Low energy effective actions and tachyon dynamics from String Field Theory

by

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Abstract

In this thesis we show how to calculate off-shell low energy effective actions and how to study the dynamics of the tachyon from string field theory. We discuss how to obtain an effective action for the massless field and we explain how to relate it to well known results. We then study the tachyon dynamics both in cubic and in boundary string field theory.

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Chapter 1

Introduction

While the standard model has been very successful in describing most of the results of the experiments that we observe with the current generation of particle accelerators, it doesn't give us a "unified" description of the universe and a microscopic description of gravity. There is a strong evidence that the couplings present in the standard model, electromagnetic, nuclear weak and nuclear strong, should all unify at an energy of about 10^{16} Gev. It is also thought that the gravitational force becomes equally important at an energy of about 10^{19} Gev. String theory should give an explanation of what happens at energies above the unification scale of the standard model and it should be considered as the best candidate for a microscopic theory of gravity. Moreover, even if string theory turns out to be wrong, it is still the only consistent model of quantum gravity, and its study has to be considered useful if we want to understand new features of gravity at a microscopic level.

Given that, since the mid-1990's people have been considering string theory as the best candidate for a theory unifying the four fundamental forces existing in nature. A big virtue of the theory is that all its formulations are related by duality transformations and are thought to be just different vacua of the same theory, M-theory. However, the theory has the big disadvantage of having an enormous number of vacua. The question that arises is: "How do we pick up the correct vacuum"? We may think to find a string vacuum that corresponds to the standard model vacuum, but what about the huge number of other vacua? It seems that a completely background independent formulation of the theory is needed to answer this question.

Since the early days people thought that string field theory (SFT) was the best way to formulate a background independent string theory. According to that theory the string's wave function has to be quantized and a Lagrangian whose Feynman rules reproduce the scattering amplitudes of the first quantized theory has to be constructed. In the 1980's people worked a lot to formulate such a theory using the BRST approach [1]-[4]. In 1984 Witten [5] constructed for the open string a covariant cubic SFT based on the Chern-Simons action. While Witten's Open String Field Theory (OSFT) is described in a simple abstract language, the subject is very complicated and practical computations are not easily performed. Moreover, the supersymmetric version of the theory [6]-[8] was found to have problems from picture-changing operators and from having a nonpolynomial action. For these reasons OSFT was restricted to the bosonic case, which however suffers from having a non stable perturbative vacuum. Despite the fact that there was a large amount of work done in SFT in the early 90's, people couldn't get a satisfactory insight in to non-perturbative physics and started to lose interest in the subject.

Research on the subject stalled until 1999, when Sen [9] realized that OSFT could be used to understand the decay of unstable D-branes[1] . The 26-dimensional bosonic string has, both in the open and the closed case, a tachyon in its spectrum indicating that the ususal perturbative vacuum of these theories is unstable. Sen pointed out that the system in the open string case is unstable and will decay into the true vacuum[2]. Sen made the three following conjectures:

1. The effective potential for the tachyon mode has a minimum, and the difference in energy between the perturbative vacuum and the minimum of the potential cancels the mass of the 25-dimensional space filling D-brane.

2. In the minimum of the potential there are no open string excitations. This is

[1]Strings can have various kinds of boundary conditions. For example closed strings have periodic boundary conditions (the string comes back onto itself). Open strings can have two different kinds of boundary conditions called Neumann and Dirichlet boundary conditions. With Dirichlet boundary conditions the endpoint is fixed to move only on some manifold and this manifold is called a D-brane.

[2]The same thing is happening in the standard model where at the perturbative vacuum we have a negative mass-squared field. This just tells us that the stable vacuum is at the minimum of the potential involving that field.

so because we expect that at the true vacuum the brane has decayed and only closed string excitations are present.

3. There should be lump solutions of the tachyon potential which correspond to lower dimensional branes.

OSFT turns out to be a good laboratory for checking these conjectures and we clearly see a new nonperturbative application of SFT. Already in 1987 Kostelecky and Samuel [10] observed that the tachyon effective potential has a minimum, and that, like in the Standard Model, the true vacuum is not the naive perturbative vacuum. However, at that time people were not thinking in terms of branes and their work wasn't that much considered.

In their paper Kostelecky and Samuel introduced the notion of *level truncation* to calculate the effective action at zero momenta (effective potential) for the tachyon. They truncate the cubic OSFT action by setting to zero all the fields that have a mass greater than some value, some cutoff to which we can refer as the *level*. In this way there are only a finite number of fields and it becomes possible to calculate the effective action. We can think about the calculated effective action as a function of the level, and usually results converge quite fast to their asymptotic values. In chapter 2 we use level truncation to calculate the effective action for the massless field present in cubic OSFT, the gauge field, and we get the first few terms in the expansion of the Born-Infeld action[3] in powers of the field strength tensor plus (covariant) derivative corrections.

While SFT involves an infinite number of space-time fields, most of these fields have masses of the order of the Planck scale and it makes sense to see what is the effective theory which has only the low energy modes, massless modes, as degrees of freedom. By integrating out the massive fields, we arrive at an effective action for

[3]The Born-Infeld action was first obtained in string theory in 1985 by Fradkin and Tseytlin [11] when they were calculating the low energy effective action for the abelian massless vector field using string's world-sheet methods. The lagrangian looks like:

$$\sqrt{\det(\delta_{\mu\nu} + F_{\mu\nu})}, \tag{1.1}$$

where $F_{\mu\nu}$ is the usual field strength tensor. In the abelian case derivative corrections to this action are well defined and they consist of derivative of powers of the field strength tensor.

a finite number of massless fields. In the case of the open string this leads to an action for the massless gauge field that we compute term-by-term using the level-truncation approximation in SFT. It is natural to expect that the effective action we compute for the massless vector field will take the form of the Born-Infeld action, including higher-derivative terms. Indeed, we show that this is the case, although some care must be taken in making this connection. In fact, this action has a gauge invariance which agrees with the usual Yang-Mills gauge invariance to leading order, but which has higher-order corrections arising from the string field star product[4]. A field redefinition analogous to the Seiberg-Witten map[5] [12, 13] is necessary to get a field which transforms in the usual fashion [14, 15]. Early work deriving the Born-Infeld action from string theory used world-sheet calculations that were on-shell calculations [11, 16, 17, 18]. In our work we start with cubic OSFT, which is a manifestly off-shell formalism. The resulting effective action is therefore also an off-shell action.

