly.

1.1 Current Research

Adaptive control has emerged as a tool for controlling partial known plant for several decades. The previous work is mainly about linearly parameterized systems, which is a quite mature area and the results summarized in books such as [1]. However, linearly parameterized systems are just a special class and in most cases ideal situation of practical systems. How to extend the adaptive estimator and controller into general nonlinearly parameterized (NLP) systems is an active research area which draws a lot of interests and efforts.

Recently, a stability framework has been established for studying estimation and control of NLP systems in [1]-[8]. In [1]-[8], various NLP systems were considered and the conditions for global stability, regulation and tracking were derived using a min-max algorithm, while in [8], stability and parameter convergence in a class of discrete-time systems was considered.

In the parameter estimation and control of dynamic systems, one commonly raised question is that if you can differentiate the output signals to obtain the information about system parameters. Measurement of output signal y does not mean $\dot{y}$ is also available from output noise and measurement error. In a digital sampling system, measurement on y just requires y to be measured in a desired precision. However, if you want to obtain $\dot{y}$, high precision time recording is also required and small error in y could be amplified. In continuous system, differentiator is not a feasible implementation since its gain reaches infinity as frequency increases. For a feasible physical implementation, we usually require the system to be stable and proper. Adaptive estimator is an differentiator free feasible implementation and it uses only the input output signals. Bounded output noise or measurement error results in bounded estimation error. In this thesis, differentiator free parameter estimators are developed for NLP systems in the presence of noise. In some special cases, such as Chapter 8 in this thesis, some stochastic properties of the noise can be exploited and the parameter can be estimated exactly.

1.2 Thesis Contributions

The thesis contributions are mainly in two areas. One is the development of a series of new adaptive estimation algorithms. Another is the adaptive control of NLP systems based on these estimation algorithms. In the first case, adaptive estimation algorithms for general NLP systems are developed and the Nonlinear Persistent Excitation (NLPE) condition for parameter convergence is established. In the second case, a polynomial adaptive controller (CPAC) is developed for a partial known plant with unknown parameters. Since there is no general control law for nonlinear systems, we develop stable controllers for special classes of NLP systems. In both estimation and control, the robustness of adaptive estimator and controller is established in the presence of output noise. A brief description of the various chapters in this thesis is given below.

In Chapter 2, we consider parameter estimation in static nonlinearly parameterized systems. The training of a neural network is basically a parameter estimation process which finds the unknown parameters from the desired input output relationship. We consider the problem of global convergence in a neural network whose parameters are unknown and are to be identified. In particular, we examine conditions under which global and local minima can occur. Two different training algorithms are considered for estimating the parameters, which include the standard gradient algorithm, and a collective gradient algorithm. In the former case, we provide some sufficient conditions under which global convergence can occur, while in the latter case, we present necessary and sufficient conditions. We conclude with several examples of neural networks with a small number of neurons, and show that these conditions are not satisfied, even in some simple examples, which leads to local minima and therefore non-global convergence.

In the past few years, a stability framework for estimation and control of NLP systems has been established. We address the issue of parameter convergence in such systems in

Chapter 3. Systems with both convex/concave and general parameterizations are considered. In the former case, sufficient conditions are derived under which parameter estimates converge to their true values using a min-max algorithm. In the latter case, to achieve parameter convergence a hierarchical min-max algorithm is proposed where the lower-level consists of a min-max algorithm and the higher-level component updates the bounds on the parameter region within which the unknown parameter is known to lie. Using this hierarchical algorithm, a necessary and sufficient condition is established for global parameter convergence in systems with a general nonlinear parameterization. In both cases, the conditions needed are shown to be stronger than linear persistent excitation conditions that guarantee parameter convergence in linearly parameterized systems. Explanations and examples of these conditions and simulation results are included to illustrate the nature of these conditions. A general definition of Nonlinear Persistent Excitation (NLPE) that leads to parameter convergence of Hierarchical min-max algorithm is proposed at the end of the paper.

Chapter 4 gives an application of Hierarchical min-max algorithm in dynamic system modeling. The goal is to establish the parameterized model of force-displacement dynamics of tissue in liver and esophagus of live animals. The raw data comes from the surgical experiments performed in Harvard Medical school between 2001-2002. The input is the displacement of a robotic end-effector to the tissue surface and the output is the force response. The first step is to establish the parameterized model structure from physical insights and simulations. The hierarchical min-max algorithm is then used for the purpose of parameter estimation. After we have a model with a set of parameters which produces the similar output as experimental data for same input signals, we can use this model to simulate and generate the virtual force response when touching tissues/skins by the robotic end-effector in virtual reality and it can be used to train new doctors for operations. It is observed from the data that the differentiator methods cannot be applied in the practical dynamic system modeling. The optimization objective here is to minimize the error between model output and experimental output for same input signals.

In Chapter 5, we propose a new Polynomial Adaptive Estimator(PAE) to estimate parameters that occur nonlinearly. The estimator is based on a polynomial nonlinearity in the

Lyapunov function which is chosen so that the nonlinearity in the unknown parameter is accommodated as accurately as possible while maintain stability and parameter convergence. We further extend the PAE algorithm to Discretized-parameter Polynomial Adaptive Estimator(DPAE) and establish the Nonlinear Persistent Excitation Condition, which is similar to linear persistent excitation condition and serve as a sufficient condition for parameters to be identified in nonlinearly parameterized system. We show in this Chapter that the DPAE algorithm has the ability to estimate parameters in any Lipschitz continuous nonlinear function if the input and system variables satisfies the NLPE condition. The advantage of DPAE over Hierarchical min-max is that it relaxes the parameter convergence conditions. The NLPE condition for DPAE is much less restrictive than that associated with Hierarchical min-max algorithm.

In Chapter 6, we propose a Hierarchical Discretized-parameter Polynomial Adaptive Estimator (HDPAE) to estimate unknown parameters in Lipschitz continuous systems. It is shown that under the same NLPE Condition, HDPAE has the ability to estimate unknown parameters globally same as DPAE in Chapter 5, and is able to greatly reduce the computation complexity. Different parameter estimation algorithms for both static and dynamic systems are given and comparison among them is discussed. It is shown that Hierarchical Search algorithm for static systems and HDPAE for dynamic systems have the ability to guarantee a globally convergent estimation however the gradient algorithms do not.

In Chapter 7, we focus on parameter estimation in systems with output noise. By adding a dead-zone to the Polynomial Adaptive Estimator, it is shown that statistically the bounded output noise can be filtered out and that the true unknown parameters are estimated exactly under some conditions. This time-domain noise filter which applies to systems with unknown parameters is denoted as filtered dead-zone estimator and it is later extended to situation where the output noise is white noise. The difference between model disturbance and output noise is discussed and the extension to situation where both of them exist is proposed. It is noted that the same dead-zone technique to deal with output noise can be applied to linear adaptive estimator, DPAE and HDPAE as well.

In Chapter 8, an adaptive controller for NLP systems is proposed. We propose a continuous polynomial adaptive controller (CPAC) which deals with piece-wise linearly parameterized functions as the same as traditional adaptive controller for linear ones. Since most of the commonly encountered NLP systems can be piece-wise linearly approximated through local linearization, the CPAC serves as a general tool for them. Stability of CPAC with bounded output noise, disturbance and approximation error is also established. Control laws for several typical classes of NLP systems are provided to demonstrate the applications of the CPAC. The CPAC extends the traditional linear adaptive control theory to general nonlinearly parameterized (NLP) systems and much more subsequent progress is expected.

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Chapter 2

Conditions for Existence of Global and Local Minima in Neural Networks

2.1 Introduction

A neural network is a parameterized function which has been used for many years as a universal approximation method to model an unknown static function. Assuming that the underlying unknown static function is a mapping represented by

$$y = f(x) \tag{2.1}$$

where x, y are inputs and outputs, the neural network is basically a parametric function

$$y = h(x, \theta)$$

which can approximate the function in (2.1) by choosing appropriate parameters θ. The training of a neural network is the process by which we find parameters that makes it approximate the function in (2.1) as closely as possible. It is well known that several networks such as multilayered neural networks [1, 2], and radial basis functions [3, 4] exist that have such a universal approximation ability.

In this chapter, we address a simpler question than the above, which is the following.

Suppose that the underlying unknown static function has the *same* structure as that of a neural network, where the unknown components are simply restricted to the parameter θ, but otherwise h is known. Hence, the modeling of the unknown function in this context reduces to estimation of the unknown parameter θ^*. That is, we start with a system of the form

$$y = h(x, \theta^*) \tag{2.2}$$

where θ^* is an unknown parameter in $\mathbb{R}^N$. The goal is to estimate θ^* using an estimator of the form

$$y = h(x, \hat{\theta}(t))$$

starting from arbitrary initial conditions $\hat{\theta}(t_0)$, and determine the conditions under which global convergence is possible.

A necessary condition in any parameter convergence problem is identifiability. We assume that h is identifiable throughout this chapter. To define this precisely, we denote the input-output training data of the neural network as

$$(x_i, y_i), i = 1, .., M, \tag{2.3}$$

where M is the sample size. We now state the identifiability assumption.

Assumption 1 *For the training data as in (2.3), if*

$$h(x_i, \theta) = h(x_i, \theta^*), \qquad i = 1, .., M,$$

then

$$\theta = \theta^*.$$

Assuming that the underlying neural network satisfies assumption 1, we consider the standard gradient algorithm and a collective gradient algorithm where the training errors from multiple inputs are collectively used to determine the gradient. These algorithms are used to generate a recursive estimate $\hat{\theta}$ of θ. We then examine conditions under which $\hat{\theta}$ converges to θ^* starting from arbitrary initial conditions, using both of these algorithms.

This chapter is organized as follows. In Section 2, we introduce the gradient algorithm which is used to find unknown parameters θ^* and the global convergence condition associated with it is also proposed. In section 3, we present both cases where the collective gradient algorithm leads to global convergence and case with no guarantee of the global convergence. Section 4 summarizes the results and concludes the chapter.

2.2 Global Convergence in Neural Networks

2.2.1 Statement of the Problem

The system under consideration is assumed to be of the form

$$y(x, \theta^*) = h(x, \theta^*) \tag{2.4}$$

where $x, y : \mathbb{R} \to \mathbb{R}$, x and y denote the input and output of the neural network, respectively, $\theta_i^* \in \mathbb{R}$, and $\theta^* \in \mathbb{R}^N$ is the unknown parameter to be identified. For example, θ^* represents the weights and biases in a single-layered neural network. We propose to identify θ^* using an estimator of the form

$$\hat{y}(x, \widehat{\theta}) \quad = \quad = h(x, \widehat{\theta}) \tag{2.5}$$

and a recursive algorithm that generates an estimate $\widehat{\theta}(t)$ of θ^* at each instant of time. The goal of this chapter is to determine conditions under which $\widehat{\theta}(t)$ converges asymptotically to θ^* starting from *arbitrary* values in $\mathbb{R}^N$.

Generally, gradient algorithms are employed to find the weights of the neural network [5].Typically, in these algorithms, the training error

$$V = \sum_{i=1}^{M} \frac{1}{2}(y_i - h(x_i, \hat{\theta}))^2$$

is used to determine the gradient, where x_i and y_i are the training data defined in (2.3). Below, we first present an instantaneous gradient algorithm and the related convergence results in [6]. We then present a collective gradient algorithm whose convergence conditions are less restrictive.

Instantaneous Gradient Algorithm

In [6], a standard gradient algorithm (such as the back-propagation) was proposed, and is of the form

$$\dot{\hat{\theta}} = -\left(\hat{y}(x,\hat{\theta}) - y(x,\theta^*)\right) \nabla_{\hat{\theta}} h(x,\hat{\theta}). \tag{2.6}$$

The following assumptions are made regarding h:

Assumption 2 *We assume that the function $h(x,\theta)$ is differentiable and the magnitudes of the first derivatives $\nabla_\theta h(x,\theta)$ are bounded.*

We also assume that h is monotonic with respect to θ. That is, if $\lambda(x,\theta)$ denotes the gradient of h with respect to θ, i.e.

$$\lambda(x,\theta) = \nabla_\theta h(x,\theta),$$

we assume that the following holds:

Assumption 3 $\lambda(x_1,\theta)\lambda(x_2,\theta) \geq 0$ *for any x_1,x_2 and θ.*

The following definitions are useful for stating the convergence result:

Definition 1 *Let $q(a) = l(a) \otimes \{\eta_1(a), \eta_2(a), \ldots, \eta_n(a)\}$ denote the orthogonal projection of a vector l at a point* a *onto the surface whose tangent plane at* a *is defined by normals $\{\eta_1(a), \ldots, \eta_i(a)\}$. The orthogonal projection is defined as*

$$\begin{aligned} q(a) &= l(a) - \sum_{j=1}^{n} \frac{l^T \nu_j}{||\nu_j||^2} \nu_j, \qquad \textit{where} \\ \nu_j &= \eta_j(a) - \sum_{k=1}^{j-1} \frac{\eta_j^T \nu_k}{||\nu_k||^2} \nu_k, \qquad j \in \{1,\ldots,n\} \end{aligned}$$

Definition 2 *Let*

$$
\begin{aligned}
H(\theta) &= \{\, u_i \mid e(u_i,\theta)=0\} \\
H_\lambda(\theta) &= \{\, \lambda(u,\theta) \mid u \in H(\theta)\} \\
I(\theta) &= \dim\{\mathcal{L}\{H_\lambda(\theta)\}\} \qquad (2.7)\\
K_i &= \{\, \theta \mid I(\theta) \geq i\}
\end{aligned}
$$

and

$$
M(\Psi_M) \;=\; \{\theta \mid e(u(t),\theta)=0,\;\; t\in\Psi_M\} \qquad (2.8)
$$

The set $\Lambda(\Psi_M,\theta)$ is the set of all normals to the manifold M at a point θ. Let

$$
\begin{aligned}
m_u(\theta_a) &= \lambda(u,\theta_a)\otimes\Lambda(\Psi_M,\theta_a) \\
m_l(\theta_a) &= \lambda(u(t_l),\theta_a)\otimes\Lambda(\Psi_M,\theta_a), \qquad (2.9)
\end{aligned}
$$

we define a vector field $q(\theta)$ such that

$$
\begin{aligned}
q(\theta) &= m_l(\theta) \qquad \theta\in K_i \\
q(\theta) &= m_u(\theta) \qquad \theta\in K_{i+1}. \qquad (2.10)
\end{aligned}
$$

Definition 3 *Let $T_x > 0$, and let $\lambda(x,\theta)=\nabla_\theta h(x,\theta)$, with $h(x,\theta):\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$. Let*

$$
\begin{aligned}
\Omega_t &= [t_0, t_0+T_x], \qquad\qquad T_x > 0 \\
\Psi &= \{t_i\in\Omega_t,\quad i=1,\ldots,M \mid |t_{i+1}-t_i| > \varepsilon_0\}, \\
& \qquad M,\varepsilon_u > 0 \\
\Lambda(\Psi,\theta) &= \{\lambda(x(t_k),\theta) \mid t_k\in\Psi\}
\end{aligned}
$$

A function $x(t):\mathbb{R}\to\mathbb{R}$ is said to belong to the class U_{PE}^N on the interval $t\in[t_0,t_0+T_x]$ if it satisfies the three properties stated below:

(P1) linear independence is invariant: *If the set $\Lambda(\Psi_a, \theta_a)$ is linearly independent for some set $\Psi_a \in \Omega_t$ and $\theta_a \in \Omega_\theta$, then $\Lambda(\Psi_a, \theta)$ is linearly independent for all $\theta \in \Omega_\theta$.*

(P2) sufficient degree of excitation exists: *There exists a set $\Psi_b \in \Omega_t$ consisting of N elements such that $\Lambda(\Psi_b, \theta_a)$ is linearly independent.*

(P3) potential field exists: *For the vector field $q(\theta)$ constructed in (2.10), there exists a potential field $s(\theta)$ such that $\nabla s = q$.*

The main convergence result in [6] is summarized in the following Theorem.

Theorem 1 *Let Assumptions 1, 2 and 3 hold. For the system in (2.4)-(2.5), if for every $t > 0$ there exist a $t_1 > t$ and $T > 0$ such that $u(t) \in U_{PE}^N$ over the interval $[t_1, t_1 + T]$, then $\lim_{t \to \infty} \widehat{\theta}(t) = \theta^*$.*

The reader is referred to [6],[7] for the proof of Theorem 1.

As stated in the above theorem, global convergence in single-layered neural networks can be guaranteed provided the properties (P1), (P2), and (P3) are satisfied by the input x. Of the three, (P1) and (P2) are relatively easy to be guaranteed since they concern excitation properties of the input, and are qualitatively related to linear independence. However, (P3) concerns a topological property of the nonlinear system defined in (2.4), (2.5) and (2.6). This property is central to the global convergence of the neural network and is needed in order to guarantee the existence of a converging metric and therefore an associated Lyapunov function. Only when $q(\theta)$ is an irrotational field, i.e. $\nabla \times q = 0$, is it possible that a potential field s can be found to make assumption (P3) satisfied. In general, there is no guarantee that $q(\theta)$ is irrotational and it is extremely difficult if not impossible to determine what classes of x will satisfy these properties. It is therefore useful to examine if other algorithms that do not require such restrictive assumptions can be determined that can still guarantee global convergence. As shown in the next section, such an algorithm can indeed be found.

2.2.2 A Collective Gradient Algorithm

Instead of determining the gradient by using a single input and output, we take an alternative approach in this section by collecting multiple input-output pairs $(x_i, y_i), i = 1, .., M,$, where $y_i = h(x_i, \theta^*)$ and M is the number of input-output pairs. If we use the same estimator structure as in (2.5), we can define the corresponding output error as

$$V = \sum_{i=1}^{M} \frac{1}{2}(y_i - h(x_i, \hat{\theta}))^2. \tag{2.11}$$

We now determine the gradient algorithm using V, which can be viewed as a collective output error for a range of inputs $x_i, i = 1, \ldots, M$. In this collective gradient algorithm we update $\hat{\theta}$ using the negative gradient of V with respect to $\hat{\theta}$. That is,

$$\dot{\hat{\theta}} = -\nabla_{\hat{\theta}} V. \tag{2.12}$$

The question that arises is if the estimator in (2.5) together with the gradient algorithm in (2.12) can guarantee global convergence of $\hat{\theta}$ to θ^*.

We assume N to be even for ease of exposition. Then the structure of the nonlinearity h in a neural network is of the form

$$h(x, \hat{\theta}) = \sum_{j=1}^{N/2} \hat{\theta}_j g(\hat{\phi}_j x) \tag{2.13}$$

where $\hat{\theta} = \{\hat{\theta}_1, ..., \hat{\theta}_i, ..., \hat{\theta}_{N/2}, \hat{\phi}_1, ..., \hat{\phi}_i, ..., \hat{\phi}_{N/2}\}$ are the parameters, and the input x and output y belong to $\mathbb{R}$. Combining (2.11) and (2.13), (2.12) can be rewritten as

$$\dot{\hat{\theta}} = \sum_{j=1}^{M} (y_j - h(x_j, \hat{\theta})) v_j \tag{2.14}$$

where

$$v_j = [g(\hat{\phi}_1 x_j), ..., g(\hat{\phi}_i x_j), ..., g(\hat{\phi}_{N/2} x_j), \hat{\theta}_1 \frac{\partial g(\hat{\phi}_1 x_j)}{\partial \hat{\theta}_1}, ..., \hat{\theta}_i \frac{\partial g(\hat{\phi}_i x_j)}{\partial \hat{\theta}_i}, ..., \hat{\theta}_{N/2} \frac{\partial g(\hat{\phi}_{N/2} x_j)}{\partial \hat{\theta}_{N/2}}]^T. \tag{2.15}$$

From the neural network structure in (2.13), it is easy to see that Ω defined below is an invariant set

$$\Omega = \{\hat{\theta}|\hat{\theta}_i = \hat{\theta}_j, \hat{\phi}_i = \hat{\phi}_j, \forall i \neq j\}. \tag{2.16}$$

For any $\hat{\theta}(t_0) \in \Omega$, we could see from (2.14) that $\hat{\theta}(t) \in \Omega, t \geq t_0$ as well. Therefore we focus on only those initial conditions that do not lie in Ω from here onwards.

Define $E = \mathbb{R}^N \backslash \Omega$ and $\tilde{\theta} = \hat{\theta} - \theta^*$. We now state the convergence result in Theorem 2.

Theorem 2 *Under assumption 1, if*

$$V(\tilde{\theta}) \to \infty \qquad as\ \|\tilde{\theta}\| \to \infty, \tag{2.17}$$

for the system in (2.4) and estimator in (2.5), for any $\hat{\theta}(t_0) \in E$, the gradient algorithm in (2.12) leads to $\hat{\theta}(t) \to \theta^*$ as $t \to \infty$ iff

$$\nabla_\theta V = 0 \Longleftrightarrow \theta = \theta^* \qquad \theta \in E \tag{2.18}$$

where V is defined as in (2.11).

Proof of Theorem 2: From (2.4), (2.5), (2.11), and (2.12), it follows that V is an autonomous system of $\tilde{\theta}$ with

$$\dot{V}\ (\tilde{\theta}) = -\|\nabla_{\hat{\theta}} V\|^2 \leq 0. \tag{2.19}$$

Assumption 1 states $V(\tilde{\theta})$ is a positive definite function of $\tilde{\theta}$, and Assumption 2.17 implies that $V(\tilde{\theta})$ is a decrescent function of $\tilde{\theta}$. From (2.19), it therefore follows that $\hat{\theta}(t) \in L^\infty$. Equation (2.19) and condition (2.18) implies that $\dot{V}\ (\tilde{\theta})$ is a negative definite function of $\tilde{\theta}$. Therefore, it follows that $\hat{\theta}(t) \to 0$ as $t \to \infty$. Necessity of (2.18) can be proved in a similar manner. •

When condition (2.17) and (2.18) are satisfied, global convergence follows and a simple example is

$$h(x, \theta^*) = x\theta^*$$

where $x, \theta^* \in \mathbb{R}$. In general, however, condition (2.17) is not satisfied or difficult to check.

Hence, we have the following corollary which yields a convergence condition in a given set under less restrictive conditions.

Assume the minimum limit value of $V(\tilde{\theta})$ as $\tilde{\theta} \to \infty$ is C and we define a region

$$\Omega_1 = \{\theta \mid V(\theta - \theta^*) < C, \theta \in \mathbb{R}^N\}. \tag{2.20}$$

and $E_1 = \Omega_1 \backslash \Omega$. The corollary below discusses convergence in the set E_1.

Corollary 1 *Under assumption 1, for the system in (2.4) and estimator in (2.5), for any $\hat{\theta}(t_0) \in E_1$, the gradient algorithm in (2.12) leads to $\hat{\theta}(t) \to \theta^*$ as $t \to \infty$ iff*

$$\nabla_\theta V = 0 \Longleftrightarrow \theta = \theta^* \qquad \theta \in E_1 \tag{2.21}$$

where V is defined as in (2.11).

Proof of Corollary 1: It follows from (2.19) that once $\hat{\theta}(t_0) \in E_1$, i.e.

$$V(t_0) < C, \tag{2.22}$$

we have

$$\hat{\theta}(t) \in E_1, \qquad t \geq t_0. \tag{2.23}$$

Because the minimum limit of V as $\tilde{\theta} \to \infty$ is C, it follows that $\hat{\theta}(t)$ will not converge to ∞ and is bounded. Therefore, similar to the proof in Theorem 2, it is easy to show that (2.21) is a sufficient and necessary condition for $\hat{\theta}(t) \to \theta^*$ as $t \to \infty$. •

From Theorem 2 and Corollary 1, it is shown that the global convergence in region E or at least in region E_1 is equivalent to establishing condition (2.18). Since $\nabla_\theta V = 0$ holds for any θ that satisfies

$$\sum_{j=1}^{M} (y_j - h(x_j, \theta))v_j = 0, \tag{2.24}$$

it is of interest to examine conditions under which θ is an extremum of V. More precisely, we have the following property.

Property 1 *If θ exists such that (2.24) is satisfied, then one of the following holds:*

$$\begin{aligned} (i) \quad & \theta \quad \textit{is a saddle point of } V; \\ (ii) \quad & \theta \quad \textit{is a local minimum of } V; \\ (iii) \quad & \theta \quad \textit{is a local maximum of } V. \end{aligned}$$

Remark 1: Theorem 2 implies that the collective gradient algorithm guarantees global convergence if (2.18) is satisfied. It is worth noting that this condition is considerably less restrictive than those needed by the instantaneous gradient algorithm which required [P1], [P2] and [P3] to hold. In fact, we note that (2.18) is almost identical to [P1].

Remark 2: Suppose that a θ that satisfies Property 1-(iii) exists, and V has no other local extrema other than θ and θ^*. It follows that except for one initial condition where $\hat{\theta}(t_0) = \theta$, all other initial conditions will converge to θ^*. Therefore, to guarantee global convergence, we need to address only θ's that satisfy Property 1-(i) and 1-(ii).

The central question that remains to be pursued is if at a given point θ in $\mathbb{R}^N$, the gradients v_j defined as in (2.15) are linearly independent. Since the answer to this question for a general function $h(x, \theta)$ depends on h, the derivation of constructive conditions for checking linear independence at all θ is extremely difficult if not impossible. In what follows, we address specific examples of h and discuss the existence of local extremes. In particular, we provide some counterexamples of h which indeed do have local saddle points, and their implications.

2.3 Conditions of Global Convergence

Four different neural networks with very few parameters are considered in this section to further evaluate if θ that satisfies Property 1-(i) and Property 1-(ii) exists. In sections 3.1 and 3.2, we consider exponential and sigmoidal nonlinearities, respectively, with no output weights. In section 3.3 and 3.4, we consider counter examples which are a function with sin component and a sigmoidal nonlinearity with output weights, respectively.

2.3.1 Example 1: Exponential Functions

We consider a 3-node network of the form

$$h(x,\theta^*) = \sum_{i=1}^{3} e^{\theta_i^* x} \tag{2.25}$$

in this section. It follows that the collective gradient algorithm is given by

$$\dot{\hat{\theta}}_i = \sum_{j=1}^{3}(y_j - \sum_{i=1}^{3} e^{\hat{\theta}_i x_j})x_j e^{\hat{\theta}_i x_j} \qquad i = 1,2,3. \tag{2.26}$$

Here the invariant set is given by

$$\Omega = \{\theta | \theta_i = \theta_j, \forall i \neq j \text{ and } i,j = 1,2 \text{ or } 3\}.$$

Before establishing the convergence, a lemma which shows the full rank of a matrix is needed and stated below.

Lemma 1

$$\begin{bmatrix} e^{\theta_1 x_1} & e^{\theta_1 x_2} & e^{\theta_1 x_3} \\ e^{\theta_2 x_1} & e^{\theta_2 x_2} & e^{\theta_2 x_3} \\ e^{\theta_3 x_1} & e^{\theta_3 x_2} & e^{\theta_3 x_3} \end{bmatrix} \tag{2.27}$$

is full rank where

$$\theta_i \neq \theta_j, x_i \neq x_j, \qquad \forall i \neq j.$$

Proof of lemma 1: Without loss of generality, we assume

$$\begin{aligned} \theta_3 > \ \theta_2 > \ \theta_1 \\ x_3 > \ x_2 > \ x_1. \end{aligned}$$

Scale ith row by $e^{-\theta_i x_1}$ and subtract the first row from the second and third rows, matrix

(2.27) can be transformed into

$$\begin{bmatrix} 1 & e^{\theta_1 \hat{x}_1} & e^{\theta_1 \hat{x}_2} \\ 0 & e^{\theta_2 \hat{x}_1} - e^{\theta_1 \hat{x}_1} & e^{\theta_2 \hat{x}_2} - e^{\theta_1 \hat{x}_2} \\ 0 & e^{\theta_3 \hat{x}_1} - e^{\theta_1 \hat{x}_1} & e^{\theta_3 \hat{x}_2} - e^{\theta_1 \hat{x}_2} \end{bmatrix} \tag{2.28}$$

where

$$\hat{x}_2 = x_3 - x_1 > \hat{x}_1 = x_2 - x_1 > 0.$$

Scaling the second and third columns by $e^{-\theta_1 \hat{x}_1}$ and $e^{-\theta_1 \hat{x}_2}$, respectively, the matrix (2.28) is transformed into

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & e^{\tilde{\theta}_1 \hat{x}_1} - 1 & e^{\tilde{\theta}_1 \hat{x}_2} - 1 \\ 0 & e^{\tilde{\theta}_2 \hat{x}_1} - 1 & e^{\tilde{\theta}_2 \hat{x}_2} - 1 \end{bmatrix}$$

where

$$\tilde{\theta}_2 = \theta_3 - \theta_1 > \tilde{\theta}_1 = \theta_2 - \theta_1 > 0.$$

Therefore, the full rank of (2.27) is equivalent to show full rank of

$$\begin{bmatrix} e^{\tilde{\theta}_1 \hat{x}_1} - 1 & e^{\tilde{\theta}_1 \hat{x}_2} - 1 \\ e^{\tilde{\theta}_2 \hat{x}_1} - 1 & e^{\tilde{\theta}_2 \hat{x}_2} - 1 \end{bmatrix} \tag{2.29}$$

where

$$\begin{aligned} \hat{x}_2 &> \hat{x}_1 &> 0 \\ \tilde{\theta}_2 &> \tilde{\theta}_1 &> 0. \end{aligned}$$

Full rank of matrix (2.29) can be shown to be true if we can show function

$$\frac{e^{\tilde{\theta}_2 x} - 1}{e^{\tilde{\theta}_1 x} - 1}, \qquad \tilde{\theta}_2 > \tilde{\theta}_1 > 0 \tag{2.30}$$

is monotonous increasing for $x > 0$. The derivative of (2.30) is

$$\frac{\tilde{\theta}_2 e^{\tilde{\theta}_2 x}(e^{\tilde{\theta}_1 x} - 1) - \tilde{\theta}_1 e^{\tilde{\theta}_1 x}(e^{\tilde{\theta}_2 x} - 1)}{(e^{\tilde{\theta}_1 x} - 1)^2}$$

and if we can show

$$\tilde{\theta}_2 e^{\tilde{\theta}_2 x}(e^{\tilde{\theta}_1 x} - 1) > \tilde{\theta}_1 e^{\tilde{\theta}_1 x}(e^{\tilde{\theta}_2 x} - 1), \tag{2.31}$$

lemma 1 is proved. The polynomial expansion of $\tilde{\theta}_2 e^{\tilde{\theta}_2 x}$ is

$$\tilde{\theta}_2 + \tilde{\theta}_2^2 x + \frac{1}{2!}\tilde{\theta}_2^3 x^2 + \frac{1}{3!}\tilde{\theta}_2^3 x^3 + ... \tag{2.32}$$

and the polynomial expansion of $e^{\tilde{\theta}_1 x} - 1$ is

$$\tilde{\theta}_1 x + \frac{1}{2!}\tilde{\theta}_1^2 x^2 + \frac{1}{3!}\tilde{\theta}_1^3 x^3 + ... \tag{2.33}$$

Combining (2.32) and (2.33), the polynomial expansion of $\tilde{\theta}_2 e^{\tilde{\theta}_2 x}(e^{\tilde{\theta}_1 x} - 1)$ is

$$\tilde{\theta}_1 \tilde{\theta}_2 x + (\tilde{\theta}_1 \tilde{\theta}_2^2 + \frac{1}{2!}\tilde{\theta}_1^2 \tilde{\theta}_2) x^2 + ...$$

and the coefficients of x^n is

$$\frac{1}{n!}\tilde{\theta}_1^n \tilde{\theta}_2 + \frac{1}{(n-1)!}\tilde{\theta}_1^{n-1}\tilde{\theta}_2^2 + ...$$
$$+\frac{1}{2!}\tilde{\theta}_1^2 \frac{1}{(n-2)!}\tilde{\theta}_2^{n-1} + \frac{1}{1!}\tilde{\theta}_1 \frac{1}{(n-1)!}\tilde{\theta}_2^n$$

and it can be expressed compactly by

$$\sum_{i=1}^{n} \frac{1}{i!}\frac{1}{(n-i)!}\tilde{\theta}_1^i \tilde{\theta}_2^{n-i+1}. \tag{2.34}$$

In a similar way, it can be shown that the coefficient of x^n of polynomial expansion of $\tilde{\theta}_1 e^{\tilde{\theta}_1 x}(e^{\tilde{\theta}_2 x} - 1)$ is

$$\sum_{i=1}^{n} \frac{1}{i!}\frac{1}{(n-i)!}\tilde{\theta}_2^i \tilde{\theta}_1^{n-i+1}. \tag{2.35}$$

Subtract (2.35) from (2.34), we get the coefficients of x^n in polynomial expansion of

$$\tilde{\theta}_2 e^{\tilde{\theta}_2 x}(e^{\tilde{\theta}_1 x} - 1) - \tilde{\theta}_1 e^{\tilde{\theta}_1 x}(e^{\tilde{\theta}_2 x} - 1)$$

as

$$\sum_{i=1}^{m} \left(\frac{1}{i!}\frac{1}{(n-i)!} - \frac{1}{(i-1)!}\frac{1}{(n+1-i)!}\right)(\tilde{\theta}_1^i \tilde{\theta}_2^{n-i+1} - \tilde{\theta}_2^i \tilde{\theta}_2^{n-i+1})$$

where $m = (n - mod(n, 2))/2$. Because $\tilde{\theta}_2 > \tilde{\theta}_1$, it follows that

$$\tilde{\theta}_1^i \tilde{\theta}_2^{n-i+1} - \tilde{\theta}_2^i \tilde{\theta}_2^{n-i+1} > 0, \qquad i = 1, .., m. \tag{2.36}$$

For any $i = 1, .., m$, because $i \leq \frac{n}{2}$, it follows

$$\begin{aligned} i!(n-i)! &= i(i-1)!(n-i)! \\ &< (n+1-i)(i-1)!(n-i)! \\ &= (i-1)!(n+1-i)!. \end{aligned} \tag{2.37}$$

From (2.37), we have

$$\frac{1}{i!}\frac{1}{(n-i)!} - \frac{1}{(i-1)!}\frac{1}{(n+1-i)!} > 0. \tag{2.38}$$

Combining (2.36) and (2.38), we notice that all the coefficients of polynomial expansion of

$$\tilde{\theta}_2 e^{\tilde{\theta}_2 x}(e^{\tilde{\theta}_1 x} - 1) - \tilde{\theta}_1 e^{\tilde{\theta}_1 x}(e^{\tilde{\theta}_2 x} - 1)$$

are positive and it shows (2.31) is true. Therefore, the matrix in (2.27) is full rank. •

We now show that using lemma 1, we can establish global convergence.

Theorem 3 *For the function in (2.25), the collective gradient algorithm in (2.26) is globally convergent.*

Proof of Theorem 3: Assume that the output error and its gradient are given by

$$\begin{aligned} V &= \sum_{i=1}^{3}(y_i - \sum_{i=1}^{3} e^{\theta_i x_i})^2 \\ -\nabla_\theta V &= \sum_{j=1} 3\epsilon_j v_j, \end{aligned}$$

where

$$\begin{aligned} \epsilon_j &= y_j - \sum_{i=1} 3e^{\theta_i x_j}, \\ v_j &= \begin{bmatrix} e^{\theta_1 x_j} \\ e^{\theta_2 x_j} \\ e^{\theta_3 x_j} \end{bmatrix} \end{aligned} \tag{2.39}$$

Lemma 1 shows that v_1, v_2 and v_3 are linear independent. Hence if there exists some θ such that

$$-\nabla_\theta V = 0,$$

then

$$\epsilon_j = 0, \qquad j = 1, 2, 3 \tag{2.40}$$

where ϵ_j is defined as in (2.39). Equation (2.40) establishes that $V = 0$ and $\theta = \theta^*$ from Assumption 1. Hence Theorem 2 guarantees that the collective gradient algorithm is globally convergent. •

2.3.2 Example 2: Sigmoidal Neural Network with 2 unknown parameters

Here we assume that

$$h(x, \theta^*) = \sum_{i=1}^{2} \frac{e^{2\theta_i^* x} - 1}{e^{2\theta_i^* x} + 1}.$$

The invariant set in this case is

$$\Omega = \{\theta | \theta_1 = \theta_2\}.$$

Similar to the previous section, the global convergence of the collective gradient algorithm is equivalent to showing the matrix

$$\begin{bmatrix} \frac{e^{2\theta_1 x_1}}{(e^{2\theta_1 x_1}+1)^2} & \frac{e^{2\theta_2 x_1}}{(e^{2\theta_2 x_1}+1)^2} \\ \frac{e^{2\theta_1 x_2}}{(e^{2\theta_1 x_2}+1)^2} & \frac{e^{2\theta_2 x_2}}{(e^{2\theta_2 x_2}+1)^2} \end{bmatrix} \tag{2.41}$$

is of full rank and is stated in Lemma 2.

Lemma 2 *Let* $c(i)$ *be defined as*

$$\frac{(2\theta_2+\theta_1)^i + 2(\theta_2+\theta_1)^i + \theta_1^i}{(2\theta_1+\theta_2)^i + 2(\theta_2+\theta_1)^i + \theta_2^i}. \tag{2.42}$$

Then the matrix in (2.41) is full rank for any $\theta_1 \neq \theta_2$ *and* $x_1 \neq x_2$ *if* $c(i)$ *increases with* i*.*

Proof of Lemma 2: The full rankness of (2.41) is equivalent to show function

$$\frac{e^{2\theta_1 x}(e^{2\theta_2 x}+1)^2}{e^{2\theta_2 x}(e^{2\theta_1 x}+1)^2} \tag{2.43}$$

is monotonically increasing for $\theta_2 > \theta_1$ and $x > 0$. Substitute $2\theta_1$ and $2\theta_2$ with θ_1 and θ_2 separately, the increasingness of (2.43) is equivalent to show

$$\frac{e^{(2\theta_2+\theta_1)x} + 2e^{(\theta_1+\theta_2)x} + e^{\theta_1 x}}{e^{(2\theta_1+\theta_2)x} + 2e^{(\theta_1+\theta_2)x} + e^{\theta_2 x}} \tag{2.44}$$

is monotonous increasing. Derive the derivative of function (2.44) and we get

$$\frac{G_1G_2 - G_3G_4}{(e^{(2\theta_1+\theta_2)x} + 2e^{(\theta_1+\theta_2)x} + e^{\theta_2 x})^2} \tag{2.45}$$

where

$$\begin{aligned} G_1 &= (2\theta_2+\theta_1)e^{(2\theta_2+\theta_1)x} + 2(\theta_1+\theta_2)e^{(\theta_1+\theta_2)x} + \theta_1 e^{\theta_1 x} \\ G_2 &= e^{(2\theta_1+\theta_2)x} + 2e^{(\theta_1+\theta_2)x} + e^{\theta_2 x} \\ G_3 &= e^{(2\theta_2+\theta_1)x} + 2e^{(\theta_1+\theta_2)x} + e^{\theta_1 x} \\ G_4 &= (2\theta_1+\theta_2)e^{(2\theta_1+\theta_2)x} + 2(\theta_1+\theta_2)e^{(\theta_1+\theta_2)x} + \theta_2 e^{\theta_2 x} \end{aligned}$$

Now let us express these using the polynomial expansion. The coefficient of x^n in polynomial expansion of G_1 is

$$\frac{1}{n!}\left((2\theta_2+\theta_1)^{n+1}+2(\theta_1+\theta_2)^{n+1}+\theta_1^{n+1}\right). \tag{2.46}$$

The coefficient of x^n in polynomial expansion of G_2 is

$$\frac{1}{n!}\left((2\theta_1+\theta_2)^{n}+2(\theta_1+\theta_2)^{n}+\theta_2^{n}\right). \tag{2.47}$$

Combining (2.46) and (2.47), the coefficient of x^n in polynomial expansion of G_1G_2 is

$$\begin{aligned}&\frac{4}{n!}\left((2\theta_2+\theta_1)^{n+1}+2(\theta_1+\theta_2)^{n+1}+\theta_1^{n+1}\right)+\sum_{i=0}^{n-1}\\ &\frac{1}{i!(n-i)!}\left((2\theta_2+\theta_1)^{i+1}+2(\theta_1+\theta_2)^{i+1}+\theta_1^{i+1}\right)\\ &\left((2\theta_1+\theta_2)^{n-i}+2(\theta_1+\theta_2)^{n-i}+\theta_2^{n-i}\right).\end{aligned} \tag{2.48}$$

In a similar way, substituting $j=n-1-i$, the coefficient of G_3G_4 can be written as

$$\begin{aligned}&\frac{4}{n!}\left((2\theta_1+\theta_2)^{n+1}+2(\theta_2+\theta_1)^{n+1}+\theta_2^{n+1}\right)+\sum_{j=0}^{n-1}\\ &\frac{1}{(j+1)!(n-1-j)!}\left((2\theta_1+\theta_2)^{n-j}+2(\theta_2+\theta_1)^{n-j}\right.\\ &\left.+\theta_2^{n-j}\right)\left((2\theta_2+\theta_1)^{j+1}+2(\theta_2+\theta_1)^{j+1}+\theta_1^{j+1}\right).\end{aligned} \tag{2.49}$$

Subtract (2.49) from (2.48), we get the coefficient of x^n in polynomial expansion of $G_1G_2-G_3G_4$, which is

$$\begin{aligned}&\frac{4}{n!}\left((2\theta_2+\theta_1)^{n+1}+\theta_1^{n+1}-(2\theta_1+\theta_2)^{n+1}-\theta_2^{n+1}\right)+\\ &\sum_{i=0}^{n-1}\left(\frac{1}{(i)!}\frac{1}{(n-i)!}-\frac{1}{(i+1)!}\frac{1}{(n-1-i)!}\right)\\ &\left((2\theta_1+\theta_2)^{n-i}+2(\theta_2+\theta_1)^{n-i}+\theta_2^{n-i}\right)\\ &\left((2\theta_2+\theta_1)^{i+1}+2(\theta_2+\theta_1)^{i+1}+\theta_1^{i+1}\right).\end{aligned} \tag{2.50}$$

In what follows, we will show coefficient in (2.50) is positive.

Coefficient in (2.50) can be rewritten as

$$\begin{aligned}
&\frac{4}{n!}\left((2\theta_2+\theta_1)^{n+1}+\theta_1^{n+1}-(2\theta_1+\theta_2)^{n+1}-\theta_2^{n+1}\right)\\
&+\sum_{i=0}^{n-1}\left(\frac{1}{n-i}-\frac{1}{i+1}\right)\frac{1}{i!}\frac{1}{(n-1-i)!}\\
&\left((2\theta_1+\theta_2)^{n-i}+2(\theta_2+\theta_1)^{n-i}+\theta_2^{n-i}\right)\\
&\left((2\theta_2+\theta_1)^{i+1}+2(\theta_2+\theta_1)^{i+1}+\theta_1^{i+1}\right).
\end{aligned} \tag{2.51}$$

Let $M=\left[\frac{n-1}{2}\right]$, (2.51) can be transformed as

$$\begin{aligned}
&\frac{4}{n!}\left((2\theta_2+\theta_1)^{n+1}+\theta_1^{n+1}-(2\theta_1+\theta_2)^{n+1}-\theta_2^{n+1}\right)\\
&+\sum_{i=0}^{M}\left(\frac{1}{n-i}-\frac{1}{i+1}\right)\frac{1}{i!}\frac{1}{(n-1-i)!}\\
&\{\left((2\theta_1+\theta_2)^{n-i}+2(\theta_2+\theta_1)^{n-i}+\theta_2^{n-i}\right)\\
&\left((2\theta_2+\theta_1)^{i+1}+2(\theta_2+\theta_1)^{i+1}+\theta_1^{i+1}\right)-\\
&\left((2\theta_2+\theta_1)^{n-i}+2(\theta_2+\theta_1)^{n-i}+\theta_1^{n-i}\right)\\
&\left((2\theta_1+\theta_2)^{i+1}+2(\theta_2+\theta_1)^{i+1}+\theta_2^{i+1}\right)\}.
\end{aligned} \tag{2.52}$$

Because

$$\frac{1}{n-i}-\frac{1}{i+1}<0 \tag{2.53}$$

and

$$n-i\geq i+1, \forall 1\leq i\leq M\leq\frac{n-1}{2}, \tag{2.54}$$

Coefficient (2.52) is positive is equivalent to

$$(2\theta_2+\theta_1)^{n+1}+\theta_1^{n+1}-(2\theta_1+\theta_2)^{n+1}-\theta_2^{n+1}>0 \tag{2.55}$$

and

$$\left((2\theta_1+\theta_2)^j+2(\theta_2+\theta_1)^j+\theta_2^j\right)$$

$$
\begin{aligned}
&\left((2\theta_2+\theta_1)^i+2(\theta_2+\theta_1)^i+\theta_1^i\right) > \\
&\quad\left((2\theta_1+\theta_2)^i+2(\theta_2+\theta_1)^i+\theta_2^i\right) \\
&\left((2\theta_2+\theta_1)^j+2(\theta_2+\theta_1)^j+\theta_1^j\right)
\end{aligned}
\tag{2.56}
$$

for any $\theta_2 > \theta_1$ and $i > j$. It follows easily that

$$
\begin{aligned}
(2\theta_2+\theta_1)+2(\theta_2+\theta_1)+\theta_1 &= (2\theta_1+\theta_2)+2(\theta_2+\theta_1)+\theta_2 \\
(2\theta_2+\theta_1) &> (2\theta_1+\theta_2).
\end{aligned}
\tag{2.57}
$$

Now that (2.42) increases with i, we know that inequality (2.56) holds. Another fact comes from the increasness of (2.42) is

$$
\frac{(2\theta_2+\theta_1)^i+2(\theta_2+\theta_1)^i+\theta_1^i}{(2\theta_1+\theta_2)^i+2(\theta_2+\theta_1)^i+\theta_2^i} \geq 1 \forall i \geq 1. \tag{2.58}
$$

Inequality (2.58) proves (2.55) and it is established that coefficient in (2.50) is positive.

Positive coefficient in (2.50) implies that $G_1G_2 - G_3G_4$ is always positive. It follows that the derivative in equation (2.45) is positive and the full rank of (2.44) follows. This proves Lemma 2. •

About the condition that (2.42) increases with i, one observed fact is that for one number, if we participate them more evenly, the separated polynomial will increase slower and this can be verified to hold numerically.

2.3.3 Counter Example 1: Function with sin Component

We now consider a counter-example of system where Assumption 1 and condition (2.17) are satisfied, but condition (2.18) is not. This system is a network which contains sinusoidal function as

$$
h(x,\theta^*) = \frac{2}{\pi}x\theta^* + x^2\sin(\theta^*) + \sqrt{x}\sin(\theta^*)
$$

where $\theta^*, x \in \mathbb{R}$ and $x > 0$.

When we increase sample size M, we could always find a training data as in (2.3) which makes the function identifiable. Therefore, the following Lyapunov function follows as

$$\begin{aligned} V &= \sum_{i=1}^{M}\Big(\frac{2}{\pi}x_i\hat{\theta} + x_i^2\sin(\hat{\theta}) + \sqrt{x_i}\sin(\hat{\theta}) - \frac{2}{\pi}x_i\theta^* \\ &\quad - x_i^2\sin(\theta^*) + \sqrt{x_i}\sin(\theta^*)\Big)^2. \end{aligned} \tag{2.59}$$

In what follows, we will show that for this special case condition (2.17) in Theorem 2 is satisfied, but condition (2.18) is not for any training data set

$$(x_i, y_i) \qquad i = 1, ..., M. \tag{2.60}$$

where $x_i > 0$.

Equation (2.59) can be rewritten as

$$\begin{aligned} V &= \sum_{i=1}^{M}\Big(\frac{2}{\pi}x_i\tilde{\theta} + x_i^2\sin(\hat{\theta}) + \sqrt{x_i}\sin(\hat{\theta}) \\ &\quad - x_i^2\sin(\theta^*) + \sqrt{x_i}\sin(\theta^*)\Big)^2 \end{aligned} \tag{2.61}$$

and it follows from the fact $|sin(\theta)| \leq 1$ that

$$\begin{aligned} V &\geq \sum_{i=1}^{M}\Big(\frac{2}{\pi}|x_i\tilde{\theta}| - 2(x_i^2 + \sqrt{x_i})\Big)^2 \\ &\geq \left(\sum_{i=1}^{M}(\frac{2x_i}{\pi})^2\right)\tilde{\theta}^2 - \left(\sum_{i=1}^{M}2\frac{2x_i}{\pi}(x_i^2 + \sqrt{x_i})\right)|\tilde{\theta}| \\ &\quad - \left(\sum_{i=1}^{M}(x_i^2 + \sqrt{x_i})^2\right). \end{aligned} \tag{2.62}$$

As $|\tilde{\theta}| \to \infty$, the $\tilde{\theta}^2$ term in (2.62) dominates and $V \to \infty$ which proves condition (2.17).

For any x_i in (2.60), now that $x_i > 0$ and

$$y_i - h(x_i, \hat{\theta}) \geq \frac{2}{\pi}x_i(\theta^* - \hat{\theta}) - 2(x_i^2 + \sqrt{x_i}), \tag{2.63}$$

we can always find some θ^- such that

$$y_i - h(x_i, \hat{\theta}) \geq 0 \qquad \forall\, \hat{\theta} \leq \theta^-. \tag{2.64}$$

For any trainset defined as in (2.60), we can find some θ^- such that (2.64) holds for any $i = 1, .., M$. Choosing some j such that

$$-2j\pi \in (-\infty,\ \theta^-], \tag{2.65}$$

it follows that

$$\begin{aligned} h(x_i, -2j\pi) &= -4jx_i \\ h(x_i, -2j - \pi) &= (-4j - 2)x_j \end{aligned} \tag{2.66}$$

and

$$h(x_i, -2j\pi - \pi/2) = (-4j - 1)x_i + (x_i^2 + \sqrt{x_i}). \tag{2.67}$$

Because

$$x_i < x_i^2 + \sqrt{x_i}, \qquad \forall\, x_i > 0, \tag{2.68}$$

it follows from (2.65), (2.66) and (2.67) that

$$\begin{aligned} 0 &\leq y_i - h(x_i, -2j\pi - \pi/2) < y_i - h(x_i, -2j\pi) < \\ &y_i - h(x_i, -2j\pi - \pi) \qquad \forall i = 1, .., M. \end{aligned} \tag{2.69}$$

Therefore $V(\hat{\theta})$ has the property of

$$\begin{aligned} V(-2j\pi - \pi/2) &< V(-2j\pi) \\ V(-2j\pi - \pi/2) &< V(-2j\pi - \pi) \end{aligned} \tag{2.70}$$

for j chosen as in (2.65) and there must exists some $\theta \in [-2j\pi - \pi,\ -2j\pi]$ and

$$\nabla_{\hat{\theta}} V(\hat{\theta}) \,|_{\hat{\theta}=\theta} = 0. \tag{2.71}$$

Equation (2.71) means that there is some $\theta \neq \theta^*$ and

$$\nabla_{\hat{\theta}} V(\hat{\theta}) \,|_{\hat{\theta}=\theta} = 0$$

which implies that condition (2.18) is not satisfied. Hence, there exist many local minima and prevent the global convergence of the collective gradient algorithms. In Figure 2-1, we plot the Lyapunov function V w.r.t. $\hat{\theta}$. The choice of training set $x = 1, 2$ or 3 makes h identifiable because the only global minimum happens at true unknown parameter $\theta^* = 1$ and it can be seen clearly when we zoom in the Figure. It is shown that there exist many lcoal minimums which will prevent the global convergence.

2.3.4 Counter Example 2: Sigmoidal Neural Network with 4 parameters

We now consider another system where assumption 1 is satisfied, but conditions (2.17) and (2.18) are not. Here, the system is a sigmoidal neural network with 2 nodes and output weights given by

$$h(x, \theta^*) = \sum_{i=1}^{2} \theta_i^* \frac{e^{\phi_i^* x} - e^{-\phi_i^* x}}{e^{\phi_i^* x} + e^{-\phi_i^* x}} \tag{2.72}$$

where the unknown parameters $\theta^* = [\theta_1^*, \theta_2^*, \phi_1^*, \phi_2^*]^T$, the invariant set is

$$\Omega = \{\theta = [\theta_1\ \theta_2\ \phi_1\ \phi_2]^T | \theta_1 = \theta_2, \phi_1 = \phi_2\}.$$

In this case, we show that the global convergence is not guranteed in Lemma 3 by a counter example.

Lemma 3 *For the function in (2.72), the collective gradient algorithm is not globally convergent.*

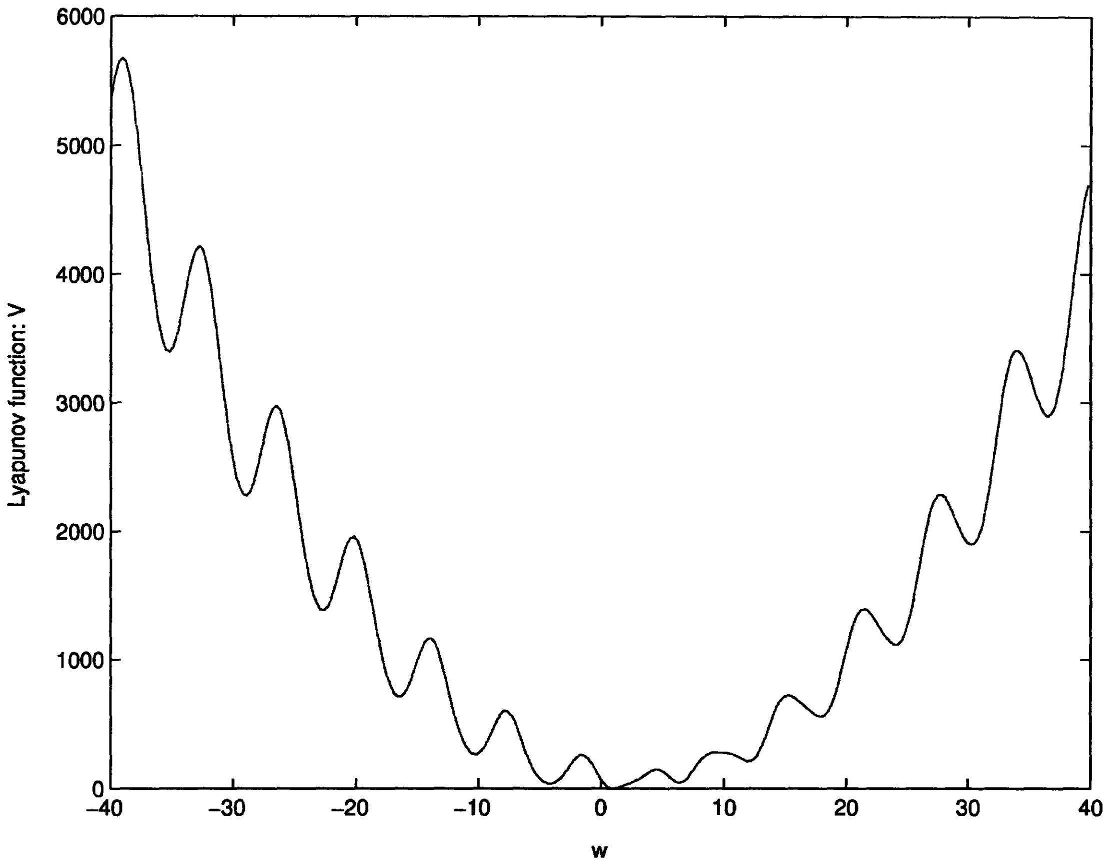

Figure 2-1: Lyapunov function $V(t)$ over $\hat{\theta}$ with $\theta^* = 1$

Proof of Lemma 3: We prove this lemma by showing that there are points $\theta_0 \neq \theta^*$ where $\nabla_{\theta_0} V = 0$. From (2.11) and (2.72), it follows that

$$-\nabla_\theta V = \sum_{j=1}^{4} \epsilon_j v_j$$

where

$$\epsilon_j = h(x_j, \theta^*) - h(x_j, \theta)$$

and $[v_1 v_2 v_3 v_4]$ forms the matrix A given by

$$\begin{bmatrix} \frac{e^{2\phi_1 x_1}-1}{e^{2\phi_1 x_1}+1} & \frac{e^{2\phi_1 x_2}-1}{e^{2\phi_1 x_2}+1} & \frac{e^{2\phi_1 x_3}-1}{e^{2\phi_1 x_3}+1} & \frac{e^{2\phi_1 x_4}-1}{e^{2\phi_1 x_4}+1} \\ \frac{e^{2\phi_2 x_1}-1}{e^{2\phi_2 x_1}+1} & \frac{e^{2\phi_2 x_2}-1}{e^{2\phi_2 x_2}+1} & \frac{e^{2\phi_2 x_3}-1}{e^{2\phi_2 x_3}+1} & \frac{e^{2\phi_2 x_4}-1}{e^{2\phi_2 x_4}+1} \\ \frac{\theta_1 x_1 e^{2\phi_1 x_1}}{(e^{2\phi_1 x_1}+1)^2} & \frac{\theta_1 x_2 e^{2\phi_1 x_2}}{(e^{2\phi_1 x_2}+1)^2} & \frac{\theta_1 x_3 e^{2\phi_1 x_3}}{(e^{2\phi_1 x_3}+1)^2} & \frac{\theta_1 x_4 e^{2\phi_1 x_4}}{(e^{2\phi_1 x_4}+1)^2} \\ \frac{\theta_2 x_1 e^{2\phi_2 x_1}}{(e^{2\phi_2 x_1}+1)^2} & \frac{\theta_2 x_2 e^{2\phi_2 x_2}}{(e^{2\phi_2 x_2}+1)^2} & \frac{\theta_2 x_3 e^{2\phi_2 x_3}}{(e^{2\phi_2 x_3}+1)^2} & \frac{\theta_2 x_4 e^{2\phi_2 x_4}}{(e^{2\phi_2 x_4}+1)^2} \end{bmatrix}.$$

Denote the ith row of matrix A as r_i. It follows that A is not full rank for $\theta \in \Theta_1, \Theta_2$ or Θ_3, where

$$\begin{aligned} \Theta_1 &= \{\theta | \theta_1 \neq \theta_2,\ \ \phi_1 = \phi_2\}, \\ \Theta_2 &= \{\theta | \theta_1 = 0\}, \\ \Theta_3 &= \{\theta | \theta_2 = 0\}. \end{aligned} \tag{2.73}$$

This is because when $\theta \in \Theta_2$, or Θ_3, the row r_3 or r_4 becomes zero, respectively and hence A loses rank. Also, it follows from the definition of Θ_1 such that

$$r_1 = r_2 \qquad r_3 = r_4. \tag{2.74}$$

We now examine if it is possible for

$$\nabla V_{\theta_0} = 0 \tag{2.75}$$

for $\theta_0 \in \Theta_1$ with $\theta_0 \neq \theta^*$. To do this, we start with a θ_0 that satisfies Eqs.(2.73), (2.74) and (2.75), and determine if a θ^* exists under the same conditions.

