Integrated Quality and Quantity Modelling of a Production Line

by

Jongyoon Kim

B.S., Seoul National University (1997)
M.S., Seoul National University (1999)

Submitted to the Department of Mechanical Engineering in partial fulfillment of the requirements for the degree of

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Author ....
Department of Mechanical Engineering
November 10, 2004

Certified by ....
Stanley B. Gershwin
Senior Research Scientist
Thesis Supervisor

Accepted by ....
Lallit Anand
Chairman, Department Committee on Graduate Students

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Abstract

The interaction of quantity and quality performance in a factory is clearly of great economic importance. However, there is very little quantitative analytical literature in this area. This thesis is an essential early research step in analyzing how production system design, quality, and productivity are inter-related in transfer lines. We develop a new Markov process model for machines with both quality and operational failures, and we identify important differences between types of quality failures. We present analytic models, solution techniques, performance evaluations, and validation of two-machine systems as well as longer transfer lines. Through numerical studies, we have investigated some of the conventional wisdom on this interaction, and we have found that the wisdom holds only under specific conditions, and we show that the conventional wisdom is wrong under other conditions. We therefore anticipate that more such research will have a dramatic effect on the performance of factories, and we propose promising research directions.

Thesis Supervisor: Stanley B. Germanin
Title: Senior Research Scientist

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Chapter 1

Introduction

1.1 Motivation

During the past three decades, the success of the Toyota Production System has spurred much research in manufacturing systems design. Numerous research papers have tried to explain the relationship between production system design and productivity, so that they can show ways to design factories to produce more products on time with less resources (including people, material, space, and equipment). At the same time, topics in quality research have also captured the attention of practitioners and researchers since the early 1980s. The recent popularity of Statistical Quality Control (SQC), Total Quality Management (TQM), and Six Sigma has demonstrated the importance of quality.

These two fields, Productivity and Quality, have been extensively studied and reported separately in both the manufacturing systems research literature and the practitioner literature, but there is a lack of research in their intersection. The need for such work was recently described by authors from the GM Corporation based on their experience [Inman *et al.*, 2003]. All manufacturers must achieve high productivity and high quality at the same time to maintain their competitiveness.

Toyota Production System advocates admonish factory designers to combine inspections with operations. In the Toyota Production System, the machines are designed to detect abnormalities and to stop automatically whenever they occur. Also, operators are equipped with means of stopping the production flow whenever they note anything unusual. (This practice is called *jidoka*.) Toyota Production System advocates argue that mechanical and human jidoka prevents the waste that would result from producing a series of defective items. Therefore jidoka is a means to improve quality and increase productivity at the same time [Shingo, 1989], [Toyota Motors Corporation, 1996]. But this statement is arguable: quality failures are often such that the quality of each part is independent of the quality of others. This is the case when the defect takes place due to common (or chance or random) causes of variations [Ledolter and Burrill, 1999]. In this case, there is no benefit to stop a machine that has made a bad part because there is no reason to believe that stopping it will reduce the number of bad parts in the future. In this case, therefore, stopping the operation does not influence quality but it does reduce productivity. On the other hand, when quality failures are such that once a bad part is produced, all subsequent parts will be bad until the machine is repaired (due to special or assignable or systematic causes of variations) [Ledolter and Burrill, 1999], detecting bad parts and stopping the machine as soon as possible is the best way to maintain high quality and productivity.

Zero inventory, or *lean* production, is another popular buzzword in manufacturing systems engineering. Some lean manufacturing professionals advocate reducing inventory on the factory floor since the reduction of work-in-process (WIP) reveals the problems in the production lines [Black, 1991]. In this way, it can help improve production quality. This is sometimes true: less inventory reduces the time between making a defect and identifying the defect; thus, it improves the traceability of the root causes of problems. But it is also true that productivity would diminish significantly without stock due to increased blockage and starvation [Burman *et al.*, 1998]. Since there is a tradeoff, there must be optimal stock levels that are specific to each manufacturing environment. In fact, Toyota recently changed their view on inventory and are trying to re-adjust their inventory levels [Fujimoto, 1999], [Benders and Morita, 2004].

What is missing in discussions of factory design, quality, and productivity is a quantitative model to show how they are inter-related. Most of the arguments about this topic are based on anecdotal evidence or qualitative reasoning that lack a sound scientific quantitative foundation. The research described here tries to establish such a foundation to investigate how production system design and operation influence productivity and product quality by developing conceptual and computational models of transfer lines and performing numerical experiments.

1.2 Background and literature review

1.2.1 Importance of quality

Since 1980, industry and academia's interest in quality has grown significantly because it has been recognized that quality is critical to the competitiveness of companies. Many studies have been conducted to estimate the importance of quality. Some studies have tried to find a linkage between high products qualities and companies' financial performances: for example Hendricks and Singhal [Hendricks and Singhal, 1997], [Hendricks and Singhal, 2001] demonstrate that companies that win quality awards outperform other firms on operating income measures as well as stock performance. Another group of studies attempt to develop economic measures of quality to find optimal operation policy to minimize total cost [Son and Park, 1987], [Son and Hsu, 1991], [Nandakumar *et al.*, 1993].

1.2.2 Quality models

Quality failures are of two extreme types, depending on the characteristics of variations that cause the failures. In the quality literature, these variations are called *common* (or chance or random) cause variations and *assignable* (or special or unusual) cause variations [Montgomery, 1991].

Figure 1-1 shows the types of quality failures and variations. Common cause failures are those in which the quality of each part is independent of that of the others. Such failures occur often when an operation is sensitive to external perturbations like a random defect in raw material or the operation uses a new technology that is difficult to control. This is inherent in the design of the process and cannot be removed. Such failures can be represented by independent Bernoulli random variables, in which a binary random variable indicating whether or not the part is good is chosen each time a part is operated on. A good part is produced with probability π, and a bad part is produced with probability $1 - \pi$. The occurrence of a bad part implies nothing about the quality of future parts, so no permanent changes can have occurred in the machine. For the sake of clarity, we call this a *Bernoulli-type quality failure.* Most of the quantitative literature on inspection allocation assumes this kind of quality failure [Raz, 1986], [Lee and Unnikrishnan, 1998]. In this case, if bad parts are destined to be scrapped, it is useful to catch them as soon as possible because the longer before they are scrapped, the more they consume the capacity of downstream machines and buffers. However, there is no reason to stop a machine that has produced a bad part due to this kind of failure.

The quality failures due to assignable cause variations are those in which a quality failure happens only after a change occurs in the machine. In that case, it is very likely that once a bad part is produced, all subsequent parts will be bad until the machine is repaired. Here, there is much more incentive to catch defective parts and stop the machine quickly. In addition to minimizing the waste of downstream capacity, this strategy minimizes the further production of defective parts. For this kind of quality failure, there is no inherent measure of yield because the fractions of parts that are good and bad depend on how soon bad parts are detected and how quickly the machine is stopped for repair. In this thesis, we call this a *persistent-type quality failure*. Most quantitative studies in Statistical Quality Control are dedicated to finding efficient inspection policies (sampling interval, sample size, and others) to detect this type of quality failure [Woodall and Montgomery, 1999]. In reality, there may also be cases where failures occur independently but at different rates, depending on what state the machine is in. These are referred to here as *multiple-yield quality failures*. Specifically, the machine may produce defective parts with a certain small probability p when it is in good working order; when it is in need of adjustment, however, it might produce defective parts with a certain probability $q > p$.

It can be argued that the quality strategy of the Toyota Production System, in which machines are stopped as soon as a defective part is detected, is implicitly based on the assumption of the persistent-type quality failure.

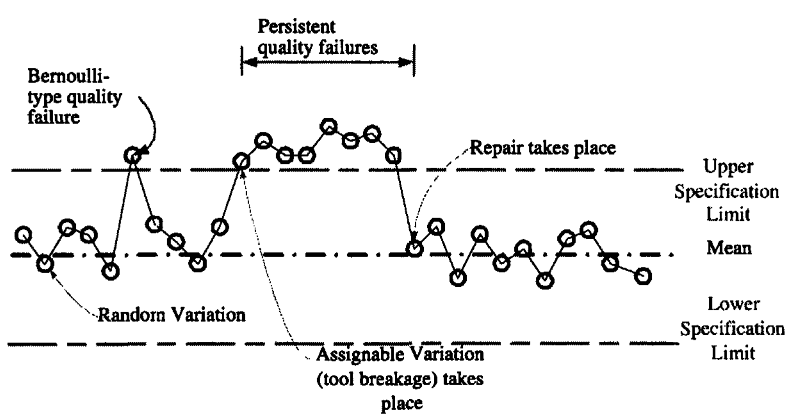


Figure 1-1: Types of Quality Failures

1.2.3 System yield

System yield is defined here as the fraction of input to a system that is transformed into output of acceptable quality. This is an important metric because customers observe the quality of products only after all the manufacturing processes are done and the products are shipped. The system yield is a complex function of how the factory is designed and operated, as well as of the characteristics of the machines. Some influencing factors include individual operation yields, inspection strategies, operation policies, and buffer sizes. Comprehensive approaches are needed to manage system yield effectively. This research aims to develop mathematical models to show how the system yield is influenced by these factors.

1.2.4 Quality improvement strategy

System yield is a complex function of various factors such as inspection, individual operation yields, buffer size, operation policies, and others. There are many ways to affect the system yield discussed in the literature.

Inspection strategy

Inspection policy has received the most attention in the literature. Research on inspection policies can be divided into optimizing inspection parameters at a single station and the inspection station allocation problem. The former topic has been investigated extensively in the Statistical Quality Control (SQC) literature [Duncan, 1956], [Montgomery, 1980], [Montgomery, 1991], [Ho and Case, 1994], [Keats *et al.*, 1997], [Wooddall and Montgomery, 1999]. Here, optimal SQC parameters such as sampling size, control limits, and frequency are sought for an optimal balance between the inspection cost and the cost of quality.

The latter research looks for the optimal location and scope of inspection along production lines [Raz, T., 1986], [Peters and Williams, 1987], [Shin *et al.*, 1995] [Lee, Unnikrishan, 1998], [Emmons and Rabinowitz, 2002]. Most of the literature on inspection allocation assumes Bernoulli quality failures. The objective of the research is to find optimal inspection locations and scopes to screen out defective parts as efficiently as possible. Existing research does not attempt to identify machines in bad states in order repair them to prevent the generation of defects in the future.

Improving individual operation yield

Improving individual operation yield is another important way to increase the system yield. Studies in this field try to stabilize the process either by finding root causes of variation and eliminating them, or by making the process insensitive to external perturbations. The former topic has numerous qualitative research papers in the fields of Total Quality Management (TQM) [Besterfield et al., 2003] and Six Sigma [Pande and Holpp, 2002]. Quantitative research is more oriented toward the latter topic. Robust engineering [Phadke, 1989] is an area that has gained substantial attention.

1.2.5 Lean manufacturing, people, and quality

The design and the operation of manufacturing systems affect the people involved in the production line. They also indirectly influence the performance of the manufacturing systems by changing the behavior of workers in the production lines [Schultz *et al.*, 1998], [Lieberman and Demeester, 1999]. Experts in lean manufacturing argue that inventory reduction is an effective means to improve quality; they assert that the reduction of inventory leads to an early detection of quality failures. Early detection prevents defective parts moving downstream in the manufacturing line from consuming capacities of the downstream machines, and facilitating the identification of the root cause of the problems [Shingo, 1989] , [Monden, 1998], [Alles et al., 2000]. This allows people in the manufacturing lines to develop a better understanding of the manufacturing processes and to gives them information required for operations improvement (i.e., kaizen). Also, with less inventory, the manufacturing lines become more vulnerable to the failures of a machine in the line: the manufacturing line stops more frequently with less inventory. Therefore, workers feel more pressure to prevent any kind of machine failures. On the other hand, it is also true that productivity would diminish significantly without inventory due to increased blockage and starvation [Burman *et al.*, 1998]. Since there is a tradeoff in the inventory reduction between vulnerability of a manufacturing line to a machine failure and workers' learning speed, there must be optimal stock levels that are specific to each manufacturing environment.

Another group of researchers and practitioners argue that U-shaped cellular manufacturing lines, which are widely used in lean manufacturing, are better than straight lines for producing higher quality products since there are more points of contact between operators. Also there is less material movement, and there are other reasons. (see Cheng, [Cheng et al., 2000].)

1.2.6 Stochastic modeling of manufacturing systems

A number of methods have been developed for analyzing production lines with unreliable machines and finite buffers. Dallery and Gershwin [Dallery and Gershwin, 1992] survey the literature on the stochastic modeling of manufacturing systems. Recent books include Buzacott and Shanthikumar [Buzacott and Shantikumar, 1993], Gershwin [Gershwin, 1994], and Altiok [Altiok, 1997]. Early analytic work focused on various two-machine models. The synchronous discrete model was first introduced by Buzacott [Buzacott, 1967]. Obtaining exact analytical solutions of asynchronous models of production lines with deterministic processing times is in general not feasible. As a result, continuous models, which were first proposed by Zimmern [Zimmern, 1956], have been used to approximate the behavior of asynchronous models. The continuous models provide a good approximation of the original asynchronous model so long as the average times to failures are significantly larger than the processing times, which is usually the case in production systems.

Analysis of longer lines is based on approximation methods. Among these methods, the decomposition method developed by Gershwin [Gershwin, 1987] in the context of the synchronous model appears to be quite accurate. The decomposition equations proposed by Gershwin were more efficiently solved by the DDX-algorithm, which was formulated by Dallery, David, and Xie [Dallery et. al, 1989]. This was not directly applicable to systems in which machines had different processing times. A decomposition technique for a continuous long line with different operation speeds and operation dependent failures was proposed by Glassey and Hong [Glassey and Hong, 1993], and it was improved by Burman [Burman, 1995]. The decomposition method was extended to assembly/disassembly systems in DiMascolo, David, and Dallery [DiMascolo *et al.*, 1991]. Recent works have extended these methods to systems with closed loops [Levantesi, 2001].

1.3 Thesis Outline

The rest of this thesis is organized as follows. In Chapter 2 we introduce a taxonomy, quality failure models, fundamental modeling assumptions, and the basic structure of the modeling techniques used throughout the thesis. Also the analysis and the validation of 2-machine-1-buffer systems with zero buffer size and infinite buffer size are presented. In Chapter 3, we provide modeling, solution techniques, performance measures evaluation, and validation of 2-machine-1-finite buffer (2M1B) systems. Discussions on the behavior of a 2M1B line based on numerical experiments are provided in Chapter 4. Chapter 5 provides long line analysis using the decomposition technique. Chapter 6 introduces a modeling technique for multiple-yield quality failures, and with this technique, the optimality of stopping policy incorporated into Jidoka practice is discussed. Future research plans are shown in Chapter 7. Chapter 8 provides summary of the contribution of this research and concludes the thesis.

Chapter 2

Fundamental Models

2.1 Taxonomy and modeling assumptions

In this section, we specify notation, terminologies, and assumptions used in this thesis to model a production line with quality failures. More detailed explanation can be found in Schick [Schick *et al.*, 2004].

2.1.1 Definition of terminology

- *A flow (or transfer) line*: a manufacturing system with a very special structure. It is a linear network of service stations or machines (M_1, M_2, ..., M_k) separated by buffer storages (B_1, B_2, ..., B_{k-1}). Material flows from outside the system to M_1, then to B_1, then to M_2, and so forth until it reaches M_k after which it leaves. Figure 2-1 depicts a flow line. The rectangles represent machines and the circles represent buffers.

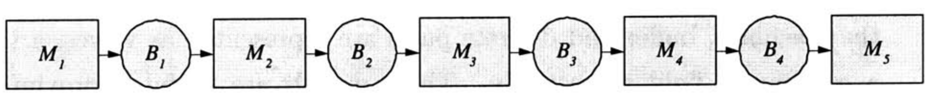


Figure 2-1: Five-Machine Flow Line

- *Stationary processes*: stationarity means that the probabilistic properties of a system do not change over time.

- *Saturated system*: a system where inexhaustible supply of workpieces is available upstream of the first machine in the line, and an unlimited storage area is present downstream of the last machine. Thus, the first machine is never starved, and the last machine is never blocked. This is a widespread assumption in the flow line literature [Dallery and Gershwin, 1992]. In reality, vendors sometimes fail to deliver, and sales are sometimes less than expected. An easy approach to handle this would be to use the first machine in the model to represent the arrivals of material and the last machine of the model could represent the demand or sales process [Dallery and Gershwin, 1992].

- *Open system*: a queuing system where arrival and departure are independent.

- *Processing time variations*: the *cycle time* is the time required for a single operation on an isolated machine. Cycle times are considered *deterministic* when they do not vary from one part to the next on a specific process. *Stochastic* cycle times vary randomly from part to part. Flow lines are usually designed to produce similar or identical products in large quantities. Unless work centers jam or fail completely, they usually perform their task with a low level of variability when operational.

- *Synchronous line*: a production line where cycle time of each machine is deterministic and identical, and operations start and stop together.

- *Asynchronous line*: a production line where cycle time of each machine may differ from machine to machine, and operations do not start and stop together.

- *Continuous model*: continuous models treat material travelling through the production system as if it were a continuous fluid. In this model, the quantity of material in a buffer is a real number ranging from zero to the capacity of the buffer. Figure 2-2 shows the two-machine-one-buffer continuous model where the machines, buffer and discrete parts are represented as valves, a tank, and a continuous fluid respectively. These models are useful approximations to discrete material systems as long as cycle times are relatively small in relation to failure and repair times and buffers are of a reasonable size. Continuous models assume deterministic cycle times.

- *Buffer transit time*: buffer transit time is the time from when a part enters an empty buffer that is not blocked by a downstream machine until that part is able to leave the buffer. Most of the flow line models in the literature as well as this study, assume a zero transit time in the buffer.

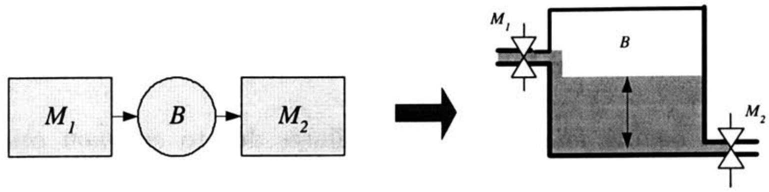


Figure 2-2: Two-Machine-One-Buffer Continuous Model

- *Conservation of flow*: workpieces are not destroyed or rejected at any stage in the line. Defective parts identified from inspection are marked and reworked or scrapped later in a specified area. This is the case with the automotive assembly lines where parts are bulky.

- *Operational failure*: failures like motor burn-outs which cause machines to stop producing parts.

- *Quality failures*: the events that a defective parts is produced. These may happen due to defective raw materials as well as failures like tool damage at the operation.

- *Operation dependent failures*: machines fail only while processing workpieces. Thus, if Machine M_i is operational but starved or blocked, it can not fail.

- *Independent operational failures*: each machine's operational failure process is assumed to be independent of the state of the rest of the system. This excludes such event as a power failure that affects the whole line.

- *Unlimited repair personnel*: the repair process at each machine depends only on the characteristics of the machine, and not on any system-wide properties (i.e., infinite number of repair persons).

- *Non-self-correcting process*: once an either of operational failure or quality failure has occurred, the process can be returned to the good condition only by human intervention.

- *Common (or chance or random) cause variation*: variation that is inherent in the design of the process and cannot be removed. Such variations occur often when an operation is sensitive to external perturbations like imperfect raw material.

- *Assignable (or special or random) cause variation*: variation due to a specific, identifiable cause which changes the process mean or variance.

- *Bernoulli quality failures*: quality failures due to *common cause variations.* Since no permanent changes have occurred in the machine, the occurrence of a bad part implies nothing about the quality of future parts.

- *Persistent quality failures*: quality failures due to *assignable cause variations.* This kind of quality failures only happen after a change occurs in the machine or raw material. In that case, once a bad part is produced, all subsequent parts will be bad until the machine is repaired.

- *Multiple-Yield failures*: quality failures that occur independently but at different rates, depending on what state the machine is in. For example, the machine may produce defective parts with a certain small probability p when it is in good working order; when it is in need of adjustment, however, it might produce defective parts with a higher probability $q > p$.

- *Statistical correlation among different quality failures*: specific failures are associated with specific features of a part. Distinct failures may or may not be correlated with each other depending on the relationship between features, as well as the sequence in which features are processed by machines:

  - *Bias (mean-shift) correlation.* when a single machine performs several tasks, or several tools are mounted on a single head, it is possible that a single misalignment could result in a consistent shift across several different features.

  - *Variance correlation.* when a single machine performs several tasks, or several tools are mounted on a single head, it is possible that a single source of imprecision (e.g. a loose arm) could result in several different features being out of specification, though not necessarily in the same direction.

  - *Cumulative effects.* in a sequence of operations, it is possible that an upstream failure results in the malfunctioning of downstream operations as well, or that a downstream failure results in the corruption of the product of upstream operations. Thus, multiple failures may occur due to a single root cause even when operations are performed by physically distinct machines.

- *Full blockage*: machine M_i is *fully blocked* at time t if one of downstream machine is down and all buffers between this machine and machine M_i are full.

- *Full starvation*: machine M_i is *fully starved* at time t if one of the upstream machines is down and all buffers between this machine and machine M_i are empty.

- *Partial blockage*: machine M_i is *partially blocked* at time t if one of downstream machine (M_j) is working slower than M_i (i.e. $\mu_j < \mu_i$) and all buffers between M_j and M_i are full. In this case, failure probability rates and inspection rates need to be reduced. (e.g. $p_i^b = p_i \frac{\mu_j}{\mu_i}$, $g_i^b = g_i \frac{\mu_j}{\mu_i}$, and $f_i^b = f_i \frac{\mu_j}{\mu_i}$). Partial blockage takes place only with continuous models.

- *Partial starvation*: machine M_i is *partially starved* at time t if one of upstream machine (M_j) is working slower than M_i (i.e. $\mu_j < \mu_i$) and all buffers between M_j and M_i are empty. In this case, failure probability rates and inspection rates need to be reduced (e.g. $p_i^b = p_i \frac{\mu_j}{\mu_i}$, $g_i^b = g_i \frac{\mu_j}{\mu_i}$, and $f_i^b = f_i \frac{\mu_j}{\mu_i}$). Partial starvation takes place only with continuous models.

- *Operation dependent inspection*: inspection is carried out only while a machine is processing workpieces. Thus, if Machine M_i is operational but starved or blocked, inspection is not performed.

- *Reliability of inspection*: there are two kinds of errors in inspection.
  - *Type I error*: error that a good item is classified as defective.
  - *Type II error*: error that a defective item is classified as good.

2.1.2 Modeling assumptions

In this thesis we assume:

- Stationary, saturated, and open systems.

- Continuous models which have deterministic cycle times and full/partial blockage/starvation.

- Buffer transit time is zero.

- Material flow is conserved: defective parts are reworked or scrapped later in a specified area. No workpieces are destroyed in the line.
- Each machine can have operational failures and quality failures and these failures are operation dependent.
- All the failures and repairs are uncorrelated.
- Nondestructive and operation dependent inspection which has *Type II errors* only.
- Only reactive actions on the failures excluding any of learning effect to people.

2.2 Single machine model

There are many possible ways to characterize the states of a machine for the purpose of simultaneously studying quality and quantity issues. Here, we model a machine as a discrete state, continuous time Markov process. Material is assumed continuous, and μ_i is the speed at which Machine i processes material while it is operating and not constrained by the other machine or the buffer. It is a constant, in that μ_i does not depend on the repair state of the other machine or the buffer level.

Figure 2-3 shows the proposed state transitions of a single machine with persistent-type quality failures. In the model, the machine has three states.

- State 1: The machine is operating and producing good parts.
- State -1: The machine is operating and producing bad parts, but the operator does not know this yet.
- State 0: The machine is not operating.

The machine therefore has two different failure modes (i.e. transition to failure states from state 1):

- *Operational failure*: transition from state 1 to state 0. The machine stops producing parts due to failures like motor burnout.
- *Quality failure*: transition from state 1 to state -1. The machine stops producing good parts (and starts producing bad parts) due to a failure like sudden tool damage.

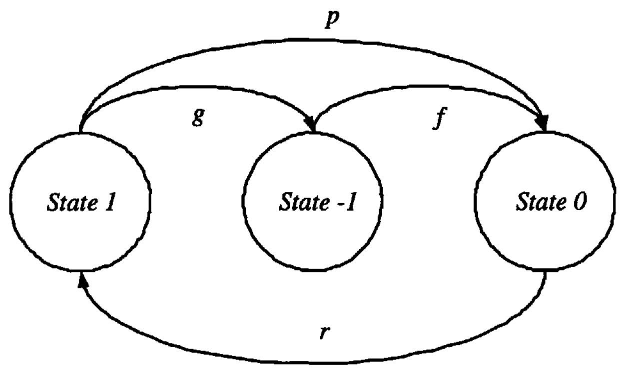


Figure 2-3: States of a Machine

When a machine is in state 1, it can fail due to a non-quality-related event. It goes to state 0 with probability rate p. After that an operator fixes it, and the machine goes back to state 1 with probability rate r. Sometimes, due to an assignable cause, the machine begins to produce bad parts, so there is a transition from state 1 to state -1 with a probability rate of g. Here g is the reciprocal of the *Mean Time To Quality Failure (MTQF)*. A more stable operation leads to a larger *MTQF* and a smaller g.

The machine, when it is in state -1, can be stopped for two reasons: it may experience the same kind of operational failure as it does when it is in state 1; or the operator may stop it for repair when he learns that it is producing bad parts. The transition from state -1 to state 0 occurs at probability rate $f = p + h$ where h is the reciprocal of the *Mean Time To Detect* (MTTD). A more reliable inspection leads to a shorter MTTD and a larger f. (The detection can take place elsewhere, for example at a remote inspection station.) Note that this implies that $f > p$. All the indicated transition times are assumed to follow exponential distributions.

The machine state definition illustrated in Figure 2-3 is a simplification of a more generalized machine state definition shown in Figure 2-4. More complex machine state definition leads to substantially more complicated internal transition equations and boundary conditions discussed in Chapter 3. Therefore, for simplicity, we assume:

- A machine in a bad condition (i.e., state -1) can be returned to the good condition (i.e., state 1) *only through repair* (i.e., state 0). Therefore, a machine does not have direct state transition from state -1 to state 1 (i.e., $q = 0$).

- When a machine is under repair (i.e., state 0, state 0', and state 0"), an operator can not tell whether the machine is down due to a quality failure or an operational failure. Therefore, whenever a machine is under repair, the operator fixes the machine completely so that the machine goes back to state 1. As a result, the repair rates of the three down states in Figure 2-4 are identical ($r = r' = r$").

- Operational failure rates does not depend upon the state of the machine (either state 1 or state -1). Thus, $p = p'$ in Figure 2-4

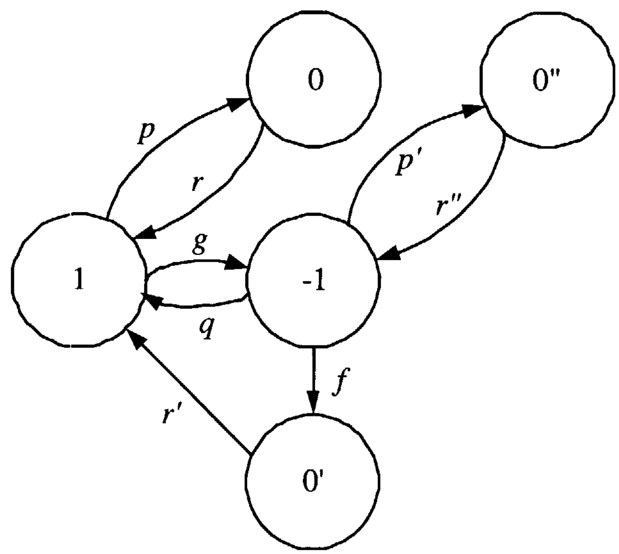


Figure 2-4: States of a Generalized Machine

To determine the production rate of a single machine, we first determine the steady-state probability distribution. This is calculated based on the probability balance principle: in steady state, the probability rate of leaving a state is the same as the probability rate of entering that state. We have

$$(g+p)P(1) = rP(0) \tag{2.1}$$

$$fP(-1) = gP(1) \tag{2.2}$$

$$rP(0) = pP(1) + fP(-1) \tag{2.3}$$

The probabilities must also satisfy the normalization equation:

$$P(0) + P(1) + P(-1) = 1 \tag{2.4}$$

The solution of (2.1)–(2.4) is

$$P(1) = \frac{1}{1 + (p+g)/r + g/f} \tag{2.5}$$

$$P(0) = \frac{(p+g)/r}{1 + (p+g)/r + g/f} \tag{2.6}$$

$$P(-1) = \frac{g/f}{1 + (p+g)/r + g/f} \tag{2.7}$$

The *total production rate*, including good and bad parts, is

$$P_T = \mu(P(1) + P(-1)) = \mu \frac{1 + g/f}{1 + (p+g)/r + g/f} \tag{2.8}$$

The *effective production rate*, the production rate of good parts only, is

$$P_E = \mu P(1) = \mu \frac{1}{1 + (p+g)/r + g/f} \tag{2.9}$$

The *yield* is

$$\frac{P_E}{P_E + P_T} = \frac{P(1)}{P(1) + P(-1)} = \frac{f}{f+g} \tag{2.10}$$

2.3 Simulation Model

Discrete event simulation models are needed for the validation of the analytic models developed from the research. A new discrete-event-simulation based on C^{++} (Qsim) has been developed and tested to ensure accuracy.

For all the numerical experiments, we used a transient period of 10,000 time units followed by 1,000,000 time units of data collection period. This ensures that statistically significant number of events are generated since the typical value of the mean time to operational failures or quality failures is around 100 time units.

2.4 Special two-machine-one-buffer (2M1B) model

2.4.1 Infinite buffer case

An infinite buffer case is a special 2M1B line in which the size of the Buffer (B) is infinite. This is an extreme case in which the first machine (M_1) never suffers from blockage. To derive expressions for the total production rate and effective production rate, we observe that when there is infinite buffer capacity between two machines (M_1, M_2), the total production rate of the 2M1B system is a minimum of the total production rates of M_1 and M_2. The total production rate of machine i is given by (2.8), so the total production rate of the 2M1B system is

$$P_T^{\infty} = \min\left[\frac{\mu_1(1+g_1/f_1)}{1+(p_1+g_1)/r_1+g_1/f_1}, \frac{\mu_2(1+g_2/f_2)}{1+(p_2+g_2)/r_2+g_2/f_2}\right] \tag{2.11}$$

The probability that machine M_i does not add non-conformities is

$$Y_i = \frac{P_i(1)}{P_i(1)+P_i(-1)} = \frac{f_i}{f_i+g_i} \tag{2.12}$$

Since there is no scrap and rework in the system, the system yield is

$$\frac{f_1 f_2}{(f_1+g_1)(f_2+g_2)} \tag{2.13}$$

As a result, the effective production rate is

$$P_E^{\infty} = \frac{f_1 f_2}{(f_1+g_1)(f_2+g_2)} P_T^{\infty} \tag{2.14}$$

The effective production rate evaluated from (2.14) has been compared with a discrete-event, discrete-part simulation. The continuous model is a good approximation since Table 2.1 shows good agreement. The parameters for theses cases are shown in Appendix A.

2.4.2 Zero buffer case

The zero buffer case is one in which there is no buffer space between the machines. This is the other extreme case where blockage and starvation take place most frequently.

Case #	$P_E^\infty(Analytic)$	$P_E^\infty(Simulation)$	%Difference
1	0.762	0.761	0.17
2	0.708	0.708	0.00
3	0.657	0.657	-0.00
4	0.577	0.580	-0.50
5	0.527	0.530	-0.42
6	0.745	0.745	0.01
7	0.762	0.760	0.30
8	1.524	1.522	0.14
9	0.762	0.762	0.00
10	1.524	1.526	-0.13

Table 2.1: Infinite Buffer Case

In the zero-buffer case in which machines have different operation times, whenever one of the machines stops, the other one is also stopped. In addition, when both of them are working, the production rate is $min[\mu_1, \mu_2]$. To calculate the production rates, consider a long time interval of length T during which M_1 fails m_1 times and M_2 fails m_2 times. If we assume that average time to repair the M_1 is $1/r_1$ and average time to repair M_2 is $1/r_2$, then the total system down time will be close to $D = \frac{m_1}{r_1} + \frac{m_2}{r_2}$. Consequently, total up time will be approximately

$$U = T - D = T - (\frac{m_1}{r_1} + \frac{m_2}{r_2}) \tag{2.15}$$

Since we assume operation-dependent failures, the rates of failure are reduced for the faster machine. Therefore,

$p_i^b = p_i \frac{\min(\mu_1,\mu_2)}{\mu_i}$, $g_i^b = g_i \frac{\min(\mu_1,\mu_2)}{\mu_i}$ and $f_i^b = g_i \frac{\min(\mu_1,\mu_2)}{\mu_i}$

The reduction of p_i is explained in detail in [Gershwin, 1994]. The reductions of g_i and f_i are done for the same reasons.

Table 2.2 lists the possible working states α_1 and α_2 of M_1 and M_2. The third column is the probability of finding the system in the indicated state. The fourth and fifth columns indicate the expected number of transitions to down states during the time interval from each of the states in column 1.

From Table (2.2), the expectations of m_1 and m_2 are

$$\begin{aligned} Em_1 &= \sum_{\alpha_1=-1}^{1} \sum_{\alpha_2=-1}^{1} Em_1(\alpha_1, \alpha_2) = \frac{Uf_1^b(p_1^b+g_1^b)}{f_1^b+g_1^b} \\ Em_2 &= \sum_{\alpha_1=-1}^{1} \sum_{\alpha_2=-1}^{1} Em_2(\alpha_1, \alpha_2) = \frac{Uf_2^b(p_2^b+g_2^b)}{f_2^b+g_2^b} \end{aligned} \tag{2.16}$$

α_1	α_2	Probability $\pi(\alpha_1, \alpha_2)$	$Em_1(\alpha_1, \alpha_2)$	$Em_2(\alpha_1, \alpha_2)$
1	1	$\frac{f_1^b}{f_1^b+g_1^b} \frac{f_2^b}{f_2^b+g_2^b}$	$p_1^b U \pi(1,1)$	$p_2^b U \pi(1,1)$
1	-1	$\frac{f_1^b}{f_1^b+g_1^b} \frac{g_2^b}{f_2^b+g_2^b}$	$p_1^b U \pi(1,-1)$	$f_2^b U \pi(1,-1)$
-1	1	$\frac{g_1^b}{f_1^b+g_1^b} \frac{f_2^b}{f_2^b+g_2^b}$	$f_1^b U \pi(-1,1)$	$p_2^b U \pi(-1,1)$
-1	-1	$\frac{g_1^b}{f_1^b+g_1^b} \frac{g_2^b}{f_2^b+g_2^b}$	$f_1^b U \pi(-1,-1)$	$f_2^b U \pi(-1,-1)$

Table 2.2: Zero-Buffer States, Probabilities, and Expected Numbers of Events

By plugging them into equation (2.15), we find the total production rate:

$$P_T^0 = \frac{Min[\mu_1, \mu_2]}{1 + \frac{f_1^b(p_1^b+g_1^b)}{r_1(f_1^b+g_1^b)} + \frac{f_2^b(p_2^b+g_2^b)}{r_2(f_2^b+g_2^b)}} \tag{2.17}$$

The effective production rate is

$$P_E^0 = \frac{f_1^b f_2^b}{(f_1^b + g_1^b)(f_2^b + g_2^b)} P_T^0 \tag{2.18}$$

The comparison with simulation is shown in Table 2.3. The parameters are in Appendix A.

Case #	$P_E^0(Analytic)$	$P_E^0(Simulation)$	%Difference
1	0.657	0.662	-0.73
2	0.620	0.627	-1.15
3	0.614	0.621	-1.03
4	0.529	0.534	-0.99
5	0.480	0.484	-0.77
6	0.647	0.651	-0.57
7	0.706	0.712	-0.91
8	1.377	1.526	-9.17
9	0.706	0.711	-0.77
10	1.377	1.380	-0.22

Table 2.3: Zero Buffer Case

Chapter 3

Two-Machine-One-Finite-Buffer (2M1B) Line

In this chapter, we present modeling, solution techniques, and validation of the two-machine-one-finite-buffer case. The two-machine line is the simplest non-trivial case of a transfer line. It is used in decomposition approximations of longer lines. (See Chapter 5.)

3.1 Modeling

3.1.1 State definition

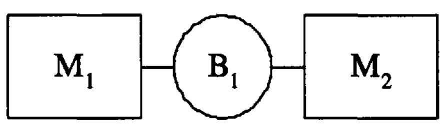


Figure 3-1: Two-machine-one-buffer line

The state of the 2M1B line illustrated in Figure 3-1 is (x, α_1, α_2) where:

- x: the total amount of material in buffer B. $0 \le x \le N$.
- α_1: the state of M_1 ($\alpha_1 = -1, 0,$ or, 1).
- α_2: the state of M_2 ($\alpha_2 = -1, 0,$ or, 1).

The parameters of machine M_i are $\mu_i, r_i, p_i, f_i, g_i$ as explained in section 2.2, and the buffer size is N. The probabilistic behavior of the 2M1B is described by probability density functions (e.g., $f(x,1,1)$) when buffer B is neither empty nor full, and by probability masses (e.g., $P(0,1,1)$) when the buffer is either empty or full. If we find all the probability density functions and the probability masses, we can calculate the performance measures of the 2M1B line, since these are expressed in terms of the probability density functions and the probability masses. The probability density functions and probability masses are to be found by solving the internal transition equations and the boundary transition equations presented below.