Background independent OSFT [19]-[22] is the other main approach used to study off-shell phenomena in string theory. Background independent OSFT has been useful for finding the classical tachyon potential energy functional and the leading derivative terms in the tachyon effective action [23]-[25]. It is formulated as a problem in boundary conformal field theory. One begins with the tree-level partition function of open-string theory where the two-dimensional world-sheet swept out by the string has the topology of a disk. An interaction in the boundary of the string's world-sheet with arbitrary operators is added. If only the tachyon field is added then the resulting 2-dimensional field theory is super-renormalizable by power counting. Renormalization fixed points, which correspond to conformal field theories, are solutions of classical equations of motion and should be viewed as the solutions of classical string field

[4]As we will discuss later, in OSFT an associative product between string field is defined to construct the action. This action posses a complicated nonlinear gauge symmetry constructed with this star product. Treating all the massive fields classically, i.e. solving their equation of motion in terms of the gauge fields, we arrive at a complicated nonlinear gauge transformation for the vector field.

[5]Seiberg and Witten have argued that certain noncommutative gauge theories are equivalent to commutative ones and that there exists a map between the two. The name of the map is Seiberg-Witten map.

theory. Witten and Shatashvili [19, 21] have argued that these equations of motion come from an action which can be derived from the disk partition function Z. More precisely, the effective action for a generic coupling constant g^i (which can be identified with the tachyon, the gauge or any other field that correspond to excitations of the open bosonic string) is related to the renormalized partition function of open string theory on the disk, $Z(g^i)$, through

$$S = \left(1 - \beta^i \frac{\delta}{\delta g^i}\right) Z(g^i) \ , \tag{1.2}$$

where β^i is the beta-function of the coupling g^i.

In chapter 3 we compute the non-linear β-function for the tachyon field up to the third order in powers of the field and to any order in derivatives of the field. From this we show that the solutions of the RG fixed point equations on the 2-dimensional world-sheet swept-out by the string generate the three and four-point open bosonic string scattering amplitudes involving tachyons and we construct the effective action for the tachyon field. Then, with the same renormalization prescription, we compute β to the leading orders in derivatives but to any power of the tachyon field and we show that the corresponding action coincides with the one found in [23]-[25]. Knowledge of the non-linear tachyon β-function is very important also for another reason. The solutions of the equation $\beta^T = 0$, T superscript stands for tachyon, give the conformal fixed points, the backgrounds that are consistent with the string dynamics. In the case of slowly varying tachyon profiles, we show that the equations of motion for the Witten-Shatashvili action, WS, can be made identical to the RG fixed point equation $\beta^T = 0$.

An important aspect of the open string tachyon which is not yet fully understood is the dynamical process through which the tachyon rolls from the unstable vacuum to the stable vacuum. A review of previous work on this problem is given in [26]. Computations using background states, RG flow analysis [27], and background string field theory (BSFT) [19]- [22] show that the tachyon should monotonically roll toward the true vacuum, but should not arrive at the true vacuum in finite time [28]-[35].

In BSFT variables, where the tachyon T goes to $T \to \infty$ in the stable vacuum, the time-dependence of the tachyon field goes as $T(t) = e^t$. This dynamics is intuitively fairly transparent[6], and follows from the fact that e^t is a marginal boundary operator [36, 37, 38, 28, 35]. Other approaches to understanding the rolling tachyon from a variety of viewpoints including DBI-type actions [39]-[42], S-branes and timelike Liouville theory [43]-[47], matrix models [48]-[53], and fermionic boundary CFT [54] lead to a similar picture of the time dynamics of the tachyon.

In CSFT, on the other hand, the rolling tachyon dynamics appears much more complicated. In [55], Moeller and Zwiebach used level truncation to analyze the time dependence of the tachyon. They found that at low levels of truncation, the tachyon rolls well past the minimum of the potential, then turns around and begins to oscillate. It was further argued by Fujita and Hata in [56] that such oscillations are a natural consequence of the form of the CSFT equations of motion, which include an exponential of time derivatives acting on the tachyon field.

These two apparently completely different pictures of the tachyon dynamics raise an obvious puzzle. Which picture is correct? Does the tachyon converge monotonically to the true vacuum, or does it undergo wild oscillations? Is there a problem with the BSFT approach? Does the CSFT analysis break down for some reason such as a branch point singularity at a finite value of t? Does the dynamics in CSFT behave better when higher-level states are included? Is CSFT an incomplete formulation of the theory?

In chapter 4 we resolve this puzzle. We carry out a systematic level-truncation analysis of the tachyon dynamics for a particular solution in CSFT. We compute the trajectory as a power series in e^t at various levels of truncation. We show that indeed the dynamics in CSFT has wild oscillations. We find, however, that the trajectory is well-defined in the sense that increasing the level of truncation in CSFT and the number of terms retained in the power series in e^t rapidly leads to a convergent tachyon trajectory in any fixed range of t. We reconcile this apparent discrepancy

[6]The tachyon potential in the open bosonic string case is $e^{-T}(1+T)$, and it has a minimum at $T = \infty$.

with the results of BSFT by demonstrating that a field redefinition which takes the CSFT action to the BSFT action also maps the wildly oscillating CSFT solution to the well-behaved BSFT exponential solution.