Since Eq. (2.75) holds if

$$r_i \epsilon = 0 \qquad \text{for } i = 1, \ldots, 4$$

for $\theta_0 \in \Theta_1$, from Eq. ((2.74)), we have that (2.75) is equivalent to

$$\epsilon(\theta_0, \theta^*) r_1(\theta_0) \;=\; 0$$

$$\epsilon(\theta_0, \theta^*) r_3(\theta_0) = 0 \quad (2.76)$$

where $\epsilon(\theta_0, \theta^*) = [\epsilon_1\ \epsilon_2\ \epsilon_3\ \epsilon_4]$. Eq. 2.76) implies that there are 2 equations while θ^* has 4 elements, and hence it implies that there exists some $\theta^* \neq \theta_0$ which satisfies (2.76). A similar procedure can be used to find a θ^* that exists for any θ_0 in Θ_2 and Θ_3 as well. That is, we have established the existence of $\theta_0 \neq \theta^*$ for which $\nabla_{\theta_0} V = 0$. Therefore from theorem 2, it follows that no global convergence can be guaranteed. •

Remark 3: It follows from (2.19) and Barbalatt's lemma that

$$\lim_{t \to \infty} \nabla_{\hat{\theta}} V = 0.$$

If we define the set B as

$$B = \{\theta \mid \theta \in E, \nabla_\theta V = 0\},$$

it follows that $\hat{\theta}(t) \to B$ as $t \to \infty$. Lemma 3 shows that B can include at least one more point other than θ^* and hence that $\hat{\theta}$ converges to some θ_0 where $\theta_0 \in B$ and $\theta_0 \neq \theta^*$.

Remark 4: It should be noted that we can not avoid convergence to a local minimum by increasing M. This is because in this case ϵ, r_1 and r_3 will have M elements instead of four in (2.76), but we still will have only two equations for four unknowns and the same conclusions follow.

2.3.5 Simulation Results

In this section, we will give numerical results of the counter-example constructed in section 2.3.4. We will consider a sigmoidal neural networks with 4 parameters as defined in (2.72) with input $U = [1\ 2\ 3\ 4]$. Choosing

$$\theta_0 = [1\ 2\ 1\ 1] \in \Phi_1, \quad (2.77)$$

using the procedure as discussed in section 2.3.4, we found

$$\theta^* = [6.4855\ \ -3.4886\ 1.1\ 1.2] \quad (2.78)$$

which satsfies

$$\nabla V_{\theta_0} = 0. \tag{2.79}$$

We plot the Lyapunov function V around equillibrium θ_0 in $[\theta_1 \;\; \theta_2]$ and $[\phi_1 \;\; \phi_2]$ space respectively in Figures 2-2 and 2-3. It is shown clearly that θ_0 is not a local maximum. Simulation results shows that there are infinite points in E which can lead $\hat{\theta}(t) \rightarrow \theta_0$ and one example is $\hat{\theta}(0) = [1.02 \; 2.02 \; 1.0068 \; 1.0095]$ with $V(t)$ plotted in Figure 2-4.

Increasing size of training data can not solve this singular problem. For example, when we increase the training data set to

$$U = [0.5 \; 1 \; 2 \; 3 \; 4], \tag{2.80}$$

using similar procedure, we found that for

$$\theta^* = [6.2717 \;\; -3.2760 \; 1.1 \; 1.2],$$

$$\nabla V_{\theta_0} = 0$$

where $\theta_0 = [1 \; 2 \; 1 \; 1]$.

2.4 Implications on the Control of Nonlinear Dynamic Systems Using Neural Networks

The discussions above clearly indicate the following: Suppose that an unknown system is in the form of a single-layer neural network whose number of nodes n is *known*, but its weights are unknown and are to be estimated. If an identical neural network is constructed whose nodes are equal to n in number, and whose weights are started from *arbitrary* locations and adjusted using the collective gradient algorithm so as to estimate the weights of the first neural network, it is quite likely that the weight-estimates will *not* converge to their true values, but to those where the tracking error reaches a *local* minimum which is larger than zero. This establishes conclusively that it is quite likely that the best result achievable

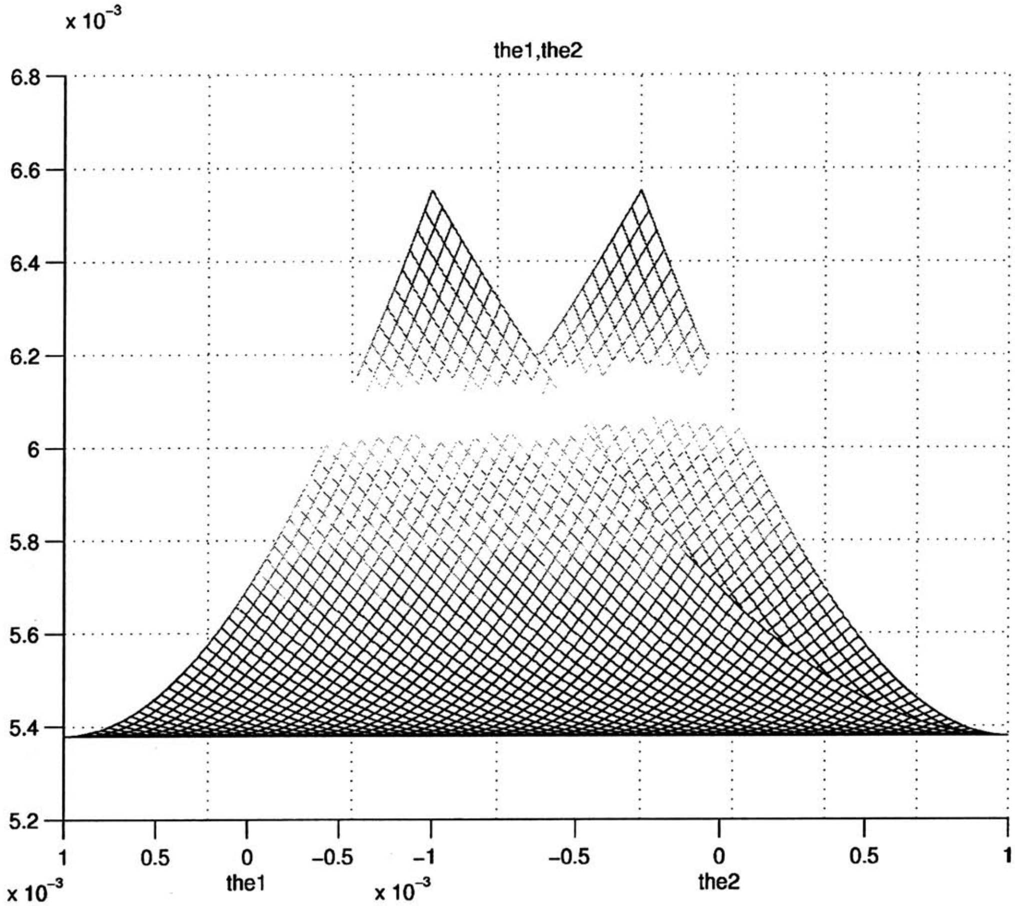

Figure 2-2: Lyapunov function V along $[\theta_1\ \theta_2]$ around $\theta_0 = [1\ 2\ 1\ 1]$.

with a neural network is a local result in such an identification problem. The question is, if a similar property will be found in the context of control of a nonlinear system using neural networks.

To address this question, we consider the control of a nonlinear system, whose closed-loop structure is such that it can be described as

$$\dot{y} = h(\omega, \theta^*) \tag{2.81}$$

where ω is a closed-loop system variable that can be measured on-line, θ^* is an unknown parameter that keeps the system in ((2.81)) bounded, and h is a known function of its arguments that satisfies assumption 1. Suppose that $\dot{y}$ can be measured at each instant of

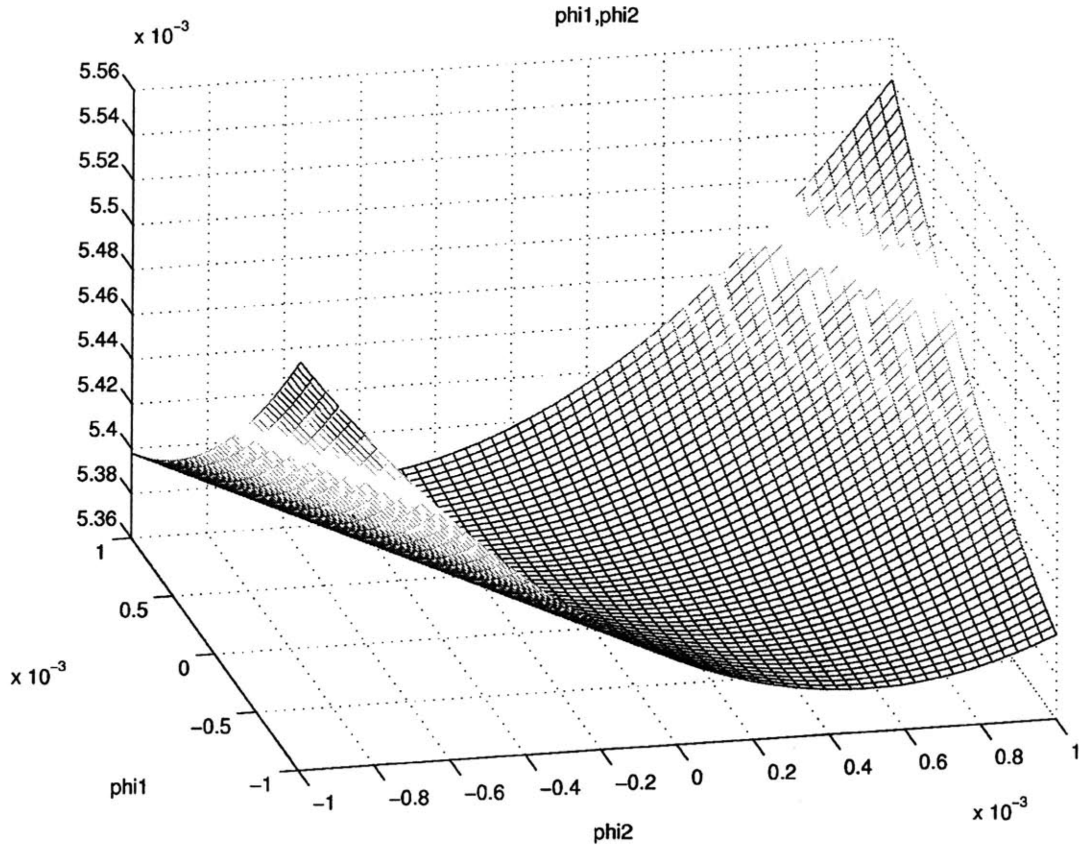

Figure 2-3: Lyapunov function V along $[\phi_1\ \phi_2]$ around $\theta_0 = [1\ 2\ 1\ 1]$.

time. This implies that the control of the system in (2.81) is equivalent to the identification problem considered in section 2.1. It is therefore clear that in such cases, the statements regarding the local behavior of neural networks are applicable to such a control problem as well. Therefore control of general nonlinear systems using neural networks needs to be approached with caution with care exercised to make sure that the local minima problems are avoided.

2.5 Summary

In this chapter, we examine conditions under which the weights of a neural network can converge starting from arbitrary values to those of another neural network with the same structure. Both a standard gradient algorithm and a collective gradient algorithm are used

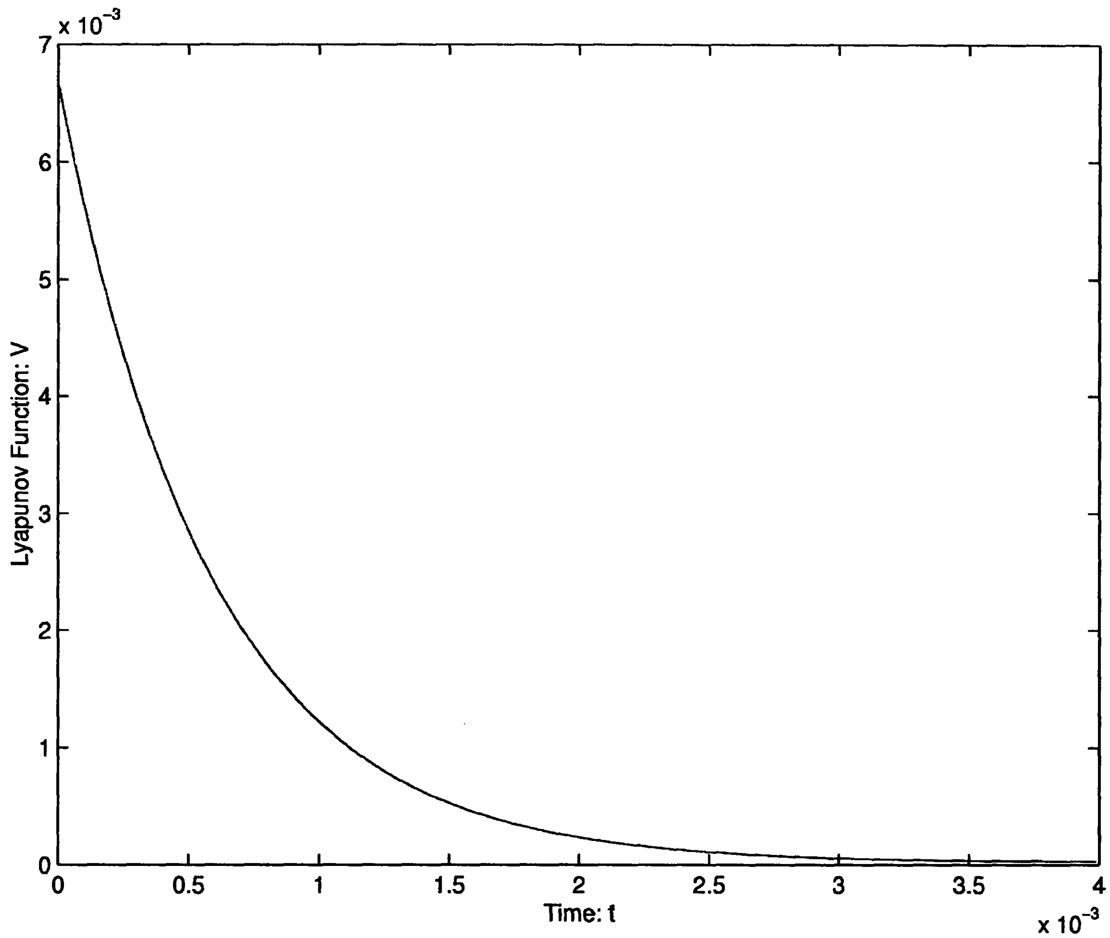

Figure 2-4: Trajectory of Lyapunov function $V(t)$ with $\hat{\theta}(0) = [1.02\ 2.02\ 1.0068\ 1.0095]$

to evaluate the convergence properties. In the former case, we provide some sufficient conditions under which global convergence can occur, while in the latter case, we present necessary and sufficient conditions for global convergence. We conclude with several examples of neural networks with a small number of neurons, and show that these conditions need not be satisfied even in some simple examples, which leads to local minima and therefore non-global convergence.

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Chapter 3

Hierarchical Min-max Algorithm

3.1 Introduction

In this chapter, we consider parameter convergence in a class of continuous-time dynamic systems. We begin with systems that have convex/concave parameterization and derive sufficient conditions under which parameter convergence can occur in such systems. These conditions are related to linear persistent excitation (LPE) conditions relevant for convergence in linearly parameterized systems [1], and are shown to be stronger, with the additional complexity being a function of the underlying nonlinearity.

We also propose a new hierarchical min-max algorithm in this chapter in order to relax the sufficient conditions for parameter convergence. The lower-level of this algorithm consists of the same min-max algorithm as in [1, 5]. An additional higher-level component is included in the hierarchical algorithm that consists of updating the bounds on the parameter region that the unknown parameter is assumed to belong to. We then show, using the hierarchical algorithm, that parameter convergence can be accomplished globally under a necessary and sufficient condition on the system variables and the underlying nonlinearity f. Examples of functions that satisfy such a condition, which we denote as a condition of Nonlinear Persistent Excitation (NLPE), and relations to LPE are also presented in this chapter.

The chapter is organized as follows. Section 2 gives the statement of the problem, the estimator based on the min-max algorithm and the properties. In section 3, parameter estimation in functions that are concave/convex is considered, and a sufficient condition for parameter convergence is derived. In section 4, a hierarchical min-max algorithm is proposed and necessary and sufficient conditions for parameter convergence are proposed. Examples and relation to LPE are also presented in this section. Simulation results are included in Section 5. Summary and concluding remarks are stated in section 6. Proofs of all properties, lemmas, and theorems can be found in Appendix A.

3.2 Statement of the Problem

The problem considered is the estimation of unknown parameters in a class of nonlinear systems of the form

$$\begin{aligned} \dot{y} &= -\alpha(y,u)y + f(\theta_0, \omega(y,u)) \\ & 0 < \alpha_{min} \leq \alpha(y,u) \leq \alpha_{max} \end{aligned} \tag{3.1}$$

where $\theta_0 \in \Omega^0 \subset \mathbb{R}^n$ are bounded unknown parameters, $u, y \in \mathbb{R}$ are input and output respectively, and the functions ω and f are given by $\omega : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^m$ and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$.

We make the following assumptions regarding ω and f.

Assumption 1: The function $\omega(y(t), u(t))$ is Lipschitz in t so that

$$||\omega(t_1) - \omega(t_2)|| \leq U_b ||t_1 - t_2||, \qquad \forall t_1, t_2 \in \mathbb{R}^+.$$

Assumption 2: f is Lipschitz with respect to its arguments, i.e.

$$|f(\theta + \Delta\theta, \omega + \Delta\omega) - f(\theta, \omega)| \leq B_\theta ||(\Delta\omega, \Delta\theta)|| \leq B_\theta(||\Delta\omega|| + ||\Delta\theta||).$$

Assumption 3: $\omega(y,u)$ is a bounded, continuous function of its arguments, and u is bounded and continuous.

Assumption 4: The system in (3.1) has bounded solutions if u is bounded.

Assumption 5: $\theta_0 \in \Omega^0 \subset \mathbb{R}^n$, and Ω^0 is a known compact set.

Let a set U_I be defined as follows:

$$U_I = \{\omega_i,\ i = 1, ..., I,\ \omega_i \neq \omega_j \text{ if } i \neq j,\ \omega_i \in \mathbb{R}^m\}. \tag{3.2}$$

We introduce the definition of an identifiable function which is necessary for parameter convergence.

Definition 1 *A function $f(\theta, \omega), \theta \in \Omega^0 \subset \mathbb{R}^n$ is identifiable over parameter region Ω^0 with respect to U_I if there does not exist $\theta_1, \theta_2 \in \Omega^0$ and $\theta_1 \neq \theta_2$ such that*

$$\lim_{\theta \to \theta_1} f(\theta, \omega_i) = \lim_{\theta \to \theta_2} f(\theta, \omega_i) \qquad \forall \omega_i \in U_I, i = 1, ..., I.$$

Definition 1 implies that identifiability follows if the system of equations

$$f(\widehat{\theta}, \omega_i) - f(\theta_0, \omega_i) = 0 \qquad \forall \omega_i \in U_I \tag{3.3}$$

has a *unique* solution $\widehat{\theta} = \theta_0$ for any $\theta_0 \in \Omega^0$. Equation (3.3) suggests a procedure for constructing U_I such that for a given Ω^0, f can become identifiable over Ω^0. That is, the number I and the value ω_i, for $i = 1, ..., I$ must be chosen such that Eq. (3.3) has a unique solution.

We also note that for a given Ω^0, identifiability of f is dependent on the choice of U_I. For example, if f is linear, then f is identifiable over any $\theta \in \mathbb{R}^n$ if elements of U_I span the entire space of $\mathbb{R}^m$; for a nonlinear f, identifiability may be possible even if these elements span only a subspace. We notice that if f is not identifiable with respect to U_I, it implies that we have no way of identifying θ_0 using any input ω_i in U_I.

In the subsequent sections, we propose a min-max parameter estimation algorithm, and its properties. For simplicity, we omit the arguments of ω, and note that it is a measurable continuous function of time that satisfies assumption 1.

3.2.1 The Min-max Parameter Estimation Algorithm

The dynamics of parameter estimation algorithm that we propose is the same as the min-max algorithm in [1] and is as follows:

$$\begin{aligned}\dot{\hat{y}} &= -\alpha(y,u)\left(\hat{y} - \epsilon sat(\frac{\tilde{y}}{\epsilon})\right) + f(\hat{\theta},\omega) - a^* sat\left(\frac{\tilde{y}}{\epsilon}\right) \\ \dot{\hat{\theta}} &= -\tilde{y}_\epsilon \phi^* \end{aligned} \tag{3.4}$$

where

$$\tilde{y} = \hat{y} - y, \qquad \tilde{y}_\epsilon = \tilde{y} - \epsilon sat\left(\frac{\tilde{y}}{\epsilon}\right), \tag{3.5}$$

ϵ is an arbitrary positive number, $sat(\cdot)$ denotes the saturation function and is given by $sat(x) = sign(x)$ if $|x| \geq 1$, and $sat(x) = x$ if $|x| < 1$, and a^* and ϕ^* come from the solution of an optimization problem

$$\begin{aligned} a^* &= \min_{\phi \in \mathbb{R}^m} \max_{\theta \in \Omega^0} g(\theta,\omega,\phi) \qquad (3.6) \\ \phi^* &= arg \min_{\phi \in \mathbb{R}^m} \max_{\theta \in \Omega^0} g(\theta,\omega,\phi) \\ g(\theta,\omega,\phi) &= sat\left(\frac{\tilde{y}}{\epsilon}\right)\left(f(\hat{\theta},\omega) - f(\theta,\omega) - \phi^T(\hat{\theta} - \theta)\right) \end{aligned}$$

The choices of ϕ^* and a^* imply the following inequality:

$$sat\left(\frac{\tilde{y}}{\epsilon}\right)\left(f(\hat{\theta},\omega) - f(\theta,\omega) - \phi^{*T}(\hat{\theta} - \theta)\right) - a^* \leq 0. \tag{3.7}$$

We define

$$\tilde{\theta} = \hat{\theta} - \theta_0,$$

and rewrite the dynamics of the whole parameter estimation algorithm as

$$\begin{aligned}\dot{\tilde{y}} &= -\alpha(y,u)\tilde{y}_\epsilon + f(\hat{\theta},\omega) - f(\theta_0,\omega) - a^* sat(\tfrac{\tilde{y}}{\epsilon}) \\ \dot{\tilde{\theta}} &= -\tilde{y}_\epsilon \phi^*. \end{aligned} \tag{3.8}$$

Let $x = [\tilde{y}_\epsilon, \ \tilde{\theta}^T]^T$. The problem is therefore to determine the conditions on ω under which the system (3.8) has uniform asymptotic stability in the large (u.a.s.l.) at $x = 0$.

3.2.2 Solutions of a^* and ϕ^*

In [1] and [5], closed form solutions to (3.6) when f is a concave/convex function of θ_0 and when f is a general function of θ_0 were derived, respectively. In both [1], [5], these solutions were derived under the assumption that $\hat{\theta} \in \Omega^0$. In this chapter, results are extended to the case when this assumption is omitted. For ease of exposition, we present the results for the cases when (a) θ is a scalar, and f is a general function of θ and (b) θ is a vector, and f is a convex/concave function of θ. We define a convex set $C(\Omega_0)$ which is constructed as follows: If $H_f(\Omega_0)$ is the convex hull, which is the smallest convex set in $\mathbb{R}^{n+1}$ that contains $\{(\theta,\ f(\theta,\omega))\ |\theta \in \Omega^0\}$, then $C(\Omega_0)$ is the projection of $H_f(\Omega_0)$ on $\mathbb{R}^n$ which contains Ω_0. Such a convex set is needed since (i) the hierarchical algorithm discussed in section 3.4.3 can allow the parameter estimate to wander outside Ω_0, and (ii) the solutions to the min-max algorithm differ depending whether $\hat{\theta}$ lies within this convex set $C(\Omega_0)$ or outside.

(a) $\theta \in \Omega^0 \subset \mathbb{R}$, and f is a general function of θ: In this case $C(\Omega^0) = [\theta_{\min},\ \theta_{\max}]$. Same as in [5], the following two definitions are useful.

Definition 2 *A point $\theta^0 \in \theta_c$ if $\theta^0 \in C(\Omega^0)$ and*

$$\nabla f_{\theta^0}(\theta - \theta^0) \leq f(\theta,\omega) - f(\theta^0,\omega), \qquad \forall\theta \in C(\Omega^0) \tag{3.9}$$

where $\nabla f_{\theta^0} = \frac{\partial f}{\partial \theta}\Big|_{\theta^0}$.

Definition 3 *$\tilde{\theta}_c = \bar{\theta}_c \bigcap C(\Omega^0)$, where $\bar{\theta}_c$ denotes the complement of θ_c.*

We now state the solutions to (3.6) in case (a), when $\tilde{y} > 0$. The solutions when $\tilde{y} < 0$ can be derived in a similar manner using the concave cover.

Denoting $\breve{\theta}_c = \{\theta^{12}, \theta^{34}, \ldots, \theta^{mn}\}$, $\theta^{ij} = [\theta^i, \theta^j]$ as in [5], we obtain that

$$\left.\begin{array}{rcl} \phi^* &=& \nabla f_{\hat{\theta}} \\ a^* &=& 0 \qquad \text{if } \hat{\theta} \in \theta_c \\ \phi^* &=& \theta^{ij} \\ a^* &=& f(\hat{\theta}, \omega) - f(\theta^i, \omega) - \phi^*(\hat{\theta} - \theta^i) \\ && \text{if } \hat{\theta} \in \theta^{ij} \end{array}\right\} \text{if } \hat{\theta} \in C(\Omega^0), \tag{3.10}$$

and if $\hat{\theta} > \theta_{max}$

$$\begin{array}{rcl} a^* &=& 0 \\ \phi^* &=& \begin{cases} \phi^*(\theta_{max}) \\ \text{if } f(\theta_{max}, \omega) + \phi^*(\theta_{max})(\hat{\theta} - \theta_{max}) \geq f(\hat{\theta}, \omega) \\ \frac{f(\theta_{max}, \omega) - f(\hat{\theta}, \omega)}{\theta_{max} - \hat{\theta}} \text{ otherwise,} \end{cases} \end{array} \tag{3.11}$$

and if $\hat{\theta} < \theta_{min}$,

$$\begin{array}{rcl} a^* &=& 0 \\ \phi^* &=& \begin{cases} \phi^*(\theta_{min}) \\ \text{if } f(\theta_{min}, \omega) + \phi^*(\theta_{min})(\hat{\theta} - \theta_{min}) \geq f(\hat{\theta}, \omega) \\ \frac{f(\theta_{min}, \omega) - f(\hat{\theta}, \omega)}{\theta_{min} - \hat{\theta}} \text{ otherwise.} \end{cases} \end{array} \tag{3.12}$$

(b) $\theta \in \Omega^0 \subset \mathbb{R}^n$, f is a concave function of θ: The solutions to (3.6) are easier to find when Ω^0 is a simplex, and are presented first:

Case (i): Ω^0 is a simplex: Very similar to [1], we have the following solutions:

$a^* = 0$ $\phi^* = \nabla f_{\hat{\theta}}$	$\tilde{y} < 0, \hat{\theta} \in C(\Omega^0)$
$a^* = 0$ $\phi^* = \nabla f_{\hat{\theta}}$	$\tilde{y} < 0, \hat{\theta} \notin C(\Omega^0)$
$a^* = A_1$ $\phi^* = A_2$	$\tilde{y} > 0, \hat{\theta} \in C(\Omega^0)$
$a^* = 0$ $\phi^* = A_2$	$\tilde{y} > 0, \hat{\theta} \notin C(\Omega^0)$

where $[A_1, A_2]^T = G^{-1}b$, $A_1 \in \mathbb{R}$, $A_2 \in \mathbb{R}^n$,

$$G = \begin{bmatrix} -1 & -(\hat{\theta} - \theta_{S1})^T \\ -1 & -(\hat{\theta} - \theta_{S2})^T \\ \vdots & \vdots \\ -1 & -(\hat{\theta} - \theta_{Sn+1})^T \end{bmatrix},$$

$$b = \begin{bmatrix} -(f(\hat{\theta}, \omega) - f_{S1}) \\ -(f(\hat{\theta}, \omega) - f_{S2}) \\ \vdots \\ -(f(\hat{\theta}, \omega) - f_{Sn+1}) \end{bmatrix}.$$

$\theta_{Si}, i = 1, \ldots, n+1$ are the vertices of Ω^0, and $f_{Si} = f(\theta_{Si}, \omega)$. •

Case (ii) Ω^0 is a compact set in $\mathbb{R}^n$: We define a polygon $P(\Omega^0)$ which contains Ω^0, whose vertices are given by $P_1, P_2, ..., P_K$. Denoting $L = \begin{pmatrix} K \\ n+1 \end{pmatrix}$, we note that L hyperplanes can be constructing using a combination of $n+1$ points from the K vertices of the polygon. Denoting the vertices of the ith hyperplane as $P_{i_1}, .., P_{i_{n+1}}$, and ϕ_i as the slope of this hyperplane, we choose J as a set of the L hyperplanes such that

$$\begin{aligned} J &= \{i \,|1 \le i \le L, f(\hat{\theta}, \omega) - f(P_{i_1}, \omega) - \phi_i^T(\hat{\theta} - P_{i_1}) \\ &\geq 0, \qquad \forall\, \hat{\theta} \in P(\Omega^0)\}. \end{aligned}$$

We can derive the solutions to (3.6) as

$a^* = 0$ $\phi^* = \nabla f_{\hat{\theta}}$	$\tilde{y} < 0, \hat{\theta} \in C(P(\Omega^0))$
$a^* = 0$ $\phi^* = \nabla f_{\hat{\theta}}$	$\tilde{y} < 0, \hat{\theta} \notin C(P(\Omega^0))$
$a^* = A_1$ $\phi^* = A_2$	$\tilde{y} > 0, \hat{\theta} \in C(P(\Omega^0))$
$a^* = 0$ $\phi^* = A_2$	$\tilde{y} > 0, \hat{\theta} \notin C(P(\Omega^0))$

where

$$\begin{aligned} A_2 &= \phi_j \\ A_1 &= f(\hat{\theta}, \omega) - f(P_{j_1}, \omega) - \phi_j^T(\hat{\theta} - P_{j_1}) \\ j &= arg \max_{i \in J} f(P_{i_1}, \omega) + \phi_i^T(\hat{\theta} - P_{i_1}) \end{aligned}$$

The solutions for the case when f is a convex function of θ can be derived in a similar manner.

(c) $\theta \in \Omega^0 \subset \mathbb{R}^n$, f is a general function of θ: Using the above two cases, and in particular, a combination of concave and convex covers, convex hull, and polygons, the solutions to (3.6) can be found.

3.2.3 Properties of the Min-max Estimator

In [1], the min-max estimator and therefore the resulting error model in (3.8) was shown to be stable. The stability properties of this error model are summarized in Property 1 and Property 2 below. In what follows, the quadratic function V is useful:

$$V = \frac{1}{2}(\tilde{y}_\epsilon^2 + \tilde{\theta}^T\tilde{\theta}) = \frac{1}{2}||x(t)||^2. \tag{3.13}$$

Property 1 summarizes the stability properties of (3.8).

Property 1

$$\dot{V} \;\leq\; -\alpha\tilde{y}_\epsilon^2. \tag{3.14}$$

Property 1 implies that the min-max estimator is stable. However whether the parameter estimates will converge to their true values, that is, whether x will converge to the origin is yet to be established. To facilitate parameter convergence discussions, an additional property of the min-max estimator is stated in Property 2.

Property 2 *If in Eq. (3.8),*

$$|\tilde{y}_\epsilon(t_1)| \geq \gamma, \tag{3.15}$$

then

$$V(t_1 + T') \leq V(t_1) - \frac{\alpha_{min}\gamma^3}{3(M + \alpha_{max}\gamma)} \tag{3.16}$$

where $T' = \frac{\gamma}{M+\alpha_{max}\gamma}$, $0 < \alpha_{min} \leq \alpha(y,u) \leq \alpha_{max}$, *and*

$$\begin{aligned} M &= \max\{|m(t)|\}, \\ m(t) &= f(\hat{\theta},\omega) - f(\theta_0,\omega) - a^* sat\left(\frac{\tilde{y}_\epsilon}{\epsilon}\right). \end{aligned} \tag{3.17}$$

Property 2 implies that for parameter convergence to occur, $\tilde{y}_\epsilon$ must become periodically large. For this in turn to occur, examining the dynamics in (3.8) and defining $\tilde{f}(\hat{\theta},\theta_0,\omega) = f(\hat{\theta},\omega) - f(\theta_0,\omega)$, (i) $\tilde{f}(\hat{\theta},\theta_0,\omega)$ must be large when $||\tilde{\theta}||$ is large and (ii) a^* must be small compared to $\tilde{f}(\hat{\theta},\theta_0,\omega)$. The condition (i) is related to persistent excitation, and is similar to parameter convergence conditions in linearly parameterized systems. Condition (ii) is specific to the min-max algorithm. In order to facilitate the latter, a few properties of a^* are worth deriving, and are enumerated below in Properties 3 and 4.

Noting that a^* is defined as in (3.6), we denote

$$\begin{aligned} a_-^*(\hat{\theta},\omega) = a^* \qquad &\text{if } \tilde{y}_\epsilon < 0, \\ a_+^*(\hat{\theta},\omega) = a^* \qquad &\text{if } \tilde{y}_\epsilon > 0. \end{aligned}$$

It follows that a_-^* and a_+^* are well defined functions of $\widehat{\theta}$ and ω. We establish the following properties about a^*, $a_-^*(\widehat{\theta},\omega)$ and $a_+^*(\widehat{\theta},\omega)$.

Property 3 $\quad -a_-^*(\widehat{\theta},\omega) \leq a^* sat(\frac{\tilde{y}}{\epsilon}) \leq a_+^*(\widehat{\theta},\omega)$

Suppose for a given ω, $f(\theta,\omega)$ retains its curvature as θ varies. We define

$$\beta(\omega) = \begin{cases} 1 & \text{if } f(\theta,\omega) \text{ is convex;} \\ -1 & \text{if } f(\theta,\omega) \text{ is concave .} \end{cases} \tag{3.18}$$

Property 4 *For a^* and β defined as in (3.6) and (3.18), respectively, the following holds:*

$$(i)\ a_-^* = 0 \ \ \textit{if}\ \beta = -1; \quad (ii)\ a_+^* = 0 \ \ \textit{if}\ \beta = 1;$$
$$(iii)\ \beta a^* \tilde{y} \leq 0, \qquad \textit{for any}\ \beta. \tag{3.19}$$

Both Properties 3 and 4 are used in the next section for the proof of Theorem 1.

3.3 Parameter Convergence in Systems with Convex/Concave Parameterization

We first focus on parameter convergence of the system (3.8) when f is convex/concave for any $\theta \in \Omega^0$. For the sake of completeness, we include the definition of a concave/convex function:

Definition 4 *A function $f(\theta)$ is said to be (i) convex on Θ if it satisfies the inequality*

$$f(\lambda\theta_1 + (1-\lambda)\theta_2) \leq \lambda f(\theta_1) + (1-\lambda) f(\theta_2) \quad \forall \theta_1, \theta_2 \in \Theta$$

and (ii) concave if it satisfies the inequality

$$f(\lambda\theta_1 + (1-\lambda)\theta_2) \geq \lambda f(\theta_1) + (1-\lambda) f(\theta_2) \quad \forall \theta_1, \theta_2 \in \Theta$$

where $0 \leq \lambda \leq 1$.

We make a few qualitative comments regarding the solutions of (3.8) and their convergence properties before establishing the main result. The main difficulty in establishing parameter convergence is due to the presence of the time-varying function a^* in (3.8). As shown in properties 3-4 in Section 3.2.3, the magnitude of a^* changes with the curvature of f. As mentioned in Section 3.2.3, in order to establish parameter convergence, in addition to $\tilde{f}(\hat{\theta}, \theta_0, \omega)$ being large when $\tilde{\theta}$ is large a^* has to remain small. Property 3 shows that for any nonzero value of $\tilde{y}$, a^* can periodically take the value zero if f switches periodically between concavity and convexity. This in turn implies that a^* can periodically become small if f continues to change its curvature, that is, β changes from $+1$ to -1. As will be shown in Section 3.3.1, the conditions for parameter convergence not only require that $\tilde{f}$ become large for a large $\tilde{\theta}$ but also require f to switch between convexity and concavity over any given interval.

Yet another feature of the min-max algorithm is the use of the error $\tilde{y}_\epsilon$ for adjusting the parameter $\hat{\theta}$ instead of the traditional estimation error $\tilde{y}$. This was introduced in the estimation algorithm to ensure a continuous estimator in the presence of a discontinuous solution that can be obtained from the min-max optimization problem. The introduction of a nonzero ϵ can cause the parameter estimation to stop if $|\tilde{y}|$ becomes smaller than ϵ. As a result, the trajectories are shown to converge to a neighborhood D_ϵ of the origin rather the origin itself.

In Section 3.3.1, we state and prove the convergence result. In Section 3.3.2, we discuss the sufficient condition that results in parameter convergence, specific examples of f and counterexamples, and the relation to persistent excitation conditions that guarantee parameter convergence in the case of linear parameterization.

3.3.1 Proof of Convergence

The first convergence result in this chapter is stated in Theorem 1.

Theorem 1 *If (i) $f(\theta, \omega(t))$ is convex (or concave) on θ for any $\omega(t) \in \mathbb{R}^m$, and (ii) for every $t_1 > t_0$, there exist positive constants T_0, ϵ_u and a time instant $t_2 \in [t_1,\ t_1 + T_0]$ such* *that for any θ*

$$\beta(\omega(t_2))\left(f(\theta,\omega(t_2)) - f(\theta_0,\omega(t_2))\right) \geq \epsilon_u \|\theta - \theta_0\|, \tag{3.20}$$

where $\beta(\omega(t_2))$ is defined as in (3.18), then all trajectories of (3.8) will converge uniformly to

$$D_\epsilon = \{x \mid V(x) \leq \gamma_1\}, \tag{3.21}$$

where

$$x = [\tilde{y}_\epsilon,\ \tilde{\theta}^T]^T, \quad \gamma_1 = \tfrac{2\epsilon}{\epsilon_u^2}\left(16B_\theta U_b + 8B_\theta B_\phi + 4B_\phi^2\right), \tag{3.22}$$

ϵ is defined as in (3.8), ϵ_u is given by (3.20), U_b and B_θ are defined as in Assumptions 1 and 2, and B_ϕ is the bound on ϕ^* in (3.4) so that

$$\|\phi^*(t)\| \leq B_\phi, \qquad \forall\, t \geq t_0. \tag{3.23}$$

The proof of Theorem 1 follows by showing that if ω and f are such that condition (3.20) is satisfied, then $\tilde{y}_\epsilon(t)$ becomes large at some time t over the interval $[t_1, t_1 + T_0]$. Once $\tilde{y}_\epsilon(t)$ becomes large, it follows from Property 2 that $V(t)$ decreases over the interval $[t_1, t_1 + T_0]$ by a finite amount.

Remark 1: If f is concave (or convex) and if f satisfies the inequality in Eq. (3.20), we shall define that f satisfies the Convex Persistent Excitation (CPE) condition with respect to ω. Theorem 1 implies that if f satisfies the CPE condition with respect to ω, then parameter convergence to a desired precision follows.

Remark 2: From the definition of D_ϵ, it automatically follows that as $\epsilon \to 0$, all trajectories converge to the region $x = 0$ and hence u.a.s.l. follows.

3.3.2 Sufficient Condition for Parameter Convergence

The CPE condition specifies certain requirements on f in order to achieve parameter convergence. For a given f, theorem 1 does not state how ω should behave over time in order to satisfy (3.20). In this section, we state some observations and examples of ω that satisfies

(3.20) for a general f.

Equation (3.20) consists of two separate requirements. Denoting $\tilde{f} = f(\theta, \omega) - f(\theta_0, \omega)$, the first requirement is that the magnitude of $\tilde{f}$ must be large. The second requirement is that $\tilde{f}$ must have the same sign as β. The first component states that for a large parameter error, there must be a large error in $\tilde{f}$. It is straightforward to demonstrate that this condition is equivalent to linear persistent excitation condition in [10], and is shown in section 3.3.2. The second requirement states what the sign of $\tilde{f}$ should be in relation to the convexity/concavity of f. If f is convex, $\tilde{f}$ should be positive, and conversely, if f is concave, $\tilde{f}$ should be negative.

The coupling of convexity/concavity and the sign of the integral of $\tilde{f}$ has the following practical implications. To ensure parameter convergence, ω must be such that one of the following occurs: At least at one instant $t_2 \in [t_1,\ t_1 + T]$,

(a) For the given $\tilde{\theta}$, ω must change in such a way that the sign of $\tilde{f}$ is reversed, while keeping the convexity/concavity of f the same, *or*

(b) For the given $\tilde{\theta}$, ω must reverse the convexity/concavity of f, while preserving the sign of $\tilde{f}$

Examples

We illustrate the above comments using specific examples of f. Suppose

$$f = e^{-\omega^T \theta} \tag{3.24}$$

where $\omega(t) : \mathbb{R} \rightarrow \mathbb{R}^n$, $\theta \in \Omega \subset \mathbb{R}^n$. It can be checked that f given in (3.24) is always convex with respect to θ for all ω. Therefore, option *(b)* is not possible. Hence, ω must be such that $\tilde{f}$ can switch sign for any $\tilde{\theta}$ as required by option *(a)*. One example of such an ω is if for any t_1, there exists $t_2 \in [t_1,\ t_1 + T]$ such that

$$\omega^T(t_2)\, v \ \geq \varepsilon_u \tag{3.25}$$

where v is any unit vector in $\mathbb{R}^n$. Another example which satisfies condition (3.20) is given by

$$f = \theta^{\omega}, \qquad \omega \in \mathbb{R}.$$

It is easy to show that for such an f, condition (b) is satisfied if ω switches between ω_1 and ω_2 where $0 < \omega_1 < 1$ and $\omega_2 > 1$.

The above examples show that the condition on ω that satisfies Eq. (3.20) varies with f.

Relation to Conditions of Linear Persistent Excitation

The relation between CPE and LPE is worth exploring. For this purpose, we consider a linearly parameterized system, which is given by Eq. (3.1) with

$$f(\theta_0, \omega) = \theta_0^T \phi(\omega)$$

where $\phi(\omega) \in \mathbb{R}^n$. In this case, it is well known that the corresponding estimator is given by equation (3.4) with $a^* = 0$ and $\phi^* = \phi$ [1]. The resulting error equations are summarized by

$$\begin{aligned} \dot{\tilde{y}} &= -\alpha(y,u)\tilde{y}_\epsilon + \tilde{\theta}^T \phi(\omega) \\ \dot{\tilde{\theta}} &= -\tilde{y}_\epsilon \phi(\omega). \end{aligned} \tag{3.26}$$

In [10], it is shown that u.a.s.l. of (3.26) follows under an LPE condition. For the sake of completeness, we state this condition below.

Definition (LPE): ϕ is said to be linearly persistently exciting (l.p.e.) if for every $t_1 > t_0$, there exists positive constants T_0, δ_0, ϵ_0 and a subinterval $[t_2,\ t_2 + \delta_0] \in [t_1,\ t_1 + T_0]$ such that

$$\left| \int_{t_2}^{t_2+\delta_0} \theta^T \phi(\omega(\tau)) d\tau \right| \geq \epsilon_0 \|\theta\|. \tag{3.27}$$

We now show the relation between the LPE condition and the CPE condition in (3.20). When $f(\theta, \omega) = \theta^T \phi(\omega)$, if Assumption 1 holds, it can be shown that the LPE condition is equivalent to the following inequality: For every $t_1 > t_0$ and θ, there exists positive constants T_0, ϵ_0 and a time instant $t_2 \in [t_1,\ t_1 + T_0]$ such that

$$|f(\theta, \omega(t_2)) - f(\theta_0, \omega(t_2))| \geq \epsilon_u ||\theta - \theta_0||. \tag{3.28}$$

Since a linear function can be considered to be either convex or concave, the inequality in (3.28) is equivalent to the CPE condition in (3.20). This equivalence is summarized in the following lemma:

Lemma 1 *When $f(\theta, \omega) = \theta^T \phi(\omega)$, if Assumption 1 holds, the CPE condition in Eq. (3.20) is equivalent to the LPE condition in Eq. (3.27).*

It should be noted that for a general nonlinear f, the CPE condition becomes more restrictive than the LPE condition. For example, for f as in (3.24), the CPE condition implies that ω must satisfy (3.25). On the other hand, if $f = \omega^T \theta$, even if ω is such that $|\omega^T(t_2)v|$ is periodically large, the LPE condition is satisfied.

A counter-example

For a general function f, it may not be possible to find a ω that satisfies either condition (a) or (b) mentioned above. A simple example is

$$f = cos(\theta\omega)$$

where $|\omega| \leq \omega_{max}$ and $\theta \in [0,\ \pi/(2\omega_{max})]$. We note that f is concave and monotonically decreasing for any ω with $|\omega| \leq \omega_{max}$. Hence neither (a) nor (b) is satisfied. That is, it is possible for the parameter estimate $\hat{\theta}$ of the min-max algorithm to get "stalled" in a region in Ω^0. This motivates the need for an improved min-max algorithm, and is outlined in section 3.4.

3.4 Parameter Convergence in Systems with a General Parameterization

In the previous section, we showed that if a function f is convex (or concave), and if f and ω satisfy the CPE condition, then parameter convergence follows. However, as we saw in section 3.3.2, not all convex/concave functions can satisfy the CPE condition. In this section, we present a new algorithm which not only allows the persistent excitation condition to be relaxed but also enables parameter convergence for non-convex and non-concave functions.

The algorithm we present in this section is hierarchical in nature, and consists of a lower-level and a higher-level. In the lower-level, for a given unknown parameter region Ω^0, the parameter estimate $\hat{\theta}$ is updated using the min-max algorithm as in (3.4). In the higher-level, using information regarding the parameter estimate $\hat{\theta}$ obtained from the lower-level, the unknown parameter region is updated as Ω^1. Iterating between the lower and higher levels, the overall hierarchical algorithm guarantees a sequence of parameter region Ω^k. The properties of these two levels are discussed in Sections 3.4.1 and 3.4.2, respectively. In Section 3.4.4, we discuss conditions under which $\hat{\theta}$ converges to θ_0. Using these conditions, the definition of persistent excitation for nonlinearly parameterized systems is introduced. In Section 3.4.5, we present examples of such a Nonlinear Persistent Excitation (NLPE). The relation between NLPE and CPE is discussed in Section 3.4.6.

3.4.1 Lower-level Algorithm

The lower-level algorithm consists of the min-max parameter estimation as in (3.4) with the unknown parameter $\theta_0 \in \Omega^k$. We show in this section that when this algorithm is used, $\tilde{y}_\epsilon(t)$ becomes small in a finite time, which is denoted as lower-level convergence. Once this occurs, the parameter estimate $\hat{\theta}$, derived from the lower-level convergence, remains nearly steady. This estimate, in turn, is used in the higher-level part of hierarchical algorithm to update the unknown parameter region from Ω^k to Ω^{k+1}. The convergence of $\tilde{y}_\epsilon$ is stated in Lemma 2, and the characterization of the unknown parameter is stated in Lemma 3.

Lemma 2 *For the system in (3.1) and the estimator in (3.4), given any positive T and δ, there exists a finite time t_1 such that*

$$|\tilde{y}_\epsilon(t)| \leq \delta \qquad \text{for } t_1 \leq t \leq t_1 + T. \tag{3.29}$$

We note that for every specific ω, a time t_1 that satisfies (3.29) exists. However the value of t_1 will depend on the choice of ω. Since our goal is parameter convergence, we require ω to assume distinct values, i.e. persistently span a set of interest. This is stated in the definition below.

Let U_I be defined as in Eq. (3.2).

Definition 5 *ω is said to persistently span U_I if for any $\omega_i \in U_I$ and any t_1, there exist a finite T_i and τ_i such that*

$$\omega(\tau_i) = \omega_i \qquad \tau_i \in [t_1,\ t_1 + T_i] \qquad i = 1, ..., I. \tag{3.30}$$

Definition 5 implies that ω periodically visits all points in U_I.

Let

$$B_t = 2B_\theta(\delta B_\phi + 2U_b) + \delta B_\phi^2, \tag{3.31}$$

where U_b, B_θ and B_ϕ are defined in Assumption 1, Assumption 2 and (3.23) and δ is any positive number. If we choose T as

$$T = \max_{1 \leq i \leq I} T_i + \frac{2\sqrt{B_t(\delta + \epsilon)}}{B_t}$$

where T_i is given by (3.30), then Lemma 2 implies that there exists a finite time t_1 such that

$$|\tilde{y}_\epsilon(t)| \leq \delta \qquad t_1 \leq t \leq t_1 + T. \tag{3.32}$$

When $\tilde{y}_\epsilon$ satisfies (3.32), we refer to it as lower-level convergence. If ω persistently spans U_I, then Definition 5 and the choice of τ_i implies that at $\tau_i \in [t_1,\ t_1 + T]$, $\omega(\tau_i) = \omega_i$,

$i = 1, ..., I$. The parameter estimate $\hat{\theta}(\tau_i)$ at time instances are defined as

$$\hat{\theta}_i^c = \hat{\theta}(\tau_i) \qquad i = 1, ..., I,$$

and are denoted as low-level convergent estimates. We characterize the region where the unknown parameters lie in lemma 3 using these lower-level convergent estimates.

Lemma 3 *For the system in (3.1) and estimator in (3.4), let Ω be the unknown parameter region and $\hat{\theta}_i^c, i = 1, ..., I$, be the lower-level convergent estimates. If the input ω persistently spans U_I, then*

$$\theta_0 \in \bigcap_{i=1}^{I} \Phi_\epsilon(\Omega, \omega_i, \epsilon, \delta, \hat{\theta}_i^c).$$

where

$$\begin{aligned} \Phi_\epsilon(\Omega, \omega_i, \epsilon, \delta, \hat{\theta}_i^c) &= \left\{\theta \in \Omega | \underline{f}_i \leq f(\theta, \omega_i) \leq \bar{f}_i\right\} \\ \underline{f}_i &= f(\hat{\theta}_i^c, \omega_i) - a_+^*(\hat{\theta}_i^c, \omega) - \alpha_{max}\delta - 2\sqrt{B_t(\delta + \epsilon)} \\ \bar{f}_i &= f(\hat{\theta}_i^c, \omega_i) + a_-^*(\hat{\theta}_i^c, \omega) + \alpha_{max}\delta + 2\sqrt{B_t(\delta + \epsilon)}\} \end{aligned} \tag{3.33}$$

and B_t as in (3.31).

Lemma 3 implies that the unknown parameter θ_0 lies in Φ_ϵ for a given ω_i. It should be noted that in general, Φ_ϵ need not be smaller than Ω. However other properties of Φ_ϵ are useful for characterizing the convergence behavior of the min-max algorithm. These are enumerated below.

(P1): For $\delta = \epsilon = 0$, if $a_+^* = a_-^* = 0$, then Φ_ϵ reduces to the manifold

$$f(\theta_0, \omega_i) = f(\hat{\theta}_i^c, \omega_i).$$

(P2): Property (P3) implies that if (i) ω is p.e. in U_I, (ii) f is identifable w.r.t. U_I, (iii) $\delta = \epsilon = 0$ and (iv) $a_+^* = a_-^* = 0$, then

$$\bigcap_{i=1}^{I} \Phi_\epsilon(\Omega^k, \omega_i, \epsilon, \delta, \hat{\theta}_i^c) = \{\theta_0\}.$$

These properties are made judicious use of in designing the higher-level algorithm in the section 4.2.

3.4.2 Higher-Level Algorithm

We now present the higher-level component of the hierarchical algorithm. Here, our goal is to start from a known parameter region Ω^k that the unknown parameter θ_0 lies in, and update it as Ω^{k+1} using all available information from the lower-level component. In particular, we use Φ_ϵ defined in Eq. (3.33) to update Ω^k. In order to reduce the parameter uncertainty, different Φ_ϵ's are computed by varying ω_i, $i = 1, ..., I$. The resulting Ω^{k+1} is therefore chosen as

$$\Omega^{k+1} = \bigcap_{i=1}^{I} \Phi_\epsilon(\Omega^k, \omega_i, \epsilon, \delta, \hat{\theta}_i^c). \tag{3.34}$$

3.4.3 The Hierarchical Algorithm

The complete hierarchical algorithm is stated in Table 3.1.

It should be noted that Steps 2 and 3 correspond to the lower-level and the higher-level parts of the hierarchical algorithm, respectively. Also, we note that Step 2 requires the closed-form solutions of a^* and ϕ^* which can be found as outlined in Section 2.1.

In order to obtain parameter convergence using the hierarchical algorithm, what remains to be shown is whether Ω^{k+1} is a strict subset of Ω^k.

3.4.4 Parameter Convergence with the Hierarchical Algorithm

We now address the parameter convergence of the hierarchical algorithm. We introduce a definition for a "stalled" parameter region Δ_i:

For any $\Omega \subseteq \Omega^0$, define $\underline{f}_i^*$ and $\bar{f}_i^*$ as

$$\underline{f}_i^* = \min_{\theta \in \Omega} f(\theta, \omega_i), \qquad \bar{f}_i^* = \max_{\theta \in \Omega} f(\theta, \omega_i). \tag{3.35}$$

Table 3.1: Hierarchical Minmax Algorithm

Step 1:	Set $k = 0$ and $\Omega^k = \Omega^0$, and $T = \max_i T_i + 2\sqrt{B_t(\delta + \epsilon)}/B_t$ where B_t is defined as in (3.31).
Step 2:	Run the estimator in (3.4) where $a^* = \min_{\phi \in \mathbb{R}^m} \max_{\theta \in \Omega^k} g(\theta, \omega, \phi)$ $\phi^* = arg \min_{\phi \in \mathbb{R}^m} \max_{\theta \in \Omega^k} g(\theta, \omega, \phi)$ $g(\theta, \omega, \phi) =$ $sat\left(\frac{\tilde{y}}{\epsilon}\right)\left(f(\hat{\theta}, \omega) - f(\theta, \omega) - \phi^T(\hat{\theta} - \theta)\right).$ Wait until time t_k^* where $\lvert\tilde{y}_\epsilon(t)\rvert \leq \delta \qquad \text{for } t \in [t_k^*,\ t_k^* + T],$ and record the low level convergent estimate $\hat{\theta}_i^c$ as $\hat{\theta}_i^c = \hat{\theta}(\tau_i)$ where $\omega(\tau_i) = \omega_i, \qquad \forall\, \tau_i \in [t_k^*,\ t_k^* + T].$
Step 3:	Calculate Ω^{k+1} from Ω^k and $\hat{\theta}_i^c, i = 1, \ldots I$, as follows: $\Omega^{k+1} = \bigcap_{i=1}^{I} \Phi_\epsilon(\Omega^k, \omega_i, \epsilon, \delta, \hat{\theta}_i^c)$ $\Phi_\epsilon(\Omega, \omega_i, \epsilon, \delta, \hat{\theta}_i^c) = \left\{\theta \in \Omega \mid \underline{f}_i \leq f(\theta, \omega_i) \leq \bar{f}_i\right\}$ $\underline{f}_i = f(\hat{\theta}_i^c, \omega_i) - a_+^*(\hat{\theta}_i^c, \omega) - Q(\delta, \epsilon)$ $\bar{f}_i = f(\hat{\theta}_i^c, \omega_i) + a_-^*(\hat{\theta}_i^c, \omega) + Q(\delta, \epsilon)$ where $Q(\delta, \epsilon) = \alpha_{max}\delta + 2\sqrt{B_t(\delta + \epsilon)}$.
Step 4:	If $\Omega^{k+1} = \Omega^k$, stop. Otherwise, set $k = k + 1$ and return to step 2.

Then we define $\Delta_i(\Omega)$ to be a "stalled" estimate-region of Ω as

$$\Delta_i(\Omega) = \{\theta \mid f(\theta, \omega_i) - a_+^*(\theta, \omega_i) - D(\epsilon, \delta) \leq \underline{f}_i^* \\ \text{and } f(\theta, \omega_i) + a_-^*(\theta, \omega_i) + D(\epsilon, \delta) \geq \bar{f}_i^*\}. \tag{3.36}$$

where

$$D(\epsilon, \delta) = \alpha_{max}\delta + 2\sqrt{B_t(\delta + \epsilon)} \tag{3.37}$$

We prove a property of $\Delta_i(\Omega)$ which explains why it corresponds to a "stalled" region in Ω.