3.1.2 Internal Transition Equations

In this section, we present equations describing behavior of the 2M1B system when buffer B is neither full nor empty. When buffer B is neither empty nor full, its level can rise or fall depending on the states of adjacent machines. Since it can change only a small amount during a short time interval, it is reasonable to use a continuous probability density $f(x, \alpha_1, \alpha_2)$ and differential equations to describe its behavior. The probability of finding both machines at state 1 with a storage level between x and $x + \delta x$ at time $t + \delta t$ is given by $f(x, 1, 1, t + \delta t)\delta x$, where

$$\begin{aligned} f(x,1,1,t+\delta t) &= \{1-(p_1+g_1+p_2+g_2)\delta t\} f(x+(\mu_2-\mu_1)\delta t, 1, 1) \\ &+ r_2 \delta t f(x-\mu_1\delta t, 1, 0) + r_1 \delta t f(x+\mu_2\delta t, 0, 1) + o(\delta t) \end{aligned} \tag{3.1}$$

Except for the factor of δx, the first term is the probability of transition from between $(x + (\mu_2 - \mu_1)\delta t, 1, 1)$ and $(x + (\mu_2 - \mu_1)\delta t + \delta x, 1, 1)$ at time t to between $(x, 1, 1)$ and $(x + \delta x, 1, 1)$ at time $t + \delta t$. This is because

- The probability of neither machine failing between t and $t + \delta t$ is

$$\{1-(p_1+g_1)\delta t\}\{1-(p_2+g_2)\delta t\} \simeq \{1-(p_1+g_1+p_2+g_2)\delta t\} \tag{3.2}$$

- If there are no failures between t and $t + \delta t$ and the buffer level is between x and $x + \delta x$ at time $t + \delta t$, then it could only have been between $x + (\mu_2 - \mu_1)\delta t$ and $x + (\mu_2 - \mu_1)\delta t + \delta x$ at time t.

The other terms, which represent the probabilities of transition from (1) machine states (1,0) with buffer level between $x - \mu_1 \delta t$ and $x - \mu_1 \delta t + \delta x$ and (2) machine states (0,1) with buffer level between $x + \mu_2 \delta t$ and $(x + \mu_2 \delta t + \delta x$ can be found similarly. No other transitions are possible. After linearizing, and letting $\delta t \rightarrow 0$, this equation becomes

$$\frac{\partial f(x,1,1)}{\partial t} = (\mu_2 - \mu_1)\frac{\partial f(x,1,1)}{\partial x} - (p_1 + g_1 + p_2 + g_2) f(x,1,1) + r_2 f(x,1,0) + r_1 f(x,0,1). \tag{3.3}$$

In steady state $\frac{\partial f}{\partial t} = 0$. Then, we have

$$(\mu_2 - \mu_1)\frac{df(x,1,1)}{dx} - (p_1 + g_1 + p_2 + g_2) f(x,1,1) + r_2 f(x,1,0) + r_1 f(x,0,1) = 0 \tag{3.4}$$

In the same way, the eight other internal transition equations for the probability density function are

$$p_2 f(x,1,1) - \mu_1 \frac{df(x,1,0)}{dx} - (p_1 + g_1 + r_2) f(x,1,0) + f_2 f(x,1,-1) + r_1 f(x,0,0) = 0 \tag{3.5}$$

$$g_2 f(x,1,1) + (\mu_2 - \mu_1)\frac{df(x,1,-1)}{dx} - (p_1 + g + f_2) f(x,1,-1) + r_1 f(x,0,-1) = 0 \tag{3.6}$$

$$p_1 f(x,1,1) + \mu_2 \frac{df(x,0,1)}{dx} - (r_1 + p_2 + g_2) f(x,0,1) + r_2 f(x,0,0) + f_1 f(x,-1,1) = 0 \tag{3.7}$$

$$p_1 f(x,1,0) + p_2 f(x,0,1) - (r_1 + r_2) f(x,0,0) + f_2 f(x,0,-1) + f_1 f(x,-1,0) = 0 \tag{3.8}$$

$$p_1 f(x,1,-1) + g_2 f(x,0,1) - (r_1 + f_2) f(x,0,-1) + \mu_2 \frac{df(x,0,-1)}{dx} + f_1 f(x,-1,-1) = 0 \tag{3.9}$$

$$g_1 f(x,1,1)-(p_2+g_2+f_1)f(x,-1,1)+(\mu_2-\mu_1)\frac{df(x,-1,1)}{dx}+r_2 f(x,-1,0)=0 \quad (3.10)$$

$$g_1 f(x,1,0)-\mu_1\frac{df(x,-1,0)}{dx}-(r_2+f_1)f(x,-1,0)+p_2 f(x,-1,1)+f_2 f(x,-1,-1)=0 \quad (3.11)$$

$$g_1 f(x,1,-1)+g_2 f(x,-1,1)+(\mu_2-\mu_1)\frac{df(x,-1,-1)}{dx}-(f_1+f_2)f(x,-1,-1)=0. \quad (3.12)$$

3.1.3 Boundary transition equations

While the internal behavior of the system can be described by probability density functions, there is a nonzero probability of finding the system in certain boundary states. For example, if $\mu_1 < \mu_2$ and both machines are in state 1, the level of storage tends to decrease. If both machines remain operational for enough time, the storage will become empty ($x = 0$). Once the system reaches state $(0, 1, 1)$, it will remain there until a machine fails. There are 18 probability masses for boundary states ($P(N, \alpha_1, \alpha_2)$ and $P(0, \alpha_1, \alpha_2)$ where $\alpha_1 = -1$, 0 or 1, and $\alpha_2 = -1$, 0 or 1).

The boundary behavior depends on which machine is faster ($\mu_1 = \mu_2$ or $\mu_1 > \mu_2$ or $\mu_1 < \mu_2$). When M_1 is faster than M_2, the probability masses corresponding to states with full buffer are greater than those with M_1 being slower than M_2 ($\mu_1 < \mu_2$). Thus, there are three different sets of boundary equations.

Boundary condition for $\mu_1 = \mu_2$

To arrive at state $(0, 0, 1)$ at time $t + \delta t$ when the $\mu_1 = \mu_2$, the system may have been in one of five states at time t:

- It could have been in state $(0, 1, 1)$ with an operational failure of M_1. Note that the M_2 could not have failed since it was starved. Therefore the transition probability is $p_1\delta t$.
- It could have been in state $(0, -1, 1)$ with a detection of a quality failure at

M_1. Again, the M_2 could not have failed since it was starved. Therefore the transition probability is $f_1\delta t$.

- It could have been in state $(0,0,1)$ without repair of M_1. The corresponding transition probability is $1-r_1\delta t$ since M_2 could not have failed due to starvation.
- It could have been in some internal state $(x,0,1)$ where $0 \le x \le \mu_2\delta t$ without repair of M_1 and failure of M_2. The corresponding transition probability is $(1-r_1\delta t)(1-(p_2+g_2)\delta t) \simeq 1-(r_1+p_2+g_2)\delta t$.
- It could have been in state $(0,0,0)$ with only repair of M_2 (not M_1). The corresponding transition probability is $(1-r_1\delta t)r_2\delta t \simeq r_2\delta t$.

If the second order terms are ignored,

$$\begin{aligned} P(0,0,1,t+\delta t) &= p_1\delta t P(0,1,1) + f_1\delta t P(0,-1,1) + (1-r_1\delta t)P(0,0,1) \\ &+ \{1-(r_1+p_2+g_2)\delta t\}\int_0^{\mu_2\delta t} f(x,0,1)dx + r_2\delta t P(0,0,0). \end{aligned} \tag{3.13}$$

After the usual analysis, (3.13) becomes

$$\frac{\partial P(0,0,1)}{\partial t} = p_1 P(0,1,1) - r_1 P(0,0,1) + \mu_2 f(0,0,1) + f_1 P(0,-1,1) + r_2 P(0,0,0). \tag{3.14}$$

In steady state, it becomes as equation (3.15)

$$p_1 P(0,1,1) - r_1 P(0,0,1) + \mu_2 f(0,0,1) + f_1 P(0,-1,1) + r_2 P(0,0,0) = 0. \tag{3.15}$$

There are 21 other boundary equations derived similarly for $\mu_1 = \mu_2$:

$$-(p_1+g_1+p_2+g_2)P(0,1,1) + r_1 P(0,0,1) = 0. \tag{3.16}$$

$$P(0,1,0) = 0 \tag{3.17}$$

$$g_2 P(0,1,1) - (p_1+g_1+f_2)P(0,1,-1) + r_1 P(0,0,-1) = 0 \tag{3.18}$$

$$-(r_1 + r_2)P(0,0,0) = 0 \tag{3.19}$$

$$p_1P(0,1,-1) - r_1P(0,0,-1) + \mu_2 f(0,0,-1) + f_1P(0,-1,-1) = 0 \tag{3.20}$$

$$g_1P(0,1,1) - (f_1 + p_2 + g_2)P(0,-1,1) = 0 \tag{3.21}$$

$$P(0,-1,0) = 0 \tag{3.22}$$

$$g_1P(0,1,-1) + g_2P(0,-1,1) - (f_1 + f_2)P(0,-1,-1) = 0 \tag{3.23}$$

$$-(p_1 + g_1 + p_2 + g_2)P(N,1,1) + r_2P(N,1,0) = 0 \tag{3.24}$$

$$p_2P(N,1,1) - r_2P(N,1,0) + \mu_1 f(N,1,0) + f_2P(N,1,-1) + r_1P(N,0,0) = 0 \tag{3.25}$$

$$g_2P(N,1,1) - (p_1 + g_1 + f_2)P(N,1,-1) = 0 \tag{3.26}$$

$$P(N,0,1) = 0 \tag{3.27}$$

$$-(r_1 + r_2)P(N,0,0) = 0 \tag{3.28}$$

$$P(N,0,-1) = 0 \tag{3.29}$$

$$g_1P(N,1,1) - (f_1 + g_2 + p_2)P(N,-1,1) + r_2P(N,-1,0) = 0 \tag{3.30}$$

$$-r_2P(N,-1,0) + \mu_1 f(N,-1,0) + f_2P(N,-1,-1) + p_2P(N,-1,1) = 0 \tag{3.31}$$

$$g_1P(N,1,-1) + g_2P(N,-1,1) - (f_1+f_2)P(N,-1,-1) = 0 \tag{3.32}$$

$$\mu_1 f(0,1,0) = r_1P(0,0,0) + p_2P(0,1,1) + f_2P(0,1,-1) \tag{3.33}$$

$$\mu_1 f(0,-1,0) = p_2P(0,-1,1) + f_2P(0,-1,-1) \tag{3.34}$$

$$\mu_2 f(N,0,1) = r_2P(N,0,0) + p_1P(N,1,1) + f_1P(N,-1,1) \tag{3.35}$$

$$\mu_2 f(N,0,-1) = p_1P(N,1,-1) + g_2P(N,0,1) + f_1P(N,-1,-1). \tag{3.36}$$

Boundary condition for $\mu_1 > \mu_2$

When $\mu_1 > \mu_2$, 26 boundary equations can be derived similarly. In this case, there are 4 more boundary equations than in the $\mu_1 = \mu_2$ cases since it is possible to reach internal states $(x,1,1)$, $(x,1,-1)$, $(x,-1,1)$, and $(x,-1,-1)$ (where $0 < x \leq (\mu_1 - \mu_2)\delta t$) at time $t + \delta t$ from the boundary states $P(0,\alpha_1,\alpha_2)$, (α_1 = -1,0,1, and α_2 = -1,0,1) at time t.

$$\mu_1 f(0,1,0) = 0 \tag{3.37}$$

$$\mu_1 f(0,-1,0) = 0 \tag{3.38}$$

$$(\mu_1 - \mu_2) f(0,1,1) = r_1P(0,0,1) \tag{3.39}$$

$$(\mu_1 - \mu_2) f(0,1,-1) = r_1P(0,0,-1) \tag{3.40}$$

$$f(0,-1,1) = 0 \tag{3.41}$$

$$f(0,-1,-1) = 0 \tag{3.42}$$

$$\mu_2 f(N,0,1) = p_1^b P(N,1,1) + f_1^b P(N,-1,1) \tag{3.43}$$

$$\mu_2 f(N,0,-1) = p_1^b P(N,1,-1) + f_1^b P(N,-1,-1) \tag{3.44}$$

$$P(0,1,1) = 0 \tag{3.45}$$

$$P(0,1,0) = 0 \tag{3.46}$$

$$P(0,1,-1) = 0 \tag{3.47}$$

$$-r_1 P(0,0,1) + \mu_2 f(0,0,1) + r_2 P(0,0,0) = 0 \tag{3.48}$$

$$P(0,0,0) = 0 \tag{3.49}$$

$$-r_1 P(0,0,-1) + \mu_2 f(0,0,-1) = 0 \tag{3.50}$$

$$P(0,-1,1) = 0 \tag{3.51}$$

$$P(0,-1,0) = 0 \tag{3.52}$$

$$P(0,-1,-1) = 0 \tag{3.53}$$

$$-(p_1^b + g_1^b + p_2 + g_2) P(N,1,1) + (\mu_1 - \mu_2) f(N,1,1) + r_2 P(N,1,0) = 0 \tag{3.54}$$

$$p_2 P(N,1,1) - r_2 P(N,1,0) + \mu_1 f(N,1,0) + f_2 P(N,1,-1) + r_1 P(N,0,0) = 0 \tag{3.55}$$

$$g_2P(N,1,1) - (p_1^b + g_1^b + f_2)P(N,1,-1) + (\mu_1 - \mu_2)f(N,1,-1) = 0 \tag{3.56}$$

$$P(N,0,1) = 0 \tag{3.57}$$

$$P(N,0,0) = 0 \tag{3.58}$$

$$P(N,0,-1) = 0 \tag{3.59}$$

$$g_1^bP(N,1,1) - (f_1^b + g_2 + p_2)P(N,-1,1) + (\mu_1 - \mu_2)f(N,-1,1) + r_2P(N,-1,0) = 0 \tag{3.60}$$

$$-r_2P(N,-1,0) + \mu_1 f(N,-1,0) + f_2P(N,-1,-1) + p_2P(N,-1,1) = 0 \tag{3.61}$$

$$g_1^bP(N,1,-1) + g_2P(N,-1,1) - (f_1^b + f_2)P(N,-1,-1) + (\mu_1 - \mu_2)f(N,-1,-1) = 0 \tag{3.62}$$

Boundary condition for $\mu_1 < \mu_2$

Here, the 26 boundary equations for the $\mu_1 < \mu_2$ case are shown.

$$\mu_1 f(0,1,0) = p_2^bP(0,1,1) + f_2^bP(0,1,-1) \tag{3.63}$$

$$\mu_1 f(0,-1,0) = p_2^bP(0,-1,1) + f_2^bP(0,-1,-1) \tag{3.64}$$

$$\mu_2 f(N,0,1) = 0 \tag{3.65}$$

$$\mu_2 f(N,0,-1) = 0 \tag{3.66}$$

$$(\mu_2 - \mu_1) f(N,1,1) = r_2 P(N,1,0) \tag{3.67}$$

$$(\mu_2 - \mu_1) f(N,-1,1) = r_2 P(N,-1,0) \tag{3.68}$$

$$f(N,1,-1) = 0 \tag{3.69}$$

$$f(N,-1,-1) = 0 \tag{3.70}$$

$$-(p_1 + g_1 + p_2^b + g_2^b) P(0,1,1) + (\mu_2 - \mu_1) f(0,1,1) + r_1 P(0,0,1) = 0 \tag{3.71}$$

$$P(0,1,0) = 0 \tag{3.72}$$

$$g_2^b P(0,1,1) - (p_1 + g_1 + f_2^b) P(0,1,-1) + (\mu_2 - \mu_1) f(0,1,-1) + r_1 P(0,0,-1) = 0 \tag{3.73}$$

$$p_1 P(0,1,1) - r_1 P(0,0,1) + \mu_2 f(0,0,1) + f_1 P(0,-1,1) + r_2 P(0,0,0) = 0 \tag{3.74}$$

$$-(r_1 + r_2) P(0,0,0) = 0 \tag{3.75}$$

$$p_1 P(0,1,-1) - r_1 P(0,0,-1) + \mu_2 f(0,0,-1) + f_1 P(0,-1,-1) = 0 \tag{3.76}$$

$$g_1 P(0,1,1) - (f_1 + p_2^b + g_2^b) P(0,-1,1) + (\mu_2 - \mu_1) f(0,-1,1) = 0 \tag{3.77}$$

$$P(0,-1,0) = 0 \tag{3.78}$$

$$g_1 P(0,1,-1) + g_2^b P(0,-1,1) - (f_1 + f_2^b) P(0,-1,-1) + (\mu_2 - \mu_1) f(0,-1,-1) = 0 \tag{3.79}$$

$$P(N,1,1) = 0 \tag{3.80}$$

$$-r_2 P(N,1,0) + \mu_1 f(N,1,0) + r_1 P(N,0,0) = 0 \tag{3.81}$$

$$P(N,1,-1) = 0 \tag{3.82}$$

$$P(N,0,1) = 0 \tag{3.83}$$

$$P(N,0,0) = 0 \tag{3.84}$$

$$P(N,0,-1) = 0 \tag{3.85}$$

$$P(N,-1,1) = 0 \tag{3.86}$$

$$-r_2 P(N,-1,0) + \mu_1 f(N,-1,0) + f_2 P(N,-1,-1) + p_2 P(N,-1,1) = 0 \tag{3.87}$$

$$P(N,-1,-1) = 0 \tag{3.88}$$

3.1.4 Normalization

In addition to these, all the probability density functions and probability masses must satisfy the normalization equation:

$$\sum_{\alpha_1=-1,0,1} \sum_{\alpha_2=-1,0,1} \left[\int_0^N f(x,\alpha_1,\alpha_2)dx + P(0,\alpha_1,\alpha_2) + P(N,\alpha_1,\alpha_2) \right] = 1. \tag{3.89}$$

3.1.5 Performance measures

After finding all probability density functions and probability masses, we can calculate the average inventory in the buffer from

$$\overline{x} = \sum_{\alpha_1=-1,0,1} \sum_{\alpha_2=-1,0,1} \left[\int_0^N x f(x,\alpha_1,\alpha_2)dx + NP(N,\alpha_1,\alpha_2) \right]. \tag{3.90}$$

The total production rate is

$$P_T = P_T^1 = \sum_{\alpha_2=-1,0,1} \mu_1 [\int_0^N \{f(x,-1,\alpha_2) + f(x,1,\alpha_2)\}dx + P(0,1,\alpha_2) + P(0,-1,\alpha_2)] \\ +\mu_2\{P(N,1,-1) + P(N,1,1) + P(N,-1,-1) + P(N,-1,1\}. \tag{3.91}$$

The rate at which machine M_1 produces good parts is

$$P_E^1 = \sum_{\alpha_2=-1,0,1} \mu_1 [\int_0^N f(x,1,\alpha_2)dx + P(0,1,\alpha_2)] + \mu_2\{P(N,1,-1) + P(N,1,1)\}. \tag{3.92}$$

The probability that the first machine produces a non-defective part is then $Y_1 = P_E^1/P_T$. The probability that the second machine finishes its operation without adding a non-conforming feature to a part is $Y_2 = P_E^2/P_T$ where

$$P_E^2 = \sum_{\alpha_1=-1,0,1} \mu_2 [\int_0^N f(x,\alpha_1,1)dx + P(N,\alpha_1,1)] + \mu_1\{P(0,-1,1) + P(0,1,1)\}. \tag{3.93}$$

Therefore, the effective production rate is

$$P_E = Y_1 Y_2 P_T. \tag{3.94}$$

3.2 Solution technique

3.2.1 Solution to internal transition equations

It is logical to assume an exponential form for the solution to the steady state density functions since (3.4)–(3.12) are coupled ordinary linear differential equations. A solution of the form $e^{\lambda x}K_1^{\alpha_1}K_2^{\alpha_2}$ worked successfully in the continuous material two-machine line with perfect quality [Gershwin, 1994]. Therefore, a solution of a form

$$f(x,\alpha_1,\alpha_2) = e^{\lambda x}G_1(\alpha_1)G_2(\alpha_2) \tag{3.95}$$

is assumed here. This form satisfies the transition equations if all of the following equations are met. Equations (3.4)–(3.12) become, after substituting (3.95) into them,

$$\{(\mu_2-\mu_1)\lambda-(p_1+g_1+p_2+g_2)G_1(1)G_2(1)\}+r_2G_1(1)G_2(0)+r_1G_1(0)G_2(1) = 0 \tag{3.96}$$

$$-\{\mu_1\lambda+(p_1+g_1+r_2)\}G_1(1)G_2(0)+p_2G_1(1)G_2(1)+f_2G_1(1)G_2(-1)+r_1G_1(0)G_2(0) = 0 \tag{3.97}$$

$$\{(\mu_2-\mu_1)\lambda-(p_1+g_1+f_2)\}G_1(1)G_2(-1)+g_2G_1(1)G_2(1)+r_1G_1(0)G_2(-1) = 0 \tag{3.98}$$

$$\{\mu_2\lambda-(r_1+p_2+g_2)\}G_1(0)G_2(1)+p_1G_1(1)G_2(1)+r_2G_1(0)G_2(0)+f_1G_1(-1)G_2(1) = 0 \tag{3.99}$$

$$p_1G_1(1)G_2(0)+p_2G_1(0)G_2(1)-(r_1+r_2)G_1(0)G_2(0)+f_2G_1(0)G_2(-1)+f_1G_1(-1)G_2(0) = 0 \tag{3.100}$$

$$\{\mu_2\lambda-(r_1+f_2)\}G_1(0)G_2(-1)+p_1G_1(1)G_2(-1)+g_2G_1(0)G_2(1)+f_1G_1(-1)G_2(-1) = 0 \tag{3.101}$$

$$\{(\mu_2-\mu_1)\lambda-(p_2+g_2+f_1)\}G_1(-1)G_2(1)+g_1G_1(1)G_2(1)+r_2G_1(-1)G_2(0)=0 \quad (3.102)$$

$$-\{\mu_1\lambda+(r_2+f_1)\}G_1(-1)G_2(0)+g_1G_1(1)G_2(0)+p_2G_1(-1)G_2(1)+f_2G_1(-1)G_2(-1)=0 \quad (3.103)$$

$$\{(\mu_2-\mu_1)\lambda-(f_1+f_2)\}G_1(-1)G_2(-1)+g_1G_1(1)G_2(-1)+g_2G_1(-1)G_2(1)=0. \quad (3.104)$$

These are nine equations with seven unknowns $(\lambda, G_1(1), G_2(0), G_1(-1), G_2(1), G_2(0),$ and $G_2(-1))$. Thus, there must be seven independent equations and two dependent ones. If we divide equations (3.96) – (3.104) by $G_1(0)G_2(0)$ and define six new variables

$$\Gamma_i = p_i\frac{G_i(1)}{G_i(0)} - r_i + f_i\frac{G_i(-1)}{G_i(0)} = p_iY_i - r_i + f_iZ_i, \quad (i=1,2) \quad (3.105)$$

$$\Psi_i = -p_i - g_i + r_i\frac{G_i(0)}{G_i(1)} = -p_i - g_i + \frac{r_i}{Y_i} \quad (i=1,2) \quad (3.106)$$

$$\Theta_i = -f_i + g_i\frac{G_i(1)}{G_i(-1)} = -f_i + g_i\frac{Y_i}{Z_i} \quad (i=1,2). \quad (3.107)$$

then equations (3.96)-(3.104) can be rewritten as

$$\Gamma_1 + \Gamma_2 = 0 \quad (3.108)$$

$$-\mu_2\lambda = \Gamma_1 + \Psi_2 \quad (3.109)$$

$$\mu_1\lambda = \Gamma_2 + \Psi_1 \quad (3.110)$$

$$(\mu_1 - \mu_2)\lambda = \Psi_1 + \Psi_2 \quad (3.111)$$

$$(\mu_1 - \mu_2)\lambda = \Theta_1 + \Theta_2 \tag{3.112}$$

$$\mu_1\lambda = \Gamma_2 + \Theta_1 \tag{3.113}$$

$$-\mu_2\lambda = \Gamma_1 + \Theta_2 \tag{3.114}$$

$$(\mu_1 - \mu_2)\lambda = \Psi_2 + \Theta_1 \tag{3.115}$$

$$(\mu_1 - \mu_2)\lambda = \Psi_1 + \Theta_2. \tag{3.116}$$

Equations (3.108) to (3.116) are reduced to seven equation.

$$\Psi_1 = \Theta_1 \tag{3.117}$$

$$\Theta_1 = \mu_1\lambda + \Gamma_1 \tag{3.118}$$

$$\Psi_2 = \Theta_2 \tag{3.119}$$

$$\Theta_2 = -\mu_2\lambda - \Gamma_1. \tag{3.120}$$

$$\Gamma_1 + \Gamma_2 = 0 \tag{3.121}$$

Combining equations (3.117), (3.106), and (3.107), we have

$$\Psi_1 = \mu_1\lambda + \Gamma_1 = -p_1 - g_1 + \frac{r_1}{Y_1} = -f_1 + g_1\frac{Y_1}{Z_1} \tag{3.122}$$

$$\Psi_2 = -\mu_2\lambda - \Gamma_1 = -p_2 - g_2 + \frac{r_2}{Y_2} = -f_2 + g_2\frac{Y_2}{Z_2}. \tag{3.123}$$

From equation (3.122), we have

$$\Psi_1 Y_1 = -(p_1 + g_1)Y_1 + r_1 \tag{3.124}$$

$$\Psi_1 Z_1 = -f_1 Z_1 + g_1 Y_1. \tag{3.125}$$

Together, equations (3.105), (3.124), and (3.125) imply

$$\Psi_1(Y_1 + Z_1) = -\Gamma_1. \tag{3.126}$$

Using the same procedure with equations (3.105) and (3.123), we find

$$\Psi_2(Y_2 + Z_2) = -\Gamma_2 = \Gamma_1. \tag{3.127}$$

Therefore, we get the new relationship

$$\Psi_1(Y_1 + Z_1) = -\Psi_2(Y_2 + Z_2). \tag{3.128}$$

From equations (3.117) and (3.126),

$$\Psi_1 = \mu_1\lambda + \Gamma_1 = \mu_1\lambda - \Psi_1.(Y_1 + Z_1) \tag{3.129}$$

Then, we have

$$\Psi_1 = \frac{\mu_1\lambda}{1 + Y_1 + Z_1}. \tag{3.130}$$

Using similar procedures,

$$\Psi_2 = \frac{-\mu_2\lambda}{1 + Y_2 + Z_2}. \tag{3.131}$$

By plugging equations (3.130) and (3.131) into equation (3.128), we have

$$\frac{\mu_1(Y_1 + Z_1)}{1 + Y_1 + Z_1} = \frac{\mu_2(Y_2 + Z_2)}{1 + Y_2 + Z_2}. \tag{3.132}$$

Now we try to rewrite the internal transition equations and all the unknowns used for the equations to two equations and two unknowns by introducing two new variables. From equations (3.105) and (3.108), we introduce a new variable U:

$$p_1 Y_1 - r_1 + f_1 Z_1 = -(p_2 Y_2 - r_2 + f_2 Z_2) = U. \tag{3.133}$$

Another variable V is introduced from equation (3.132):

$$\frac{1}{\mu_1}(1+\frac{1}{Y_1+Z_1}) = \frac{1}{\mu_2}(1+\frac{1}{Y_2+Z_2}) = V. \tag{3.134}$$

From equation (3.134), we have

$$Z_1 = \frac{1}{\mu_1 V - 1} - Y_1,\ Z_2 = \frac{1}{\mu_2 V - 1} - Y_2. \tag{3.135}$$

After plugging it into equation (3.133), we have

$$Y_1 = \frac{1}{p_1 - f_1}(U + r_1 - \frac{f_1}{\mu_1 V - 1}),\ Y_2 = \frac{1}{p_2 - f_2}(-U + r_2 - \frac{f_2}{\mu_2 V - 1}). \tag{3.136}$$

Also, from equation (3.135),

$$Y_1/Z_1 = \frac{(\mu_1 V - 1)Y_1}{1 - Y_1(\mu_1 V - 1)},\quad Y_2/Z_2 = \frac{(\mu_2 V - 1)Y_2}{1 - Y_2(\mu_2 V - 1)}. \tag{3.137}$$

By using equations (3.136) and (3.137), we can replace Y_1, Y_2, Y_1/Z_1, and Y_2/Z_2 in equations (3.122) and (3.123) and get following two equations:

$$\frac{\{(U+r_1)(\mu_1 V-1)-f_1\}^2}{(f_1-p_1)(\mu_1 V-1)} - \frac{\{(p_1+g_1-f_1)+r_1(\mu_1 V-1)\}\{(U+r_1)(\mu_1 V-1)-f_1\}}{(f_1-p_1)(\mu_1 V-1)} - r_1 = 0 \tag{3.138}$$

$$\frac{\{(-U+r_2)(\mu_2 V-1)-f_2\}^2}{(f_2-p_2)(\mu_2 V-1)} - \frac{\{(p_2+g_2-f_2)+r_2(\mu_2 V-1)\}\{(-U+r_2)(\mu_2 V-1)-f_2\}}{(f_2-p_2)(\mu_2 V-1)} - r_2 = 0. \tag{3.139}$$

Now the 9 transition equations (3.96) - (3.104) and 7 unknowns are simplified into two equations and two unknowns. By solving these quadratic equations, we can get U and V. From equations (3.133) and (3.134), we can calculate Y_1, Y_2, Z_1, and Z_2. From these a probability density function (equation (3.95)) can be found per (U, V) set. The more detailed procedure is presented in sections 3.2.2 and 3.2.3.

3.2.2 Algorithm to solve equations (3.138) and (3.139)

By solving equations (3.138) and (3.139) simultaneously, we can calculate U and V. An example of these equations is plotted in Figure 3-2. Equation (3.138) is represented as red (lighter) lines and equation (3.139) is shown as blue (darker) lines. The intersections of the two lines are the solutions of the equations.

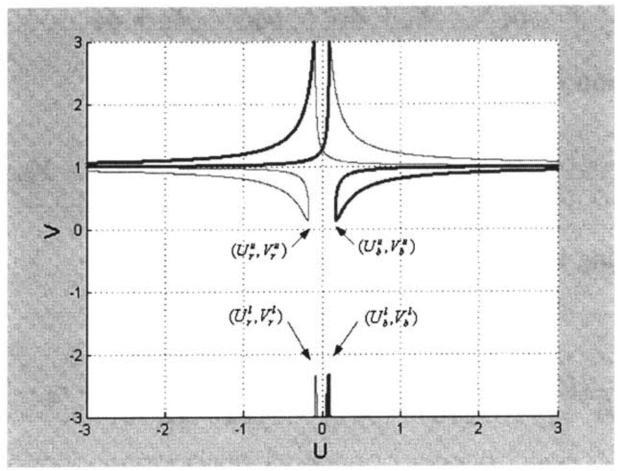


Figure 3-2: Plot of Equations (3.138) and (3.139)

These are high order equations for which no general analytical solution exists. Therefore, a numerical approach is required to find the roots of the equations. But conventional numerical solvers (e.g., the Newton-Raphson method) can not be used directly since here are discontinuous ranges in each red (lighter) line and blue (darker) line as illustrated in Figure 3-2. The coordinates of the end points of the red (lighter) line are denoted as (U_r^u, V_r^u) and (U_r^l, V_r^l). The end points of the blue (darker) lines are denoted as (U_b^u, V_b^u) and (U_b^l, V_b^l). Thus, we need to identify the number of roots of the equations and regions where roots exist. After that the Newton-Raphson method is applied to find the exact location of each root.

Characterization of the curves

For the development of an efficient algorithm to find roots of equations (3.138) and (3.139), we need to characterize their shape. As Figure 3-2 depicts, these curves have asymptotes and discontinuities. Locating these is the first step in the characterization of the shape.

Finding asymptotes As shown in Figure 3-2, there are two asymptotes perpendicular to the U axis and and one perpendicular to the V axis for each red (lighter) and blue (darker) line. We can find asymptotes perpendicular to the U axis as follows.

Equations (3.138) and (3.139) can be expressed in terms of U:

$$
\begin{aligned}
&(\mu_1 V-1)^2U^2+\{-p_1-g_1-f_1+r_1(\mu_1 V-1)\}(\mu_1 V-1)U\\
&-(g_1+f_1)(\mu_1 V-1)r_1+(p_1+g_1)f_1=0
\end{aligned}
\tag{3.140}
$$

$$
\begin{aligned}
&(\mu_2 V-1)^2U^2-\{-p_2-g_2-f_2+r_2(\mu_2 V-1)\}(\mu_2 V-1)U\\
&-(g_2+f_2)(\mu_2 V-1)r_2+(p_2+g_2)f_2=0.
\end{aligned}
\tag{3.141}
$$

After dividing equations (3.140) and (3.141) by V^2, we have

$$
\begin{aligned}
&(\mu_1^2-\tfrac{2\mu_1}{V}+\tfrac{1}{V^2})U^2+\{\tfrac{-p_1-g_1-f_1}{V}+r_1(\mu_1-\tfrac{1}{V})\}(\mu_1-\tfrac{1}{V})U\\
&-\tfrac{(g_1+f_1)(\mu_1 V-1)r_1}{V^2}+\tfrac{(p_1+g_1)f_1}{V^2}=0
\end{aligned}
\tag{3.142}
$$

$$
\begin{aligned}
&(\mu_2^2-\tfrac{2\mu_2}{V}+\tfrac{1}{V^2})U^2-\{\tfrac{-p_2-g_2-f_2}{V}+r_2(\mu_2-\tfrac{1}{V})\}(\mu_2-\tfrac{1}{V})U\\
&-\tfrac{(g_2+f_2)(\mu_2 V-1)r_2}{V^2}+\tfrac{(p_2+g_2)f_2}{V^2}=0.
\end{aligned}
\tag{3.143}
$$

As $V\to\pm\infty$, equations (3.142) and (3.143) become

$$\mu_1^2 U(U+r_1)=0 \tag{3.144}$$

$$\mu_2^2 U(U-r_2)=0. \tag{3.145}$$

Therefore, the asymptotes of the red lines that are perpendicular to the U axis are $U=0$ and $U=-r_1$. And the asymptotes of the blue lines that are perpendicular to the U axis are $U=0$ and $U=r_2$.

Similarily, equations (3.138) and (3.139) can be expressed in terms of V to find the asymptotes that are perpendicular to V axis:

$$
\begin{aligned}
&\mu_1^2 U(U+r_1)V^2+\mu_1\{-U(2U+2r_1+p_1+g_1+f_1)-r_1(g_1+f_1)\}V\\
&+U(U+r_1+p_1+g_1+f_1)+r_1(g_1+f_1)+f_1(p_1+g_1)=0
\end{aligned}
\tag{3.146}
$$

$$
\begin{aligned}
&\mu_2^2 U(U-r_2)V^2+\mu_2\{-U(2U-2r_2-p_2-g_2-f_2)-r_2(g_2+f_2)\}V\\
&+U(U-r_2-p_2-g_2-f_2)+r_2(g_2+f_2)+f_2(p_2+g_2)=0.
\end{aligned}
\tag{3.147}
$$

After dividing equations (3.146) and (3.147) by U^2, we get

$$\begin{aligned}&\mu_1^2(1+\tfrac{r_1}{U})V^2+\mu_1\{-(2+\tfrac{2r_1+p_1+g_1+f_1}{U})-\tfrac{r_1(g_1+f_1)}{U^2}\}V\\&+(1+\tfrac{r_1+p_1+g_1+f_1}{U})+\tfrac{r_1(g_1+f_1)+f_1(p_1+g_1)}{U^2}=0\end{aligned} \tag{3.148}$$

$$\begin{aligned}&\mu_2^2(1-\tfrac{r_2}{U})V^2+\mu_2\{-(2+\tfrac{-2r_2-p_2-g_2-f_2}{U})-\tfrac{r_2(g_2+f_2)}{U^2}\}V\\&+(1+\tfrac{-r_2-p_2-g_2-f_2}{U})+\tfrac{r_2(g_2+f_2)+f_2(p_2+g_2)}{U^2}=0\end{aligned} \tag{3.149}$$

As $U \to \pm\infty$, equations (3.148) and (3.149) become

$$(\mu_1 V - 1)^2 = 0 \tag{3.150}$$

$$(\mu_2 V - 1)^2 = 0. \tag{3.151}$$

Therefore, the asymptote of the red lines that is perpendicular to V is $V = \frac{1}{\mu_1}$. And the asymptote of the blue lines is $V = \frac{1}{\mu_2}$.

Finding discontinuous range The solutions of equation (3.138) are two red lines whose equations can be re-written as

$$V = f_r(U) = \tfrac{1}{\mu_1}[1+\tfrac{\{(p_1+g_1+f_1)U+(g_1+f_1)r_1\}+\sqrt{\{(p_1+g_1+f_1)U+(g_1+f_1)r_1\}^2-4U(U+r_1)(p_1+g_1)f_1}}{2U(U+r_1)}]. \tag{3.152}$$

$$V = g_r(U) = \tfrac{1}{\mu_1}[1+\tfrac{\{(p_1+g_1+f_1)U+(g_1+f_1)r_1\}-\sqrt{\{(p_1+g_1+f_1)U+(g_1+f_1)r_1\}^2-4U(U+r_1)(p_1+g_1)f_1}}{2U(U+r_1)}]. \tag{3.153}$$

A discontinuous range in the red lines appears where $V = f_r(U)$ and $V = g_r(U)$ have complex values. Therefore the U-coordinates of the two end points of the discontinuous range in the red lines are the solutions of the equation:

$$\begin{aligned}&\{(p_1+g_1+f_1)U+(g_1+f_1)r_1\}^2-4U(U+r_1)(p_1+g_1)f_1\\&=(p_1+g_1+f_1)^2U^2+2r_1\{(p_1+g_1)(g_1-f_1)+f_1(g_1+f_1)\}U+(g_1+f_1)^2r_1^2=0.\end{aligned} \tag{3.154}$$

If we set $a_R = (p_1+g_1+f_1)^2$, $b_R = 2r_1\{(p_1+g_1)(g_1-f_1)+f_1(g_1+f_1)\}$, and $c_R = (g_1+f_1)^2 r_1^2$, then,

$$U_r^u = \frac{-b_R - \sqrt{b_R^2 - 4a_Rc_R}}{2a_R}, U_r^l = \frac{-b_R + \sqrt{b_R^2 - 4a_Rc_R}}{2a_R}. \tag{3.155}$$

Corresponding V coordinates of the points are given as

$$V_r^u = \frac{1}{\mu_1}[1 + \frac{(p_1 + g_1 + f_1)U_r^u + (g_1 + f_1)r_1}{2U_r^u(U_r^u + r_1)}]$$

$$V_r^l = \frac{1}{\mu_1}[1 + \frac{(p_1 + g_1 + f_1)U_r^l + (g_1 + f_1)r_1}{2U_r^l(U_r^l + r_1)}]. \tag{3.156}$$

The (U_r^u, V_r^u) and (U_r^l, V_r^l) are the coordinates of the end points of the red line as illustrated in Figure 3-2.