Chapter 2

Effective action for the massless field from cubic open string field theory

In this chapter, we discuss our work with I. Sigalov and W. Taylor [57]. We compute the leading terms in the tree-level effective action for the massless fields of the bosonic open string by integrating out all massive fields. In both the abelian and nonabelian theories, field redefinitions make it possible to express the effective action in terms of the conventional field strength.

2.1 Introduction

An important feature of SFT, which allows it to transcend the usual limitations of local quantum field theories, is its essential nonlocality. SFT is a theory which can be defined with reference to a particular background in terms of an infinite number of space-time fields, with highly nonlocal interactions. The nonlocality of SFT is similar in spirit to that of noncommutative field theories which have been the subject of much recent work [58], but in SFT the nonlocality is much more extreme. In order to understand how string theory encodes a quantum theory of gravity at short distance scales, where geometry becomes poorly defined, it is clearly essential to achieve a better understanding of the nonlocal features of string theory.

Integrating out the massive fields present in OSFT, we obtain an effective action for the massless fields. In the case of a closed string field theory, performing such an integration would give an effective action for the usual multiplet of gravity/supergravity fields. This action will, however, have a complicated nonlocal structure which will appear through an infinite family of higher-derivative terms in the effective action. In the case of the open string, integrating out the massive fields leads to an action for the massless gauge field. Again, this action is highly nonlocal and contains an infinite number of higher-derivative terms. This nonlocal action for the massless gauge field in the bosonic open string theory is the subject of this paper. By explicitly integrating out all massive fields in Witten's open string field theory (including the tachyon), we arrive at an effective action for the massless open string vector field. We compute this effective action term-by-term using the level-truncation approximation in string field theory, which gives us a very accurate approximation to each term in the action.

We expect that the effective action we compute for the massless vector field will take the form of the Born-Infeld action, including higher-derivative terms. Indeed, we show that this is the case, although some care must be taken in making this connection. Early work deriving the Born-Infeld action from string theory [11, 16] used world-sheet methods [17, 18]. More recently, in the context of the supersymmetric nonabelian gauge field action, other approaches, such as κ-symmetry and the existence of supersymmetric solutions, have been used to constrain the form of the action (see [59] for a recent discussion and further references). In this work we take a different approach. We start with string field theory, which is a manifestly off-shell formalism. Our resulting effective action is therefore also an off-shell action. This action has a gauge invariance which agrees with the usual Yang-Mills gauge invariance to leading order, but which has higher-order corrections arising from the string field star product. A field redefinition analogous to the Seiberg-Witten map [12, 13] is necessary to get a field which transforms in the usual fashion [14, 15]. We identify the leading terms in this transformation and show that after performing the field redefinition our action indeed takes the Born-Infeld form in the abelian theory. In the nonabelian theory, there is an additional subtlety, which was previously encountered in related contexts in [14, 15]. Extra terms appear in the form of the gauge transformation which cannot be removed by a field redefinition. These additional terms, however, are trivial and can be dropped, after which the standard form of gauge invariance can be restored by a field redefinition. This leads to an effective action in the nonabelian theory which takes the form of the nonabelian Born-Infeld action plus derivative correction terms.

It may seem surprising that we integrate out the tachyon as well as the fields in the theory with positive mass squared. This is, however, what is implicitly done in previous work such as [11, 16] where the Born-Infeld action is derived from bosonic string theory. The abelian Born-Infeld action can similarly be derived from recent proposals for the coupled tachyon-vector field action [60, 61, 62, 63] by solving the equation of motion for the tachyon at the top of the hill. In the supersymmetric theory, of course, there is no tachyon on a BPS brane, so the supersymmetric Born-Infeld action should be derivable from a supersymmetric open string field theory by only integrating out massive fields. Physically, integrating out the tachyon corresponds to considering fluctuations of the D-brane in stable directions, while the tachyon stays balanced at the top of its potential hill. While open string loops may give rise to problems in the effective theory [64], at the classical level the resulting action is well-defined and provides us with an interesting model in which to understand the nonlocality of the Born-Infeld action. The classical effective action we derive here must reproduce all on-shell tree-level scattering amplitudes of massless vector fields in bosonic open string theory. To find a sensible action which includes quantum corrections, it is probably necessary to consider the analogue of the calculation in this paper in the supersymmetric theory, where there is no closed string tachyon.

The structure of this chapter is as follows: In Section 2.2 we review the formalism of string field theory, set notation and make some brief comments regarding the Born-Infeld action. In Section 2.3 we introduce the tools needed to calculate terms in the effective action of the massless fields. Section 2.4 contains a calculation of the effective action for all terms in the Yang-Mills action. Section 2.5 extends the analysis to include the next terms in the Born-Infeld action in the abelian case and Section 2.6 does the same for the nonabelian analogue of the Born-Infeld action. Section 2.7 contains concluding remarks. Some useful properties of the Neumann matrices appearing in the 3-string vertex of Witten's string field theory are included in the Appendix.

2.2 Review of formalism

Subsection 2.2.1 summarizes our notation and the basics of string field theory. In subsection 2.2.2 we review the method of [65] for computing terms in the effective action. The last subsection, 2.2.3, contains a brief discussion of the Born-Infeld action.