Lemma 4 *For some k, if $\hat{\theta}_i^c \in \Delta_i(\Omega^k)$, $\forall\, i = 1, .., I$, then*

$$\Omega^{k+1} = \Omega^k.$$

In order to establish parameter convergence, we first characterize the region L that the parameter estimate converges to in Lemma 5, and then establish the conditions under which L simply coincides with the true parameter θ_0 in Theorem 2. The set L is defined as follows:

$$L(\Omega^0, \epsilon, \delta) = \bigcup_{\forall\, \Omega \in \Omega^0,\ \theta_0 \in \Omega} \left(\bigcap_{i=1,..,I} B(\Delta_i(\Omega)) \right), \tag{3.38}$$

where $B(X)$ is a box that contains any set X and is defined as

$$B(X) = \{\theta \mid ||\theta - \bar{\theta}|| \leq \delta T B_\phi,\ \forall\, \bar{\theta} \in X\}. \tag{3.39}$$

Lemma 5 *For the system in (3.1) and estimator in (3.4), under assumptions 1-4, the hierarchical algorithm outlined in Table 3.1 guarantees that*

$$\lim_{t \to \infty} \hat{\theta}(t) \in L(\Omega^0, \epsilon, \delta). \tag{3.40}$$

Since ϵ and δ are arbitrary positive numbers, they can be chosen to be as small as possible. When $\epsilon \to 0,\ \delta \to 0$, it follows directly from (3.36), (3.39) and (3.38) that

$$\lim_{\epsilon\to 0,\delta\to 0} L(\Omega^0,\epsilon,\delta) = \bigcup_{\forall\,\Omega\in\Omega^0,\text{ with }\theta_0\in\Omega}\ \bigcap_{i=1,..,I} \bar{\Delta}_i(\Omega) \tag{3.41}$$

where

$$\begin{aligned}\bar{\Delta}_i(\Omega) &= \{\theta \mid f(\theta,\omega_i) - a^*_+(\theta,\omega_i) \le \underline{f}^*_i \\ &\quad \text{and } f(\theta,\omega_i) + a^*_-(\theta,\omega_i) \ge \bar{f}^*_i\}.\end{aligned} \tag{3.42}$$

with $\underline{f}^*_i$ and $\bar{f}^*_i$ defined as in (3.35). From (3.41), we have the following theorem:

Theorem 2 *For the system in (3.1) and estimator in (3.4), under assumptions 1-4,*

$$\lim_{t\to\infty,\epsilon\to 0,\delta\to 0} \hat{\theta} = \theta_0 \tag{3.43}$$

if and only if for any $\Omega\subseteq\Omega_0$ where $\theta_0 \in \Omega$,

$$\bigcap_{i=1,\ldots,I} \bar{\Delta}_i(\Omega) = \phi \text{ or } \bigcap_{i=1,\ldots,I} \bar{\Delta}_i(\Omega) = \{\theta_0\}. \tag{3.44}$$

where ϕ denotes the null set and $\bar{\Delta}$ is defined as in (3.42).

Theorem 2 gives us a method to check if the hierarchical algorithm can estimate the true parameters to any desired precision when we set ϵ and δ small enough for a specific problem. We note that $L(\theta_0,\epsilon,\delta)$ is a continuous function of ϵ and δ, and that as ϵ and δ becomes small, L becomes arbitrarily close to the set $\{\theta_0\}$. Hence the parameter estimate converges to the true value with a desired precision.

Remark 3: If $f(\theta,\omega)$ is identifiable over Ω with respect to U_I, ω persistently spans U_I, and f satisfies the inequality (3.44), we shall define that f satisfies the Nonlinear Persistent Excitation (NLPE) condition with respect to ω. Theorem 2 implies that NLPE of f with respect to ω is necessary and sufficient for parameter convergence to take place.

Remark 4: The requirement on ω for f to satisfy the NLPE can sometimes be less stringent than that on ω for LPE. An example of this fact is for the parameter $\theta = [\theta_1,\theta_2]^T$, and the cases (i) $l(\theta) = \theta^T \omega$, and (ii) $f(\theta) = \theta_1 \omega_1 \cos(\theta_2 \omega_2)$ where ω_1 and ω_2 are the elements of ω. Clearly, for a ω such that $\omega_1 = k\omega_2$, where k is a constant, ω does not satisfy LPE, but f does satsify NLPE with respect to ω. As shown in Section 3.3.2, NLPE can impose more stringent conditions on ω as well.

Remark 5: It should be noted that the NLPE condition guarantees parameter convergence for any general nonlinear function f that is identifiable. This implies that the min-max algorithm outlined in [5], which is applicable for even a non-convex (or a non-concave) function, can be used to establish parameter convergence. We include simulation results of such an example in Section 3.5.

Remark 6: It should be noted that a fairly extensive treatment of conditions of persistent excitation has been carried out in [11, 12] for a class of nonlinear systems. The systems under consideration in this chapter do not belong to this class. The most distinct features of the system (3.1) is the presence of the quantity a^* and the quantity $f(\widehat{\theta}, \omega) - f(\theta_0, \omega)$, where the former can introduce equilibrium points other than zero and the latter is not Lipschitz with respect to $\widehat{\theta} - \theta$. As a result, an entirely different set of conditions and properties have had to be derived to establish parameter convergence.

Remark 7: The closed-form solutions of a^* and ϕ^* can be calculated as shown in Section 2.1. It should be noted that these solutions have been derived without requiring that $\hat{\theta} \in \Omega^k$, thereby expanding the results of [1]. Since $\hat{\theta}$ can lie anywhere, subsequent iterations of the hierarchical algorithm can be carried out during which time the corresponding min-max solutions can be derived.

As is evident from (3.44), (3.35) and (3.42), to check if indeed the NLPE condition is satisfied for every $\Omega \subseteq \Omega_0$ for a given f and ω is a difficult task. In the following section, we show that when $\theta \in \mathbb{R}^2$, if f is monotonic function of θ, identifiable with respect to U_I, and f is convex/concave, then the NLPE condition is satisfied.

3.4.5 Parameter Convergence when $\theta \in \mathbb{R}^2$: An Example

When $\theta = [\theta_1, \ \theta_2] \in \mathbb{R}^2$, the following lemma provides sufficient conditions for Eq. (3.44) to hold and hence for the hierarchical algorithm to guarantee convergence.

Lemma 6 *For system in (3.1), estimator in (3.4) where $\theta \in \mathbb{R}^2$, let $\theta_0 \in \Omega^0$ and f be identifiable over Ω^0 with respect to U_2. If*

$$(i) \quad f(\theta, \omega_i) \text{ is convex (or concave) over all } \theta \text{ in } \Omega^0$$

$$\omega_1, \omega_2 \in U_I \tag{3.45}$$

$$(ii) \quad f(\theta, \omega_i) \text{ is monotonic with respect to } \theta \text{ in } \Omega^0$$

$$\omega_1, \omega_2 \in U_I \tag{3.46}$$

then equation (3.44) holds for any $\Omega \subseteq \Omega^0$ where $\theta_0 \in \Omega$.

The reader is refered to [13] for the proof.

3.4.6 Relation between NLPE and CPE

In what follows we compare the NLPE and the CPE conditions. In order to facilitate this comparison, we restate the CPE condition in a simpler form:

Definition 6 *f is said to satisfy the CPE′ condition with respect to ω if (i) $f(\theta, \omega(t))$ is convex (or concave) for any $\omega(t) \in \mathbb{R}^m$, and (ii) ω is persistently spanning with respect to U_I, and (iii) for any θ, there exists $\omega_i \in U_I$ such that*

$$\beta(\omega_i)\left(f(\theta, \omega_i) - f(\theta_0, \omega_i)\right) \geq \epsilon_u \|\theta - \theta_0\|. \tag{3.47}$$

We note that the only distinction between the inequalities in (3.20) and (3.47) is in the value taken by $\omega(t_2)$ for some t_2 in the interval $[t, t+T]$. In (3.47) it implies that $\omega(t_2)$ assumes one of the finite values ω_i in U_I while in (3.20), the corresponding U_I can consist of infinite values. If ω is "ergodic" in nature so that it visits all typical values that it will assume for all t over one interval, then it implies that the two conditions (3.20) and (3.47) are equivalent. We shall assume in the following that the input is "ergodic."

Lemma 7 *Let $f(\theta, \omega_i)$ be convex (or concave) for all $\theta \in \Omega^0$, then the CPE′ condition implies the NLPE condition.*

Remark 8: Lemma 7 shows that the CPE′ condition is sufficient for the NLPE to hold if f is convex (or concave). Clearly, the CPE′ condition is not necessary, as shown by the counterexample in Section 3.3.2. The NLPE condition therefore represents the most general definition of persistent excitation in nonlinearly parameterized systems.

3.5 Simulation Results

We consider the system in (3.1) and the estimator in (3.4) to evaluate the performance of the hierarchical algorithm. The system parameters are chosen as follows:

$$f = \left(\theta_0 - \frac{\omega}{8}\right)^2 + 12exp\left\{-5\left(\theta_0 - 2 + \frac{\omega}{4}\right)^2\right\}$$

where θ_0 is an unknown parameter that belongs to a known interval $\Omega^0 = [0, 5]$. System variable ω is chosen as a sinusoidal function $\omega = 1.1\sin(2t)$ and the true unknown parameter θ_0 equals 2. We note that the function f is non-convex (and non-concave), whose values are shown in Figure 1 for $\omega = 1, -1, 0$. It can be shown that f is identifiable with respect to Ω^0 and that ω is persistently spanning with respect to $U_I = \{1, -1, 0\}$. The hierarchical algorithm in Table 3.1 was implemented to estimate θ_0. The parameters $\epsilon = 0.001$ and $\delta = 0.02$. Since ω is a sinusoid, the parameter T was set to the corresponding period π. The resulting output error $\tilde{y}_\epsilon$, parameter estimate $\widehat{\theta}$, and the update of the parameter region Ω^k are shown in Figures 2-4, respectively. The evolutions of the lower and upper bounds $\underline{f}_i^k$ and $\bar{f}_i^k$, $i = 1, 2, 3$ with respect to t are also shown in Figure 5. A similar convergence was observed to occur for any θ_0 in Ω^0. These figures show that the update of Ω^k is not necessarily periodic. Once $\tilde{y}_\epsilon$ becomes smaller than δ over an interval T, the corresponding parameter estimates and the upper and lower bounds on f_i and therefore the unknown parameter region are computed. It was also observed that just the min-max algorithm without the higher level component did not result in parameter convergence.

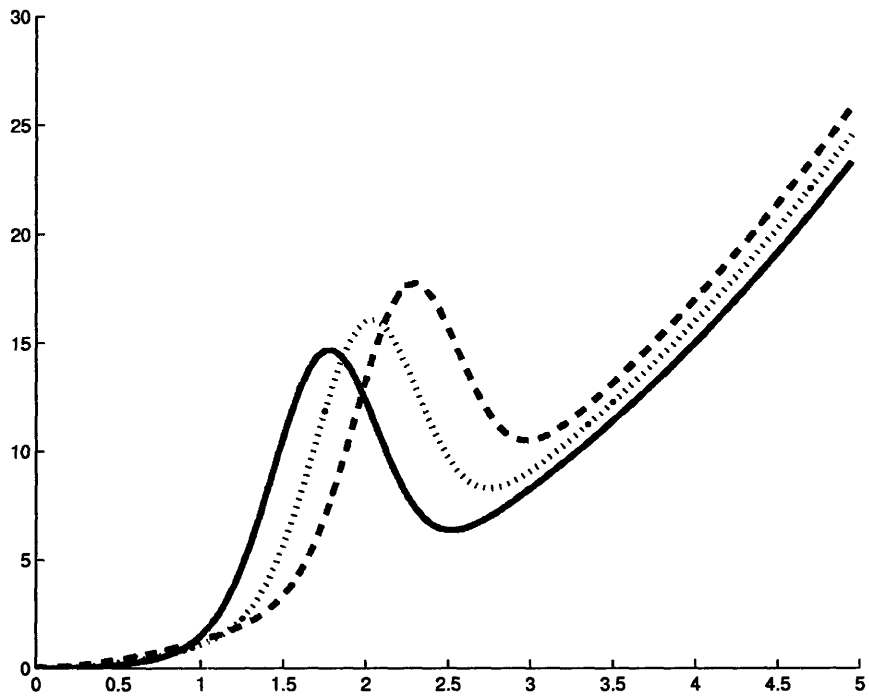

Figure 3-1: A non-concave (and non-convex) function $f(\theta, u)$ vs. θ, for $u = 1, 0, -1$. $f(\theta, 1)$:—, $f(\theta, -1)$:- - -, $f(\theta, 0)$:.....

3.6 Summary

In this chapter, the problem of parameter estimation in systems with general nonlinear parameterization is considered. In systems with convex/concave parameterization, sufficient conditions are derived under which parameter estimates converge to their true values using a min-max algorithm as in [1]. In systems with a general nonlinear parameterization, a hierarchical min-max algorithm is proposed where the lower-level consists of a min-max algorithm and the higher-level component updates the bounds on the parameter region within which the unknown parameter is known to lie. Using this algorithm, a necessary and sufficient condition is established for parameter convergence in systems with a general nonlinear parameterization. In both cases, the conditions needed are shown to be stronger than linear persistent excitation conditions that guarantee parameter convergence in linearly parameterized systems, thereby leading to a general definition of nonlinear persistent excitation (NLPE).

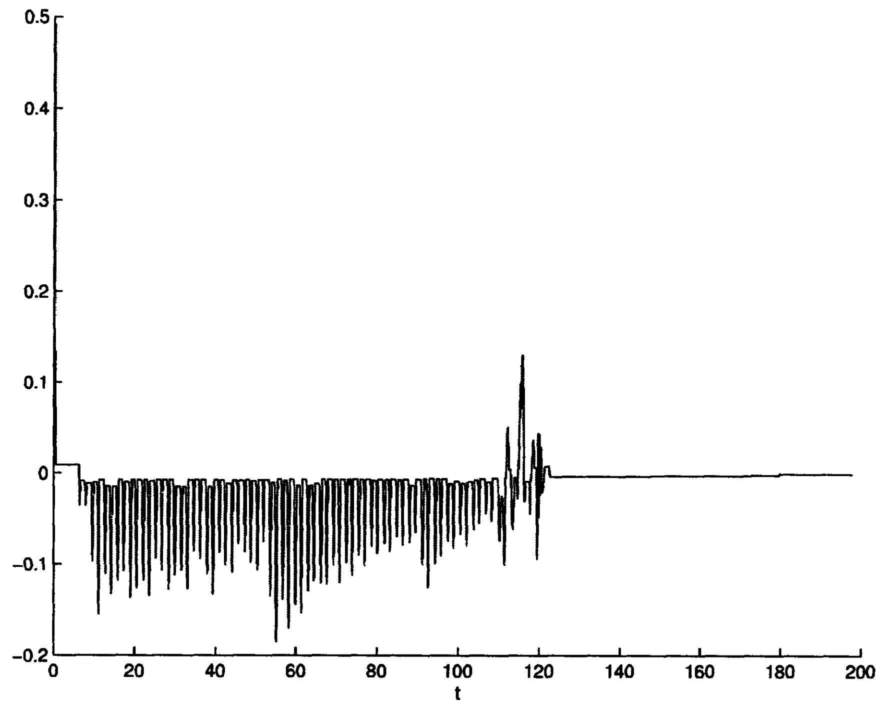

Figure 3-2: The output error $\tilde{y}_\epsilon(t)$ with t using the hierarchical algorithm. $\epsilon = 0.001$ and $\delta = 0.02$.

The results in this chapter establish parameter estimation in a system of the form (3.1). Even though the output is a scalar, as is shown in [5], a wide variety of adaptive control and estimation problems can be reduced to an error model of the form of (3.1). As a result, the persistent excitation conditions presented in this chapter are applicable to all of these problems.

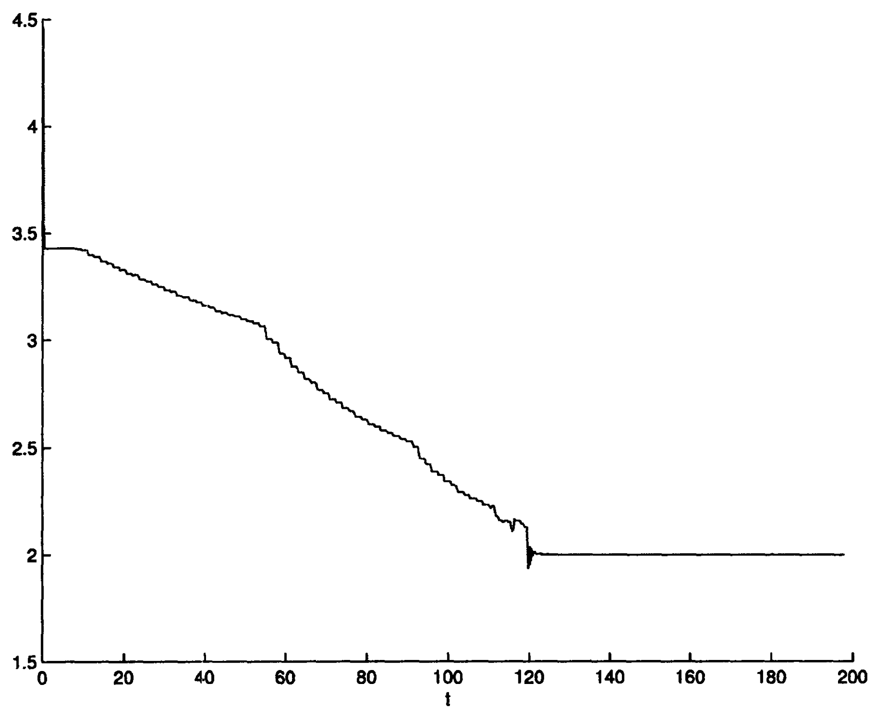

Figure 3-3: The parameter estimate $\widehat{\theta}(t)$ with t using the hierarchical algorithm. True parameter value $\theta_0 = 2$.

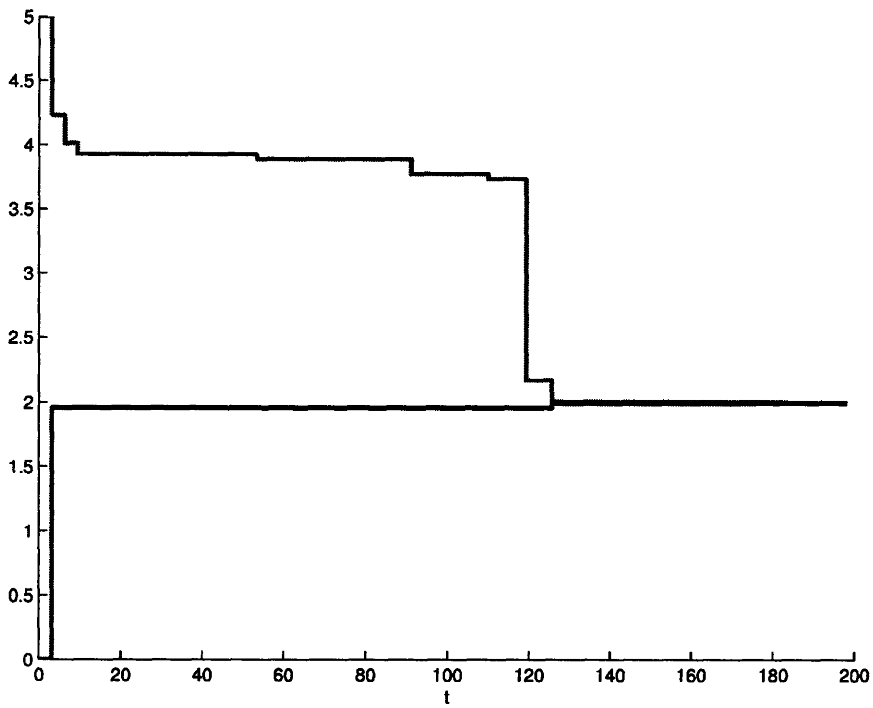

Figure 3-4: The evolution of the parameter region Ω^k with t, using the hierarchical algorithm. Note that Ω^k is updated at instants t_k^* such that $|\tilde{y}_\epsilon(t)| \leq \delta$ for $t \in [t_k^* - T, t_k^*]$.

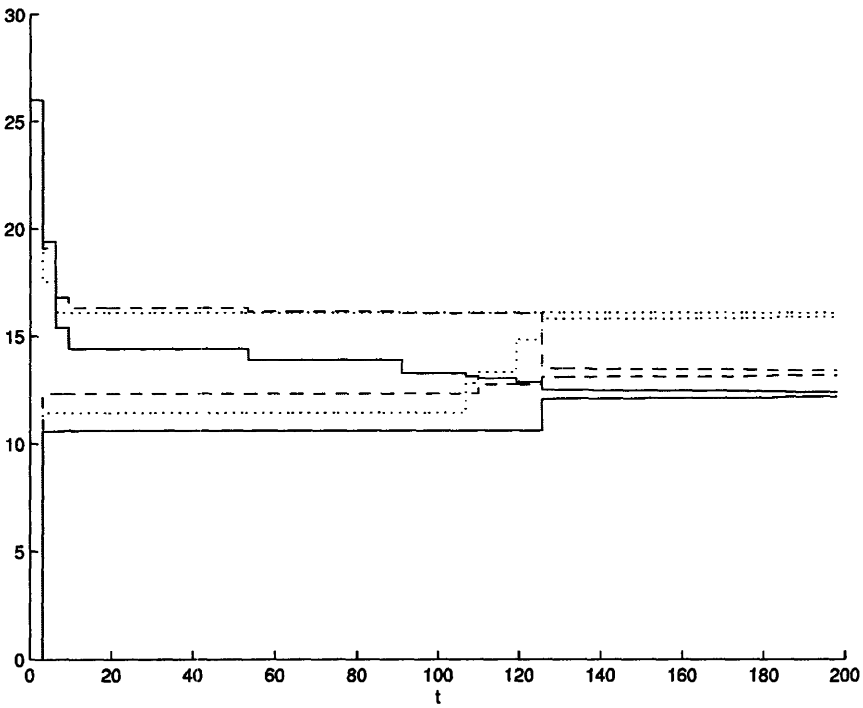

Figure 3-5: The upper-bounds $\bar{f}_i^k$ and lower-bounds $\underline{f_i^k}$ of $f(\theta, u_i)$ with t using the hierarchical algorithm, for $u_i = 1, -1, 0$. $\bar{f}_1$, $\underline{f_1}$:—, $\bar{f}_2$,$\underline{f_2}$:- - -, $\bar{f}_3$,$\underline{f_3}$:....

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3.7 Appendix

Proof of Property 1: From (3.8) and (3.13), it follows that

$$\begin{aligned} \dot{V} &= -\alpha(y,u)\tilde{y}_\epsilon^2 + \tilde{y}_\epsilon \\ &\quad \left(f(\widehat{\theta},\omega) - f(\theta_0,\omega) - \phi^{*T}(\widehat{\theta}-\theta_0) - a^* sat(\frac{\tilde{y}}{\epsilon})\right). \end{aligned} \tag{3.48}$$

When $|\tilde{y}| \leq \epsilon$, it follows that $\tilde{y}_\epsilon = 0$ and hence, $\dot{V}= 0$. When $|\tilde{y}| > \epsilon$, it follows that $sat(\frac{\tilde{y}}{\epsilon}) = sign(\tilde{y})$. Then (3.48) is transformed into

$$\begin{aligned} &\dot{V}= -\alpha(y,u)\tilde{y}_\epsilon^2 + \tilde{y}_\epsilon sat(\frac{\tilde{y}}{\epsilon}) \left(sat(\frac{\tilde{y}}{\epsilon})\right. \\ &\left. + \left(f(\widehat{\theta},\omega) - f(\theta_0,\omega) - \phi^{*T}(\widehat{\theta}-\theta)\right) - a^*\right). \end{aligned} \tag{3.49}$$

Combining (3.7) and (3.49), Property 1 is established. •

Proof of Property 2: To prove Property 2, the following sublemma is needed.

Sublemma 2.1: For given systems

$$\begin{aligned} \dot{x} &= -k(t)x + z(t) \\ \dot{x}_m &= -k_m x_m + z_m \end{aligned}$$

where $k(t) > 0,\ k_m > 0$ and

$$|z(t)| \le z_m \qquad \forall\, t \ge t_0,$$

$$\begin{aligned} \text{if} \quad & x(t_0) \le x_m(t_0) < 0,\ k(t) \le k_m, \\ \text{then} \quad & x(t) \le x_m(t), \forall\, t \ge t_0 \text{ where } x_m(t) \le 0. \end{aligned}$$

The proof of the sublemma is straight forward and is omitted. Now let us prove Property 2.

Without loss of generality, we assume that

$$\tilde{y}_\epsilon(t_1) \le -\gamma. \tag{3.50}$$

From (3.8), it follows that

$$\dot{\tilde{y}}_\epsilon = -\alpha(y,u)\tilde{y}_\epsilon + m(t) \tag{3.51}$$

where $m(t)$ is defined as in (3.17). From Assumption 2, because Ω^0 is bounded, $|\hat{a}f(u,\hat{\omega}) - f(u,\omega)|$, a^* and therefore $m(t)$ are also bounded, with $|m(t)|$ bounded by M. Let $y_m(t)$ be specified as the solution of the following differential equation for $t \ge t_1$:

$$\dot{y}_m = -\alpha_{max} y_m + M, \qquad y_m(t_1) = -\gamma. \tag{3.52}$$

From (3.50), (3.51) and (3.52), *Sublemma 2.1* implies that

$$\tilde{y}_\epsilon(t_1+\tau) \leq y_m(t_1+\tau), \qquad \forall\, \tau \geq 0 \text{ and } y_m(t_1+\tau) \leq 0. \tag{3.53}$$

From (3.52), it follows that

$$y_m(t_1+\tau) = \left(-\frac{M}{\alpha_{max}} - \gamma\right) e^{-\alpha_{max}\tau} + \frac{M}{\alpha_{max}}$$

We note that $y_m(t_1+\tau)$ is a concave function of τ for $\tau \geq 0$. From properties of concave functions, it can be shown that $y_m(t_1+\tau)$ satisfies the inequality

$$y_m(t_1+\tau) \leq y_m(t_1) + \nabla_\tau y_m(t_1+\tau)\mid_{\tau=0}. \tag{3.54}$$

From (3.53) and (3.54), we obtain that

$$\tilde{y}_\epsilon(t_1+\tau) \leq -\gamma + (M+\alpha_{max}\gamma)\tau, \qquad \text{for } \tau \geq 0. \tag{3.55}$$

For $T' = \frac{\gamma}{M+\alpha_{max}\gamma}$ we can verify easily from (3.55) that

$$\tilde{y}_\epsilon(t) \leq 0 \qquad \forall\, t \in [t_1,\ t_1+T'].$$

From (3.55), we have that

$$\int_{t_1}^{t_1+T'} |\tilde{y}_\epsilon(\tau)|^2 d\tau \geq \frac{\gamma^3}{3(M+\alpha_{max}\gamma)}.$$

Integrating (3.14) over $[t_1, t_1+T']$, we have that

$$V(t_1+T') \leq V(t_1) - \frac{\alpha_{min}\gamma^3}{3(M+\alpha_{max}\gamma)}.$$

For

$$\tilde{y}_\epsilon(t_1) \geq \gamma,$$

we can obtain a similar result. This proves Property 2. •

Proof of Property 3: Let us first prove that

$$-a_{-}^{*}(\widehat{\theta},\omega) \leq a^{*}sat(\frac{\tilde{y}}{\epsilon}). \tag{3.56}$$

Since $g(\hat{\theta},\omega,\phi) = 0$, it follows that for at least one value of θ in Ω^0, $g(\theta,\omega,\phi) = 0$. This proves

$$a^{*} \geq 0. \tag{3.57}$$

If $\tilde{y} \geq 0$, from (3.57), (3.56) holds. If $\tilde{y} < 0$, it follows that

$$a^{*}sat(\frac{\tilde{y}}{\epsilon}) = a_{-}^{*}(\widehat{\theta},\omega)sat(\frac{\tilde{y}}{\epsilon}) \geq -a_{-}^{*}(\widehat{\theta},\omega).$$

Similarly, we can prove $a^{*} \leq a_{+}^{*}(\widehat{\theta},\omega)$ and Property 3 is established. •

Proof of Property 4: Since $\beta = -1$, $f(\theta,\omega)$ is concave. It follows from the solutions of the min-max algorithm in section 2.1 that

$$a^{*} = 0, \qquad \text{if } \tilde{y} < 0, \tag{3.58}$$

which proves Property 4-(i). When $\tilde{y} > 0$, it follows from the solutions of the min-max algorithm that a^{*} is nonnegative, hence

$$\beta a^{*}\tilde{y} \leq 0, \qquad \text{if } \tilde{y} > 0. \tag{3.59}$$

When $\beta = 1$, proceeding in the same manner as above, it follows that

$$a^{*} = 0, \qquad \text{if } \beta = 1, \text{ and } \tilde{y} > 0, \tag{3.60}$$

and

$$\beta a^{*}\tilde{y} \leq 0, \qquad \text{if } \beta = 1, \text{ and } \tilde{y} < 0. \tag{3.61}$$

Equations (3.58)-(3.61) prove Property 4-(ii) and 4-(iii). •

Proof of Theorem 1: For any t_1 and $\hat{\theta}(t_1)$, it follows from (3.20) that there exists

$t_2 < t_1 + T_0$ such that

$$\beta(\omega(t_2))\left(f(\hat{\theta}(t_1), \omega(t_2)) - f(\theta_0, \omega(t_2))\right) \geq \epsilon_u \|\hat{\theta}(t_1) - \theta_0\| \tag{3.62}$$

Without loss of generality, we assume that $\beta(\omega(t_2)) = 1$ which means that $f(\theta, \omega(t_2))$ is convex (or linear) over θ. The proof can be given in a similar manner if $\beta(t_2) = -1$. When $\beta(t_2) = 1$, (3.62) can be rewritten as

$$f(\hat{\theta}(t_1), \omega(t_2)) - f(\theta_0, \omega(t_2)) \geq \bar{\epsilon} \tag{3.63}$$

where

$$\bar{\epsilon} = \epsilon_u \|\hat{\theta}(t_1) - \theta_0\|. \tag{3.64}$$

If $x(t_1) \in D_\epsilon$, we note that $x(t) \in D_\epsilon$ for all $t \geq t_1$, since V is a Lyapunov function. Hence we assume that $x(t_1) \notin D_\epsilon$. It follows from the definition of V in (3.13) that either

$$(i)\|\tilde{\theta}(t_1)\| > \sqrt{\gamma_1} \qquad \text{or} \qquad (ii)|\tilde{y}_\epsilon(t_1)| > \sqrt{\gamma_1}. \tag{3.65}$$

If (3.65)-(ii) holds, it is easy to show that V decreases. If (3.65)-(i) holds, we show below that $\tilde{y}_\epsilon(t)$ will become large for some $t > t_1$. Using the definitions of γ_1 in (3.22), it follows from (3.64) and (3.65)-(i) that

$$\bar{\epsilon}^2 > 2\epsilon(16B_\theta U_b + 8B_\theta B_\phi + 4B_\phi^2). \tag{3.66}$$

We shall show that if (3.66) holds, there exists $t_3 \in [t_2,\ t_2 + T_1]$ such that

$$|\tilde{y}_\epsilon(t_3)| \geq \min\{1,\ \bar{\delta}\} \tag{3.67}$$

where

$$\bar{\delta} \quad = \quad \min\left\{\frac{\bar{\epsilon}}{2(B_\theta B_\phi T_0 + \alpha_{max})},\right. \tag{3.68}$$

$$
\left.\frac{\bar{\epsilon}^2 - \epsilon(16B_\theta U_b + 8B_\theta B_\phi + 4B_\phi^2)}{2\bar{\epsilon}B_\theta B_\phi T_0 + 2\bar{\epsilon}\alpha_{max} + 16B_\theta U_b + 8B_\theta B_\phi + 4B_\phi^2}\right\}
$$

$$
T_1 = \frac{\bar{\epsilon} - (B_\theta B_\phi T_0 + \alpha_{max})\bar{\delta}}{4B_\theta U_b + 2B_\theta B_\phi + B_\phi^2}. \tag{3.69}
$$

From (3.66), we can verify easily that both $\bar{\delta}$ and T_1 are positive numbers. We prove (3.67) by contradiction.

Suppose (3.66) holds and (3.67) is not true. Then it follows that

$$
\text{(a)}\ |\tilde{y}_\epsilon(t_2+\tau)| < 1 \quad \text{and} \quad \text{(b)}\ |\tilde{y}_\epsilon(t_2+\tau)| < \bar{\delta} \tag{3.70}
$$

for any $\tau \in [0,\ T_1]$. From (3.8) and (3.70)(b), it follows that

$$
\dot{\tilde{y}} \geq -\alpha_{max}\bar{\delta} + \left[f(\hat{\theta},\omega) - f(\theta_0,\omega)\right] - a^* sat(\frac{\tilde{y}}{\epsilon}). \tag{3.71}
$$

We prove that $\tilde{y}_\epsilon(t)$ must become large over $[t_2, t_2+T_1]$ by establishing lower bounds on the bracketed term and the last term on the right hand side of Eq. (3.71).

It follows from equations (3.4), (3.70)(b), Assumption 2, and the fact that $t_2 - t_1 \leq T_0$ that

$$
|f(\hat{\theta}(t_2),\omega(t_2)) - f(\hat{\theta}(t_1),\omega(t_2))| \leq B_\theta B_\phi T_0 \bar{\delta}. \tag{3.72}
$$

Combining (3.63) and (3.72), we have that

$$
f(\hat{\theta}(t_2),\omega(t_2)) - f(\theta_0,\omega(t_2)) \geq \bar{\epsilon} - B_\theta B_\phi T_0 \bar{\delta}. \tag{3.73}
$$

From Assumption 1, it follows that

$$
||\omega(t_2+\tau) - \omega(t_2)|| \leq U_b \tau. \tag{3.74}
$$

For $\tau \in [0,\ T_1]$, since $|\tilde{y}_\epsilon(t_2+\tau)| \leq 1$ from (3.70(a)), by integrating (3.4) over $[t_2,\ t_2+\tau]$, we obtain that

$$
||\hat{\theta}(t_2+\tau) - \hat{\theta}(t_2)|| \leq B_\phi \tau. \tag{3.75}
$$

By combining (3.74), (3.75) and Assumption 2, it follows that

$$
\begin{aligned}
&|\hat{f}_{2\tau} - \hat{f}_2 - (f(\theta_0, \omega(t_2+\tau)) - f(\theta_0, \omega(t_2)))| \\
&\quad \leq B_\theta(2U_b + B_\phi)\tau
\end{aligned}
\tag{3.76}
$$

which can be rewritten as

$$
\begin{aligned}
&\hat{f}_{2\tau} - f(\theta_0, \omega(t_2+\tau)) \geq \\
&\qquad \hat{f}_2 - f(\theta_0, \omega(t_2)) - B_\theta(2U_b + B_\phi)\tau
\end{aligned}
\tag{3.77}
$$

where

$$
\begin{aligned}
\hat{f}_{2\tau} &= f(\hat{\theta}(t_2+\tau), \omega(t_2+\tau)) \\
\hat{f}_2 &= f(\hat{\theta}(t_2), \omega(t_2)).
\end{aligned}
\tag{3.78}
$$

Combining (3.73) and (3.77), it follows that for any $\tau \in [0, T_1]$,

$$
\begin{aligned}
&\hat{f}_{2\tau} - f(\theta_0, \omega(t_2+\tau)) \geq \\
&\qquad \bar{\epsilon} - B_\theta B_\phi T_0 \bar{\delta} - B_\theta(2U_b + B_\phi)\tau
\end{aligned}
\tag{3.79}
$$

which establishes a lower bound for the bracketed term in (3.71).

We now derive a lower bound for the third term in (3.71). For any θ, using the same procedure as for equation (3.76), it can be shown that

$$
\begin{aligned}
&|\hat{f}_{2\tau} - \hat{f}_2 - (f(\theta, \omega(t_2+\tau)) - f(\theta, \omega(t_2)))| \\
&\quad \leq B_\theta(2U_b + B_\phi)\tau
\end{aligned}
\tag{3.80}
$$

where $\hat{f}_{2\tau}, \hat{f}_2$ are defined in (3.78). It follows from (3.75) that

$$
|\phi^*(\tau_i)(\hat{\theta}(t_2+\tau) - \hat{\theta}(t_2))| \leq B_\phi^2 \tau. \tag{3.81}
$$

We know that at t_2, because $\beta(t_2) = 1$, it follows from Property $4-(ii)$ that

$$a_+^*(\hat{\theta}(t_2), \omega(t_2)) = 0. \tag{3.82}$$

From the definition of a_+^*, and the optimization problem in (3.6), we obtain that

$$\begin{aligned}
&a_+^*(\hat{\theta}(t_2), \omega(t_2)) \\
&= \max_{\theta \in \Omega^0} \left(\hat{f}_2 - f(\theta, \omega(t_2)) - \phi^*(t_2)(\hat{\theta}(t_2) - \theta)\right) \\
&a_+^*(\hat{\theta}(t_2 + \tau), \omega(t_2 + \tau)) = \max_{\theta \in \Omega^0} \left(\hat{f}_{2\tau}\right. \\
&\left. -f(\theta, \omega(t_2 + \tau)) - \phi^*(t_2 + \tau)(\hat{\theta}(t_2 + \tau) - \theta)\right)
\end{aligned} \tag{3.83}$$

where $\hat{f}_{2\tau}, \hat{f}_2$ are defined as in (3.78). Because $\phi^*(\tau_i + t)$ is the value that result in the minimum value of $a_+^*(\hat{\theta}(\tau_i + t), \omega(\tau_i + t))$, it follows that

$$\begin{aligned}
&a_+^*(\hat{\theta}(t_2 + \tau), \omega(t_2 + \tau)) \leq \max_{\theta \in \Omega^0} \left(\hat{f}_{2\tau}\right. \\
&\left. -f(\theta, \omega(t_2 + \tau)) - \phi^*(t_2)(\hat{\theta}(t_2 + \tau) - \theta)\right).
\end{aligned} \tag{3.84}$$

Combining (3.83) and (3.84), it follows that

$$\begin{aligned}
&a_+^*(\hat{\theta}(t_2 + \tau), \omega(t_2 + \tau)) \leq a_+^*(\hat{\theta}(t_2), \omega(t_2)) \\
&+ \max_{\theta \in \Omega^0} \left(\hat{f}_{2\tau} - f(\theta, \omega(t_2 + \tau))\right. \\
&\left. - (\hat{f}_2 - f(\theta, \omega(t_2))) - \phi^*(\tau_i)(\hat{\theta}(t_2 + \tau) - \hat{\theta}(t_2))\right)
\end{aligned} \tag{3.85}$$

where $\hat{f}_{2\tau}, \hat{f}_2$ are defined in (3.78). From Eqs. (3.81), (3.85), and (3.80) it follows that

$$\begin{aligned}
&a_+^*(\hat{\theta}(t_2 + \tau), \omega(t_2 + \tau)) \leq \\
&\qquad a_+^*(\hat{\theta}(t_2), \omega(t_2)) + B_\theta(2U_b + B_\phi)\tau + B_\phi^2\tau.
\end{aligned} \tag{3.86}$$

Combining (3.82) and (3.86), it follows that

$$-a_+^*(\hat{\theta}(t_2 + \tau), \omega(t_2 + \tau))sat(\frac{\tilde{y}}{\epsilon}) \geq -B_\theta(2U_b + B_\phi)\tau - B_\phi^2\tau. \tag{3.87}$$

It follows from (3.87) and Property 3 that for all $\tau \in [0, T_1]$,

$$-a^*(\hat{\theta}(t_2+\tau), \omega(t_2+\tau))sat(\frac{\tilde{y}}{\epsilon}) \geq -B_\theta(2U_b + B_\phi)\tau - B_\phi^2\tau \tag{3.88}$$

which establishes a lower bound on the last term on the right hand side of Eq. (3.71).

Using (3.79) and (3.88), Eq. (3.71) leads to the inequality

$$\begin{aligned} \dot{\tilde{y}}\,(t_2+\tau) \;\; &\geq \;\; \bar{\epsilon} - (B_\theta B_\phi T_0 + \alpha_{max})\bar{\delta} \\ &\quad -(4B_\theta U_b + 2B_\theta B_\phi + B_\phi^2)\tau \qquad \forall\, \tau \in [0, T_1]. \end{aligned} \tag{3.89}$$

Integrating both sides of (3.89) over $[t_2,\ t_2+T_1]$ where T_1 is defined in (3.69), we have

$$\begin{aligned} \tilde{y}(t_2+T_1) - \tilde{y}(t_2) \;\; \geq \;\; &\int_{t_2}^{t_2+T_1} \Big(\bar{\epsilon} - (B_\theta B_\phi T_0 + \alpha_{max})\bar{\delta} - \\ &(4B_\theta U_b + 2B_\theta B_\phi + B_\phi^2)\tau\Big)\, d\tau \end{aligned}$$

which can be simplified as

$$\tilde{y}(t_2+T_1) - \tilde{y}(t_2) \geq \frac{(\bar{\epsilon} - (B_\theta B_\phi T_0 + \alpha_{max})\bar{\delta})^2}{2(4B_\theta U_b + 2B_\theta B_\phi + B_\phi^2)}. \tag{3.90}$$

Since (3.70) holds for all $\tau \in [0,\ T_1]$, we have that

$$\tilde{y}(t_2) > -\epsilon - \bar{\delta}. \tag{3.91}$$

It follows from the definition of $\bar{\delta}$ in equation (3.68) that

$$\bar{\delta} \leq \frac{\bar{\epsilon}^2 - \epsilon(16B_\theta U_b + 8B_\theta B_\phi + 4B_\phi^2)}{2\bar{\epsilon}B_\theta B_\phi T_0 + 2\bar{\epsilon}\alpha_{max} + 16B_\theta U_b + 8B_\theta B_\phi + 4B_\phi^2} \tag{3.92}$$

Equation (3.92) can be rewritten as

$$\frac{\bar{\epsilon}^2 - 2\bar{\epsilon}(B_\theta B_\phi T_0 + \alpha_{max})\bar{\delta}}{2(4B_\theta U_b + 2B_\theta B_\phi + B_\phi^2)} \geq 2\epsilon + 2\bar{\delta}. \tag{3.93}$$

It follows from (3.90) and (3.91) that

$$\tilde{y}(t_2+T_1) \geq \frac{\bar{\epsilon}^2 - 2\bar{\epsilon}(B_\theta B_\phi T_0 + \alpha_{max})\bar{\delta}}{2(4B_\theta U_b + 2B_\theta B_\phi + B_\phi^2)} - \epsilon - \bar{\delta}. \tag{3.94}$$

From (3.93), equation (3.94) can be simplified as

$$\tilde{y}(t_2+T_1) \geq \epsilon + \bar{\delta}. \tag{3.95}$$

Equation (3.95) implies $\tilde{y}_\epsilon(t_2+T_1) \geq \bar{\delta}$ which contradicts (3.70). Thus we have shown that (3.67) must hold.

In summary, we have shown that if $V(t_1) > \gamma_1$, then either

$$\begin{aligned} &(i) \quad |\tilde{y}_\epsilon(t_3)| \geq \min\{1,\ \bar{\delta}\} \quad t_3 \in [t_1,\ t_1+T_0+T_1], \qquad \text{or} \\ &(ii) \quad |\tilde{y}_\epsilon(t_1)| > \sqrt{\gamma_1}. \end{aligned} \tag{3.96}$$

where $t_3 = t_2 + T_1$. From Property 2, it follows that if (3.96)-(i) holds, then there exists $T_1' = \frac{\bar{\delta}}{M+\alpha_{max}\bar{\delta}}$ such that

$$V(t_3+T_1') \leq V(t_3) - \frac{\alpha_{min}\bar{\delta}^3}{3(M+\alpha_{max}\bar{\delta})}. \tag{3.97}$$

Similarly, if (3.96)-(ii) holds, then

$$V(t_1+T_2') \leq V(t_1) - \frac{\alpha_{min}\sqrt{\gamma_1}^3}{3(M+\alpha_{max}\sqrt{\gamma_1})} \tag{3.98}$$

where $T_2' = \frac{\sqrt{\gamma_1}}{M+\alpha_{max}\sqrt{\gamma_1}}$. Because $V(t)$ is non-increasing, it follows from (3.97) and (3.98) that for any $V(t_1) > \gamma_1$,

$$V(t_1+T_3') \leq V(t_1) - \Delta V \tag{3.99}$$

where

$$T_3' = \max\{T_0+T_1+T_1',\ T_2'\}$$

$$\Delta V = \min\{\frac{\alpha_{min}\bar{\delta}^3}{3(M+\alpha_{max}\bar{\delta})}, \frac{\alpha_{min}\gamma_1^3}{3(M+\alpha_{max}\gamma_1)}\}.$$

This implies that $V(t)$ decreases by a finite amount over erery interval T_3' until trajectories reach D_ϵ. This proves Theorem 1. •

Proof of Lemma 2 For any m, if

$$|\tilde{y}_\epsilon(t)| \leq \delta \qquad \forall t \in [t_0 + mT,\ t_0 + (m+1)T], \tag{3.100}$$

we are done. Otherwise, it means there exists $t_1 \in [t_0 + mT,\ t_0 + (m+1)T]$ such that

$$|\tilde{y}_\epsilon(t_1)| \geq \delta.$$

It follows from Property 2 that there exists $T' = \frac{\delta}{M+\alpha_{max}\delta}$ such that

$$V(t_1 + T') \leq V(t_1) - \frac{\alpha_{min}\delta^3}{3(M+\alpha_{max}\delta)}.$$

This implies that every time when $|\tilde{y}_\epsilon(t)| \geq \delta$, Lyapunov function will decrease a small amount. Now that $V(t_0)$ is finite, these kind of situation can only happen finite times. It means that we can find a finite m^* such that

$$|\tilde{y}_\epsilon(t)| \leq \delta \qquad \forall t \in [t_0 + m^*T,\ t_0 + (m^*+1)T].$$

This establishes Lemma 2. •

Proof of Lemma 3: We shall prove by contradiction that Lemma 3 holds. Assume that $\theta_0 \notin \Phi_\epsilon(\Omega, \omega_i, \epsilon, \delta, \hat{\theta}_i^c)$ for some $1 \leq i \leq I$. That is

$$(i) f(\theta_0, \omega_i) < \underline{f}_i, \qquad \text{or} \qquad (ii) f(\theta_0, \omega_i) > \bar{f}_i,$$

for some $1 \leq i \leq I$. Suppose (i) is true. Since $\hat{\theta}_i^c = \hat{\theta}(\tau_i)$, case (i) implies that

$$f(\hat{\theta}(\tau_i), \omega) - f(\theta_0, \omega) - a_+^*(\hat{\theta}(\tau_i), \omega) - \alpha_{max}\delta > 2\sqrt{B_t(\delta + \epsilon)}. \tag{3.101}$$

From (3.8), Property 3 and the fact $|\tilde{y}_\epsilon| \leq \delta$, it follows that

$$\dot{\tilde{y}} \geq y_l \tag{3.102}$$

where

$$y_l = -\alpha_{max}\delta + f(\hat{\theta}, \omega) - f(\theta_0, \omega) - a_+^*(\hat{\theta}, \omega) \tag{3.103}$$

represents the lower bound of $\dot{\tilde{y}}$. Combining (3.101) and (3.103), it follows that

$$y_l(\tau_i) > 2\sqrt{B_t(\delta + \epsilon)}. \tag{3.104}$$

From the definition of a_+^*, and the optimization problem in (3.6), we obtain that

$$\begin{aligned}
&a_+^*(\hat{\theta}(\tau_i), \omega(\tau_i)) = \\
&\max_{\theta \in \Omega^0} \left(\hat{f}_\tau - f(\theta, \omega(\tau_i)) - \phi^*(\tau_i)(\hat{\theta}(\tau_i) - \theta)\right) \\
&a_+^*(\hat{\theta}(\tau_i + t), \omega(\tau_i + t)) = \max_{\theta \in \Omega^0} \Big(\hat{f}_{\tau t} \\
&-f(\theta, \omega(\tau_i + t)) - \phi^*(\tau_i + t)(\hat{\theta}(\tau_i + t) - \theta)\Big)
\end{aligned} \tag{3.105}$$

where

$$\begin{aligned}
\hat{f}_\tau &= f(\hat{\theta}(\tau_i), \omega(\tau_i)) \\
\hat{f}_{\tau t} &= f(\hat{\theta}(\tau_i + t), \omega(\tau_i + t)).
\end{aligned} \tag{3.106}$$

Because $\phi^*(\tau_i + t)$ is the value that result in the minimum value of $a_+^*(\hat{\theta}(\tau_i + t), \omega(\tau_i + t))$, it follows that

$$\begin{aligned}
a_+^*(\hat{\theta}(\tau_i + t), \omega(\tau_i + t)) \leq \max_{\theta \in \Omega^0} \Big(\hat{f}_{\tau t} - f(\theta, \omega(\tau_i + t)) \\
-\phi^*(\tau_i)(\hat{\theta}(\tau_i + t) - \theta)\Big).
\end{aligned} \tag{3.107}$$

Combining (3.105) and (3.107), it follows that

$$a_+^*(\hat{\theta}(\tau_i + t), \omega(\tau_i + t)) \leq a_+^*(\hat{\theta}(\tau_i), \omega(\tau_i))$$

$$+\max_{\theta\in\Omega^0}\Big(\hat{f}_{\tau t} - f(\theta,\omega(\tau_i+t)) \\ - (\hat{f}_\tau - f(\theta,\omega(\tau_i))) - \phi^*(\tau_i)(\hat{\theta}(\tau_i+t)-\hat{\theta}(\tau_i))\Big) \tag{3.108}$$

where $\hat{f}_\tau$ and $\hat{f}_{\tau t}$ are defined as in (3.106). From Assumption 1, it follows that

$$\|\omega(\tau_i+t)-\omega(\tau_i)\| \leq U_b t. \tag{3.109}$$

Since $|\tilde{y}_\epsilon(\tau_i+t)| \leq \delta$, by integrating (3.4) over $[\tau_i;\ \tau_i+t]$, we obtain that

$$\|\hat{\theta}(\tau_i+t)-\hat{\theta}(\tau_i)\| \leq \delta B_\phi t. \tag{3.110}$$

By combining (3.109), (3.110) and Assumption 2, it follows that

$$|\hat{f}_{\tau t} - \hat{f}_\tau - (f(\theta,\omega(\tau_i+t)) - f(\theta,\omega(\tau_i)))| \\ \leq B_\theta(2U_b+\delta B_\phi)t \tag{3.111}$$

where $\hat{f}_\tau$ and $\hat{f}_{\tau t}$ are defined as in (3.106). From (3.110), (3.111) and (3.108), we get

$$a_+^*(\hat{\theta}(\tau_i+t),\omega(\tau_i+t)) \leq a_+^*(\hat{\theta}(\tau_i),\omega(\tau_i)) + \\ B_\theta(2U_b+\delta B_\phi+\delta B_\phi^2)t. \tag{3.112}$$

Incorporating (3.111) and (3.112) into (3.103), we have that

$$y_l(\tau_i+t) - y_l(\tau_i) > -(4B_\theta U_b + 2\delta B_\theta B_\phi + \delta B_\phi^2)t.$$

which can be simplified using (3.31) and (3.104) as

$$y_l(\tau_i+t) > 2\sqrt{B_t(\delta+\epsilon)} - B_t t. \tag{3.113}$$

It follows from (3.102) and (3.113) that

$$\dot{\tilde{y}}\ (\tau_i+t) > 2\sqrt{B_t(\delta+\epsilon)} - B_t t. \tag{3.114}$$

Integrating both sides of (3.114) over $[\tau_i,\ \tau_i + \tau_B]$ where $\tau_B = 2\sqrt{B_t(\delta+\epsilon)}/B_t$,

$$\tilde{y}(\tau_i + \tau_B) - \tilde{y}(\tau_i) = \int_{\tau_i}^{\tau_i+\tau_B} 2\sqrt{B_t(\delta+\epsilon)} - B_t t. \tag{3.115}$$

Since $\tilde{y}(\tau_i) \geq -(\delta+\epsilon)$, we can rewrite (3.115) as

$$\tilde{y}(\tau_i + \tau_B) > (\delta + \epsilon). \tag{3.116}$$

Equation (3.116) implies

$$\tilde{y}_\epsilon(\tau_i + \tau_B) > \delta$$

and this contradicts the fact that $|\tilde{y}_\epsilon(t)| \leq \delta$ over $[t_1,\ t_1 + T]$. Thus we conclude that the assumption (i) that $f < \underline{f}_i$ for some $1 \leq i \leq I$ is not true and hence

$$f(\theta_0, \omega) \geq \underline{f}_i = f(\hat{\theta}_i^c, \omega) - a_+^*(\hat{\theta}_i^c, \omega) - \alpha_{max}\delta - 2\sqrt{B_t(\delta+\epsilon)}$$

$$\forall i = 1, .., I. \tag{3.117}$$

In the same manner, we can prove that

$$f(\theta, \omega) \leq \bar{f}_i = a_-^*(\hat{\theta}_i^c, \omega) + f(\hat{\theta}_i^c, \omega) + \alpha_{max}\delta + 2\sqrt{B_t(\delta+\epsilon)}$$

$$\forall i = 1, .., I. \tag{3.118}$$

(3.117) and (3.118) concludes the proof of Lemma 3. •

Proof of Lemma 4: This proof follows directly from the definition of Δ_i in (3.36) and the construction of Ω^{k+1} in (3.34).

Proof of Lemma 5: We start with the hierarchical algorithm shown in Table 3.1. Because $\Omega^{k+1} \subseteq \Omega^k$, there exists Ω^l such that

$$\Omega^l = \lim_{k \to \infty} \Omega^k. \tag{3.119}$$

Corresponding to Ω^l, if the lower-level convergent estimate of θ_0 is given by $\hat{\theta}^l$, it follows that

$$\lim_{t\to\infty} \hat{\theta}(t) = \hat{\theta}^l. \tag{3.120}$$

Suppose (3.40) does not hold, it implies that

$$\hat{\theta}^l \notin \bigcap_{i=1}^{I} B(\Delta_i(\Omega^l)). \tag{3.121}$$

Then there exists an $i,\ 1 \le i \le I$ such that

$$\hat{\theta}_i^l \notin \Delta_i(\Omega^l) \tag{3.122}$$

where $\hat{\theta}_i^l = \hat{\theta}(\tau_i)$ with $\omega(\tau_i) = \omega_i$. We can prove (3.122) by contradiction. We assume that

$$\hat{\theta}_i^l \in \Delta_i(\Omega^l) \qquad \forall\, i = 1, .., I.$$

Because

$$\|\hat{\theta}^l - \hat{\theta}_i^l\| \le \delta T B_\phi,$$

combining the definition of $L(\Omega^0, \epsilon, \delta)$ in (3.38), it follows that

$$\hat{\theta}^l \in L(\Omega^0, \epsilon, \delta)$$

which is a contradiction to (3.121). Thus (3.122) must be true if (3.40) does not hold.

Let $\underline{f}_i^l$ and $\bar{f}_i^l$ be lower and upper bounds in Ω^l specified as

$$\underline{f}_i^l = \min_{\theta\in\Omega^l} f(\theta, \omega_i), \qquad \bar{f}_i^l = \max_{\theta\in\Omega^l} f(\theta, \omega_i).$$

If we define

$$\underline{\hat{f}}_i = f(\hat{\theta}_i^l, \omega_i) - a_+^*(\hat{\theta}_i^l, \omega_i) - D(\epsilon, \delta)$$
$$\hat{\bar{f}}_i = f(\hat{\theta}_i^l, \omega_i) + a_-^*(\hat{\theta}_i^l, \omega_i) + D(\epsilon, \delta)$$

where $D(\epsilon, \delta)$ is defined as in (3.37), equation (3.122) together with the definition of Δ_i

imply that

$$\underline{\hat{f}}_i > \underline{f}_i^l \qquad \text{or} \qquad \hat{\bar{f}}_i < \bar{f}_i^l. \tag{3.123}$$

Equation (3.123) implies that tighter bounds $\underline{\hat{f}}_i$ or $\hat{\bar{f}}_i$ can be found for $f(\theta, \omega_i)$ for $\theta \in \Omega^l$ which implies that a smaller set Ω^{l+1} can be found using $\underline{\hat{f}}_i$ or $\hat{\bar{f}}_i$. This contradicts the assumption in (3.119) and Lemma 5 is proved. •

Proof of Theorem 2: Sufficiency follows directly from Lemma 5 and equation (3.41).

To prove necessity, we assume that (3.44) does not hold. That is, there exists $\Omega \subseteq \Omega^0$ where $\theta_0 \in \Omega$ such that

$$\begin{aligned} (i) \quad & \bigcap_{i=1,\ldots,I} \bar{\Delta}_i(\Omega) \neq \phi, \text{ and} \\ (ii) \quad & \bigcap_{i=1,\ldots,I} \bar{\Delta}_i(\Omega) \neq \{\theta_0\}. \end{aligned} \tag{3.124}$$

It implies that there exists some $\bar{\theta} \in \bigcap_{i=1,\ldots,I} \bar{\Delta}_i(\Omega)$ and $\bar{\theta} \neq \theta_0$. Assume that at iteration k, the unknown parameter region $\Omega^k = \Omega$ and the lower-level convergent parameter estimate at this iteration is given by $\hat{\theta}^{ck} = \bar{\theta}$. Then condition (ii) in (3.124) implies that

$$\hat{\theta}^{ck} \in \bigcap_{i=1,\ldots,I} \bar{\Delta}_i(\Omega)$$

since $\bigcap_{i=1,\ldots,I} \bar{\Delta}^i(\Omega)$ is not empty. From Lemma 4, it follows that

$$\Omega^j = \Omega^k = \Omega \qquad j = k+1, k+2, \ldots..$$

and $\hat{\theta}$ will remains at $\hat{\theta}^{ck}$ always. Since $\hat{\theta}^{ck} \neq \theta_0$, the parameter estimate will not converge to θ_0 even ϵ and δ approaches 0. This implies that (3.44) is a necessary condition for (3.43).