The solutions of equation (3.139) are two blue lines whose equations can be re-written as:

$$V = f_b(U) = \frac{1}{\mu_2}[1 + \frac{-\{(p_2+g_2+f_2)U-(g_2+f_2)r_2\}+\sqrt{\{(p_2+g_2+f_2)U-(g_2+f_2)r_2\}^2-4U(U-r_2)(p_2+g_2)f_2}}{2U(U-r_2)}] \tag{3.157}$$

$$V = g_b(U) = \frac{1}{\mu_2}[1 + \frac{-\{(p_2+g_2+f_2)U-(g_2+f_2)r_2\}-\sqrt{\{(p_2+g_2+f_2)U-(g_2+f_2)r_2\}^2-4U(U-r_2)(p_2+g_2)f_2}}{2U(U-r_2)}]. \tag{3.158}$$

Using the same procedure, we can find (U_b^u, V_b^u) and (U_b^l, V_b^l), which are the coordinates of the end points of the blue lines as illustrated in Figure 3-2.

$$U_b^u = \frac{-b_B + \sqrt{b_B^2 - 4a_Bc_B}}{2a_B}, U_b^l = \frac{-b_B - \sqrt{b_B^2 - 4a_Bc_B}}{2a_B} \tag{3.159}$$

where $a_B = (p_2 + g_2 + f_2)^2$, $b_B = -2r_2\{(p_2 + g_2)(g_2 - f_2) + f_2(g_2 + f_2)\}$, and $c_B = (g_2 + f_2)^2 r_2^2$

$$V_b^u = \frac{1}{\mu_2}[1 + \frac{-(p_2 + g_2 + f_2)U_b^u + (g_2 + f_2)r_2}{2U_b^u(U_b^u - r_2)}]$$

$$V_r^l = \frac{1}{\mu_2}[1 + \frac{-(p_2 + g_2 + f_2)U_b^l + (g_2 + f_2)r_2}{2U_b^l(U_b^l - r_2)}] \tag{3.160}$$

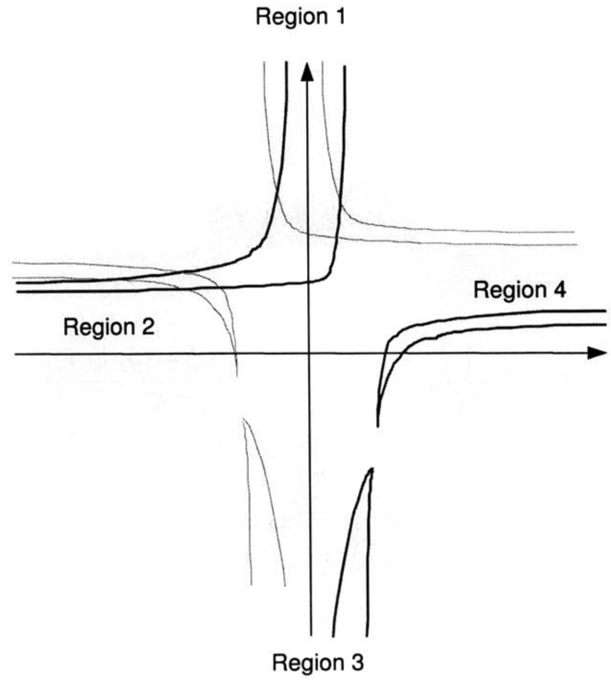


Figure 3-3: Typical shape of the solutions of equations 3.138 and 3.139

Root finding algorithm

Figure 3-3 shows a typical shape of the simplified internal transition equations (i.e., equations (3.138) and (3.139)). But the number and the locations of roots vary depending on machine parameters (i.e. $\mu_1, \mu_2, r_1, r_2, p_1, p_2, g_1, g_2, f_1$, and f_2) as shown in Figures 3-4, 3-5, and 3-6.

As illustrated in Figure 3-3, there are four regions in the (U, V) space in which roots could possibly exist:

Region 1 There are 3 roots in this region regardless of machine parameters as depicted in Figure 3-7. The asymptotes of the blue lines are located at $U = 0, U = r_2$, and the blue lines approach the asymptotes from the left. On the other hand, the red curves have asymptotes at $U = 0, U = -r_1$ and approach them from the right. Therefore, one blue curve approaching the asymptote $U = r_2$ meets two of the red curves. The other blue curve approaching the asymptote $U = 0$ meets only one red curve, which approaches the asymptote $U = -r_1$.

Let us define ϵ as a small number (e.g., order of 10^{-9}) and U_{big} as a large number (e.g., order of 10^6). We can find these roots in region 1 as follows:

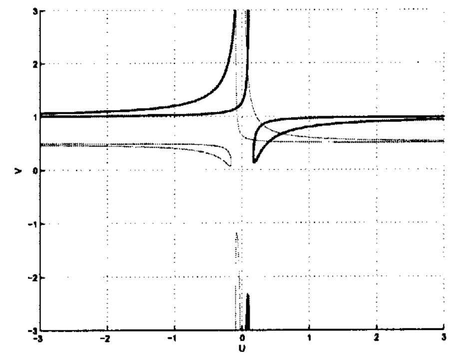


Figure 3-4: Plot of the simplified internal transition equations with $\mu_1 > \mu_2$

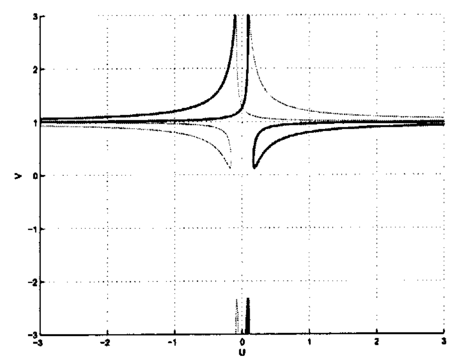


Figure 3-5: Plot of the simplified internal transition equations with $\mu_1 = \mu_2$

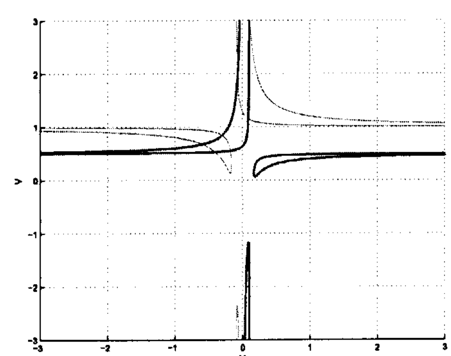


Figure 3-6: Plot of the simplified internal transition equations with $\mu_1 < \mu_2$

- One root at the intersection of $g_b(U)$ and $g_r(U)$ (Root 1 in Figure 3-7) is located in $[-r_1 + \epsilon, r_2 - \epsilon]$. It can be found by solving $g_b(U) - g_r(U) = 0$ using the Newton-Raphson method.

- Another root at intersection of $f_b(U)$ and $g_r(U)$ (Root 2 in Figure 3-7) is located in $[-r_1+\epsilon, -\epsilon]$. It can be found by solving $f_b(U)-g_r(U) = 0$ using the Newton-Raphson method.

- The other root at the intersection of $f_r(U)$ and $g_b(U)$ is (Root 3 in Figure 3-7) located in $[\epsilon, r_2 - \epsilon]$. It can be found by solving $f_r(U) - g_b(U) = 0$ with the Newton-Raphson method.

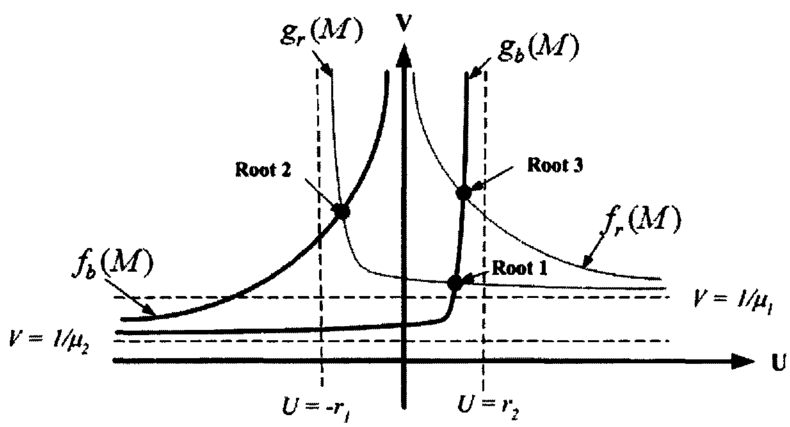


Figure 3-7: Root finding in region 1

Region 3 There are no roots in this area regardless of machine parameters. This is because the blue lines come from the right hand side and approach the asymptotes located at $U = 0, U = r_2$. The red lines come from the left hand side and approach the asymptotes located at $U = 0, U = -r_1$. Thus, the blue lines and red lines can not intersect with each other.

Region 2 If $\mu_1 \geq \mu_2$, then there is no root in region 2. Because the blue curves approach $V = \frac{1}{\mu_2}$ from above and the red curves approach $V = \frac{1}{\mu_1}$ from below. But if $\mu_1 < \mu_2$, there are many cases to consider:

(1) If $f_1 > p_1$ and $g_1 > 0$, there is a gap in the red lines as shown in Figure (3-8). In this case, the number of roots depends on the location of the gap in the red lines. In Figure (3-8), (U_r^u, V_r^u) and (U_r^l, V_r^l) are the Cartesian coordinates of the end points of the gap in the red lines.

Here, let us define a new function $H_b(U, V)$ as

$$H_b(U, V) = (V - f_b(U))(V - g_b(U)). \tag{3.161}$$

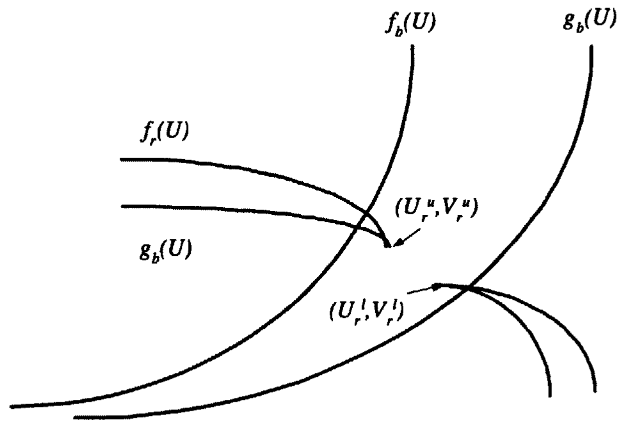
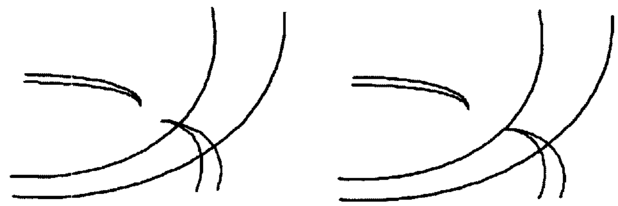


Figure 3-8: Plot of red lines and blue lines with $\mu_1 < \mu_2$ in region 2

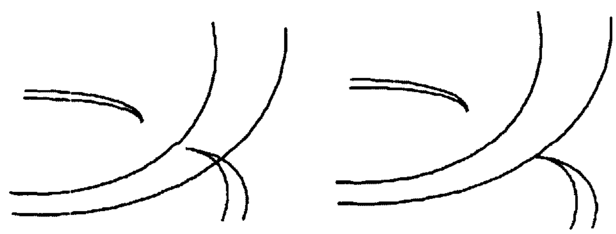
The number and the location of roots in region 2 when $f_1 > p_1$ and $g_1 > 0$ are as follows and these roots can be found through the Newton-Raphson method:

- *Case 1*: If $V_r^l - f_b(U_r^l) > 0$, there are *four* roots in the region:
  - One from the equation $f_b(U) - f_r(U) = 0$ is located in $[U_r^l + \epsilon, -\epsilon]$.
  - Another from the equation $f_b(U) - g_r(U) = 0$ is located in $[U_r^l + \epsilon, -r_1 - \epsilon]$.
  - Another from the equation $g_b(U) - g_r(U) = 0$ is located in $[U_r^l + \epsilon, -r_1 - \epsilon]$.
  - The other from the equation $g_b(U) - f_r(U) = 0$ is located in $[U_r^l + \epsilon, -\epsilon]$.
- *Case 2* : If $V_r^l - f_b(U_r^l) = 0$, there are *three* roots in the region:
  - One root is located at (U_r^L, V_r^L).
  - Another from the equation $g_b(U) - g_r(U) = 0$ is located in $[U_r^l + \epsilon, -r_1 - \epsilon]$.
  - The other from the equation $g_b(U) - f_r(U) = 0$ is located in $[U_r^l + \epsilon, -\epsilon]$.



Case 1 Case 2

Figure 3-9: Case 1 and Case 2



Case 3 Case 4

Figure 3-10: Case 3 and Case 4

- *Case 3*: If $V_r^u - f_b(U_r^u) > 0$ and $H_b(U_r^l, V_r^l) < 0$, there are *two* roots in the region:
  - One from the equation $g_b(U) - g_r(U) = 0$ is located in $[U_r^l + \epsilon, -r_1 - \epsilon]$.
  - The other from the equation $g_b(U) - f_r(U) = 0$ is located in $[U_r^l + \epsilon, -\epsilon]$.
- *Case 4*: If $V_r^u - f_b(U_r^u) > 0$ and $V_r^l - g_b(U_r^l) = 0$, then *one* root is located at (U_r^l, V_r^l).

Case 5 Case 6

Figure 3-11: Case 5 and Case 6

- *Case 5*: If $V_r^u - f_b(U_r^u) > 0$ and $V_r^l - g_b(U_r^l) < 0$, then there is *no root* in the region.

- *Case 6*: If $V_r^u - f_b(U_r^u) = 0$ and $H_b(U_r^l, V_r^l) < 0$, there are *three* roots in the region:
  - One at (U_r^u, V_r^u).
  - The other from the equation $g_b(U) - g_r(U) = 0$ is located in $[U_r^l + \epsilon, -r_1 - \epsilon]$.
  - The other from the equation $g_b(U) - f_r(U) = 0$ is located in $[U_r^l + \epsilon, -\epsilon]$.

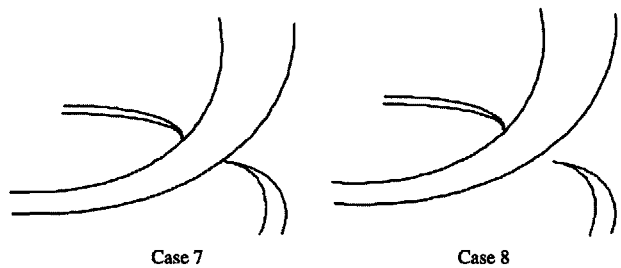


Figure 3-12: Case 7 and Case 8

- *Case 7*: If $V_r^l - f_b(U_r^u) = 0$ and $V_r^l - g_b(U_r^l) = 0$, there are *two* roots in the region:
  - One is at (U_r^u, V_r^u).
  - The other one is at (U_r^l, V_r^l).
- *Case 8*: If $V_r^u - f_b(U_r^u) = 0$ and $V_r^l - g_b(U_r^l) < 0$, *one* root is located at (U_r^u, V_r^u).

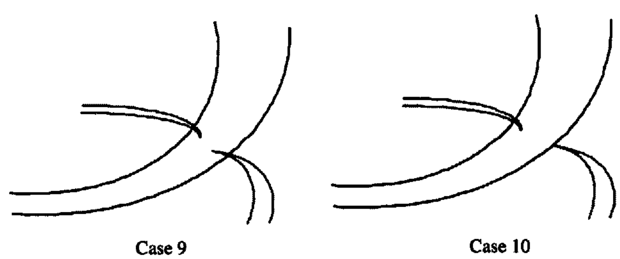


Figure 3-13: Case 9 and Case 10

- *Case 9*: If $H_b(U_r^u, V_r^u) < 0$ and $H_b(U_r^l, V_r^l) < 0$, there are *four* roots in the region;
  - One from the equation $f_b(U) - g_r(U) = 0$ is located in $[U_r^u - U_{big}, U_r^u - \epsilon]$.

- Another from the equation $f_b(U) - f_r(U) = 0$ is located in $[U_r^u - U_{big}, U_r^u - \epsilon]$.
- Another from the equation $g_b(U) - g_r(U) = 0$ is located in $[U_r^u + \epsilon, -r_1 - \epsilon]$.
- The other from the equation $g_b(U) - f_r(U) = 0$ is located in $[U_r^u + \epsilon, -\epsilon]$.

- *Case 10*: If $H_b(U_r^u, V_r^u) < 0$ and $V_r^l - g_b(U_r^l) = 0$, there are *three* in the region:
  - One is at (U_r^u, V_r^u).
  - Another from the equation $f_b(U) - g_r(U) = 0$ is located in $[U_r^u - U_{big}, U_r^u - \epsilon]$.
  - The other from the equation $f_b(U) - f_r(U) = 0$ is located in $[U_r^u - U_{big}, U_r^u - \epsilon]$.

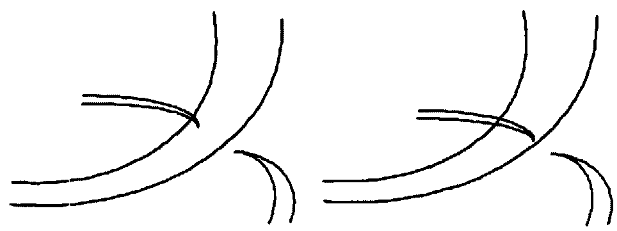


Figure 3-14: Case 11 and Case 12

- *Case 11*: If $H_b(U_r^u, V_r^u) < 0$ and $V_r^l - g_b(U_r^l) < 0$, there are *two* roots in the region:
  - One from the equation $f_b(U) - g_r(U) = 0$ is located in $[U_r^u - U_{big}, U_r^u - \epsilon]$.
  - The other from the equation $f_b(U) - f_r(U) = 0$ is located in $[U_r^u - U_{big}, U_r^u - \epsilon]$.

- *Case 12*: If $V_r^u - g_b(U_r^u) = 0$, there are *three* roots in the region:
  - One at (U_r^u, V_r^u).
  - Another from the equation $f_b(U) - g_r(U) = 0$ is located in $[U_r^u - U_{big}, U_r^u - \epsilon]$.
  - The other from the equation $f_b(U) - f_r(U) = 0$ is located in $[U_r^u - U_{big}, U_r^u - \epsilon]$.

- *Case 13*: If $V_r^u - g_b(U_r^u) < 0$, there are *four* roots in the region:
  - One from $f_b(U) - g_r(U) = 0$ is located in $[U_r^u - U_{big}, U_r^u - \epsilon]$.

- Another from the equation $f_b(U)-f_r(U) = 0$ is located in $[U_r^u-U_{big}, U_r^u-\epsilon]$.
- Another from the equation $g_b(U)-f_r(U) = 0$ is located in $[U_r^u-U_{big}, U_r^u-\epsilon]$.
- The other from the equation $g_b(U)-g_r(U) = 0$ is located in $[U_r^u-U_{big}, U_r^u-\epsilon]$.

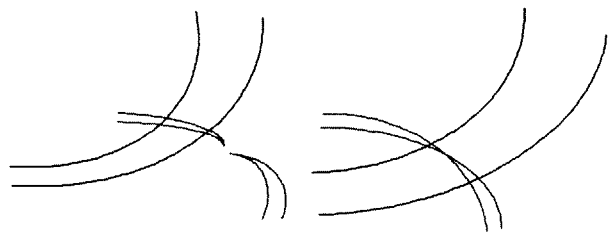


Figure 3-15: Case 13 and Case 14

(2) If $f_1 = g_1$ or $g_1 = 0$, then there is no gap thus, the equations have 3 or 4 roots. In this case the two red curves intersect at $(U_r^u, V_r^u) = (U_r^l, V_r^l)$

- *Case 14*: If $H_b(U_r^u, V_r^u) < 0$, there are *four* roots in the region:
  - One from the equation $f_b(U) - g_r(U) = 0$ is located in $[U_r^u - U_{big}, -r_1 - \epsilon]$.
  - Another from the equation $f_b(U) - f_r(U) = 0$ is located in $[U_r^u - U_{big}, -\epsilon]$.
  - Another from the equation $g_b(U) - f_r(U) = 0$ is located in $[U_r^u + \epsilon, -\epsilon]$.
  - The other from the equation $g_b(U)-g_r(U) = 0$ is located in $[U_r^u+\epsilon, -r_1-\epsilon]$.
- *Case 15*: If $V_r^u - g_b(U_r^u) = 0$, there are *three* in the region:
  - One is at (U_r^u, V_r^u).
  - Another from the equation $f_r(U) = f_b(U)$ is located in $[U_r^u - U_{big}, U_r^u - \epsilon]$.
  - The other from the equation $g_r(U) = f_b(U)$ is located in $[U_r^u - U_{big}, U_r^u - \epsilon]$.
- *Case 16*: If $V_r^u - f_b(U_r^u) = 0$, there are *three* roots in the region:
  - One is at (U_r^u, V_r^u).
  - Another from the equation $f_r(u) = g_b(u)$ is located in $[U_r^u + \epsilon, -\epsilon]$.
  - The other from the equation $g_r(U) = g_b(U)$ is located in $[U_r^u-U_{big}, -r_1-\epsilon]$.

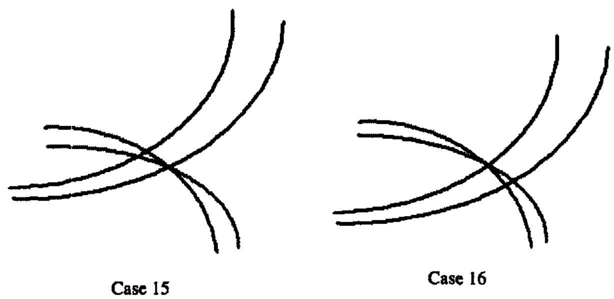


Figure 3-16: Case 15 and Case 16

Region 4 If $\mu_1 \leq \mu_2$, then there is no root in region 4 because the blue curves approach $V = \frac{1}{\mu_2}$ from below and the red curves approach to $V = \frac{1}{\mu_1}$ from above. But if $\mu_1 > \mu_2$, there are many cases to consider;

(1) If $f_2 > p_2$ **and** $g_2 > 0$, there is a gap in the blue lines as shown in Figure 3-17. In this case, the number of roots depends on the location of the gap in the blue lines. In Figure 3-17, (U_b^u, V_b^u) and (U_b^l, V_b^l) are the Cartesian coordinates of the end points of the gap in the blue lines.

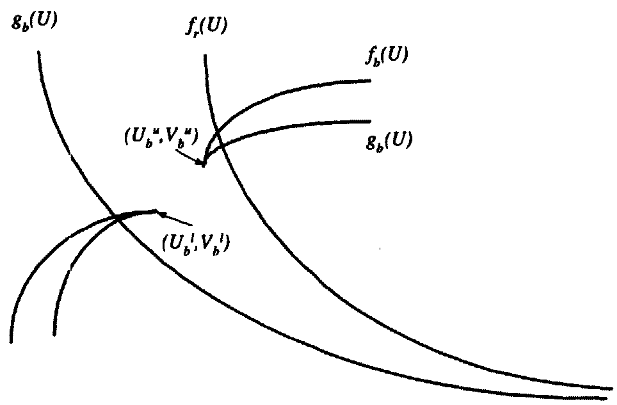


Figure 3-17: Plot of red lines and blue lines with $\mu_1 > \mu_2$ in region 4

Here, let us define a new function $H_r(U, V)$ as

$$H_r(U, V) = (V - f_r(U))(V - g_r(U)). \tag{3.162}$$

The number and the location of roots in region 4 are as follows and these roots can be found using the Newton-Raphson method:

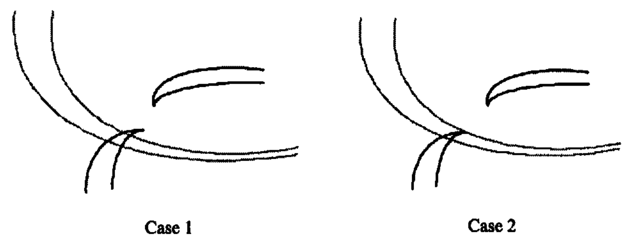


Figure 3-18: Case 1 and Case 2

- *Case 1*: If $V_b^l - f_r(U_b^l) > 0$, there are *four* roots in the region:
  - One from the equation $f_r(U) - f_b(U) = 0$ is located in $[\epsilon, U_b^l - \epsilon]$.
  - Another from the equation $f_r(U) - g_b(U) = 0$ is located in $[r_2 + \epsilon, U_b^l - \epsilon]$.
  - Another from the equation $f_r(U) - f_b(U) = 0$ is located in $[\epsilon, U_b^l - \epsilon]$.
  - The other from the equation $g_r(U) - g_b(U) = 0$ is located in $[r_2 + \epsilon, U_b^l - \epsilon]$.
- *Case 2*: If $V_b^l - f_r(U_b^l) = 0$, there *three* roots in the region:
  - One is at (U_b^l, V_b^l).
  - Another from the equation $g_r(U) - f_b(U) = 0$ is located in $[\epsilon, U_b^l - \epsilon]$.
  - The other from the equation $g_r(U) - g_b(U) = 0$ is located in $[r_2 + \epsilon, U_b^l - \epsilon]$.

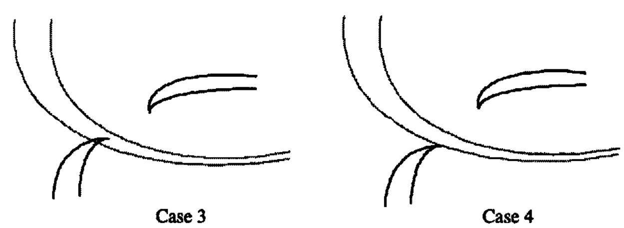


Figure 3-19: Case 3 and Case 4

- *Case 3*: If $V_b^u - f_r(U_b^u) > 0$ and $H_r(U_b^l, V_b^l) < 0$, there are *two* roots in the region:
  - One from the equation $g_r(U) - f_b(U) = 0$ is located in $[\epsilon, U_b^l - \epsilon]$.
  - The other from the equation $g_r(U) - g_b(U) = 0$ is located in $[r_2 + \epsilon, U_b^l - \epsilon]$.

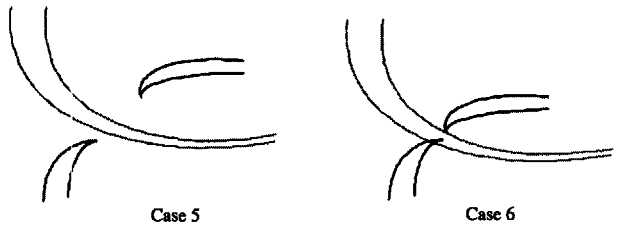


Figure 3-20: Case 5 and Case 6

- *Case 4*: If $U_b^u - f_r(U_b^u) > 0$ and $V_b^l - g_r(U_b^l) = 0$, then *one* root is located at (U_b^l, V_b^l).
- *Case 5*: If $V_b^u - f_r(U_b^u) > 0$ and $V_b^l - g_r(U_b^l) < 0$, then there is *no* root in the region.
- *Case 6*: If $V_b^u - f_r(U_b^u) = 0$ and $H_r(U_b^l, V_b^l) < 0$, there are *three* roots in the region:
  - One is at (U_b^u, V_b^u).
  - Another from the equation $g_r(U) - f_b(U) = 0$ is located in $[\epsilon, U_b^l - \epsilon]$.
  - The other from the equation $g_r(U) - g_b(U) = 0$ is located in $[r_2 + \epsilon, U_b^l - \epsilon]$.

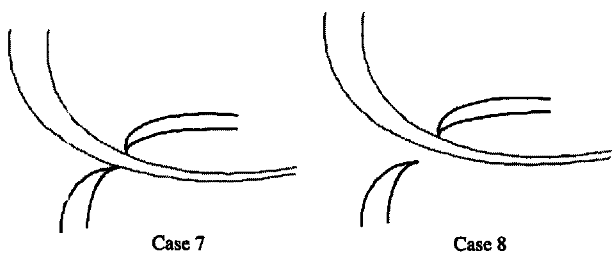


Figure 3-21: Case 7 and Case 8

- *Case 7*: If $V_b^u - f_r(U_b^u) = 0$ and $V_b^l - g_r(U_b^l) = 0$, then *two* roots are located in the region:
  - One is at (U_b^u, V_b^u).
  - The other is at (U_b^l, V_b^l).
- *Case 8*: If $V_b^u - f_r(U_b^u) = 0$ and $V_b^l - g_r(U_b^l) < 0$, *one* root is located at (U_b^u, V_b^u).

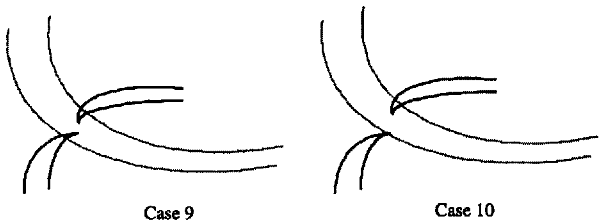


Figure 3-22: Case 9 and Case 10

- *Case 9*: If $H_r(U_b^u, V_b^u) < 0$ and $H_r(U_b^l, V_b^l) < 0$, there are *four* roots in the region:
  - One from the equation $f_r(U) - g_b(U) = 0$ is located in $[U_b^u + \epsilon, U_b^u + U_{big}]$.
  - Another from the equation $f_r(U) - f_b(U) = 0$ is located in $[U_b^u + \epsilon, U_b^u + U_{big}]$.
  - Another from the equation $g_r(U) - f_b(U) = 0$ is located in $[\epsilon, U_b^l - \epsilon]$.
  - The other from the equation $g_r(U) - g_b(U) = 0$ is located in $[r_2 + \epsilon, U_b^l - \epsilon]$.
- *Case 10*: If $H_r(U_b^u, V_b^u) < 0$ and $V_b^l - g_r(U_b^l) = 0$, then there are *three* roots in the region:
  - One is at (U_b^l, V_b^l).
  - One from the equation $f_r(U) - g_b(U) = 0$ is located in $[U_b^u + \epsilon, U_b^u + U_{big}]$.
  - Another from the equation $f_r(U) - f_b(U) = 0$ is located in $[U_b^u + \epsilon, U_b^u + U_{big}]$.

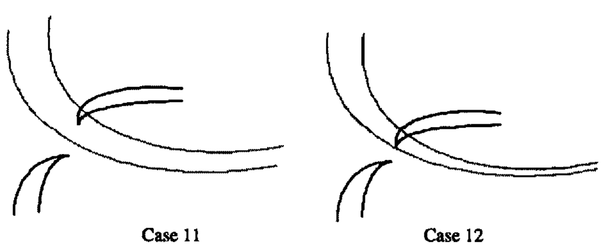


Figure 3-23: Case 11 and Case 12

- *Case 11*: If $H_r(U_b^u, V_b^u) < 0$ and $V_b^l - g_r(U_b^l) < 0$, there are *two* roots in the region:
  - One from the equation $f_r(U) - g_b(U) = 0$ is located in $[U_b^u + \epsilon, U_b^u + U_{big}]$.

– Another from the equation $f_r(U) - f_b(U) = 0$ is located in $[U_b^u + \epsilon, U_b^u + U_{big}]$.

- *Case 12*: If $V_b^u - g_r(U_b^u) = 0$,*three* roots are located in the region:
  - One is at (U_b^u, V_b^u).
  - Another from the equation $f_r(U) - g_b(U) = 0$ is located in $[U_b^u + \epsilon, U_b^u + U_{big}]$.
  - The other from the equation $f_r(U) - f_b(U) = 0$ is located in $[U_b^u + \epsilon, U_b^u + U_{big}]$.

- *Case 13*: If $V_b^u - g_r(U_b^u) < 0$, there are *four* roots in the region:
  - One from the equation $f_r(U) - g_b(U) = 0$ is located in $[U_b^u + \epsilon, U_b^u + U_{big}]$.
  - Another from the equation $f_r(U) - f_b(U) = 0$ is located in $[U_b^u + \epsilon, U_b^u + U_{big}]$.
  - Another from the equation $g_r(U) - f_b(U) = 0$ is located in $[U_b^u + \epsilon, U_b^u + U_{big}]$.
  - The other from the equation $g_r(U) - g_b(U) = 0$ is located in $[U_b^u + \epsilon, U_b^u + U_{big}]$.

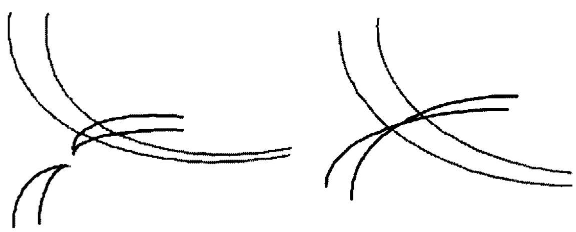


Figure 3-24: Case 13 and Case 14

(2) If $f_2 = g_2$ or $g_2 = 0$, then there is no gap in the blue lines. Thus, the equations have either 3 or 4 roots. In this case the two blue curves intersect at $(U_b^u, V_b^u) = (U_b^l, V_b^l)$.

- *Case 14* If $H_r(U_b^u, V_b^u) < 0$, there are four roots in the region:
  - One from the equation $f_b(U) - g_r(U) = 0$ is located in $[\epsilon, U_b^u - \epsilon]$.
  - Another from the equation $f_b(U) - f_r(U) = 0$ is located in $[U_b^u + \epsilon, U_b^u + U_{big}]$.
  - Another from the equation $g_b(U) - f_r(U) = 0$ is located in $[R2 + \epsilon, U_b^u + U_{big}]$.
  - The other from the equation $g_b(U) - g_r(U) = 0$ is located in $[R2 + \epsilon, U_b^u - \epsilon]$.

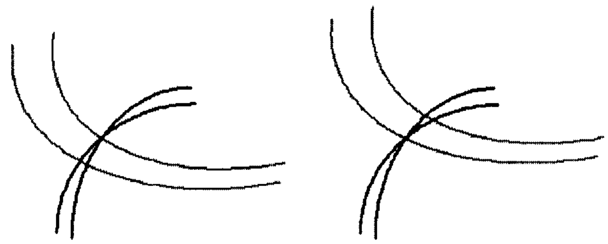


Figure 3-25: Case 15 and Case 16

- *Case 15*: If $V_b^u - f_r(U_b^u) = 0$, there are *three* located in the region:
  - One is at (U_b^u, V_b^u).
  - Another from the equation $g_r(U) - f_b(U) = 0$ is located in $[\epsilon, U_b^u - \epsilon]$.
  - The other from the equation $g_r(U) - g_b(U) = 0$ is located in $[r_2 + \epsilon, U_b^u - \epsilon]$.
- *Case 16*: If $V_b^u - g_r(U_b^u) = 0$, there are *three* in the region;
  - One is at (U_b^u, V_b^u).
  - Another from the equation $f_r(U) - f_b(U) = 0$ is located in $[U_b^u + \epsilon, U_b^u + U_{big}]$
  - The other from $f_r(U) - g_b(U) = 0$ is located in $[U_b^u + \epsilon, U_b^u + U_{big}]$

3.2.3 Building the probability density function

Once we find the roots of equations (3.138) and (3.139), we can get Y_i and Z_i ($i = 1, 2$) from equations (3.133) and (3.134). From $Y_i = \frac{G_i(1)}{G_i(0)}$, $Z_i = \frac{G_i(-1)}{G_i(0)}$, we can get $G_1(1), G_1(-1), G_2(1)$, and $G_2(-1)$ by setting $G_1(0) = G_2(0) = 1$ since only the ratios matter. Then, from equations (3.105) and (3.122), λ is

$$\lambda = \frac{-p_1 - g_1 + {}^{r_1}\!/_{G_1(1)} - p_1 G_1(1) + r_1 - f_1 G_1(-1)}{\mu_1}. \tag{3.163}$$

As a result, we can get the probability density function $f_i(x, \alpha_1, \alpha_2)$ corresponding to a (U_i, V_i) pair. Therefore, the general expression of the probability density function is

$$f(x, \alpha_1, \alpha_2) = \sum_{i=1}^{RN} c_i f_i(x, \alpha_1, \alpha_2) \tag{3.164}$$

where RN is the number of roots of equations (3.138) and (3.139) which is found in 3.2.2.

The remaining unknowns, including coefficients c_i $i = 1, 2..., RN$ and probability masses at the boundaries, can be calculated by solving the boundary transition equations and the normalization equation.

3.2.4 Methods to solve boundary conditions

$\mu_1 = \mu_2$ Case

The boundary equations (3.15) – (3.36) are linear equations in which the unknowns are the probability masses and the coefficients in equation (3.164). Some of the probability masses are 0 according to the equations, and functions $f_i(x, \alpha_1, \alpha_2)$ are found by solving the internal transition equations in section 3.2.2. Note that when $\mu_1 = \mu_2$, the internal transition equations have 3 roots. The boundary equations can be simplified as follows:

- Drop off the probability masses which are set to 0.
- Temporarily set $P(0, 1, 1) = 1$.
- Substitute

$$f(x, \alpha_1, \alpha_2) = c_1 f_1(x, \alpha_1, \alpha_2) + c_2 f_2(x, \alpha_1, \alpha_2) + c_3 f_3(x, \alpha_1, \alpha_2)$$

  where $f_i(x, \alpha_1, \alpha_2) = e^{\lambda_i x} G_1^i(\alpha_1) G_2^i(\alpha_2)$ and $G_j^i(1) = Y_j^i$,$G_j^i(0) = 1$, and $G_j^i(-1) = Z_j^i$ ($i = 1, 2, 3$ and $j = 1, 2$).