2.2.1 Basics of string field theory

In this subsection we review the basics of Witten's open string field theory [5]. For further background information see the reviews [66, 67, 68, 69]. The degrees of freedom of string field theory (SFT) are functionals $\Phi[x(\sigma); c(\sigma), b(\sigma)]$ of the string configuration $x^\mu(\sigma)$ and the ghost and antighost fields $c(\sigma)$ and $b(\sigma)$ on the string at a fixed time. String functionals can be expressed in terms of string Fock space states, just as functions in $\mathcal{L}^2(\mathbb{R})$ can be expressed as linear combinations of harmonic oscillator eigenstates. The Fock module of a single string of momentum p is obtained by the action of the matter, ghost and antighost oscillators on the (ghost number one) highest weight vector $|p\rangle$. The action of the raising and lowering oscillators on $|p\rangle$ is defined by the creation/annihilation conditions and commutation relations

$$\begin{aligned}
a^\mu_{n\geq 1}\,|p\rangle &= 0, & [a^\mu_m, a^\nu_{-n}] &= \eta^{\mu\nu}\delta_{m,n}, \\
p^\mu\,|k\rangle &= k^\mu\,|k\rangle\,, & & \\
b_{n\geq 0}\,|p\rangle &= 0, & \{b_m, c_{-n}\} &= \delta_{m,n}, \\
c_{n\geq 1}\,|p\rangle &= 0. & &
\end{aligned} \tag{2.1}$$

Hermitian conjugation is defined by $a_n^{\mu\dagger} = a_{-n}^{\mu}$, $b_n^{\dagger} = b_{-n}$, $c_n^{\dagger} = c_{-n}$. The single-string Fock space is then spanned by the set of all vectors $|\chi\rangle = \cdots a_{n_2} a_{n_1} \cdots b_{k_2} b_{k_1} \cdots c_{l_2} c_{l_1} |p\rangle$ with $n_i, k_i < 0$ and $l_i \leq 0$. String fields of ghost number 1 can be expressed as linear combinations of such states $|\chi\rangle$ with equal number of b's and c's, integrated over momentum.

$$|\Phi\rangle = \int d^{26}p \; \left(\phi(p) + A_\mu(p)\, a_{-1}^{\mu} - i\alpha(p) b_{-1} c_0 + B_{\mu\nu}(p) a_{-1}^{\mu} a_{-1}^{\nu} + \cdots\right) |p\rangle \,. \quad (2.2)$$

The Fock space vacuum $|0\rangle$ that we use is related to the $SL(2,\mathbb{R})$ invariant vacuum $|1\rangle$ by $|0\rangle = c_1 |1\rangle$. Note that $|0\rangle$ is a Grassmann odd object, so that we should change the sign of our expression whenever we interchange $|0\rangle$ with a Grassmann odd variable. The bilinear inner product between the states in the Fock space is defined by the commutation relations and

$$\langle k|\, c_0\, |p\rangle = (2\pi)^{26} \delta(k+p). \quad (2.3)$$

The SFT action can be written as

$$S = -\frac{1}{2} \langle V_2|\, \Phi, Q_B \Phi\rangle - \frac{g}{3} \langle V_3|\, \Phi, \Phi, \Phi\rangle \quad (2.4)$$

where $|V_n\rangle \in \mathcal{H}^n$. This action is invariant under the gauge transformation

$$\delta\, |\Phi\rangle = Q_B\, |\Lambda\rangle + g\big(\langle \Phi, \Lambda|\, V_3\rangle - \langle \Lambda, \Phi|\, V_3\rangle\big) \quad (2.5)$$

with Λ a string field gauge parameter at ghost number 0. Explicit oscillator representations of $\langle V_2|$ and $\langle V_3|$ are given by [70, 71, 72, 73]

$$\langle V_2| = \int d^{26}p \; \langle p|^{(1)} \otimes \langle -p|^{(2)} \left(c_0^{(1)} + c_0^{(2)}\right) \exp\left(a^{(1)} \cdot C \cdot a^{(2)} - b^{(1)} \cdot C \cdot c^{(2)} - b^{(1)} \cdot C \cdot c^{(2)}\right) \quad (2.6)$$

and

$$\langle V_3| = \mathcal{N} \int \prod_{i=1}^{3} \left(d^{26} p_i \langle p|^{(i)} c_0^{(i)} \right) \delta(\sum p_j)$$
$$\times \exp\left(\frac{1}{2} a^{(r)} \cdot V^{rs} \cdot a^{(s)} - p^{(r)} V_{0\cdot}^{rs} \cdot a^{(s)} + \frac{1}{2} p^{(r)} V_{00}^{rr} p^{(r)} - b^{(r)} \cdot X^{rs} \cdot c^{(s)} \right) \quad (2.7)$$

where all inner products denoted by $\cdot$ indicate summation from 1 to ∞ except in $b \cdot X$, where the summation includes the index 0. The contracted Lorentz indices in a_n^μ and p_μ are omitted. $C_{mn} = (-1)^n \delta_{mn}$ is the BPZ conjugation matrix. The matrix elements V_{mn}^{rs} and X_{mn}^{rs} are called Neumann coefficients. Explicit expressions for the Neumann coefficients and some relevant properties of these coefficients are summarized in the Appendix. The normalization constant $\mathcal{N}$ is defined by

$$\mathcal{N} = \exp(-\frac{1}{2} \sum_r V_{00}^{rr}) = \frac{3^{9/2}}{2^6}, \quad (2.8)$$

so that the on-shell three-tachyon amplitude is given by $2g$. We use units where $\alpha' = 1$.

2.2.2 Calculation of effective action

String field theory can be thought of as a (nonlocal) field theory of the infinite number of fields that appear as coefficients in the oscillator expansion (2.2). In this paper, we are interested in integrating out all massive fields at tree level. This can be done using standard perturbative field theory methods. Recently an efficient method of performing sums over intermediate particles in Feynman graphs was proposed in [65]. We briefly review this approach here; an alternative approach to such computations has been studied recently in [74].

In this paper, while we include the massless auxiliary field α appearing in the expansion (2.2) as an external state in Feynman diagrams, all the massive fields we integrate out are contained in the Feynman-Siegel gauge string field satisfying

$$b_0 \left|\Phi\right\rangle = 0, \tag{2.9}$$

This means that intermediate states in the tree diagrams we consider do not have a c_0 in their oscillator expansion. For such states, the propagator can be written in terms of a Schwinger parameter τ as

$$\frac{b_0}{L_0} = b_0 \int_0^\infty d\tau\; e^{-\tau L_0}, \tag{2.10}$$

In string field theory, the Schwinger parameters can be interpreted as moduli for the Riemann surface associated with a given diagram [68, 75, 76, 77, 78].