•

Proof of Lemma 7: For any $\Omega \subseteq \Omega^0$ where $\theta_0 \in \Omega$, if (3.44) does not hold, it follows that

$$\bigcap_{i=1}^{I} \bar{\Delta}_i(\Omega) \neq \phi \qquad \text{and} \qquad \bigcap_{i=1}^{I} \bar{\Delta}_i(\Omega) \neq \{\theta_0\}. \tag{3.125}$$

From (3.125), it follows that there exists $\theta \neq \theta_0$ such that

$$\theta \in \bigcap_{i=1}^{I} \bar{\Delta}_i(\Omega). \tag{3.126}$$

For this choice of θ, from (3.47) we have that there exists a $\omega_i(\theta)$ such that

$$\beta(f(\theta, \omega_i) - f(\theta_0, \omega_i)) > 0. \tag{3.127}$$

Without loss of generality, we assume that $\beta = -1$. It follows from Property 4-(i) that

$$a_-^*(\theta, \omega_i) = 0, \tag{3.128}$$

It follows from (3.128) and the definition of $\bar{\Delta}_i(\Omega)$ in (3.42) that

$$f(\theta, \omega_i) \geq \bar{f}_i^*, \qquad \forall\, \theta \in \bigcap_{i=1}^{I} \bar{\Delta}_i(\Omega). \tag{3.129}$$

From the definition of $\bar{f}_i^*$ in (3.35) and the fact that $\theta_0 \in \Omega$, it follows that

$$\bar{f}_i^* \geq f(\theta_0, \omega_i). \tag{3.130}$$

Combining (3.129) and (3.130), it follows that

$$f(\theta, \omega_i) \geq f(\theta_0, \omega_i) \tag{3.131}$$

which is a contradiction to (3.127) since $\beta = -1$. This proves Lemma 7. •

Chapter 4

A Nonlinear Force-Displacement Model of Intra-abdominal Tissues

4.1 Introduction

Advances in medical technology have necessitated the need for accurate medical simulation techniques in general and virtual-reality based medical trainers in particular. The latter allows physicians to be suitably trained via virtual patients before the actual clinical practice. Towards this end, the creation of datasets that corresponds to a "Palpable Human" similar to the Visible Human project initiated by the National Library Medicine, is extremely useful. For example, force feedback from various parts of the anatomy in response to different types of deformation will enable the characterization of Intra-abdominal tissues. An additional advantage that can stem from this dataset is the classification of these soft tissues according to their state of normalcy or malignancy.

With the above goals in mind, in this chapter, we embark on the characterization of force-displacement relations of various intra-abdominal tissues. In [ ], an extensive characterization of intra-abdominal tissues via experimental investigations was reported, thereby enabling an overall framework for obtaining in vivo mechanical properties of these tissues. In particular, force response for various displacement stimuli such as ramp-and-hold were obtained. This data shed light on the isotropic properties of the organs, stiffness properties before and after the death of the animal, the impedance properties of the organs, and their power dissipation properties. A closer examination of this data reveals that the relationship between the input stimulus and the output response is in fact nonlinear. In this chapter, we derive a nonlinear model that quantifies this relationship.

The field of systems theory deals with the temporal response of relevant variables to input stimulus. Depending upon the nature of the output response to inputs, the underlying system is typically characterized as linear or nonlinear. If the property of superposition is satisfied, then the system is referred to as linear and nonlinear, otherwise. Depending upon whether the output responds to the input stimulus with a time-lag or not, the system is characterized as dynamic or static. Finally, if the system parameters appear linearly or nonlinearly, the underlying system is denoted as linearly parameterized or nonlinearly parameterized. As is shown in the subsequent sections, we demonstrate that the most accurate model of intra-abdominal tissues is a dynamic, nonlinearly parameterized system.

This chapter is organized as follows. In section 2, the experimental details and the data collected are briefly discussed. In section 3, the underlying model is presented. Various features of the model and the reason for their inclusion are detailed. In section 4, the ability of the model to predict the experimental data is demonstrated. In section 5, an algorithm for automatic determination of the system parameters is derived and studied via simulations. Summary and concluding remarks are included in section 6.

4.2 The Experiment

The experimental data for liver under in-vivo condition is shown in Figure 4-1 which shows the ramp-and-hold displacement-input, the corresponding force-output, and both of their filtered values. The filtering was carried out using a low-pass filter with a corner frequency of 2 Hz. It is clear from Figure 4-1 that the heart-beat is present in the force-output, which can be filtered out for the purposes of tissue-modeling. In what follows, we restrict our attention only to the filtered output signals. As mentioned earlier, for any experiment i, the ramp-hold tests were performed using five different input signals. For the same value of the input, it was observed that different runs yielded different outputs. In figure 4-2, we

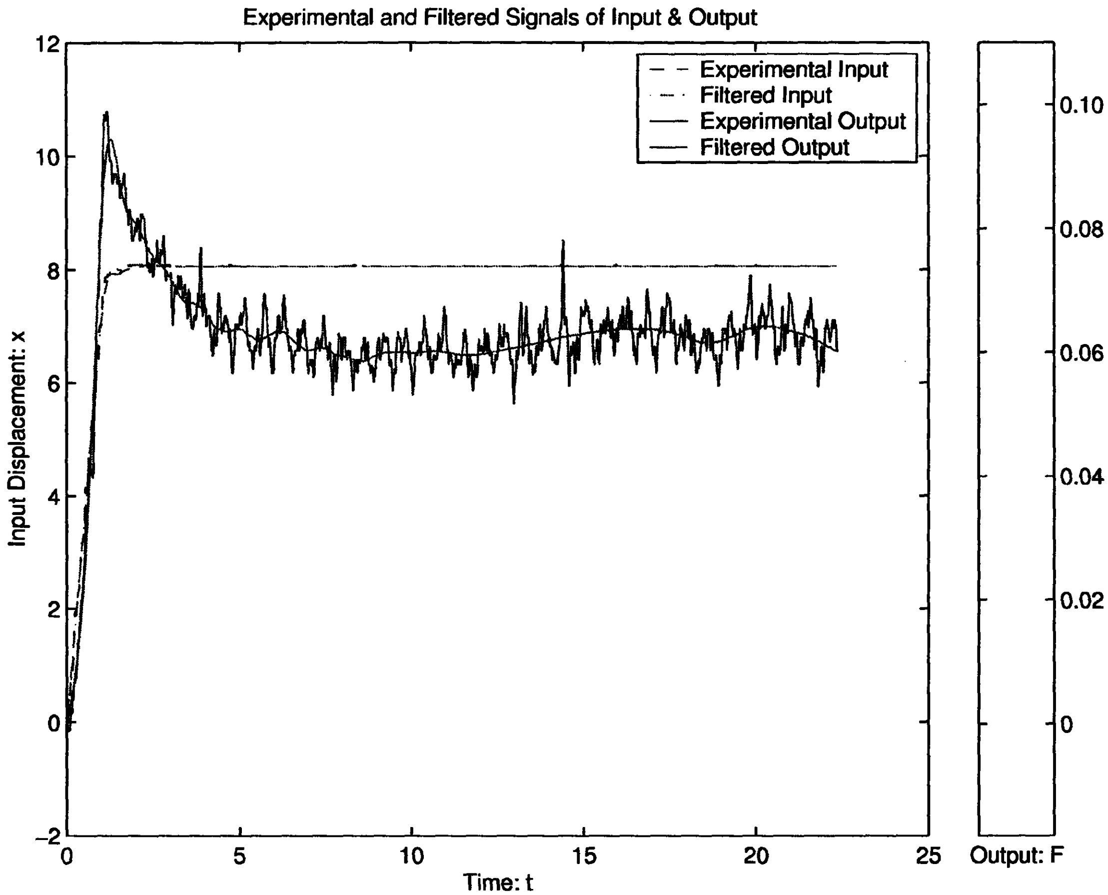

Figure 4-1: Experimental and Filtered Signals of Input and Output

present the averaged output signals for each of the inputs, which was computed as

$$\bar{F}_j = \frac{1}{N_j} \sum_{k=1} N_j F_{jk}. \tag{4.1}$$

where N_j is the number of runs with input j. Figure 4-2 also indicates the range of variation in the outputs for a given input. The variance for each experiment was computed as

$$V_i = \frac{\sum_{j=1}^{M} \sum_{k=1}^{N_j} \int (\bar{F}_j - F_{jk})^2 dt}{\sum_{j=1}^{M} \sum_{k=1}^{N_j} \int F_{jk}^2 dt} 100\% \tag{4.2}$$

where $M = 5$.

For esophagus under in-vivo and ex-vivo conditions, the output force responses as in

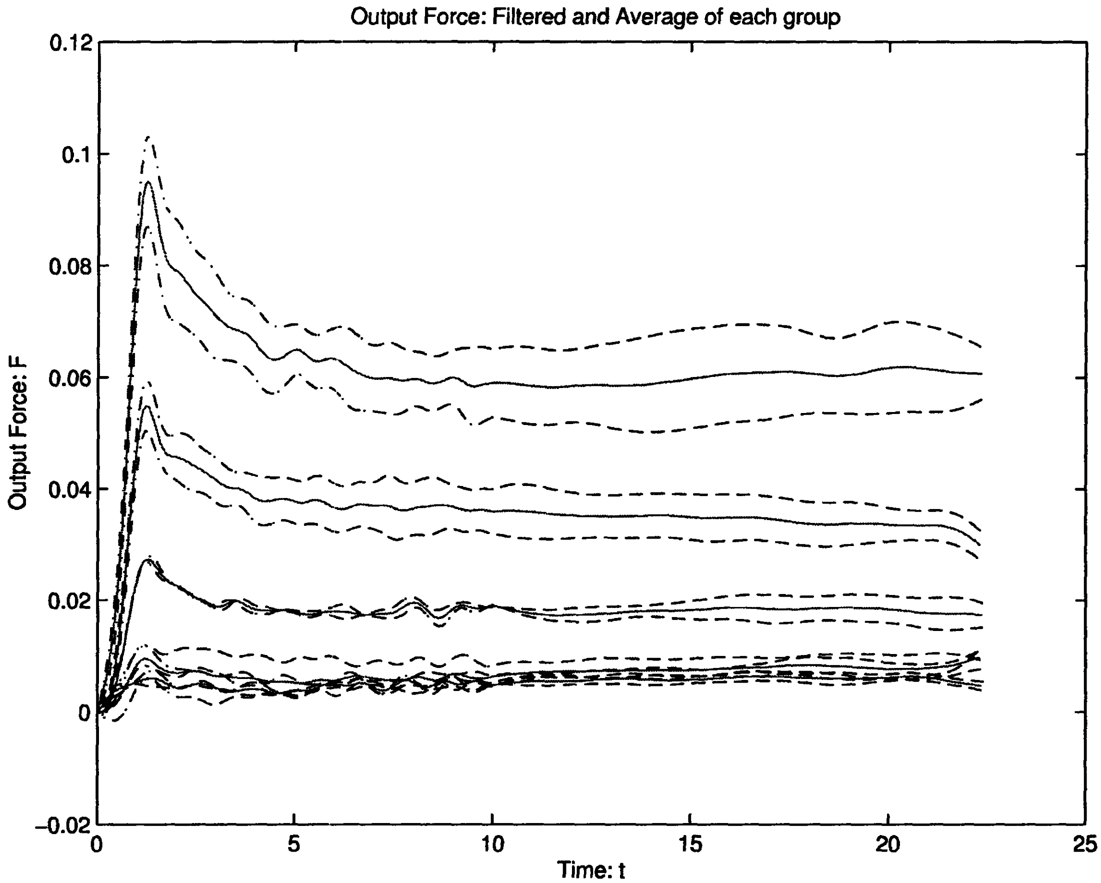

Figure 4-2: Output Force F: Average of each group of filtered signals

Figure 4-3 for ramp-hold input signals has two overshoot peaks and are quite different from above situations.

In Table 4.1, we list V_i for twenty-five experimental conditions. It should be noted that the experiments were conducted using three different animals, where experiments 1-9, 10-19, and 20-25 corresponds to the first, second, and third animal, respectively.

4.3 The Model

In this section we derive a dynamic model that predicts the tissue behavior shown in figures 4-1 and 4-2. Figure 4-2 illustrates many interesting tissue properties. First, we see that corresponding to ramp-hold inputs, the outputs have overshoot which decays and finally

Table 4.1: Variance for Different Experimental Conditions

No.	Organ	Cond.	Input	$V_i(\%)$
1	Liver	In-Vivo	RH-Z	1.3
2	Esophagus	Ex-Vivo	RH-Z	0.0068
3	Liver	Ex-Vivo	RH-Z	1.3
4	Liver	In-Vitro	RH-Z	0.92
5	Esophagus	In-Vitro	RH-Z	0.75
6	Liver	In-Vitro	RH-Z	2.1
7	Esophagus	In-Vitro	RH-Z	0.96
8	Liver	In-Vivo	RH-Z	0.62
9	Esophagus	In-Vitro	RH-Z	1.4
10	Liver	In-Vivo	RH-Z	2.3
11	Esophagus	In-Vivo	RH-Z	4.0
12	Esophagus	Ex-Vivo	RH-Z	3.1
13	Liver	Ex-Vivo	RH-Z	1.6
14	Liver	In-Vitro	RH-Z	0.62
15	Esophagus	In-Vitro	RH-Z	0.47
17	Esophagus	In-Vitro	RH-Z	0.88
18	Liver	In-Vitro	RH-Z	0.7
19	Esophagus	In-Vitro	RH-Z	2.8
20	Esophagus	In-Vivo	RH-Z	3.0
21	Liver	In-Vivo	RH-Z	1.5
22	Esophagus	Ex-Vivo	RH-Z	4.3
24	Esophagus	Ex-Vivo	RH-Z	2.1
25	Liver	Ex-Vivo	RH-Z	2.6

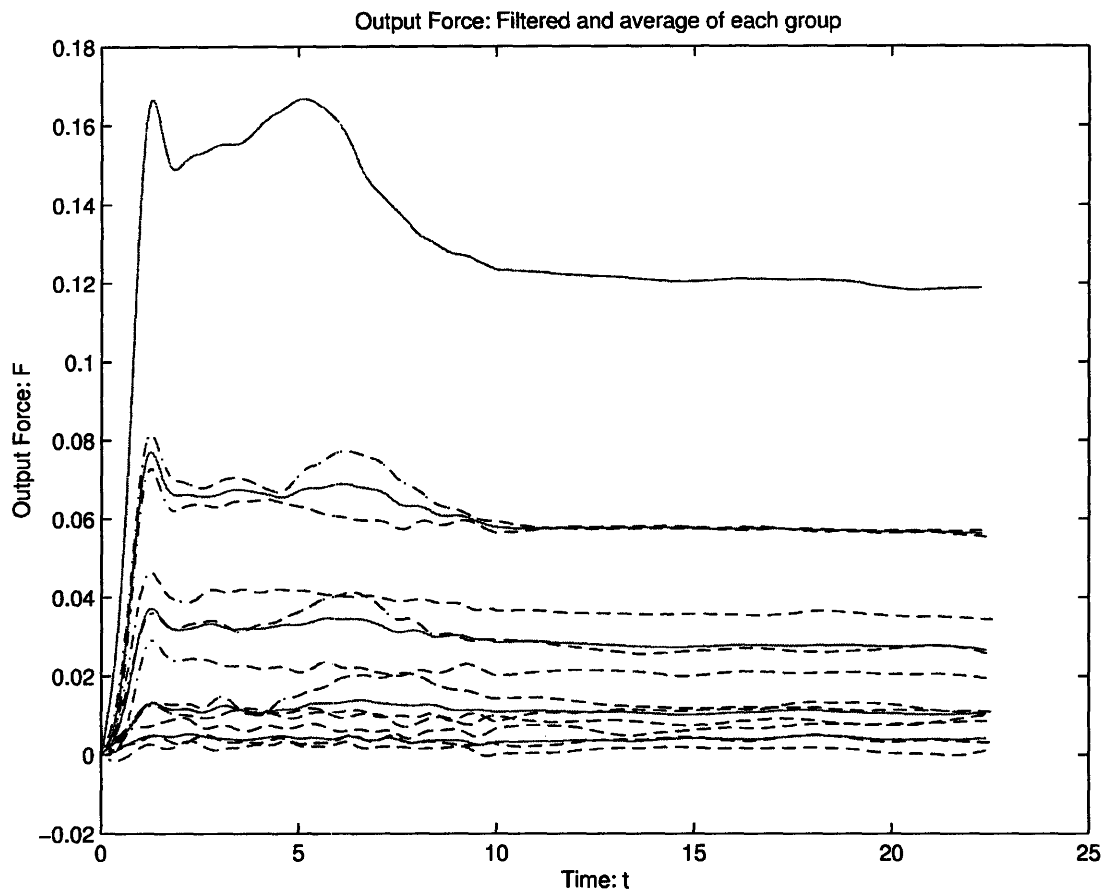

Figure 4-3: Output Force F: Average of each group of filtered signals

settles at a steady constant. Second, we see that the output decays even when the ramp-and-hold input is a constant. This implies that the tissue model must be dynamic in nature. Third, the relationship between output steady values and input hold values are nonlinear. The fourth property is that the output overshoot values are nonlinear with respect to the input hold values.

To satisfy the tissue properties above, we introduce Model I, which is described as follows:

$$\begin{aligned} F &= F_0 + k_{11} F_1 \\ F_0 &= k_{01} x^{k_{02}} \\ F_1 &= k_{13}(x - x_1)^{k_{12}} \end{aligned}$$

$$\begin{aligned}\dot{x_1} &= x_2\\ \dot{x_2} &= k_{13}(x - x_1)^{k_{12}} - x_2.\end{aligned} \tag{4.3}$$

In (4.3), x is the displacement input to the tissue, and F is the measured force. The objective here is to predict the input-output tissue behavior exhibited in Figure 4-1. The structure of Model I is shown in Figure 4-4, where it can be seen that the output force F is a linear combination of two components F_0, which is from a nonlinear spring, and F_1, which is from a nonlinear spring-friction system. For a ramp-hold input signal, spring-friction system output F_1 will experience a pulse response which increases quickly during the input-ramp period because the spring is compressed and decays gradually to zero in the input-hold period because the compressed spring gradually come back to its natural length and the energy is compensated in friction. Therefore, the steady output force is completely determined by static spring F_0 because the steady state of F_1 will approaches zero for ramp-hold inputs. The nonlinear spring contribution to F_0 characterizes the nonlinear steady input output relationship in tissue dynamics while nonlinear spring in spring-friction component F_1 yields several of the nonlinear overshoot characteristics. Thus, it can be seen that Model I captures the tissue properties. It will be shown later that using the above combination of nonlinear springs where the nonlinearities are polynomial Model I is capable of predicting the output accurately.

To show that Model I is the simplest spring-friction model which accurately characterizes the tissue properties, we compare Model I with four other models denoted as Model 1 through Model 4 whose structures are shown in Figure 4-4. These models can be described as follows. The output prediction of different models for same input signal are plotted in Figure 4-5.

$$\begin{aligned}F &= F_0\\ F_0 &= k_{01}x^{k_{02}}.\end{aligned} \tag{4.4}$$

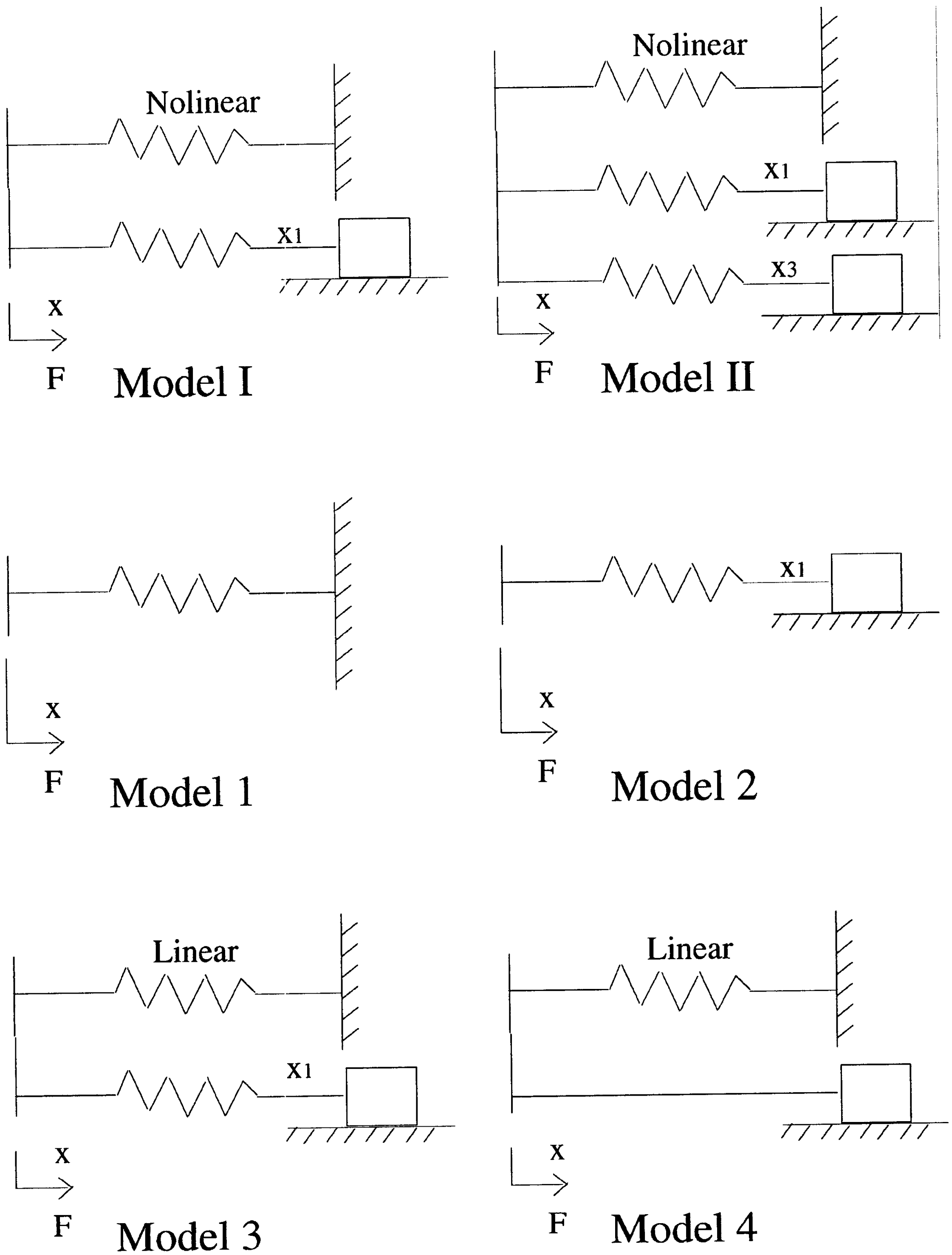

Figure 4-4: Different Model Structures

$$
\begin{aligned}
F_1 &= k_{11}k_{13}(x - x_1)^{k_{12}} \\
\dot{x_1} &= x_2 \\
\dot{x_2} &= k_{13}(x - x_1)^{k_{12}} - x_2.
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
F &= F_0 + k_{11}F_1 \\
F_0 &= k_{01}x \\
F_1 &= k_{13}(x - x_1)^{k_{12}} \\
\dot{x_1} &= x_2 \\
\dot{x_2} &= k_{13}(x - x_1)^{k_{12}} - x_2.
\end{aligned} \tag{4.6}
$$

$$
\begin{aligned}
F &= F_0 + F_1 \\
F_0 &= k_{01}x \\
F_1 &= k_{11}\,\dot{x}\,.
\end{aligned} \tag{4.7}
$$

In Model 1, the dynamic component which generates F_1 is omitted, while the nonlinear static spring that generates F_0 is retained. In Model 2, the F_0 component is omitted while the F_1 component is retained. In Model 3,a linear spring is used to replace the nonlinear spring that contributes F_0 in model I, while the effect of F_1 is identical to that in Model I. In Model 4, the nonlinear aspects of both F_0 and F_1 are neglected, and the entire system is modeled using a linear spring and a linear mass-friction dynamic system.

Since F_1 is neglected in Model 1, it has no ability to generate a overshoot; similarly, since F_0 is omitted in Model 2, it always has a zero steady-state output for a ramp-and-hold input. This is illustrated in figure 4-5 which shows the inability of both models to accurately predict the tissue dynamics. While the trends of overshoot and non-zero steady-state output are captured by Model 3, since it contains a linear spring, Model 3 has no ability to represent the fundamentally nonlinear properties of the tissue. That is, Model 3 results in a fairly large steady-state errors for certain inputs, as shown in figure 4-5. Similarly, Model 4 is unable to predict the trends of decay (between 2 and 7 secs.) in addition to the steady-state error, due to its linear components.

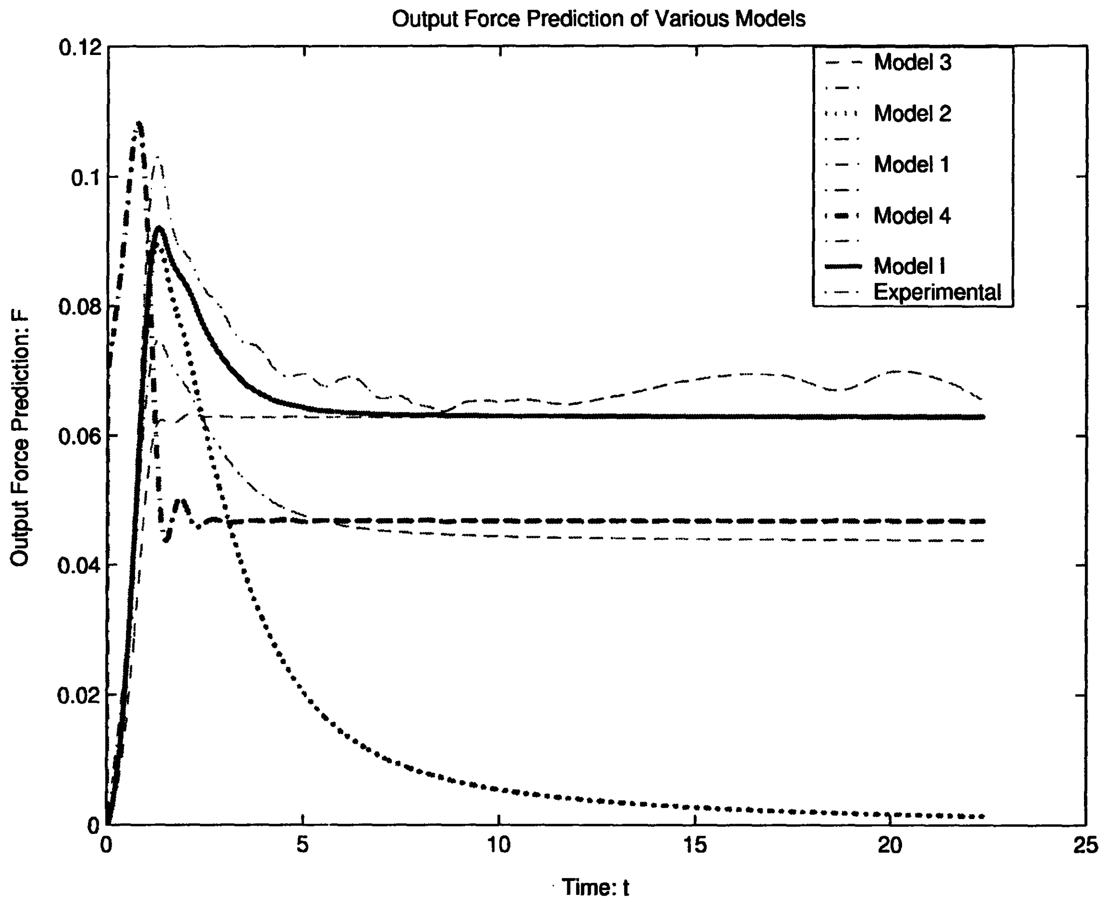

Figure 4-5: Output Force Prediction of Various Models

We also compare the performance of Model I with models 1 through 4 using a prediction-error measure. This measure, E_i is defined as

$$E_i = \frac{\sum_{j=1}^{M} \sum_{k=1}^{N_j} \int (\hat{F}_j - F_{jk})^2 dt}{\sum_{j=1}^{M} \sum_{k=1}^{N_j} \int F_{jk}^2 dt} 100\% \tag{4.8}$$

where $j = 1, \ldots, M$ are the inputs, and N_j is the number of experiments carried out with input j. $\hat{F}_j$ is the model-output prediction and F_{jk} is the actual filtered output for kth run of input j. For experimental data of liver under in-vivo condition, we list the model prediction errors for different models in Table 4.2 below. From Figure 4-5 and Table 4.2, we notice that Model I can qualitatively characterize the tissue dynamic properties well and achieves

Table 4.2: Comparision of Prediction Errors for Various Models

	Model I (%)	Model 1 (%)	Model 2 (%)	Model 3 (%)	Model 4 (%)
E	1.75	2.66	62.57	6.04	13.78

the minimum model prediction error. Any further simplication of the model as in model 1-4 will make the performance deteriorate or lose some key features of the tissue dynamics.

While Model I was found to be adequate for the experiment discussed above, in a few cases, it was found that an additional component needed to be added to Model I, which corresponded to the esophagus-tissue under in-vivo and ex-vivo conditions shown in Figure 4-3. It can be observed that the output force response corresponding to ramp-hold input signals has two overshoots. The augmented model, denoted as Model II consists of the components F_0 and F_1 present in Model I, and has an additional mass-spring-friction system whose input is delayed by a few seconds. This model is described below:

$$\begin{aligned} F(t) &= F_0(t) + k_{11}F_1(t) + k_{21}F_2(t-T) \\ F_0 &= k_{01}x^{k_{02}} \\ F_1 &= k_{13}(x-x_1)^{k_{12}} \\ \dot{x_1} &= x_2 \\ \dot{x_2} &= k_{13}(x-x_1)^{k_{12}} - x_1 \\ F_2 &= k_{23}(x-x_3)^{k_{22}} \\ \dot{x_3} &= x_4 \\ \dot{x_4} &= k_{23}(x-x_3)^{k_{22}} - x_4. \end{aligned} \tag{4.9}$$

4.4 Model Prediction

In the previous section, we developed two models, Model I and Model II to predict the tissue properties, with Model I representing the liver tissue at all conditions and the esophagus tissue only under in-vitro conditions, and Model II representing the esophagus tissue under in-vivo and ex-vivo conditions. We validate this model in this section by comparing its predicted output values with actual outputs under various conditions.

Determining any dynamic model includes two steps, where the first step is the selection of a suitable model structure, and the second step is the selection of the parameters of that model. As mentioned above, we carried out the first step in the previous section, while we proceed to the second step in this section.

As mentioned in the previous section, Model I includes five parameters k_{01}, k_{02}, k_{11}, k_{12} and k_{13}. Of these, the steady-state value of the output is a function of F_0 whereas the transient properties of the output, such as overshoot and decay, are functions of F_1. It therefore follows that k_{01} and k_{02} can be determined by studying the steady-state component, while the last three parameters can be found by monitoring the transient characteristics. Using the functional form of F_0 shown in Eq. (4.3), k_{01} and k_{02} were calculated. To determine the remaining three parameters, we first note that x_1 is nearly zero during the "ramp" portion of the input, and hence, $F_1 = k_{11}k_{13}x^{k_{12}}$, which corresponds to the overshoot magnitude. Therefore, using the overshoot values at different inputs, k_{12} and the product $k_{11}k_{13}$ can be determined. We also note that only the parameter k_{13} controls the decay of the output exhibited over the period [2sec., 9 sec.], and is a parameter of the linear first-order equation

$$\dot{x}_2 = k_{13}(x - x_1)^{k_{12}} - x_2 \tag{4.10}$$

Since the remaining parameters are known, using the input x and the measured output F, x_1 and x_2 can be calculated as functions of time. Using these values in turn, the parameter k_{13} can be determined using an adaptive estimation algorithm as in [1]

$$\begin{aligned} \dot{\hat{x}}_2 &= -\hat{x}_2 + \hat{k}_{13}(x - x_1)^{k_{12}} \\ \dot{\hat{k}}_{13} &= -(\hat{x}_2 - x_2)(x - x_1)^{k_{12}} \end{aligned} \tag{4.11}$$

It is also possible to identify all the five parameters simultaneously using a nonlinear persistent excitation algorithm as in [2]. The same procedure can be used for determining the parameters of Model II, where T is determined empirically by observing the lag between the two times where the overshoots occur. The parameters thus obtained are shown in Table 4.3.

Once the parameters are determined using the above method, we obtain a complete model of the tissue properties. In Figure 4-6, the predicted output of this model is plotted and compared with the corresponding experimental output for various inputs. The specific case of the liver tissue under in-vivo condition is shown in Figure 4-6.

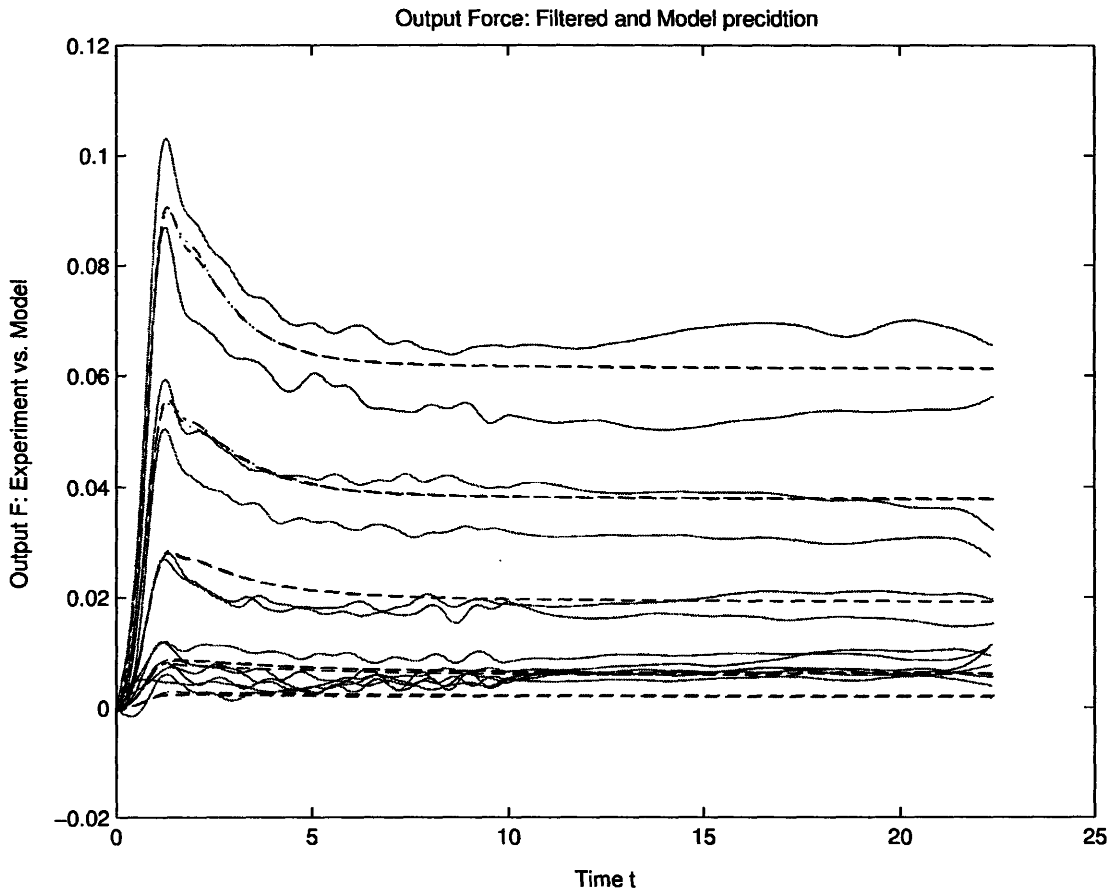

Figure 4-6: Output Force F: Model Prediction with the Parameter Identified

The figure shows that the model prediction is quite accurate for all inputs. Similarly, the predictive ability of Model II is shown in Figure 4-7. The model prediction error E_i, defined in Eq. (4.8), is calculated for all cases of liver-tissue and esophagus-tissue and shown in Table 4.4.

We carried out another set of experiments to further test Models I and II. We varied the

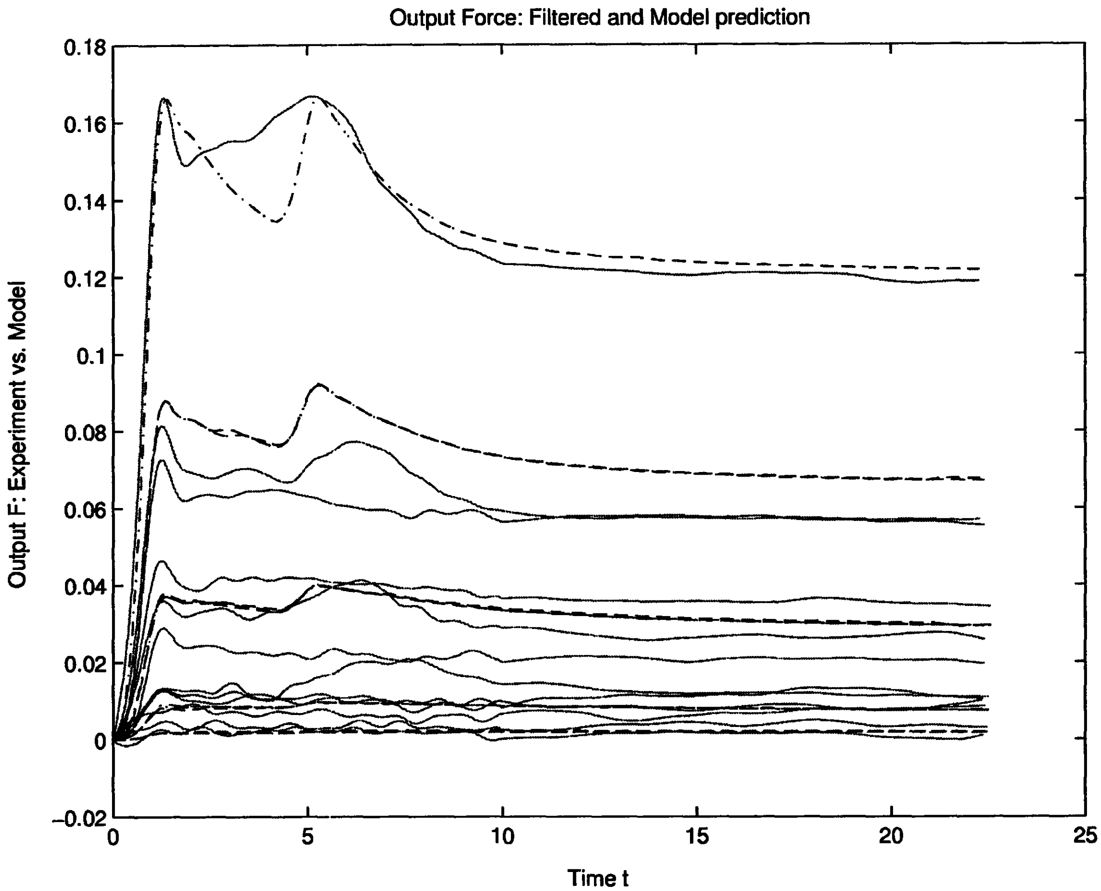

Figure 4-7: Output Force F: Model Prediction with the Parameter Identified

input from a ramp-and-hold to a sinusoid, and tested these two models under the sinusiodal input, keeping the parameters the same as in Figures 4-6 and 4-7, respectively. The corresponding model output is shown in Figures 4-8 and 4-9 for the case of liver and esophagus at the in-vivo condition and compared with the corresponding experimental outputs. These figures show that the model prediction continues to be quite accurate. – discuss the bias with Srini –

4.5 A Nonlinear Parameter Estimation Algorithm

As mentioned in the previous section, the parameters of Model I and Model II can alternately be identified using a nonlinear parameter estimation algorithm. This procedure is

Table 4.3: Parameters Identified of Various Experiments

No. i	Model	k_{01}	k_{02}	k_{12}	k_{11}	k_{13}	k_{21}	k_{22}	k_{23}	T
1	I	0.00175	1.69	1.82	0.000765	0.1				
2	II	0.00154	2.09	2.21	0.000481	0.03	2.39	0.000306	0.015	4
3	I	0.00666	1.48	1.86	0.00362	0.03				
4	I	0.00157	2.31	2.48	0.000701	0.007				
5	I	0.00103	2.24	2.19	0.00113	0.025				
6	I	0.00124	2.73	2.62	0.000671	0.007				
7	I	0.000902	2.46	2.68	0.000464	0.007				
8	I	0.00095	2.68	2.91	0.000375	0.004				
9	I	0.00111	2.61	2.64	0.000602	0.007				
10	I	0.00279	1.54	1.91	0.00103	0.03				
11	II	0.00319	1.49	1.13	0.00436	0.04	2.37	0.000304	0.015	2
12	II	0.00367	1.52	0.934	0.0109	0.15	2.04	0.00053	0.015	2
13	I	0.00870	1.29	1.28	0.0102	0.15				
14	I	0.00187	1.96	2.34	0.000605	0.007				
15	I	0.000563	2.58	2.47	0.000777	0.007				
17	I	0.00177	2.20	2.28	0.000873	0.005				
18	I	0.000992	2.53	2.60	0.00062	0.005				
19	I	0.00219	2.34	2.30	0.00140	0.005				
20	II	0.00482	1.45	1.58	0.00178	0.009	2.04	0.000534	0.015	2
21	I	0.00200	1.70	2.14	0.000779	0.015				
22	I	0.000883	2.52	2.58	0.000726	0.007				
24	I	0.00627	1.45	1.79	0.00321	0.06				
25	I	0.000946	2.39	2.70	0.000325	0.006				

Table 4.4: Model Prediction Error for Different Conditions

No. i	Organ	Condition	Input	$V_i(\%)$	$E_i(\%)$
1	Liver	In-Vivo	RH-Z	1.9	1.3
2	Esophagus	Ex-Vivo	RH-Z	2.5	0.68
3	Liver	Ex-Vivo	RH-Z	1.9	1.3
4	Liver	In-Vitro	RH-Z	1.3	0.92
5	Esophagus	In-Vitro	RH-Z	1.9	0.75
6	Liver	In-Vitro	RH-Z	2.8	2.1
7	Esophagus	In-Vitro	RH-Z	1.6	0.96
8	Liver	In-Vitro	RH-Z	1.4	0.62
9	Esophagus	In-Vitro	RH-Z	1.8	1.4
10	Liver	In-Vivo	RH-Z	4.2	2.3
11	Esophagus	In-Vivo	RH-Z	8.1	4.0
12	Esophagus	Ex-Vivo	RH-Z	6.6	3.1
13	Liver	Ex-Vivo	RH-Z	2.6	1.6
14	Liver	In-Vitro	RH-Z	0.87	0.62
15	Esophagus	In-Vitro	RH-Z	3.1	0.47
17	Esophagus	In-Vitro	RH-Z	1.3	0.88
18	Liver	In-Vitro	RH-Z	1.5	0.7
19	Esophagus	In-Vitro	RH-Z	3.1	2.8
20	Esophagus	In-Vivo	RH-Z	7.3	3.0
21	Liver	In-Vivo	RH-Z	2.3	1.5
22	Esophagus	Ex-Vivo	RH-Z	6.1	4.3
24	Esophagus	Ex-Vivo	RH-Z	2.8	2.1
25	Liver	Ex-Vivo	RH-Z	3.0	2.6

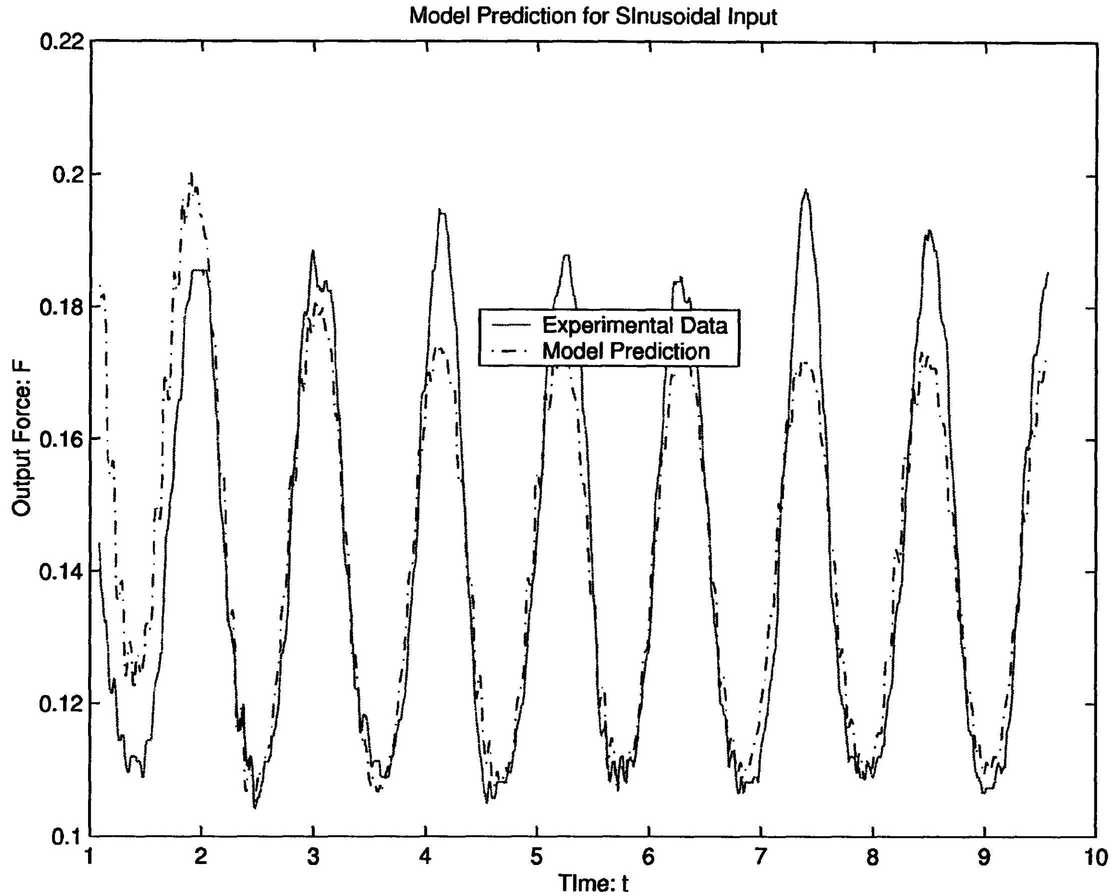

Figure 4-8: Output Force F: Model Prediction of Sinusoidal Input with the Parameter Identified

outlined in this section. While the procedure below is described in the context of Model I, the same is applicable for Model II. It is assumed that the model is of the form of Eq. (4.3) where the input x and F are available through measurement at each instant of time. Unknown parameters k_{01} and k_{02} can be identified from static input output relationship and then it can be assumed that the intermediate state variables $\dot{x}$, $\ddot{x}$, F_0 and therefore $\bar{F}_1 = k_{11}F_1$, $\dot{\bar{F}}_1$ are available. The goal is then to identify the unknown parameters k_{11}, k_{12} and k_{13}. The algorithm that is used for their estimation is described below.

From (4.3), it follows that

$$x_1 = x - (\frac{\bar{F}_1}{k_{11}k_{13}})^{1/k_{12}}. \tag{4.12}$$

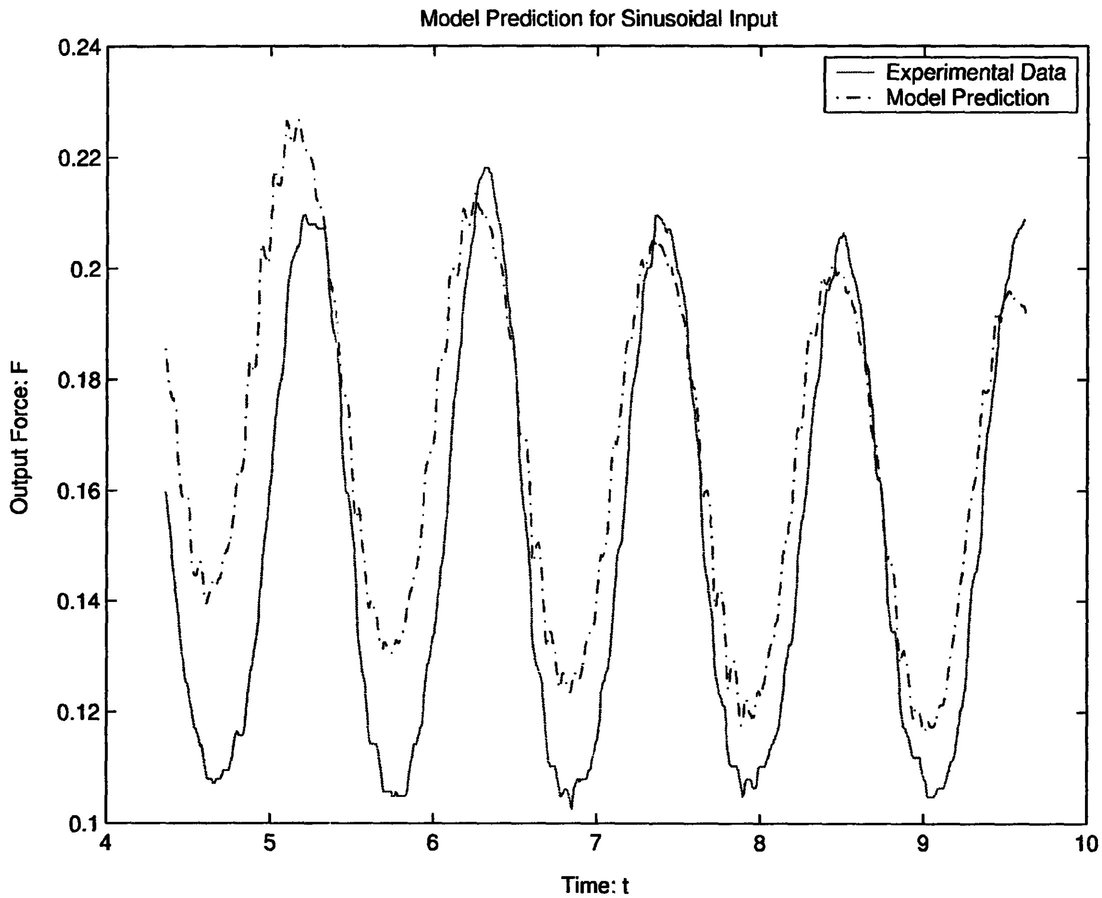

Figure 4-9: Output Force F: Model Prediction of Sinusoidal Input Signal with the Parameter Identified

It follows from (4.12) that

$$\dot{x}_1 = \dot{x} - \left(\frac{\bar{F}_1}{k_{11}k_{13}}\right)^{\frac{1}{k_{12}-1}} \frac{\dot{\bar{F}}_1}{k_{11}k_{13}} \tag{4.13}$$

and

$$\ddot{x}_1 = \ddot{x} - \left(\frac{\bar{F}_1}{k_{11}k_{13}}\right)^{\frac{1}{k_{12}-1}} \frac{\ddot{\bar{F}}_1}{k_{11}k_{13}} - \left(\frac{\dot{\bar{F}}_1}{k_{11}k_{13}}\right)^2 \left(\frac{\bar{F}_1}{k_{11}k_{13}}\right)^{\frac{1}{k_{12}}-2} . \frac{\dot{\bar{F}}_1}{k_{11}k_{13}}. \tag{4.14}$$

It follows from (4.3) that

$$\ddot{x}_1 = \dot{x}_2 = \frac{\bar{F}_1}{k_{11}} - \dot{x}_1 = \frac{\bar{F}_1}{k_{11}} - \dot{x} + \left(\frac{\bar{F}_1}{k_{11}k_{13}}\right)^{\frac{1}{k_{12}-1}} \frac{\dot{\bar{F}}_1}{k_{11}k_{13}}. \tag{4.15}$$

Combining (4.14) and (4.15), we establish an equation between $\dot{x}$, $\ddot{x}$, $\bar{F}_1$, $\dot{\bar{F}}_1$ and $\ddot{\bar{F}}_1$ and it can be simplified as

$$\ddot{\bar{F}}_1 = -\dot{\bar{F}}_1 - \frac{(\dot{\bar{F}}_1)^2}{\bar{F}_1} + \frac{k_{11}k_{13}}{(\frac{\bar{F}_1}{k_{11}k_{13}})^{\frac{1}{k_{12}-1}}}(\dot{x} + \ddot{x}) - \frac{\bar{F}_1 k_{13}}{(\frac{\bar{F}_1}{k_{11}k_{13}})^{\frac{1}{k_{12}-1}}}. \tag{4.16}$$

Since $\dot{x}$, $\ddot{x}$, $\bar{F}_1$ and $\dot{\bar{F}}_1$ are available, equation (4.16) can be viewed as a nonlinearly parameterized system with inputs and outputs available where k_{11}, k_{12} and k_{13} as unknown parameters that need to be estimated. Applying the min-max algorithm as in [2], we have the following estimator

$$\begin{aligned}
\ddot{\hat{F}}_1 &= -\dot{\hat{F}}_1 - \frac{(\dot{\bar{F}}_1)^2}{\bar{F}_1} + \frac{\hat{k}_{11}\hat{k}_{13}}{(\frac{\bar{F}_1}{\hat{k}_{11}\hat{k}_{13}})^{\frac{1}{\hat{k}_{12}-1}}}(\dot{x} + \ddot{x}) - \frac{\bar{F}_1 \hat{k}_{13}}{(\frac{\bar{F}_1}{\hat{k}_{11}\hat{k}_{13}})^{\frac{1}{\hat{k}_{12}-1}}} - a^* sat(\frac{\tilde{F}}{\epsilon}) \\
\dot{K} &= -\tilde{F}_\epsilon \phi^* \\
\tilde{F} &= \hat{F}_1 - \bar{F}_1 \\
\tilde{F}_\epsilon &= \tilde{F} - \epsilon sat(\frac{\tilde{F}}{\epsilon}).
\end{aligned} \tag{4.17}$$

where a^* and ϕ^* comes from a optimization problem

$$\begin{aligned}
a^* &= \min \phi \in \mathbb{R}^3 \max_{K \in \Omega} g \\
\phi^* &= arg \min \phi \in \mathbb{R}^3 \max_{K \in \Omega} g \\
g &= sat(\frac{\tilde{F}}{\epsilon}) \left((-\frac{(\dot{\bar{F}}_1)^2}{\bar{F}_1} + \frac{\hat{k}_{11}\hat{k}_{13}}{(\frac{\bar{F}_1}{\hat{k}_{11}\hat{k}_{13}})^{\frac{1}{\hat{k}_{12}-1}}}(\dot{x} + \ddot{x}) - \frac{\bar{F}_1 \hat{k}_{13}}{(\frac{\bar{F}_1}{\hat{k}_{11}\hat{k}_{13}})^{\frac{1}{\hat{k}_{12}-1}}}) - \right. \\
&\quad \left. (-\frac{(\dot{\bar{F}}_1)^2}{\bar{F}_1} + \frac{\hat{k}_{11}\hat{k}_{13}}{(\frac{\bar{F}_1}{\hat{k}_{11}\hat{k}_{13}})^{\frac{1}{\hat{k}_{12}-1}}}(\dot{x} + \ddot{x}) - \frac{\bar{F}_1 \hat{k}_{13}}{(\frac{\bar{F}_1}{\hat{k}_{11}\hat{k}_{13}})^{\frac{1}{\hat{k}_{12}-1}}}) \right).
\end{aligned} \tag{4.18}$$

4.5.1 NLPE simulation results

To evaluate the algorithm in (4.17), we use the experimental data for the liver tissue under in-vivo condition over the time interval $[2sec,\ 14sec]$. The quantities $\bar{F}_1$ and $\dot{\bar{F}}_1$ in eq. (4.16) can be calculated at each instant of time using x and the experimental output

F, numerical realization of the derivative $\dot{\hat{F}}_1$ is not straight forward. Therefore, in order to evaluate the parameter estimation algorithm, a model as in (4.16) was numerically simulated. For this simulation, the parameters k_{01} and k_{02} were assumed to be known and the remaining parameters were chosen as [0.000375 2.91 0.004], which corresponds to the liver tissue under in-vivo condition (see Table 4.3). We implemented the NLPE algorithm in (4.17)-(4.18) with the initial parameter values as

$$\hat{K}(0) = [\hat{k}_{11}\ \hat{k}_{12}\ \hat{k}_{13}]^T = [0.000375\ 3.3\ 0.004]^T. \tag{4.19}$$

We then used a ramp-and-hold input x to generate the inputs and output in (16) for 12 seconds over which $\hat{K}$ is estimated. This process is iterated repeatedly, with the parameter estimates at the end of the iteration used as the initial values of the next iteration. Using this process, we observed that the parameters converged, with the final parameter values given by

$$\hat{K}(45) = [\hat{k}_{11}\ \hat{k}_{12}\ \hat{k}_{13}]^T = [0.000375\ 2.91\ 0.004]^T. \tag{4.20}$$

We illustrate the nature of the parameter convergence in Figure 4-11. The estimated force F based on the initial values $\hat{K}(0)$ and $\hat{K}(45)$ are shown in Figure 4-10. It is found that at the beginning the initial parameter results in a large output error and this error is gradually reduced with more iterations.