Then, we have an equation $AX = B$ (3.165), which is in matrix form. For example, the first row of A and B are from equation (3.15). After plugging $f(0, 0, 1) = c_1 Y_2^1 + c_2 Y_2^2 + c_3 Y_2^3$, into equation (3.15), the equation becomes $\mu_2(c_1 Y_2^1 + c_2 Y_2^2 + c_3 Y_2^3) - r_1 P(0, 0, 1) + +f_1 P(0, -1, 1) + p_1 = 0$. The unknowns ($c_1$, c_2, c_3, $P(0, 0, 1)$, and $P(0, -1, 1)$) are placed at the matrix X, p_1 is at B, and the others are at A. After solving the equation (3.165) using a linear equation solver, all the unknowns are expressed as multiples of $P(0, 1, 1)$. Then, the value of $P(0, 1, 1)$ can be calculated from the normalization equation (3.89).

$$A = \begin{pmatrix}
\mu Y_2^1 & \mu Y_2^2 & \mu Y_2^3 & 0 & -r_1 & 0 & f_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu Y_1^1 & \mu Y_1^2 & \mu Y_1^3 & -f_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu Z_1^1 & \mu Z_1^2 & \mu Z_1^3 & 0 & 0 & 0 & -p_2 & -f_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu e^{\lambda_1 N} Y_2^1 & \mu e^{\lambda_2 N} Y_2^2 & \mu e^{\lambda_3 N} Y_2^3 & 0 & 0 & 0 & 0 & 0 & -p_1 & 0 & 0 & -f_1 & 0 & 0 \\
\mu e^{\lambda_1 N} Z_2^1 & \mu e^{\lambda_2 N} Z_2^2 & \mu e^{\lambda_3 N} Z_2^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p_1 & 0 & 0 & -f_1 \\
0 & 0 & 0 & 0 & r_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -(p_1 + g_1 + f_2) & 0 & r_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu Y_2^1 & \mu Y_2^2 & \mu Y_2^3 & 0 & -r_1 & 0 & f_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu Z_2^1 & \mu Z_2^2 & \mu Z_2^3 & p_1 & 0 & -r_1 & 0 & f_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & f_1 + p_2 + g_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & g_1 & 0 & 0 & g_2 & -(f_1 + f_2) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -(p_1 + g_1 + p_2 + g_2) & r_2 & 0 & 0 & 0 & 0 \\
\mu e^{\lambda_1 N} Y_1^1 & \mu e^{\lambda_2 N} Y_1^2 & \mu e^{\lambda_3 N} Y_1^3 & 0 & 0 & 0 & 0 & 0 & p_2 & -r_2 & f_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_2 & 0 & -(p_1 + g_1 + f_2) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_1 & 0 & 0 & -(f_1 + g_2 + p_2) & r_2 & 0 \\
\mu e^{\lambda_1 N} Z_1^1 & \mu e^{\lambda_2 N} Z_1^2 & \mu e^{\lambda_3 N} Z_1^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_2 & -r_2 & f_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g_1 & g_2 & 0 & -(f_1 + f_2)
\end{pmatrix}$$

$$X = [c_1, c_2, c_3, P(0,1,-1), P(0,0,1), P(0,0,-1), P(0,-1,1), P(0,-1,-1), P(N,1,1), P(N,1,0), P(N,1,-1), P(N,-1,1), P(N,-1,0), P(N,-1,-1)]^T$$

$$B = [-p_1, p_2, 0, 0, 0, p_1 + g_1 + p_2 + g_2, -g_2, 0, g_1, 0, 0, 0, 0, 0, 0, 0]^T \tag{3.165}$$

$\mu_1 > \mu_2$ Case

When $\mu_1 > \mu_2$, the number of roots from the internal transition equations varies from three to seven depending on machine parameters. The number of roots and the corresponding probability density functions are found through the algorithm presented at 3.2.2.

The boundary equations (3.37) – (3.62) can be simplified:

- Drop off the probability masses which are set to 0.
- Temporarily set $P(0,0,1) = 1$.
- Substitute

$$f(x,\alpha_1,\alpha_2) = \sum_{i=1}^{RN} c_i f_i(x,\alpha_1,\alpha_2)$$

  where RN is the number of roots, $f_i(x,\alpha_1,\alpha_2) = e^{\lambda_i x} G_1^i(\alpha_1) G_2^i(\alpha_2)$ and $G_j^i(1) = Y_j^i$,$G_j^i(0) = 1$, and $G_j^i(-1) = Z_j^i$ ($i = 1,2,3$ and $j = 1,2$).

When there are seven roots from the equations (3.138) and (3.139), we have an equation $AX = B$ (3.166), which is a matrix form. After solving the equation (3.166) using a linear equation solver, all the unknowns are expressed as multiples of $P(0,0,1)$. Then, the value of $P(0,0,1)$ can be calculated from the normalization equation (3.89).

When there are six roots from the equations (3.138) and (3.139), the number of unknowns is reduced to 13 from 14. (c_7 no longer exists). In this case we can use the same equation (3.166) and the same procedure after setting $c_7 = 0$. We can do the same procedure when the internal transition equations have 3, 4, or 5 roots:

- In case of 5 roots, set $c_7 = 0$, $c_6 = 0$.
- In case of 4 roots, set $c_7 = 0$, $c_6 = 0$, and $c_5 = 0$.
- In case of 3 roots, set $c_7 = 0$, $c_6 = 0$, $c_5 = 0$, and , $c_4 = 0$.

Matrix A in equation (3.166) contains elements which can be different by several orders of magnitude (e.g., $e^{\lambda_i N} Z_i^j Y_i^j$ and Y_i^j). This may cause the reduction of the apparent rank of matrix A, which will lead to errors. Techniques to prevent this kind of numerical error are presented in Appendix C.

$$A = \begin{pmatrix}
Y_1^1 & Y_1^2 & Y_1^3 & Y_1^4 & Y_1^5 & Y_1^6 & Y_1^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Z_1^1 & Z_1^2 & Z_1^3 & Z_1^4 & Z_1^5 & Z_1^6 & Z_1^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Y_1^1Y_2^1 & Y_1^2Y_2^2 & Y_1^3Y_2^3 & Y_1^4Y_2^4 & Y_1^5Y_2^5 & Y_1^6Y_2^6 & Y_1^7Y_2^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Y_1^1Z_2^1 & Y_1^2Z_2^2 & Y_1^3Z_2^3 & Y_1^4Z_2^4 & Y_1^5Z_2^5 & Y_1^6Z_2^6 & Y_1^7Z_2^7 & -\frac{r_1}{\mu_D} & 0 & 0 & 0 & 0 & 0 & 0 \\
Z_1^1Y_2^1 & Z_1^2Y_2^2 & Z_1^3Y_2^3 & Z_1^4Y_2^4 & Z_1^5Y_2^5 & Z_1^6Y_2^6 & Z_1^7Y_2^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Z_1^1Z_2^1 & Z_1^2Z_2^2 & Z_1^3Z_2^3 & Z_1^4Z_2^4 & Z_1^5Z_2^5 & Z_1^6Z_2^6 & Z_1^7Z_2^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e^{\lambda_1 N}Y_2^1 & e^{\lambda_2 N}Y_2^2 & e^{\lambda_3 N}Y_2^3 & e^{\lambda_4 N}Y_2^4 & e^{\lambda_5 N}Y_2^5 & e^{\lambda_6 N}Y_2^6 & e^{\lambda_7 N}Y_2^7 & 0 & -\frac{p_1^b}{\mu_2} & 0 & 0 & -\frac{f_1^b}{\mu_2} & 0 & 0 \\
e^{\lambda_1 N}Z_2^1 & e^{\lambda_2 N}Z_2^2 & e^{\lambda_3 N}Z_2^3 & e^{\lambda_4 N}Z_2^4 & e^{\lambda_5 N}Z_2^5 & e^{\lambda_6 N}Z_2^6 & e^{\lambda_7 N}Z_2^7 & 0 & 0 & 0 & -\frac{p_1^b}{\mu_2} & 0 & 0 & -\frac{f_1^b}{\mu_2} \\
Y_2^1 & Y_2^2 & Y_2^3 & Y_2^4 & Y_2^5 & Y_2^6 & Y_2^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Z_2^1 & Z_2^2 & Z_2^3 & Z_2^4 & Z_2^5 & Z_2^6 & Z_2^7 & -\frac{r_1}{\mu_2} & 0 & 0 & 0 & 0 & 0 & 0 \\
e^{\lambda_1 N}Y_1^1Y_2^1 & e^{\lambda_2 N}Y_1^2Y_2^2 & e^{\lambda_3 N}Y_1^3Y_2^3 & e^{\lambda_4 N}Y_1^4Y_2^4 & e^{\lambda_5 N}Y_1^5Y_2^5 & e^{\lambda_6 N}Y_6^1Y_2^6 & e^{\lambda_7 N}Y_1^7Y_2^7 & 0 & -\frac{p_1^b+g_1^b+p_2+g_2}{\mu_D} & \frac{r_2}{\mu_D} & 0 & 0 & 0 & 0 \\
e^{\lambda_1 N}Y_1^1 & e^{\lambda_2 N}Y_1^2 & e^{\lambda_3 N}Y_1^3 & e^{\lambda_4 N}Y_1^4 & e^{\lambda_5 N}Y_1^5 & e^{\lambda_6 N}Y_1^6 & e^{\lambda_7 N}Y_1^7 & 0 & \frac{p_2}{\mu_1} & -\frac{r_2}{\mu_1} & \frac{f_2}{\mu_1} & 0 & 0 & 0 \\
e^{\lambda_1 N}Y_1^1Z_2^1 & e^{\lambda_2 N}Y_1^2Z_2^2 & e^{\lambda_3 N}Y_1^3Z_2^3 & e^{\lambda_4 N}Y_1^4Z_2^4 & e^{\lambda_5 N}Y_1^5Z_2^5 & e^{\lambda_6 N}Y_1^6Z_2^6 & e^{\lambda_7 N}Y_1^7Z_2^7 & 0 & \frac{g_2}{\mu_D} & 0 & -\frac{p_1^b+g_1^b+f_2}{\mu_D} & 0 & 0 & 0 \\
e^{\lambda_1 N}Z_1^1Y_2^1 & e^{\lambda_2 N}Z_1^2Y_2^2 & e^{\lambda_3 N}Z_1^3Y_2^3 & e^{\lambda_4 N}Z_1^4Y_2^4 & e^{\lambda_5 N}Z_1^5Y_2^5 & e^{\lambda_6 N}Z_1^6Y_2^6 & e^{\lambda_7 N}Z_1^7Y_2^7 & 0 & \frac{g_1^b}{\mu_D} & 0 & 0 & -\frac{f_1^b+g_2+p_2}{\mu_D} & \frac{r_2}{\mu_D} & 0 \\
e^{\lambda_1 N}Z_1^1 & e^{\lambda_2 N}Z_1^2 & e^{\lambda_3 N}Z_1^3 & e^{\lambda_4 N}Z_1^4 & e^{\lambda_5 N}Z_1^5 & e^{\lambda_6 N}Z_1^6 & e^{\lambda_7 N}Z_1^7 & 0 & 0 & 0 & 0 & \frac{p_2}{\mu_1} & -\frac{r_2}{\mu_1} & \frac{f_2}{\mu_1} \\
e^{\lambda_1 N}Z_1^1Z_2^1 & e^{\lambda_2 N}Z_1^2Z_2^2 & e^{\lambda_3 N}Z_1^3Z_2^3 & e^{\lambda_4 N}Z_1^4Z_2^4 & e^{\lambda_5 N}Z_1^5Z_2^5 & e^{\lambda_6 N}Z_1^6Z_2^6 & e^{\lambda_7 N}Z_1^7Z_2^7 & 0 & 0 & 0 & \frac{g_1^b}{\mu_D} & \frac{g_2}{\mu_D} & 0 & -\frac{f_1^b+f_2}{\mu_D}
\end{pmatrix}$$

$$X = [c_1, c_2, c_3, c_4, c_5, c_6, c_7, P(0,0,-1), P(N,1,1), P(N,1,0), P(N,1,-1), P(N,-1,1), P(N,-1,0), P(N,-1,-1)]^T$$

$$B = [0, 0, \frac{r_1}{\mu_D}, 0, 0, 0, 0, 0, \frac{r_1}{\mu_2}, 0, 0, 0, 0, 0, 0, 0]^T \tag{3.166}$$

where $\mu_D = \mu_1 - \mu_2$.

$\mu_1 < \mu_2$ Case

When $\mu_1 < \mu_2$, the number of roots from the internal transition equations varies from three to seven depending on machine parameters. The number of roots and the corresponding probability density functions are found through the algorithm presented at 3.2.2.

The boundary equations (3.63) - (3.88) can be simplified:

- Drop off the probability masses which are set to 0.
- Temporarily set $P(0,1,1) = 1$.
- Substitute

$$f(x,\alpha_1,\alpha_2) = \sum_{i=1}^{RN} c_i f_i(x,\alpha_1,\alpha_2)$$

where RN is the number of roots, $f_i(x,\alpha_1,\alpha_2) = e^{\lambda_i x} G_1^i(\alpha_1) G_2^i(\alpha_2)$ and $G_j^i(1) = Y_j^i$,$G_j^i(0) = 1$, and $G_j^i(-1) = Z_j^i$ ($i = 1,2,3$ and $j = 1,2$).

When there are seven roots from the equations (3.138) and (3.139), we have a equation $AX = B$ (3.167), which is a matrix form. After solving the equation (3.167) using a linear equation solver, all the unknowns are expressed as multiples of $P(0,1,1)$. Then, the value of $P(0,1,1)$ can be calculated from the normalization equation (3.89).

When there are six roots from the equations (3.138) and (3.139), the number of unknowns is reduced to 13 from 14. (c_7 no longer exists). In this case we can use the same equation (3.167) and the same procedure after setting $c_7 = 0$. We can follow the same procedure when the internal transition equations have 3, 4, or 5 roots:

- In case of 5 roots, set $c_7 = 0$, $c_6 = 0$.
- In case of 4 roots, set $c_7 = 0$, $c_6 = 0$, and $c_5 = 0$.
- In case of 3 roots, set $c_7 = 0$, $c_6 = 0$, $c_5 = 0$, and , $c_4 = 0$.

$$
A = \begin{pmatrix}
Y_1^1 & Y_1^2 & Y_1^3 & Y_1^4 & Y_1^5 & Y_1^6 & Y_1^7 & -\frac{f_2^b}{\mu_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
Z_1^1 & Z_1^2 & Z_1^3 & Z_1^4 & Z_1^5 & Z_1^6 & Z_1^7 & 0 & 0 & 0 & -\frac{p_2^b}{\mu_1} & -\frac{f_2^b}{\mu_1} & 0 & 0 \\
e^{\lambda_1 N} Y_2^1 & e^{\lambda_2 N} Y_2^2 & e^{\lambda_3 N} Y_2^3 & e^{\lambda_4 N} Y_2^4 & e^{\lambda_5 N} Y_2^5 & e^{\lambda_6 N} Y_2^6 & e^{\lambda_7 N} Y_2^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e^{\lambda_1 N} Z_2^1 & e^{\lambda_2 N} Z_2^2 & e^{\lambda_3 N} Z_2^3 & e^{\lambda_4 N} Z_2^4 & e^{\lambda_5 N} Z_2^5 & e^{\lambda_6 N} Z_2^6 & e^{\lambda_7 N} Z_2^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e^{\lambda_1 N} Y_1^1 Y_2^1 & e^{\lambda_1 N} Y_1^2 Y_2^2 & e^{\lambda_3 N} Y_1^3 Y_2^3 & e^{\lambda_4 N} Y_1^4 Y_2^4 & e^{\lambda_5 N} Y_1^5 Y_2^5 & e^{\lambda_6 N} Y_1^6 Y_2^6 & e^{\lambda_7 N} Y_1^7 Y_2^7 & 0 & 0 & 0 & 0 & 0 & -\frac{r_2}{\mu_M} & 0 \\
e^{\lambda_1 N} Z_1^1 Y_2^1 & e^{\lambda_2 N} Z_1^2 Y_2^2 & e^{\lambda_3 N} Z_1^3 Y_2^3 & e^{\lambda_4 N} Z_1^4 Y_2^4 & e^{\lambda_5 N} Z_1^5 Y_2^5 & e^{\lambda_6 N} Z_1^6 Y_2^6 & e^{\lambda_7 N} Z_1^7 Y_2^7 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{r_2}{\mu_M} \\
e^{\lambda_1 N} Y_1^1 Z_2^1 & e^{\lambda_1 N} Y_2^2 Z_2^2 & e^{\lambda_3 N} Y_1^3 Z_2^3 & e^{\lambda_4 N} Y_1^4 Z_2^4 & e^{\lambda_5 N} Y_1^5 Z_2^5 & e^{\lambda_6 N} Y_1^6 Z_2^6 & e^{\lambda_7 N} Y_1^7 Z_2^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e^{\lambda_1 N} Z_1^1 Z_2^1 & e^{\lambda_2 N} Z_1^2 Z_2^2 & e^{\lambda_3 N} Z_1^3 Z_2^3 & e^{\lambda_4 N} Z_1^4 Z_2^4 & e^{\lambda_5 N} Z_1^5 Z_2^5 & e^{\lambda_6 N} Z_1^6 Z_2^6 & e^{\lambda_7 N} Z_1^7 Z_2^7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
Y_1^1 Y_2^1 & Y_1^2 Y_2^2 & Y_1^3 Y_2^3 & Y_1^4 Y_2^4 & Y_1^5 Y_2^5 & Y_1^6 Y_2^6 & Y_1^7 Y_2^7 & 0 & \frac{r_1}{\mu_M} & 0 & 0 & 0 & 0 & 0 \\
Y_1^1 Z_2^1 & Y_1^2 Z_2^2 & Y_1^3 Z_2^3 & Y_1^4 Z_2^4 & Y_1^5 Z_2^5 & Y_1^6 Z_2^6 & -\frac{f_1+p_2^b+g_2^b}{\mu_M} & 0 & 0 & 0 & & & & \\
Y_2^1 & Y_2^2 & Y_2^3 & Y_2^4 & Y_2^5 & Y_2^6 & Y_2^7 & 0 & -\frac{r_1}{\mu_2} & 0 & \frac{f_1}{\mu_2} & 0 & 0 & 0 \\
Z_2^1 & Z_2^2 & Z_2^3 & Z_2^4 & Z_2^5 & Z_2^6 & Z_2^7 & \frac{p_1}{\mu_2} & 0 & -\frac{r_1}{\mu_2} & 0 & \frac{f_1}{\mu_2} & 0 & 0 \\
Z_1^1 Y_2^1 & Z_1^2 Y_2^2 & Z_1^3 Y_2^3 & Z_1^4 Y_2^4 & Z_1^5 Y_2^5 & Z_1^6 Y_2^6 & Z_1^7 Y_2^7 & 0 & 0 & 0 & -\frac{f_1+p_2^b+g_2^b}{\mu_M} & 0 & 0 & 0 \\
Z_1^1 Z_2^1 & Z_1^2 Z_2^2 & Z_1^3 Z_2^3 & Z_1^4 Z_2^4 & Z_1^5 Z_2^5 & Z_1^6 Z_2^6 & Z_1^7 Z_2^7 & \frac{g_1}{\mu_M} & 0 & 0 & \frac{g_2^b}{\mu_M} & -\frac{f_1+f_2^b}{\mu_M} & 0 & 0 \\
e^{\lambda_1 N} Y_1^1 & e^{\lambda_2 N} Y_1^2 & e^{\lambda_3 N} Y_1^3 & e^{\lambda_4 N} Y_1^4 & e^{\lambda_5 N} Y_1^5 & e^{\lambda_6 N} Y_1^6 & e^{\lambda_7 N} Y_1^7 & 0 & 0 & 0 & 0 & 0 & -\frac{r_2}{\mu_1} & 0 \\
e^{\lambda_1 N} Z_1^1 & e^{\lambda_2 N} Z_1^2 & e^{\lambda_3 N} Z_1^3 & e^{\lambda_4 N} Z_1^4 & e^{\lambda_5 N} Z_1^5 & e^{\lambda_6 N} Z_1^6 & e^{\lambda_7 N} Z_1^7 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{r_2}{\mu_1}
\end{pmatrix}
$$

$$X = [c_1, c_2, c_3, c_4, c_5, c_6, c_7, P(0,1,-1), P(0,0,1), P(0,0,-1), P(0,-1,1), P(0,-1,-1), P(N,1,0), P(N,-1,0)]^T$$

$$B = [\frac{p_2^b}{\mu_1}, 0, 0, 0, 0, 0, 0, 0, 0, \frac{p_1 + g_1 + p_2^b + g_2^b}{\mu_M}, -\frac{g_2^b}{\mu_M}, -\frac{p_1}{\mu_2}, 0, -\frac{g_1}{\mu_M}, 0, 0, 0]^T \tag{3.167}$$

where $\mu_M = \mu_2 - \mu_1$.

3.2.5 Methods to evaluate performance measures

Normalization Equation

From $f(x,\alpha_1,\alpha_2) = \sum_{i=1}^{RN} c_1 e^{\lambda_i x} G_1^i(\alpha_1) G_2^i(\alpha_2)$, we have

$$\int_0^N f(x,\alpha_1,\alpha_2)dx = \sum_{i=1}^{RN} \frac{c_i}{\lambda_i}(e^{\lambda_i N} - 1) G_1^i(\alpha_1) G_2^i(\alpha_2). \tag{3.168}$$

Then,

$$\begin{aligned} \sum_{\alpha_1=-1,0,1} \sum_{\alpha_2=-1,0,1} \int_0^N f(x,\alpha_1,\alpha_2)dx &= \tfrac{c_1(e^{\lambda_1 N}-1)}{\lambda_1} \sum_{\alpha_1=-1,0,1} \sum_{\alpha_2=-1,0,1} \sum_{i=1}^{RN} G_1^i(\alpha_1) G_2^i(\alpha_2) \\ &= \tfrac{c_1(e^{\lambda_1 N}-1)}{\lambda_1} \sum_{i=1}^{RN} \sum_{\alpha_1=-1,0,1} G_1^i(\alpha_1) \sum_{\alpha_2=-1,0,1} G_2^i(\alpha_2) \\ &= \tfrac{c_1(e^{\lambda_1 N}-1)}{\lambda_1} \sum_{i=1}^{RN} (Z_1^i + 1 + Y_1^i)(Z_2^i + 1 + Y_2^i). \end{aligned} \tag{3.169}$$

Therefore, the normalization equation (3.89) becomes

$$\begin{aligned} &\sum_{i=1}^{RN} \tfrac{c_i(e^{\lambda_i N}-1)}{\lambda_i} (Z_1^i + 1 + Y_1^i)(Z_2^i + 1 + Y_2^i) \\ &+ \sum_{\alpha_1=-1,0,1} \sum_{\alpha_2=-1,0,1} \{P(0,\alpha_1,\alpha_2) + P(N,\alpha_1,\alpha_2)\} = 1. \end{aligned} \tag{3.170}$$

Production Rates

In equation (3.91)

$$\begin{aligned} &\sum_{\alpha_1=-1,1} \sum_{\alpha_2=-1,0,1} \int_0^N f(x,\alpha_1,\alpha_2)dx = \sum_{\alpha_1=-1,1} \sum_{\alpha_2=-1,0,1} \int_0^N \sum_{i=1}^{RN} c_i e^{\lambda_i x} G_1^i(\alpha_1) G_2^i(\alpha_2)dx \\ &= \sum_{i=1}^{RN} \tfrac{c_i(e^{\lambda_i N}-1)}{\lambda_i} \sum_{\alpha_1=-1,1} G_1^i(\alpha_1) \sum_{\alpha_2-1,0,1} G_2^i(\alpha_2) = \sum_{i=1}^{RN} \tfrac{c_i(e^{\lambda_i N}-1)}{\lambda_i} (Y_1^i + Z_1^i)(Y_2^i + 1 + Z_2^i). \end{aligned} \tag{3.171}$$

Therefore, the total production rate equation (3.91) becomes

$$P_T^1 = \mu_1 \sum_{i=1}^{RN} \frac{c_i(e^{\lambda_i N}-1)}{\lambda_i}(Y_1^i + Z_1^i)(Y_2^i + 1 + Z_2^i) + \mu_1 \sum_{\alpha_2=-1,0,1} \{P(0,1,\alpha_2) + P(0,-1,\alpha_2)\}$$
$$+\mu_2\{P(N,-1,-1) + P(N,-1,1) + P(N,1,-1) + P(N,1,1)\}. \tag{3.172}$$

Finally the same procedure, P_E^1 (equation (3.92)) is expressed as

$$\mu_1[\sum_{i=1}^{RN} c_i Y_1^i(Y_2^i + 1 + Z_2^i)\frac{e^{\lambda_i N}-1}{\lambda_i} + \mu_1 \sum_{\alpha_2=-1,0,1} P(0,1,\alpha_2) + \mu_2\{P(N,1,-1) + P(N,1,1)\} \tag{3.173}$$

and P_E^2 (equation (3.93)) becomes

$$\mu_2[\sum_{i=1}^{RN} c_i Y_2^i(Y_1^i + 1 + Z_1^i)\frac{e^{\lambda_i N}-1}{\lambda_i} + \mu_2 \sum_{\alpha_1=-1,0,1} P(N,\alpha_1,1) + \mu_1\{P(0,1,-1) + P(0,1,1)\} \tag{3.174}$$

Average Inventory

In equation (3.90),

$$\begin{aligned}
\sum_{\alpha_1=-1,0,1}\sum_{\alpha_2=-1,0,1}\int_0^N x f(x,\alpha_1,\alpha_2)dx &= \sum_{\alpha_1=-1,0,1}\sum_{\alpha_2=-1,0,1}\int_0^N [x\sum_{i=1}^{RN} c_i e^{\lambda_i x} G_1^i(\alpha_1) G_2^i(\alpha_2)]dx \\
&= \sum_{i=1}^{RN}[c_i \int_0^N x e^{\lambda_i x}dx \sum_{\alpha_1=-1,0,1} G_1^i(\alpha_1) \sum_{\alpha_2=-1,0,1} G_2^i(\alpha_2)] \\
&= \sum_{i=1}^{RN}[c_i(Y_1^i + 1 + Z_1^i)(Y_2^i + 1 + Z_2^i)\int_0^N x e^{\lambda_i x}dx].
\end{aligned} \tag{3.175}$$

From integration by parts, $\int f'g = fg - \int fg'$, we can set $f' = e^{\lambda x}$, $g = x$. Then, we have

$$\begin{aligned}
\int_0^N x e^{\lambda x}dx &= [\tfrac{x}{\lambda}e^{\lambda x}]_0^N - \int_0^N \tfrac{1}{\lambda}e^{\lambda x}dx = \tfrac{N}{\lambda}e^{\lambda N} - \tfrac{1}{\lambda^2}(e^{\lambda N}-1) \quad (if\ \lambda \neq 0) \\
&= N^2/2 \quad (if\ \lambda = 0)
\end{aligned} \tag{3.176}$$

Finally, the average inventory equation (equation (3.90)) becomes

$$\overline{X} = \frac{\sum_{i=1}^{RN} [c_i(Y_1^i + 1 + Z_1^i)(Y_2^i + 1 + Z_2^i)(\frac{N}{\lambda_i} e^{\lambda_i N} - \frac{e^{\lambda_i N} - 1}{\lambda_i^2})]}{N \sum_{\alpha_1=-1,0,1} \sum_{\alpha_2=-1,0,1} P(N, \alpha_1, \alpha_2)} \tag{3.177}$$

3.3 Validation

A mathematical model for the two-machine-one-finite-buffer system has been solved. But as we have indicated, we present discrete parts in this model as a continuous fluid and time as a continuous variable. On the other hand, in simulation and in most real systems, both material and time are discrete. We compare analytical and simulation results in this section. For simulation, transient period of 10,000 time units and 1,000,000 time units of data collection period are used. For each case, the half widths of the 95% confidence intervals on the total production rate and the effective production rates fall below 1% of their mean values. Also, the half widths of the 95% confidence intervals on the average inventory become less than 3% of the nominal value of average inventory for all cases.

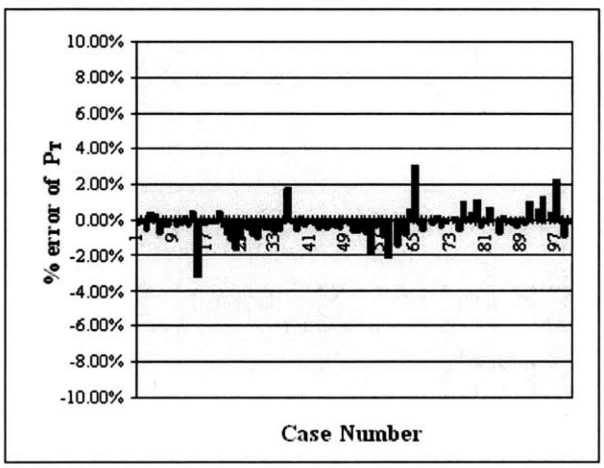


Figure 3-26: Validation of the total production rate

Figures 3-26, 3-27, and 3-28 illustrate the comparison of the total production rate, the effective production rate, and the average inventory from the analytic model and the simulation respectively. By changing machine and buffer parameters, 100 cases are generated and % errors are plotted in the vertical axis. The parameters for these cases are given in Appendix A. The % errors in the production rates are calculated

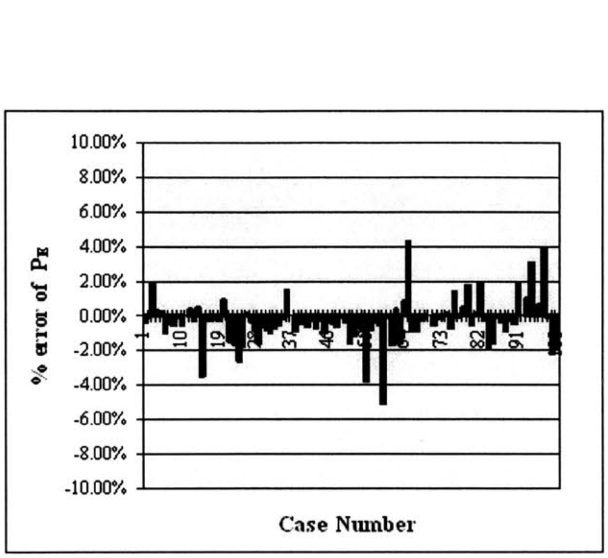


Figure 3-27: Validation of the effective production rate

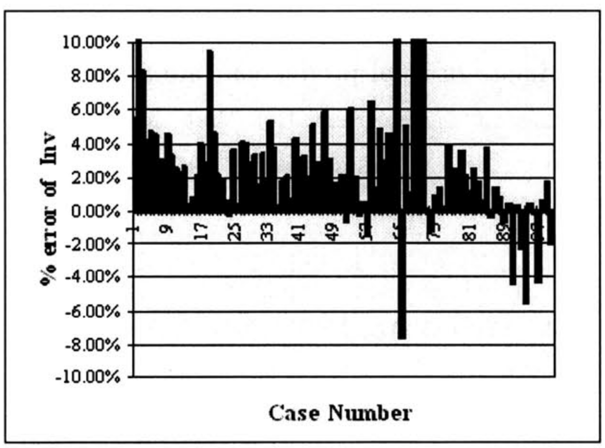


Figure 3-28: Validation of average inventory

from

$$P_T \quad \%error = \frac{P_T(A) - P_T(S)}{P_T(S)} \times 100(\%)$$

$$P_E \quad \%error = \frac{P_E(A) - P_E(S)}{P_E(S)} \times 100(\%) \tag{3.178}$$

where $P_T(A)$ and $P_E(A)$ are the total production rate and the effective production rate calculated from the analytical model, and $P_T(S)$ and $P_E(S)$ are the total production rate and the effective production rate estimated from the simulation. But the % error in the average inventory is calculated from

$$Inv \quad \%error = \frac{Inv(A) - Inv(S)}{0.5 \times N} \times 100(\%) \tag{3.179}$$

where $Inv(A)$ and $Inv(S)$ are average inventory estimated from the analytical model and the simulation respectively and N is buffer size. This equation is an unbiased way to calculate the error in average inventory. If it were calculated in the same way as the error in the production rates, the error would depend on the relative speeds of the machines. This is because there will be a lower error when the buffer is mostly full (i.e., when M_1 is faster than M_2) and a higher error when the buffer is empty (i.e., when M_1 is slower than M_2).

The average absolute value of the % errors in the total production rate, the effective production rate, and the average inventory are 0.49%, 0.92%, and 3.22% respectively. The estimate of total production rate shows less error than that of the effective production rate. The observation that the production rates estimates are better than average buffer levels is consistent with the rest of literature [Dallery and Gershwin, 1992], [Burman, 1995].

3.4 Quality information feedback

Factory designers and managers know that it is ideal to have inspection after every operation. However, it is often costly to do this. As a result, factories are usually designed so that multiple inspections are performed at a small number of stations. In this case, inspection at downstream operations can detect bad features made by upstream machines. (We call this *quality information feedback.*). A simple example of the quality information feedback in 2M1B systems is when M_1 produces defective features but does not have inspection, and M_2 has inspection and it can detect bad features made by M_1. In this situation, as we demonstrate below, the yield of a line is a function of the size of the buffer. When the buffer gets larger, more material can accumulate between an operation (M_1) and the inspection of that operation (M_2). All such material will be defective if a persistent quality failure takes place. In other words, if the buffer is larger, there tends to be more material in the buffer and consequently more material is produced before detection occurs. In addition, it takes longer to have inspections after finishing operations. We can capture this phenomenon with the adjustment of the transition probability rate of M_1 from state -1 to state 0.

Let us define f_1^q as the transition rate of M_1 from state -1 to state 0 when there is quality information feedback and f_1 as the transition rate without quality information feedback. The adjustment can be done in a way that the yield of M_1 becomes the same as $\frac{K_1^g}{K_1^g+K_1^b}$ where:

- K_1^b : the expected number of bad parts generated by M_1 from the time it enters state -1 until it leaves state -1.
- K_1^g : the expected number of good parts produced by M_1 from the moment when M_1 leaves the -1 state to the next time it arrives at state -1.

From equations (2.5), (2.6), and (2.5), the yield of M_1 is

$$\frac{P(1)}{P(1)+P(-1)} = \frac{f_1^q}{f_1^q + g_1}. \tag{3.180}$$

Suppose that M_1 has been in state 1 for a long time. Then all parts in the buffer B are non-defective. Suppose that M_1 then goes to state -1. Defective parts will then begin to accumulate in the buffer. Until all the parts in the buffer are defective, the only way that M_1 can go to state 0 is due to its own inspection or its own operational failures. Therefore, the probability of a transition to 0 before M_1 finishes a part is

$$\chi_{11} = \frac{f_1}{\mu_1}. \tag{3.181}$$

Note that $f_1 \leq \mu_1$ since the detection of bad feature can not be done before the completion of making the feature.

Eventually all the parts in the buffer are bad, so that defective parts reach M_2. Then, there is another way that M_1 can move to state 0 from state -1: *quality* *information feedback.* The probability that inspection at M_2 detects a nonconformity made by M_1 is

$$\chi_{21} = \frac{h_{21}}{\mu_2} \tag{3.182}$$

where $\frac{1}{h_{21}}$ is the mean time until the inspection at M_2 detects a bad part made by M_1 after M_2 receives the bad part.

The expected number of bad parts produced by M_1 before it is stopped by either operational failures or quality information feedback is

$$\begin{aligned} K_1^b = &[\chi_{11} + 2\chi_{11}(1-\chi_{11}) + 3\chi_{11}(1-\chi_{11})^2 + \ldots + w\chi_{11}(1-\chi_{11})^{w-1}] \\ &+[(w+1)(1-\chi_{11})^w\chi_{21} + (w+2)(1-\chi_{11})^{w+1}\chi_{21}(1-\chi_{21}) + \ldots] \end{aligned} \tag{3.183}$$

where w is average inventory in the buffer B. This is an approximation since we simply use the average inventory rather than averaging the expected number of bad parts produced by M_1 depending on different inventory levels w_i. After some mathematical manipulation,

$$K_1^b = \frac{1-(1-\chi_{11})^w}{\chi_{11}} - w(1-\chi_{11})^w + \frac{(1-\chi_{11})^w\chi_{21}[(w+1) - w(1-\chi_{11})(1-\chi_{21})]}{[1-(1-\chi_{11})(1-\chi_{21})]^2} \tag{3.184}$$

On the other hand, K_1^g is given as

$$K_1^g = \frac{\mu_1}{p_1+g_1} - \frac{p_1}{p_1+g_1}\frac{\mu_1}{p_1+g_1} + \left(\frac{p_1}{p_1+g_1}\right)^2\left(\frac{\mu_1}{p_1+g_1}\right)\ldots = \frac{\mu_1}{g_1} \tag{3.185}$$

By setting $\frac{f_1^q}{f_1^q+g_1} = \frac{K_1^g}{K_1^g+K_1^b}$ we have

$$f_1^q = \frac{\mu_1}{\frac{1-(1+w\chi_{11})(1-\chi_{11})^w}{\chi_{11}} + \frac{(1-\chi_{11})^w\chi_{21}[1+w(\chi_{21}+\chi_{11}-\chi_{21}\chi_{11})]}{[1-(1-\chi_{11})(1-\chi_{21})]^2}} \tag{3.186}$$

Since the average inventory is a function of f_1^q and f_1^q is dependent on the average inventory, an iterative method is used to get these values.