In field theory one computes amplitudes by contracting vertices with external states and propagators. Using the quadratic and cubic vertices (2.6), (2.7) and the propagator (2.10) we can do same in string field theory. To write down the contribution to the effective action arising from a particular Feynman graph we include a vertex $\langle V_3| \in \mathcal{H}^{*3}$ for each vertex of the graph and a vertex $|V_2\rangle$ for each internal edge. The propagator (2.10) can be incorporated into the quadratic vertex through[1]

$$|P\rangle = -\int_0^\infty d\tau\; e^{\tau(1-p^2)} \left|\tilde{V}_2\right\rangle. \tag{2.13}$$

where in the modified vertex $\left|\tilde{V}_2(\tau)\right\rangle$ the ghost zero modes c_0 are canceled by the b_0

[1]Consider the tachyon propagator as an example. We contract $c_0\,|p_1\rangle$ and $c_0\,|p_2\rangle$ with $\langle P|$ to get

$$\langle P|\, c_0\,|p_1\rangle\, c_0\,|p_2\rangle = -\int_0^\infty d\tau e^{\tau(1-p_1^2)}\delta(p_1+p_2) = -\frac{\delta(p_1+p_2)}{p_1^2-1}. \tag{2.11}$$

This formula assumes that both momenta are incoming. Setting $p_1 = -p_2 = p$ and using the metric with $(-,+,+,...,+)$ signature we have

$$-\frac{1}{p^2+m^2} = \frac{1}{p_0^2-\vec{p}^2-m^2} \tag{2.12}$$

thus (2.11) is indeed the correct propagator for the scalar particle of mass $m^2 = -1$.

in (2.10) and the matrix C_{mn} is replaced by

$$\tilde{C}_{mn}(\tau) = e^{-m\tau}(-1)^m \delta_{mn} \,. \tag{2.14}$$

With these conventions, any term in the effective action can be computed by contracting the three-vertices from the corresponding Feynman diagram on the left with factors of $|P\rangle$ and low-energy fields on the right (or vice-versa, with $|V_3\rangle$'s on the right and $\langle P|$'s on the left). Because the resulting expression integrates out *all* Feynman-Siegel gauge fields along interior edges, we must remove the contribution from the intermediate massless vector field by hand when we are computing the effective action for the massless fields. Note that in [65], a slightly different method was used from that just described; there the propagator was incorporated into the three-vertex rather than the two-vertex. Both methods are equivalent; we use the method just described for convenience.

States of the form

$$\exp\left(\lambda \cdot a^\dagger + \frac{1}{2} a^\dagger \cdot S \cdot a^\dagger\right) |p\rangle \tag{2.15}$$

are called squeezed states. The vertex $|V_3\rangle$ and the propagator $|P\rangle$ are (linear combinations of) squeezed states and thus are readily amenable to computations. The inner product of two squeezed states is given by [79]

$$\begin{aligned}
\langle 0| \exp(\lambda \cdot a + \frac{1}{2} a \cdot S \cdot a) \exp(\mu \cdot a^\dagger + \frac{1}{2} a^\dagger \cdot V \cdot a^\dagger) |0\rangle \\
= \mathrm{Det}(1 - S \cdot V)^{-1/2} \exp\big[\lambda \cdot (1 - V \cdot S)^{-1} \cdot \mu \\
+ \frac{1}{2} \lambda \cdot (1 - V \cdot S)^{-1} \cdot V \cdot \lambda + \frac{1}{2} \mu \cdot S \cdot (1 - V \cdot S)^{-1} \cdot \mu\big]
\end{aligned} \tag{2.16}$$

and (neglecting ghost zero-modes)

$$\begin{aligned}
\langle 0| \exp(b \cdot \lambda_b - \lambda_c \cdot c - b \cdot S \cdot c) \exp(b^\dagger \cdot \mu_b + \mu_c \cdot c^\dagger + b^\dagger \cdot V \cdot c^\dagger) |0\rangle \\
= \mathrm{Det}(1 - S \cdot V) \exp\big[-\lambda_c \cdot (1 - V \cdot S)^{-1} \cdot \mu_b - \mu_c \cdot (1 - S \cdot V)^{-1} \cdot \lambda_b \\
+ \lambda_c \cdot (1 - V \cdot S)^{-1} \cdot V \cdot \lambda_b + \mu_c \cdot S \cdot (1 - V \cdot S)^{-1} \cdot \mu_b\big].
\end{aligned} \tag{2.17}$$

Using these expressions, the combination of three-vertices and propagators associated with any Feynman diagram can be simply rewritten as an integral over modular (Schwinger) parameters of a closed form expression in terms of the infinite matrices $V_{nm}, X_{nm}, \tilde{C}_{nm}(\tau)$. The schematic form of these integrals is

$$
\begin{aligned}
(\langle V_3|)^v(|P\rangle)^i \sim & \left(\prod_{j=1}^{i} \int d\tau^j\right) \frac{\mathrm{Det}(1-\hat{C}\hat{X})}{\mathrm{Det}(1-\hat{C}\hat{V})^{13}} \\
& \times (\langle 0|)^{3v-2i} \exp\left(\frac{1}{2} a^\dagger \cdot S \cdot a^\dagger + \mu \cdot a^\dagger + b^\dagger \cdot U \cdot c^\dagger + \mu_c \cdot c^\dagger + b^\dagger \cdot \mu_b\right) \qquad (2.18)
\end{aligned}
$$

where $\hat{C}, \hat{X}, \hat{V}$ are matrices with blocks of the form $\tilde{C}, X, V$ arranged according to the combinatorial structure of the diagram. The matrix $\hat{C}$ and the squeezed state coefficients S, U, μ, μ_b, μ_c depend implicitly on the modular parameters τ^i.