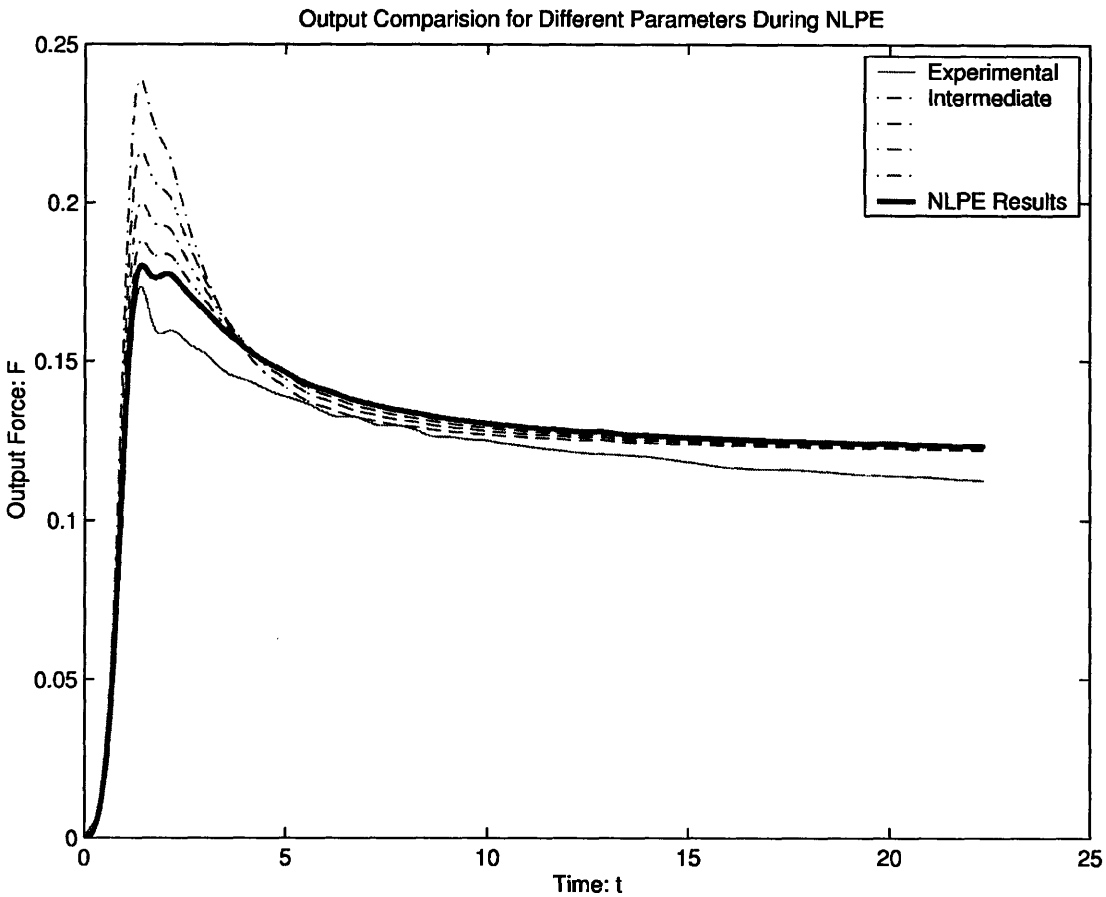

Figure 4-10: Output Force F: Model Prediction of Parameters before NLPE and after NLPE

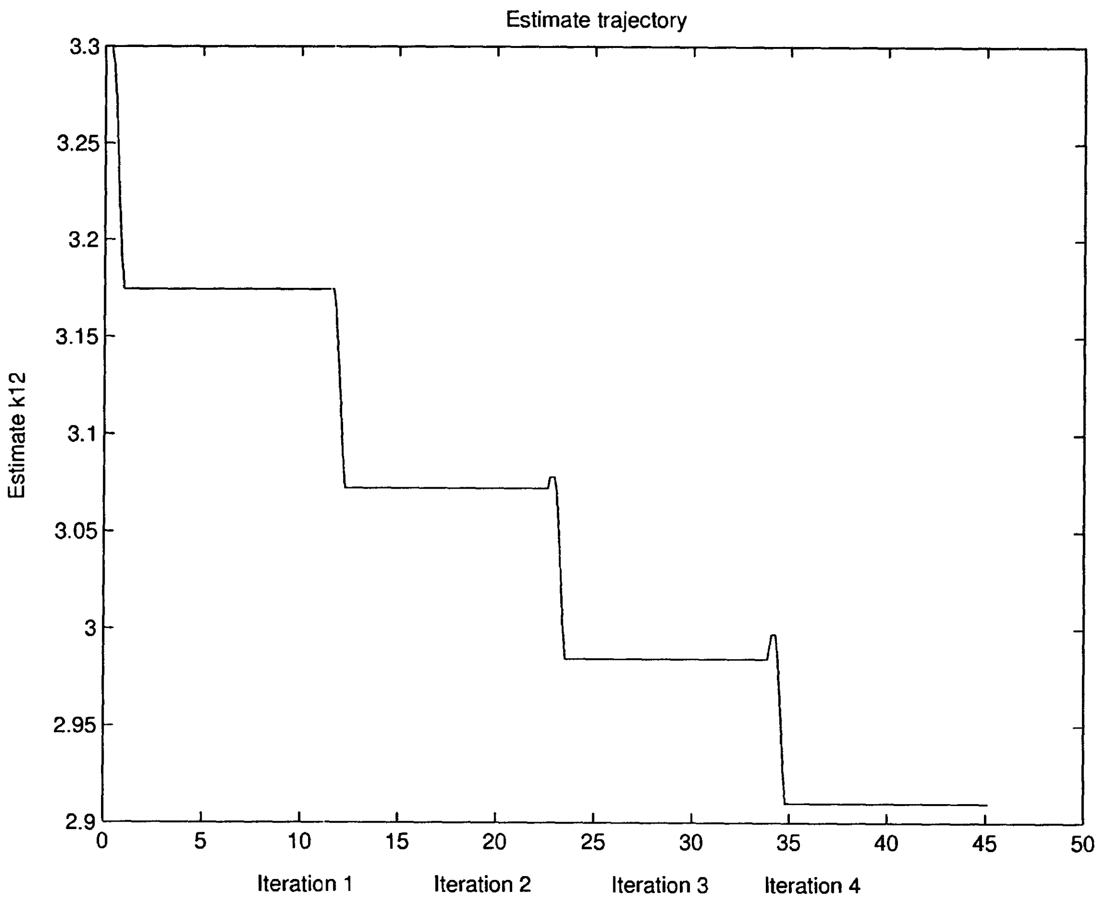

Figure 4-11: Trajectory of Parameter Estimate $\hat{k}_{12}$ in NLPE Algorithm

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Chapter 5

Polynomial Adaptive Estimator

5.1 Introduction

The problem of parameter estimation in a nonlinearly parameterized system can be stated as follows:

$$\dot{y} = f(y, u, \theta_0) \tag{5.1}$$

where f is nonlinear in the unknown parameter θ_0. The goal is to develop an estimator

$$\dot{\hat{y}} = f(\hat{y}, u, \hat{\theta}) \tag{5.2}$$

with $\hat{\theta}$ adjusted so that $\hat{\theta} \to \theta_0$.

A stability framework has been established for studying estimation and control of nonlinearly parameterized systems in [1]-[9]. In [1, 2], for example, stability and parameter convergence with suitable NLPE conditions have been established. The problem however is that the NLPE condition is quite restrictive, and requires a certain property to be satisfied by all possible subsets in the parameter space and is rather difficult to check. One of the reasons for this is that the unknown parameter is estimated using a quadratic nonlinearity in the Lyapunov function which essentially generates a linear function in the parameter error. For example, for the system in (5.1), and the estimator in (5.2), suppose the parameter estimation is chosen as

$$\dot{\tilde{\theta}} = -\tilde{y}_\epsilon \phi^*,$$

a Lyapunov function of the form

$$V = \tilde{y}_\epsilon^2/2 + \tilde{\theta}^2/2$$

leads to a time-derivative

$$\dot{V} = \tilde{y}_\epsilon \left(f(y,u,\widehat{\theta}) - \phi^* \tilde{\theta} - f(y,u,\theta_0) \right).$$

The term $f(y,u,\widehat{\theta}) - \phi^* \tilde{\theta}$ is clearly linear in $\tilde{\theta}$ and therefore in θ_0. Since f is not linear in θ_0, it is clear that there are not enough degrees of freedom in the estimator. This is the motivation for choosing a polynomial Lyapunov function

$$V = \tilde{y}_\epsilon^2/2 + \sum_{i=1}^{N} p_i(\tilde{\theta}_i)$$

where

$$\dot{\tilde{\theta}}_i = -\tilde{y}_\epsilon \phi_i^*, \qquad i = 1, .., N.$$

By choosing a $p(\cdot)$ in V and multiple parameter estimates $\widehat{\theta}_i$, we will generate a Lyapunov derivative which gives us more degrees of freedom.

The chapter is organized as follows. Section 2 includes the statement of the problem and description of the PAE algorithm. In Section 3, the PAE and its stability properties are discussed in the simple case when the parametric nonlinearity is polynomial in nature. In section 4, a DPAE algorithm is introduced to address general nonlinearities and the expansion to higher dimension is explained. In Section 5, the NLPE condition and parameter convergence are presented. It is shown that the NLPE condition is identical to the linear persistent excitation condition. In section 6, simulation results are provided. Summary and conclusions are stated in section 7.

5.2 The Structure of PAE

5.2.1 Statement of the Problem

We start with the simplest problem of a first order plant with a scalar unknown parameter while the extension to unknown parameters and systems in higher dimension is discussed later in this chapter. This plant can be described as

$$\dot{y} = -\alpha y + f(y, u, \theta_0) \tag{5.3}$$

where $\theta_0 \in \Theta \subset \mathbb{R}$ is unknown parameter, Θ is the known compact set where the unknown parameter θ_0 belongs to, $y \in \mathbb{R}$ is state variable, $u \in \mathbb{R}^m$ includes inputs, measurable system variables and even system time t. We note that problem formulation in (5.3) also include plants of the form

$$\dot{y} = \bar{f}(y, u, \theta_0)$$

since they can be transformed into (5.3) with $f(y, u, \theta_0) = \alpha y + \bar{f}(y, u, \theta_0)$. Secondly, we note that there exist multiple unknown parameters for nonlinear dynamic systems for the same input-output relationship, which is different from linear systems. One simple example of multiple true parameters is a periodic function which has $f(y, u, \theta_0) = f(y, u, \theta_0 + \Delta)$. In this situation, we have no way to distinguish θ_0 from $\theta_0 + \Delta$. In nonlinear parameter estimation, we denote Θ as the set of the unknown parameters where

$$\Theta = \{\theta \mid f(y, u, \theta) = f(y, u, \theta_0), \forall\, y, u, \theta \in \Theta\}.$$

Remark 1: We note that for any parameter estimation algorithm, it has no ability to distinguish the points in Θ just by using input and output information because their performance are identical. Therefore, for a general nonlinear parameter estimation algorithm, unlike the Linear Adaptive Estimation algorithm, it must have the ability to identify all the points in Θ if it is globally convergent.

In this chapter, for all the situations where just the value of $f(y, u, \theta)$ matters, we use θ_0 to represent any point in Θ and we note that any result achieved for θ_0 holds for any point in Θ. We make the following assumptions regarding function f.

Assumption 1: The function $f(y, u, \theta)$ is Lipschitz with its arguments $x = [y, u, \theta]^T$, i.e. there exists positive constant B such that

$$|f(x + \Delta x) - f(x)| \leq B||\Delta x||. \tag{5.4}$$

Assumption 2: Input signal $u(t)$ is Lipschitz with respect to t, i.e. there exists constant U such that

$$||u(t_1) - u(t_2)|| \leq U|t_1 - t_2|.$$

Assumption 3: Function f is bounded, i.e.

$$|f(y, u, \theta_0)| \leq F_1$$

Assumption 4: y is bounded by F_2.

Assumption 3 and 4 mean that $\dot{y}$, the derivative of state variable, is also bounded by

$$F = F_1 + \alpha F_2. \tag{5.5}$$

and therefore y will not jump and the maximum change rate of y is bounded by F. We define the Lipschitz continuity of dynamic system as follows.

Definition 1 *System in (5.3) is a Lipschitz continuous system if it satisfies Assumptions* 1-4.

It is noted that nearly all the systems we encounter in practice are Lipschitz continuous. It just requires the signal change rate in a nonlinear dynamic system to be bounded. Through this chapter, we will restrict our attention to Lipschitz continuous systems.

5.2.2 Structure of Polynomial Adaptive Estimator

The Polynomial Adaptive Estimator(PAE) that we propose include several new features. PAE expands the commonly used quadratic form Lyapunov functions and adopts a new approach of auxiliary estimates, which uses $\widehat{\theta}_1, .., \widehat{\theta}_N$ for one unknown parameter θ_0. The

PAE is of the form

$$\begin{aligned}
\dot{\hat{y}} &= -\alpha(\hat{y} - \epsilon sat(\frac{\tilde{y}}{\epsilon})) + \phi_0^* - a^* sat(\frac{\tilde{y}}{\epsilon}) \\
\dot{\hat{\theta}}_i &= -\tilde{y}_\epsilon \phi_i^*, \qquad i = 1, ..., N
\end{aligned} \tag{5.6}$$

where

$$\tilde{y} = \hat{y} - y \qquad \tilde{y}_\epsilon = \tilde{y} - \epsilon sat\left(\frac{\tilde{y}}{\epsilon}\right)$$

ϵ is an arbitrary positive number, $sat(.)$ denote the saturation function and is given by $sat(x) = sign(x)$ if $|x| \geq 1$ and $sat(x) = x$ if $|x| < 1$, and the calculation of a^* and ϕ^* will be discussed later.

Combining (5.3) and (5.6), we rewrite the dynamics of the entire system as

$$\begin{aligned}
\dot{\tilde{y}} &= -\alpha\tilde{y}_\epsilon + \phi_0^* - f(y, u, \theta_0) - a^* sat(\frac{\tilde{y}}{\epsilon}) \\
\dot{\tilde{\theta}}_i &= -\tilde{y}_\epsilon \phi_i^*, \qquad i = 1, .., N.
\end{aligned}$$

where

$$\tilde{\theta}_i = \hat{\theta}_i - \theta_0.$$

To consider stability, we introduce a Lyapunov function V as

$$V = \tilde{y}_\epsilon^2 + \sum_{i=1}^{N} p_i(\tilde{\theta}_i) \tag{5.7}$$

where $p_i(.)$ is a polynomial function. Therefore, the derivative of $p_i(.)$ is also a polynomial function and denoted as g_i where

$$g_i(x) = \frac{dp_i(x)}{dx}, \qquad \forall\, i = 1, ..., N.$$

For V to become a Lyapunov function, the choices of p_i needs to satisfies the following conditions

$$
\begin{aligned}
(1) \quad g_i(\tilde{\theta}_i) & < 0 \text{ if } \tilde{\theta}_i < 0 \\
(2) \quad g_i(\tilde{\theta}_i) & > 0 \text{ if } \tilde{\theta}_i > 0 \\
(3) \quad p_i(0) & = 0 \\
(4) \quad g_i(0) & = 0
\end{aligned} \tag{5.8}
$$

for any $i = 1, .., N$ and all possible values of $\tilde{\theta}_i$. If $p_i(\tilde{\theta}_i)$ satisfies (5.8), it can be shown easily that $p_i(\tilde{\theta}_i)$ is nonnegative with $p_i(\tilde{\theta}_i) = 0$ iff $\tilde{\theta}_i = 0$ and $p_i(\tilde{\theta}_i)$ increases as $|\tilde{\theta}_i|$ increases.

To make V a Lyapunov function, we need to make sure that $\dot{V}$ is nonpositive. Because

$$
\dot{V} = -\alpha \tilde{y}_\epsilon^2 + \tilde{y}_\epsilon \left(\phi_0^* - f(y, u, \theta_0) - \sum_{i=1}^{N} g_i(\tilde{\theta}_i)\phi_i^* - a^* sat(\frac{\tilde{y}}{\epsilon}) \right), \tag{5.9}
$$

if we choose $\phi_i^*, i = 1, ..., N$ and a^* to make

$$
\tilde{y}_\epsilon \left(\phi_0^* - f(y, u, \theta_0) - \sum_{i=1}^{N} g_i(\tilde{\theta}_i)\phi_i^* - a^* sat(\frac{\tilde{y}}{\epsilon}) \right) \leq 0, \tag{5.10}
$$

it follows that

$$
\dot{V} = -\alpha \tilde{y}_\epsilon^2 \leq 0 \tag{5.11}
$$

and V serve as a Lyapunov function.

We note that if $\tilde{y}_\epsilon = 0$, inequality (5.10) holds always. If $\tilde{y}_\epsilon \neq 0$, it implies that $|\tilde{y}| > \epsilon$ and hence

$$
sat(\frac{\tilde{y}}{\epsilon}) = sign(\tilde{y}_\epsilon).
$$

Inequality (5.10) can be transformed into

$$
\tilde{y}_\epsilon sign(\tilde{y}_\epsilon) \left(sign(\tilde{y}_\epsilon)(\phi_0^* - f(y, u, \theta_0) - \sum_{i=1}^{N} g_i(\tilde{\theta}_i)\phi_i^*) - a^* \right) \leq 0.
$$

Because $\tilde{y}_\epsilon sign(\tilde{y}_\epsilon) \geq 0$, to achieve (5.11), we just need to choose ϕ^* and a^* to make the following inequality holds

$$sign(\tilde{y}_\epsilon)(e(y,u,\theta_0^*) - f(y,u,\theta_0)) - a^* \leq 0 \tag{5.12}$$

where

$$\begin{aligned} e(y,u,\theta_0,\phi^*) &= \phi_0^* - \sum_{i=1}^{N} g_i(\tilde{\theta}_i)\phi_i^* \\ \phi^* &= [\phi_0^*, ..., \phi_N^*]^T. \end{aligned}$$

Because ϕ^* can be chosen arbitrarily, once they are determined, e becomes a function of y,u and unknown parameters θ_0.

Now we establish the definition of a Polynomial Adaptive Estimator. Firstly, we need to determine the order N of PAE and choose appropriate Lyapunov function components p_i. Secondly, in the running of the algorithm, design a methodology to find ϕ^* and a^* which satisfies (5.12). The definition of a PAE is as follows.

Definition 2 *The Polynomial Adaptive Estimator(PAE) is an adaptive estimation algorithm in (5.6) which satisfies conditions (5.8) and (5.12).*

This definition leaves a lot of room for constructing different PAE algorithms with two fixed properties. One is the construction of a Lyapunov function which satisfies (5.8) and another is the calculation of ϕ^* and a^* that satisfy (5.12). In section 5.2.3, we will propose a method to construct such a Lyapunov function which is used through this chapter. In section 5.2.4, we will discuss the calculation of ϕ^* and a^*.

5.2.3 Construction of A Polynomial Lyapunov function

We choose $p(.)$ in the Lyapunov function in (5.7) as

$$\begin{aligned} p_i(\tilde{\theta}_i) &= \frac{1}{i+1}\tilde{\theta}_i^{i+1} \qquad \text{if } i \text{ is odd;} \\ p_i(\tilde{\theta}_i) &= \frac{1}{i}\tilde{\theta}_i^{i} + \frac{k_i}{i+1}\tilde{\theta}_i^{i+1} \qquad \text{if } i \text{ is even} \end{aligned} \tag{5.13}$$

for $i = 1, .., N$, where k_i is to be chosen appropriately. The corresponding g_i is therefore given by

$$
\begin{aligned}
g_i(\tilde{\theta}_i) &= \tilde{\theta}_i^i \qquad \text{if } i \text{ is odd;} \\
g_i(\tilde{\theta}_i) &= \tilde{\theta}_i^{i-1} + k_i \tilde{\theta}_i^i \qquad \text{if } i \text{ is even.}
\end{aligned} \tag{5.14}
$$

In what follows we will show that (5.8) is satisfied with these choice of p_i. Conditions 3 and 4 follow immediately. Conditions 1 and 2 in (5.8) follow as well when i is odd, as does condition 2 in (5.8) when i is even. Hence, what remains to be shown is condition 1 when i is even, which is not true for any $\tilde{\theta}_i$. However, the feature we can exploit is that the range of $\tilde{\theta}_i$ is constrained by Lyapunov function V defined as in (5.7) and we just need to choose k_i which makes condition 1 in (5.8) holds for any possible $\widehat{\theta}_i$. For any choice of initial $\widehat{\theta}_i$ and $\hat{y}$ at $t = 0$, the Lyapunov function is $V(0)$. From (5.11), it follows that

$$V(t) < V(0) \tag{5.15}$$

for any $t \geq 0$. Equation (5.15) implies that $\tilde{\theta}_i$ is bounded and the bounds can be calculated easily. Assume the lower bound of $\tilde{\theta}_i$ is $\tilde{\theta}_i^b$, to make condition 1 in (5.8) satisfied, we just need to choose k_i which satisfies

$$k_i < -\frac{1}{\tilde{\theta}_i^b} \tag{5.16}$$

Choosing Lyapunov function V as in (5.13) and an appropriate k_i that satisfies (5.16), we establish stability of the PAE algorithm if (5.12) can be satisfied. Throughout the rest of the chapter, we will choose Lyapunov function as in (5.13).

5.2.4 Implementation of PAE

One choice of ϕ^* and a^* that satisfy (5.12) so that V is nonincreasing is as follows:

$$
\begin{aligned}
\phi^* &= arg \min_{\phi \in \mathbb{R}^N} \max_{\theta \in \Omega_0} h(y, \theta, u) \\
a^* &= \min_{\phi \in \mathbb{R}^N} \max_{\theta \in \Omega_0} h(y, \theta, u)
\end{aligned}
$$

$$h(y,\theta,u) \;\; = \;\; sign(\tilde{y}_\epsilon)(\phi_0^* - f(y,u,\theta_0) - \sum_{i=1}^{N} g_i(\tilde{\theta}_i)\phi_i^*) \qquad (5.17)$$

When conditions (5.17) and (5.8) are satisfied, it follows that the PAE is stable. However, similar to the min-max algorithm in [1], this implies that a nonlinear optimization problem has to be solved to obtain ϕ^* at every time step. Given that the optimal ϕ^* can lie anywhere in an unconstrained region $\mathbb{R}^N$, this is an extremely difficult task if not impossible. In the next section, we will consider two examples where the solution to the above problem can be obtained in a straightforward manner.

The motivation for the structure of the PAE in (5.6) and (5.17) and the polynomial Lyapunov function in (5.13) is as follows: The nonlinear function f of the unknown parameter θ_0 is estimated using ϕ_0^*, and a^*, which is added so as to guarantee stability, needs to be made to approach zero. If a quadratic Lyapunov function and a single parameter estimation for θ_0 is used, the Lyapunov derivative is of the form

$$\dot{V} = -\alpha\tilde{y}_\epsilon^2 + \tilde{y}_\epsilon\left[f(y,u,\widehat{\theta}) - \phi^*\tilde{\theta} - f(y,u,\theta_0)\right] - a^* sat(\frac{\tilde{y}}{\epsilon}). \qquad (5.18)$$

The term $f(y,u,\widehat{\theta}) - \phi^*\tilde{\theta}$ is clearly linear in $\tilde{\theta}$ and therefore in θ_0. Since f is not linear in θ_0, it is clear that there are not enough degrees of freedom in the estimator. In the PAE, by choosing a $p(\cdot)$ in V and multiple parameter estimates $\widehat{\theta}_i$, instead of Eq. (5.18), we obtain a Lyapunov derivative as in (5.9) which gives us more degrees of freedom. In particular, the term

$$\phi_0^* - \sum_{i=1}^{N} g_i(\tilde{\theta}_i)\phi_i^*$$

provides us with a *Polynomial* approximation of f.

It should be noted that even with these additional degrees of freedom, two problems remain. The first is the requirement that a^* has to approach zero. The second is the construction of ϕ^* such that Eq. (5.12) is satisfied. The introduction of $p(.)$ and the additional degrees of freedom makes the latter somewhat more complex. However, as shown in section 5.5, the condition on persistent excitation leading to parameter convergence is much simpler.

5.3 Polynomial Adaptive Estimator

We will consider two cases in this section where we will show how to choose ϕ^* and a^* to form an Nth order polynomial expansion for the function f in (5.3) so that (5.12) is satisfied. In case i, f is exactly an Nth order polynomial function over θ_0, and in case ii, f can be approximated by a Nth order polynomials.

<u>*Case i:*</u> When f is exactly a Nth order polynomial, it implies that

$$f(y, u, \theta_0) = \sum_{i=0}^{N} c_i \theta_0^i \tag{5.19}$$

where c_i is a known of y and u, while θ_0 is unknown. For the choice of V as in (5.13), $e(y, u, \theta_0^*)$ is written as

$$e(y, u, \theta_0^*) = \phi_0^* - \sum_{i=1}^{N} \phi_i^* g_i(\tilde{\theta}_i)$$

where g_i is defined as in (5.14). We will choose ϕ^* in a way that

$$\phi_0^* - \sum_{i=1}^{N} \phi_i^* g_i(\tilde{\theta}_i) \;\; = \;\; \sum_{i=0}^{N} c_i \theta_0^i, \tag{5.20}$$

and

$$a^* = 0. \tag{5.21}$$

Because $e(y, u, \theta_0^*)$ can perfectly match f, there is no need to introduce a^* or ϵ. It can be checked easily that choice of ϕ^* and a^* as in (5.20) and (5.21) satisfies (5.12).

We notice that $\tilde{\theta}_i = \widehat{\theta}_i - \theta_0$ and g_i is a ith order polynomial function of θ_0 and it can be expressed as

$$g_i = \sum_{j=0}^{i} d_{ij}(\widehat{\theta}_i) {\theta_0}^j. \tag{5.22}$$

With known $\widehat{\theta}_i$ and k_i, the calculation of coefficients of d_{ij} follows easily, with especially

$$\begin{aligned} d_{00} &= 1 \\ d_{ii} &= -1 \qquad \text{if } i \text{ is odd} \\ d_{ii} &= k_i \qquad \text{if } i \text{ is even.} \end{aligned} \tag{5.23}$$

It follows from (5.22) that equation (5.20) is equivalent to

$$A\phi^* = C$$

where

$$A = \begin{bmatrix} d_{00} & * & * & .. & * \\ 0 & d_{11} & * & .. & * \\ 0 & 0 & d_{22} & .. & * \\ : & : & : & :: & : \\ 0 & 0 & 0 & .. & d_{NN} \end{bmatrix} \tag{5.24}$$

and

$$C = [c_0 \; c_1 \; ... c_N]^T. \tag{5.25}$$

The element of ith row and jth column of matrix A in (5.24) is

$$A_{ij} = \begin{cases} 0 & i > j; \\ d_{ji} & i \leq j \end{cases}$$

where d_{ji} is defined as in (5.22) and (5.23). We notice that A is a upper-triangular matrix and it must be full rank. It means that $e(y, u, \theta_0)$ has the ability to match any Nth order polynomial functions. Calculation of ϕ^* follows easily as

$$\phi^* = A^{-1}C.$$

In fact, for a upper-triangular matrix, we even do not need to perform the matrix inversion and we can get ϕ_N^* through ϕ_0^* one by one by simple manipulation.

<u>*Case ii:*</u> When function f is not exactly a Nth order polynomial however can be approximated by a Nth order polynomial, it follows that

$$f(y, u, \theta_0) = \sum_{i=0}^{N} c_i \theta_0^i + r(y, u, \theta_0) \tag{5.26}$$

where $r(y, u, \theta_0)$ is the residual error between objective function f and the Nth order polynomial approximation and

$$|r(y,u,\theta_0)| \leq a^*_{max}. \tag{5.27}$$

Using the same method as in case i to calculate ϕ^* as

$$\begin{aligned}\phi_0^* - \sum_{i=1}^{N} \phi_i^* g_i(\tilde{\theta}_i) &= \sum_{i=0}^{N} c_i \theta_0^i \\ a^* &= a^*_{max}\end{aligned}$$

and it can be checked easily that such choice of a^* and ϕ^* satisfies (5.12). The PAE algorithm is stated as

$$\begin{aligned}\dot{\hat{y}} &= -\alpha(\hat{y} - \epsilon sat(\frac{\tilde{y}}{\epsilon})) + \phi_0^* - a^* sat(\frac{\tilde{y}}{\epsilon}) \\ \dot{\hat{\theta}}_i &= -\tilde{y}_\epsilon \phi_i^*, \qquad i = 1, ..., N \\ \tilde{y} &= \hat{y} - y \\ \tilde{y}_\epsilon &= \tilde{y} - \epsilon sat\left(\frac{\tilde{y}}{\epsilon}\right) \\ a^* &= a^*_{max} \\ \phi^* &= A^{-1}C\end{aligned} \tag{5.28}$$

where

$$\phi^* = [\phi_0^*, \ \phi_1^*, ..., \phi_N^*]^T,$$

$sat(.)$ denote the saturation function, A, C and a^*_{max} are defined in (5.24), (5.25) and (5.27) respectively.

5.3.1 Properties of the PAE

In this section, we will establish some properties of the PAE algorithm in (5.28). For the PAE in $Case\ i$ of section 5.3, same properties and lemmas follow if we assume that $a^*_{max} = 0$. All the proofs of the properties and lemmas can be found in the Appendix.

First, we will establish some properties of ϕ^*. We will show that ϕ^* is bounded and Lipschitz w.r.t. time t.

Property 1

ϕ^* is bounded.

Property 2

$$|\phi_0^*(t_2) - \phi_0^*(t_1)| \leq Q_1|t_2 - t_1|.$$

In PAE, ϕ_0^* is a known variable in the algorithm and the maximum change rate Q_1 can be measured and kept on line. Unlike ϕ^* which is calculated by solve a group of linear equations, a^* in PAE will be at a constant nonnegative value a^*_{max}. About a^*_{max}, we have the following property.

Property 3

$$-a^*_{max} \leq a^* sat(\frac{\tilde{y}}{\epsilon}) \leq a^*_{max}.$$

The proof of this property is obvious now that $|sat(\frac{\tilde{y}}{\epsilon})| \leq 1$. If we define

$$m(t) = \phi_0^* - f(y, u, \theta_0) - a^* sat(\frac{\tilde{y}}{\epsilon}), \tag{5.29}$$

Property 4 shows that $m(t)$ is bounded.

Property 4 *There exists a finite positive M such that*

$$|m(t)| \leq M \tag{5.30}$$

where $m(t)$ is defined as in (5.29).

Define

$$n(t) = \phi_0^* - f(y, u, \theta_0), \tag{5.31}$$

we conclude that $n(t)$ is Lipschitz w.r.t. t in the Property 5.

Property 5

$$|n(t + \tau) - n(t)| \leq Q|\tau| \tag{5.32}$$

where

$$Q = B(U + F) + Q_1, \tag{5.33}$$

with B, U, F, Q_1 defined as in Assumptions 1, 2, Eq. (5.5) and Property 2 respectively.

Remark 2: In fact, the estimator variables ϕ^*, a^* and $\widehat{\theta}$ are associated by a non-singular matrix. From Assumptions 1-3, all the system variables u, y are Lipschitz w.r.t. time t, therefore, all the variables in the algorithm are Lipschitz w.r.t. time t, i.e. change rate bounded.

Next, we will show several lemmas related with the PAE. In the following lemma, it is shown that when the output error is large, the Lyapunov function will decrease by a finite amount.

Lemma 1 *For the system in (5.3) and PAE as in (5.28), if*

$$|\tilde{y}_\epsilon(t_1)| \geq \gamma,$$

then

$$V(t_1 + T') \leq V(t_1) - \frac{\alpha\gamma^3}{3(M + \alpha\gamma)}$$

where $T' = \gamma/(M + \alpha\gamma)$ and M is defined as in (5.30).

The proof of lemma 1 is shown in [2]. In the following Lemma, we show the relationship between $n(t)$ in (5.31) and output error $\tilde{y}_\epsilon$.

Lemma 2 *For the system in (5.3) and PAE as in (5.28), if*

$$\begin{aligned} n(t_1) &> \alpha\gamma + 2\sqrt{Q(\gamma+\epsilon)} + 2a^*_{max} \text{ or} \\ n(t_1) &< -\alpha\gamma - 2\sqrt{Q(\gamma+\epsilon)} - 2a^*_{max} \end{aligned}$$

for any positive constant γ at some time instant t_1, then there exists some $t_2 \in [t_1,\ t_1 + T_1]$ and

$$|\tilde{y}_\epsilon(t_2)| \geq \gamma,$$

where

$$T_1 = 2\sqrt{(\gamma+\epsilon)/Q}. \tag{5.34}$$

and Q is defined as in (5.33).

The following lemma shows that for any time interval T and output error criteria γ, the output convergence over interval T will happen.

Lemma 3 *For any T, there exists positive integer s such that*

$$|\tilde{y}_\epsilon| \leq \gamma \tag{5.35}$$

for any $t \in [sT,\ (s+1)T]$.

5.4 Discretized-parameter Polynomial Adaptive Estimator

In PAE discussed in section 5.3, the function f is approximated by a polynomial function and we assume the coefficients in (5.26) which includes c_i, $i = 1, .., N$ and a^*_{max} are known. To extend the PAE to arbitrary f, and when θ_0 is a vector, we will introduce a Discretized-parameter Polynomial Adaptive Estimator(DPAE) in this section.

5.4.1 DPAE Algorithm

For a compact unknown parameter region $\Omega = [\theta_{min},\ \theta_{max}]$, we discretize the unknown parameter region and represent them as a discrete set D of evenly distributed N points as

$$\begin{aligned} D &= [x_1, .., x_i, .., x_N] \\ x_i &= \theta_{min} + \frac{\theta_{max} - \theta_{min}}{2N} + \frac{\theta_{max} - \theta_{min}}{N}(i-1) \qquad i = 1, .., N. \end{aligned} \tag{5.36}$$

The minimum distance l of point $\theta \in \Omega$ towards the set D follows

$$\begin{aligned} l(\theta) &= \min_{x \in D} ||\theta - x|| \\ d(\theta) &= arg \min_{x \in D} ||\theta - x|| \end{aligned}$$

where $d(\theta)$ is the projection of θ on D and

$$l(\theta) \le \frac{\theta_{max} - \theta_{min}}{2N}. \tag{5.37}$$

From the Lipschitz property of function f and therefore $f(y, u, \theta_0)$, it follows from (5.4) and (5.37) that

$$|f(y, u, \theta) - f(y, u, d(\theta))| \le \frac{B(\theta_{max} - \theta_{min})}{2N}. \tag{5.38}$$

Choosing Lyapunov function same as discussed in section 2, we replace the unknown parameter region Ω with discrete set D and it follows from (5.38) that the new system is

$$\begin{aligned} \dot{y} &= -\alpha y + f(y, u, \bar{\theta}_0) + r \\ r &\le a^*_{max} \\ a^*_{max} &= \frac{B(\theta_{max} - \theta_{min})}{2N} \\ \bar{\theta}_0 &\in \bar{\Theta} \subset D \end{aligned} \tag{5.39}$$

where D is defined as in (5.36), B is defined as in Assumption 1, and $\bar{\Theta}$ is the new unknown parameter set where new defined unknown parameter $\bar{\theta}_0$ belongs to, and is defined as

$$\bar{\Theta} = \{\theta \mid |f(y, u, \theta_0) - f(y, u, \theta)| \le a^*_{max}, \forall\, y, u, \theta \in D\} \tag{5.40}$$

Our goal is to construct an estimation set $\widehat{\Theta}$ which can estimate $\bar{\Theta} \subset D$. Combining (5.38) and (5.40), it follows that

$$d(\theta_0) \in \bar{\Theta}, \qquad \forall\, \theta_0 \in \Theta$$

even $\bar{\Theta}$ may also include other points.

For this new system in (5.39), we choose a $N-1$ order PAE which satisfies

$$\begin{aligned} \phi_0^* - \sum_{i=1}^{N-1} \phi_i^* g_i(\widehat{\theta}_i - \theta) &= f(y, u, \theta) \qquad \forall \theta \in D \\ a^* &= a^*_{max} \end{aligned} \tag{5.41}$$

and construct the DPAE algorithm exactly same with PAE as in section 5.3 except the determination of ϕ^*, which needs to satisfy (5.41). To satisfy (5.41), we choose

$$\phi^* \ = \ A^{-1}C$$

where A is an N by N matrix given by

$$A \ = \ \begin{bmatrix} 1 & .. & .. & :: \\ : & :: & : & :: \\ 1 & .. & a_{ij} & .. \\ : & :: & : & :: \end{bmatrix} \qquad (5.42)$$

with the ith row and jth column element a_{ij} as

$$\begin{aligned} a_{i1} &= 1 \qquad 1 \le i \le N \\ a_{ij} &= -g_{j-1}(\widehat{\theta}_{j-1} - x_i) \qquad 1 \le i \le N, 2 \le j \le N \end{aligned}$$

where g_i is defined as in (5.14) and C is an N by 1 vector given by

$$C = [f(y,u,x_1) \, .. \, f(y,u,x_i) \, .. f(y,u,x_N)]^T. \qquad (5.43)$$

with the ith element b_i as

$$b_i = f(y,u,x_i).$$

It is straightforward to show that such choice of A and B satisfies equation (5.41). We could check easily that the calculation of ϕ^* and a^* in DPAE algorithm satisfies (5.12) from (5.41). To guarantee that the above DPAE has the same properties and lemmas as PAE in section 5.3, one requirement is that A is full rank, which is stated in the following lemma.

Lemma 4 *The matrix A as defined in (5.42) is full rank with D chosen as in (5.36) and g_i as defined in (5.14).*

The proof of Lemma 4 can be found in the Appendix.

Because A is full rank, all the conditions in DPAE is the same as those with PAE as in section 5.3 and therefore all the properties and lemmas in section 5.3.1 can be proved in a similar manner with $\bar{\theta}_0 \in \bar{\Theta}$ instead of $\theta_0 \in \Theta$. Therefore, bounds in DPAE like Q in (5.33) and T_1 in (5.34) follow in a similar way as in PAE. We state the complete DPAE algorithm below:

For any positive number γ, ϵ and T,

$$\begin{aligned}
\dot{\hat{y}} &= -\alpha(\hat{y} - \epsilon sat(\frac{\tilde{y}}{\epsilon})) + \phi_0^* - a^* sat(\frac{\tilde{y}}{\epsilon}) \\
\dot{\hat{\theta}}_i &= -\tilde{y}_\epsilon \phi_i^*, \qquad i = 1, ..., N-1 \\
\tilde{y} &= \hat{y} - y \\
\tilde{y}_\epsilon &= \tilde{y} - \epsilon sat\left(\frac{\tilde{y}}{\epsilon}\right) \\
a^* &= a_{max}^* \\
\phi^* &= A^{-1}C \\
\hat{\Theta} &= \{\theta | \theta \in D, \phi_0^*(\tau_1) - \beta \leq f(y(\tau_1), u(\tau_1), \theta) \leq \phi_0^*(\tau_1) + \beta, \forall\, \tau_1 \in [t_1, t_1 + T], \\
&\quad |\tilde{y}_\epsilon(\tau_2)| \leq \gamma, \forall\, \tau_2 \in [t_1, t_1 + T + T_1]\} \qquad (5.44)
\end{aligned}$$

where

$$\begin{aligned}
\beta &= \alpha\gamma + 2\sqrt{Q(\gamma + \epsilon)} + 2a_{max}^*, \\
\phi^* &= [\phi_0^*,\ \phi_1^*, ..., \phi_{N-1}^*]^T,
\end{aligned}$$

$sat(.)$ denote the saturation function, A, C and a_{max}^* are defined in (5.42), (5.43), and (5.39) respectively.

Assume that $\hat{\theta} \in \hat{\Theta}$, first we need to find a time interval $[t_1,\ t_1 + T + T_1]$ where the output error convergence happens, i.e.

$$|\tilde{y}_\epsilon| \leq \gamma \qquad (5.45)$$

over this interval. Then, $\hat{\Theta}$ includes the set of all points in Ω which satisfies

$$\phi_0^*(\tau_1) - \beta \leq f(y(\tau_1), u(\tau_1), \theta) \leq \phi_0^*(t) + \beta, \forall\, \tau_1 \in [t_1, t_1 + T].$$

Lemma 3 implies that the output convergence will always happen, which means there will always exist time interval $[t_1,\ t_1 + T + T_1]$ where (5.45) holds. From Lemma 2, it follows that

$$\phi_0^*(\tau_1) - \beta \leq f(y(\tau_1), u(\tau_1), \theta_0) \leq \phi_0^*(t) + \beta, \forall\, \tau_1 \in [t_1, t_1 + T]. \tag{5.46}$$

Combining (5.46) and the definition of $\widehat{\Theta}$ in (5.44), we have that

$$\theta_0 \in \widehat{\Theta}$$

and hence

$$\Theta \subseteq \widehat{\Theta}.$$

5.4.2 Extension to Higher Dimension

One important feature in the DPAE algorithm is that when the unknown parameter set is a discretized set $X \in \mathbb{R}$, what really matter are the values of $f(y, u, x)$ where $x \in X$ but not the locations of x in the $\mathbb{R}$ axis. It means that even in section 5.4.1 we choose x as its original value, we also can adjust the distance between different x or even adjust the sequence of x just if we keep a mapping between points x with its original points in Ω space. The auxiliary estimates will run in the new X space and finally $\widehat{\Theta}$ will give some x where $f(y, u, x)$ matches $f(y, u, \theta_0)$. When we map x back to its original value in Ω, we still have an estimation of θ_0. What is affected here is the Lipschitz bounds of the underlying function in X. If we put two points with different $f(y, u, x)$ close to each other, it implies that we have a very large B. From the discussion above, we notice that if we apply DPAE, now that we use a discretized set D to represent the compact unknown parameter set Ω, it does not matter whether Ω belongs to $\mathbb{R}$ or $\mathbb{R}^n$ and we can extend the DPAE into higher dimension of unknown parameters easily.

For an unknown parameter set $\Omega \in \mathbb{R}^n$ and any given precision a^*_{max}, we can construct a discrete set $D \in \Omega$ with N points and for any $\theta \in \Omega$, there exists $D_i \in D$ such that

$$|f(y, u, \theta) - f(y, u, D_i)| \leq a^*_{max}.$$

The new unknown parameter set $\bar{\Theta}$ is defined as in (5.40). The most common choice of D is the grid points in Ω. From Assumption 1, for any a^*_{max}, we can find such a discrete set D while N increases as a^*_{max} decreases. After that, we map the discrete set D into a discrete set $X \in \mathbb{R}$ with the same data points and establish a one to one mapping between these two sets and we assume that

$$f(y, u, x_i) = f(y, u, D_i) \tag{5.47}$$

where x_i are the mapping of D_i on X. We perform the same DPAE as in section 5.4.1 with X as the unknown parameter set instead of D. In DPAE algorithm we note that we just need the value of $f(y, u, x_i)$ which can be achieved from (5.47) and there is no need for $f(y, u, x)$ where $x \notin X$. That is also the reason why we can choose X freely.

When we get an estimation $\widehat{\Theta}$ in X, performing simple mapping, it is equivalent to an estimation in D where

$$\begin{aligned} \widehat{\Theta} &= \{\theta | \theta \in D, \phi^*_0(\tau_1) - \beta \leq f(y(\tau_1), u(\tau_1), \theta) \leq \phi^*_0(\tau_1) + \beta, \forall\, \tau_1 \in [t_1, t_1 + T], \\ &\quad |\tilde{y}_\epsilon(\tau_2)| \leq \gamma, \forall\, \tau_2 \in [t_1, t_1 + T + T_1]\}. \end{aligned}$$

Therefore, DPAE can be applied if unknown parameters in higher dimension nearly the same way as for one-dimension.

When system variables is higher dimension, i.e.

$$\dot{Y} = F(Y, u, \theta_0)$$

where $Y \in \mathbb{R}^L$ are measurable, we can treat them as L one-dimension system variable systems and apply the DPAE algorithm above to perform the parameter estimation. By this way, we can extend the DPAE algorithm to a more general Lipschitz nonlinear dynamic system.

5.5 Nonlinear Persistent Excitation Condition

In previous sections, we introduced PAE and DPAE algorithms which generate $\widehat{\Theta}$ which estimate the set of unknown parameters Θ, and showed that $\Theta \in \widehat{\Theta}$. In this section, we will establish the Nonlinear Persistent Excitation condition which guarantee the convergence of $\widehat{\Theta}$ to Θ in PAE and DPAE. The relationship between LPE and NLPE is also discussed in section 5.5.2.

5.5.1 Nonlinear Persistent Excitation Condition

First, we will define the distance between two sets. For two sets Θ and $\widehat{\Theta}$, we define the distance $||.||_d$ between these two sets as below.

Definition 3

$$||\widehat{\Theta} - \Theta||_d = \max_{\widehat{\theta}\in\widehat{\Theta}} \min_{\theta\in\Theta} ||\widehat{\theta} - \theta||.$$

It can be checked easily that

$$||\widehat{\Theta} - \Theta||_d = ||\Theta - \widehat{\Theta}||_d.$$

$||\widehat{\Theta} - \Theta||_d = \epsilon_0$ has two meanings. One is that for every point $\theta \in \Theta$, there is a point $\widehat{\theta} \in \widehat{\Theta}$ such that

$$||\widehat{\theta} - \theta|| \leq \epsilon_0$$

and another is that there is no point $\widehat{\theta} \in \widehat{\Theta}$ such that

$$||\widehat{\theta} - \theta|| > \epsilon_0 \qquad \forall\, \theta \in \Theta.$$

Global convergence of PAE and DPAE is said to follow if

$$||\widehat{\Theta} - \Theta||_d \to 0.$$

Now we introduce the NLPE condition which can guarantee the convergence of PAE and DPAE.

Definition 4 NLPE: *For problem formulation as in (5.3) under assumptions 1-4,* y, u *is said to have nonlinearly persistent excitation if for any* t*, there exists time constant* T*,* ϵ_0 *and a time instant* $t_1 \in [t,\ t+T]$ *such that*

$$|\,(f(y(t_1), u(t_1), \theta) - f(y(t_1), u(t_1), \theta_0))\,| \geq \epsilon_0 \min_{\theta_0 \in \Omega} ||\theta - \theta_0|| \qquad \forall \theta_0 \in \Theta.$$

In the following, we will prove that under NLPE, DPAE can lead to global convergence. For PAE, just the choice of polynomial estimator and ϕ^* has the ability to make a^* approaches 0, the global convergence can be proved easily using a similar way. We use DPAE as the example just because DPAE involves set $\bar{\Theta}$ and it is more difficult than PAE to prove the convergence.

Theorem 1 *For problem formulation as in (5.3) under assumptions 1-4, under NLPE condition as in definition 4, for any* ϵ_1*, there exists a DPAE as in (5.44) which leads to*

$$||\widehat{\Theta} - \Theta||_d \leq \epsilon_1. \tag{5.48}$$

The proof of Theorem 1 can be found in Appendix.

5.5.2 Comparison to LPE

For linear systems, when the system variables u is Lipschitz w.r.t. time t, $u(t)$ will change gradually and any error introduced by $|\theta_0^T u - \theta^T u|$ will not disappear soon and it will cause perturbation in the integral of the error. Thus the Linear Persistent Excitation is transformed into the following one as in [2].

Definition 5 LPE: *Assume* u *is Lipschitz w.r.t. time* t*,* u *is said to be linearly persistent excitation if for any* t*, there exists time constant* T*,* ϵ_0 *and a time instant* $t_1 \in [t,\ t+T]$ *such that*

$$|\theta_0^T u - \theta^T \phi| \geq \epsilon_0 ||\theta - \theta_0||.$$

Remark 6: We notice here that when objective function is linear, the NLPE is just transformed into LPE. The difference in right part is just because for linear function, there is only one true unknown parameter. In fact, Linear Persistent Estimator can also be treated as a first order PAE.

5.6 Simulation Results

We consider a simple example with nonlinearly-parameterized parameter such as

$$\dot{y} = -4y + u^2\theta_0 + u^3\theta_0^2$$

with the true unknown parameter $\theta_0 = 1$. The known parameter region $\Omega = [0, \ \ 2]$ where $\theta_0 \in \Omega$. The input signal u is

$$u(t) = \sin(2t + \pi/2).$$

We choose Lyapunov function as

$$V = \frac{1}{2}\tilde{y}^2 + \frac{1}{2}\tilde{\theta}_1^2 + \frac{1}{2}\tilde{\theta}_2^2 + \frac{1}{3}k\tilde{\theta}_2^3.$$

To design the Polynomial Adaptive Estimator, we first will determine the value of k according to the discussion as in previous section. Choose $k = 0.2$ and it is shown that it satisfies the requirement as in (5.8). The PAE estimator algorithm is stated below:

$$\begin{aligned}
\dot{\hat{y}} &= -4\hat{y} + \phi_0^* \\
\dot{\hat{\theta}}_1 &= -\tilde{y}\phi_1^* \\
\dot{\hat{\theta}}_2 &= -\tilde{y}\phi_2^* \\
\phi_2^* &= -u^3/k \\
\phi_1^* &= u^2 - \phi_2^* - 2k\theta_2\phi_2^* \\
\phi_0^* &= \phi_1^*\theta_1 + \phi_2^*\theta_2 + k\theta_2^2\phi_2^* \\
k &= 0.2.
\end{aligned}$$

The simulation results are shown in Figures 5-1-5-3. Figure 5-1 shows the output error $\tilde{y}_\epsilon$ and Figure 5-2 shows the trajectories of estimates $\widehat{\theta}_1$ and $\widehat{\theta}_2$. Because it can be checked easily that the input signal satisfies the NLPE condition, it has the ability to identify the true parameters. Convergent auxiliary estimates indicate the common convergent value is the true unknown parameter. In Figure 5-3, lyapunov function V is plotted.

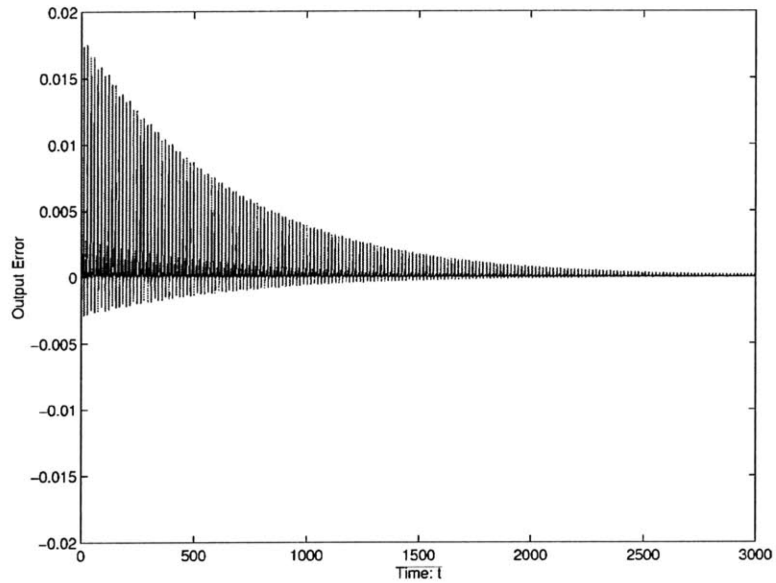

Figure 5-1: Output error $\tilde{y}_\epsilon$

5.7 Summary

In this chapter, we establish the structure of Polynomial Adaptive Estimator(PAE) and its various implementation PAE, and DPAE which are used to estimate unknown parameters in a general Lipschitz continuous dynamic system. The related Nonlinear Persistent Excitation is also established and the comparison to Linear Persistent Excitation is discussed. It has been shown that the PAE and DPAE have the ability to estimate the unknown parameters globally if we can make a^*_{max} approach 0.

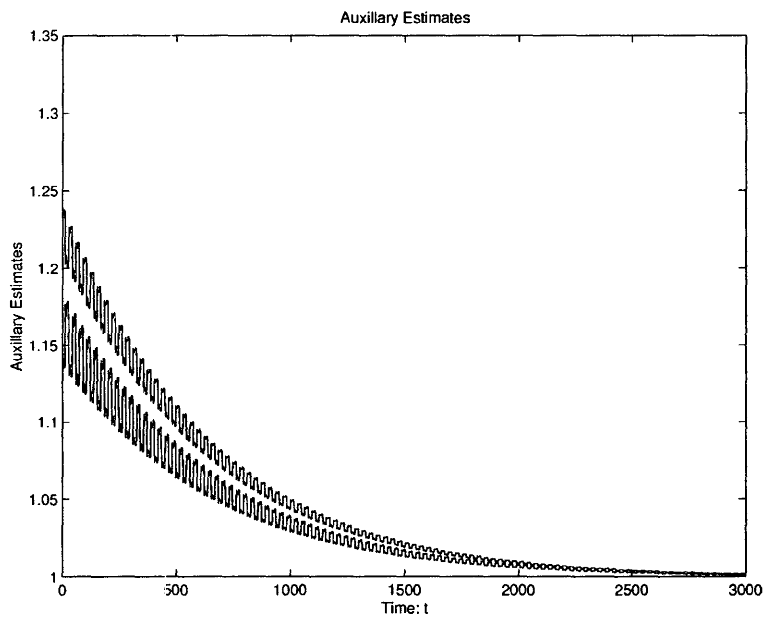

Figure 5-2: Trajectory of auxiliary estimates $\widehat{\theta}_1$ and $\widehat{\theta}_2$

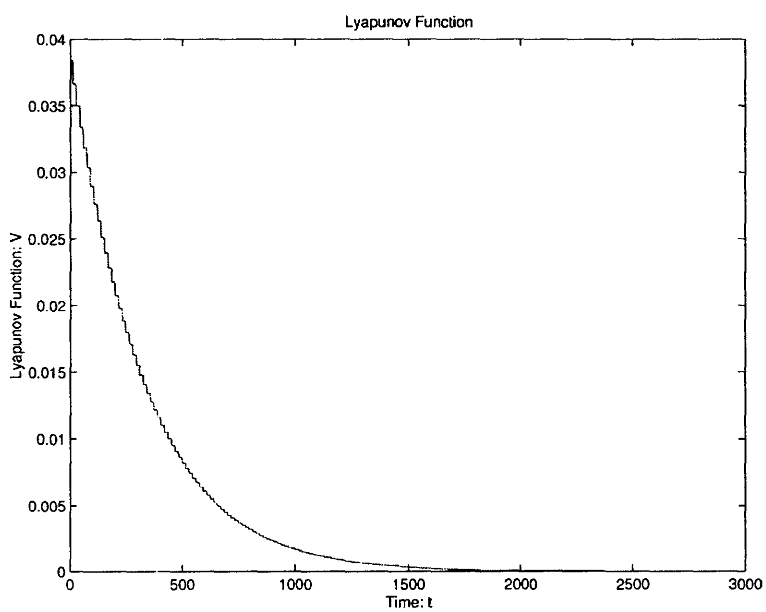

Figure 5-3: Trajectory of Lyapunov function $V(t)$

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5.8 Appendix

Proof of Property 1: In PAE, ϕ^* is achieved by

$$\phi^* = A^{-1}B.$$

From Assumption 3, B is bounded. We note that A is a regular full rank matrix not approaching singular. Thus ϕ^* must be bounded. •

Proof of Property 2: From Assumption 2 and 3, u and y is Lipschitz w.r.t. t. In PAE, it follows from Assumptions 1-3 such that the function f will not change abruptly. So the polynomial coefficients c_i of f in vector C will change gradually and be Lipschitz w.r.t. t. ϕ^* is bounded means that $\widehat{\theta}_i$ is also Lipschitz w.r.t. t and therefore the elements of matrix A is Lipschitz w.r.t. time. ϕ_0^* is calculated by these quantities and therefore is Lipschitz w.r.t. t. •

Proof of Property 4: From Assumption 3 and 4, it follows that $f(y, u, \theta_0)$ is bounded. From Property 1, it follows ϕ_0^* is bounded. Property 3 shows that $a^* sat(\frac{\tilde{y}}{\epsilon})$ is bounded by a^*_{max}. Combing all these, it shows that the bound of $m(t)$ exists and we denote it as M. •

Proof of Property 5: From Assumptions 1-4, it follows that $f(y, u, \theta_0)$ is Lipschitz w.r.t. t and it can be expressed as

$$|f(y(t+\tau), u(t+\tau), \theta_0) - f(y(t), u(t), \theta_0)| \leq Q_2\tau \tag{5.49}$$

where

$$Q_2 = B(U + F).$$

Combining (5.49) and Property 2, Property 5 is established. •

Proof of Lemma 2: Firstly, let us consider situation where

$$n(t_1) > \alpha\gamma + 2\sqrt{Q(\gamma + \epsilon)} + 2a^*_{max}. \tag{5.50}$$

It follows from (5.32) that

$$n(t_1 + \tau) > n(t_1) - Q\tau. \tag{5.51}$$

Combining (5.26) and (5.28), it follows that

$$\dot{\tilde{y}}_\epsilon = -\alpha\tilde{y}_\epsilon + \phi_0^* - f(y, u, \theta_0) - r - a^* sat(\frac{\tilde{y}}{\epsilon}) \tag{5.52}$$

where

$$|r| \leq a^*_{max}. \tag{5.53}$$

Combining (5.50), (5.51), and (5.53), we have

$$n(t_1 + \tau) - a^*_{max} > \alpha\gamma + 2\sqrt{Q(\gamma + \epsilon)} - Q\tau \qquad \forall\, \tau \geq 0. \tag{5.54}$$

It follows Property 3 and (5.54) that

$$n(t_1 + \tau) - a^* sat(\frac{\tilde{y}}{\epsilon}) > \alpha\gamma + 2\sqrt{Q(\gamma + \epsilon)} - Q\tau \qquad \forall\, \tau \geq 0. \tag{5.55}$$

If there exists some time constant $t_2 \in [t_1,\ t_1 + T']$ such that

$$\tilde{y}_\epsilon(t_2) \geq \gamma, \tag{5.56}$$

then the lemma is true. If (5.56) is not true, which means

$$\tilde{y}_\epsilon(t_2) \leq \gamma, \qquad \forall\, t \in [t_1,\ t_1 + T'], \tag{5.57}$$

then it follows from (5.55) that

$$-\alpha\tilde{y}_\epsilon(t_1 + \tau) + n(t_1 + \tau) - r - a^* sat(\frac{\tilde{y}}{\epsilon}) \geq 2\sqrt{Q(\gamma + \epsilon)} - Q\tau \qquad \forall\, \tau \geq 0. \tag{5.58}$$

Integrating (5.52) from t_1 to $t_1 + T'$ and it follows from (5.58) that

$$\begin{aligned} \tilde{y}(t_1 + T') - \tilde{y}(t_1) &= \int_0^{T'} \left(-\alpha\tilde{y}_\epsilon(t_1 + \tau) + n(t_1 + \tau) - r - a^* sat(\frac{\tilde{y}}{\epsilon})\right) d\tau \\ &\geq \int_0^{T'} \left(2\sqrt{Q(\gamma + \epsilon)} - Q\tau\right) d\tau \\ &= 2(\gamma + \epsilon). \end{aligned} \tag{5.59}$$

Equation (5.57) implis that

$$\tilde{y}(t_1) > -\gamma - \epsilon \tag{5.60}$$

otherwise $|\tilde{y}_\epsilon(t_1)| \geq \gamma$. Combining (5.59) and (5.60), it follows that

$$\tilde{y}(t_1 + T') \geq \gamma + \epsilon$$

and therefore

$$|\tilde{y}_\epsilon(t_1)| \geq \gamma. \tag{5.61}$$

Equation (5.61) contradicts (5.57) and it proves that the lemma is true. For situation where

$$n(t_1) < -\alpha\gamma - 2\sqrt{Q(\gamma+\epsilon)} - 2a^*_{max}, \tag{5.62}$$

similar results follows. •

Proof of Lemma 3: If there exists some s which satisfies (5.35), Lemma 3 is proved. If there does not exists such s, from Lemma 1, Lyapunov function will decrease a small amount S for every time interval where $\tilde{y}_\epsilon \geq \gamma$ at some time instant. Because $V(0)$ is finite, at most after $[V(0)/S]$ times (5.35) will happen. •

Proof of Lemma 4: Performing linear transformation to matrix will not affect its rankness. In what follows, we show that by a series of column scale and add/subtract operation, matrix A can be transformed into a Vendemonts's matrix which is known to be full rank.