Figures 3-29, 3-30, and 3-31 show the comparison of the total production rate, the effective production rate and the average inventory from the analytic model and the simulation. By selecting different machines and buffer parameters, 50 cases are generated, and % errors are plotted in the vertical axes. The parameters for these

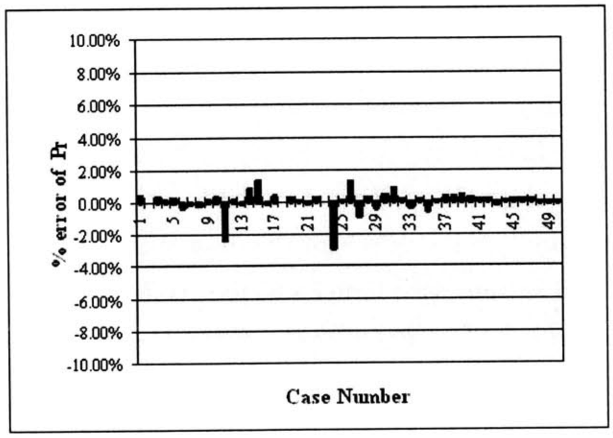


Figure 3-29: Quality information feedback: total production rate

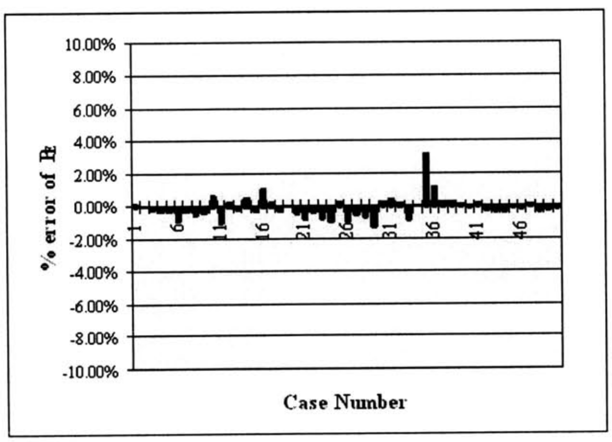


Figure 3-30: Quality information feedback: effective production rate

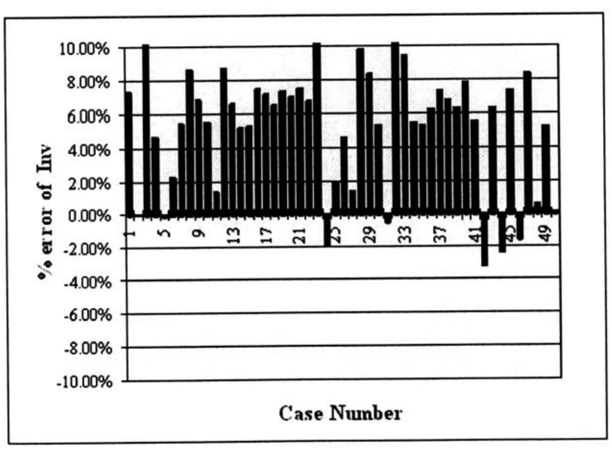


Figure 3-31: Quality information feedback: average inventory

cases are given in Appendix A. % errors in the effective production rate and average inventory are calculated using equations (3.178) and (3.179) respectively. The average absolute value of the % error in P_T, P_E and $\overline{x}$ estimates are 0.38%, 0.46%, and 5.64% respectively. Comparisons of average inventory (Figures 3.179 and 3-31) reveal that the estimation from the analytical model have positive bias. This bias happens because the minimum buffer size is 1 for the simulation whereas it is 0 for the analytic model; when 2M1B system with buffer size N is compared, the simulation treats the system as if it has $N-1$ effective buffer space, but the analytic model uses N effective buffer space.

Chapter 4

Insights From Numerical Experimentation

In this chapter, we describe a set of numerical experiments that provide intuitive insight into the behavior of 2M1B systems with quality and productivity issues. The parameters of all the cases are presented in Appendix A.

4.1 Beneficial buffer case

In this section, we describe a case in which a larger buffer leads to the higher effective production rate as well as the more total production rate.

4.1.1 Production rates

Having quality feedback means having more inspections than otherwise. Therefore, machines tend to stop more frequently. As a result, the total production rate of the line decreases. However, the effective production rate can increase since added inspections prevent the making of defective parts. This phenomenon is shown in Figures 4-1 and 4-2. Note that the total production rate P_T without quality information feedback is consistently higher than P_T with quality information feedback regardless of buffer size, and the opposite is true for the effective production rate P_E. In the beneficial buffer case, it should be noted that both the total production rate and the effective production rate increase with buffer size, with or without quality information feedback.

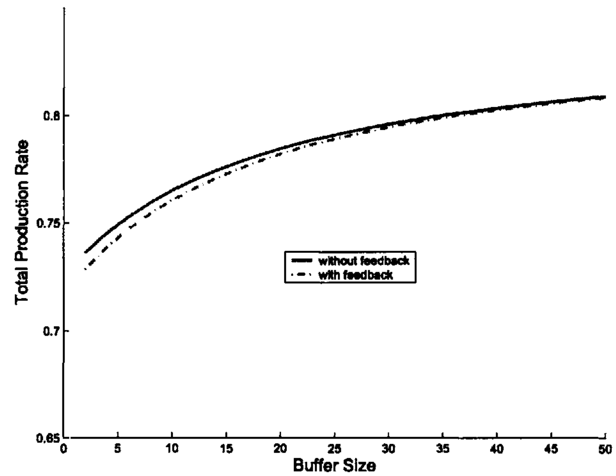


Figure 4-1: Beneficial Buffer Case: Total Production Rate

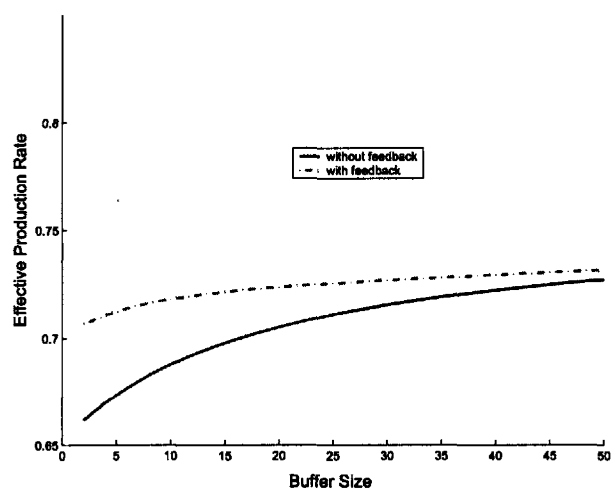


Figure 4-2: Beneficial Buffer Case: Effective Production Rate

4.1.2 System yield and buffer size

Even though a larger buffer increases both total and effective production rates in this case, it decreases yield. As explained in Section 3.4, the system yield is a function of the buffer size if there is quality information feedback. Figure 4-3 depicts system yield decreasing as buffer size increases when there is quality information feedback. This relationship happens because when the buffer gets larger, more material accumulates between an operation and the inspection of that operation. All such material will be defective when the first machine is at state -1 but the inspection at the first machine does not find it. This is a case in which *a smaller buffer improves quality,* which is widely believed to be generally true. If there is no quality information feedback, then the system yield is independent of the buffer size (and is substantially less).

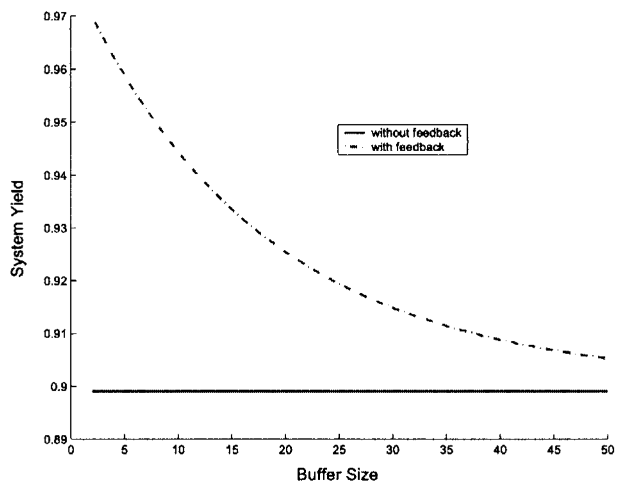


Figure 4-3: Beneficial Buffer Case: System Yield as a Function of Buffer Size

4.2 Harmful buffer case

4.2.1 Production rates

Typically, increasing the buffer size leads to higher effective production rate. This relationship is illustrated in Figure 4-2. But under certain conditions, the effective production rate can actually decrease as buffer size increases. This phenomenon can happen when:

- The first machine produces bad parts frequently: this means g_1 is large.

- The inspection at the first machine is poor or non-existent and inspection at the second machine is reliable: this means $h_1 << h_2$ or $f_1 - p_1 << f_2 - p_2$.

- There is quality information feedback.

- The isolated production rate of the first machine is higher than that of the second machine:

$$\frac{\mu_1(1+g_1/f_1)}{1+(p_1+g_1)/r_1+g_1/f_1} > \frac{\mu_2(1+g_2/f_2)}{1+(p_2+g_2)/r_2+g_2/f_2}.$$

Figure 4-4 presents a case in which a buffer size increase leads to a lower effective production rate. Note that even in this case the total production rate monotonically increases as buffer size increases (See Figure 4-5).

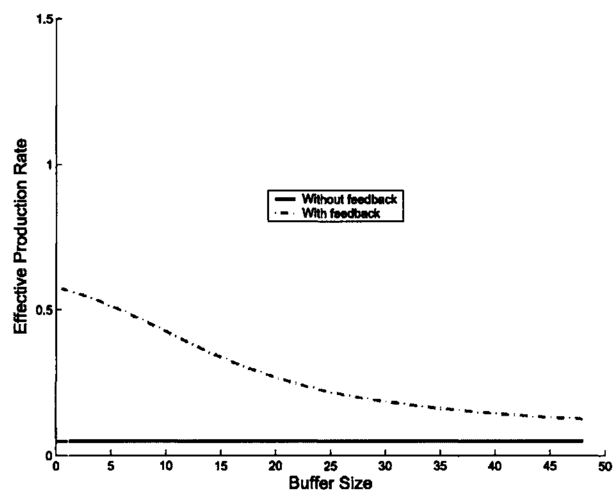


Figure 4-4: Harmful Buffer Case: Effective Production Rate

4.2.2 System Yield

The system yield for this case is shown in Figure 4-6. Note that the yield decreases dramatically as the buffer size increases. In this case, the decrease of the system yield is more than the increase of the total production rate so that the effective production rate monotonically decreases as buffer gets bigger.

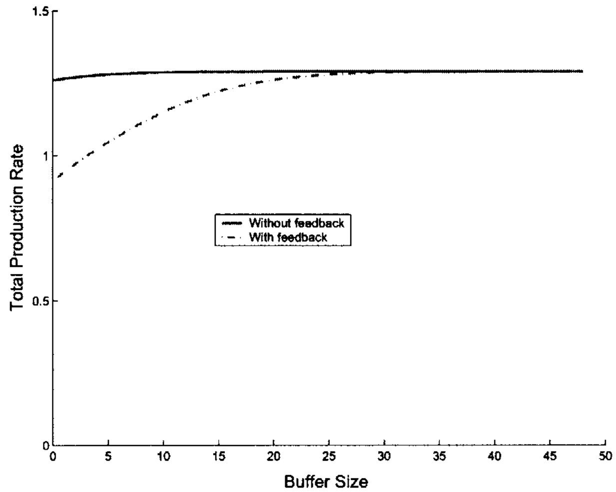


Figure 4-5: Harmful Buffer Case: Total Production Rate

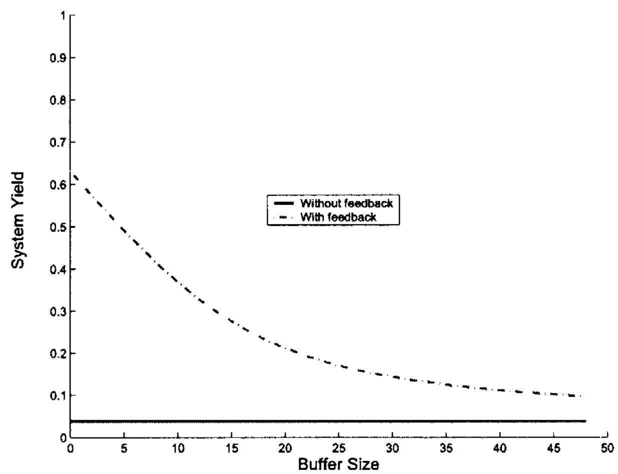


Figure 4-6: Harmful Buffer Case: System Yield as a Function of Buffer Size

4.3 Optimal buffer case

As demonstrated in Figure 4-7, there are cases in which the effective production rate increases up to a certain level then decreases as the buffer size (N) increases. In this situation, we have an optimal buffer level N^* that maximizes the effective production rate. When $N < N^*$, as the buffer size gets bigger, the increase of the total production rate is more than the decrease of the system yield. Therefore, the effective production rate, which is a multiplication of the total production rate and the buffer size, increases. But it decreases as the buffer size gets bigger when $N > N^*$

since the decrease of the system yield excels the increase of the total production rate. The behaviors of the total production rate and the system yield are shown in Figures 4-8 and 4-9.

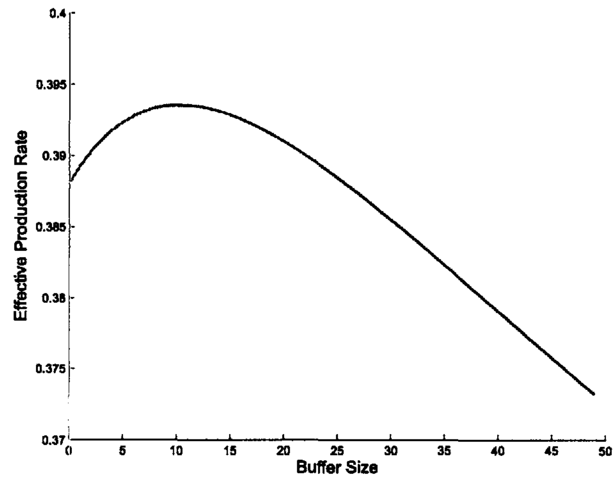


Figure 4-7: Optimal Buffer Size Case: Effective Production Rate

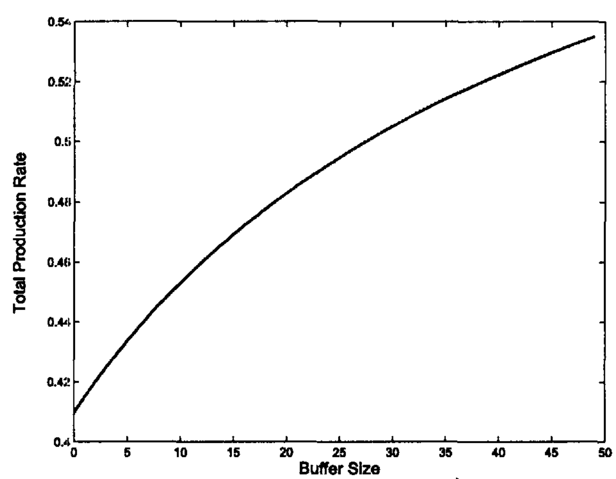


Figure 4-8: Optimal Buffer Size Case: Total Production Rate

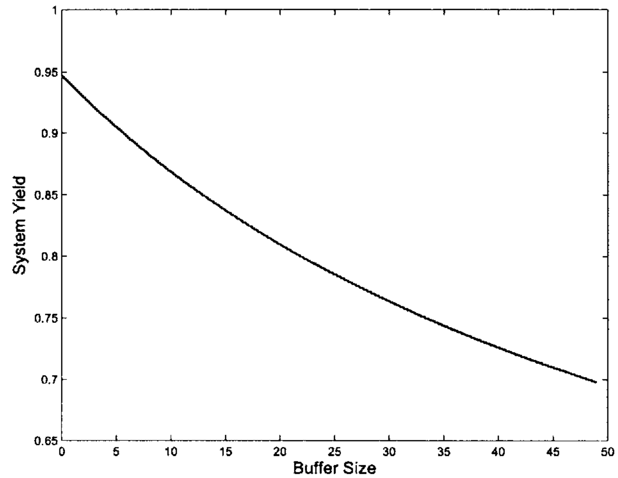


Figure 4-9: Optimal Buffer Size Case: System Yield

4.4 How to improve quality in a line with persistent quality failures

Quality can be improved in two major ways. One way is to increase the yield of individual operations, and the other is to perform more rigorous inspection. Performing extensive preventive maintenance on manufacturing equipment and using robust engineering techniques to stabilize operations have been suggested as tools to increase yield of individual operations. Both approaches increase the mean time to quality failure ($MTQF$) (i.e., decrease g). On the other hand, the inspection policy aims to detect bad parts as soon as possible and to prevent their flow toward downstream operations. More rigorous inspection decreases the mean time to detect ($MTTD$) (i.e., increases h and therefore increases f). It is reasonable to believe that using only one kind of method to achieve a target quality level would not give the most cost efficient quality assurance policy. Figure 4-10 indicates that the impact of individual operation stabilization on the system yield decreases as the operation becomes more stable. Figure 4-11 shows that effect of improving inspection ($MTTD$) on the system yield decreases as inspection becomes more reliable. Therefore, it is optimal to use a combination of both methods to improve quality.

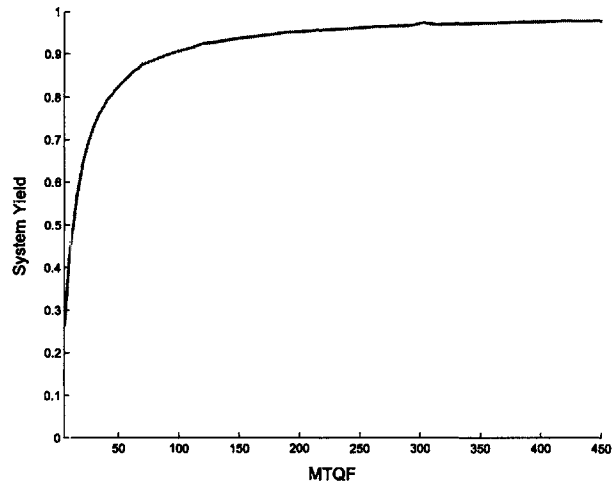


Figure 4-10: Quality Improvement Through Increase of MTQF

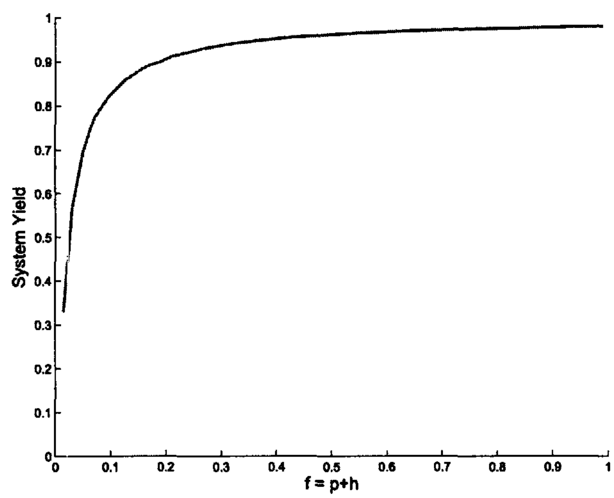


Figure 4-11: Quality Improvement Through Increase of f

4.5 How to increase the effective production rate

Improving the stand-alone throughput of each operation and increasing buffer space are typical ways to increase the production rates of manufacturing systems. If operations are apt to have quality failures, however, there may be other ways to increase the effective production rate: increasing the yield of each operation and conducting more reliable inspections. Stabilizing operations, thus improving the yield of individual operations, will increase effective throughput of a manufacturing system regardless of the type of quality failure. On the other hand, reducing the mean time to detect ($MTTD$) will *increase* the effective production rate only if the quality failure is persistent, but it will *decrease* the effective production rate if the quality failure is Bernoulli. This phenomenon occurs because the quality of each part is independent of the others when the quality failure is Bernoulli. Therefore, stopping the line does not reduce the number of bad parts in the future.

In a situation in which machines produce defective parts frequently and inspection is poor, increasing inspection reliability is more effective than increasing buffer size to boost the effective production rate. Figure 4-12 demonstrates this. Also, in other situations in which machines produce defective parts frequently and inspection is reliable, increasing machine stability is more effective than increasing buffer size to enhance effective production rate. Figure 4-13 depicts this phenomenon.

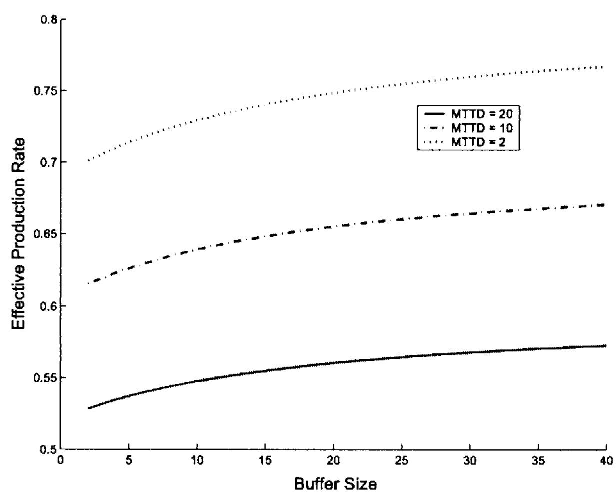


Figure 4-12: Mean Time to Detect and Effective Production Rate

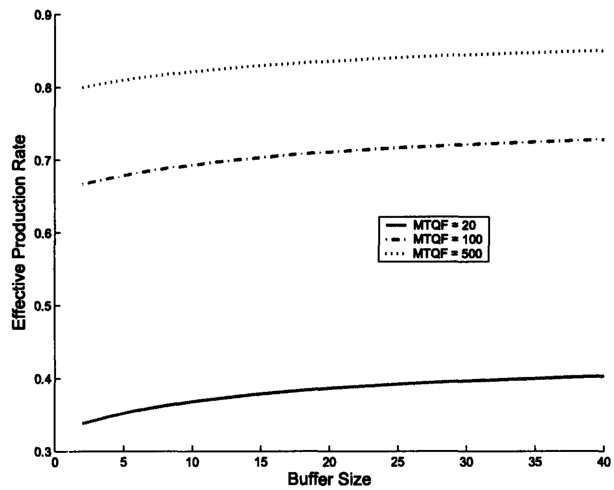


Figure 4-13: Quality Failure Frequency and Effective Production Rate

Chapter 5

Long Line Analysis

The two-machine lines of Chapter 3 can be solved analytically, and this means that fast computer programs can be written to determine performance measures. However, no such exact analytical solution exists for longer lines. In this chapter, we describe approximate techniques for long transfer lines with quality failures. Many different kinds of long manufacturing lines can be analyzed: different topologies (e.g, tandem, parallel, assembly/disassembly, and closed loops), different quality failures, different inspection policies and so on. However, in this chapter we focus only on three non-trivial long manufacturing line tasks to provide fundamental solution methods. Analysis of various long manufacturing lines is a promising topic for future research. (See Chapter 7.)

5.1 Introduction

5.1.1 Approximation techniques in long line analysis

Analysis of a line with more than two machines is more difficult than the analysis of a two-machine system since it leads to a higher state space dimension and increases the number of boundary conditions. Gershwin and Schick [Gershwin and Schick, 1983] derived an exact solution for the three-machine version of the discrete model. However, they recognized that it is not extendable to larger systems since it is difficult to program and ill-behaved.

In case of the continuous model, the analysis of longer lines increases the dimensions of the partial differential equations used in the internal transition equations. Thus, it appears that obtaining a solution for transfer lines with more than two machines requires approximations. Two different types of approximate techniques have been proposed so far: *decomposition methods* and *aggregation methods.*

The use of decomposition techniques for the analysis of long transfer line was proposed by Zimmern [Zimmern, 1956] for the machines with operation-dependent failures and by Sevast'yanov [Sevast'yanov, 1962] for the machines with time-dependent failures.

The idea of the decomposition technique is to decompose a long line into a set of two-machine lines. Both authors used the continuous model and considered only the case of homogeneous lines in which all machines have the same repair rates. For the analysis of the discrete model of long homogeneous lines, approximate decomposition equations were proposed by Gershwin [Gershwin, 1987]. The decomposition equations proposed by Gershwin was efficiently solved by the DDX-algorithm, which was formulated by Dallery, David, and Xie [Dallery et. al, 1989].

Aggregation techniques for the approximate analysis of transfer lines have been independently proposed by Ancelin and Semery [Ancelin and Semery, 1987], and Terracol and David [Terracol and David, 1987] in the case of operation-dependent failures, and De Koster [De Koster, 1987] in the case of time-dependent failures.

The basic idea of the aggregation method is to replace a two-machine-one-buffer section of the line by a single equivalent machine. Sections of the line are repeatedly aggregated until only one two-stage system remains. The major deficiency of the aggregation models is that they only account for a unidirectional propagation of events. For example, when the first two machines are aggregated, there is no accounting for the effects that downstream blocking might have on the parameters of a previously aggregated stage. Therefore, the result from aggregating from the first two machines can be quite different from the result from the aggregating from the last two machines.

The rest of the chapter focuses only on the decomposition methods.

5.1.2 Decomposition techniques for continuous models without quality failures

For two-machine lines, we can find an exact solution to calculate performance measures. However, for a long line, it appears to be impossible to find such a solution. Therefore, an approximation technique is needed. The *decomposition* technique is the most popular approximation technique that decomposes the K machine line L into a set of $K-1$ two-machine lines $L(i)$ $(i = 1, 2, ..., K-1)$. Each line $L(i)$ is composed of

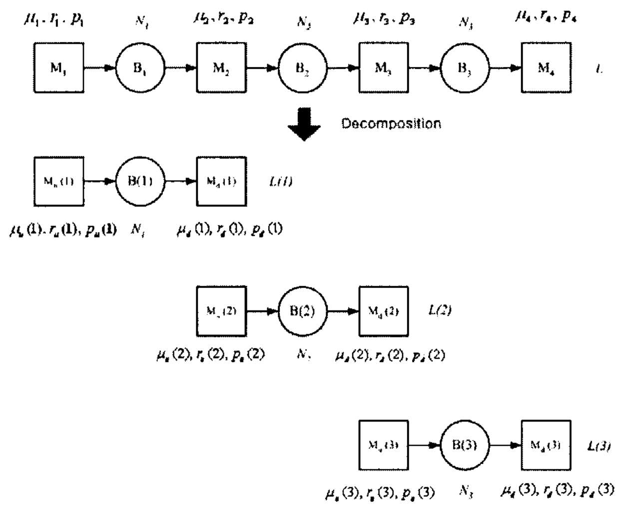


Figure 5-1: Decomposition of a four-machine line into three two-machine lines

an upstream machine $M_u(i)$ and a downstream machine $M_d(i)$, separated by a buffer $B(i)$. This decomposition is illustrated in Figure 5-1 for a four-machine line.

The principle of the decomposition is that the behavior of the material flow in buffer $B(i)$ closely matches that of the flow in buffer B_i of line L. Machine $M_u(i)$ represents the part of the line L upstream of B_i and machine $M_d(i)$ represents the part of the line L downstream from B_i.

Decomposition techniques for continuous long lines with operation dependent failure is more complex than other models (e.g., a deterministic processing time long line) since it assumes different machine speeds, and the speeds of machines can be slowed down due to partial blockage and partial starvation. A decomposition technique for a continuous long line with different operation speeds and operation dependent failures was first proposed by Glassey and Hong [Glassey and Hong, 1993]. Another method was developed by Burman [Burman, 1995].

The Accelerated DDX algorithm (ADDX), which was formulated by Burman [Burman, 1995], converges faster and gives more accurate estimates than Glassey and Hong's algorithm. The algorithm works as follows:

(1) Initialization

Provide the following initial guesses for the parameters of each two-stage line:

$$p_u(i) = p_i$$

$$r_u(i) = r_i$$

$$\mu_u(i) = \mu_i$$

$$p_d(i) = p_{i+1}$$

$$r_d(i) = r_{i+1}$$

$$\mu_d(i) = \mu_{i+1} \qquad i = 1, \ldots, k-1 \tag{5.1}$$

(2) Iteration

Perform Step 1 and Step 2 until the Termination Condition is satisfied.

Step 1 Let i range over values from 2 to $k-1$. Evaluate $L(i-1)$ using the continuous two-machine-one-buffer model with the most recent values of $r_d(i-1)$, $p_d(i-1)$, $\mu_d(i-1)$, $r_u(i-1)$, $p_u(i-1)$, $\mu_u(i-1)$. Then substitute these parameters and the resulting $P(i-1)$ into the upstream decomposition equations (5.2) – (5.4), in that order.

$$p_u(i) = \frac{p_i K_2 K_3 + r_i p_i + r_i K_1 K_3}{r_i + K_2 K_3 - K_1 K_3} \tag{5.2}$$

$$r_u(i) = \frac{p_i K_2 K_3 + r_i p_i + r_i K_1 K_3}{p_i + K_1 K_3 - K_2 K_3} \tag{5.3}$$

$$\mu_u(i) = \frac{K_3(p_i + r_i)}{r_i + K_2 K_3 - K_1 K_3} \tag{5.4}$$

where

$$K_1 = p_1 \left(\frac{\mathbf{p}_{i-1}(0,1,1)}{P(i-1)} \left(\frac{\mu_u(i-1)}{\mu_d(i-1)} - 1 \right) \right) + \left(\frac{\mathbf{p}_{i-1}(0,0,1)}{P(i-1)} \right) r_u(i-1) \tag{5.5}$$

$$K_2 = (r_u(i-1) - r_i) \left(\frac{\mathbf{p}_{i-1}(0,0,1)}{P(i-1)} \right) \tag{5.6}$$

$$K_3 = \frac{1}{\frac{1}{P(i-1)} + \frac{1}{e_i \mu_i} - \frac{1}{e_d(i-1)\mu_d(i-1)}} \tag{5.7}$$

Here $\mathbf{p_i}(\mathbf{x}, \alpha_1, \alpha_2)$ is the probability that the decomposed two-machine system i is at state (x, α_1, α_2), $P(i)$ is the production rate of the decomposed two-machine system i, and e_i is the isolated production rate of the two-machine line.

Step 2 Let i range over values from $k-2$ to 1. Evaluate $L(i+1)$ using the continuous two-machine-one-buffer model with the most recent values of $r_d(i+1)$, $p_d(i+1)$, $\mu_d(i+1)$, $r_u(i+1)$, $p_u(i+1)$, $\mu_u(i+1)$. Then substitute these parameters and the resulting $P(i+1)$ into the downstream decomposition equations (5.8) – (5.10).

$$p_d(i) = \frac{p_{i+1}K_5K_6 + r_{i+1}p_{i+1} + r_{i+1}K_4K_6}{r_{i+1} + K_5K_6 - K_4K_6} \tag{5.8}$$

$$r_d(i) = \frac{p_{i+1}K_5K_6 + r_{i+1}p_{i+1} + r_{i+1}K_4K_6}{p_{i+1} + K_4K_6 - K_5K_6} \tag{5.9}$$

$$\mu_d(i) = \frac{K_6(p_{i+1} + r_{i+1})}{r_{i+1} + K_5K_6 - K_4K_6} \tag{5.10}$$

where

$$K_4 = p_{i+1}\left(\frac{\mathbf{p}_{i+1}(N_{i+1},1,1)}{P(i+1)}\right)\left(\frac{\mu_d(i+1)}{\mu_u(i+1)} - 1\right) + \left(\frac{\mathbf{p}_{i+1}(N_{i+1},1,0)}{P(i+1)}\right) r_d(i+1) \tag{5.11}$$

$$K_5 = (r_d(i+1) - r_{i+1}) + \left(\frac{\mathbf{p}_{i+1}(N_{i+1},1,0)}{P(i+1)}\right) \tag{5.12}$$

$$K_6 = \frac{1}{\frac{1}{P(i+1)} + \frac{1}{e_{i+1}\mu_{i+1}} - \frac{1}{e_u(i+1)\mu_u(i+1)}} \tag{5.13}$$

(3) Termination Condition Terminate the algorithm when

$$\|P(i) - P(1)\| \tag{5.14}$$

is smaller than a pre-defined small number ϵ.

5.2 Long line analysis case 1

5.2.1 Introduction

Many different kinds of long manufacturing lines with quality and operational failures can be analyzed: different topologies, different quality failures, different inspection policies, and so on. However, there has been no analytical model of these in the literature. In this thesis, we try to take the first and fundamental research step in building analytic models of long manufacturing lines with quality and operational failures, by focusing only on three non-trivial long manufacturing line tasks. Analysis of various long manufacturing lines is a promising topic for future research. (See Chapter 7.)

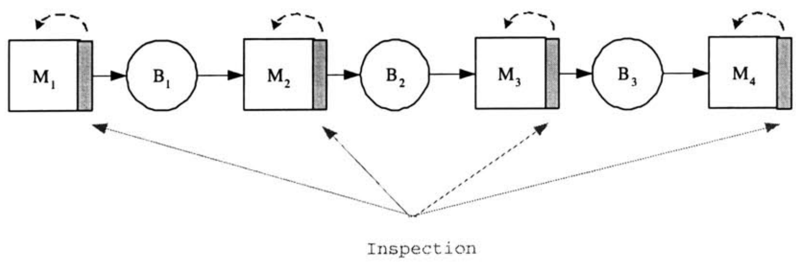


Figure 5-2: The first long line analysis task

The first task is the *ubiquitous inspection case* illustrated in Figure 5-2. This is the most fundamental but non-trivial model for the long line analysis since the manufacturing line is long (i.e., more than two machines) and the machines have both quality and operational failures. In this case, we assume that every machine undergoes both quality failures and operational failures, and inspection is done at every operation. More detailed assumptions of the case are as follows:

- Each machine has both operational failures and quality failures.
- Each operation works on different features (e.g., holes, grooves). Thus, quality failures at an operation do not influence the quality of other operations.
- Inspection at machine M_i can detect defective features made by M_i, not others.
- There is no scrap or rework in the line; defective parts are marked, and scrapped or reworked later.

5.2.2 Solution method

Transformation technique

Since machines with quality and operational failures have five parameters, μ_i, r_i, p_i, g_i, and f_i, the analysis of a K-machine line with quality failures requires equations for $10(K-1)$ pseudo-machine parameters and a efficient algorithm to solve them.

For the ubiquitous inspection case, *quality failures at an operation do not influence the quality of other operations* because of the assumption that each operations works on different features. As a result, g_i is independent of other machines' parameters. Therefore, we have

$$g_u(i) = g_i$$
$$g_d(i) = g_{i+1}. \tag{5.15}$$

Another fundamental assumption of the model is that inspection can only identify bad features made by its own operation. Therefore, for each decomposed line $L(i)$, $(i = 1, 2, ...K - 1)$ the incoming parts from upstream machines are treated as non-defective since the inspections at the decomposed line $L(i)$ can not detect defective parts from the upstream machines. In addition, outgoing defective parts from $L(i)$ are not detected by the inspections at downstream machines. Thus, f_i is also independent of other machines' parameters. Therefore, we get

$$f_u(i) = f_i$$
$$f_d(i) = f_{i+1}. \tag{5.16}$$

$4(K - 1)$ equations are developed from equations (5.15) and (5.16). The remaining $6(K - 1)$ equations are for the determination of $\mu_u(i)$, $\mu_d(i)$, $r_u(i)$, $r_d(i)$, $p_u(i)$, and $p_d(i)$, which are the parameters for machines with operational failures only. To determine these parameters, we propose that three-state-machines (state 1, state -1, and state 0) can be approximated by two-state-machines (state 1' and state 0), as depicted in Figure (5-3).

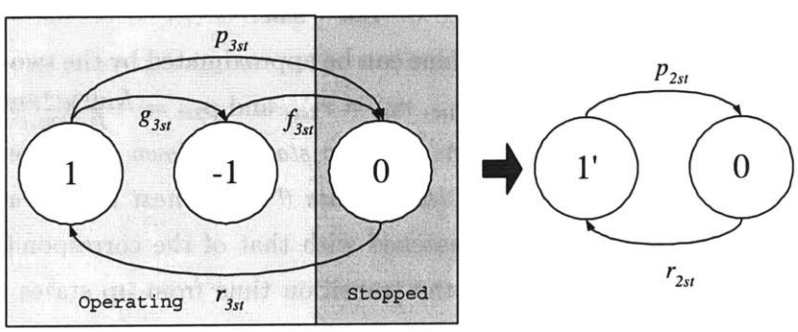


Figure 5-3: Three-state-machine and corresponding two-state-machine

In Figure 5-3, two up states (state 1 and state -1) of the three-state-machine are consolidated into one up state (state 1') of the two-state-machine.

For a three-state-machine in isolation, the probability of a machine being at each state is

$$P_{3st}(1) = \frac{1}{1 + (p_{3st} + g_{3st})/r_{3st} + g_{3st}/f_{3st}}$$

$$P_{3st}(0) = \frac{(p_{3st} + g_{3st})/r_{3st}}{1 + (p_{3st} + g_{3st})/r_{3st} + g_{3st}/f_{3st}}$$

$$P_{3st}(-1) = \frac{g_{3st}/f_{3st}}{1 + (p_{3st} + g_{3st})/r_{3st} + g_{3st}/f_{3st}}. \tag{5.17}$$

On the other hand, for a two-state-machine in isolation, the probability of a machine being at each state is

$$P_{2st}(1') = \frac{r_{2st}}{p_{2st} + r_{2st}}$$

$$P_{2st}(0) = \frac{p_{2st}}{p_{2st} + r_{2st}}. \tag{5.18}$$

The probability of state 1' of the two-state-machine is the sum of the probability of state 1 and state -1 of the three-state-machine. Therefore,

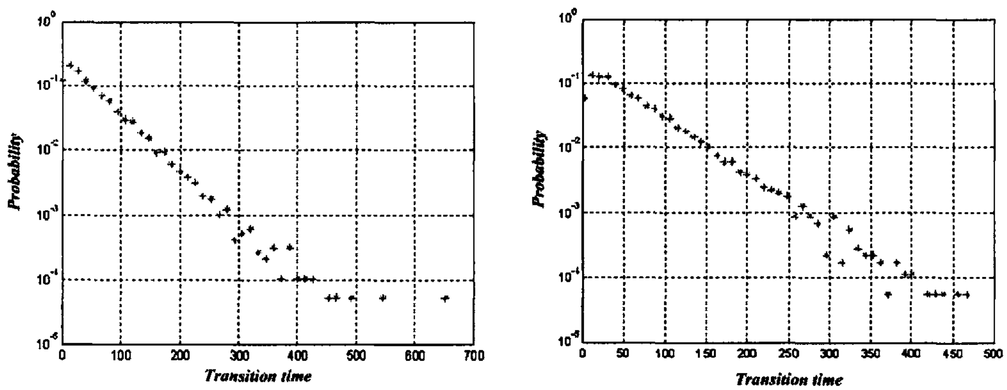
$$P_{3st}(1) + P_{3st}(-1) = P_{2st}(1'). \tag{5.19}$$

From equations (5.17), (5.18), and (5.19), we have

$$p_{2st} = \frac{f_{3st}(p_{3st} + g_{3st})}{f_{3st} + g_{3st}}. \tag{5.20}$$

As a result, the three-state machine can be approximated by the two-state machine with machine parameters $\mu_{2st} = \mu_{3st}$, $r_{3st} = r_{2st}$, and $p_{2st} = \frac{f_{3st}(p_{3st}+g_{3st})}{f_{3st}+g_{3st}}$. From these equations, the mean *transition time from up states to down state* (i.e., *the expected time from the moment a machine leaves state 0 to the next time it arrives at state 0*) of the three-state-machine is matched with that of the corresponding two-state-machine. But the distribution of the transition time from up states to down state of the three-state-machine and that of the corresponding two-state-machine are not similarly adjusted. The distributions may not match well; once a three-state-machine gets to state 1, it has two paths to move to state 0: directly to state 0 with the rate of p, or through state -1 with the rate of g and then to state 0 with the rate f. However, the corresponding two-state-machine has only one path to get to state 0. Therefore, the validity of the approximation depends on whether the distribution of the transition time from up states to down state of the three-state-machine follows closely that of the two-state-machine, which is an exponential distribution.