2.2.3 The effective vector field action and Born-Infeld

In this subsection we describe how the effective action for the vector field is determined from SFT and we discuss the Born-Infeld action [80] which describes the leading terms in this effective action. For a more detailed review of the Born-Infeld action, see [81]

As discussed in subsection 2.1, the string field theory action is a space-time action for an infinite set of fields, including the massless fields $A_\mu(x)$ and $\alpha(x)$. This action has a very large gauge symmetry, given by (2.5). We wish to compute an effective action for $A_\mu(x)$ which has a single gauge invariance, corresponding at leading order to the usual Yang-Mills gauge invariance. We compute this effective action in several steps. First, we use Feynman-Siegel gauge (2.9) for all *massive* fields in the theory. This leaves a single gauge invariance, under which A_μ and α have linear components in their gauge transformation rules. This partial gauge fixing is described more precisely in section 2.5.2. Following this partial gauge fixing, all massive fields in the theory, including the tachyon, can be integrated out using the method described in the previous subsection, giving an effective action

$$\check{S}[A_\mu(x), \alpha(x)] \qquad (2.19)$$

depending on A_μ and α. We can then further integrate out the field α, which has no kinetic term, to derive the desired effective action

$$S[A_\mu(x)]\,. \tag{2.20}$$

The action (2.20) still has a gauge invariance, which at leading order agrees with the Yang-Mills gauge invariance

$$\delta A_\mu(x) = \partial_\mu \lambda(x) - i g_{\rm YM}[A_\mu(x), \lambda(x)] + \cdots \tag{2.21}$$

The problem of computing the effective action for the massless gauge field in open string theory is an old problem, and has been addressed in many other ways in past literature. Most methods used in the past for calculating the effective vector field action have used world-sheet methods. While the string field theory approach we use here has the advantage that it is a completely off-shell formalism, as just discussed the resulting action has a nonstandard gauge invariance [15]. In world-sheet approaches to this computation, the vector field has the standard gauge transformation rule (2.21) with no further corrections. A general theorem [82] states that there are no deformations of the Yang-Mills gauge invariance which cannot be taken to the usual Yang-Mills gauge invariance by a field redefinition. In accord with this theorem, we identify in this paper field redefinitions which take the massless vector field A_μ in the SFT effective action (2.20) to a gauge field $\hat{A}_\mu$ with the usual gauge invariance. We write the resulting action as

$$\hat{S}[\hat{A}_\mu(x)]\,. \tag{2.22}$$

This action, written in terms of a conventional gauge field, can be compared to previous results on the effective action for the open string massless vector field.

Because the mass-shell condition for the vector field $A_\mu(p)$ in Fourier space is $p^2 = 0$, we can perform a sensible expansion of the action (2.20) as a double expansion in p and A. We write this expansion as

$$S[A_\mu] = \sum_{n=2}^{\infty} \sum_{k=0}^{\infty} S^{[k]}_{A^n} \tag{2.23}$$

where $S^{[k]}_{A^n}$ contains the contribution from all terms of the form $\partial^k A^n$. A similar expansion can be done for $\hat{S}$, and we similarly denote by $\check{S}^{[k]}_{\alpha^m A^n}$ the sum of the terms in $\check{S}$ of the form $\partial^k \alpha^m A^n$.

Because the action $\hat{S}[\hat{A}]$ is a function of a gauge field with conventional gauge transformation rules, this action can be written in a gauge invariant fashion; *i.e.* in terms of the gauge covariant derivative $\hat{D}_\mu = \partial_\mu - ig_{\mathrm{YM}}[\hat{A}, \cdot]$ and the field strength $\hat{F}_{\mu\nu}$. For the abelian theory, $\hat{D}_\mu$ is just ∂_μ, and there is a natural double expansion of $\hat{S}$ in terms of p and F. It was shown in [11, 16] that in the abelian theory the set of terms in $\hat{S}$ which depend only on $\hat{F}$, with no additional factors of p (*i.e.*, the terms in $\hat{S}^{[n]}_{\hat{A}^n}$) take the Born-Infeld form (dropping hats)

$$S_{BI} = -\frac{1}{(2\pi g_{YM})^2} \int dx \sqrt{-\det\left(\eta_{\mu\nu} + 2\pi g_{YM} F_{\mu\nu}\right)} \tag{2.24}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.25}$$

is the gauge-invariant field strength. Using $\log(\det M) = \mathrm{tr}\,(\log(M))$ we can expand in F to get

$$\begin{aligned} S_{BI} = -\frac{1}{(2\pi g_{YM})^2} \int dx \Bigg(1 &+ \frac{(2\pi g_{YM})^2}{4} F_{\mu\nu} F^{\mu\nu} \\ &- \frac{(2\pi g_{YM})^4}{8} \left(F_{\mu\nu} F^{\nu}{}_{\lambda} F^{\lambda}{}_{\sigma} F^{\sigma\mu} - \frac{1}{4} (F_{\mu\nu} F^{\mu\nu})^2 \right) + \cdots \Bigg). \end{aligned} \tag{2.26}$$

We expect that after the appropriate field redefinition, the result we calculate from string field theory for the effective vector field action (2.20) should contain as a leading part at each power of $\hat{A}$ terms of the form (2.26), as well as higher-derivative terms of the form $\partial^{n+k} A^n$ with $k > 0$. We show in section 5 that this is indeed the case.