We denote the ith column of A as A_i. First, let us consider A_2 which is

$$[-(\widehat{\theta}_1 - x_1) \; .. \; -(\widehat{\theta}_1 - x_i) \; .. \; -(\widehat{\theta}_1 - x_N)]^T.$$

Add $\widehat{\theta}_1 A_1$ from A_2 and the new column 2 is

$$\bar{A}_2 = [x_1 \; .. \; x_i \; .. \; x_N]^T.$$

We can continue this process through column 3 to N. For $(j+1)$th column, we assume that the columns through 1 to j has already been transformed into

$$[\bar{A}_1 \; .. \; \bar{A}_j] = \begin{bmatrix} 1 & x_1 & .. & x_1^{j-1} \\ : & : & :: & : \\ 1 & x_i & .. & x_i^{j-1} \\ : & : & :: & : \\ 1 & x_N & .. & x_N^{j-1} \end{bmatrix}$$

A_{j+1} can be expressed as sum of vectors $\bar{A}_1$ through $\bar{A}_i$ with coefficients as function of $\widehat{\theta}_i$.

Therefore, by subtract weighted $\bar{A}_1$ through $\bar{A}_i$ from A_{i+1}, the new $i+1$th column is

$$\bar{A}_{i+1} = [x_1^j \ .. \ x_i^j \ .. \ x_N^j]^T.$$

Repeat this process until the last column, and by simple matrix operation, the new matrix becomes

$$[\bar{A}_1 \ .. \ \bar{A}_N] \quad = \quad \begin{bmatrix} 1 & x_1 & .. & x_1^{N-1} \\ : & : & :: & : \\ 1 & x_i & .. & x_i^{N-1} \\ : & : & :: & : \\ 1 & x_N & .. & x_N^{N-1} \end{bmatrix} \tag{5.63}$$

Since (5.63) is Vandemeonts' matrix and is full rank, A if of full rank as well. •

Proof of Theorem 1: We choose parameter T in DPAE same as the T in the NLPE condition which is defined in definition 4. For problem formulation in (5.39) and DPAE as in (5.44), it follows from Lemma 3 that the output error $\tilde{y}_\epsilon$ will converge for some interval $[t, \ t+T+T_1]$ for any γ where

$$|\tilde{y}_\epsilon(\tau_2)| \leq \gamma, \forall \ \tau_2 \in [t, t+T+T_1].$$

When output convergence happens we construct the $\widehat{\Theta}$ by

$$\widehat{\Theta} = \{\theta | \theta \in D, \phi_0^*(\tau_1) - \beta \leq f(y(\tau_1), u(\tau_1), \theta) \leq \phi_0^*(\tau_1) + \beta, \forall \ \tau_1 \in [t, t+T]\} \tag{5.64}$$

where

$$\beta = \alpha\gamma + 2\sqrt{Q(\gamma+\epsilon)} + 2a_{max}^*.$$

It follows from Lemma 2 that

$$|f(y(\tau_1), u(\tau_1), \bar{\theta}) - \phi_0^*| \leq \beta \qquad \forall \ \tau_1 \in [t, t+T] \tag{5.65}$$

for any $\bar{\theta} \in \bar{\Theta}$. From NLPE condition, for any $\theta \in \widehat{\Theta}$ there exists $t_1 \in [t, \ t+T]$ such that

$$|f(y(t_1), u(t_1), \theta) - f(y(t_1), u(t_1), \theta_0)| \geq \epsilon_0 \min_{\theta_0 \in \Omega^0} \|\theta - \theta_0\| \qquad \forall \theta_0 \in \Theta. \tag{5.66}$$

It follows from (5.64) and (5.65), respectively, that

$$\begin{aligned} |f(y(t_1), u(t_1), \theta) - \phi_0^*(t_1)| &\leq \beta, \quad \forall \theta \in \widehat{\Theta} \\ |f(y(t_1), u(t_1), \bar{\theta}) - \phi_0^*(t_1)| &\leq \beta, \quad \forall \bar{\theta} \in \bar{\Theta}. \end{aligned} \tag{5.67}$$

From (5.67), we have

$$|f(y(t_1), u(t_1), \theta) - f(y(t_1), u(t_1), \bar{\theta})| \leq 2\beta \tag{5.68}$$

for any $\theta \in \widehat{\Theta}$ and $\bar{\theta} \in \bar{\Theta}$. It follows from equation (5.40) that

$$|f(y(t_1), u(t_1), \theta_0) - f(y(t_1), u(t_1), \bar{\theta})| \leq a^*_{max} \tag{5.69}$$

for any $\theta_0 \in \Theta$ and $\bar{\theta} \in \bar{\Theta}$. Combining (5.68) and (5.69), it follows that

$$|f(y(t_1), u(t_1), \theta_0) - f(y(t_1), u(t_1), \theta)| \leq 2\beta + a^*_{max} \tag{5.70}$$

for any $\theta_0 \in \Theta$ and $\theta \in \widehat{\Theta}$. Combining (5.66) and (5.70), it follows that for any $\theta \in \widehat{\Theta}$

$$\min_{\theta_0 \in \Omega^0} \|\theta - \theta_0\| \leq \frac{2\beta + a^*_{max}}{\epsilon_0} \tag{5.71}$$

which implies

$$\|\widehat{\Theta} - \Theta\| \leq \frac{2\beta + a^*_{max}}{\epsilon_0}.$$

Now that we can choose γ, ϵ and N to make a^*_{max} and β arbitrary small, when we choose $2\beta + a^*_{max} = \epsilon_0\epsilon_1$, (5.48) is satisfied and this proves Theorem 1. •

Chapter 6

Hierarchical Polynomial Adaptive Estimator

6.1 Introduction

Although Linear Adaptive Estimator and its related Linear Persistent Excitation(LPE) Condition have been established for many years, the corresponding algorithm for general nonlinearly parameterized dynamic system is still out of reach. Even a lot of efforts have been made, such as min-max algorithm and hierarchical min-max algorithm, there still exist some restrictions which prevent them to be general approaches for nonlinearly parameterized systems. A stability framework has been established for studying estimation and control of nonlinearly parameterized systems in [1]. In [2], the min-max algorithm in [1] is further extended into a hierarchical one and its associated Nonlinear Persistent Excitation Condition associated is established. No matter in min-max algorithm or its hierarchical one, it involves a nonlinear optimization problem at every time step. However, the computation complexity is still not the main disadvantage which prevent these algorithms to becomes a general solution for nonlinear parameterized system if we notice that they can not guarantee parameter convergence. The Nonlinear Persistent Excitation (NLPE) condition established in [2] describes the condition under which the hierarchical min-max algorithm achieves parameter convergence and it turns out this condition consists of too rigorous requirements. First, they require the input u to be "ergodic", which means that u must keep revisiting some interesting points. This condition is not required in LPE and it will be hard to be satisfied in practical problems. If u belongs to higher dimension, it is possible that u will never come back to same values even it is bounded. It makes situation worse if the nonlinear part contains both u and y now that there is no guarantee that (u, y) can be "ergodic" even u is "ergodic". Secondly, unlike the LPE condition, the NLPE condition in [2] has an additional requirement which is a property needed to be satisfied by any subset of the compact set that the unknown parameters belong to. This condition is extremely difficult to be verified for a general nonlinear function and it is not true for all nonlinear functions even they are Lipschitz and continuous.

In [1], a new algorithm named Polynomial Adaptive Estimator(PAE) is proposed. In that chapter, it uses the technique of auxiliary estimates to form a polynomial approximation of nonlinear function instead of linear approximation as in Linear Adaptive Estimator and min-max algorithms. To make PAE a general approach for nonlinear Lipschitz continuous systems and not just polynomial functions, Discretized-parameter Polynomial Adaptive Estimator(DPAE) is proposed as a further development in [1]. The disadvantage of DPAE is the low computation efficiency. In this chapter, we will propose a Hierarchical Discretized-parameter Polynomial Adaptive Estimator (HDPAE) which has the same convergence property as DPAE however greatly reduce the computation complexity through the introduction of a hierarchical structure. Same as DPAE in [1], Nonlinear Persistent Excitation(NLPE) condition which is no more restrictive than LPE for linearly parameterized systems is established for HDPAE. In this chapter, we will show that under the NLPE condition the HDPAE can lead to global parameter convergence for any nonlinear dynamic system that is Lipschitz continuous over a compact unknown parameter region, which covers most of the practical systems that we encountered. With the introduction of HDPAE in this chapter, a computation efficient parameter estimation algorithm for general nonlinear parameterized system is established for the first time.

With a long history, all kinds of gradient algorithms are widely applied to estimate parameters in nonlinear systems just because there is no other choice. With the introduction of HDPAE, we will give a generalization of parameter estimation in both static and dynamic systems. Hierarchical Search method, the counterpart of HDPAE in the static domain, is proposed to estimate unknown parameter in Lipschitz continuous static systems. In the comparison among different algorithms, we clearly show that the gradient algorithms which just exploit the local information have no guarantee for global convergence for nonlinearly parameterized systems.

This chapter consists of two parts. Part I, which covers section 6.2 to 6.5, introduce the HDPAE algorithm for Lipschitz continuous dynamic systems. Section 6.2 gives the problem formulation. In section 6.3, HDPAE is given and the properties and lemmas of HDPAE are also proposed. Nonlinear Persistent Excitation(NLPE) condition under which HDPAE leads to global convergence is proposed in section 6.4. In section 6.5, simulation results of HDPAE is given. Part II consists of two sections. In section 6.6, different methods to estimate unknown parameters in static systems are given. They consist of gradient algorithm, brute search and Hierarchical Search algorithm. In section 6.7, generalization and comparison of different parameter estimation algorithms for both static and dynamic systems are given and it is shown that gradient algorithms do not apply to general nonlinearly parameterized systems.

6.2 Problem Formulation

We consider a problem formulation with one dimension state variable and the extension to higher dimension can be easily derived as in [1].

The problem considered is the estimation of unknown parameters in nonlinear systems of the form

$$\dot{y} = -\alpha y + f(y, u, \omega^*) \tag{6.1}$$

where $\omega^* \in \Omega^0 \subset \mathbb{R}^n$ are unknown parameters, $y \in \mathbb{R}$ is measurable state variable, $u \in \mathbb{R}^m$ includes inputs, measurable system variables and even system time t, Ω^0 is the known compact set where the unknown parameters belongs to. Without loss of generality, we assume Ω^0 is a cubic region, i.e.

$$\Omega^0 = \{\theta | \theta \in \mathbb{R}^n, \omega_{min_i} \leq \theta_i \leq \omega_{max_i}, \ i = 1, ..., n\}. \tag{6.2}$$

First, we note that problem formulation in (6.1) is a general problem formulation now that

$$\dot{y} = \bar{f}(y, u, \omega^*)$$

can be transformed into (6.1) with

$$f(y, u, \omega^*) = \alpha y + \bar{f}(y, u, \omega^*).$$

Secondly, we note that there exist multiple unknown parameters for nonlinear systems, which is different from linear ones. One simple example is to consider a periodic function where $f(y, u, \omega^*) = f(y, u, \omega^* + \Delta)$. In this situation, there is no way to distinguish ω^* with $\omega^* + \Delta$. Therefore, we denote Ω^* as the set of the unknown parameters where

$$\Omega^* = \{\omega \mid f(y,u,\omega) = f(y,u,\omega^*), \forall\, y,u,\omega \in \Omega^0\}. \tag{6.3}$$

Remark 1: We note that there is no way to distinguish the points in Ω^* just by input and output information because their performance are identical. Therefore, a general nonlinear parameter estimation algorithm, unlike the Linear Adaptive Estimator, must have the ability to identify all the points in Ω^*.

In the situations where just the value of $f(y,u,\omega)$ matters, we use ω^* to represent any point in Ω^* and we note that any result achieved for ω^* holds for any $\omega \in \Omega^*$.

About function f, we make the following assumptions.

Assumption 1: The function $f(y,u,\omega)$ is Lipschitz with its arguments $x = [y,u,\omega]^T$, i.e. there exists positive constant B such that

$$|f(x+\Delta x) - f(x)| \leq B\|\Delta x\|. \tag{6.4}$$

Assumption 2: Input signal $u(t)$ is Lipschitz with respect to time t, i.e. there exists constant U such that

$$\|u(t_1) - u(t_2)\| \leq U|t_1 - t_2|.$$

Assumption 3: Function f is bounded, i.e.

$$|f(y,u,\omega)| \leq F_1, \qquad \forall\, \omega \in \Omega^0.$$

Assumption 4: y is bounded by F_2.

Assumption 3 and 4 imply that $\dot{y}$, the derivative of state variable, is also bounded by

$$F = F_1 + \alpha F_2 \tag{6.5}$$

and therefore y is Lipschitz with respect to time and the maximum change rate of y is bounded by F. We define a Lipschitz continuous dynamic system as follows.

Definition 1 *System in (6.1) is a Lipschitz continuous system if it satisfies Assumptions* 1-4.

It is noted that nearly all the systems we encounter in practice are Lipschitz continuous. It just requires the signal change rate in a nonlinear dynamic system to be bounded. Throughout this chapter, we will restrict our attention to Lipschitz continuous systems.

6.3 Hierarchical Discretized-parameter Polynomial Adaptive Estimator (HDPAE)

In this section, the HDPAE is proposed. In section 6.3.1, we will introduce the method to discretize the unknown parameter region. In section 6.3.2, Discretized-parameter Polynomial Adaptive Estimator(DPAE), which is the element of HDPAE, is proposed. The properties of DPAE are summarized in section 6.3.3. The complete HDPAE is proposed in section 6.3.4.

6.3.1 Discretized-parameter Representation

One important property of Lipschitz continuous funtion is that discrete points can represent its adjacent region with finite precision. In section 6.3.1, we will give a systematic method of using discrete set to represent continuous parameter region.

Let us introduce several useful definitions. First, we define the distance of one point to a discrete set as follows.

Definition 2 *For any point $x_1 \in \mathbb{R}^n$ and a discrete set $X_2 \subset \mathbb{R}^n$, the distance of x_1 to X_2 is defined as*

$$\|x_1, X_2\|_d = \min_{x_2 \in X_2} \|x_1 - x_2\|. \tag{6.6}$$

Then, we define the distance between two discrete sets as follows.

Definition 3 *For any sets $X_1, X_2 \subset \mathbb{R}^n$, the distance between them is defined as*

$$\|X_1, X_2\|_d = \max_{x_1 \in X_1} \|x_1, X_2\|_d = \max_{x_1 \in X_1} \min_{x_2 \in X_2} \|x_1 - x_2\|. \tag{6.7}$$

We could check easily that the norm definition is communicative, i.e.

$$\|X_1, X_2\|_d = \|X_2, X_1\|_d. \tag{6.8}$$

Finally, for any discrete set Z and distance ϵ_0, we define its ϵ_0-net E as follows.

Definition 4 *The ϵ_0-net of a discrete set $Z \in \mathbb{R}^n$ is defined as*

$$E(Z, \epsilon_0) = \{x | x \in \mathbb{R}^n, \|x - Z\|_d \leq \epsilon_0\}. \tag{6.9}$$

For compact region Ω^0 and any distance d^0, we can find a discrete set Z^0 which satisfies

$$\Omega^0 \subseteq E(Z^0, d^0) \tag{6.10}$$

where the d^0-net of set Z^0 is defined as in (6.9). In what follows, we will show how to choose Z^0 which satisfies (6.10). In every dimension of $\mathbb{R}^n$, assume ith dimension, we choose a set $Z_i^0 \in \mathbb{R}$ which consists of J uniformly distributed points

$$Z_i^0 = \{x_{i1}, \ldots x_{ij}, \ldots x_{iJ}\} \tag{6.11}$$

where

$$x_{i(j+1)} - x_{ij} = d^0/\sqrt{n} \tag{6.12}$$

$$\omega_{min_i} < x_{i1} \leq \omega_{min_i} + d^0/\sqrt{n} \tag{6.13}$$

$$\omega_{max_i} - d^0/\sqrt{n} \leq x_{iJ} < \omega_{max_i}. \tag{6.14}$$

We could check easily that in $\mathbb{R}$ space

$$[\omega_{min_i}, \omega_{max_i}] \subseteq E(Z_i^0, d^0/\sqrt{n}). \tag{6.15}$$

In order to achieve an organized discrete set Z^0, we choose Z^0 as a grid in $\mathbb{R}^n$ as follows:

$$Z^0 = \{(x_1, .., x_i, .., x_n) | x_i \in Z_i^0\} \tag{6.16}$$

where Z_i^0 is defined as in (6.11). About Z^0 we have the following two properties.

Property 1

$$\Omega^0 \subseteq E(Z^0, d^0). \tag{6.17}$$

Proof of Property 1: For any $x = [x_1, .., x_i, .., x_n] \in \mathbb{R}^n$, it follows from (6.15) and (6.16) that there exists point $Z_x \in D$ such that

$$|Z_{xi} - x_i| \le d^0/\sqrt{n}, \qquad \forall i = 1, .., n. \tag{6.18}$$

Therefore, we have

$$\|x - Z_x\| \le d^0 \tag{6.19}$$

and property 1 is proved. •

Combining Property 1 and Assumption 1, the following property follows directly.

Property 2 *For any $x_1 \in \Omega^0$, there exists $x_2 \in Z^0$ such that*

$$|f(y, u, x_1) - f(y, u, x_2)| \le Bd^0 \tag{6.20}$$

for any y, u.

From Property 2, we note that for any true unknown parameter $\omega^* \in \Omega^* \subset \Omega^0$, there must exist some $z_0^* \in Z^0$ such that

$$|f(y, u, z_0^*) - f(y, u, \omega^*)| \le Bd^0 \qquad \forall y.u. \tag{6.21}$$

We denote the set of such z_0^* as Z_0^* which is stated as

$$Z_0^* = \{z \in Z^0| \ |f(y, u, z) - f(y, u, \omega^*)| \le Bd^0, \ \forall \ y, u \ \ \forall \ \omega^* \in \Omega^*\}. \tag{6.22}$$

Now we can transform problem formulation in (6.1) into a problem formulation with true unknown parameters Z_0^* as in (6.22) and unknown parameter region as a discrete set Z^0. It follows that

$$\begin{aligned} \dot{y} &= -\alpha y + f(y, u, z_0^*) + r(y, u, z_0^*) \\ z_0^* &\in Z_0^* \subseteq Z^0 && (6.23) \\ r(y, u, z_0^*) &\le a_{max}^* = Bd^0 && (6.24) \end{aligned}$$

where Z_0^* and Z^0 are defined as in (6.22) and (6.16).

We assume the discrete set Z^0 contains N^0 points. For unknown parameter region which belongs to a discrete set, what really matters are the value of $f(y, u, z)$ and it does not matter which space it belongs to originally. we map $Z^0 \in \mathbb{R}^n$ into a discrete set $\Theta^0 \in \mathbb{R}$ and establish a one-to-one mapping H between them, i.e.

$$\begin{aligned} Z^0 &= H(\Theta^0) \\ \Theta^0 &= H^{-1}(Z^0). \end{aligned} \tag{6.25}$$

We note that both Z^0 and Θ^0 contains N^0 discrete points and the commonly choice of Θ^0 is to distribute N^0 points uniformly between an interval $[\Theta^0_{min},\ \Theta^0_{max}]$, for example $[0,\ 1]$. Define

$$\Theta_0^* = H^{-1}(Z_0^*) \tag{6.26}$$

and the system in (6.24) is equivalent to the following system:

$$\begin{aligned} \dot{y} &= -\alpha y + f(y, u, \theta_0^*) + r \\ \theta_0^* &\in \Theta_0^* \subseteq \Theta^0 = [\theta_1, ..., \theta_{N^0}] \\ r &\leq a^*_{max} = Bd^0 \\ f(y, u, \theta_i) &= f(y, u, H(\theta_i)) \qquad i = 1, .., N^0 - 1. \end{aligned} \tag{6.27}$$

In section 6.3.2, we will give the DPAE algorithm corresponding to the system in (6.27).

6.3.2 DPAE

To estimate unknown parameter in (6.27), we need a DPAE with order $N^0 - 1$ and the dynamics of the DPAE is stated below.

$$\begin{aligned} \dot{\hat{y}} &= -\alpha(\hat{y} - \epsilon sat(\frac{\tilde{y}}{\epsilon})) + \phi_0^* - a^* sat(\frac{\tilde{y}}{\epsilon}) \\ \dot{\hat{\theta}}_i &= -\tilde{y}_\epsilon \phi_i^*, \qquad i = 1, ..., N^0 - 1 \end{aligned} \tag{6.28}$$

where

$$\tilde{y} = \hat{y} - y \qquad \tilde{y}_\epsilon = \tilde{y} - \epsilon sat\left(\frac{\tilde{y}}{\epsilon}\right) \tag{6.29}$$

ϵ is an arbitrary positive number, $sat(.)$ denote the saturation function and is given by $sat(x) = sign(x)$ if $|x| \geq 1$ and $sat(x) = x$ if $|x| < 1$, and the calculation of a^* and ϕ^* will be discussed later.

Combining (6.27) and (6.28), we rewrite the dynamics of the entire system as

$$\dot{\tilde{y}} = -\alpha\tilde{y}_\epsilon + \phi_0^* - f(y, u, \theta_0^*) - r - a^* sat(\frac{\tilde{y}}{\epsilon}) \tag{6.30}$$

$$\dot{\tilde{\theta}}_i = -\tilde{y}_\epsilon \phi_i^*, \qquad i = 1, .., N^0 - 1. \tag{6.31}$$

where

$$\tilde{\theta}_i = \widehat{\theta}_i - \theta_0^*.$$

Same as the DPAE proposed in [1], we will first construct a polynomial Lyapunov function which is stated in section 6.3.2

Polynomial Lyapunov Function

For system in (6.27) and the DPAE algorithm in (6.28), we introduce a polynomial Lyapunov function V as

$$V = \tilde{y}_\epsilon^2 + \sum_{i=1}^{N^0-1} p_i(\tilde{\theta}_i) \tag{6.32}$$

where $p_i(.)$ is a polynomial function. Therefore, the derivative of $p_i(.)$ is also a polynomial function and denoted as g_i where

$$g_i(x) = \frac{dp_i(x)}{dx}, \qquad \forall\, i = 1, ..., N^0 - 1. \tag{6.33}$$

For V to become a Lyapunov function, p_i needs to satisfy the following conditions

$$\begin{aligned}
(1) \quad & g_i(\tilde{\theta}_i) && < 0 \text{ if } \tilde{\theta}_i < 0 \\
(2) \quad & g_i(\tilde{\theta}_i) && > 0 \text{ if } \tilde{\theta}_i > 0 \\
(3) \quad & p_i(0) && = 0 \\
(4) \quad & g_i(0) && = 0
\end{aligned} \tag{6.34}$$

for any $i = 1, .., N^0 - 1$ and all possible values of $\tilde{\theta}_i$. If $p_i(\tilde{\theta}_i)$ satisfies (6.34), it can be shown easily that $p_i(\tilde{\theta}_i)$ is nonnegative with $p_i(\tilde{\theta}_i) = 0$ iff $\tilde{\theta}_i = 0$ and $p_i(\tilde{\theta}_i)$ increases as $|\tilde{\theta}_i|$ increases.

To make V a Lyapunov function, we need to make sure that $\dot{V}$ is nonpositive, i.e.

$$\dot{V} = -\alpha\tilde{y}_\epsilon^2 + \tilde{y}_\epsilon\left(\phi_0^* - f(y,u,\theta_0^*) - \sum_{i=1}^{N} g_i(\tilde{\theta}_i)\phi_i^* - r - a^* sat(\frac{\tilde{y}}{\epsilon})\right) \leq 0. \tag{6.35}$$

We note that if $\tilde{y}_\epsilon = 0$, inequality (6.35) holds always. If $\tilde{y}_\epsilon \neq 0$, it implies that $|\tilde{y}| > \epsilon$ and hence

$$sat(\frac{\tilde{y}}{\epsilon}) = sign(\tilde{y}_\epsilon).$$

Inequality (6.35) is equivalent to

$$\tilde{y}_\epsilon sign(\tilde{y}_\epsilon)\left(sign(\tilde{y}_\epsilon)(\phi_0^* - f(y,u,\theta_0^*) - \sum_{i=1}^{N^0-1} g_i(\tilde{\theta}_i)\phi_i^*) - rsign(\tilde{y}_\epsilon) - a^*\right) \leq 0. \tag{6.36}$$

Because $\tilde{y}_\epsilon sign(\tilde{y}_\epsilon) \geq 0$, to achieve (6.36), we need to choose ϕ^* and a^* as follows

$$\begin{aligned} \phi_0^* - \sum_{i=1}^{N^0-1} \phi_i^* g_i(\widehat{\theta}_i - \theta) &= f(y,u,\theta) \qquad \forall\theta \in \Theta^0 && (6.37)\\ a^* &= a_{max}^*. && (6.38) \end{aligned}$$

The choice of a^* is straightforward. If the choice of ϕ^* satisfies equation (6.38), it follows that

$$\dot{V} \leq -\alpha\tilde{y}_\epsilon^2 \leq 0 \tag{6.39}$$

and V serves as a Lyapunov function.

Let us firstly find Lyapunov function which satisfies (6.34). We choose

$$\begin{aligned} p_i(\tilde{\theta}_i) &= \frac{1}{i+1}\tilde{\theta}_i^{i+1} \qquad \text{if } i \text{ is odd;}\\ p_i(\tilde{\theta}_i) &= \frac{1}{i}\tilde{\theta}_i^i + \frac{k_i}{i+1}\tilde{\theta}_i^{i+1} \qquad \text{if } i \text{ is even} && (6.40) \end{aligned}$$

for $i = 1, .., N^0 - 1$, where k_i is to be chosen appropriately. The corresponding g_i follows

$$\begin{aligned} g_i(\tilde{\theta}_i) &= \tilde{\theta}_i^i \qquad \text{if } i \text{ is odd;}\\ g_i(\tilde{\theta}_i) &= \tilde{\theta}_i^{i-1} + k_i\tilde{\theta}_i^i \qquad \text{if } i \text{ is even.} && (6.41) \end{aligned}$$

In what follows we will show that (6.34) is satisfied with these choice of p_i. Conditions 3 and 4 follow immediately. Conditions 1 and 2 in (6.34) follow as well when i is odd, as does condition 2 in (6.34) when i is even. Hence, what remains to be shown is condition 1 when i is even, which is not true for any $\tilde{\theta}_i$, however the feature we can exploit is that the range of $\tilde{\theta}_i$ is constrained by Lyapunov function V defined as in (6.32) and we just need to choose k_i which makes condition 1 in (6.34) holds for any possible $\widehat{\theta}_i$. For any choice of initial $\widehat{\theta}_i$ and $\hat{y}$ at $t = 0$, the Lyapunov function is $V(0)$. From (6.39), it follows that

$$V(t) < V(0) \tag{6.42}$$

for any $t \geq 0$. Equation (6.42) implies that $\tilde{\theta}_i$ is bounded and the bounds can be calculated easily. Assume the lower bound of $\tilde{\theta}_i$ is $\tilde{\theta}_i^b$, to make condition 1 in (6.34) satisfied, we just need to choose k_i which satisfies

$$k_i < -\frac{1}{\tilde{\theta}_i^b}. \tag{6.43}$$

Choosing Lyapunov function V as in (6.40) and an appropriate k_i as in (6.43), (6.34) is satisfied and we establish a stability framework of the DPAE algorithm if (6.37) can be satisfied.

Implementation of DPAE

In DPAE, what remains is how to choose ϕ^* to make (6.37) satisfied, which is stated as follows:

$$\phi^* = A^{-1}C \tag{6.44}$$

where A is an N^0 by N^0 matrix given by

$$A = \begin{bmatrix} 1 & .. & .. & :: \\ : & :: & : & :: \\ 1 & .. & a_{ij} & .. \\ : & :: & : & :: \end{bmatrix} \tag{6.45}$$

with the ith row and jth column element a_{ij} as

$$\begin{aligned} a_{i1} &= 1 \qquad 1 \leq i \leq N^0 \\ a_{ij} &= -g_{j-1}(\widehat{\theta}_{j-1} - \theta_i) \qquad 1 \leq i \leq N^0, 2 \leq j \leq N^0 \end{aligned}$$

where g_i is defined as in (6.41), θ_i is defined as in (6.27), and C is an N^0 by 1 vector given by

$$C = [f(y, u, \theta_1) \, .. \, f(y, u, \theta_i) \, .. f(y, u, \theta_{N^0})]^T \tag{6.46}$$

with the ith element c_i as

$$c_i = f(y, u, \theta_i).$$

It is straightforward that such choices of A and B satisfy equation (6.37). In the lemma follows, we will show A is full rank and therefore we will not encounter singular problem in (6.44).

Lemma 1 *Matrix A as defined in (6.45) is full rank.*

Proof of Lemma 1 is in the Appendix.

6.3.3 Properties of PAE

In this section, we will establish some properties and lemmas related to DPAE proposed in section 6.3.2. All the proofs of the properties and lemmas are in the Appendix.

First we will establish some properties about ϕ^*. We will show that ϕ^* is bounded and Lipschitz w.r.t. time t.

Property 3

$$\phi^* \textit{ is bounded.} \tag{6.47}$$

Property 4

$$|\phi_0^*(t_2) - \phi_0^*(t_1)| \leq Q_1 |t_2 - t_1|. \tag{6.48}$$

In DPAE, ϕ_0^* is a known variable in the algorithm and the maximum change rate Q_1 can be measured and kept on line. ϕ^* is calculated by solving a group of linear equations and a^* will keep constant value a^*_{max}. About a^*_{max}, we have the following property.

Property 5

$$-a^*_{max} \leq a^* sat(\frac{\tilde{y}}{\epsilon}) \leq a^*_{max}. \tag{6.49}$$

The proof of this property is obvious now that $|sat(\frac{\tilde{y}}{\epsilon})| \leq 1$.

Define

$$m(t) = \phi_0^* - f(y, u, \theta_0^*) - a^* sat(\frac{\tilde{y}}{\epsilon}), \tag{6.50}$$

the following Property 6 shows that $m(t)$ is bounded.

Property 6 *There exists finite positive M such that*

$$|m(t)| \leq M \tag{6.51}$$

where $m(t)$ is defined as in (6.50).

Define

$$n(t) = \phi_0^* - f(y, u, \theta_0^*), \tag{6.52}$$

we conclude that $n(t)$ is Lipschitz w.r.t. t in the Property 7.

Property 7

$$|n(t+\tau) - n(t)| \leq Q|\tau| \tag{6.53}$$

where

$$Q = B(U + F) + Q_1, \tag{6.54}$$

with B, U, F, Q_1 defined as in Assumptions 1, 2, Eq. (6.5) and Property 4 respectively.

Remark 3: In fact, the estimator variables ϕ^*, a^* and $\widehat{\theta}$ are related through a non-singular matrix. Assumptions 1-3 state that signals u, y are Lipschitz w.r.t. t, which implies that all variables in the algorithm are Lipschitz w.r.t. time.

Next, we will show several lemmas related with the DPAE. In the following lemma, it is shown that when output error happens, the Lyapunov function will decrease.

Lemma 2 *For the system in (6.27) and DPAE in (6.28), if*

$$|\tilde{y}_\epsilon(t_1)| \geq \gamma, \tag{6.55}$$

then

$$V(t_1 + T') \leq V(t_1) - \frac{\alpha\gamma^3}{3(M + \alpha\gamma)} \tag{6.56}$$

where $T' = \gamma/(M + \alpha\gamma)$ and M is defined as in (6.51).

The proof of lemma 2 is shown in [2]. In the following Lemma, we show the relationship between $n(t)$ in (6.52) and the output error $\tilde{y}_\epsilon$.

Lemma 3 *For the system in (6.27) and PAE as in (6.28), if*

$$\begin{aligned} n(t_1) &> \alpha\gamma + 2\sqrt{Q(\gamma+\epsilon)} + 2a^*_{max} \text{ or} \\ n(t_1) &< -\alpha\gamma - 2\sqrt{Q(\gamma+\epsilon)} - 2a^*_{max} \end{aligned}$$

for any positive constant γ *at some time instant* t_1*, then there exists some* $t_2 \in [t_1,\ t_1 + T_1]$ *such that*

$$|\tilde{y}_\epsilon(t_2)| \geq \gamma,$$

where

$$T_1 = 2\sqrt{(\gamma+\epsilon)/Q}. \tag{6.57}$$

and Q *is defined in (6.54).*

The following lemma shows that for any time interval T and output error criteria γ, the output convergence over interval T will happen.

Lemma 4 *For any constant* T*,* γ*, there exists a finite positive integer* s *such that*

$$|\tilde{y}_\epsilon| \leq \gamma \tag{6.58}$$

for any $t \in [sT,\ (s+1)T]$.

In what follows, we will state the complete DPAE for problem formulation as in (6.27).

For any positive number γ, ϵ and T,

$$\begin{aligned} \dot{\hat{y}} &= -\alpha(\hat{y} - \epsilon sat(\frac{\tilde{y}}{\epsilon})) + \phi_0^* - a^* sat(\frac{\tilde{y}}{\epsilon}) \\ \dot{\hat{\theta}}_i &= -\tilde{y}_\epsilon \phi_i^*, \qquad i = 1, ..., N^0 - 1 \\ \tilde{y} &= \hat{y} - y \\ \tilde{y}_\epsilon &= \tilde{y} - \epsilon sat\left(\frac{\tilde{y}}{\epsilon}\right) \\ a^* &= a^*_{max} \\ \phi^* &= A^{-1}C \\ \hat{\Theta}^0 &= \Big\{\theta | \theta \in \Theta^0, \phi_0^*(\tau_1) - \beta \leq f(y(\tau_1), u(\tau_1), \theta) \leq \phi_0^*(t) + \beta, \forall\, \tau_1 \in [t_1, t_1 + T], \\ & \quad |\tilde{y}_\epsilon(\tau_2)| \leq \gamma, \forall\, \tau_2 \in [t_1, t_1 + T + T_1]\} \end{aligned} \tag{6.59}$$

where

$$\beta = \alpha\gamma + 2\sqrt{Q(\gamma+\epsilon)} + 2a^*_{max}, \tag{6.60}$$

$$\phi^* = [\phi^*_0, \phi^*_1, ..., \phi^*_{N^0-1}]^T, \tag{6.61}$$

$sat(.)$ denote the saturation function, A, C, Q, T_1 and a^*_{max} are defined in (6.45), (6.46), (6.54), (6.57) and (6.24) respectively. Here $\hat{\Theta}^0$ is the estimation of the true unknown parameter set Θ^*_0. When we are interested in $\mathbb{R}^n$ space, we need to map it back to obtain

$$\hat{Z}^0 = H^{-1}(\hat{\Theta}^0). \tag{6.62}$$

which is an estimation of Z^*_0. About the relationship between Ω^* and $\hat{Z}^0$, we have the following lemma which serves as the basis for a hierarchical iteration.

Theorem 1 *For any* $\omega^* \in \Omega^*$,

$$(i) \quad \omega^* \in E(\hat{Z}^0, d^0); \tag{6.63}$$

$$(ii) \quad \omega^* \notin E(Z^0/\hat{Z}^0, d^0). \tag{6.64}$$

Proof of Lemma 1: Lemma 4 implies that the output convergence will always happen, which means there exists time interval $[t_1, \ t_1 + T + T_1]$ where

$$|\tilde{y}_\epsilon| \leq \gamma. \tag{6.65}$$

During this convergence interval, it follows from lemma 3 that for any $\theta^*_0 \in \Theta^*_0$,

$$\phi^*_0(\tau_1) - \beta \leq f(y(\tau_1), u(\tau_1), \theta^*_0) \leq \phi^*_0(t) + \beta, \forall\, \tau_1 \in [t_1, t_1 + T]. \tag{6.66}$$

From the construction of $\hat{\Theta}^0$ as in (6.59), it follows that

$$\begin{aligned} \theta^*_0 &\in \widehat{\Theta}^0 \\ \theta^*_0 &\notin \Theta^0/\widehat{\Theta}^0. \end{aligned} \tag{6.67}$$

Because there exists one to one mapping between Θ^0 and Z^0, it follows from (6.26), (6.62) and

(6.67) that

$$\begin{aligned} z_0^* &\in Z(\widehat{\Theta}^0) \\ z_0^* &\notin Z^0/Z(\widehat{\Theta}^0) \end{aligned} \tag{6.68}$$

for any $z_0^* \in Z_0^*$. Combing Property 1 and (6.68), it follows that

$$\begin{aligned} \omega^* &\in E(\hat{Z}^0, d^0) \\ \omega^* &\notin E(Z^0/\hat{Z}^0, d^0) \end{aligned} \tag{6.69}$$

for any $\omega^* \in \Omega^*$. This proves the Theorem. •

6.3.4 Complete HDPAE

From Theorem 1, we can establish a systematic method to reduce the unknown parameter region and therefore establish the HDPAE. From $\hat{Z}^0$, a region Ω^1 is constructed as follows:

$$\Omega^1 = \{z \in \mathbb{R}^n | \; |z_i - \hat{z}_i^0| \le d^0/\sqrt{n} \text{ for all } i = 1, .., n, \; \forall \; \hat{z}^0 \in \hat{Z}^0\}. \tag{6.70}$$

We define the near complementaty set of Ω^1 in Ω^0 as

$$\bar{\Omega}^1 = \{z \in \mathbb{R}^n | \; |z_i - \hat{z}_i^0| \le d^0/\sqrt{n} \text{ for all } i = 1, .., n, \; \forall \; \hat{z}^0 \in Z^0/\hat{Z}^0\} \tag{6.71}$$

and we can easily verify that

$$\Omega^0 \subset \Omega^1 \bigcup \bar{\Omega}^1. \tag{6.72}$$

We note that $\bar{\Omega}^1$ is not exactly the complementary of Ω^1 because they share some of the common borders. For every single point $z \in \mathbb{R}^n$,

$$|z_i - \hat{z}_i^0| \le d^0/\sqrt{n}, \qquad \forall i = 1, .., n$$

means a cubic centered at z and

$$|z_i - \hat{z}_i^0| \le d^0$$

means a ball centered at z and contains the cubic exactly. Thus, the following property can be shown to be true straightforward.

Property 8

$$\Omega^1 \subset E(\hat{Z}^0, d^0) \tag{6.73}$$

$$\bar{\Omega}^1 \subset E(Z^0/\hat{Z}^0, d^0). \tag{6.74}$$

One corollary from Theorem 1 is as follows

Corollary 1

$$\omega^* \in \Omega^1, \qquad \forall\, \omega^* \in \Omega^* \tag{6.75}$$

where Θ^1 *is defined as in (6.70).*

Proof of Corollary 1: It follows from (6.74) and (ii) of Theorem 1 that

$$\omega^* \notin \bar{\Omega}^1, \qquad \forall\, \omega^* \in \Omega^*. \tag{6.76}$$

It follows from (6.72) that

$$\Omega^0/\bar{\Omega}^1 \subseteq \Omega^1. \tag{6.77}$$

Combining (6.76) and (6.77), it follows

$$\omega^* \in \Omega^1, \qquad \forall\, \omega^* \in \Omega^* \tag{6.78}$$

which proves the corollary. •

Corollary 1 establishes that we can reduce the unknown parameter region from Ω^0 to Ω^1, which leads to a hierarchical algorithm. For new unknown parameter region Ω^1, we reduce d^0 half to

$$d^1 = d^0/2 \tag{6.79}$$

and construct a new discrete set Z^1 as follows:

$$Z^1 = \{(z_1', .., z_i', .., z_n') \mid z_i' = z_i + d^0/(2*\sqrt{n}),\ \text{or}\ z_i' = z_i - d^0/(2*\sqrt{n}),\ \forall\, z = (z_1, .., z_i, .., z_n) \in \hat{Z}^0\}. \tag{6.80}$$

We note that for every point in $\hat{Z}^0$, we construct 2^n points. It can be verified easily that

$$\Omega^1 \subseteq E(Z^1, d^1). \tag{6.81}$$

Now we have exactly the same DPAE problem formulation as before except using Ω^1, d^1 and Z^1 to replace Ω^0, d^0 and Z^0. Repeating this iteration and we establish the HDPAE.

In what follows, we state the complete HDPAE algorithm for problem formulation as in (6.1).

Step 1: Choose positive constant d^0 arbitrarily, construct Z^0 as in (6.16). Set $j = 0$.

Step 2: Establish a one-to-one mapping as in (6.25) to map Z^i to Θ^i.

Step 3: Perform DPAE algorithm as in (6.59) with Z^0 and d^0 replaced by Z^j and d^j, which is stated as follows.

For any positive number γ, ϵ and T,

$$\begin{aligned}
\dot{\hat{y}} &= -\alpha(\hat{y} - \epsilon sat(\frac{\tilde{y}}{\epsilon})) + \phi_0^* - a^* sat(\frac{\tilde{y}}{\epsilon}) \\
\dot{\hat{\theta}}_i &= -\tilde{y}_\epsilon \phi_i^*, \qquad i = 1, ..., N-1 \\
\tilde{y} &= \hat{y} - y \\
\tilde{y}_\epsilon &= \tilde{y} - \epsilon sat\left(\frac{\tilde{y}}{\epsilon}\right) \\
a^* &= a^*_{max} \\
\phi^* &= A^{-1}C \\
\hat{\Theta}^i &= \Big\{\theta | \theta \in \Theta^i, \phi_0^*(\tau_1) - \beta \le f(y(\tau_1), u(\tau_1), \theta) \le \phi_0^*(t) + \beta, \forall\, \tau_1 \in [t_1, t_1 + T], \\
&\quad |\tilde{y}_\epsilon(\tau_2)| \le \gamma, \forall\, \tau_2 \in [t_1, t_1 + T + T_1]\}
\end{aligned} \tag{6.82}$$

where

$$\beta = \alpha\gamma + 2\sqrt{Q(\gamma+\epsilon)} + a^*_{max}, \tag{6.83}$$

$$\phi^* = [\phi_0^*,\ \phi_1^*, ..., \phi_{N-1}^*]^T, \tag{6.84}$$

$sat(.)$ denote the saturation function, A, C and a^*_{max} are defined in (6.45), (6.46), and (6.24) respectively.

Step 3: Let $d^{j+1} = d^j/2$ and construct new discrete set Z^{j+1} from $\hat{Z}^j$ as in (6.80) when $j = 0$. Set $j = j + 1$.

Step 4: If achieve desired precision, stop. Otherwise, go back to step 2.

6.4 Nonlinear Persistent Excitation Condition

In previous sections, we introduce the HDPAE algorithm which generate $\hat{Z}^j$ as an estimation of the unknown parameters ω^*, however whether $\hat{Z}^j$ can converges to ω^* remains unknown. In this section, we will establish the Nonlinear Persistent Excitation condition which guarantee the global convergence in the HDPAE. The relationship between LPE and NLPE is also discussed.

6.4.1 Nonlinear Persistent Excitation Condition

For the HDPAE algorithm in section 6.3.4, $\hat{Z}^j$ serve as an estimation of the unknown parameter set Ω^* and we record the final $\hat{Z}^j$ when HDPAE stops as $\hat{Z}$ which serves as the estimation of Ω^*. Now that both of $\hat{Z}$ and Ω^* can be a discrete set, a rigous definition of global convergence of the HDPAE is that

$$\|\hat{Z} - \Omega^*\|_d \to 0 \tag{6.85}$$

as $j \to \infty$.

First, we state the NLPE condition as what follows.

Definition 5 NLPE: *For problem formulation as in (6.1) under assumptions 1-4, y, u is said to be nonlinearly persistent excitation if for any t, there exist constant T, ϵ_0 and a time instant $t_1 \in [t,\ t+T]$ such that*

$$|\,(f(y(t_1), u(t_1), \omega) - f(y(t_1), u(t_1), \omega^*))\,| \geq \epsilon_0 \min_{\omega^* \in \Omega^*} \|\omega - \omega^*\| \qquad \forall \omega \in \Omega^0.$$

In the following, we will prove that under NLPE, the HDPAE can lead to globally convergent estimation.

Theorem 2 *Under NLPE condition as in (5), for any ϵ_1, there exists a j in the HDPAE such that $\hat{Z} = \hat{Z}^j$ and*

$$\|\hat{Z} - \Omega^*\|_d \leq \epsilon_1. \tag{6.86}$$

The proof of Theorem 2 can be found in the Appendix.

Table 6.1: HDPAE Simulation results

Iter	d	a^*_{max}	Z
0	1.25	34.3302	[1.25 3.75]
1	0.6250	17.1651	[0.6250 1.8750 3.1250 4.3750]
2	0.3125	8.5825	[1.5625 2.1875 2.8125 3.4375 4.0625 4.6875]
3	0.1563	4.2913	[3.2813 3.5938 3.9063 4.2188 4.5313 4.8438]
4	0.0781	2.1456	[3.8281 3.9844 4.1406 4.2969 4.4531 4.6094]
5	0.0391	1.0728	[4.2578 4.3359 4.4141 4.4922 4.5703 4.6484]
6	0.0195	0.5364	[4.3945 4.4336 4.4727 4.5117 4.5508 4.5898]
7	0.0098	0.2682	[4.4629 4.4824 4.5020 4.5215 4.5410 4.5605]

6.5 Simulation Results of HDPAE

In this section, we give a specific example which illustrate the implementation of the HDPAE. The plant is of

$$\dot{y} = -4y + u^2\omega^* + \sin(\sqrt{u}\omega^*) - \frac{u}{1+\omega^*} \tag{6.87}$$

where $\omega^* \in \Omega^0 = [0,\ 5]$. It can be calculated that

$$B = 12 + \sqrt{3}. \tag{6.88}$$

Input signal is of

$$u(t) = 1 + sin(0.2t).$$

and the true unknown parameter is 4.5. Choosing

$$\gamma = 0.0001, \tag{6.89}$$

Table 6.1 shows the sample points Z^j at every iteration.

It is noted that during this simulation, the maximum size of Z is 6, which means that what need to be done at every time step is just to invert a 6 by 6 matrix with the determinant already known. There is no optimization problem involved anymore and the final result obtained is $\omega^* \in [4.45,\ 4.57]$.

6.6 Parameter Estimation in Static Systems

6.6.1 Parameter Estimation Algorithms for Static System

Parameter Estimation in static systems is an easier problem comparing to dynamic ones. We propose the problem formulation as of

$$y = f(u, \omega^*) \tag{6.90}$$

where $\omega^* \in \Omega^0 \subset \mathbb{R}^n$ is the unknown parameter, $y \in \mathbb{R}$ is output, and $u \in \mathbb{R}^m$ is input. For a nonlinear function, there may exist multiple ω^* and we record the set of all such ω^* as Ω^*, i.e.

$$\Omega^* = \{\omega \mid f(u, \omega) = f(u, \omega^*), \forall\, u, \omega \in \Omega^0\}. \tag{6.91}$$

The extension to higher dimension of y can be easily made.

For system as in (6.90), we make the following assumption.

Assumption 5: Function $f(x, \omega^*)$ is Lipschitz with its arguments $x = [u, \omega]$, i.e. there exists constant B such that

$$|f(x + \Delta x) - f(x)| \leq B\|\Delta x\|. \tag{6.92}$$

Definition 6 *System in (6.90) is a Lipschitz static system if it satisfies assumption* 5.

In system (6.90), signals $u(t)$ and $y(t)$ can be obtained online and we can sample the signals at any time instant. For a series of input-output pairs

$$[u(i), y(i)] \qquad \text{for } i = 1, 2, 3....$$

where

$$f(u(i), \omega^*) = y(i) \qquad \forall\, i, \text{ and } \forall\, \omega^* \in \Omega^*, \tag{6.93}$$

unknown parameter ω^* is identifiable if it satisfies the following condition.

Definition 7 *For a set U_I of input output pairs*

$$(u(i), y(i)), \qquad i = 1, ..., I, \tag{6.94}$$

if

$$y(i) = f(u(i), \omega), i = 1, .., I \;\Rightarrow \omega \in \Omega^*, \tag{6.95}$$

U_I *makes* Ω^* *identifiable.*

We note that U_I defined as in (6.94) gives us enough information to identify Ω^*. The existence of such input-output set U_I is the sufficient condition to estimate the unknown parameters, which is similar to the role of NLPE for nonlinear dynamic systems.

For U_I defined as in (6.94), problem in (6.90) is equivalent to estimate all $\omega \in \Omega^0$ such that

$$f(u(i), \omega) = y(i), \qquad \forall\, (u(i), y(i)) \in U_I. \tag{6.96}$$

and such ω is exactly ω^*. To estimate ω^* in system (6.96), we will introduce several methods which are listed below.

Hierarchical Search Algorithm

Using the same iteration technique as in the HDPAE, we propose a Hierarchical Search algorithm below.

Step 1: Choose arbitrary d^0, construct discrete set Z^0 as in (6.16), set $j = 0$.

Step 2: Construct $\hat{Z}^0 = Z^0$. For any $(u(i), y(i)) \in U_I$, if it satisfies

$$|f(u(i), z^0) - y(i)| > d^0, \tag{6.97}$$

for some $z^0 \in \hat{Z}^0$, remove z^0 from $\hat{Z}^0$.

Step 3: When the size of $\hat{Z}^0$ can not be reduced anymore, set $d^{j+1} = d^j/2$ and construct new discrete set Z^{j+1} from $\hat{Z}^j$ same as in the HDPAE.

Step 4: If d^j achieve desired precision, stop and $\hat{Z} = \hat{Z}^j$ is the estimation of unknown parameter Ω^*. Otherwise, set $j = j + 1$ and go back to step 2.

Brute Force Method

This method just search all the points in the unknown parameter region Ω^0 and check if there is some points which can match all the input-output pairs $[u(i), y(i)]$. The rigorous description is just the first step in hierarchical search when $j = 0$ and we set the desired precision to be d^0. Hence, d^0 means the final precision that Brute Search Mehtod can obtain. If we want to achieve fine precision, we need to set d^0 small and the points needed to be searched is huge.

Gradient Algorithm

We construct a Lyapunov function as of

$$V = \sum_{i=1}^{I}(y(i) - f(u(i), \omega))^2/2 \tag{6.98}$$

where $(u(i), y(i))$ belongs to U_I defined as in (6.94). The process to find the optimum ω^* is the routine descent gradient algorithm as follows:

$$\dot{\omega} = -\nabla V = -\sum_{i=1}^{I}(y(i) - f(u(i), \omega))\nabla_\omega f(u(i), \omega). \tag{6.99}$$

Stochastic Gradient Algorithm

This algorithm is similar to gradient algorithm however it just apply one component of the gradient at every time instant, i.e.

$$\dot{\omega} = -\nabla V = -(y(j(t)) - f(u(j(t)), \omega))\nabla_\omega f(u(j(t)), \omega), \qquad j(t) \in U_I \tag{6.100}$$

and the choice of $j(t)$ is randomly w.r.t. time t. If we just choose U_I be $u[0, \infty)$ and choose j serially, it becomes

$$\dot{\omega} = -\nabla V = -(y(t) - f(u(t), \omega))\nabla_\omega f(u(t), \omega). \tag{6.101}$$

6.6.2 Global Convergence Result

About the global convergence of the Hierarchical Search algorithm, we note that it is exactly the counterpart of the HDPAE in static systems. Using the same derivation as in Theorem 1 and Property 1, we can establish easily that at iteration j, Ω^* always belong to a cubic centered at $\hat{Z}^j$ at step 3 with half of the edge as $d^j/\sqrt{n}$. Because d_j keep reducing, we can achieve any desired precision and therefore the Hierarchical Search algorithm is globally convergent. Same result holds for the Brute Search algorithm if we discretize the unknown parameter using desired precision.

About gradient algorithm, it can stop prematurely at any point where

$$\nabla V = 0 \tag{6.102}$$

and a more detailed discussion about the convergence of gradient algorithm is proposed in [4].

About stochastic gradient algorithm, there even do not exist a Lyapunov function and you just go around to try your luck. Even for some situations where the gradient algorithm works, there exists some specific designed input u which can lead stochastic gradient algorithm into ∞. Hence, for general Lipschitz continuous nonlinear function, there is no guarantee of convergence for gradient and stochastic gradient algorithms.

6.6.3 Dealing with Noise

In realistic situations where the given model f can not exactly describe the input output mapping or there exist measurement errors in y and u, problem formulation in (6.96) is transformed into the following one

$$y = f(u, \omega^*) + o(t) \tag{6.103}$$

$$|o(t)| \leq O \tag{6.104}$$

where y, u, ω^* are same as described in problem formulation (6.90) and $o(t)$ is the uncertainty or noise added. For the problem formulation as in (6.103), the hierarchical search algorithm can be modified as follows.

Step 1: Choose arbitrary d^0, construct Z^0 as in (6.16), set $j = 0$.

Step 2: Construct $\hat{Z}^0 = Z^0$. For any series $(u(i), y(i))$, $i = 1, 2...$, if

$$|f(u(i), z^0) - y(i)| > d^0 + O \tag{6.105}$$

for some $z^0 \in \hat{Z}^0$, remove z^0 from $\hat{Z}^0$.

Step 3: When the size of $\hat{Z}^0$ can not be reduced anymore, set $d^{j+1} = d^j/2$ and construct new discrete set Z^{j+1} from $\hat{Z}^j$ same as in HDPAE. Set $j = j + 1$.

Step 4: If achieved desired precision, stop. Otherwise, go back to step 2.

Because of the existence of uncertainty, the final precision is not d^j which can approach zero but $d^j + O$ which is lower-bounded by O.

The effects of noise and uncertainty are unpredictable in gradient algorithm because they just exploit the local information. Even in cases where the gradient algorithm works, it is possible that the algorithm divergence if noise and uncertainty perturb the dynamics at some critical point.

6.6.4 Simulation Results

For function

$$y = \frac{2u\omega^*}{\pi} + (u^2 + \sqrt{u})\sin(\omega^*) \tag{6.106}$$

where $y, u, \omega^* \in \mathbb{R}$, true unknown parameter $\omega^* = 1$ and unknown parameter region $\Omega^0 = [-40,\ 40]$. We choose $U_I = [1,\ 2,\ 3]$ which will make ω^* identifiable. Considering gradient algorithm firstly, the Lyapunov function

$$V = \sum_{i=1}^{3} \left(y_i - \frac{2u\omega}{\pi} + (u^2 + \sqrt{u})\sin(\omega + 2u) \right)^2 /2$$

over ω is plotted in Figure 6-1 and it is shown that there exists only one global minimum at ω^* which verifies that U_I makes ω^* idnetifiable. From Figure 6-1, we note that there exist many local minima which prevent the gradient algorithm to converge. For stochastic gradient algorithm, we even can easily choose a sequence of u which leads estimate to ∞.

Now we apply the Hierarchical search algorithm. For this example, it can be easily derived that

$$B = 6/\pi + 9 + \sqrt{3}.$$

Choosing $d^0 = 20$ and $Z^0 = [-20,\ 20]$, the hierarchical algorithm gives the result of

$$\hat{Z} = [0.9999943,\ 1.000013]$$

with $d^j = 1.9073 \times 10^{-5}$ after $j = 21$ iterations. Combining $\hat{Z}$ and d^j, the true unknown parameter should satisfy

$$\omega^* \in [0.999975,\ 1.000032].$$

The totally searched points are 84. If we want to use brute search algorithm to achieve same precision, the points visited will be

$$80/d^j = 4.2 \times 10^6.$$

This simulation illustrated that the Hierarchical Search algorithm is globally convergent and more efficient than Brute Search algorithm.

Table 6.2: Comparison of different approaches

Static System	Dynamic System	Global Convergence	Computation Complexity	Noise Tolerance	Multiple Optima
Brute Search	DPAE	Yes	High	Good	Yes
Hierarchical Search	HDPAE	Yes	Medium	Good	Yes
Gradient Algorithm	Auxiliary Convergence of PAE	No	Low	Poor	No

6.7 Comparison of different approaches

In this section, we will give a generalization of parameter estimation in nonlinear systems which is summarized in Table 6.2.

Remark 4 The illustration of 3 different algorithms for static systems are shown in Figure 6-2. The Hierarchical Search is like a process of continuous zooming and its advantage over brute search is that it can detect region which does not contain the optima earlier and just zoom in the promising region.