To check this, a simple simulation model of a three-state-machine is developed and the transition times from up states to down state are recorded. For statistical significance,1,000,000 cycles of run were used.



(a) p=0.01, g=0.01, f=1   (a) p=0.01, g=0.01, f=0.1

Figure 5-4: Distribution of transition time from up states to down state: three-state machines

Figure 5-4 represents the observed distributions of the transition times for two different parameter settings where operational failures and quality failures take place equally frequently. The horizontal axis represents the transition time and the vertical axis represents the corresponding probability in a logarithmic scale. Both cases in Figure 5-4 show a close to linear relationship between transition times and the corresponding probabilities in logarithmic scales. This means that the transition time closely follows the exponential distribution.

The equivalent two-state-machine gives the total production rate and average inventory. But the effective production rate should be estimated indirectly since the two-state-machine can not tell the difference between 'good' state and 'bad' state. Since there is no scrap in the system, the yield of a machine is $\frac{P_{3st}(1)}{P_{3st}(1)+P_{3st}(-1)} = \frac{f_{3st}}{f_{3st}+g_{3st}}$. For multiple machine lines, the system yield becomes a product of the individual yields. Thus, the effective production rate can be calculated by multiplying the system yield by the total production rate.

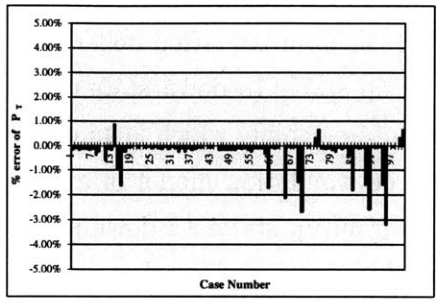


Figure 5-5: 3-state-machine vs. 2-state-machine - comparison of P_T

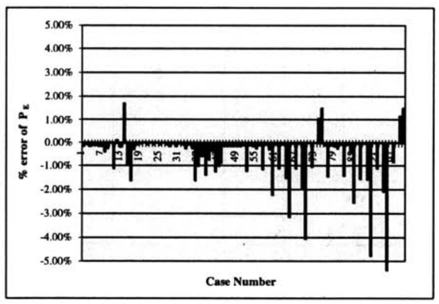


Figure 5-6: 3-state-machine vs. 2-state-machine - comparison of P_E

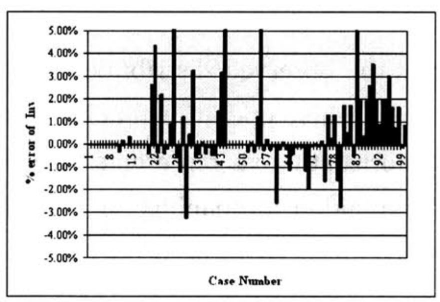


Figure 5-7: 3-state-machine vs. 2-state-machine - comparison of Inv

The performance measures of the 2M1B system with the three-state-machines and the corresponding two-state-machines are compared in Figures 5-5, 5-6, and 5-7. By changing machines and buffer parameters, 100 cases are generated, and % errors are plotted in the vertical axis. The parameters for these cases are given in Appendix

A. The average absolute errors in the total production rate, the effective production rate, and the average inventory are 0.34 %, 0.68%, and 1.07%, respectively.

Analysis procedure

The four-machine ubiquitous inspection case, presented in Figure 5-2, can be analyzed by using the following procedure:

- *Step 1*: Calculate the system yield

$$Y_{sys} = \frac{f_1}{f_1 + g_1} \times \frac{f_2}{f_2 + g_2} \times \frac{f_3}{f_3 + g_3} \times \frac{f_4}{f_4 + g_4}.$$

- *Step 2*: Transform the original line L with 3-state-machines into an equivalent line L' with 2-state-machines by setting
  - $\mu_i' = \mu_i$, $r_i' = r_i$, $p_i' = \frac{f_i(p_i+g_i)}{f_i+g_i}$ $(i = 1, 2, 3, 4)$.
  - $N_i' = N_i$ $(i = 1, 2, 3)$.

- *Step 3*: The total production rate and average inventory levels for B_i $(i = 1, 2, 3)$ of the 2-state-machines line L' is calculated from ADDX algorithm.

- *Step 4*: Evaluate the effective production rate by multiplying the system yield by the total production rate.

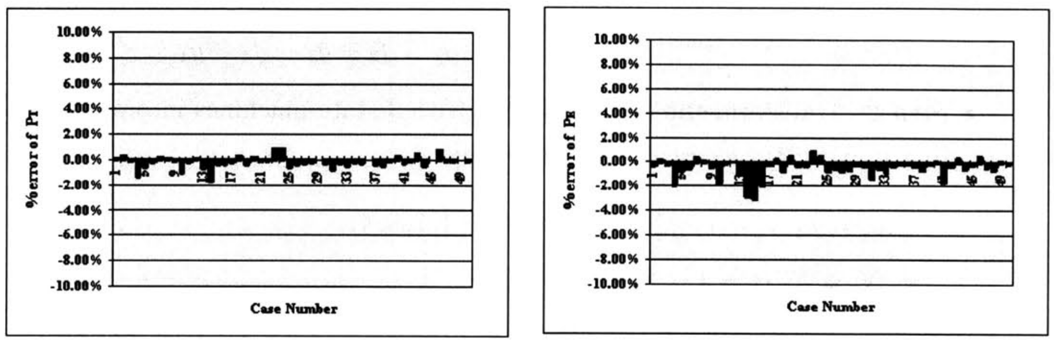
The same procedure can be used for the analysis of a general K-machine line.

5.2.3 Performance Evaluation

Figures 5-8, 5-9, and 5-10 illustrate the comparison of the performance measures of a large number of four-machine lines with ubiquitous inspections, between the decomposition algorithm result and simulation result. By choosing machine and buffer parameters, 50 cases are generated, and % errors are plotted on the vertical axes. The parameters used for these cases are given in Appendix B. The average absolute errors, which is the average of the absolute value of the % errors, are presented in Table 5.1. As observed in the two-machine lines, (Figures 3-26, 3-27, and 3-28), the estimate of total production rate shows less error than that of the effective production rate. The observation that the production rates estimates are better than average buffer levels is consistent with the rest of literature [Dallery and Gershwin, 1992], [Burman, 1995].

Table 5.1: Average absolute errors in long line analysis case 1

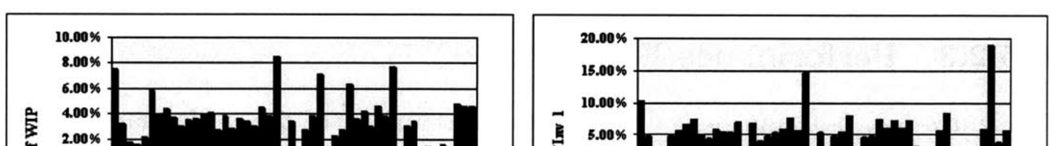
	P_T	P_E	WIP	Inv_1	Inv_2	Inv_3
Average absolute error (%)	0.37	0.64	3.41	5.46	4.51	2.34



(a) Total Production Rate

(b) Effective Production Rate

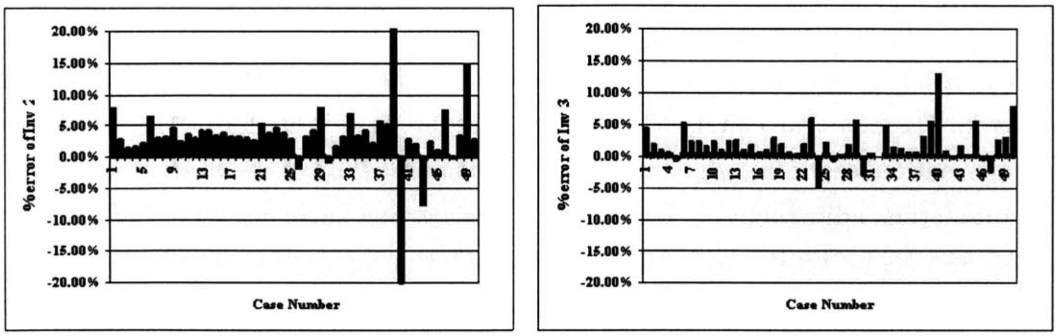
Figure 5-8: Validation – total production rate and effective production rate



(a) Average WIP

(b) Average Inventory at B1

Figure 5-9: Validation – WIP (Work-In-Process) and average inventory at B_1



(a) Average Inventory at B2

(b) Average Inventory at B3

Figure 5-10: Validation – average inventory at B_2 and B_3

5.3 Long Line Analysis Case 2

5.3.1 Introduction

The second long manufacturing line analysis task is an extended quality information feedback (*EQIF*) case with four machines as illustrated in Figure 5-11. This is an extension of the 2M1B quality information feedback model to a longer line. This is a good approximation of a real situation where operations in the manufacturing line are reliable in terms of quality, whereas incoming raw material causes major quality problems, and the defect in the raw material can only identified at the end of line.

Assumptions of the model are as follows:

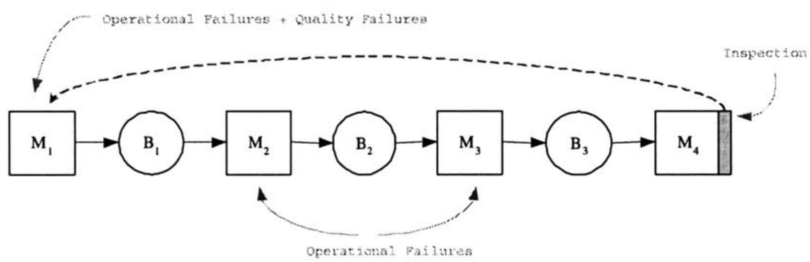


Figure 5-11: The second long line analysis task

- The first machine (M_1) has both operational failures and quality failures.
- The other machines have only operational failures.
- The only inspection is located at the end of line and it can detect non-conformities made by M_1.

5.3.2 Solution method

The four-machine *EQIF* case is an extension of 2M1B with quality information feedback. Therefore, we can use the similar procedure that is used for 2M1B with quality information feedback: adjustment of transition probability rate of M_1 from state -1 to state 0 (i.e., adjusting f_1). The only difference is that there are three buffers between M_1 and M_4 for EQIF case rather than one. Thus, w in equation (3.186) is replaced by Work-In-Process ($WIP = Inv_1 + Inv_2 + Inv_3$) in the EQIF line. After f_1 is adjusted, EQIF case becomes ubiquitous inspection case so that the similar solution method is used.

The four-machine *EQIF* case shown in Figure 5-11 can be analyzed by using the procedure as follows:

- *Step 1*: Estimate WIP ($=Inv_1 + Inv_2 + Inv_3$) to get an initial estimate of f_1^q.
- *Step 2*: Adjust f_1^q by using the quality information feedback formula

$$f_1^q = \frac{\mu_1}{\frac{1-(1+WIP\chi_{11})(1-\chi_{11})^{WIP}}{\chi_{11}} + \frac{(1-\chi_{11})^{WIP}\chi_{41}[1+WIP(\chi_{41}+\chi_{11}-\chi_{41}\chi_{11})]}{[1-(1-\chi_{11})(1-\chi_{41})]^2}}. \quad (5.21)$$

  where $\chi_{11} = \frac{f_1}{\mu_1}$, and $\chi_{41} = \frac{f_4 - p_4}{\mu_4}$.

- *Step 3*: Calculate the system yield

$$Y_{sys} = \frac{f_1}{f_1+g_1} \times \frac{f_2}{f_2+g_2} \times \frac{f_3}{f_3+g_3} \times \frac{f_4}{f_4+g_4}.$$

- *Step 4*: Transform the original line L with 3-state-machines into an equivalent line L' with 2-state-machines by setting
  - $\mu_i' = \mu_i$, $r_i' = r_i$, $p_i' = \frac{f_i(p_i+g_i)}{f_i+g_i}$ $(i = 1, 2, 3, 4)$.
  - $N_i' = N_i$ $(i = 1, 2, 3)$.
- *Step 5*: Use the ADDX algorithm to calculate the total production rate (P_T) and average inventory at each buffer B_i $(i = 1, 2, 3)$.
- *Step 6*: Estimate the effective production rate (P_E) by multiplying the total production rate by the system yield.
- *Step 7*: If new P_T, P_E, Inv are close enough to previous values, then stop. Otherwise go to *Step 2* and repeat the procedure.

Figure 5-12 illustrates this solution method.

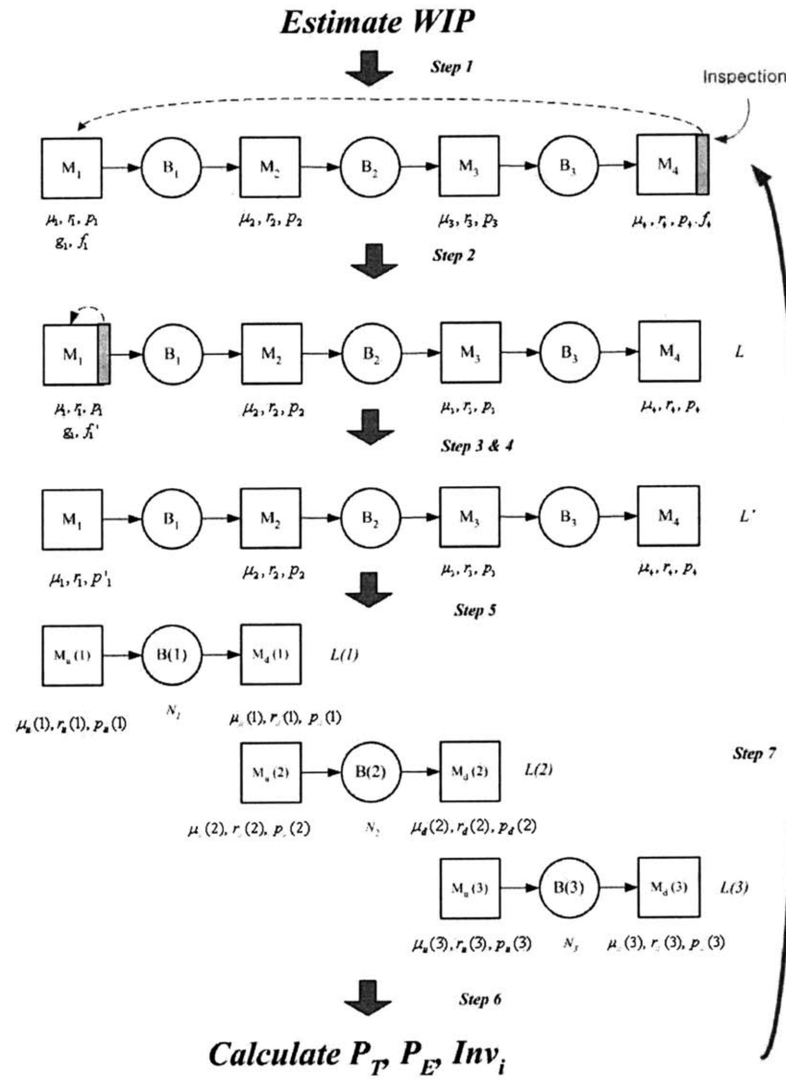


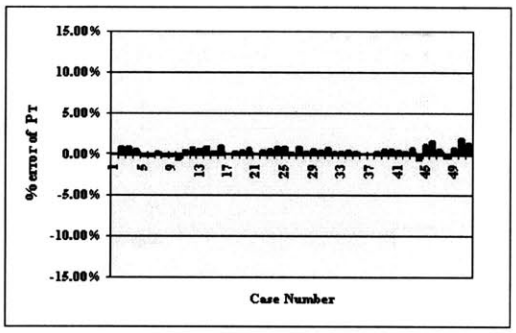
Figure 5-12: Procedure of long line analysis task 2

5.3.3 Performance Evaluation

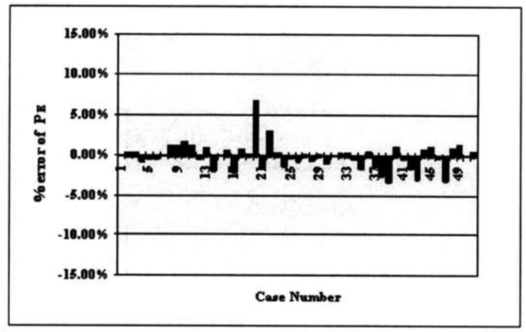
Figures 5-13, 5-14, and 5-15 illustrate the comparison of the performance measures of a large number of four-machine EQIF lines, between the decomposition algorithm result and simulation result. By changing machines and buffer parameters, 50 cases are generated, and % errors are plotted in the vertical axes. The parameters used for these cases are given in Appendix B. The average absolute errors are presented in Table 5.2. As observed in the two-machine lines and the ubiquitous inspection case, the estimate of total production rate shows less error than that of the effective production rate. The observation that the production rates estimates are better than average buffer levels is consistent with 2M1B systems and the ubiquitous inspection case. Note that performance estimates of EQIF case are slightly worse than these of ubiquitous inspection case since the quality information feedback equation is an approximate formula as discussed in Chapter 3.

Table 5.2: Average absolute errors in long line analysis case 2

	P_T	P_E	WIP	Inv_1	Inv_2	Inv_3
Average absolute error (%)	0.52	1.02	3.47	5.16	5.56	2.43

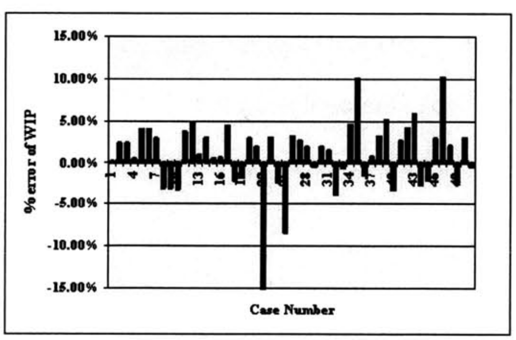


(a) Total Production Rate

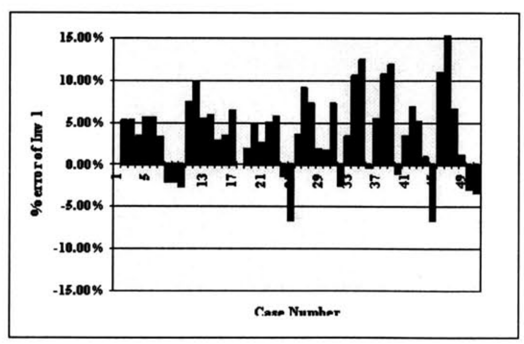


(b) Effective Production Rate

Figure 5-13: Validation - P_T and P_E



(a) Average WIP



(b) Average Inventory at B1

Figure 5-14: Validation - WIP and Average Inventory at B_1

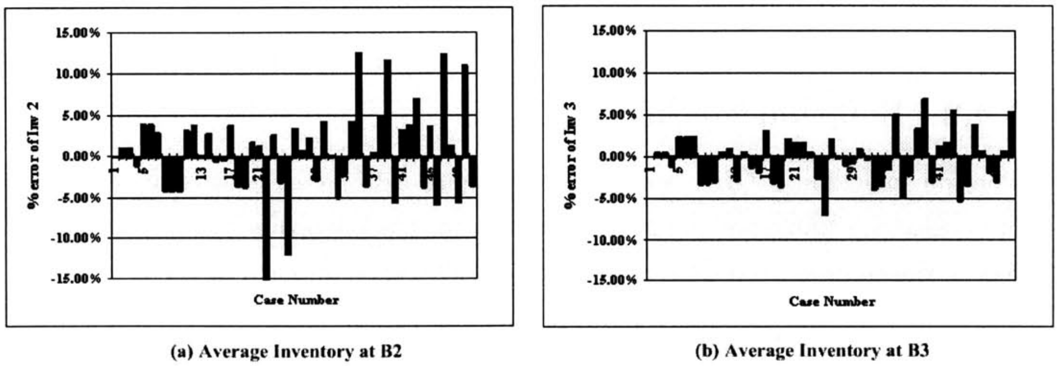


Figure 5-15: Validation - Average Inventory at B_2 and B_3

5.4 Long Line Analysis Case 3

5.4.1 Introduction

Due to the cost of inspection stations, factories are often designed so that multiple inspections are performed at a small number of stations. The inspection stations are usually located at the end of a (sub) line to guarantee that outgoing parts are defect-free. This is a typical example of a multiple quality information feedback ($MQIF$) case, which is illustrated in Figure 5-16. The assumptions of the model are as follows:

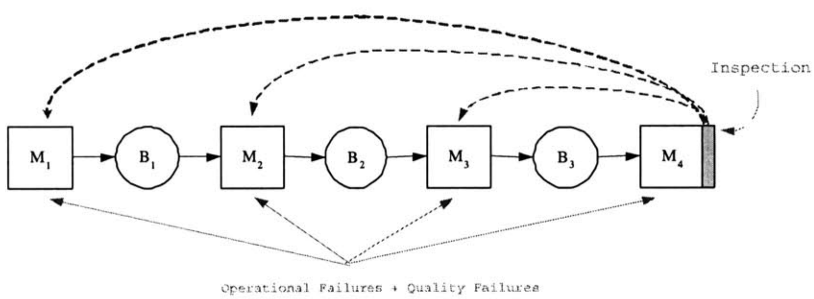


Figure 5-16: The third long line analysis task

- All the machines have both operational failures and quality failures.
- The only inspection is located at the end of line, and it can detect non-conformities made by any of the machines (M_1, M_2, M_3, and M_4).
- Each operation works on different features. Quality failures at an operation do not influence the quality of other operations.

- There is no scrap in the line; defective parts are marked and reworked later.

5.4.2 Solution method

The four-machine *MQIF* case shown in Figure 5-16, is an extension of EQIF case in a sense that multiple quality information feedback loops exist. Therefore, we can repeat the same procedure that is used for EQIF case for each of the loop. The only difference is that w in equation (3.186) is replaced by:

- $Inv_1 + Inv_2 + Inv_3$ for the adjustment of f_1.
- $Inv_2 + Inv_3$ for the adjustment of f_2.
- Inv_3 for the adjustment of f_3.

The four-machine MQIF case can be analyzed by using the procedure as follows:

- *Step 1*: Estimate the average inventory of each buffer (Inv_1, Inv_2, and Inv_3).
- *Step 2*: Adjust f_i^q ($i = 1, 2, 3$) by using the QIF formula:

$$f_i^q = \frac{\mu_i}{\frac{1-(1-\chi_{ii})^{w_i}}{\chi_{ii}} - w_i(1-\chi_{ii})^{w_i} + \frac{(1-\chi_{ii})^{w_i}\chi_{4i}[(w_i+1)-w_i(1-\chi_{ii})(1-\chi_{4i})]}{[1-(1-\chi_{ii})(1-\chi_{4i})]^2}}$$

  where $\chi_{ii} = \frac{f_i}{\mu_i}$, $\chi_{4i} = \frac{f_4 - p_4}{\mu_4}$, $w_1 = Inv_1 + Inv_2 + Inv_2$, $w_2 = Inv_2 + Inv_3$, and $w_3 = Inv_3$.

- *Step 3*: Calculate the system yield

$$Y_{sys} = \frac{f_1}{f_1 + g_1} \times \frac{f_2}{f_2 + g_2} \times \frac{f_3}{f_3 + g_3} \times \frac{f_4}{f_4 + g_4}.$$

- *Step 4*: Transform the original line L with 3-state-machines into an equivalent line L' with 2-state-machines by setting
  - $\mu_i' = \mu_i$, $r_i' = r_i$, $p_i' = \frac{f_i(p_i+g_i)}{f_i+g_i}$ ($i = 1, 2, 3, 4$).
  - $N_i' = N_i$ ($i = 1, 2, 3$).

- *Step 5*: Use the ADDX algorithm to calculate the total production rate (P_T) and average inventory at each buffer B_i ($i = 1, 2, 3$).

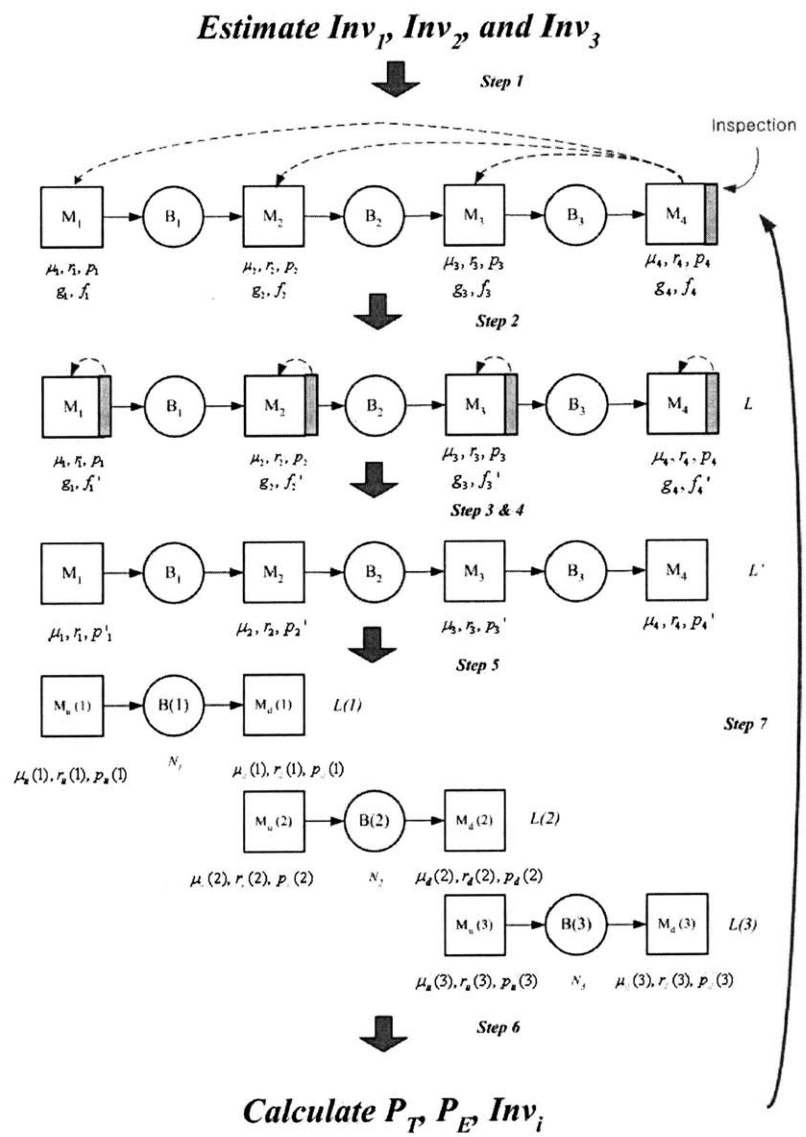


Figure 5-17: Procedure of long line analysis task 3

- *Step 6*: Estimate the effective production rate (P_E) by multiplying the total production rate by the system yield.

- *Step 7*: If the new P_T, P_E, Inv are close enough to their previous values, then stop. Otherwise go to *Step 2* and repeat the procedure..

Figure 5-17 illustrates this solution method.

5.4.3 Performance Evaluation

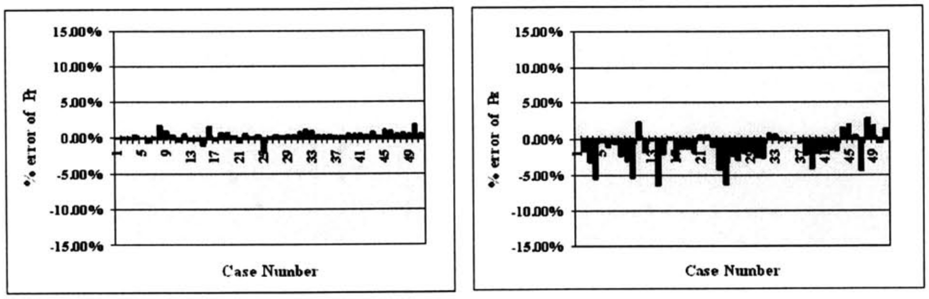
Figures 5-18, 5-19, and 5-20 illustrate the comparison of the performance measures of a large number of four-machine MQIF lines, between the decomposition algorithm result and simulation result. By changing machine and buffer parameters, 50 cases are generated, and % errors are plotted in the vertical axes. The parameters used for these cases are given in Appendix B. The average absolute errors are presented in Table 5.3.

As observed in the two-machine lines, the ubiquitous inspection case, and the EQIF case, the estimate of total production rate shows less error than that of the effective production rate. The observation that the production rates estimates are better than average buffer levels is consistent with the other cases. Note that performance estimates of MQIF case are slightly worse than these of EQIF case since the approximate quality information feedback equation is used multiple times. This deterioration of performance estimates suggests that the errors tend to increase as the manufacturing line gets longer (thus, more quality information feedback loops exist).

Table 5.3: Average absolute errors in long line analysis case 3

	P_T	P_E	WIP	Inv_1	Inv_2	Inv_3
Average absolute error (%)	0.55	1.95	8.62	5.97	4.75	2.76

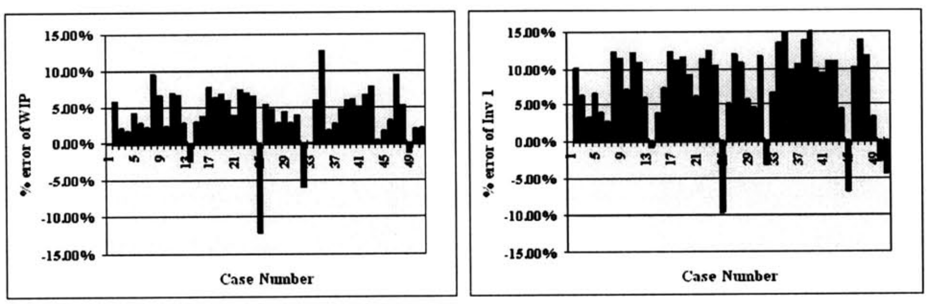
In this chapter, we propose the solution methods for the three non-trivial long manufacturing line tasks as a first step in analyzing long manufacturing lines with quality and operational failures. The comparison with simulations show that the solution methods provide the reliable performance estimates of long manufacturing lines. Analysis of various long manufacturing lines is a promising topic for future research. (See Chapter 7.)



(a) Total Production Rate

(b) Effective Production Rate

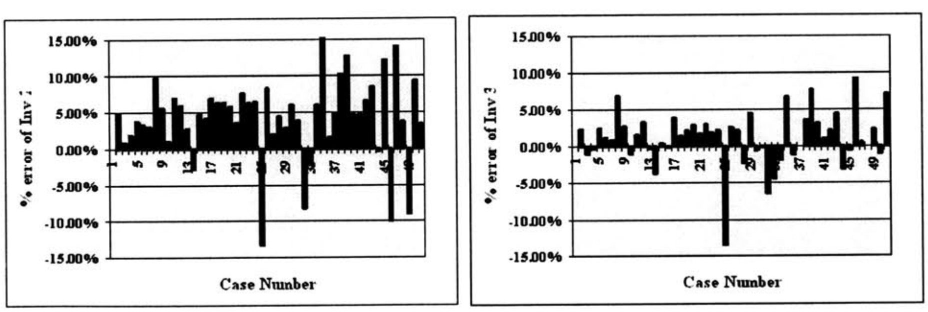
Figure 5-18: Validation - P_T and P_E



(a) Average WIP

(b) Average Inventory at B1

Figure 5-19: Validation - WIP and average inventory at B_1



(a) Average Inventory at B2

(b) Average Inventory at B3

Figure 5-20: Validation - average inventory at B_2 and B_3

Chapter 6

Jidoka

6.1 Jidoka practice in Toyota Production System

A significant portion of the Toyota Production System is traceable to an automatic loom invented early in the 20th century at Toyota Spinning & Weaving, the parent company of the Toyota Motors Corporation. The loom was designed to stop working immediately whenever thread snapped. The principle of stopping an operation when a problem occurs and preventing the production of defective items is fundamental to the Toyota Production System. This principle is called *jidoka* [Togo and Waterman, 1993]. In the Toyota Production System, equipment is designed to detect abnormalities and to stop automatically and immediately whenever they occur. Operators at assembly lines are provided means of stopping the production flow (*andon* cords) whenever they note anything unusual.

Experts in the Toyota Production System argue that the Jidoka practice has brought several benefits: The most significant of them is that it eliminates the need for the workers to oversee machine operations. As a result, an operator can handle multiple machines. It is not unusual that one operator handle 7 to 10 machines in the Toyota Production System. The man-machine separation led to significant direct labor cost saving and made it possible to use cellular manufacturing systems.

Another widely mentioned advantage of "stopping a line when abnormalities take place" is that it motivates *kaizen* (continuous improvement) since operators can clearly see the painful outcome of producing defects: the line stoppage. And it is easier to find the root cause of a problem right after the problem takes place. Through the use of systematic ways of resolving problems (e.g. asking "Why?" five times) which are widely accepted in Toyota, operators' learning speed accelerated [Fujimoto, 1999]. It has been known that operators' learning can significantly improve productivity and quality [Henderson, 1982], [Sandberg, 1995].

The other benefit of jidoka that Toyota Production System advocates claim is that it prevents the waste that would result from producing a series of defective items. Therefore, jidoka is considered to be a means to improve quality and increase productivity at the same time [Toyota Motors Corporation, 1996], [Monden, 1998]. When quality failures are persistent, in which once a bad part is produced, all subsequent parts will be bad until the machine is repaired, catching bad parts and stopping the machine as soon as possible is the best way to maintain high quality and productivity. This is the case with breakage of thread, which caused the invention of jidoka practice a century ago at Toyota Spinning & Weaving [Togo and Waterman, 1993]. On the other hand, defects are often from Bernoulli quality failures in which the quality of each part is independent of the others. In this case, there is no benefit to stop a machine that has made a bad part because there is no reason to believe that upcoming parts are bad; thus stopping the machine would reduce the number of bad parts in the future. In this case, therefore, stopping the operation does not improve quality but it reduces productivity by losing working time.

In reality, most of machines have multiple-yield quality failures. When a machine is in good shape and operating without any assignable cause variations (in control or high-quality state), it may produce a defective part with a very small probability, not because of any internal change in the machine but because of random external perturbations. However, when a machine is operating under assignable cause variations (out of control or low-quality state), it is likely that many of upcoming parts are bad. In this situation, the optimal stopping policy is, therefore to stop the machine only if the machine is in low quality state. But in many cases it is not easy to tell whether the machine is in high-quality state or low-quality-state (in other words, whether quality failure is from random variation or assignable cause variation) with one sample. Matters even get more complicated when inspection is not reliable.

In this case, what would be an optimal stopping policy? Qualitatively speaking, stopping immediately after detecting a bad part may not be optimal when Bernoulli quality failures is more frequent, mean time to repair (MTTR) is long and inspection is not reliable. Analytic models developed in Chapter 3 can give more precise and quantitative answers to the question.

6.2 Modeling of Multiple-Yield Quality Failures

6.2.1 Multiple-Yield Quality Failures

Jidoka practice means stopping a machine or a manufacturing line immediately when a defective part is made. Basically, people adopt jidoka assuming that inspection is 100% reliable and that all the defects are from persistent quality failures. In that case, it is clear that jidoka improves quality and productivity at the same time. But when there are Bernoulli quality failures and multiple-yield quality failures, there is no guarantee that subsequent parts will be defective after finding a non-conformity. In this case, it may be better to stop a machine when the machine produces two defective parts in a row since it is not likely to have two Bernoulli-type quality failures consecutively. To check the optimality of the jidoka stopping policy, we need to model multiple-yield quality failures. Figure 6-1 shows a modified state definition of a machine for the simplest multiple-yield quality failure model:

- State 1: The machine is in good shape and operating without any assignable cause variations. It may produce defective parts with probability $1-\pi(1)$, which is close to 0, due to random variations.
- State -1: The machine is operating under assignable cause variations and producing bad parts with probability $1 - \pi(-1)$, which is close to 1. But the operator does not know this yet.
- State 0: The machine is not operating.

Therefore, $\pi(1)$ is the yield of a machine when the machine is at state 1. And $\pi(-1)$ is the yield of the machine when it is at state -1. When the machine is either in state 1 or -1, it can be stopped for two reasons: operational failures with probability rate p^j and quality failures with transition rate q^j $(j = -1, 1)$. Here, $1/q^j$ is a *Mean Time to Stop due to Quality failures (MTSQ)* which depends on frequency of quality failures, inspection reliability, and machine stopping policies. Since we assume that the occurrence of operational failures is independent of machine states, $p^1 = p^{(-1)} = p$.

The transition from state 1 to state 0 occurs with probability rate $s = p^1 + q^1$, and the transition from state -1 to state 0 occurs with probability rate $f = p^{-1} + q^{-1}$. A system that has persistent quality failures only is a special case where:

- $\pi(1) = 1$ and $\pi(-1) = 0$.