The nonabelian theory is more complicated. In the nonabelian theory we must include covariant derivatives, whose commutators mix with field strengths through relations such as

$$[D_\mu, D_\nu]F_{\lambda\sigma} = [F_{\mu\nu}, F_{\lambda\sigma}]\,. \tag{2.27}$$

In this case, there is no systematic double expansion in powers of D and F. It was pointed out by Tseytlin in [83] that when F is taken to be constant, and both commutators $[F, F]$ and covariant derivatives of field strengths DF are taken to be negligible, the nonabelian structure of the theory is irrelevant. In this case, the action reduces to the Born-Infeld form (2.24), where the ordering ambiguity arising from the matrix nature of the field strength F is resolved by the symmetrized trace (STr) prescription whereby all possible orderings of the F's are averaged over. While this observation is correct, it seems that the symmetrized trace formulation of the nonabelian Born-Infeld action misses much of the important physics of the full vector field effective action. In particular, this simplification of the action gives the wrong spectrum around certain background fields, including those which are T-dual to simple intersecting brane configurations [84, 85, 86, 87]. It seems that the only systematic way to deal with the nonabelian vector field action is to include all terms of order F^n at once, counting D at order $F^{1/2}$. The first few terms in the nonabelian vector field action for the bosonic theory were computed in [88, 89, 90]. The terms in the action up to F^4 are given by

$$S_{\text{nonabelian}} = \int -\frac{1}{4}\mathrm{Tr}\, F^2 + \frac{2ig_{\mathrm{YM}}}{3}\,\mathrm{Tr}\left(F^3\right) + \frac{(2\pi g_{YM})^2}{8}\mathrm{STr}\,\left(F^4 - \frac{1}{4}(F^2)^2\right) + \cdots \tag{2.28}$$

In section 6, we show that the effective action we derive from string field theory agrees with (2.28) up to order F^3 after the appropriate field redefinition.

2.3 Computing the effective action

In this section we develop some tools for calculating low-order terms in the effective action for the massless fields by integrating out all massive fields. Section 2.3.1

describes a general approach to computing the generating functions for terms in the effective action and gives explicit expressions for the generating functions of cubic and quartic terms. Section 2.3.2 contains a general derivation of the quartic terms in the effective action for the massless fields. Section 2.3.3 describes the method we use to numerically approximate the coefficients in the action.

2.3.1 Generating functions for terms in the effective action

A convenient way of calculating SFT diagrams is to first compute the off-shell amplitude with generic external coherent states

$$|G\rangle = \exp\left(J_{m\mu} a^{\mu}_{-m} - b_{-m}\mathcal{J}_{bm} + \mathcal{J}_{cm} c_{-m}\right)|p\rangle \tag{2.29}$$

where the index m runs from 1 to ∞ in J^{μ}_{m} and $\mathcal{J}_{bm}$ and from 0 to ∞ in $\mathcal{J}_{cm}$.

Let $\Omega_M(p^i, J^i, \mathcal{J}^i_b, \mathcal{J}^i_c;\ 1 \leq i \leq M)$ be the sum of all connected tree-level diagrams with M external states $|G^i\rangle$. Ω_M is a generating function for all tree-level off-shell M-point amplitudes and can be used to calculate all terms we are interested in in the effective action. Suppose that we are interested in a term in the effective action whose j'th field $\psi^{(j)}_{\mu_1,\ldots,\mu_N}(p)$ is associated with the Fock space state

$$\prod_{m,n,q} a^{\mu_m}_{i_m} b_{k_n} c_{l_q} |p\rangle\,. \tag{2.30}$$

We can obtain the associated off-shell amplitude by acting on Ω_M with the corresponding differential operator for each j

$$\int dp\, \psi^{(j)}_{\mu_1,\ldots,\mu_N}(p) \prod_{m,n,q} \frac{\partial}{\partial J^j_{i_m \mu_m}} \frac{\partial}{\partial \mathcal{J}^j_{bk_n}} \frac{\partial}{\partial \mathcal{J}^j_{cl_q}} \tag{2.31}$$

and setting J^j, $\mathcal{J}^j_b$, and $\mathcal{J}^j_c$ to 0. Thus, all the terms in the effective action which we are interested in can be obtained from Ω_M.

When we calculate a certain diagram with external states $|G^i\rangle$ by applying formulae (2.16) and (2.17) for inner products of coherent and squeezed states the result has the general form

$$\Omega_M = \delta(\sum p^r) \int \prod_{\ell=1}^{N_{\rm prop}} d\tau_\ell \, \mathcal{F}(p,\tau)$$
$$\times \exp\left(\frac{1}{2} J^i_m \Delta^{ij}_{mn}(\tau) J^j_n - p^i \Delta^{ij}_{0m}(\tau) J^j_m + p_i \Delta^{ij}_{00}(\tau) p_j + \text{ghosts}\right). \quad (2.32)$$

A remarkable feature is that (2.32) depends on the sources $J^j, \mathcal{J}^j_b, \mathcal{J}^j_c$ only through the exponent of a quadratic form. Wick's theorem is helpful in writing the derivatives of the exponential in an efficient way. Indeed, the theorem basically reads

$$\prod_{i=1}^{M} \frac{\partial}{\partial J^i_n} \exp\left(\frac{1}{2} J^j_m \Delta^{jk}_{mn} J^k_n\right)\Bigg|_{J^i_n=0} = \text{Sum over all contraction products} \quad (2.33)$$

where the sum is taken over all pairwise contractions, with the contraction between (n, i) and (m, j) carrying the factor Δ^{ij}_{nm}.

Note that Ω_M includes contributions from all the intermediate fields in Feynman-Siegel gauge. To compute the effective action for A_μ we must project out the contribution from intermediate A_μ's.