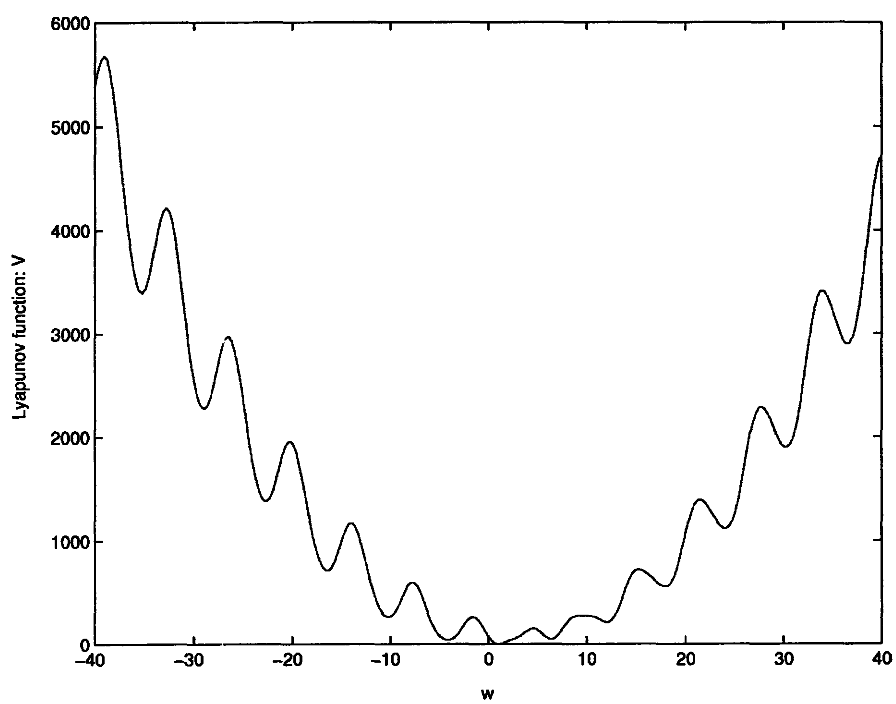

Figure 6-1: Comparison of Different Algorithms

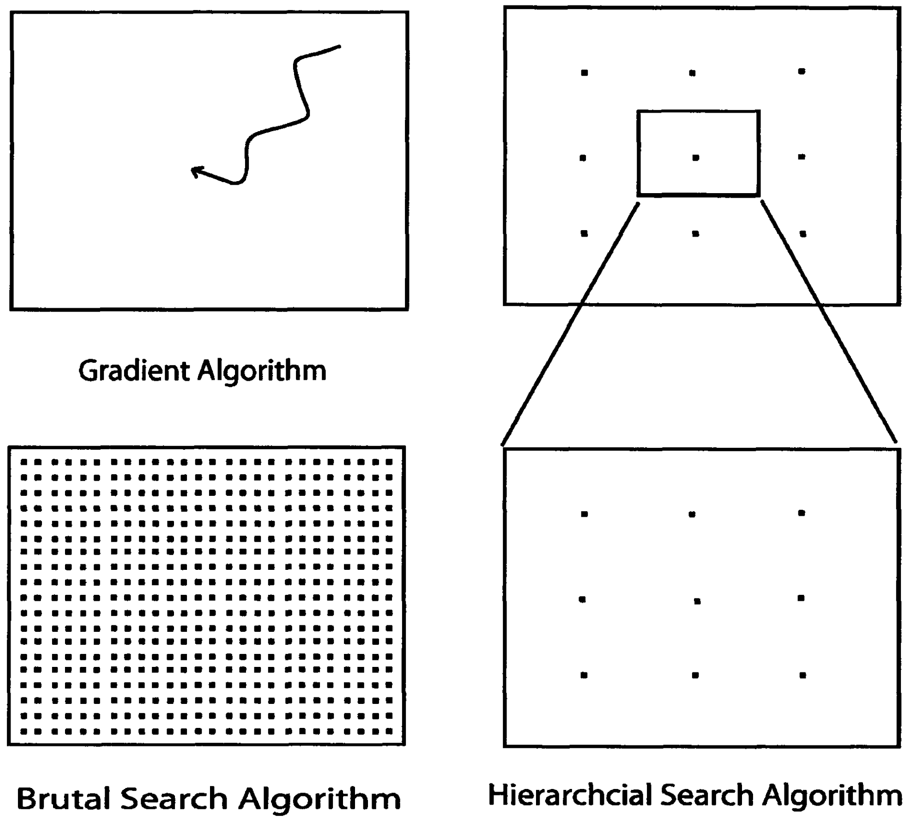

Figure 6-2: Comparison of Different Algorithms

Remark 5 Gradient algorithms are extensively used, however no matter in static systems or dynamic systems, there is no guarantee for global convergence even there just exists one global optimum. From the nature of gradient algorithm, it has no ability to estimate multiple global optima. The phenomena that gradient algorithms can not guarantee global convergence happens not rare in nonlinear systems. Like in the sigmoidal neural network, we show in [4] that even for a two-node network, there exists local equilibrium that prevent the establishment of the global convergence. The result of gradient algorithms strongly depends on the choice of the initial values and the step-size strategy however in many papers they are not explicitly stated. In engineering systems, all the results should be repeatable which requires a systematic way to choose the initial values and not the flip of coin if they can result in totally different results.

Remark 6: Here we put the computation complexity of gradient algorithm low just because at every time step the calculation of gradient increases linearly with dimension. It is quite doubtable for the whole algorithm because we do not know if it can arrive the goal and how long it will take. Even the Hierarchical search essentially still face the problem of dimension explosion as Brute Search algorithm, it is highly parallel unlike the gradient algorithm which is serial and each step must wait after its previous step finished. Our belief is that for general parameter estimation in nonlinear Lipschitz continuous systems, there is no shortcut. When applying the Hierarchical Search algorithm it is just a matter of computation ability which is increasing fast with new parallel computers.

Remark 7 One question raised here for dynamic system is that if we can measure and calculate $\dot{y}$ directly, then we can transform parameter identification in dynamic system into a static problem. The advantage of PAE, which includes PAE, DPAE and HDPAE, over the direct derivative method is that they do not need $\dot{y}$ and are not sensitive to the observation error or noise in y. The amplitude of noise in y will cause error in the evaluation of $\dot{y}$. However, for HDPAE, just the integral of noise will perturb the estimator and the estimator can correct this perturbation. In [], it is clearly shown that the correct way is to apply Adaptive Estimator to estimate the unknown parameters first and then obtain $\dot{y}$ from estimated parameters while it is wrong to calculate $\dot{y}$ and then using $\dot{y}$ to estimate the unknown parameters.

6.8 Summary

In this chapter, we propose a general and efficient parameter estimation algorithm for nonlinear Lipschitz continuous system for the first time. Generalization of parameter estimation in both static systems and dynamic systems are given and it is shown that gradient algorithms do not apply to general nonlinear systems.

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Appendix

Proof of Lemma 1: Performing linear transformation to matrix will not affect its rankness and determinant. In what follows, we show that by a series of column scale and add/subtract operation, matrix A can be transformed into a Vendemonts's matrix which is known to be full rank.

We denote the ith column of A as A_i. Firslty, let us consider A_2 which is

$$[-(\widehat{\theta}_1 - \theta_1) \ .. \ -(\widehat{\theta}_1 - \theta_i) \ .. \ -(\widehat{\theta}_1 - \theta_{N^0})]^T. \tag{6.107}$$

Add $\widehat{\theta}_1 A_1$ from A_2 and the new column 2 is

$$\bar{A}_2 = [\theta_1 \ .. \ \theta_i \ .. \ \theta_{N^0}]^T. \tag{6.108}$$

We can continue this process through column 3 to N. For $j+1$th column, we assume that the

$1,..,j$ column has already been transformed into

$$[\bar{A}_1 \ .. \ \bar{A}_i] \ = \ \begin{bmatrix} 1 & \theta_1 & .. & \theta_1^{j-1} \\ : & : & :: & : \\ 1 & \theta_i & .. & \theta_i^{j-1} \\ : & : & :: & : \\ 1 & \theta_{N^0} & .. & \theta_{N^0}^{j-1} \end{bmatrix} \tag{6.109}$$

A_{j+1} can be expressed as sum of vectors $\bar{A}_1$ through $\bar{A}_i$ with coefficients as function of $\widehat{\theta}_i$. Therefore, by subtract weighted $\bar{A}_1$ through $\bar{A}_i$ from A_{i+1}, the new $i+1$th column is

$$\bar{A}_{i+1} = [\theta_1^j \ .. \ \theta_i^j \ .. \ \theta_{N^0}^j]^T. \tag{6.110}$$

Repeat this process until the last column, and by simple matrix oprtation, the new matrix becomes

$$[\bar{A}_1 \ .. \ \bar{A}_N] \ = \ \begin{bmatrix} 1 & \theta_1 & .. & \theta_1^{N^0-1} \\ : & : & :: & : \\ 1 & \theta_i & .. & \theta_i^{N^0-1} \\ : & : & :: & : \\ 1 & \theta_{N^0} & .. & \theta_{N^0}^{N^0-1} \end{bmatrix} \tag{6.111}$$

This matrix is Vandemeonts' matrix and it is full rank. Therefore, A must be full rank too. •

Proof of Property 3: In DPAE, ϕ^* is achieved by

$$\phi^* = A^{-1}B. \tag{6.112}$$

From Assumption 3, B is bounded. We note that A is a regular full rank matrix not approaching singular from Lemma 1. Thus ϕ^* must be bounded. •

Proof of Property 4: From Assumption 2 and 3, u and y is Lipschitz w.r.t. t. Property 3 implis that ϕ^* is bounded and hence $\widehat{\theta}_i$ is also Lipschitz w.r.t. t. Therefore the elements of matrix A and B are Lipschitz w.r.t. time. ϕ_0^* is calculated by these quantities and therefore is Lipschitz w.r.t. t. •

Proof of Property 6: From Assumption 3 and 4, it follows that $f(y, u, \theta_0^*)$ is bounded. From Property 3, it follows ϕ_0^* is bounded. Property 5 shows that $a^* sat(\frac{\tilde{y}}{\epsilon})$ is bounded by a_{max}^*. Combing all these, it shows that the bound of $m(t)$ exists and we denote it as M. •

Proof of Property 7: From Assumptions 1-4, it follows that $f(y,u,\theta_0^*)$ is Lipschitz w.r.t. t and it can be expressed as

$$|f(y(t+\tau),u(t+\tau),\theta_0^*)-f(y(t),u(t),\theta_0^*)|\leq Q_2\tau \tag{6.113}$$

where

$$Q_2=B(U+F).$$

Combining (6.113) and Property 2, Property 7 is established. •

Proof of Lemma 3: Firstly, let us consider situation where

$$n(t_1)>\alpha\gamma+2\sqrt{Q(\gamma+\epsilon)}+2a_{max}^*. \tag{6.114}$$

It follows from (6.53) that

$$n(t_1+\tau)>n(t_1)-Q\tau. \tag{6.115}$$

Combining (6.27) and (6.28), it follows that

$$\dot{\tilde{y}}_\epsilon=-\alpha\tilde{y}_\epsilon+\phi_0^*-f(y,u,\theta_0^*)-r-a^*sat(\frac{\tilde{y}}{\epsilon}) \tag{6.116}$$

where

$$|r|\leq a_{max}^*. \tag{6.117}$$

Combining (6.114), (6.115), and (6.117), we have

$$n(t_1+\tau)-a_{max}^*>\alpha\gamma+2\sqrt{Q(\gamma+\epsilon)}-Q\tau \qquad \forall\,\tau\geq 0. \tag{6.118}$$

It follows Property 5 and (6.118) that

$$n(t_1+\tau)-a^*sat(\frac{\tilde{y}}{\epsilon})>\alpha\gamma+2\sqrt{Q(\gamma+\epsilon)}-Q\tau \qquad \forall\,\tau\geq 0. \tag{6.119}$$

If there exists some time constant $t_2\in[t_1,\ t_1+T']$ such that

$$\tilde{y}_\epsilon(t_2)\geq\gamma, \tag{6.120}$$

then the lemma is true. If (6.120) is not true, which means

$$\tilde{y}_\epsilon(t_2) \leq \gamma, \qquad \forall\, t \in [t_1,\ t_1 + T'], \tag{6.121}$$

then it follows from (6.119) that

$$-\alpha\tilde{y}_\epsilon(t_1+\tau) + n(t_1+\tau) - r - a^* sat(\frac{\tilde{y}}{\epsilon}) \geq 2\sqrt{Q(\gamma+\epsilon)} - Q\tau \qquad \forall\, \tau \geq 0. \tag{6.122}$$

Integrating (6.116) from t_1 to $t_1 + T'$ and it follows from (6.122) that

$$\begin{aligned} \tilde{y}(t_1+T') - \tilde{y}(t_1) &= \int_0^{T'} \left(-\alpha\tilde{y}_\epsilon(t_1+\tau) + n(t_1+\tau) - r - a^* sat(\frac{\tilde{y}}{\epsilon})\right) d\tau \\ &\geq \int_0^{T'} \left(2\sqrt{Q(\gamma+\epsilon)} - Q\tau\right) d\tau \\ &= 2(\gamma+\epsilon). \end{aligned} \tag{6.123}$$

Equation (6.121) implis that

$$\tilde{y}(t_1) > -\gamma - \epsilon \tag{6.124}$$

otherwise $|\tilde{y}_\epsilon(t_1)| \geq \gamma$. Combining (6.123) and (6.124), it follows that

$$\tilde{y}(t_1 + T') \geq \gamma + \epsilon$$

and therefore

$$|\tilde{y}_\epsilon(t_1)| \geq \gamma. \tag{6.125}$$

Equation (6.125) contradicts (6.121) and it proves that the lemma is true. For situation where

$$n(t_1) < -\alpha\gamma - 2\sqrt{Q(\gamma+\epsilon)} - 2a^*_{max}, \tag{6.126}$$

similar results follows. •

Proof of Lemma 4: If there exists some s which satisfies (6.58), Lemma 4 is proved. If there does not exists such s, from Lemma 2, Lyapunov function will decrease a small amount S for every time interval where $\tilde{y}_\epsilon \geq \gamma$ at some time instant. Because $V(0)$ is finite, at most after $[V(0)/S]$ times (6.58) will happen. •

Proof of Theorem 2: We choose parameter T in the HDPAE same as the T in the NLPE condition defined in Definition 5. For problem formulation in (6.1) and jth iteration of the HDPAE as in (6.59), it follows from Lemma 4 that the output error $\tilde{y}_\epsilon$ will converge for some interval $[t,\ t+T+T_1]$ for any γ where

$$|\tilde{y}_\epsilon(\tau_2)| \leq \gamma, \forall\, \tau_2 \in [t, t+T+T_1].$$

When output convergence happens we construct the $\hat{Z}$ by

$$\hat{Z} = \left\{z | z \in Z^j, \phi_0^*(\tau_1) - \beta \leq f(y(\tau_1), u(\tau_1), z) \leq \phi_0^*(\tau_1) + \beta, \forall\, \tau_1 \in [t, t+T]\right\} \tag{6.127}$$

where

$$\beta = \alpha\gamma + 2\sqrt{Q(\gamma+\epsilon)} + 2a^*_{max}.$$

It follows from Lemma 3 that

$$|f(y(\tau_1), u(\tau_1), \bar{z}) - \phi_0^*| \leq \beta \qquad \forall\, \tau_1 \in [t, t+T] \tag{6.128}$$

for any $\bar{z} \in Z_j^*$. From NLPE condition, for any $z \in \hat{Z}$ there exists $t_1 \in [t,\ t+T]$ such that

$$|f(y(t_1), u(t_1), z) - f(y(t_1), u(t_1), \omega^*)| \geq \epsilon_0 \min_{\omega^* \in \Omega^*} \|z - \omega^*\| \qquad \forall \omega^* \in \Omega^*. \tag{6.129}$$

It follows from (6.127) and (6.128), respectively, that

$$\begin{aligned} |f(y(t_1), u(t_1), z) - \phi_0^*(t_1)| &\leq \beta, \qquad \forall z \in \hat{Z} \\ |f(y(t_1), u(t_1), \bar{z}) - \phi_0^*(t_1)| &\leq \beta, \qquad \forall \bar{z} \in Z_j^*. \end{aligned} \tag{6.130}$$

From (6.130), we have

$$|f(y(t_1), u(t_1), z) - f(y(t_1), u(t_1), \bar{z})| \leq 2\beta \tag{6.131}$$

for any $z \in \hat{Z}$ and $\bar{z} \in Z_j^*$. It follows from the definition of Z_j^* as in (6.22) that

$$|f(y(t_1), u(t_1), \omega^*) - f(y(t_1), u(t_1), \bar{z})| \leq a^*_{max} \tag{6.132}$$

for any $\omega^* \in \Omega^*$ and $\bar{z} \in Z_j^*$. Combining (6.131) and (6.132), it follows that

$$|f(y(t_1), u(t_1), \omega^*) - f(y(t_1), u(t_1), z)| \leq 2\beta + a_{max}^* \tag{6.133}$$

for any $\omega^* \in \Omega^*$ and $z \in \hat{Z}$. Combining (6.129) and (6.133), it follows that for any $z \in \hat{Z}$

$$\min_{\omega^* \in \Omega^*} \|z - \omega^*\| \leq \frac{2\beta + a_{max}^*}{\epsilon_0} \tag{6.134}$$

which implies

$$\|\hat{Z} - \Omega^*\|_d \leq \frac{2\beta + a_{max}^*}{\epsilon_0}.$$

As j increases and we can choose γ, ϵ arbitrarily, a_{max}^* and β can become arbitrarily small. If there is some j such that $2\beta + a_{max}^* \leq \epsilon_0\epsilon_1$, (6.86) is satisfied and this proves Theorem 2. •

Chapter 7

Dead-zone Based Adaptive Filter

7.1 Introduction

Adaptive estimation algorithms have been developed for dynamic systems where the unknown parameters occur both linearly and nonlinearly over the past several errors. While stability properties of these estimators have been studied in [10]-[5], parameter convergence properties have been studied in [10]-[8]. In the presence of external disturbances and noise, it is well known that for linearly parameterized systems, either modifications in the adaptive law or persistently exciting reference inputs have to be introduced to establish robustness. The same however has not been established for nonlinearly parameterized systems thus far, and is addressed in this chapter. In particular, we establish that when output noise is present, a modified algorithm that include a deadzone, similar to that in [10], can be used to establish boundedness. We also show that the deadzone algorithm filters the output noise statistically and guarantees the asymptotic convergence of the estimates to true unknown parameters, and is denoted as the filtered deadzone estimator (FDE). The chapter is organized as follows. In section 7.2, problem formulation is proposed and the inability of the adaptive estimator to deal with output noise, without any modifications, is discussed. In section 7.3, the FDE is proposed. Proof of asymptotic convergence is also given. In section 7.4, comparison between output noise and model disturbance is discussed and the extension to situation where both of them exist is made. Section 7.5 shows simulation results.

7.2 Problem Formulation

We consider a nonlinearly parameterized dynamic system with bounded output noise such as

$$\begin{aligned} \dot{y} &= -\alpha y + \sum_{i=0}^{N} c_i(\omega^*)^i \\ y_n &= y + n(t) \end{aligned} \tag{7.1}$$

where c_i are measurable signals, $\omega^* \in \mathbb{R}$ is unknown parameter, $y \in \mathbb{R}$ is inaccessible state variable, output noise $n(t)$ is a stationary stochastic process and y_n is measured output signal.

We make the following assumptions regarding the stationary stochastic process $n(t)$.

Assumption 1: $|n(t)| \leq n_{max}, \quad \forall\, t \geq 0$ where n_{max} is a known positive constant.

It can be shown that a typical measurement noise due to effects of quantization satisfies assumptions 1. About the statistical properties of $n(t)$, we will introduce later in the discrete-time approximation.

7.2.1 Polynomial Adaptive Estimator (PAE)

In this section, we examine the properties of a Nth order PAE with N auxiliary estimates $\widehat{\omega}_i, .., \widehat{\omega}_N$ that was proposed in [8] for (7.1) in the absence of noise.

Suppose the Lyapunov function candidate is chosen as

$$V = y_n^2/2 + \sum_{i=1}^{N} p_i(\tilde{\omega}_i), \qquad \tilde{\omega}_i = \widehat{\omega}_i - \omega^* \tag{7.2}$$

where

$$\begin{aligned} p_i(\tilde{\theta}_i) &= \frac{1}{i+1}\tilde{\omega}_i^{i+1} \qquad \text{if } i \text{ is odd;} \\ p_i(\tilde{\theta}_i) &= \frac{1}{i}\tilde{\omega}_i^{i} + \frac{k_i}{i+1}\tilde{\omega}_i^{i+1} \qquad \text{if } i \text{ is even} \end{aligned} \tag{7.3}$$

for $i = 1, .., N$, and k_i is to be chosen appropriately as in [8]. The corresponding g_i is the derivative of p_i w.r.t. $\tilde{\omega}_i$ as of

$$g_i(\tilde{\theta}_i) = \tilde{\omega}_i^i \qquad \text{if } i \text{ is odd;}$$

$$g_i(\tilde{\theta}_i) \quad = \quad \tilde{\omega}_i^{i-1} + k_i\tilde{\theta}_i^i \qquad \text{if } i \text{ is even.} \tag{7.4}$$

We note that $\tilde{\theta}_i = \widehat{\theta}_i - \omega^*$ and g_i is a ith order polynomial function of ω^* and it can be expressed as

$$g_i = \sum_{j=0}^{i} d_{ij}(\widehat{\omega}_i)(\omega^*)^j. \tag{7.5}$$

The PAE is of the form

$$\begin{aligned} \dot{\hat{y}} &= -\alpha\hat{y} + \phi_0^* \\ \dot{\hat{\theta}}_i &= -\tilde{y}_n\phi_i^*, \qquad i = 1, ..., N \\ \tilde{y}_n &= \hat{y} - y_n \\ \phi^* &= A^{-1}C \end{aligned} \tag{7.6}$$

where $\phi^* = [\phi_0^*,\ \phi_1^*, ..., \phi_N^*]^T$, $sat(.)$ denote the saturation function, A is a non-singular $(N+1) \times (N+1)$ matrix as of

$$A = \begin{bmatrix} d_{00} & * & * & .. & * \\ 0 & d_{11} & * & .. & * \\ 0 & 0 & d_{22} & .. & * \\ : & : & : & :: & : \\ 0 & 0 & 0 & .. & d_{NN} \end{bmatrix} \tag{7.7}$$

and

$$C = [c_0\ c_1\ ...c_N]^T. \tag{7.8}$$

The element of ith row and jth column of matrix A in (7.7) is

$$A_{ij} = \begin{cases} 0 & i > j; \\ d_{(j-1)(i-1)} & i \leq j \end{cases}$$

where d_{ji} is defined as in (7.5).

We assume n is differential. Combining (7.1), (7.6) and (7.2), using the same derivation as in [8], the derivative of V follows as

$$\dot{V} = \tilde{y}_n(-\alpha\tilde{y} - \dot{n})$$

and hence

$$\dot{V} = -\alpha\tilde{y}\tilde{y}_n - \tilde{y}_n\,\dot{n}\,.$$

Since V cannot be guaranteed to be nonpositive in a compact set, it follows that V need not be bounded. Therefore modifications in the adaptive law are needed.

7.3 Filtered Deadzone Estimator

The result of no convergence of PAE with output noise in section 7.2.1 raises a problem for its application. To overcome this difficulty, we introduce a filtered deadzone estimator (FDE) as

$$\begin{aligned}
\dot{\hat{y}} &= -\alpha\hat{y} + \phi_0^* \\
\dot{\hat{\theta}}_i &= -\tilde{y}_{n\epsilon}\phi_i^*, \qquad i = 1, ..., N \\
\tilde{y}_n &= \hat{y} - y_n \\
\tilde{y}_{n\epsilon} &= \tilde{y}_n - n_{max} sat\left(\frac{\tilde{y}_n}{n_{max}}\right) \\
\phi^* &= A^{-1}C \\
\hat{n} &= y_n - \hat{y}
\end{aligned} \tag{7.9}$$

where $\hat{n}$ is the filted out output noise, $\phi^* = [\phi_0^*,\ \phi_1^*, ..., \phi_N^*]^T$, A and C are defined as in (7.7) and (7.8), and $sat(.)$ denotes the saturation function and is given by $sat(x) = sign(x)$ if $|x| \geq 1$ and $sat(x) = x$ if $|x| < 1$. In fact, the relationship between $\tilde{y}_{n\epsilon}$ and $\tilde{y}_n$ is of

$$\tilde{y}_{n\epsilon} = \begin{cases} \tilde{y}_n - n_{max} & \text{if} \quad \tilde{y}_n > n_{max}; \\ 0 & \text{if} \quad -n_{max} \leq \tilde{y}_n \leq n_{max}; \\ \tilde{y}_n + n_{max} & \text{if} \quad \tilde{y}_n < -n_{max} \end{cases} \tag{7.10}$$

and we use the Lyapunov function candidate

$$V = \tilde{y}_{n\epsilon}^2/2 + \sum_{i=1}^{N} p_i(\tilde{\omega}_i), \qquad \tilde{\omega}_i = \hat{\omega}_i - \omega^* \tag{7.11}$$

where $p_i()$ are the same as in (7.3).

The following properties can be derived for the FDE:

Property 1

$$(i)\ \tilde{y}_{n\epsilon} > 0 \Rightarrow \tilde{y} > \tilde{y}_{n\epsilon}$$

$$(ii)\ \tilde{y}_{n\epsilon} < 0 \Rightarrow \tilde{y} < \tilde{y}_{n\epsilon}$$

Proof of Property 1: First, let us consider case (i) in Property 1. $\tilde{y}_{n\epsilon} > 0$ implies from (7.10) that

$$\tilde{y}_n > n_{max} \text{ and } \tilde{y}_{n\epsilon} = \tilde{y}_n - n_{max}. \tag{7.12}$$

Because $\tilde{y}_n = \tilde{y} + n$ and $|n| \leq n_{max}$, it follows from (7.12) that

$$\tilde{y} > \tilde{y}_{n\epsilon}$$

which proves Case (i). Case (ii) of Property 1 can be proved in a similar manner. •

Assumption 1 just requires the output noise is bounded. The ouput noise only arises when you measure the output y and it is in fact a discrete event process no matter in the algorithm simulation in computer or realistic AD sampling process. The discrete time approximation of plant in (7.1) and dead-zone filter in (7.9) are of

$$\begin{aligned} \frac{y((\tau+1)\Delta) - y(\tau\Delta)}{\Delta} &= -\alpha y(\tau\Delta) + \sum_{i=0}^{N} c_i(\omega^*)^i \\ y_n(\tau\Delta) &= y(\tau\Delta) + n(\tau\Delta) \end{aligned} \tag{7.13}$$

and

$$\begin{aligned} \frac{\hat{y}((\tau+1)\Delta) - \hat{y}(\tau\Delta)}{\Delta} &= -\alpha\hat{y}(\tau\Delta) + \phi_0^*(\tau\Delta) \\ \frac{\widehat{\theta}_i((\tau+1)\Delta) - \widehat{\theta}_i(\tau\Delta)}{\Delta} &= -\tilde{y}_{n\epsilon}(\tau\Delta)\phi_i^*, \qquad i = 1, ..., N \end{aligned} \tag{7.14}$$

where the algebraic relationships of $\phi^*(\tau\Delta)$, $\tilde{y}_n(\tau\Delta)$ and $\hat{n}(\tau\Delta)$ on state variables $y(\tau\Delta)$, $\hat{y}(\tau\Delta)$ $\widehat{\theta}_i(\tau\Delta)$ are the same as in section 7.3.

Throughout this chapter, we assume the sampling time-interval Δ is very small and the discrete-time approximation matches the continuous system well such that we can omit the effects of discrete time approximation.

About output noise $n(\tau\Delta)$, we have the following assumption.

Assumption 2: $n(\tau\Delta)$, $\tau = 0, 1, 2, ...$ are i.i.d.

About $\tilde{y}_{n\epsilon}$, we have the following theorem.

Theorem 1 *For problem formulation in (7.13) and FDE as in (7.14) under assumptions 1 and 2, omitting the effects of discrete time approximation, it follows that*

$$Prob[\lim_{t\to\infty} \tilde{y}_{n\epsilon} = 0] = 1. \tag{7.15}$$

Proof of Theorem 1: Lyapunov function candidate V in (7.11) is expressed as

$$V(\tau\Delta) = \tilde{y}_{n\epsilon}(\tau\Delta)^2/2 + \sum_{i=1}^{N} p_i(\tilde{\omega}_i), \qquad \tilde{\omega}_i = \hat{\omega}_i(\tau\Delta) - \omega^* \tag{7.16}$$

in discrete time approximation. Since we assume Δ is small enough and the discrete time approximation matches the continuous system, we just keep the zero and first order of Δ and omit the higher order terms in the expression of $V((\tau+1)\Delta) - V(\tau\Delta)$. We know $y_{n\epsilon}(\tau\Delta)$ can always be decomposed into intervals where in each interval $y_{n\epsilon}(\tau\Delta)$ is a series with the same sign. Without loss of generally, we consider an interval $[\tau_1,\ \tau_2]$ where τ_2 can be finite or infinite such that

$$\tilde{y}_{n\epsilon}(\tau\Delta) > 0, \qquad \tau \in [\tau_1,\ \tau_2].$$

It follows from (7.16), (7.13) and (7.14) that

$$V((\tau+1)\Delta) - V(\tau\Delta) = \frac{\tilde{y}_{n\epsilon}((\tau+1)\Delta) + \tilde{y}_{n\epsilon}(\tau\Delta))}{2}(-\alpha\Delta(\hat{y}(\tau\Delta) - y(\tau\Delta)) + n(\tau\Delta) - n((\tau+1)\Delta)) \tag{7.17}$$

and therefore

$$\begin{aligned}\sum_{\tau=\tau_1}^{\tau_2}(V((\tau+1)\Delta) - V(\tau\Delta)) \quad &= \quad \sum_{\tau=\tau_1}^{\tau_2} -\alpha\Delta\tilde{y}_{n\epsilon}(\tau\Delta)\frac{\tilde{y}(\tau\Delta) + \tilde{y}((\tau-1)\Delta)}{2} + \\ &\quad \sum_{\tau=\tau_1}^{\tau_2} \frac{\tilde{y}_{n\epsilon}((\tau+1)\Delta) + \tilde{y}_{n\epsilon}(\tau\Delta))}{2}(n(\tau\Delta) - n((\tau+1)\Delta)).\end{aligned} \tag{7.18}$$

Omitting the higher order item of Δ,

$$\frac{\tilde{y}(\tau\Delta) + \tilde{y}((\tau-1)\Delta)}{2} = \tilde{y}(\tau\Delta)$$

and (7.18) can be written as

$$\sum_{\tau=\tau_1}^{\tau_2}(V((\tau+1)\Delta)-V(\tau\Delta)) = \sum_{\tau=\tau_1}^{\tau_2}-\alpha\Delta\tilde{y}_{n\epsilon}(\tau\Delta)\tilde{y}(\tau\Delta)+ \sum_{\tau=\tau_1}^{\tau_2}\frac{\tilde{y}_{n\epsilon}((\tau+1)\Delta)+\tilde{y}_{n\epsilon}(\tau\Delta))}{2}(n(\tau\Delta)-n((\tau+1)\Delta)). \quad (7.19)$$

Since

$$\tilde{y}_{n\epsilon}(\tau\Delta)=\tilde{y}(\tau\Delta)-n(\tau\Delta)-n_{max}, \quad (7.20)$$

it follows from (7.19) that

$$\sum_{\tau=\tau_1}^{\tau_2}(V((\tau+1)\Delta)-V(\tau\Delta))=\sum_{\tau=\tau_1}^{\tau_2}-\alpha\Delta\tilde{y}_{n\epsilon}(\tau\Delta)\tilde{y}(\tau\Delta)+ \sum_{\tau=\tau_1}^{\tau_2}\frac{\tilde{y}((\tau+1)\Delta)+\tilde{y}(\tau\Delta))-2n_{max}}{2}(n(\tau\Delta)-n((\tau+1)\Delta))-\sum_{\tau=\tau_1}^{\tau_2}\frac{n(\tau\Delta)^2-n((\tau+1)\Delta)^2}{2}.$$

Now let us look at the propagation of $n(\tau\Delta)$ in the entire system. $n(\tau\Delta)$ will affect $\widehat{\theta}_i((\tau+1)\Delta)$ and this will further affect $\hat{y}((\tau+2)\Delta)$. The values of $\tilde{y}((\tau+2)\Delta)$ and $\tilde{y}_{n\epsilon}((\tau+2)\Delta)$ depends on $\hat{y}((\tau+2)\Delta)$ and therefore depends on $n(\tau\Delta)$. Since

$$n(\tau\Delta) \text{ is independent of } \tilde{y}((\tau+i)\Delta), i\leq 1, \quad (7.21)$$

it follows that

$$\tilde{y}((\tau+1)\Delta)+\tilde{y}(\tau\Delta))-2n_{max} \text{ is independent of } n(\tau\Delta)-n((\tau+1)\Delta) \quad (7.22)$$

and therefore

$$E[\sum_{\tau=\tau_1}^{\tau_2}\frac{\tilde{y}((\tau+1)\Delta)+\tilde{y}(\tau\Delta))-2n_{max}}{2}(n(\tau\Delta)-n((\tau+1)\Delta))]= \sum_{\tau=\tau_1}^{\tau_2}\frac{\tilde{y}((\tau+1)\Delta)+\tilde{y}(\tau\Delta))-2n_{max}}{2}E[n(\tau\Delta)-n((\tau+1)\Delta)]. \quad (7.23)$$

It follows from Assumption 2 that

$$\begin{aligned} E[n(\tau\Delta)-n((\tau+1)\Delta)] &= 0 \\ E[n(\tau\Delta)^2-n((\tau+1)\Delta)^2] &= 0 \end{aligned} \quad (7.24)$$

and therefore

$$E[\sum_{\tau=\tau_1}^{\tau_2} \frac{\tilde{y}((\tau+1)\Delta)+\tilde{y}(\tau\Delta))-2n_{max}}{2}(n(\tau\Delta)-n((\tau+1)\Delta))-\sum_{\tau=\tau_1}^{\tau_2} \frac{n(\tau\Delta)^2-n((\tau+1)\Delta)^2}{2}]=0. \tag{7.25}$$

For intervals where $\tilde{y}_{n\epsilon} < 0$, the only difference is (7.20) becomes

$$\tilde{y}_{n\epsilon}(\tau\Delta) = \tilde{y}(\tau\Delta) - n(\tau\Delta) + n_{max} \tag{7.26}$$

and same results as in (7.25) follows. It can be checked easily from (7.25) that

$$E[\sum_{\tau=0}^{\infty} \frac{\tilde{y}_{n\epsilon}((\tau+1)\Delta)+\tilde{y}_{n\epsilon}(\tau\Delta))}{2}(n(\tau\Delta)-n((\tau+1)\Delta))]=0$$

and therefore

$$Prob[\sum_{\tau=0}^{\infty} \frac{\tilde{y}_{n\epsilon}((\tau+1)\Delta)+\tilde{y}_{n\epsilon}(\tau\Delta))}{2}(n(\tau\Delta)-n((\tau+1)\Delta))=\infty]=0 \tag{7.27}$$

since its elements are continuous distributed p.d.f. and achieves maximum at zero. Because

$$\sum_{\tau=0}^{\infty}(V((\tau+1)\Delta)-V(\tau\Delta)) = V(\infty)-V(0) \tag{7.28}$$

and the fact $V(0) < \infty$ and $V(\infty) > 0$, it follows that

$$\sum_{\tau=0}^{\infty}(V((\tau+1)\Delta)-V(\tau\Delta)) < \infty. \tag{7.29}$$

It follows from (7.27) and (7.29) that

$$Prob[\sum_{\tau=0}^{\infty} \alpha\Delta\tilde{y}_{n\epsilon}(\tau\Delta)\tilde{y}(\tau\Delta) < \infty] = 1. \tag{7.30}$$

It follows from Property 1 that

$$\tilde{y}_{n\epsilon}(\tau\Delta)\tilde{y}(\tau\Delta) > \tilde{y}_{n\epsilon}(\tau\Delta)^2 \tag{7.31}$$

and therefore

$$Prob[\sum_{\tau=0}^{\infty} \alpha\Delta\tilde{y}_{n\epsilon}(\tau\Delta)^2 < \infty] = 1. \tag{7.32}$$

Equation (7.32) implies that

$$Prob[\lim_{\tau\to\infty} \tilde{y}_{n\epsilon}(\tau\Delta) = 0] = 1 \tag{7.33}$$

which proves the Theorem. •

We note here that probability 1 implies that it is always true of

$$\lim_{t\to\infty} \tilde{y}_{n\epsilon} = 0.$$

7.3.1 Parameter Convergence of FDE

Theorem 1 established that output error $\tilde{y}_{n\epsilon}$ will converge to zero and parameter estimate is steady. What remains is whether $\hat{\omega}$ will converge asymptotically to ω^*. First, we note that once Lyapunov function V, which is defined as in (7.11), reaches zero, it will rest there and never left. This is different from PAE where the noise will drive V away from zero. In this section, we will discuss if (7.15) implies $V = 0$ and under what conditions.

In the system in (7.1), we have no assumption about the statistical properties of the output noise $n(t)$. Now we assume $n(t)$ is of

$$n(t) = U[n_L,\ n_H] \qquad \forall\, t \tag{7.34}$$

where $U[n_L,\ n_H]$ is the uniform distribution in region $[n_L,\ n_H]$. Of course it will satisfy

$$|n_L| \le n_{max} \qquad |n_H| \le n_{max} \qquad n_H > n_L. \tag{7.35}$$

We define a signal x which is a function of $\tilde{y} = \hat{y} - y$ when

$$x = \begin{cases} \tilde{y} - n_{max} + n_H \text{ if } \tilde{y} > n_{max} - n_H; \\ 0 \text{ if } -n_{max} - n_L \le \tilde{y} \le n_{max} - n_H; \\ -\tilde{y} - n_{max} - n_L \text{ if } \tilde{y} < -n_{max} - n_L. \end{cases} \tag{7.36}$$

About x, we have the following lemma.

Lemma 1 *For problem formulation in (7.1), FDE in (7.9) and output noise as in (7.34), it follows that*

$$Prob[\lim_{t\to\infty} x = 0] = 1. \tag{7.37}$$

Proof of Lemma 1: If (7.37) does not hold, it implis that

$$Prob[x(t) \neq 0] > 0 \tag{7.38}$$

as $t \to \infty$. $x(t) \neq 0$ implies that

$$\tilde{y} > n_{max} - n_H \qquad \text{or } \tilde{y} < -n_{max} - n_L. \tag{7.39}$$

Combining (7.34), (7.38) and (7.39), it follows that

$$Prob[\tilde{y}_{n\epsilon} \neq 0] > 0 \tag{7.40}$$

as $t \to \infty$ which contradicts Theorem 1. Therefore, lemma 1 must hold. •

Lemma 1 implies that

$$-n_{max} - n_L \leq \tilde{y} \leq n_{max} - n_H \tag{7.41}$$

if (7.34) and (7.35) hold.

In what follows, we will discuss the convergence of estimates for several cases.

Case 1: $n_L = -n_{max}$, $n_H = n_{max}$

It follows from (7.36) that $x = |\tilde{y}|$ and Lemma 1 states that

$$\lim_{t \to \infty} \tilde{y} = 0.$$

Thus, just the input signals satisfy the Nonlinear Persistent Condition established in [8], $\widehat{\omega}$, which is derived from $\sum_{i=0}^{N} c_i \widehat{\omega}^i = \phi_0^*$, will converge to ω^* asymptotically. In case 1 of the simulation results in section 7.5, the asymptotic convergence is illustrated.

Case 2: $n_L > -n_{max}$, $n_H = n_{max}$

It follows from Theorem 1 that $\tilde{y}_{n\epsilon}$ will converge to zero as $t \to \infty$ and hence $\widehat{\omega}$ come to some steady value $\widehat{\omega}^c$. It follows from Lemma 1 that when $t \to \infty$, $\tilde{y}$ will always be nonpositive. Instead of the NLPE condition, if for any t, there exists time constant T, ϵ_0 such that

$$\int_{\tau=t}^{t+T} \left(\sum_{i=0}^{N} c_i (\omega^i - (\omega^*)^i) \right) d\tau \geq \epsilon_0 \|\omega - \omega^*\|, \tag{7.42}$$

it will guarantee the asymptotic convergence of $\widehat{\omega}$ to ω^*. The reason is that for any $\widehat{\omega}^c \neq \omega^*$, if the input satisfies (7.42), $\tilde{y}$ will always become significantly positive. Hence, $\widehat{\omega}$ must convege to ω^*. In case 2 of the simulation results in section 7.5, the asymptotic convergence with biased output noise is illustrated.

Case 3: White Noise

When n is white noise which is not bounded, for a given n_{max} we can decompose n into 2 components

$$n = n_1 + n_2$$

where

$$\begin{array}{lll} n_1 = n & n_2 = 0 & |n| \leq n_{max} \\ n_1 = n_{max} & n_2 = n - n_1 & n > n_{max} \\ n_1 = -n_{max} & n_2 = n - n_1 & n < -n_{max}. \end{array}$$

For white noise, n can be very large however just at very small measure in time. Choosing approprite threshold value n_{max} to make

$$\lim_{T\to\infty} \frac{1}{T} \int_{t=0}^{T} |n_2| dt$$

small, we can treat the effects of additional n_2 as a disturbance which will perturb the convergence of $\widehat{\omega}$. We note that FDE does not depend on the initial value of $\widehat{\omega}$ and will correct the disturbance. Thus, $\widehat{\omega}$ is not steady but perturbed at some amplitude. Choosing n_{max} which makes $\widehat{\omega}$ perturbed in the desired precision and we are done. The tradeoff here is just if we want higher precision of $\widehat{\omega}$, we need to set the value of n_{max} bigger and therefore the time needed for convergence is longer.

7.3.2 Output Noise Filter

After $\widehat{\omega}$ converges to ω^*, $\tilde{y}$ converges to zero as well and the output noise can be evaluated exactly. We introduce the concept of the Filtered Deadzone Estimator as what follows.

Definition 1 *In dynamic system with unknown parameters, the Filterd Deadzone Estimator (FDE) is the method which applies the deadzone adaptive estimator to estimate the unknown parameters and then filter out the output noise at the same time when parameter estimation converges.*

In FDE, the estimation of output noise n is simply

$$\hat{n} = y_n - \hat{y}$$

and it follows that

$$\hat{n} - n = y - \hat{y}$$

which means that the errors of $\hat{y}$ and $\hat{n}$ to y and n are of the same amplitude and different sign. The convergence of them happens at the same time. Now that both y and n are not accessible, the indication of the convergence of $\hat{y}$ and $\hat{n}$ is that $\tilde{y}_{n\epsilon}$ converges to 0 and $\widehat{\omega}$ keeps steady.

Another information which can be derived from the FDE is the derivative of y. Now that $\widehat{\omega} \to \omega^*$ and $\hat{y} \to y$, it follows naturally that the estimation of $\dot{y}$ is of

$$\hat{\dot{y}} = -\alpha\hat{y} + \phi_0^*.$$

It is noted that this estimation will converge to the true derivative $\dot{y}$ and it is stable and free of noise. If we want to calculate the derivative directly from measured y_n, the uncertainty always exists and the derivative could be very noisy.

7.4 Model Disturbance

In [10], same structure of FDE is used to deal with model disturbance, which is

$$\dot{y} = -\alpha y + \sum_{i=0}^{N} c_i(\omega^*)^i + o \tag{7.43}$$

where $|o| \leq O$.

For systems where both output noise and model disturbance exist, i.e.

$$\begin{aligned} \dot{y} &= -\alpha y + \sum_{i=0}^{N} c_i(\omega^*)^i + o \\ y_n &= y + n \\ o &\leq O \\ n &\leq n_{max}, \end{aligned} \tag{7.44}$$

the modified FDE is of

$$
\begin{aligned}
\dot{\hat{y}} &= -\alpha\hat{y} + \phi_0^* - a_{max}^* sat\left(\frac{\tilde{y}_n}{n_{max}}\right) \\
\dot{\hat{\omega}}_i &= -\tilde{y}_{n\epsilon}\phi_i^* \qquad i = 1, .., N \\
\tilde{y}_{n\epsilon} &= \tilde{y}_n - n_{max} sat\left(\frac{\tilde{y}_n}{n_{max}}\right) \\
\tilde{y}_n &= \hat{y} - y_n \\
\phi^* &= A^{-1}C \\
a_{max} &= O, \qquad (7.45)
\end{aligned}
$$

where $\phi^* = [\phi_0^*,\ \phi_1^*, ..., \phi_N^*]^T$, A and C are defined as in (7.7) and (7.8), and $sat(.)$ denotes the saturation function and is given by $sat(x) = sign(x)$ if $|x| \geq 1$ and $sat(x) = x$ if $|x| < 1$.

Choosing Lyapunov candidate V as

$$V = \tilde{y}_{n\epsilon}^2/2 + \sum_{i=1}^{N} p_i(\tilde{\theta}_i) \qquad (7.46)$$

where $p_i()$ is defined as in (7.3), it follows that

$$\dot{V} = -\alpha\tilde{y}_{n\epsilon}\tilde{y} - \tilde{y}_{n\epsilon}\,\dot{n} + \tilde{y}_{n\epsilon}\left(-o - a_{max}^* sat\left(\frac{\tilde{y}_n}{n_{max}}\right)\right). \qquad (7.47)$$

It can be checked easily that

$$\tilde{y}_{n\epsilon}\left(-o - a_{max}^* sat\left(\frac{\tilde{y}_n}{n_{max}}\right)\right) \leq 0 \qquad (7.48)$$

since $|o| \leq a_{max}^*$. It follows from (7.47) and (7.48) that

$$\dot{V} \leq -\alpha\tilde{y}_{n\epsilon}\tilde{y} - \tilde{y}_{n\epsilon}\,\dot{n}\,. \qquad (7.49)$$

Using the same derivation as in Theorem 1, we have

$$Prob[\lim_{t\to\infty} \tilde{y}_{n\epsilon} = 0] = 1. \qquad (7.50)$$

Therefore,

$$\lim_{t\to\infty} \tilde{y}_{n\epsilon} = 0$$

for problem formulation in (7.44) and modified FDE in (7.45).

We note that the difference of (7.45) from the FDE in (7.9) is the additional item $-a^*_{max} sat\left(\frac{\tilde{y}_n}{n_{max}}\right)$ which is used to balance the model disturbance.

7.5 Simulation Results

We consider a simple example

$$\dot{y} = -4y + u\omega^* + (u^2 - u)(\omega^*)^2$$

where $\omega^* = 1$ and input $u = \sin(0.2 * t)$. For the following two cases

$$\begin{aligned} Case\,1 \quad n(t) = &\quad U[0.5, 0.5] \\ Case\,2 \quad n(t) = &\quad 0.1 + 0.01U[-1, 1] \end{aligned}$$

where $U[a,\ b]$ is uniformly distributed random variable in $[a,\ b]$, we run simulations for both PAE and FDE and compare the results. We note here that in Case 1, the mean value of noise is zero however in Case 2, it is a biased noise with mean at 0.1. In the simulation, we choose initial values

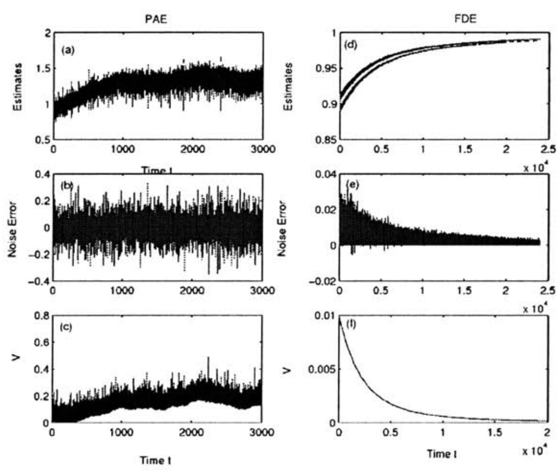

Figure 7-1: Comparison of PAE and FDE in Case 1: - Unbiased Noise. Figures (a)-(c) show the trajectories of estimates $\widehat{\omega}_1$ and $\widehat{\omega}_2$, Noise filter error as of $\hat{n} - n$, and Lyapunov function V in PAE. Figures (d)-(f) show the trajectories of estimates $\widehat{\omega}_1$ and $\widehat{\omega}_2$, Noise filter error as of $\hat{n} - n$, and Lyapunov function V in FDE

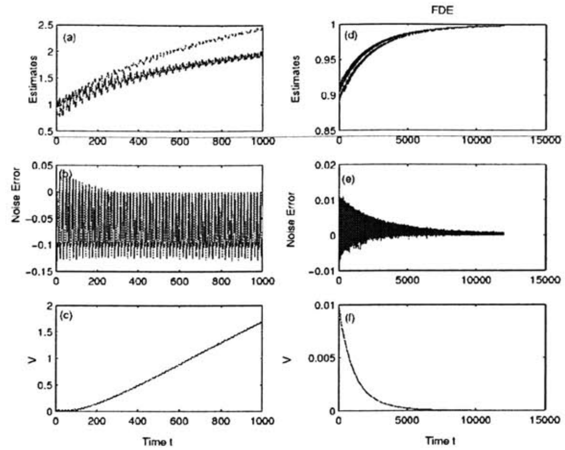

Figure 7-2: Comparison of PAE and FDE in Case 2: - Biased Noise. Figures (a)-(c) show the trajectories of estimates $\widehat{\omega}_1$ and $\widehat{\omega}_2$, Noise filter error as of $\hat{n} - n$, and Lyapunov function V in PAE. Figures (d)-(f) show the trajectories of estimates $\widehat{\omega}_1$ and $\widehat{\omega}_2$, Noise filter error as of $\hat{n} - n$, and Lyapunov function V in FDE

of $[\widehat{\omega}_1;\ \widehat{\omega}_2]$ as $[\widehat{\omega}_1;\ \widehat{\omega}_2] = [0.9;\ 0.9]$. Figures 7-1 and 7-2 show simulation results for case 1 and 2 respectively. In each figure, for both PAE in (7.6) and FDE in (7.9), it plots the trajectories of parameter estimates $\widehat{\omega}_1$ and $\widehat{\omega}_2$, noise error which is defined as

$$-\tilde{y} = y - \hat{y} = \hat{n} - n,$$

and Lyapunov function V which is defined as in (7.2) for PAE and (7.11) for FDE. The simulation results show clearly that for both cases, the FDE leads to asymptotic convergence of $\widehat{\omega}$, $\tilde{y}$, V and $\tilde{y}_{n\epsilon}$. For PAE, none of these variables converges.

The ability that FDE can filter out biased measurement or noise is extremely useful in practical applications. In the on-line measurement of dynamic systems, unlike the unbiased measurement uncertainty which is always unavoidable and restricted by measurement precision, measurement offset often means a quality problem and it is important that it can be detected on-line without perturbing the normal process of the plant.

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Chapter 8

Continuous Polynomial Adaptive Controller

8.1 Introduction

Adaptive control is an important area in the control theory which deals with the partial known plant. The structured unknown information is usually expressed by unknown or time-changing parameters. So far, the traditional adaptive controller can only deal with linearly parameterized systems. A lot of efforts [1]-[10] have been made to extend the adaptive control to NLP systems, however they are still restricted to some special situations and can not serve as the same tool as linear adaptive controller for linearly parameterized systems.

The linear adaptive controller establishes a Lyapunov function as the quadratic form of output error and parameter error, and the associated adaptive law guarantees the derivative of this Lyapunov function is non positive. In [3], a polynomial adaptive estimator is proposed which modifies the Lyapunov function candidate and associated adaptive laws for polynomially parameterized functions. In [4], a dead-zone based filter is proposed which provides a tool for adaptive estimators to deal with output noise in some plants. As a further development of the polynomial adaptive estimator, a Continuous Polynomial Adaptive Controller (CPAC) is proposed in this chapter. Using a new choice of polynomial Lyapunov function and adaptive law, we can deal with piece-wise linearly parameterized systems as the traditional adaptive controller for the linearly parameterized ones. Since most of the commonly encountered NLP systems can be approximated by piece-wise linear functions, it provide a general tool to deal with them. Since disturbance always exists, the approximation error between piece-wise linear function and original nonlinear function can also be treated as disturbance. In this chapter, we also established the stability of CPAC under bounded disturbance.

The CPAC consists of two parts. A companion adaptive system and a control law. The companion adaptive systems can be applied to any NLP systems which can be approximated by piece-wise linear functions. It consists of two parts, a companion model and an adaptive law. For a piece-wise linearly parameterized systems with all states measured, the asymptotic convergence of outputs of the plant and the companion model is established for the first time as the same as the linear adaptive theory for linearly parameterized ones. Unlike the plant which involved the unknown parameters, the companion model is deterministic without any unknown information. Therefore, what required for the control law is to find a controller for the companion model. Same control signal is applied to the plant and results in the same stability and tracking properties as the companion model. Unlike the companion adaptive system which is a general tool for NLP systems, the design of control law is case by case since there does not exist a general control law for a nonlinear system even it is deterministic. In this chapter, for several commonly encountered NLP systems, the control laws are given as the examples of how to construct a complete CPAC, including some cases with unmeasured states which requires an adaptive observer.

This chapter is organized as follows. Section 8.2 gives a general problem formulation which states the approximation of a nonlinear function using piece-wise linear functions. Section 8.3 states the companion adaptive system corresponding to the plant in section 8.2. Extension to higher dimension is also made. The stability and convergence properties for the error model and adaptive law are established, and bounded disturbance is also considered. In section 8.4, control laws for several classes of NLP systems are proposed and they demonstrate how to construct the complete CPAC. Section 8.5 provides simulation results of CPAC for a specific NLP system and section 8.6 concludes the chapter.

8.2 Problem Formulation

In this chapter, we consider the adaptive control of nonlinearly parameterized systems. For simplicity, we will consider the scalar case first and the extension to higher dimension is introduced later.

The problem formulation is of

$$\dot{y} = -\alpha y + f(y, u, \omega) \tag{8.1}$$

where $\alpha > 0$ is a known constant, $y \in \mathbb{R}$ is state variable, $u \in \mathbb{R}^m$ is control signal to be determined, ω is unknown parameter which belongs to a continuous compact set $\Omega = [\Omega_{min},\ \Omega_{max}] \subset \mathbb{R}$.

About f, we have the following assumption.

Assumption 1: $\forall y(t), u(t)$, f can be approximated by a piece-wise linear function over Ω, i.e. there exists a constant $d_{max} > 0$, N regions

$$\Omega_i = [\underline{\Omega}_i,\ \bar{\Omega}_i], \qquad i = 1, .., N \tag{8.2}$$

and $m_i(y, u)$, $r_i(y, u)$ such that

$$\Omega \subseteq \bigcup_{i=1}^{N} \Omega_i \tag{8.3}$$

$$|d(t)| = |m_i(y, u) + r_i(y, u)(\omega - \bar{\omega}) - f(y, u, \omega)| \leq d_{max}, \ \ \omega \in \Omega_i, \forall\, i = 1, .., N \tag{8.4}$$

$$\bar{\omega} = \frac{\bar{\Omega}_i + \underline{\Omega}_i}{2}. \tag{8.5}$$

First, we note that a piece-wise linear function is a typical nonlinear function which implies the extension of adaptive control theory from linearly parameterized systems to nonlinear ones. Secondly, a large class of commonly encountered functions can be piece-wise linearly approximated since any smooth function can be linearized locally. For example, if f is differentiable and its second order derivative w.r.t. ω is bounded, i.e.

$$\left|\frac{\partial^2 f(y, u, \omega)}{\partial \omega^2}\right| \leq q, \tag{8.6}$$

we divide Ω uniformly into N regions and we have

$$|m_i(y, u) + r_i(y, u)(\omega - \bar{\omega}) - f(y, u, \omega)| \leq \frac{q(\Omega_{max} - \Omega_{min})^2}{8N^2}, \forall\, i = 1, .., N \tag{8.7}$$

where

$$m_i(y,u) = f(y,u,\bar{\omega}) \tag{8.8}$$

$$r_i(y,u) = \left.\frac{\partial f(y,u,\omega)}{\partial \omega}\right|_{\bar{\omega}} \tag{8.9}$$

$$\bar{\omega} = \frac{\bar{\Omega}_i + \underline{\Omega}_i}{2}. \tag{8.10}$$

We now map the unknown parameter $\omega \in \Omega$ into a new pair of unknown parameters $[\theta,\ \zeta]$ as of

$$\begin{aligned} \theta = \theta_i, &\quad \text{if} \quad \omega \in \Omega_i \\ \zeta = \omega - \frac{\bar{\Omega}_i + \underline{\Omega}_i}{2} &\quad \text{if} \quad \omega \in \Omega_i \end{aligned} \tag{8.11}$$

$$\begin{aligned} \zeta &\in [-\zeta_{max}, \zeta_{max}] \\ \theta &\in \Theta = \{\theta_1, .., \theta_i, ..\theta_N\} \\ \theta_i &= \frac{i-1}{(N-1)\Theta_{max}} \\ \zeta_{max} &= max_{i=1,..,N} \frac{\bar{\Omega}_i - \underline{\Omega}_i}{2} \end{aligned} \tag{8.12}$$

and Θ_{max} is an arbitrary positive constant.

With the unknown parameter transformation, the problem formulation in (8.1) under Assumption 1 is equivalent to the following one:

$$\begin{aligned} \dot{y} &= -\alpha y + m(y,u,\theta) + r(y,u,\theta)\zeta + d(t) \\ d(t) &\leq d_{max} \end{aligned} \tag{8.13}$$

where unknown parameter $[\theta, \zeta]$ is defined as in (8.12) and

$$\begin{aligned} m(y,u,\theta) &= m_i(y,u), \qquad i = (N-1)\Theta_{max}\theta + 1 \\ r(y,u,\theta) &= r_i(y,u), \qquad i = (N-1)\Theta_{max}\theta + 1 \\ |d(t)| &= f(y,u,\omega) - m(y,u,\theta) - r(y,u,\theta)\zeta. \end{aligned} \tag{8.14}$$

We note here that even we do not know $m(y,u,\theta)$ and $r(y,u,\theta)$ since θ is unknown, $m(y,u,\theta_i)$ and $r(y,u,\theta_i)$ are available for specific θ_i.

The structure of the CPAC for plant in (8.13) is illustrated in Figure 8-1. We note that the CPAC consists of two parts: a companion adaptive system which is introduced in section 8.3 and a control law, which is proposed in section 8.4. Companion adaptive systems can be established for any nonlinear function which can be piece-wise linearizable w.r.t. ω in Ω and it consists of two parts: a companion model and an adaptive law. In the companion model, every $f(y, u, \omega) \in \mathbb{R}$ which includes unknown parameter is estimated as $\phi_0(y, u)$ which is a deterministic function w.r.t. y, u. Since $m(y, u, \theta) + r(y, u, \theta)\zeta$ has $2N$ freedoms: N offset values and N slope rate, $2N - 1$ auxiliary estimates are needed and the adaptive law governs the updates of these auxiliary estimates. This is in coincidence with linear parameterized systems. We need N estimates for a linear function $\omega^T u$ which has $N + 1$ degrees of freedom, one offset value and N slope rates.

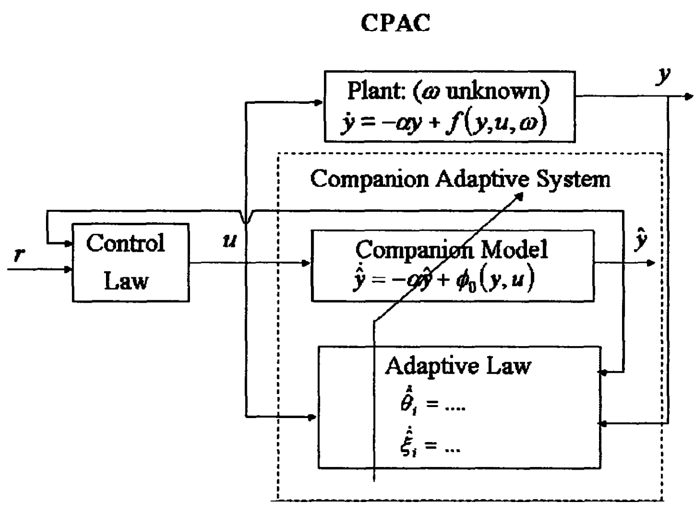

Figure 8-1: Structure of CPAC

The stability and convergence results in section 8.3 indicate that the output of the companion model $\hat{y}$ will converge asymptotically to y in the absence of disturbances and this gives us a tool to control the original plant. In section 8.4, we will show how to design the control law in the CPAC for several classes of NLP systems.