- $q^1 = 0$, $q^{-1} = h$.

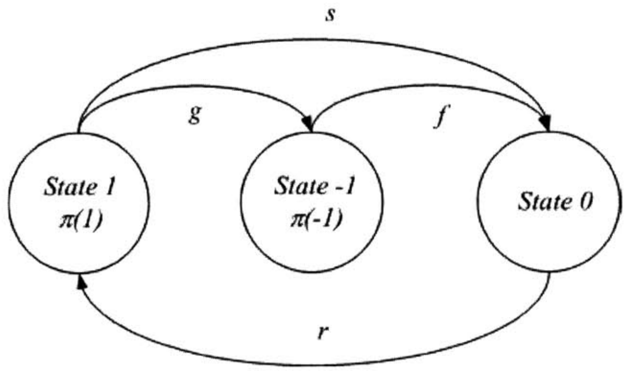


Figure 6-1: Machine states

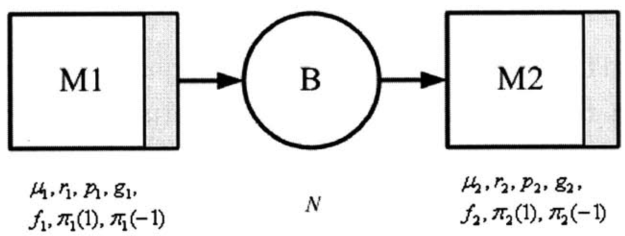


Figure 6-2: Two-Machine-One-Buffer system with multiple-yield quality failures

Figure 6-2 shows a 2M1B system with multiple-yield quality failures. Each machine has 7 parameters as shown in the figure. The analysis of 2M1B systems with multiple-yield quality failures is the same as 2M1B systems with the persistent quality failures except for a modification of the effective production rate formula. Internal transition equations, boundary conditions, total production rate, and average inventory are independent of $\pi_i(1)$ and $\pi_i(-1)$. The effective production rate of M_1 is

$$\begin{aligned}P_E^1 = &\sum_{\alpha_2=-1,0,1} \mu_1 [\int_0^N \{\pi_1(-1) f(x,-1,\alpha_2) + \pi_1(1) f(x,1,\alpha_2)\} dx + \pi_1(1) P(0,1,\alpha_2)\\ &+\pi_1(-1) P(0,-1,\alpha_2)] + \mu_2[\pi_1(1)\{P(N,1,-1) + P(N,1,1)\}\\ &+\pi_1(-1)\{P(N,-1,-1) + P(N,-1,1)\}].\end{aligned} \tag{6.1}$$

Similarly, the effective production rate of M_2 is

$$P_E^2 = \sum_{\alpha_1=-1,0,1} \mu_2[\int_0^N \{\pi_2(-1)f(x,\alpha_1,-1) + \pi_2(1)f(x,\alpha_1,1)\}dx + \pi_2(-1)P(N,\alpha_1,-1)$$
$$+\pi_2(1)P(N,\alpha_1,1)] + \mu_1[\pi_2(1)\{P(0,-1,1) + P(0,1,1)\}$$
$$+\pi_2(-1)\{P(0,-1,-1) + P(0,1,-1)\}]. \tag{6.2}$$

Equations (6.1) and (6.2) become equations (3.92) and (3.93) when $\pi_i(1) = 1$ and $\pi_i(-1) = 0$.

6.2.2 Modeling of Stopping Policies

As discussed earlier, we define q_i^j as the probability rate that a machine i (M_i) is stopped due to quality failures when M_i is in state j ($j = -1, 1$). The value of q_i^j depends on frequency of quality failures, inspection reliability, and stopping policies. Jidoka is based on 100% reliable inspection, and the frequency of quality failures is intrinsic to operations. Therefore, we only need to consider stopping policies to check the optimality of jidoka practice. For the sake of simplicity, we consider two different stopping policies:

- *Policy 1*: Stop a machine when a bad part is produced.
- *Policy 2*: Stop a machine when two bad parts are produced consecutively.

Stopping policy 1 is the policy that is incorporated in jidoka practice. If the persistent quality failures are the only quality failures, stopping a machine immediately after it produces a defect is better than stopping with two consecutive defects. But, in the case of Bernoulli quality failures and multiple-yield quality failures, the performance of the two stopping policies depends on many factors (e.g. $g, \pi(1), r$, and others).

Modeling of stopping policy 1

The probability of making a bad part when M_i is in state j is $1 - \pi_i(j)$. For stopping policy 1, we assume 100% reliable inspection and an immediate stoppage of a machine after a detection of a defect. As a result, the MTSQ ($=\frac{1}{q_i^j}$) is the same as the mean time for a quality failure to occur:

$$\frac{1}{q_i^j} = \frac{1}{\mu_i}[(1-\pi_i(j))+2(1-\pi_i(j))\pi_i(j)+3(1-\pi_i(j))(\pi_i(j))^2+...] = \frac{1}{\mu_i(1-\pi_i(j))}, (j=-1,1). \quad (6.3)$$

Here, $\frac{1}{\mu_i}$ is a cycle time. Therefore,

$$q_i^j = \mu_i(1-\pi_i(j)), (j=-1,1) \quad (6.4)$$

The 2M1B systems with multiple-yield quality failures and stopping policy 1 can be analyzed with the 2M1B systems developed in Chapter 3 with some modifications:

- Modify machine parameters p_i with $s_i = p_i + \mu_i(1-\pi_i(1))$ and f_i with $f_i = p_i + \mu_i(1-\pi_i(-1))$. Other parameters, g_i, r_i, N $(i=1,2)$, are unchanged.
- P_E^1 and P_E^2 are calculated from equations (6.1) and (6.2).

Modeling of stopping policy 2

For stopping policy 2, MTSQ is an expected time for two quality failures to occur in a row. Therefore, *MTSQ* can be estimated through the *expected time to absorption problem.* [Bertsekas and Tsitsiklis, 2002].

In Figure 6-3, there are three states:

- *State 0*: There is no defect in the last two products.
- *State 1*: There is one defect in the last two products.
- *State 2*: There are two defects in the last two products.

β is the probability of producing a good part. When two bad parts are produced in a row, the machine is stopped according to stopping policy 2. Therefore, state 2 is absorbing. If we define v_i as the expected times to absorption from a transient state i, then MTSQ is v_0/μ_i.

We can construct equations for the expected time to absorption problem as follows [Bertsekas and Tsitsiklis, 2002]:

$$v_0 = (1-\beta)v_1 + \beta v_0 + 1 \quad (6.5)$$

$$v_1 = \beta v_0 + 1 \quad (6.6)$$

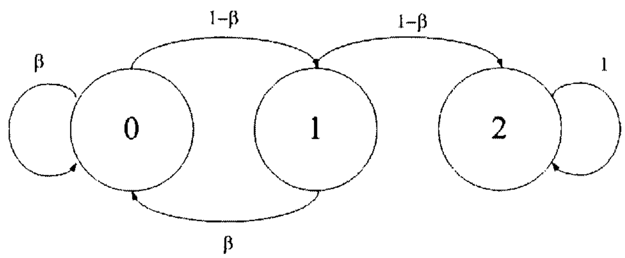


Figure 6-3: MTSQ estimation through an expected time to absorbtion problem

By solving equations (6.5) and (6.6), we get

$$v_0 = \frac{2-\beta}{(1-\beta)^2} \tag{6.7}$$

Since $\beta = \pi_i(j)$, $(j = -1, 1)$ for machine i (M_i),

$$q_i^j = \frac{\mu_i(1-\pi_i(j))^2}{2-\pi_i(j)} \tag{6.8}$$

Therefore,

$$\begin{aligned} s_i &= p_i + \tfrac{\mu_i(1-\pi_i(j))^2}{2-\pi_i(j)} \\ f_i &= p_i + \tfrac{\mu_i(1-\pi_i(j))^2}{2-\pi_i(j)} \end{aligned} \tag{6.9}$$

Again, the 2M1B system with multiple-yield quality failures and stopping policy 2 can be analyzed with the 2M1B systems developed in Chapter 3 with modification of machine parameters as shown in equation (6.9) and P_E^1 and P_E^2 being calculated from equations (6.1) and (6.2).

6.2.3 Optimality of stopping with one defect

The effective production rates of 2M1B systems with two different stopping policies are compared with varying machine parameters to see under what operating conditions, 'stopping with one defect', which is used by jidoka practice is effective.

We should note that the comparison of the two policies does not give the an exact answer to the question "under what conditions, is the stopping with one defect optimal?" In fact, guaranteeing the optimality would be a difficult task since there would be a large number of policies to be examined (e.g. 'stop a line when n out of m recent parts are bad'). But this numerical experiment gives a good idea on under what operating conditions, stopping with one defect would be close to optimal.

Base input parameters are shown in Table 6.1

Table 6.1: Base Machine Parameters

μ_1	r_1	p_1	g_1	$\pi_1(1)$	$\pi_1(-1)$
1	0.12	0.01	0.01	0.997	0
μ_2	r_2	p_2	g_2	$\pi_2(1)$	$\pi_2(-1)$
1	0.2	0.01	0.01	0.997	0

Figure 6-4 shows the effective production rate of 2M1B systems with the two stopping policies by changing $\pi_i(1)$ ($i = 1, 2$). As the figure indicates, the effectiveness of the stopping policies depends significantly on $\pi_i(1)$. Stopping policy 1 is better than stopping policy 2 only when $\pi_i(1)$ is very close to 1 (i.e. $\pi_i \geq 0.997$). This is a case in which a machine seldom produces a defect unless it is in low-quality state since there is very little random variation in the operation.

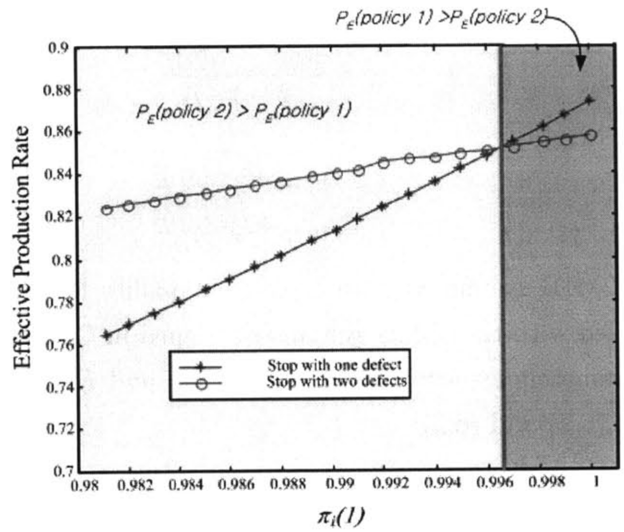


Figure 6-4: Effectiveness of stopping policy vs. $\pi(1)$

g determines the frequency of the transition from high-quality state (state 1) to low-quality state (state -1). Figure 6-5 shows the impact of g on the relative performance of the two stopping policies. Note that the influence of g on the the relative performance of each stopping policy seems to be smaller than that of $\pi(1)$.

Stopping policy 1 is better where g is large since large g means more frequent transition to low-quality state from high-quality state; thus, when a bad part is detected, it is likely that the machine has been in low-quality state.

Quicker repair means a reduction of capacity loss caused by the inappropriate

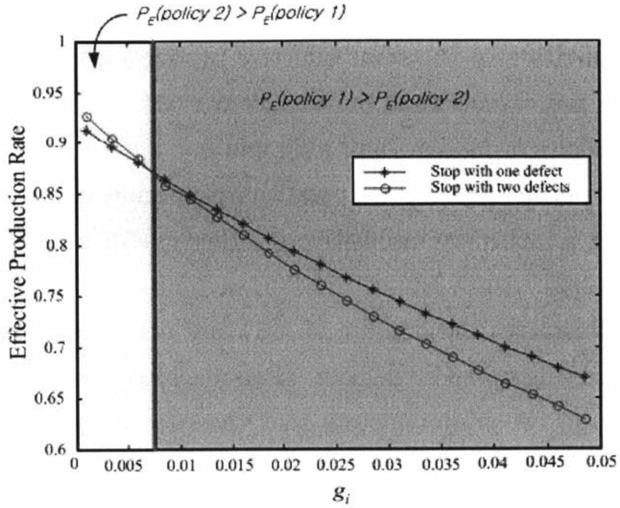


Figure 6-5: Effectiveness of stopping policy vs. g

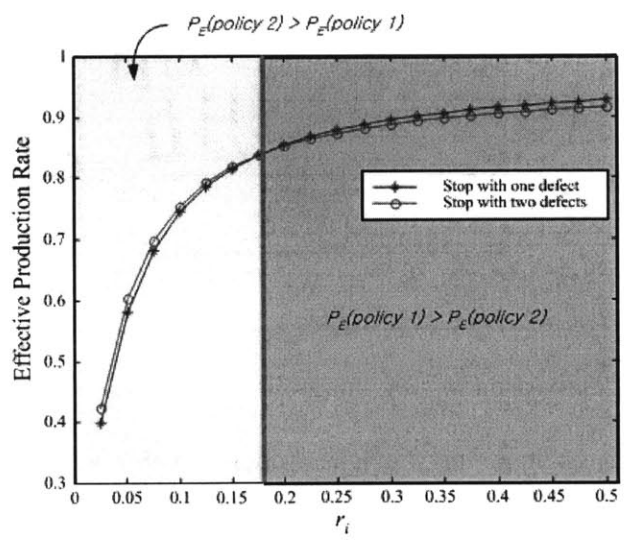


Figure 6-6: Effectiveness of stopping policy vs. repair rate

stoppage of a machine. Therefore, stopping with one defect outperforms stopping with two defects in a row, when r is large which is shown in Figure 6-6. More numerical experiments show that a higher value of $\pi_i(1)$ leads to a higher r value where the two stopping policies give the same effective production rate. The comparison with Figures 6-4 and 6-5 reveals that the impact of r on the relative effectiveness of the two stopping policies is weaker than $\pi(1)$ and g.

Numerical experiments show that the relative performances of 2M1B systems with the two stopping policies are insensitive to other machine parameters (e.g. p_i, f_i, μ_i, and $\pi_i(-1), i = 1, 2$).

Figure 6-7 illustrates the domain of $\pi(1)$ and g where 'stopping with one defect' outperforms 'stopping with two defects'. Two machine parameters $\pi(1)$ and g are used since these are the two major factors that the effective production rate is sensitive to. Standard process capability used at Toyota is $C_p = 1.33$, which means more than 99.99% of yield in operations [Monden, 1993]. If we assume that typical value of r at factories in the automotive industry is around 0.2, and that g is usually less than 0.01, this operating condition is in the domain where 'stopping with one defect' is better as shown in Figure 6-7.

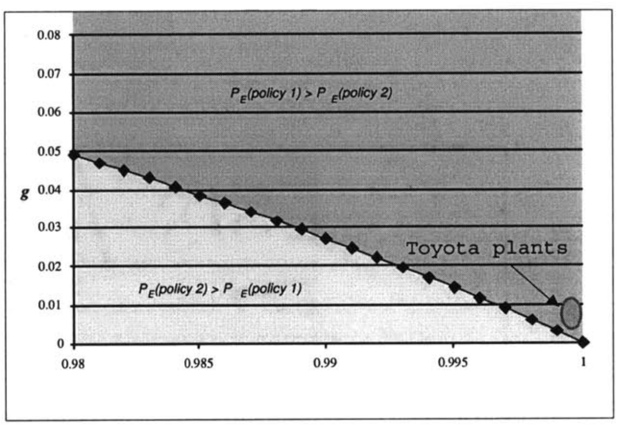


Figure 6-7: Comparison of stopping policy and operation range of Toyota plants

In other words, it seems that the stopping policy with jidoka practice is close to optimal at Toyota plants under the assumption that inspection is reliable. Note that all the equations are based on the 'perfectly reliable inspection' assumption. When inspection is not reliable, stopping with one defect is less likely be optimal since it is even not certain whether the machine actually produced nonconformity. We conclude that jidoka is likely to be optimal when factories are operating under desirable conditions (e.g., high process capability ($C_p > 1$), infrequent occurrence of assignable causes, and short repair time). However, more research is needed to determine the influence of other realistic factors such as imperfect inspection.

Chapter 7

Future Research

Throughout the thesis work, we have observed the lack of prior research on the intersection of quality, productivity, and manufacturing systems design. Discussions with automotive companies have revealed the industry's strong need for the research in this field. This thesis lays a cornerstone for the quantitative research in the area but there still remain many research opportunities. These opportunities are identified and some promising research strategies are described in this chapter.

7.1 Two-machine-one-buffer systems

7.1.1 Part scrapping at each operation

We have focused on manufacturing systems with no scrapping within the line (i.e., scrapping can take place at the end of the line). This is a reasonable assumption for an automotive assembly line where parts are big and heavy so that removing a part from the middle of the line is economically infeasible. But scrapping in the middle of the line may happen frequently in the manufacturing of small parts, such as electronic parts. In this case, the analytic modeling of the 2M1B systems becomes completely different. A new state definition may be needed since we need to differentiate good parts from defective parts in the buffer to scrap the bad ones only. In addition, a whole new set of internal transition equations and boundary conditions would have to be developed. The change in the buffer level, when the buffer is neither empty nor full, is no longer infinitesimal during a short time interval when scrapping takes place. Also, arrival to and departure from the boundaries become more complicated. New solution methods to solve the internal transition equations and boundary conditions would be needed.

7.1.2 Part rework

In a situation where defective parts are reworked at each operation, correcting the defects may require some additional operations. Therefore, it may alter the operation cycle time (i.e., machine speed) temporarily. As a result, the speed of a machine becomes dependent on the state of the machine. Again, a new set of internal transition equations, boundary conditions, and a solution method to solve these equations should be developed.

7.1.3 Correlation among different quality failures

Each quality failure is associated with a specific feature of a part. Throughout this thesis, we have assumed that each machine works on a different feature. This allows us to assume that the quality failures of the machines are independent. However, if a feature is the product of a sequence of operations (e.g., two machines work on the same hole: the first machine (M_1) does a roughing operation and the second machine (M_2) does a finishing operation), quality failures of M_2 are influenced by the operation of M_1. In this case, the state of M_2 is a function of the state of M_1. (i.e., M_2 is more likely to go to state -1 if M_1 is in state -1). In addition, the traceability of the root cause of a quality failure would be an issue; a defective feature made from a sequence of operation may contain a defect due to any one of the operations in the sequence. Then, determining which machine to stop for repair would be an important problem.

7.1.4 Reliability of inspection

Typically, testing is an imperfect process. The reliability of an inspection depends on many factors, including inspection equipment, sampling frequency, sample size, and others, but completely eliminating the errors in inspection is impossible. There are two types of errors:

- Type 1 error: errors where the part is good, but the test concludes that it is defective.
- Type 2 error: errors where the part is defective, but the test concludes that it is good.

In this thesis, 2M1B models include Type 2 errors in inspection; when f is small, a machine may produce multiple bad parts before it is stopped by operator for repair. But Type 1 errors are not considered. Including the Type 1 error may need modification of the transition rate between state 1 and state 0 or it may need a different machine state definition.

7.1.5 Productivity reduction due to inspection

In this thesis, we assume that inspection does not consume time. In many cases, however, inspection is time-consuming and tends to slow down the manufacturing process. Therefore, adding inspection may not only incur more cost (e.g., floor space, equipments, and labor) but may also reduce the capacity of the manufacturing system. The time-consuming inspection problem may be solved simply by adding a machine whose cycle time corresponds to the inspection time.

7.1.6 Aging

A machine may produce conforming parts with probability $\pi(t)$. If the machine were to get progressively out of tune due to tool wear, then $\pi(t)$ is a decreasing function of time. This phenomenon is called an *aging* process, and it is not unusual in manufacturing processes. The continuous aging process can be approximated with a machine with numerous discrete states with decreasing yields ($\pi_1 > \pi_2 \cdots > \pi_N$) as shown in Figure 7-1.

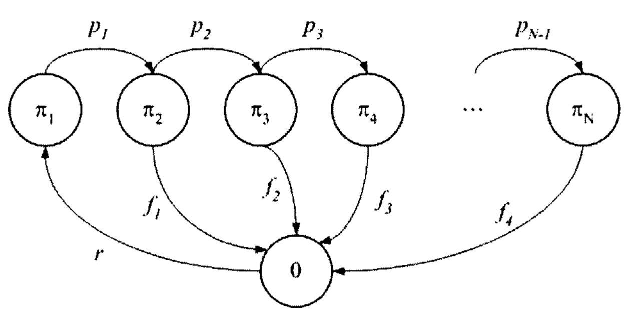


Figure 7-1: Modeling of aging process

7.2 Large systems

7.2.1 Topology of manufacturing systems

The modeling and analysis of quality and productivity issues in a manufacturing system are fundamentally influenced by its topology. Production lines come in widely different forms: serial lines, assembly/disassembly lines, parallel lines, and closed-loop systems. Decomposition techniques to analyze these topologies have been developed for production systems without quality failures [Gershwin, 1994], [Levantesi, 2001]. It is not clear whether the same kind of solution techniques, which are used in this thesis for the analysis of serial lines, can be applied to analyzing production systems with different topologies.

A split-merge line illustrated in Figure 7-2 is widely used when operations in the parallel lines (e.g.,M_3) are substantially slower than operations in the serial line (e.g., M_7). In the parallel lines that are designed to perform identical functions, normal everyday operation may lead to small amounts of variability in *nominally* identical machines, and therefore in the parts produced by those machines. Such variability may have an effect on quality of products after the end of the line after the parallel lines merge especially when a high precision assembly or fabrication operation may follow. Estimating the influence of parallel lines on the quality of products would be an important research topic.

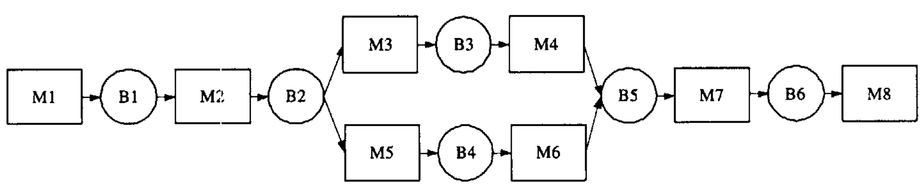


Figure 7-2: Split-merge line

7.2.2 Location and domain of inspection

Ideally, ubiquitous inspection (i.e., the placement of an inspection station after each machine) would result in the immediate detection and isolation of quality failures, simplifying root-cause traceability and minimizing the waste of downstream production capacity. However, inspection stations are expensive, in that they consume floor space, capital, and labor. Therefore, it is necessary to choose the number and location of inspection stations carefully, so as to place them as sparsely as possible while meeting quality goals.

An inspection station placed after a certain sequence of machines may be designed to detect quality failures produced by all the machines in that sequence (Figure 7-3), or only a subset of them (Figure 7-4), (Figure 7-5). Some features produced by certain operations may undergo multiple inspections (Figure 7-6). Different configurations of the inspection domain may require different decomposition procedures. The accuracy of the long line analysis may significantly depend on the the number, the location, and the configuration of inspections.

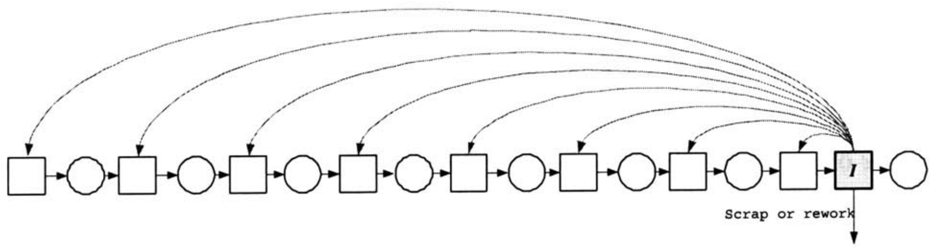


Figure 7-3: Single downstream inspection

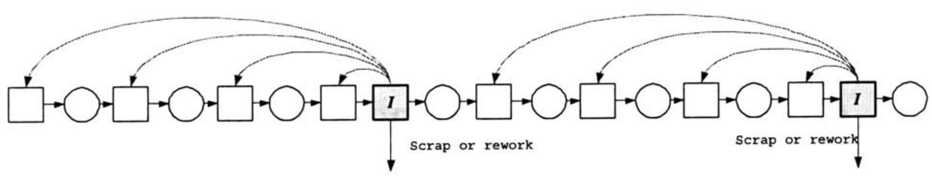


Figure 7-4: Contiguous inspection regions

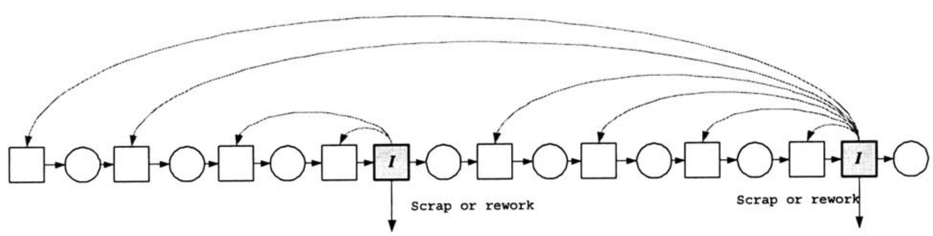


Figure 7-5: Non-contiguous inspection regions

7.2.3 Behavior of long lines

Through the numerical experiments, the behaviors of 2M1B systems were studied in Chapter 4. Although much of the qualitative behavior of long manufacturing lines

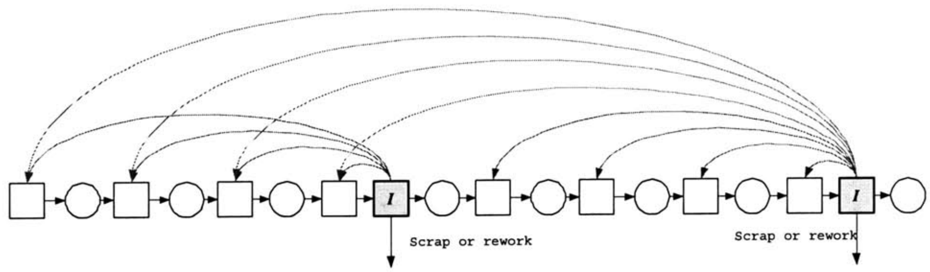


Figure 7-6: Overlapping inspection regions

can be inferred from that of the 2M1B system (e.g., beneficial, harmful, and optimal buffer cases), there may be some special behaviors of the long lines that can not be conjectured from the 2M1B systems. Looking for interesting behaviors of the long lines through conducting numerical experiments is time-consuming since the number of parameters in the system grows rapidly as the lines get longer (e.g., for K machine lines, there are $6K-1$ machine and buffer parameters). The analytic models provided in this thesis have substantial advantage over simulation, in searching for special behavior of the long manufacturing lines due to substantial computation time savings.

7.3 Optimal manufacturing system design

The optimal design of manufacturing systems is a vast research area in which a substantial number of papers have been published [Raz, 1986], [Gershwin and Schore, 2000], [Daya and Rahim, 2003]. Under this topic, two major sub-problems have received significant attentions: *the optimal inspection allocation problems*, and *the optimal buffer space allocation problems.* These two problems have been extensively studied, but there is a lack of research in their intersection since the two fields have been considered separate. In the optimal inspection allocation problems, previous authors have assumed Bernoulli-type quality failures [Raz, 1986]. Therefore, the role of inspection is to screen out defective parts, not to identify machines in bad states and fix them. Therefore, the system yield has nothing to do with average inventories and buffer sizes. In the optimal buffer space allocation problems, no quality failures are considered; machines are assumed to produce conforming parts and they undergo operational failures only. However, when persistent or multiple-yield quality failures exists and quality information feedback is used, the system yields become a function of buffer sizes as demonstrated in Chapter 4. As a result, *the optimal inspection allocation problems and the optimal buffer size allocation problems become coupled and should be solved simultaneously.* The analytical models of serial long lines presented in Chapter 5 can be used for this end, combined with a proper optimization technique.

7.4 Worker motivation and learning

The design and operation of manufacturing systems may affect the behavior of the workers on the production line, thereby indirectly influencing the quality and productivity of the manufacturing system. The relationship between the design of production lines, and workers' motivation and learning speed has been out side of the scope in this thesis. However, there has been significant research conducted on this relationship [Schultz *et al.*, 1998], [Lieberman and Demeester, 1999], [Fujimoto, 1999], [Alles *et al.*, 2000]. This research suggest that the reduction of inventory leads to an early detection of quality failures; thus, it facilitates the identification of the root cause of the problems. This allows people on manufacturing lines to develop a better understanding of the manufacturing processes and to feel more motivation for operations improvement (i.e., kaizen).

In a manufacturing system where manual labor is heavily used, (e.g., assembly lines), this relationship between the system design and the workers' behavior becomes more important. The increase of buffer size in the *beneficial buffer case* (Figure 4-2) presented in Chapter 4 may not actually be beneficial (i.e., leading to higher productivity) in fact, if the detrimental influence of large buffer on workers' learning is considered. Stopping with one defect, which is incorporated in the jidoka practice, might be the optimal stopping policy even for the less desirable operating conditions described in section 6.2.3 (see Figure 6-7) in the long run, if the workers' motivation and learning are taken into consideration.

Most quantitative research on the manufacturing system design has neglected the *human* issue. Manufacturing system design principles derived from this kind of research are useful for designing and operating factories where automated machines are heavily used (e.g., flexible manufacturing systems). But these principles would be inappropriate if they are applied to labor intensive factories. Therefore, more holistic research combining the issue of the interaction between people and manufacturing system design with quantitative modeling and analysis of manufacturing system is needed.

Chapter 8

Conclusion

This thesis takes an essential early research step in analyzing how production system design, quality, and productivity are inter-related. There was very little quantitative analytical literature that explores this area, even though the effects of the interaction are recognized on the plant floor anecdotally.

Throughout the thesis, we identify important differences among types of quality failures, and develop a new Markov process model for machines with both quality and operational failures. Based on the single-machine analysis, we present analytic models, solution techniques, performance evaluations, and validation of two-machine systems, as well as longer production lines.

Numerical studies using two-machine models show that when the first machine has quality failures and the inspection occurs only at the second machine, there are cases in which the effective production rate *increases* as buffer size increases, and there are cases in which the effective production rate *decreases* for larger buffers. We present various methods of improving quality and productivity, and demonstrate that the effectiveness of each method is greatly dependent upon the particularities of factories. Therefore, the need to find the most effective combination of method in each case is identified, and the usefulness of the quantitative tools developed in this thesis is shown. We also investigate the effectiveness of jidoka practice, and find that jidoka is useful only when machines are operating under stable conditions. We reaffirm the importance of and the urgent need for research in this field, and we propose promising research directions.

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Appendix A

2M1B parameters

Table A.1: Machine and buffer parameters for infinite buffer case and zero buffer case validation

Case #	μ_1	μ_2	r_1	r_2	p_1	p_2	g_1	g_2	f_1	f_2
1	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2
2	1.0	1.0	0.3	0.3	0.005	0.005	0.05	0.05	0.5	0.5
3	1.0	1.0	0.2	0.05	0.01	0.01	0.01	0.01	0.2	0.2
4	1.0	1.0	0.1	0.1	0.05	0.005	0.01	0.01	0.2	0.2
5	1.0	1.0	0.1	0.1	0.01	0.01	0.05	0.005	0.2	0.2
6	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.5	0.1
7	2.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.5	0.1
8	3.0	2.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2
9	1.0	2.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2
10	2.0	3.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2

Table A.2: Machine and buffer parameters for intermediate buffer case validation

Case #	μ_1	μ_2	r_1	r_2	p_1	p_2	g_1	g_2	f_1	f_2	N
1	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30
2	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	5
3	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	10
4	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	15
5	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	20
6	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	25
7	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30
8	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	35
9	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	40
10	0.5	0.5	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30
11	1.5	1.5	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30
12	2.0	2.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30
13	2.5	2.5	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30
14	3.0	3.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30
15	1.0	1.0	0.01	0.01	0.01	0.01	0.01	0.01	0.2	0.2	30
16	1.0	1.0	0.05	0.05	0.01	0.01	0.01	0.01	0.2	0.2	30
17	1.0	1.0	0.2	0.2	0.01	0.01	0.01	0.01	0.2	0.2	30
18	1.0	1.0	0.5	0.5	0.01	0.01	0.01	0.01	0.2	0.2	30
19	1.0	1.0	0.8	0.8	0.01	0.01	0.01	0.01	0.2	0.2	30
20	1.0	1.0	0.1	0.1	0.001	0.001	0.01	0.01	0.2	0.2	30
21	1.0	1.0	0.1	0.1	0.005	0.005	0.01	0.01	0.2	0.2	30
22	1.0	1.0	0.1	0.1	0.02	0.02	0.01	0.01	0.2	0.2	30
23	1.0	1.0	0.1	0.1	0.05	0.05	0.01	0.01	0.2	0.2	30
24	1.0	1.0	0.1	0.1	0.1	0.1	0.01	0.01	0.2	0.2	30
25	1.0	1.0	0.1	0.1	0.01	0.01	0.001	0.001	0.2	0.2	30
26	1.0	1.0	0.1	0.1	0.01	0.01	0.005	0.005	0.2	0.2	30
27	1.0	1.0	0.1	0.1	0.01	0.01	0.02	0.02	0.2	0.2	30
28	1.0	1.0	0.1	0.1	0.01	0.01	0.05	0.05	0.2	0.2	30
29	1.0	1.0	0.1	0.1	0.01	0.01	0.1	0.1	0.2	0.2	30
30	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.02	0.02	30
31	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.05	0.05	30
32	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.1	0.1	30
33	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.5	0.5	30
34	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.95	0.95	30
35	1.0	1.0	0.5	0.1	0.010	0.010	0.010	0.010	0.2	0.2	30
36	1.0	1.0	0.01	0.1	0.010	0.010	0.010	0.010	0.2	0.2	30
37	1.0	1.0	0.1	0.5	0.010	0.010	0.010	0.010	0.2	0.2	30
38	1.0	1.0	0.1	0.01	0.010	0.010	0.010	0.010	0.2	0.2	30
39	1.0	1.0	0.1	0.1	0.100	0.010	0.010	0.010	0.2	0.2	30
40	1.0	1.0	0.1	0.1	0.001	0.010	0.010	0.010	0.2	0.2	30

Table A.3: Machine and buffer parameters for intermediate buffer case validation - continued

Case #	μ_1	μ_2	r_1	r_2	p_1	p_2	g_1	g_2	f_1	f_2	N
41	1.0	1.0	0.1	0.1	0.010	0.100	0.010	0.010	0.2	0.2	30
42	1.0	1.0	0.1	0.1	0.010	0.001	0.010	0.010	0.2	0.2	30
43	1.0	1.0	0.1	0.1	0.010	0.010	0.100	0.010	0.2	0.2	30
44	1.0	1.0	0.1	0.1	0.010	0.010	0.001	0.010	0.2	0.2	30
45	1.0	1.0	0.1	0.1	0.010	0.010	0.010	0.100	0.2	0.2	30
46	1.0	1.0	0.1	0.1	0.010	0.010	0.010	0.001	0.2	0.2	30
47	1.0	1.0	0.1	0.1	0.010	0.010	0.010	0.010	0.9	0.2	30
48	1.0	1.0	0.1	0.1	0.010	0.010	0.010	0.010	0.05	0.2	30
49	1.0	1.0	0.1	0.1	0.010	0.010	0.010	0.010	0.2	0.9	30
50	1.0	1.0	0.1	0.1	0.010	0.010	0.010	0.010	0.2	0.05	30
51	1.0	1.0	0.5	0.5	0.1	0.1	0.01	0.01	0.2	0.2	30
52	1.0	1.0	0.010	0.010	0.001	0.001	0.010	0.010	0.200	0.200	30
53	1.0	1.0	0.500	0.500	0.010	0.010	0.050	0.050	0.200	0.200	30
54	1.0	1.0	0.010	0.010	0.010	0.010	0.001	0.001	0.200	0.200	30
55	1.0	1.0	0.500	0.500	0.010	0.010	0.010	0.010	0.950	0.950	30
56	1.0	1.0	0.010	0.010	0.010	0.010	0.010	0.010	0.050	0.050	30
57	1.0	1.0	0.500	0.500	0.010	0.010	0.010	0.010	0.200	0.200	50
58	1.0	1.0	0.010	0.010	0.010	0.010	0.010	0.010	0.200	0.200	5
59	1.0	1.0	0.100	0.100	0.100	0.100	0.050	0.050	0.200	0.200	30
60	1.0	1.0	0.100	0.100	0.001	0.001	0.001	0.001	0.200	0.200	30
61	1.0	1.0	0.100	0.100	0.100	0.100	0.010	0.010	0.950	0.950	30
62	1.0	1.0	0.100	0.100	0.001	0.001	0.010	0.010	0.050	0.050	30
63	1.0	1.0	0.100	0.100	0.100	0.100	0.010	0.010	0.200	0.200	50
64	1.0	1.0	0.100	0.100	0.001	0.001	0.010	0.010	0.200	0.200	5
65	1.0	1.0	0.100	0.100	0.010	0.010	0.050	0.050	0.950	0.950	30
66	1.0	1.0	0.100	0.100	0.010	0.010	0.001	0.001	0.050	0.050	30
67	1.0	1.0	0.100	0.100	0.010	0.010	0.050	0.050	0.200	0.200	50
68	1.0	1.0	0.100	0.100	0.010	0.010	0.001	0.001	0.200	0.200	5
69	1.0	1.0	0.100	0.100	0.010	0.010	0.010	0.010	0.950	0.950	50
70	1.0	1.0	0.100	0.100	0.010	0.010	0.010	0.010	0.050	0.050	5
71	1.2	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30
72	1.2	1.0	0.1	0.2	0.02	0.01	0.01	0.005	0.2	0.2	30
73	1.2	1.0	0.1	0.2	0.02	0.01	0.05	0.005	0.2	0.2	30
74	1.333	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30
75	1.333	1.0	0.1	0.2	0.02	0.01	0.03	0.01	0.3	0.1	30
76	1.333	1.0	0.1	0.3	0.02	0.01	0.03	0.01	0.3	0.1	30
77	1.5	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30
78	1.5	1.0	0.1	0.4	0.05	0.01	0.01	0.01	0.2	0.2	30
79	1.5	1.0	0.1	0.4	0.05	0.01	0.05	0.01	0.2	0.2	30
80	2	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30

Table A.4: Machine and buffer parameters for intermediate buffer case validation - continued

Case #	μ_1	μ_2	r_1	r_2	p_1	p_2	g_1	g_2	f_1	f_2	N
81	2	1.0	0.08	0.3	0.05	0.01	0.05	0.008	0.4	0.2	30
82	2	1.0	0.05	0.3	0.05	0.01	0.05	0.01	0.2	0.2	30
83	3	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30
84	3	1.0	0.08	0.18	0.2	0.015	0.01	0.01	0.1	0.2	30
85	3	1.0	0.08	0.18	0.3	0.015	0.01	0.01	0.1	0.2	30
86	1.0	1.2	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30
87	1.0	1.2	0.2	0.1	0.01	0.02	0.005	0.01	0.2	0.2	30
88	1.0	1.2	0.2	0.1	0.01	0.02	0.005	0.05	0.2	0.2	30
89	1.0	1.333	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30
90	1.0	1.333	0.2	0.1	0.01	0.02	0.01	0.03	0.1	0.3	30
91	1.0	1.333	0.3	0.1	0.01	0.02	0.01	0.03	0.1	0.3	30
92	1.0	1.5	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30
93	1.0	1.5	0.4	0.1	0.01	0.05	0.01	0.01	0.2	0.2	30
94	1.0	1.5	0.4	0.1	0.01	0.05	0.01	0.05	0.2	0.2	30
95	1.0	2	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30
96	1.0	2	0.3	0.08	0.01	0.05	0.008	0.05	0.2	0.4	30
97	1.0	2	0.3	0.05	0.01	0.05	0.01	0.05	0.2	0.2	30
98	1.0	3	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30
99	1.0	3	0.18	0.08	0.015	0.2	0.01	0.01	0.2	0.1	30
100	1.0	3	0.18	0.08	0.015	0.3	0.01	0.01	0.2	0.1	30

Table A.5: Machine and buffer parameters for quality information feedback validation

Case #	μ_1	μ_2	r_1	r_2	p_1	p_2	g_1	g_2	f_1	f_2	N
1	1	1	0.1	0.1	0.01	0.01	0.01	0.01	0.01	1	10
2	1	1	0.1	0.1	0.01	0.01	0.01	0.01	0.01	1	0
3	1	1	0.1	0.1	0.01	0.01	0.01	0.01	0.01	1	5
4	1	1	0.1	0.1	0.01	0.01	0.01	0.01	0.01	1	20
5	1	1	0.1	0.1	0.01	0.01	0.01	0.01	0.01	1	30
6	1	1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	1	10
7	1	1	0.05	0.05	0.01	0.01	0.01	0.01	0.01	1	10
8	1	1	0.5	0.5	0.01	0.01	0.01	0.01	0.01	1	10
9	1	1	0.1	0.1	0.001	0.001	0.01	0.01	0.01	1	10
10	1	1	0.1	0.1	0.03	0.03	0.01	0.01	0.01	1	10
11	1	1	0.1	0.1	0.1	0.1	0.01	0.01	0.01	1	10
12	1	1	0.1	0.1	0.01	0.01	0.001	0.001	0.001	1	10
13	1	1	0.1	0.1	0.01	0.01	0.005	0.005	0.005	1	10
14	1	1	0.1	0.1	0.01	0.01	0.02	0.02	0.02	1	10
15	1	1	0.1	0.1	0.01	0.01	0.05	0.05	0.05	1	10
16	0.5	0.5	0.1	0.1	0.01	0.01	0.01	0.01	0.01	1	10
17	1.5	1.5	0.1	0.1	0.01	0.01	0.01	0.01	0.01	1	10
18	2	2	0.1	0.1	0.01	0.01	0.01	0.01	0.01	1	10
19	1	1	0.1	0.1	0.01	0.01	0.01	0.01	0.02	1	10
20	1	1	0.1	0.1	0.01	0.01	0.01	0.01	0.05	1	10
21	1	1	0.1	0.1	0.01	0.01	0.01	0.01	0.1	1	10
22	1	1	0.1	0.1	0.01	0.01	0.01	0.01	0.2	1	10
23	1	1	0.5	0.1	0.01	0.01	0.01	0.01	0.01	1	10
24	1	1	0.01	0.1	0.01	0.01	0.01	0.01	0.01	1	10
25	1	1	0.1	0.5	0.01	0.01	0.01	0.01	0.01	1	10
26	1	1	0.1	0.01	0.01	0.01	0.01	0.01	0.01	1	10
27	1	1	0.1	0.1	0.1	0.01	0.01	0.01	0.01	1	10
28	1	1	0.1	0.1	0.001	0.01	0.01	0.01	0.01	1	10
29	1	1	0.1	0.1	0.01	0.1	0.01	0.01	0.01	1	10
30	1	1	0.1	0.1	0.01	0.001	0.01	0.01	0.01	1	10
31	1	1	0.1	0.1	0.01	0.01	0.05	0.01	0.05	1	10
32	1	1	0.1	0.1	0.01	0.01	0.001	0.01	0.001	1	10
33	1	1	0.1	0.1	0.01	0.01	0.01	0.05	0.01	1	10
34	1	1	0.1	0.1	0.01	0.01	0.01	0.001	0.01	1	10
35	1	1	0.1	0.1	0.01	0.01	0.01	0.01	0.01	0.05	10
36	1	1	0.1	0.1	0.01	0.01	0.01	0.01	0.01	0.1	10
37	1	1	0.1	0.1	0.01	0.01	0.01	0.01	0.01	0.3	10
38	1	1	0.1	0.1	0.01	0.01	0.01	0.01	0.01	0.5	10
39	1	1	0.1	0.1	0.01	0.01	0.01	0.01	0.01	0.7	10
40	1	1	0.1	0.1	0.01	0.01	0.01	0.01	0.01	0.9	10

Table A.6: Machine and buffer parameters for quality information feedback validation - continued

Case #	μ_1	μ_2	r_1	r_2	p_1	p_2	g_1	g_2	f_1	f_2	N
41	1.2	1	0.1	0.1	0.01	0.01	0.01	0.01	0.01	1	10
42	1	1.2	0.1	0.1	0.01	0.01	0.01	0.01	0.01	1	10
43	1.333	1	0.1	0.1	0.01	0.01	0.01	0.01	0.01	1	10
44	1	1.333	0.1	0.1	0.01	0.01	0.01	0.01	0.01	1	10
45	1.5	1	0.1	0.1	0.01	0.01	0.01	0.01	0.01	1	10
46	1	1.5	0.1	0.1	0.01	0.01	0.01	0.01	0.01	1	10
47	2	1	0.1	0.1	0.01	0.01	0.01	0.01	0.01	1	10
48	1	2	0.1	0.1	0.01	0.01	0.01	0.01	0.01	1	10
49	3	1	0.1	0.1	0.01	0.01	0.01	0.01	0.01	1	10
50	1	3	0.1	0.1	0.01	0.01	0.01	0.01	0.01	1	10

Table A.7: Machine and buffer parameters for 3-state-machine and 2-state-machine comparison (Figures 5-5, 5-6, and 5-7)

Case #	μ_1	μ_2	r_1	r_2	p_1	p_2	g_1	g_2	f_1	f_2	N	QIF
1	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
2	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	5	N
3	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	20	N
4	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	50	N
5	0.5	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
6	2.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
7	3.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
8	1.0	1.0	0.01	0.01	0.01	0.01	0.01	0.01	0.2	0.2	30	N
9	1.0	1.0	0.05	0.05	0.01	0.01	0.01	0.01	0.2	0.2	30	N
10	1.0	1.0	0.5	0.5	0.01	0.01	0.01	0.01	0.2	0.2	30	N
11	1.0	1.0	0.1	0.1	0.001	0.001	0.01	0.01	0.2	0.2	30	N
12	1.0	1.0	0.1	0.1	0.05	0.05	0.01	0.01	0.2	0.2	30	N
13	1.0	1.0	0.1	0.1	0.1	0.1	0.01	0.01	0.2	0.2	30	N
14	1.0	1.0	0.1	0.1	0.01	0.01	0.001	0.001	0.2	0.2	30	N
15	1.0	1.0	0.1	0.1	0.01	0.01	0.05	0.05	0.2	0.2	30	N
16	1.0	1.0	0.1	0.1	0.01	0.01	0.1	0.1	0.2	0.2	30	N
17	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.02	0.02	30	N
18	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.5	0.5	30	N
19	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.95	0.95	30	N
20	1.0	1.0	0.5	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
21	1.0	1.0	0.01	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
22	1.0	1.0	0.1	0.5	0.01	0.01	0.01	0.01	0.2	0.2	30	N
23	1.0	1.0	0.1	0.01	0.01	0.01	0.01	0.01	0.2	0.2	30	N
24	1.0	1.0	0.1	0.1	0.1	0.01	0.01	0.01	0.2	0.2	30	N
25	1.0	1.0	0.1	0.1	0.001	0.01	0.01	0.01	0.2	0.2	30	N
26	1.0	1.0	0.1	0.1	0.01	0.1	0.01	0.01	0.2	0.2	30	N
27	1.0	1.0	0.1	0.1	0.01	0.001	0.01	0.01	0.2	0.2	30	N
28	1.0	1.0	0.1	0.1	0.01	0.01	0.1	0.01	0.2	0.2	30	N
29	1.0	1.0	0.1	0.1	0.01	0.01	0.001	0.01	0.2	0.2	30	N
30	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.1	0.2	0.2	30	N
31	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.001	0.2	0.2	30	N
32	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.9	0.2	30	N
33	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.05	0.2	30	N
34	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.9	30	N
35	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.05	30	N
36	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	5	N
37	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	20	N
38	1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	50	N
39	1.0	1.0	0.2	0.05	0.01	0.01	0.01	0.01	0.2	0.2	5	N
40	1.0	1.0	0.2	0.05	0.01	0.01	0.01	0.01	0.2	0.2	20	N

Table A.8: Machine and buffer parameters for 3-state-machine and 2-state-machine comparison (Figures 5-5, 5-6, and 5-7)- continued

Case #	μ_1	μ_2	r_1	r_2	p_1	p_2	g_1	g_2	f_1	f_2	N	QIF
41	1.0	1	0.2	0.05	0.01	0.01	0.01	0.01	0.2	0.2	50	N
42	1.0	1	0.05	0.2	0.01	0.01	0.01	0.01	0.2	0.2	5	N
43	1.0	1	0.05	0.2	0.01	0.01	0.01	0.01	0.2	0.2	20	N
44	1.0	1	0.05	0.2	0.01	0.01	0.01	0.01	0.2	0.2	50	N
45	1.0	1	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
46	1.0	1	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
47	1.0	1	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
48	1.0	1	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
49	1.0	1	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
50	1.0	1	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
51	1.2	1	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
52	1.2	1	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	10	Y
53	1.2	1	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
54	1.2	1	0.1	0.2	0.02	0.01	0.01	0.005	0.2	0.2	30	N
55	1.2	1	0.1	0.2	0.02	0.01	0.05	0.005	0.2	0.2	30	N
56	1.33	1	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
57	1.33	1	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	10	Y
58	1.33	1	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
59	1.33	1	0.1	0.2	0.02	0.01	0.03	0.01	0.3	0.1	30	N
60	1.33	1	0.1	0.3	0.02	0.01	0.03	0.01	0.3	0.1	30	N
61	1.50	1	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
62	1.5	1	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	10	Y
63	1.5	1	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
64	1.5	1	0.1	0.4	0.05	0.01	0.01	0.01	0.2	0.2	30	N
65	1.5	1	0.1	0.4	0.05	0.01	0.05	0.01	0.2	0.2	30	N
66	2.0	1	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
67	2.0	1	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	10	Y
68	2.0	1	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
69	2.0	1	0.08	0.3	0.05	0.01	0.05	0.008	0.4	0.2	30	N
70	2.0	1	0.05	0.3	0.05	0.01	0.05	0.01	0.2	0.2	30	N
71	3.0	1	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
72	3.0	1	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	10	Y
73	3.0	1	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
74	3.0	1	0.08	0.18	0.2	0.015	0.01	0.01	0.1	0.2	30	N
75	3.0	1	0.08	0.18	0.3	0.015	0.01	0.01	0.1	0.2	30	N
76	1.0	1.2	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
77	1.0	1.2	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	10	Y
78	1.0	1.2	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
79	1.0	1.2	0.2	0.1	0.01	0.02	0.005	0.01	0.2	0.2	30	N
80	1.0	1.2	0.2	0.1	0.01	0.02	0.005	0.05	0.2	0.2	30	N

Table A.9: Machine and buffer parameters for 3-state-machine and 2-state-machine comparison (Figures 5-5, 5-6, and 5-7)- continued

Case #	μ_1	μ_2	r_1	r_2	p_1	p_2	g_1	g_2	f_1	f_2	N	QIF
81	1.0	1.333	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
82	1.0	1.333	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	10	Y
83	1.0	1.333	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
84	1.0	1.333	0.2	0.1	0.01	0.02	0.01	0.03	0.1	0.3	30	N
85	1.0	1.333	0.3	0.1	0.01	0.02	0.01	0.03	0.1	0.3	30	N
86	1.0	1.5	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
87	1.0	1.5	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	10	Y
88	1.0	1.5	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
89	1.0	1.5	0.4	0.1	0.01	0.05	0.01	0.01	0.2	0.2	30	N
90	1.0	1.5	0.4	0.1	0.01	0.05	0.01	0.05	0.2	0.2	30	N
91	1.0	2	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
92	1.0	2	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	10	Y
93	1.0	2	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
94	1.0	2	0.3	0.08	0.01	0.05	0.008	0.05	0.2	0.4	30	N
95	1.0	2	0.3	0.05	0.01	0.05	0.01	0.05	0.2	0.2	30	N
96	1.0	3	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
97	1.0	3	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	10	Y
98	1.0	3	0.1	0.1	0.01	0.01	0.01	0.01	0.2	0.2	30	N
99	1.0	3	0.18	0.08	0.015	0.2	0.01	0.01	0.2	0.1	30	N
100	1.0	3	0.18	0.08	0.015	0.3	0.01	0.01	0.2	0.1	30	N

Table A.10: Machine and buffer parameters for Figures 4-1, 4-2, and 4-3

μ_1	μ_2	r_1	r_2	p_1	p_2	g_1	g_2	f_1	f_2
1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	0.1	0.9

Table A.11: Machine and buffer parameters for Figures 4-4, 4-5 and 4-6

μ_1	μ_2	r_1	r_2	p_1	p_2	g_1	g_2	f_1	f_2
2.0	2.0	0.5	0.1	0.005	0.05	0.5	0.005	0.02	0.9

Table A.12: Machine and buffer parameters for Figures 4-7, 4-8 and 4-9

μ_1	μ_2	r_1	r_2	p_1	p_2	g_1	g_2	f_1	f_2
1.0	1.0	0.05	0.05	0.01	0.01	0.05	0.005	0.01	1.0

Table A.13: Machine and buffer parameters for Figure 4-10

μ_1	μ_2	r_1	r_2	p_1	p_2	f_1	f_2	N
1.0	1.0	0.1	0.1	0.01	0.01	0.2	0.2	30

Table A.14: Machine and buffer parameters for Figure 4-11

μ_1	μ_2	r_1	r_2	p_1	p_2	g_1	g_2	N
1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01	30

Table A.15: Machine parameters for Figure 4-12

μ_1	μ_2	r_1	r_2	p_1	p_2	g_1	g_2
1.0	1.0	0.1	0.1	0.01	0.01	0.01	0.01

Table A.16: Machine parameters for Figure 4-13

μ_1	μ_2	r_1	r_2	p_1	p_2	f_1	f_2
1.0	1.0	0.1	0.1	0.01	0.01	0.2	0.2

Appendix B

Long Line Task Parameters

B.1 Ubiquitous inspection case

Table B.1: Machine and buffer parameters for ubiquitous inspection validation

Case #	1	2	3	4	5	6	7	8	9	10
μ_1	1	1	1	1	1	1	1	1	1	1
r_1	0.1	0.1	0.1	0.02	0.05	0.5	0.1	0.1	0.1	0.1
p_1	0.01	0.01	0.01	0.01	0.01	0.01	0.001	0.005	0.02	0.05
g_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_1	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
μ_2	1	1	1	1	1	1	1	1	1	1
r_2	0.1	0.1	0.1	0.02	0.05	0.5	0.1	0.1	0.1	0.1
p_2	0.01	0.01	0.01	0.01	0.01	0.01	0.001	0.005	0.02	0.05
g_2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
μ_3	1	1	1	1	1	1	1	1	1	1
r_3	0.1	0.1	0.1	0.02	0.05	0.5	0.1	0.1	0.1	0.1
p_3	0.01	0.01	0.01	0.01	0.01	0.01	0.001	0.005	0.02	0.05
g_3	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_3	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
μ_4	1	1	1	1	1	1	1	1	1	1
r_4	0.1	0.1	0.1	0.02	0.05	0.5	0.1	0.1	0.1	0.1
p_4	0.01	0.01	0.01	0.01	0.01	0.01	0.001	0.005	0.02	0.05
g_4	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_4	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
N_1	10	20	50	20	20	20	20	20	20	20
N_2	10	20	50	20	20	20	20	20	20	20
N_3	10	20	50	20	20	20	20	20	20	20

Table B.2: Machine and buffer parameters for ubiquitous inspection validation-continued

Case #	11	12	13	14	15	16	17	18	19	20
μ_1	1	1	1	1	1	1	1	1	0.5	1.5
r_1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g_1	0.001	0.005	0.02	0.05	0.01	0.01	0.01	0.01	0.01	0.01
f_1	0.2	0.2	0.2	0.2	0.02	0.05	0.5	0.9	0.2	0.2
μ_2	1	1	1	1	1	1	1	1	0.5	1.5
r_2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g_2	0.001	0.005	0.02	0.05	0.01	0.01	0.01	0.01	0.01	0.01
f_2	0.2	0.2	0.2	0.2	0.02	0.05	0.5	0.9	0.2	0.2
μ_3	1	1	1	1	1	1	1	1	0.5	1.5
r_3	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_3	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g_3	0.001	0.005	0.02	0.05	0.01	0.01	0.01	0.01	0.01	0.01
f_3	0.2	0.2	0.2	0.2	0.02	0.05	0.5	0.9	0.2	0.2
μ_4	1	1	1	1	1	1	1	1	0.5	1.5
r_4	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_4	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g_4	0.001	0.005	0.02	0.05	0.01	0.01	0.01	0.01	0.01	0.01
f_4	0.2	0.2	0.2	0.2	0.02	0.05	0.5	0.9	0.2	0.2
N_1	20	20	20	20	20	20	20	20	20	20
N_2	20	20	20	20	20	20	20	20	20	20
N_3	20	20	20	20	20	20	20	20	20	20

Table B.3: Machine and buffer parameters for ubiquitous inspection validation-continued

Case #	21	22	23	24	25	26	27	28	29	30
μ_1	2	1	1	1	1	1	1	1	1	1
r_1	0.1	0.3	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_1	0.01	0.01	0.01	0.01	0.01	0.05	0.01	0.01	0.01	0.01
g_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.05
f_1	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
μ_2	2	1	1	1	1	1	1	1	1	1
r_2	0.1	0.1	0.3	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_2	0.01	0.01	0.01	0.01	0.01	0.01	0.05	0.01	0.01	0.01
g_2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
μ_3	2	1	1	1	1	1	1	1	1	1
r_3	0.1	0.1	0.1	0.3	0.1	0.1	0.1	0.1	0.1	0.1
p_3	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.05	0.01	0.01
g_3	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_3	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
μ_4	2	1	1	1	1	1	1	1	1	1
r_4	0.1	0.1	0.1	0.1	0.3	0.1	0.1	0.1	0.1	0.1
p_4	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.05	0.01
g_4	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_4	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
N_1	20	20	20	20	20	20	20	20	20	20
N_2	20	20	20	20	20	20	20	20	20	20
N_3	20	20	20	20	20	20	20	20	20	20

Table B.4: Machine and buffer parameters for ubiquitous inspection validation-continued

Case #	31	32	33	34	35	36	37	38	39	40
μ_1	1	1	1	1	1	1	1	2	1	1
r_1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_1	0.2	0.2	0.2	0.05	0.2	0.2	0.2	0.2	0.2	0.2
μ_2	1	1	1	1	1	1	1	1	2	1
r_2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g_2	0.01	0.05	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_2	0.2	0.2	0.2	0.2	0.05	0.2	0.2	0.2	0.2	0.2
μ_3	1	1	1	1	1	1	1	1	1	2
r_3	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_3	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g_3	0.01	0.01	0.05	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_3	0.2	0.2	0.2	0.2	0.2	0.05	0.2	0.2	0.2	0.2
μ_4	1	1	1	1	1	1	1	1	1	1
r_4	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_4	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g_4	0.01	0.01	0.01	0.05	0.01	0.01	0.01	0.01	0.01	0.01
f_4	0.2	0.2	0.2	0.2	0.2	0.2	0.05	0.2	0.2	0.2
N_1	20	20	20	20	20	20	20	20	20	20
N_2	20	20	20	20	20	20	20	20	20	20
N_3	20	20	20	20	20	20	20	20	20	20

Table B.5: Machine and buffer parameters for ubiquitous inspection validation-continued

Case #	41	42	43	44	45	46	47	48	49	50
μ_1	1	3	3	3	1	1	1	1	1	1
r_1	0.1	0.05	0.05	0.05	0.1	0.1	0.1	0.1	0.1	0.1
p_1	0.01	0.03	0.03	0.03	0.01	0.01	0.01	0.01	0.01	0.01
g_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_1	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
μ_2	1	3	1	1	3	3	1	1	1	1
r_2	0.1	0.05	0.1	0.1	0.05	0.05	0.1	0.1	0.1	0.1
p_2	0.01	0.03	0.01	0.01	0.03	0.03	0.01	0.01	0.01	0.01
g_2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
μ_3	1	1	3	1	3	1	3	1	1	1
r_3	0.1	0.1	0.05	0.1	0.05	0.1	0.05	0.1	0.1	0.1
p_3	0.01	0.01	0.03	0.01	0.03	0.01	0.03	0.01	0.01	0.01
g_3	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_3	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
μ_4	2	1	1	3	1	3	3	1	1	1
r_4	0.1	0.1	0.1	0.05	0.1	0.05	0.05	0.1	0.1	0.1
p_4	0.01	0.01	0.01	0.03	0.01	0.03	0.03	0.01	0.01	0.01
g_4	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_4	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
N_1	20	20	20	20	20	20	20	5	20	20
N_2	20	20	20	20	20	20	20	20	5	20
N_3	20	20	20	20	20	20	20	20	20	5

B.2 Extended quality information feedback case

Table B.6: Machine and buffer parameters for EQIF validation-continued

Case #	1	2	3	4	5	6	7	8	9	10
μ_1	1	1	1	1	1	1	1	1	1	1
r_1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_1	0.01	0.01	0.01	0.05	0.05	0.05	0.002	0.002	0.002	0.01
g_1	0.01	0.01	0.01	0.002	0.002	0.002	0.05	0.05	0.05	0.01
f_1	0.01	0.01	0.01	0.05	0.05	0.05	0.002	0.002	0.002	0.01
μ_2	1	1	1	1	1	1	1	1	1	1
r_2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_2	0.01	0.01	0.01	0.05	0.05	0.05	0.002	0.002	0.002	0.01
g_2	0	0	0	0	0	0	0	0	0	0
f_2	0.01	0.01	0.01	0.05	0.05	0.05	0.002	0.002	0.002	0.01
μ_3	1	1	1	1	1	1	1	1	1	1
r_3	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_3	0.01	0.01	0.01	0.05	0.05	0.05	0.002	0.002	0.002	0.01
g_3	0	0	0	0	0	0	0	0	0	0
f_3	0.01	0.01	0.01	0.05	0.05	0.05	0.002	0.002	0.002	0.01
μ_4	1	1	1	1	1	1	1	1	1	1
r_4	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_4	0.01	0.01	0.01	0.05	0.05	0.05	0.002	0.002	0.002	0.01
g_4	0	0	0	0	0	0	0	0	0	0
f_4	1	1	1	1	1	1	1	1	1	0.1
N_1	10	20	30	10	20	30	10	20	30	10
N_2	10	20	30	10	20	30	10	20	30	10
N_3	10	20	30	10	20	30	10	20	30	10

Table B.7: Machine and buffer parameters for EQIF validation-continued

Case #	11	12	13	14	15	16	17	18	19	20
μ_1	1	1	1	1	1	1	1	1	1	1
r_1	0.1	0.1	0.1	0.02	0.05	0.5	0.1	0.1	0.1	0.1
p_1	0.01	0.01	0.01	0.01	0.01	0.01	0.001	0.005	0.02	0.05
g_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_2	1	1	1	1	1	1	1	1	1	1
r_2	0.1	0.1	0.1	0.02	0.05	0.5	0.1	0.1	0.1	0.1
p_2	0.01	0.01	0.01	0.01	0.01	0.01	0.001	0.005	0.02	0.05
g_2	0	0	0	0	0	0	0	0	0	0
f_2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_3	1	1	1	1	1	1	1	1	1	1
r_3	0.1	0.1	0.1	0.02	0.05	0.5	0.1	0.1	0.1	0.1
p_3	0.01	0.01	0.01	0.01	0.01	0.01	0.001	0.005	0.02	0.05
g_3	0	0	0	0	0	0	0	0	0	0
f_3	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_4	1	1	1	1	1	1	1	1	1	1
r_4	0.1	0.1	0.1	0.02	0.05	0.5	0.1	0.1	0.1	0.1
p_4	0.01	0.01	0.01	0.01	0.01	0.01	0.001	0.005	0.02	0.05
g_4	0	0	0	0	0	0	0	0	0	0
f_4	0.5	0.1	0.5	1	1	1	1	1	1	1
N_1	10	20	20	20	20	20	20	20	20	20
N_2	10	20	20	20	20	20	20	20	20	20
N_3	10	20	20	20	20	20	20	20	20	20

Table B.8: Machine and buffer parameters for EQIF validation-continued

Case #	21	22	23	24	25	26	27	28	29	30
μ_1	1	1	1	1	1	1	1	1	1	1
r_1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g_1	0.001	0.005	0.02	0.05	0.01	0.01	0.01	0.01	0.01	0.01
f_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_2	1	1	1	1	1	1	1	1	1	1
r_2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g_2	0	0	0	0	0	0	0	0	0	0
f_2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_3	1	1	1	1	1	1	1	1	1	1
r_3	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_3	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g_3	0	0	0	0	0	0	0	0	0	0
f_3	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_4	1	1	1	1	1	1	1	1	1	1
r_4	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_4	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g_4	0	0	0	0	0	0	0	0	0	0
f_4	1	1	1	1	1	1	1	1	1	1
N_1	20	20	20	20	20	10	10	20	20	10
N_2	20	20	20	20	10	20	10	20	10	20
N_3	20	20	20	20	10	10	20	10	20	20

Table B.9: Machine and buffer parameters for EQIF validation-continued

Case #	31	32	33	34	35	36	37	38	39	40
μ_1	2	3	3	3	1	1	1	1	1	1
r_1	0.1	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.1	0.1
p_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.01
g_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_2	3	2	3	3	1	1	1	1	1	1
r_2	0.1	0.1	0.1	0.1	0.2	0.1	0.2	0.2	0.1	0.1
p_2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02
g_2	0	0	0	0	0	0	0	0	0	0
f_2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_3	3	3	2	3	1	1	1	1	1	1
r_3	0.1	0.1	0.1	0.1	0.2	0.2	0.1	0.2	0.1	0.1
p_3	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g_3	0	0	0	0	0	0	0	0	0	0
f_3	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_4	3	3	3	2	1	1	1	1	1	1
r_4	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.1	0.1	0.1
p_4	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g_4	0	0	0	0	0	0	0	0	0	0
f_4	1	1	1	1	1	1	1	1	1	1
N_1	20	20	20	20	20	20	20	20	20	20
N_2	20	20	20	20	20	20	20	20	20	20
N_3	20	20	20	20	20	20	20	20	20	20

Table B.10: Machine and buffer parameters for EQIF validation-continued

Case #	41	42	43	44	45	46	47	48	49	50
μ_1	1	1	3	2	2	2	3	3	2	2
r_1	0.1	0.1	0.05	0.2	0.2	0.2	0.05	0.05	0.2	0.2
p_1	0.01	0.01	0.03	0.01	0.01	0.01	0.03	0.03	0.01	0.01
g_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_1	0.01	0.01	0.03	0.01	0.01	0.01	0.03	0.03	0.01	0.01
μ_2	1	1	2	3	2	2	3	2	3	3
r_2	0.1	0.1	0.2	0.05	0.2	0.2	0.05	0.2	0.05	0.05
p_2	0.01	0.01	0.01	0.03	0.01	0.01	0.03	0.01	0.03	0.03
g_2	0	0	0	0	0	0	0	0	0	0
f_2	0.01	0.01	0.01	0.03	0.01	0.01	0.03	0.01	0.03	0.03
μ_3	1	1	2	2	3	2	2	3	2	3
r_3	0.1	0.1	0.2	0.2	0.05	0.2	0.2	0.05	0.2	0.05
p_3	0.02	0.01	0.01	0.01	0.03	0.01	0.01	0.03	0.01	0.03
g_3	0	0	0	0	0	0	0	0	0	0
f_3	0.01	0.01	0.01	0.01	0.03	0.01	0.01	0.03	0.01	0.03
μ_4	1	1	2	2	2	3	2	2	3	2
r_4	0.1	0.1	0.2	0.2	0.2	0.05	0.2	0.2	0.05	0.2
p_4	0.01	0.02	0.01	0.01	0.01	0.03	0.01	0.01	0.03	0.01
g_4	0	0	0	0	0	0	0	0	0	0
f_4	1	1	1	1	1	1	1	1	1	1
N_1	20	20	20	20	20	20	20	20	20	20
N_2	20	20	20	20	20	20	20	20	20	20
N_3	20	20	20	20	20	20	20	20	20	20

B.3 Multiple quality information feedback case

Table B.11: Machine and buffer parameters for MQIF validation

Case #	1	2	3	4	5	6	7	8	9	10
μ_1	1	1	1	1	1	1	1	1	1	1
r_1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_1	0.01	0.01	0.01	0.05	0.05	0.05	0.002	0.002	0.002	0.01
g_1	0.01	0.01	0.01	0.002	0.002	0.002	0.02	0.02	0.02	0.01
f_1	0.01	0.01	0.01	0.05	0.05	0.05	0.002	0.002	0.002	0.01
μ_2	1	1	1	1	1	1	1	1	1	1
r_2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_2	0.01	0.01	0.01	0.05	0.05	0.05	0.002	0.002	0.002	0.01
g_2	0.01	0.01	0.01	0.002	0.002	0.002	0.02	0.02	0.02	0.01
f_2	0.01	0.01	0.01	0.05	0.05	0.05	0.002	0.002	0.002	0.01
μ_3	1	1	1	1	1	1	1	1	1	1
r_3	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_3	0.01	0.01	0.01	0.05	0.05	0.05	0.002	0.002	0.002	0.01
g_3	0.01	0.01	0.01	0.002	0.002	0.002	0.02	0.02	0.02	0.01
f_3	0.01	0.01	0.01	0.05	0.05	0.05	0.002	0.002	0.002	0.01
μ_4	1	1	1	1	1	1	1	1	1	1
r_4	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_4	0.01	0.01	0.01	0.05	0.05	0.05	0.002	0.002	0.002	0.01
g_4	0.01	0.01	0.01	0.002	0.002	0.002	0.02	0.02	0.02	0.01
f_4	1	1	1	1	1	1	1	1	1	0.1
N_1	10	20	30	10	20	30	5	10	15	10
N_2	10	20	30	10	20	30	5	10	15	10
N_3	10	20	30	10	20	30	5	10	15	10

Table B.12: Machine and buffer parameters for MQIF validation - continued

Case #	11	12	13	14	15	16	17	18	19	20
μ_1	1	1	1	1	1	1	1	1	1	1
r_1	0.1	0.1	0.1	0.02	0.05	0.5	0.1	0.1	0.1	0.1
p_1	0.01	0.01	0.01	0.01	0.01	0.01	0.001	0.005	0.02	0.05
g_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_2	1	1	1	1	1	1	1	1	1	1
r_2	0.1	0.1	0.1	0.02	0.05	0.5	0.1	0.1	0.1	0.1
p_2	0.01	0.01	0.01	0.01	0.01	0.01	0.001	0.005	0.02	0.05
g_2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_3	1	1	1	1	1	1	1	1	1	1
r_3	0.1	0.1	0.1	0.02	0.05	0.5	0.1	0.1	0.1	0.1
p_3	0.01	0.01	0.01	0.01	0.01	0.01	0.001	0.005	0.02	0.05
g_3	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_3	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_4	1	1	1	1	1	1	1	1	1	1
r_4	0.1	0.1	0.1	0.02	0.05	0.5	0.1	0.1	0.1	0.1
p_4	0.01	0.01	0.01	0.01	0.01	0.01	0.001	0.005	0.02	0.05
g_4	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_4	0.5	0.1	0.5	1	1	1	1	1	1	1
N_1	10	20	20	10	10	10	10	10	10	10
N_2	10	20	20	10	10	10	10	10	10	10
N_3	10	20	20	10	10	10	10	10	10	10

Table B.13: Machine and buffer parameters for MQIF validation - continued

Case #	21	22	23	24	25	26	27	28	29	30
μ_1	1	1	1	1	1	1	1	1	1	1
r_1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g_1	0.001	0.005	0.02	0.05	0.01	0.01	0.01	0.01	0.01	0.01
f_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_2	1	1	1	1	1	1	1	1	1	1
r_2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g_2	0.001	0.005	0.02	0.05	0.01	0.01	0.01	0.01	0.01	0.01
f_2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_3	1	1	1	1	1	1	1	1	1	1
r_3	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_3	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g_3	0.001	0.005	0.02	0.05	0.01	0.01	0.01	0.01	0.01	0.01
f_3	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_4	1	1	1	1	1	1	1	1	1	1
r_4	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
p_4	0.01	0.005	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g_4	0.001	0.01	0.02	0.05	0.01	0.01	0.01	0.01	0.01	0.01
f_4	1	1	1	1	1	1	1	1	1	1
N_1	10	10	10	10	20	10	10	20	20	10
N_2	10	10	10	10	10	20	10	20	10	20
N_3	10	10	10	10	10	10	20	10	20	20

Table B.14: Machine and buffer parameters for MQIF validation - continued

Case #	31	32	33	34	35	36	37	38	39	40
μ_1	2	3	3	3	1	1	1	1	1	1
r_1	0.1	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.1	0.1
p_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.01
g_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_2	3	2	3	3	1	1	1	1	1	1
r_2	0.1	0.1	0.1	0.1	0.2	0.1	0.2	0.2	0.1	0.1
p_2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02
g_2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_3	3	3	2	3	1	1	1	1	1	1
r_3	0.1	0.1	0.1	0.1	0.2	0.2	0.1	0.2	0.1	0.1
p_3	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g_3	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_3	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_4	3	3	3	2	1	1	1	1	1	1
r_4	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.1	0.1	0.1
p_4	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
g_4	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_4	1	1	1	1	1	1	1	1	1	1
N_1	10	10	10	10	10	10	10	10	10	10
N_2	10	10	10	10	10	10	10	10	10	10
N_3	10	10	10	10	10	10	10	10	10	10

Table B.15: Machine and buffer parameters for MQIF validation - continued

Case #	41	42	43	44	45	46	47	48	49	50
μ_1	1	1	3	2	2	2	3	3	2	2
r_1	0.1	0.1	0.05	0.2	0.2	0.2	0.05	0.05	0.2	0.2
p_1	0.01	0.01	0.03	0.01	0.01	0.01	0.03	0.03	0.01	0.01
g_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_1	0.01	0.01	0.03	0.01	0.01	0.01	0.03	0.03	0.01	0.01
μ_2	1	1	2	3	2	2	3	2	3	3
r_2	0.1	0.1	0.2	0.05	0.2	0.2	0.05	0.2	0.05	0.05
p_2	0.01	0.01	0.01	0.03	0.01	0.01	0.03	0.01	0.03	0.03
g_2	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_2	0.01	0.01	0.01	0.03	0.01	0.01	0.03	0.01	0.03	0.03
μ_3	1	1	2	2	3	2	2	3	2	3
r_3	0.1	0.1	0.2	0.2	0.05	0.2	0.2	0.05	0.2	0.05
p_3	0.02	0.01	0.01	0.01	0.03	0.01	0.01	0.03	0.01	0.03
g_3	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_3	0.01	0.01	0.01	0.01	0.03	0.01	0.01	0.03	0.01	0.03
μ_4	1	1	2	2	2	3	2	2	3	2
r_4	0.1	0.1	0.2	0.2	0.2	0.05	0.2	0.2	0.05	0.2
p_4	0.01	0.02	0.01	0.01	0.01	0.03	0.01	0.01	0.03	0.01
g_4	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
f_4	1	1	1	1	1	1	1	1	1	1
N_1	10	10	10	10	10	10	10	10	10	10
N_2	10	10	10	10	10	10	10	10	10	10
N_3	10	10	10	10	10	10	10	10	10	10

Appendix C

Matrix manipulation technique

Matrix A in equations (3.165), (3.166), and (3.167) contains elements which can be different by several orders of magnitude (e.g., $e^{\lambda_i N} Z_i^j Y_i^j$ and Y_i^j). When $\lambda > 0$ and N is large, this may cause the reduction of the apparent rank of matrix A, which will lead to errors. To prevent this, following technique is used for solving equations (3.165), (3.166), and (3.167).

- Build a new matrix A' by doing:
  - For each column i in matrix A, check if the column contains $e^{\lambda_i N}$.
  - For the column that contains $e^{\lambda_i N}$ terms, divide all the elements in the column by $e^{\lambda_i N}$ if $\lambda_i > 0$.
- Calculate X' which is a solution of $A'X' = B$ using numerical methods.
- Divide the elements in X' by $e^{\lambda_i N}$ if corresponding column in A is modified. This gives X which is the solution of the original equation $AX = B$.