8.3 The Companion Adaptive System

In what follows, we will introduce the companion adaptive system for (8.13). The companion adaptive systems consists of two parts: the companion model as of

$$\dot{\hat{y}} = -\alpha\hat{y} + \phi_0 \tag{8.15}$$

and the adaptive law as of

$$\begin{aligned}
\dot{\hat{\theta}}_i &= \begin{cases} 0 & \text{if } \tilde{y}\phi_i > 0 \text{ and } \hat{\theta}_i \geq \Theta_{max} \\ 0 & \text{if } \tilde{y}\phi_i < 0 \text{ and } \hat{\theta}_i \leq 0 \\ \tilde{y}\phi_i & \text{otherwise.} \end{cases} \\
& \quad \forall i = 1, .., N-1 \\
\dot{\hat{\zeta}}_i &= \begin{cases} 0 & \text{if } \tilde{y}\eta_i > 0 \text{ and } \hat{\zeta}_i \geq \zeta_{max} \\ 0 & \text{if } \tilde{y}\eta_i < 0 \text{ and } \hat{\zeta}_i \leq -\zeta_{max} \\ \tilde{y}\eta_i & \text{otherwise.} \end{cases} \\
& \quad \forall i = 0, .., N-1 \\
C_r &= [r(y,u,\theta_1), .., r(y,u,\theta_i), .., r(y,u,\theta_N)]^T \\
\eta &= [\eta_0, ..., \eta_{N-1}] = -A_r^{-1} C_r \\
C_m &= [m(y,u,\theta_1), .., m(y,u,\theta_i), .., m(y,u,\theta_N)]^T \\
C_\eta &= [\hat{\zeta}_0\eta_0, .., \hat{\zeta}_i\eta_i, .., \hat{\zeta}_{N-1}\eta_{N-1}]^T \\
\Phi &= [\phi_0, ..., \phi_{N-1}] = A_m^{-1}(C_m - A_r C_\eta) \\
\tilde{y} &= \hat{y} - y
\end{aligned} \tag{8.16}$$

where A_r is an N by N matrix given by

$$A_r = \begin{bmatrix} 1 & .. & .. & :: \\ : & :: & : & :: \\ 1 & .. & a_{ij} & .. \\ : & :: & : & :: \end{bmatrix} \tag{8.17}$$

with the ith row and jth column element a_{ij} as

$$a_{ij} = \theta_i^{j-1}, \tag{8.18}$$

and A_m is an N by N matrix with the ith row and jth column element a_{ij} as

$$\begin{aligned} a_{i1} &= 1 \qquad 1 \leq i \leq N \\ a_{ij} &= -g_{j-1}(\widehat{\theta}_{j-1} - \theta_i) \qquad 1 \leq i \leq N, 2 \leq j \leq N \end{aligned} \tag{8.19}$$

with

$$g_i(x) = \begin{cases} x^{i-1} & \text{if } i \text{ is even} \\ kx^{i-1} + x^{i-2} & \text{if } i \text{ is even} \end{cases} \tag{8.20}$$

and

$$k = \frac{N-1}{N\Theta_{max}}. \tag{8.21}$$

8.3.1 Properties of the Companion Adaptive System

The following property is useful since the non-singularity of A_r and A_m is needed.

Property 1

$$\begin{aligned} det(A_r) &\neq 0 \\ det(A_m) &\neq 0 \end{aligned} \tag{8.22}$$

Proof of Property 1: It can be verified directly that A_r is a Vandemeonts' matrix and it is full rank.

In what follows, we show that by a series of column scale and add/subtract operation, matrix A_m can be transformed into a Vendemonts's matrix which is full rank.

We denote the ith column of A_m as A_{m_i}. First, let us consider A_{m_2} which is

$$[-(\widehat{\theta}_1 - \theta_1) \;..\; -(\widehat{\theta}_1 - \theta_i) \;..\; -(\widehat{\theta}_1 - \theta_N)]^T.$$

Add $\widehat{\theta}_1 Am_1$ from A_{m_2} and the new column 2 is

$$\bar{A}_{m_2} = [\theta_1 \;..\; \theta_i \;..\; \theta_N]^T.$$

We can continue this process through column 3 to N. For $(j+1)$th column, we assume that the columns through 1 to j have already been transformed into

$$[\bar{A}_{m_1} \; .. \; \bar{A}_{m_j}] = \begin{bmatrix} 1 & \theta_1 & .. & \theta_1^{j-1} \\ : & : & :: & : \\ 1 & \theta_i & .. & \theta_i^{j-1} \\ : & : & :: & : \\ 1 & \theta_N & .. & \theta_N^{j-1} \end{bmatrix}$$

$A_{m_{j+1}}$ can be expressed as sum of vectors $\bar{A}_{m_1}$ through $\bar{A}_{m_i}$ with coefficients as function of $\widehat{\theta}_i$. Therefore, by subtract weighted $\bar{A}_{m_1}$ through $\bar{A}_{m_i}$ from $A_{m_{i+1}}$, the new $i+1$th column is

$$\bar{A}_{m_{i+1}} = [\theta_1^j \; .. \; \theta_i^j \; .. \; \theta_N^j]^T.$$

Repeat this process until the last column, and by simple matrix operation, the new matrix becomes

$$[\bar{A}_{m_1} \; .. \; \bar{A}_{m_N}] = \begin{bmatrix} 1 & \theta_1 & .. & \theta_1^{N-1} \\ : & : & :: & : \\ 1 & \theta_i & .. & \theta_i^{N-1} \\ : & : & :: & : \\ 1 & \theta_N & .. & \theta_N^{N-1} \end{bmatrix} \tag{8.23}$$

Since (8.23) is the Vandemeonts' matrix and is full rank, A_m is of full rank as well. •

In fact, $det(A_r)$ and $det(A_m)$ are fixed constants once Θ_{max} and Θ are determined. Even A_m is a function of $\widehat{\theta}_i$, $det(A_m)$ does not depend on them. The following Property establishes the boundness of auxiliary estimates $\widehat{\theta}_i$ and $\hat{\zeta}_i$.

Property 2

$$0 \leq \; |\hat{\theta}_i(t)| \; \leq \Theta_{max} \qquad i = 1, .., N-1 \tag{8.24}$$

$$-\zeta_{max} \leq \; |\hat{\zeta}_i(t)| \; \leq \zeta_{max} \qquad i = 0, .., N-1. \tag{8.25}$$

Proof of Property 2: For any $i = 1, .., N-1$, it follows from (8.16) that

$$\begin{aligned} \dot{\hat{\theta}}_i &\leq 0 \text{ if } \hat{\theta}_i \geq \Theta_{max} \\ \dot{\hat{\theta}}_i &\geq 0 \text{ if } \hat{\theta}_i \leq 0 \end{aligned} \tag{8.26}$$

and (8.24) is established. Using a similar method, (8.25) can be established. •

8.3.2 Stability Analysis

We first consider the ideal case where $d_{max} = 0$, which means that we have a function with piecewise linear parameterization. The extension to nonzero disturbance $d(t)$ is discussed in section 8.3.3.

The adaptive laws in (8.16) can be rewritten as

$$\begin{aligned} \dot{\hat{\theta}}_i &= \tilde{y}\phi_i + v_i, \qquad i = 1, .., N-1 \\ \dot{\hat{\zeta}}_i &= \tilde{y}\eta_i + w_i, \qquad i = 0, .., N-1 \end{aligned} \tag{8.27}$$

where

$$\begin{cases} v_i = 0 & \text{if } \quad \hat{\theta}_i \in (0, \Theta_{max}), \\ v_i \leq 0 & \text{if } \quad \hat{\theta}_i \geq \Theta_{max}, \\ v_i \geq 0 & \text{if } \quad \hat{\theta}_i \leq 0. \end{cases} \qquad i = 1, .., N-1$$

$$\begin{cases} w_i = 0 & \text{if } \quad \hat{\zeta}_i \in (-\zeta_{max}, \zeta_{max}), \\ w_i \leq 0 & \text{if } \quad \hat{\zeta}_i \geq \zeta_{max}, \\ w_i \geq 0 & \text{if } \quad \hat{\zeta}_i \leq -\zeta_{max}. \end{cases} \qquad i = 0, .., N-1. \tag{8.28}$$

and the other algebraic relationships in the adaptive law are the same as in (8.16).

Define

$$\begin{aligned} &\tilde{\theta}_i = \hat{\theta}_i - \theta, i = 1, .., N-1 \\ &\tilde{\zeta}_i = \hat{\zeta}_i - \zeta, i = 0, .., N-1, \end{aligned} \tag{8.29}$$

we introduce a polynomial Lyapunov function as of

$$V = \tilde{y}^2/2 + \sum_{i=1}^{N-1} p_i(\tilde{\theta}_i) + \sum_{i=0}^{N-1} \theta_i^i(\tilde{\zeta}_i)^2/2 \tag{8.30}$$

where

$$p_i(x) = \begin{cases} x^i/i & \text{if} \quad i \text{ is even;} \\ k_i x^i/i + x^{i-1}/(i-1) & \text{if} \quad i \text{ is odd.} \end{cases} \tag{8.31}$$

Combining (8.20) and (8.31), the derivative of $p_i(\tilde{\theta}_i)$ w.r.t. $\tilde{\theta}_i$ is of

$$\frac{dp_i(\tilde{\theta}_i)}{d\tilde{\theta}_i} = g_i(\tilde{\theta}_i) = \begin{cases} \tilde{\theta}_i^{i-1} & \text{if} \quad i \text{ is even;} \\ k_i\tilde{\theta}_i^{i-1} + \tilde{\theta}_i^{i-2} & \text{if} \quad i \text{ is odd.} \end{cases} \tag{8.32}$$

Property 2 implies that $\tilde{\theta}_i(t) \in [-\Theta_{max},\ \Theta_{max}]$ for any $t \geq 0$. It can be checked that $p_i(x)$ is a well-posed Lyapunov function candidate over $[-\Theta_{max},\ \Theta_{max}]$ since

$$\begin{aligned} p_i(0) &= 0 \\ g_i(x) &= \frac{dp_i(x)}{dx} < 0, x \in [-\Theta_{max},\ 0) \\ g_i(x) &= \frac{dp_i(x)}{dx} > 0, x \in (0,\ \Theta_{max}] \end{aligned} \tag{8.33}$$

from the definition of k as in (8.21).

Lemma 1 *For plant in (8.13) and the companion adaptive systems in (8.15) and (8.16), if* $d_{max} = 0$, *then*

$$\dot{V} \leq -\alpha\tilde{y}^2. \tag{8.34}$$

Proof of Lemma 1: It follows from (8.13), (8.16) and $d_{max} = 0$ that the error model of the plant and the companion model can be written as

$$\dot{\tilde{y}} = \alpha\tilde{y} + \phi_0 - (m(y,u,\theta) + r(y,u,\theta)\zeta)\,. \tag{8.35}$$

Combining (8.35), (8.27) and (8.32), we have

$$\dot{V} \;=\; -\alpha\tilde{y}^2 + \sum_{i=1}^{N-1} g_i(\tilde{\theta})v_i + \sum_{i=0}^{N-1} \theta_i^i(\hat{\zeta}_i - \zeta)w_i$$

$$+\tilde{y}\left(\phi_0 - (m(y,u,\theta) + r(y,u,\theta)\zeta) + \sum_{i=2}^{N} g_i(\tilde{\theta})\phi_i + \sum_{i=0}^{N-1} \theta_i^i(\hat{\zeta}_i - \zeta)\eta_i\right). \quad (8.36)$$

For a well-posed Lyapunov function, it follows from (8.33) that

$$g_i(\tilde{\theta}_i) \geq 0 \quad (8.37)$$

when

$$\widehat{\theta}_i = \Theta_{max}. \quad (8.38)$$

It follows from (8.28) that

$$g_i(\tilde{\theta}_i)v_i \leq 0 \qquad \widehat{\theta}_i = \Theta_{max}. \quad (8.39)$$

Using the same methodology, it can be shown that

$$g_i(\tilde{\theta}_i)v_i \leq 0 \qquad \widehat{\theta}_i = 0. \quad (8.40)$$

It can be verified easily that

$$g_i(\tilde{\theta}_i)v_i = 0 \qquad \widehat{\theta}_i \in (0,\ \Theta_{max}) \quad (8.41)$$

since $v_i = 0$ when $0 < \widehat{\theta}_i < \Theta_{max}$. It follows from Property 2 that

$$\widehat{\theta}_i(t) \in [0,\ \Theta_{max}], \qquad \forall i = 0,..,N-1, \text{ and } t \geq 0. \quad (8.42)$$

Combining (8.39), (8.40), (8.41) and (8.42), we have

$$g_i(\tilde{\theta}_i)v_i \leq 0 \qquad \forall i = 1,..,N-1 \quad (8.43)$$

and hence

$$\sum_{i=1}^{N-1} g_i(\tilde{\theta})v_i \leq 0. \quad (8.44)$$

Using the same method, it can be verified easily that

$$\sum_{i=0}^{N-1} \theta_i^i(\hat{\zeta}_i - \zeta)w_i \leq 0. \quad (8.45)$$

Combining (8.36), (8.44) and (8.45), we have

$$\dot{V} \leq -\alpha\tilde{y}^2 + \tilde{y}\left(\phi_0 - (m(y,u,\theta) + r(y,u,\theta)\zeta) + \sum_{i=2}^{N} g_i(\tilde{\theta})\phi_i + \sum_{i=0}^{N-1} \theta_i^i(\hat{\zeta}_i - \zeta)\eta_i\right). \tag{8.46}$$

The equation

$$-A_r\eta = C_r \tag{8.47}$$

in (8.16) implies that

$$r(y,u,\theta_j) = -\sum_{i=0}^{N-1} \theta_j^i \eta_i \tag{8.48}$$

for any θ_j, $j = 1,..,N-1$ and therefore

$$r(y,u,\theta)\zeta = -\sum_{i=0}^{N-1} \theta_i^i \eta_i \zeta \qquad \forall\theta \in \Theta, \zeta \in [-\zeta_{max}, \zeta_{max}] \tag{8.49}$$

Similarly, it can be verified that equation

$$A_m\Phi = C_m - A_r C_\eta \tag{8.50}$$

in (8.16) and the definition of A_m in (8.19) implies that

$$m(y,u,\theta_j) + \sum_{i=0}^{N-1} \theta_j^i \eta_i \hat{\zeta} + \sum_{i=1}^{N-1} g_i(\widehat{\theta}_i - \theta_j)\phi_i - \phi_0 = 0 \qquad \forall j = 1,..,N-1. \tag{8.51}$$

and thus

$$m(y,u,\theta) + \sum_{i=0}^{N-1} \theta^i \eta_i \hat{\zeta} + \sum_{i=1}^{N-1} g_i(\tilde{\theta})\phi_i - \phi_0 = 0 \qquad \forall\theta \in \Theta. \tag{8.52}$$

Combining (8.49) and (8.52), it follows that

$$\phi_0 - (m(y,u,\theta) + r(y,u,\theta)\zeta) + \sum_{i=2}^{N} g_i(\tilde{\theta})\phi_i + \sum_{i=0}^{N-1} \theta_i^i(\hat{\zeta}_i - \zeta)\eta_i = 0, \ \forall\theta \in \Theta, \zeta \in [-\zeta_{max}, \zeta_{max}] \tag{8.53}$$

Combining (8.46) and (8.53), we have

$$\dot{V} \leq -\alpha\tilde{y}^2 \tag{8.54}$$

which proves Lemma 1. •

The following lemma shows that $\hat{y}$ will track y with the L_2 error bounded.

Lemma 2 *For plant in (8.13) and the companion adaptive systems in (8.15) and (8.16), if* $d_{max} = 0$*, then*

$$\int_0^\infty \tilde{y}^2 dt \leq \frac{V(0)}{\alpha} \tag{8.55}$$

Proof of Lemma 2: It follows from Lemma 1 that

$$\int_0^\infty \dot{V}\, dt \leq \int_0^\infty -\alpha\tilde{y}^2 dt \tag{8.56}$$

and therefore

$$V(\infty) - V(0) \leq \int_0^\infty -\alpha\tilde{y}^2 dt. \tag{8.57}$$

Equation (8.57) implies that

$$\int_0^\infty \alpha\tilde{y}^2 dt \leq V(0) - V(\infty). \tag{8.58}$$

Since

$$V(t) \geq 0, \qquad \forall t \geq 0, \tag{8.59}$$

it follows from (8.58) that (8.55) holds which proves the Lemma. •

8.3.3 Stability with disturbance

When $d_{max} \neq 0$, lemma 1 is replaced by the following lemma.

Lemma 3 *For plant in (8.13) and the companion adaptive systems in (8.15) and (8.16), we have*

$$\dot{V} \leq -\alpha\tilde{y}^2 + \tilde{y}d(t). \tag{8.60}$$

Proof of Lemma 3: It follows from (8.13) and (8.16) that the error model of the plant and companion model can be written as

$$\dot{\tilde{y}} = \alpha\tilde{y} + \phi_0 - (m(y,u,\theta) + r(y,u,\theta)\zeta) + d(t) \tag{8.61}$$

instead of (8.35). Using the same deviation as in the proof of lemma 1, (8.60) can be proved. •

With disturbance, Property 2 still holds which means that all auxiliary estimates are bounded and they are stable in nature. In the following lemma, we will show that $\tilde{y}$ is also bounded.

Lemma 4 *For the plant in (8.13) and the companion adaptive systems in (8.15) and (8.16), we have*

$$\begin{aligned} \dot{V} &\leq 0 \qquad \text{if } V \geq V_{max} \\ V_{max} &= (\frac{d_{max}}{\alpha})^2/2 + \sum_{i=1}^{N-1} p_i(\Theta_{max}) + \sum_{i=0}^{N-1} \Theta^i_{max}(2\zeta_{max})^2/2 \end{aligned} \tag{8.62}$$

Proof of Lemma 3: It follows from Property 2 that

$$\begin{aligned} |\widehat{\theta}_i(t) - \theta| &\in [-\Theta_{max}, \Theta_{max}] \qquad \forall t \geq 0, i = 1, .., N-1 \\ |\hat{\zeta}_i(t) - \zeta| &\leq 2\zeta_{max} \qquad \forall t \geq 0, i = 0, .., N-1. \end{aligned} \tag{8.63}$$

The definition of $p()$ in (8.31) implies that

$$p_i(\Theta_{max}) = \max_{x \in [-\Theta_{max}, \Theta_{max}]} p_i(x), \qquad i = 1, .., N-1. \tag{8.64}$$

Combining (8.63) and (8.64), we have

$$\sum_{i=1}^{N-1} p_i(\widehat{\theta}(t) - \theta) + \sum_{i=0}^{N-1} \theta^i_i(\hat{\zeta}(t) - \zeta)^2/2 \leq \sum_{i=1}^{N-1} p_i(\Theta_{max}) + \sum_{i=0}^{N-1} \Theta^i_{max}(2\zeta_{max})^2/2. \tag{8.65}$$

If

$$V(t) \geq V_{max}, \tag{8.66}$$

it follows from (8.30) and (8.65) that

$$\tilde{y}(t)^2/2 \geq (\frac{d_{max}}{\alpha})^2/2. \tag{8.67}$$

It follows from (8.67) and the fact $d(t) \leq d_{max}$ in (8.13) that

$$-\alpha\tilde{y}^2 + \tilde{y}d(t) \leq 0 \tag{8.68}$$

which implies that

$$\dot{V}\ (t) \leq 0, \text{ if } V(t) \geq V_{max}. \tag{8.69}$$

•

It follows from (8.30) that

$$\frac{\tilde{y}(t)^2}{2} \leq V_{max} \tag{8.70}$$

if $V(t) \leq V_{max}$ and $\tilde{y}$ is bounded.

With the disturbance, we can not talk about asymptotic convergence of $\tilde{y}$ to zero anymore. However we still want to have some criteria about $\tilde{y}$ and some average quantity of $\tilde{y}$ could serve this purpose, which is stated in the following lemma.

Lemma 5

$$\lim_{T\to\infty} \frac{\int_{t=0}^{T} z(t)^2 dt}{T} \leq \frac{d_{max}^2}{4\alpha^2} \tag{8.71}$$

where

$$z(t) = \tilde{y} - \frac{d(t)}{2\alpha}. \tag{8.72}$$

Proof of Lemma 3: It follows from Lemma 3 that

$$\dot{V} \leq -(\sqrt{\alpha}\tilde{y} - \frac{d(t)}{2\sqrt{\alpha}})^2 + \frac{d(t)^2}{4\alpha}. \tag{8.73}$$

Since $|d(t)| \leq d_{max}$, we have

$$\int_{t=0}^{T} \dot{V}\, dt \leq -\alpha \int_{t=0}^{T} (\tilde{y} - \frac{d(t)}{2\alpha}) dt + \frac{d_{max}^2}{4\alpha} \tag{8.74}$$

and thus

$$\frac{V(T) - V(0)}{\alpha T} \leq -\int_{t=0}^{T} (\tilde{y} - \frac{d(t)}{2\alpha}) dt + \frac{d_{max}^2}{4\alpha^2}. \tag{8.75}$$

It follows from (8.30) and lemma 4 that both $V(0)$ and $V(T)$ are finite and

$$\lim_{T\to\infty} \frac{\int_{t=0}^{T} (\tilde{y} - \frac{d(t)}{2\alpha}) dt}{T} \leq \frac{d_{max}^2}{4\alpha^2} \tag{8.76}$$

which proves lemma 5. •

8.3.4 Stability under Bounded Output Noise

In practical systems, output noise which prevents the using of the differentiator always exists. In this section, we will establish the stability of the companion adaptive system under output noise.

With bounded output noise, plant (8.13) is transformed into

$$
\begin{aligned}
\dot{y} &= -\alpha y + m(y,u,\theta) + r(y,u,\theta)\zeta + d(t) \\
d(t) &\leq d_{max} \\
y_n &= y + n \\
|n| &\leq n_{max}
\end{aligned}
\tag{8.77}
$$

where unknown parameter $[\theta, \zeta]$ is defined as in (8.12), d_{max} and n_{max} are the upperbounds of absolute values of the disturbance and output noise. About plant in (8.77), we make the following assumptions.

Assumption 2: $\forall\ y(t), u(t)$, $m(y(t), u(t), \theta)$ and $r(y(t), u(t), \theta)$ are Lipschitz continuous w.r.t. y.

It follows from Assumption 2 and (8.77) that there eixsts N_b such that

$$|n_x(t)| \leq N_b|n(t)| \leq N_b n_{max}. \tag{8.78}$$

where

$$n_x(t) = m(y_n, u, \theta) + r(y_n, u, \theta)\zeta - (m(y, u, \theta) + r(y, u, \theta)\zeta) \tag{8.79}$$

The companion model is the same as in (8.15) and the adaptive law is of

$$
\begin{aligned}
\dot{\hat{\theta}}_i &= \begin{cases} 0 & \text{if } \tilde{y}_n\phi_i > 0 \text{ and } \hat{\theta}_i \geq \Theta_{max} \\ 0 & \text{if } \tilde{y}_n\phi_i < 0 \text{ and } \hat{\theta}_i \leq 0 \\ \tilde{y}_n\phi_i & \text{otherwise.} \end{cases} \\
& \quad \forall i = 1, .., N-1 \\
\dot{\hat{\zeta}}_i &= \begin{cases} 0 & \text{if } \tilde{y}_n\eta_i > 0 \text{ and } \hat{\zeta}_i \geq \zeta_{max} \\ 0 & \text{if } \tilde{y}_n\eta_i < 0 \text{ and } \hat{\zeta}_i \leq -\zeta_{max} \\ \tilde{y}_n\eta_i & \text{otherwise.} \end{cases} \\
& \quad \forall i = 0, .., N-1 \\
C_r &= [r(y,u,\theta_1), .., r(y,u,\theta_i), .., r(y,u,\theta_N)]^T \\
\eta &= [\eta_0, ..., \eta_{N-1}] = -A_r^{-1}C_r \\
C_m &= [m(y,u,\theta_1), .., m(y,u,\theta_i), .., m(y,u,\theta_N)]^T \\
C_\eta &= [\hat{\zeta}_0\eta_0, .., \hat{\zeta}_i\eta_i, .., \hat{\zeta}_{N-1}\eta_{N-1}]^T
\end{aligned}
$$

$$\begin{aligned}\Phi &= [\phi_0, ..., \phi_{N-1}] = A_m^{-1}(C_m - A_r C_\eta) \\ \tilde{y} &= \hat{y} - y \end{aligned} \tag{8.80}$$

where the other definitions are the same as in (8.16). It is noted that all y are replace by y_n in the adaptive law since y is not measurable.

Lemma 6 *For plant in (8.77) and the companion adaptive systems in (8.15) and (8.80) under Assumption 2, we have*

$$\dot{V} \leq -\alpha\tilde{y}^2 + \tilde{y}(d(t) + n_x(t)) \tag{8.81}$$

where V is defined in (8.30) and $n_x(t)$ is defined in (8.79).

Proof of Lemma 6: It follows from (8.77) and (8.80) that the error model of the plant and companion model can be written as

$$\dot{\tilde{y}} = \alpha\tilde{y} + \phi_0 - (m(y, u, \theta) + r(y, u, \theta)\zeta) + d(t) \tag{8.82}$$

instead of (8.35). Combining (8.82) and (8.78), it follows that

$$\dot{\tilde{y}} = \alpha\tilde{y} + \phi_0 - (m(y_n, u, \theta) + r(y_n, u, \theta)\zeta) + d(t) + n_x(t)$$

where $n_x(t)$ is defined in (8.79). Using the same deviation as in the proof of lemma 1, we have

$$\dot{V} \leq -\alpha\tilde{y}^2 + \tilde{y}(d(t) + n_x(t)) + \tilde{y}_n \left(\phi_0 - (m(y_n, u, \theta) + r(y_n, u, \theta)\zeta) + \sum_{i=2}^{N} g_i(\tilde{\theta})\phi_i + \sum_{i=0}^{N-1} \theta_i^i(\hat{\zeta}_i - \zeta)\eta_i \right). \tag{8.83}$$

Using the same method as in Lemma 3, it follows from (8.80) that

$$\left(\phi_0 - (m(y_n, u, \theta) + r(y_n, u, \theta)\zeta) + \sum_{i=2}^{N} g_i(\tilde{\theta})\phi_i + \sum_{i=0}^{N-1} \theta_i^i(\hat{\zeta}_i - \zeta)\eta_i \right) = 0. \tag{8.84}$$

Combining (8.83) and (8.84), we have

$$\dot{V} \leq -\alpha\tilde{y}^2 + \tilde{y}(d(t) + n_x(t)) \tag{8.85}$$

and Lemma 6 is proved. •

About the plant with bounded disturbance and output noise, we have the following lemmas which can be derived from lemma 6 using the same methods as in section 8.3.3.

Lemma 7 *For plant in (8.77) and the companion adaptive systems in (8.15) and (8.80) under assumption 2, we have*

$$\begin{aligned} |\tilde{y}(t)| &\leq \sqrt{\frac{2V_{max}}{\alpha}} \\ V_{max} &= (\frac{d_{max} + N_b n_{max}}{\alpha})^2/2 + \sum_{i=1}^{N-1} p_i(\Theta_{max}) + \sum_{i=0}^{N-1} \Theta^i_{max}(2\zeta_{max})^2/2 \end{aligned}$$

Lemma 8

$$\lim_{T\to\infty} \frac{\int_{t=0}^{T} z(t)^2 dt}{T} \leq \frac{(d_{max} + N_b n_{max})^2}{4\alpha^2} \tag{8.86}$$

where

$$z(t) = \tilde{y} - \frac{d(t) + n_x(t)}{2\alpha}. \tag{8.87}$$

8.3.5 Extension to Higher Dimension

When ω belongs to $\mathbb{R}^n$ instead of $\mathbb{R}$, if the plant is N piece-wise linear over ω, it can be written as

$$\dot{y} = -\alpha y + m(y, u, \theta) + \sum_{j=1}^{n} r_j(y, u, \theta)\zeta_j \tag{8.88}$$

where

$$\begin{aligned} \theta &\in \Theta = \{\theta_1, .., \theta_i, ..\theta_N\} \\ \zeta_j &\in [-\zeta_{j_{max}}, \zeta_{j_{max}}] \\ \theta_i &= \frac{i-1}{(N-1)\Theta_{max}}. \end{aligned}$$

The companion model is the same as in (8.15) and the adaptive law is of

$$\dot{\hat{\theta}}_i = \begin{cases} 0 & \text{if } \tilde{y}\phi_i > 0 \text{ and } \hat{\theta}_i \geq \Theta_{max} \\ 0 & \text{if } \tilde{y}\phi_i < 0 \text{ and } \hat{\theta}_i \leq 0 \\ \tilde{y}\phi_i & \text{otherwise.} \end{cases}$$
$$\forall i = 1, .., N-1$$

$$
\begin{aligned}
\dot{\hat{\zeta}}_{ji} &= \begin{cases} 0 & \text{if } \tilde{y}\eta_{j_i} > 0 \text{ and } \hat{\zeta}_{j_i} \geq \zeta_{j_{max}} \\ 0 & \text{if } \tilde{y}\eta_{j_i} < 0 \text{ and } \hat{\zeta}_{j_i} \leq -\zeta_{j_{max}} \\ \tilde{y}\eta_{j_i} & \text{otherwise.} \end{cases} \\
& \qquad \forall i = 0,..,N-1, j = 1,..,n \\
C_{r_j} &= [r_j(y,u,\theta_1),..,r_j(y,u,\theta_i),..,r_j(y,u,\theta_N)]^T, \qquad j = 1,..,n \\
\eta_j &= [\eta_{j0},...,\eta_{j_{N-1}}] = -A_r^{-1}C_{r_j}, \qquad j = 1,..,n \\
C_m &= [m(y,u,\theta_1),..,m(y,u,\theta_i),..,m(y,u,\theta_N)]^T \\
C_{\eta_j} &= [\hat{\zeta}_{j0}\eta_{j0},..,\hat{\zeta}_{j_i}\eta_{j_i},..,\hat{\zeta}_{j_{N-1}}\eta_{j_{N-1}}]^T, \qquad j = 1,..,n \\
\Phi &= [\phi_0,...,\phi_{N-1}] = A_m^{-1}(C_m - \sum_{j=1}^{n} A_r C_{\eta_j}) \\
\tilde{y} &= \hat{y} - y
\end{aligned}
\tag{8.89}
$$

where the other definitions are the same as in the scalar case. Modify the Lyapunov function in (8.30) into

$$
V = \tilde{y}^2/2 + \sum_{i=1}^{N-1} p_i(\tilde{\theta}_i) + \sum_{j=1}^{n}\sum_{i=0}^{N-1} \theta_i^i(\hat{\zeta}_{j_i} - \zeta_j)^2/2, \tag{8.90}
$$

using the similar methodology as in the proof of Lemma 1, it can be shown that

$$
\dot{V} \leq -\alpha\tilde{y}^2. \tag{8.91}
$$

The effects of nonzero disturbance is the same as in scalar case.

The extension of state variables and nonlinear functions to higher dimension, i.e.

$$
\dot{X} = AX + I[f_1(X,u,\omega_1) \, ..f_2(X,u,\omega_2),..,f_2(X,u,\omega_2) \tag{8.92}
$$

is straightforward since we can deal with every nonlinear function with unknown parameters separately.

8.4 Control Law

In section 8.3, we introduced the companion adaptive system and showed that the adaptive law and the error model between plat and companion model are globally stable under Assumption 1.

However how to choose control signal u is still an open problem. We note that comparing to the plant in (8.13) which is partial known and contains unknown information, the companion model

$$\dot{\hat{y}} = -\alpha\hat{y} + \phi_0(y, u) \tag{8.93}$$

involves no uncertain information. Therefore, the control law of the CPAC is simply to find a controller for companion model (8.93). If the problem formulation is of

$$\dot{y} = -\alpha y + f(y) + u, \tag{8.94}$$

equation (8.93) becomes

$$\dot{\hat{y}} = -\alpha\hat{y} + \phi_0(y) + u \tag{8.95}$$

and we simplly choose control law to be

$$u = -\phi_0(y) + \alpha\hat{y} - \beta(\hat{y} - r), \qquad \beta > 0. \tag{8.96}$$

Combining (8.95) and (8.96), the closed-loop system is

$$\dot{\hat{y}} = -\beta(\hat{y} - r). \tag{8.97}$$

Theorem 1 *For plant in (8.94), the companion adaptive systems in (8.95) (8.16), and control law in (8.96), if* $d_{max} = 0$ *and* r *is bounded, then*

$$\lim_{t\to\infty} \tilde{y}(t) = 0 \tag{8.98}$$

Proof of Lemma 1: Bounded r implies bounded $\hat{y}$ from (8.97). It follows from Lemma 4 that $\tilde{y}_\epsilon$ is bounded and therefore y is bounded. It is noted that all variables in the adaptive law and the error model between the plant and the companion model are related through non-singular relationships and therefore $\dot{\tilde{y}}$ is bounded for bounded y. Since $\tilde{y} \in L^2$ from Lemma 2, it follows from Barbalat's lemma that (8.98) holds, which is the same as the proof for the linear parameterized systems. •

If we find the control law which stabilizes the companion model, same control signal stabilizes the plant too. The reason we use plant (8.13) is to demonstrate that the companion adaptive system in section 8.3 is universal for any nonlinear systems under Assumption 1. The design of control law for nonlinear companion model is case by case even it is deterministic. In what follows, we will introduce several more classes of nonlinearly parameterized systems where the control law is given and we will keep in mind that the CPAC will not be restricted to these classes.

8.4.1 Class 1:

The plant is of

$$\dot{x}= Ax + B_1 u + B_2 f(x,\omega) \tag{8.99}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^{n_u}$, $f = [f_1(X,\omega),..,f_{n_f}(X,\omega)]^T$ and A, B_1 and B_2 are matrices with appropriate dimensions.

In addition to Assumption 1 for f, we make the following assumptions regarding the plant in (8.99):

Assumption 3: (A, B_1) is controllable.

Assumption 4: $B_2 \in span(B_1)$.

It follows from Assumption 3 that there exists $K \in \mathbb{R}^{n_u} \times \mathbb{R}^n$ such that $A_c = A + B_1 K$ is stable. Let

$$\begin{aligned} u &= u_1 + u_2 \\ u_1 &= Kx, \end{aligned} \tag{8.100}$$

the system is transformed into

$$\dot{x}= A_c x + B_1 u_2 + B_2 f(x,\omega) \tag{8.101}$$

and the companion model is of

$$\dot{\hat{x}}= A_c \hat{x} + B_1 u_2 + B_2 \bar{\Phi} \tag{8.102}$$

where $\bar{\Phi} = [\phi_{01},..,\phi_{0n_f}]^T$ and the adaptive law for specific nonlinear function f_i is the same as in (8.16). It follows from assumption 4 that there exist $B_3 \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_f}$ such that

$$B_2 = B_1 B_3. \tag{8.103}$$

Choosing

$$u_2 = -B_3\bar{\bar{\Phi}} \tag{8.104}$$

and the dynamics of the companion model under this control law is of

$$\dot{\hat{X}} = A_c\hat{X}_k \tag{8.105}$$

which is stable and the effects of unknown parameter is totally eliminated.

The entire CPAC is summarized as the companion model in (8.102), the adaptive law as in (8.16), and the control law as in (8.100) and (8.104).

8.4.2 Class 2

The plant is of

$$\dot{y} = -\alpha y + uf(y,\omega) \tag{8.106}$$

where $\alpha > 0$ is known. α can also be unknown and the traditional linear adaptive control methods can be applied for unknown α.

The stabilization of this system is easy and you just need to set $u = 0$. However if the purpose is tracking a bounded reference signal r, you have to have the knowledge of $f(y,\omega)$ whose sign and value are unknown. This makes the problem difficult and is possible to be unstable in the control process.

In addition to Assumption 1, we assume $d_{max} = 0$ in this case for simplicity and we make the following assumption about plant in (8.106).

Assumption 5:

$$f(0,\omega) \neq 0, \tag{8.107}$$

$$f(r,\omega) \neq 0. \tag{8.108}$$

The necessity of Assumption 5 is straightforward. If (8.107) is not satisfied, $y = 0$ is an equilibrium point for the system and the controller loose control of the plant when $y = 0$ since $uf(0,\omega) = 0$ always. If (8.108) is not satisfied, there is no u can make $y = r$.

The companion model is of

$$\dot{\hat{y}} = -\alpha\hat{y} + \phi_0(y,u). \tag{8.109}$$

Since u is linear and its linearity is kept in calculation of $\phi_0(y, u)$, it follows that

$$\phi_0(y, u) = u\bar{\phi}(y). \tag{8.110}$$

If we choose the control law as of

$$u = \frac{\alpha r}{\bar{\phi}(y)} \tag{8.111}$$

the closed loop companion model will be

$$\dot{\hat{y}} = -\alpha\hat{y} + \alpha r \tag{8.112}$$

and $\hat{y}$ will converge asymptotically to r.

However we note that since $\bar{\phi}(y)$ could be zero and there is a problem in choosing the control law as in (8.110). When $\bar{\phi}(y)$ approaches zero, we just choose u as an arbitrary nonzero constant and $u\bar{\phi}$ approaches zero too, which implies that $\hat{y}$ will approaches zero. However we note that when $\hat{y}$ approaches zero, V will keep decreasing and $\hat{y}$ can not stay in zero for a long time. The reason is that for a finite u, it follows from Assumption 5 that y will not approaches zero and the output error in $\tilde{y}$ will reduce V and drive $\bar{\phi}$ away from zero. When V is reduced to zero, $\bar{\phi}$ will approach $f(y, \omega)$ and we do not need to worry about the singular problem anymore.

For a more rigorous statement, we assume f is continuous and (8.107) implies that there exist positive constants δ and y_δ such that

$$|f(y, \omega)| \geq \delta, \qquad \forall y \in [-y_\delta,\ y_\delta]. \tag{8.113}$$

Assume $r \leq r_{max}$, since u can not be infinite and we set some bounded value for u as

$$U_{max} = \frac{\alpha r_{max}}{\bar{\delta}/2}. \tag{8.114}$$

where

$$\bar{\delta} = min\{\delta, y_\delta\}. \tag{8.115}$$

The control law is of

$$u = \begin{cases} \frac{\alpha r}{\bar{\phi}(y)}, & if \quad |\frac{\alpha r}{\bar{\phi}(y)}| \leq U_{max} \\ U_{max}, & if \quad \frac{\alpha r}{\bar{\phi}(y)} > U_{max} \\ -U_{max}, & if \quad \frac{\alpha r}{\bar{\phi}(y)} < -U_{max}. \end{cases} \tag{8.116}$$

If $|\frac{\alpha r}{\bar{\phi}(y)}| \leq U_{max}$, the control law in (8.116) is equivalent to (8.111). Otherwise, it follows from (8.116) and (8.114) that

$$|\bar{\phi}(y)| \leq \frac{\bar{\delta}}{2}. \tag{8.117}$$

If $|y(t)| > y_\delta$, it follows from (8.115) and (8.116) that

$$|y(t) - \hat{y}(t)| \geq \frac{\bar{\delta}}{2} \tag{8.118}$$

and output error exists. If $|y(t)| \leq y_\delta$, it follows from (8.113) that

$$|f(y, \omega)| \geq \delta \tag{8.119}$$

and

$$|f(y, \omega) - \bar{\phi}| \geq \frac{\bar{\delta}}{2} \tag{8.120}$$

which will also result in output error. Therefore, the control law in (8.116) keeps reducing Lyapunov function if

$$|\frac{\alpha r}{\bar{\phi}(y)}| \geq U_{max}. \tag{8.121}$$

Since $V(0)$ is finite, control law in (8.116) is stable and will guarantee the asymptotic convergence of y to any constant r.

8.4.3 Class 3

Now we consider a case where not all states measured. The plant is of

$$\begin{aligned} \dot{x} &= Ax + bu + bf(y, \omega) \\ y &= c^T X \end{aligned} \tag{8.122}$$

where $x \in \mathbb{R}^n$ are state variables, $y \in \mathbb{R}$ is measured output signal, $u \in \mathbb{R}$ is control signal to be determined and A, b, c are in appropriate dimensions. In addition to Assumption 1 about $f(y, \omega)$, we make the following assumptions about (8.122).

Assumption 6: (A, b) is controllable.

Assumption 7: $\exists k_o$ such that the $A_o = A + k_o c^T$ is stable and $c^T(sI - A_o)b$ is strictly positive real.

It follows from Assumption 6-7 that there exist k_c and k_o such that A_c and A_o are stable matrices where

$$\begin{aligned} A_c &= A - bk_c \\ A_o &= A - k_o c^T. \end{aligned} \tag{8.123}$$

It follows from Assumption 7 and K-Y-P lemma that there exists $P > 0$ such that

$$\begin{aligned} Q &= PA_o + A_o^T P < 0 \\ Pb &= c. \end{aligned} \tag{8.124}$$

We construct the companion model as of

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} - k_o\tilde{y} + bu + b\phi_0 \\ \hat{y} &= c^T\hat{x} \end{aligned} \tag{8.125}$$

and the adaptive law is exactly the same one as in (8.16) using $\tilde{y}$ in the adaptation of auxiliary estimates.

The error model between (8.122) and (8.125) now becomes

$$\dot{\tilde{x}}= A_o\tilde{x} + bu + b\phi_0 \tag{8.126}$$

and we construct the Lyapunov function as of

$$V = \tilde{x}^T P\tilde{x}/2 + \sum_{i=1}^{N-1} p_i(\widehat{\theta}_i - \theta) + \sum_{i=0}^{N-1} \theta_i^i(\hat{\zeta}_i - \zeta)^2/2. \tag{8.127}$$

It follows from (8.124) that

$$\tilde{x}^T Pb = \tilde{x}^T c = \tilde{y} \tag{8.128}$$

and therefore (8.127) implies that

$$\dot{V}= \tilde{x}^T Q\tilde{x} + \tilde{y}d(t) + \tilde{y}\left(\phi_0 - (m(y,u,\theta) + r(y,u,\theta)\zeta) + \sum_{i=2}^{N} g_i(\tilde{\theta})\phi_i + \sum_{i=0}^{N-1} \theta_i^i(\hat{\zeta}_i - \zeta)\eta_i\right). \tag{8.129}$$

Using the similar method as in Lemma 2, it can be shown that

$$\dot{V} \leq \tilde{x}^T Q \tilde{x} \leq 0 \tag{8.130}$$

if $d_{max} = 0$ and

$$\dot{V} \leq \tilde{x}^T Q \tilde{x} + \tilde{y} d(t) \tag{8.131}$$

with disturbance. The discussion of the stability and output error is exactly the same as in section 8.3.2.

Since we establish the companion adaptive system, what left is to find the control law to control companion model in (8.125), which is of

$$u = -k_c^T \hat{x} - \phi_0 + k_g r \tag{8.132}$$

and the closed loop of the companion model is of

$$\dot{\hat{x}} = A_c \hat{x} + b k_g r \tag{8.133}$$

where A_c is stable from (8.123) and k_g is the gain which satisfies

$$-c^T A_c^{-1} b k_g = 1 \tag{8.134}$$

and makes $y = r$ in steady state. Thus, under assumptions 1,6,7, the complete CPAC includes the companion model in (8.125), the adaptive law in (8.16) and the control law in (8.132).

Remark 4: We assume in Assumption 1 that a nonlinearly parameterized system can be approximated by a piece-wise linear function for any $y(t)$, this condition can be relaxed a little bit since $y(t)$ is bounded from stability. For a bounded reference signal $r(t)$, $\hat{y}(t)$ is also bounded from the control law and the companion model. It follows from Lemma 4 that for any given d_{max}, the bounds of y can be determined such that

$$y(t) \in Y^b(d_{max}) \tag{8.135}$$

where Y^b is some bounded region. Therefore, we just choose N big enough such that the disturbance due to approximation error is smaller than d_{max} for any $y \in Y^b$ and do not need to consider unrestrained y which is sometimes difficult to establish for some functions like $f = |y|^\omega$. We see here the CPAC serve as a robust stable controller which is in contrast to the unconditional application of traditional adaptive controller for NLP systems. Since linear adaptive controller uses a linear function to approximate a global nonlinear function in the unknown parameter region, we have no control of the amplitude of the approximation error or disturbance at given y. Therefore, big approximation error results in big y, which in turn results in bigger approximation error and this could drive the whole system unstable. Under the CPAC structure, robustness of adaptive controller for nonlinearly parameterized systems can be established under desired margin. In fact we can set desired approximation error under the guaranteed stability and the trade-off is just between precision and complexity.

8.5 Simulation Results

We will use a plant in case 3 of section 8.4 as our simulation example. The plant is of

$$\begin{aligned} \dot{x} &= Ax + bu + bf(y, \omega) \\ y &= c^T x \end{aligned} \tag{8.136}$$

where

$$\begin{aligned} A &= \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} \\ b &= [2 \;\; 1]^T \\ c &= [1 \;\; 0]^T. \end{aligned} \tag{8.137}$$

From the results in case 3 of section 8.4, the companion model is of

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} - k_o\tilde{y} + bu + b\phi_0 \\ \hat{y} &= c^T\hat{x}, \end{aligned}$$

the adaptive law is the same as in (8.16) and the control law is of

$$u = -k_c\hat{x} - \phi_0 + k_g r \tag{8.138}$$

where

$$\begin{aligned} k_o &= [6 \;\; 3]^T \\ k_c &= [2.6 \;\; 0.8]^T \\ k_g &= 2.4 \end{aligned} \tag{8.139}$$

It can be checked easily that the choice in (8.139) makes A_c and A_o stable and guarantees the existence of

$$P = \begin{bmatrix} 0.5802 & -0.1604 \\ -0.1604 & 0.3208 \end{bmatrix} > 0 \tag{8.140}$$

which satisfies $Q = PA_o + A_o^T P < 0$ and $Pb = c$.

What left is just the choice of N and $m_i(y), r_i(y)$ as in (8.7). We will consider two cases for different f, Ω and reference signal r.

<u>Case 1</u> Piece-wise linear function

$$\begin{aligned} f(y,\omega) &= \begin{cases} y^3\omega, & \text{if } \omega \in [0,\;5] \\ -y^3\omega + y^2\omega, & \text{if } \omega \in [-2.5,\;0] \\ -y^3 + y^2(-5-\omega), & \text{if } \omega \in [-5\;\;-2.5] \end{cases} \\ \Omega &= [-5,\;5] \\ r &= 1.5. \end{aligned} \tag{8.141}$$

In this situation, we choose $N = 4$ and divide Ω evenly into N regions

$$\Omega_i = [\underline{\Omega}_i,\;\bar{\Omega}_i]. \tag{8.142}$$

The choice of $m_i(y), r_i(y)$ is of

$$m_i(y,u) = f(y,u,\bar{\omega}) \tag{8.143}$$

$$r_i(y,u) = \left.\frac{\partial f(y,u,\omega)}{\partial \omega}\right|_{\bar{\omega}} \tag{8.144}$$

$$\bar{\omega} = \frac{\bar{\Omega}_i + \underline{\Omega}_i}{2}. \tag{8.145}$$

The simulation results is shown in figure 8-2 and the asymptotic convergence of $\hat{y}$ and y to r is illustrated.

Case 2 Nonlinear function

$$\begin{aligned} f(y,\omega) &= |y|^{\omega} \\ \Omega &= [1,\ 5] \\ r &= 1.5sin(t). \end{aligned} \tag{8.146}$$

For this nonlinear function, we choose $N = 7$ following the discussion in Remark 4 and divide Ω evenly into N regions

$$\Omega_i = [\underline{\Omega}_i,\ \bar{\Omega}_i]. \tag{8.147}$$

What remains is the determination of $m_i(y)$ and $r_i(y)$, which is of

$$\begin{aligned} m_i(y) &= f(y, \frac{\underline{\Omega}_i + \bar{\Omega}_i}{2}) \\ r_i(y) &= \frac{f(y,\bar{\Omega}_i) - f(y,\underline{\Omega}_i)}{\bar{\Omega}_i - \underline{\Omega}_i} \end{aligned} \tag{8.148}$$

The simulation results is shown in figure 8-3. It is noted that $\hat{y}$ tracks r with desired transient error, as well as y.

To demonstrate the necessity of CPAC, we also include the results of nominal controller for plant in (8.136) and (8.146). The nominal controller is of

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} - k_o(\hat{y} - y) + bu + bf(y,\hat{\omega}) \\ \hat{y} &= c^T\hat{x} \\ u &= -k_c\hat{x} - f(y,\hat{\omega}) + k_g r \end{aligned} \tag{8.149}$$

where the values of k_c, k_o and k_g are the same as in (8.139). If ω is known, we set $\hat{\omega} = \omega$ and this nominal controller controls the plant well which is shown in figure 8-4. Figure 8-5 and 8-6 plots the performance of nominal controller when $\hat{\omega} = \omega(1 + 2.2\%)$ and $\hat{\omega} = \omega(1 - 2.2\%)$ respectively. It can be seen that they are unstable which shows that the nominal controller is very sensitive to the unknown parameter ω. The comparison of the simulations shows that to control the plant as in (8.136) and (8.146), we have to know the information of ω and the CPAC is necessary.

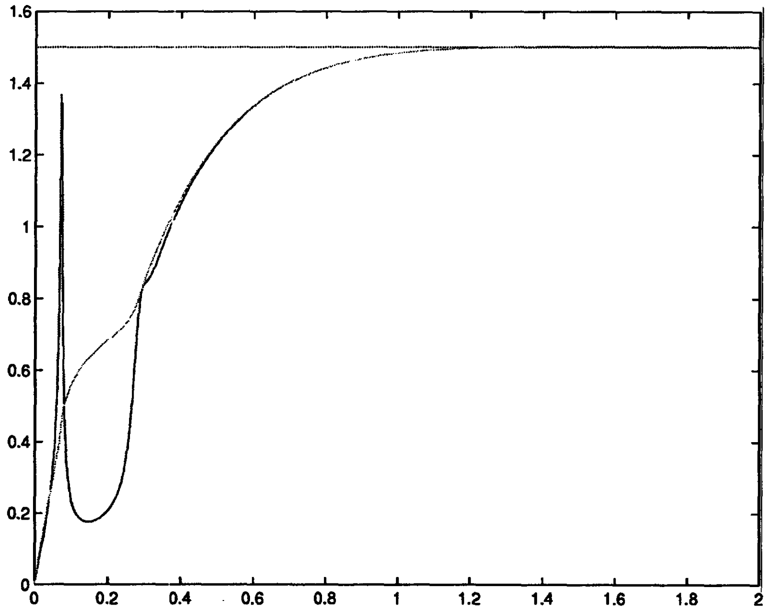

Figure 8-2: CPAC - (Case 1): Trajectory of y, $\hat{y}$ and reference signal r

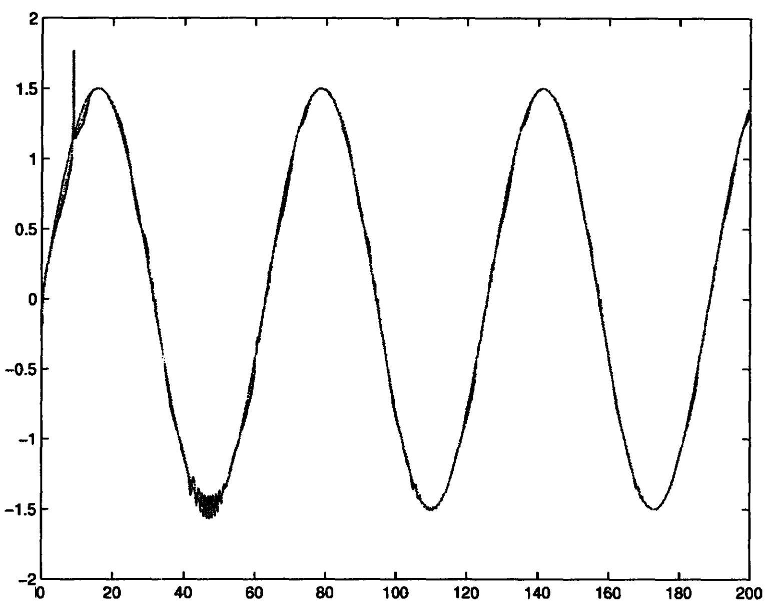

Figure 8-3: CPAC - (Case 2): Trajectory of y, $\hat{y}$ and reference signal r

8.6 Conclusion

The adaptive law in the CPAC to deal with nonlinear parameters opens the door for a general non-linear adaptive controller. The CPAC, which transforms the controller design from a partial known

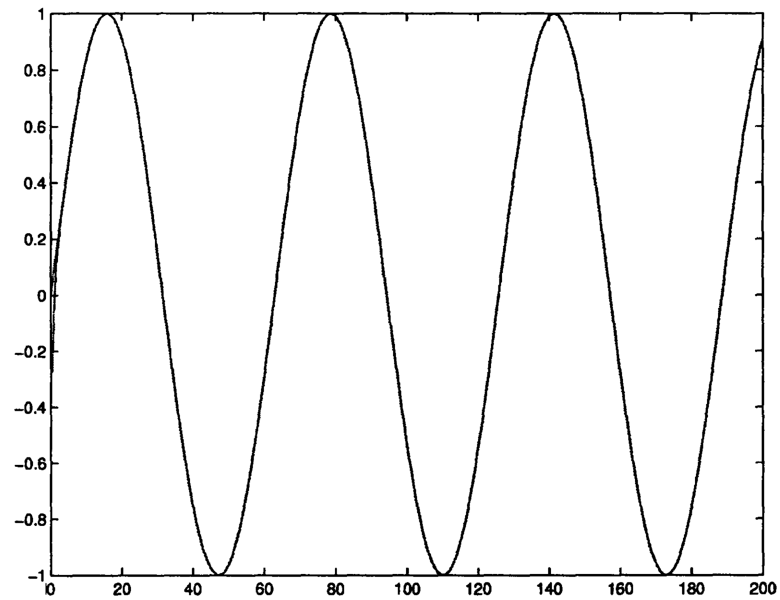

Figure 8-4: Nominal Controller - (Case 2, $\hat{\omega} = \omega$): Trajectory of y, and reference signal r

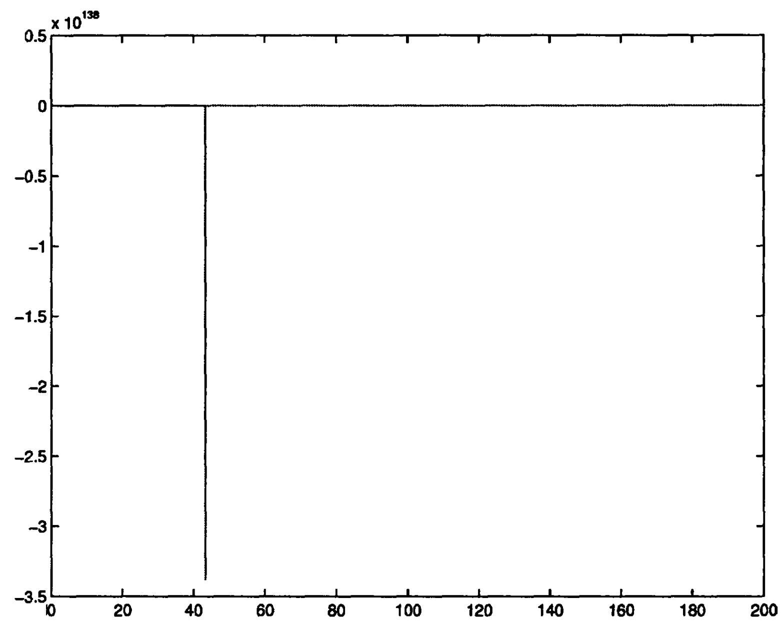

Figure 8-5: Nominal Controller - (Case 2, $\hat{\omega} = \omega(1 + 2.2\%)$): Trajectory of y, and reference signal r

plant into a deterministic one, serves as a general tool to deal with nonlinearly parameterized systems. Stability of the CPAC with bounded disturbance is also established. For several classes of

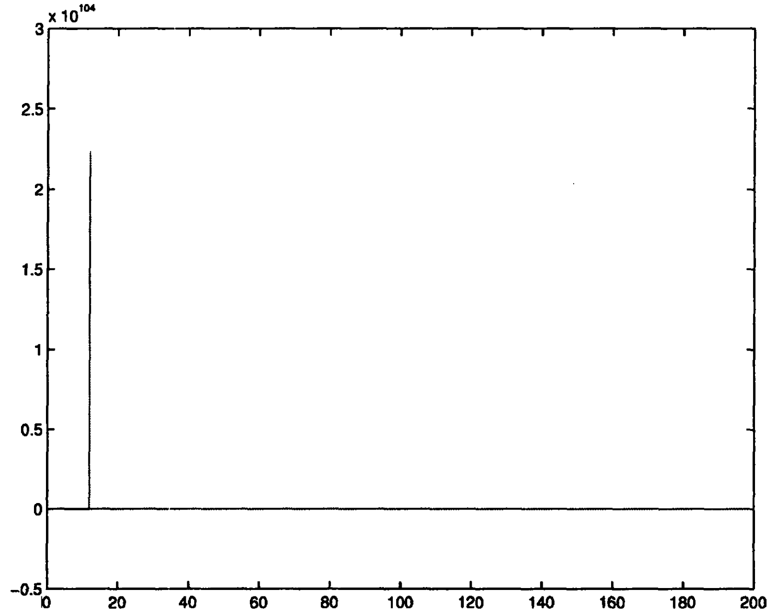

Figure 8-6: Nominal Controller - (Case 2, $\widehat{\omega} = \omega(1-2.2\%)$): Trajectory of y, and reference signal r

NLP systems, the control laws are given and the complete CPACs are constructed. We note that the CPAC is not restricted to these classes and more applications are worthy to be explored.

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