Three-point generating function

Here we illustrate the idea sketched above with the simple example of the three-point generating function. This generating function provides us with an efficient method of computing the coefficients of the SFT action and the SFT gauge transformation. Plugging $|G^i\rangle$, $1 \leq i \leq 3$ into the cubic vertex (2.7) and using (2.16), (2.17) to evaluate the inner products we find

$$\Omega_3 = -\frac{\mathcal{N} g}{3} \delta(\sum_r p^r) \exp\Big(\frac{1}{2} p^r V^{rs}_{00} p^s - p^r V^{rs}_{0n} J^s_n + \frac{1}{2} J^r_m V^{rs}_{mn} J^s_n - \mathcal{J}^r_{cm} X^{rs}_{mn} \mathcal{J}^s_{bn}\Big). \quad (2.34)$$

As an illustration of how this generating function can be used consider the three-tachyon term in the effective action. The external tachyon state is $\int dp \, \phi(p) \, |p\rangle$. The three-tachyon vertex is obtained from (2.34) by simple integration over momenta and setting the sources to 0. No differentiations are necessary in this case. The three-tachyon term in the action is then

$$-\frac{g}{3}\langle V_3|\,\phi,\phi,\phi\rangle = -\frac{\mathcal{N}g}{3}\int \delta(\sum_s p^s)\prod_r dp^r\phi(p_r)\exp\left(\frac{1}{2}p^r V_{00}^{rs}p^s\right) = -\frac{\mathcal{N}g}{3}\int dx\,\tilde{\phi}(x)^3 \tag{2.35}$$

where

$$\tilde{\phi}(x) = \exp\left(-\frac{1}{2}V_{00}^{11}\partial^2\right)\phi(x). \tag{2.36}$$

For on-shell tachyons, $\partial^2\phi(x) = -\phi(x)$, so that we have

$$-\frac{g}{3}\langle V_3|\,\phi,\phi,\phi\rangle = -\frac{g}{3}\mathcal{N}e^{\frac{3}{2}V_{00}^{11}}\int dx\phi(x)^3 = -\frac{g}{3}\int dx\phi(x)^3. \tag{2.37}$$

The normalization constant cancels so that the on-shell three-tachyon amplitude is just $2g$, in agreement with conventions used here and in [91].

Four-point generating function

Now let us consider the generating function for all quartic off-shell amplitudes (see Figure 2-1). The amplitude Ω_4 after contracting all indices can be written as

$$\Omega_4 = \frac{\mathcal{N}^2 g^2}{2}\int_0^\infty d\tau\; e^{\tau(1-(p_1+p_2)^2)}\left\langle \tilde{V}_2\right| |R(1,2)\rangle\, |R(3,4)\rangle \tag{2.38}$$

where

$$|R(i,j)\rangle^{(k)} = \left\langle G^i\right| \left\langle G^j\right| |V_3\rangle^{(ijk)}. \tag{2.39}$$

Applying (2.16), (2.17) to the inner products in (2.39) we get

$$\begin{aligned}|R(1,2)\rangle = \exp\Big(&\frac{1}{2}p^\alpha_\mu U^{\alpha\beta}_{00}p^{\mu\beta} - p^\alpha_\mu U^{\alpha\beta}_{0n}J^{\beta\mu}_n + \frac{1}{2}J^\alpha_{m\mu}U^{\alpha\beta}_{mn}J^{\beta\mu}_n\\ &+ a^{(3)}_{-m\mu}U^{33}_{mn}a^{\mu(3)}_{-n} + (J^\alpha_{m\mu}U^{\alpha 3}_{mn} - p^\alpha_\mu U^{\alpha 3}_{0n})a^{\mu(3)}_{-n} - \mathcal{J}^\alpha_{cm}X^{\alpha\beta}_{mn}\mathcal{J}^\beta_{bn}\\ &+ b^{(3)}_{-m}X^{3\alpha}_{mn}\mathcal{J}^\alpha_{bn} - \mathcal{J}^\alpha_{cm}X^{\alpha 3}_{mn}c^{(3)}_{-n} - b^{(3)}_{-m}X^{33}_{mn}c^{(3)}_{-n}\Big)c_0\left|-p^1-p^2\right\rangle.\end{aligned} \tag{2.40}$$

Here $\alpha, \beta \in 1, 2$ and

$$U^{rs} = \begin{pmatrix} V_{00}^{rs} - V_{00}^{r3} - V_{00}^{3s} + V_{00}^{33} & V_{0n}^{rs} - V_{0n}^{3s} \\ V_{n0}^{rs} - V_{n0}^{r3} & V_{mn}^{rs} \end{pmatrix}. \tag{2.41}$$

Using (2.16), (2.17) one more time to evaluate the inner products in (2.38) we obtain

$$\begin{aligned} \Omega_4 = \frac{\mathcal{N}^2 g^2}{2} \delta(\sum_i p^i) \int_0^\infty d\tau e^\tau \operatorname{Det} \left(\frac{1 - \tilde{X}^2}{(1 - \tilde{V}^2)^{13}} \right) \\ \times \exp \left(\frac{1}{2} p_\mu^i Q_{00}^{ij} p^{j\mu} - p_\mu^i Q_{0n}^{ij} J_n^{j\mu} + \frac{1}{2} J_{m\mu}^i Q_{mn}^{ij} J_n^{j\mu} - \mathcal{J}_{cm}^i \mathcal{Q}_{mn}^{ij} \mathcal{J}_{bn}^j \right). \end{aligned} \tag{2.42}$$

Here $i, j \in 1, 2, 3, 4$. the matrices $\tilde{V}$ and $\tilde{X}$ are defined by

$$\tilde{V}_{mn} = e^{-\frac{m}{2}\tau} V_{mn} e^{-\frac{n}{2}\tau}, \qquad \tilde{X}_{mn} = e^{-\frac{m}{2}\tau} X_{mn} e^{-\frac{n}{2}\tau}. \tag{2.43}$$

The matrices Q^{ij} and $\mathcal{Q}^{ij}$ are defined through the tilded matrices $\tilde{Q}^{ij}$ and $\tilde{\mathcal{Q}}^{ij}$

$$\tilde{Q}_{mn}^{ij} = e^{-\frac{m}{2}\tau} Q_{mn}^{ij} e^{-\frac{n}{2}\tau}, \qquad \tilde{\mathcal{Q}}_{mn}^{ij} = e^{-\frac{m}{2}\tau} \mathcal{Q}_{mn}^{ij} e^{-\frac{n}{2}\tau} \tag{2.44}$$

where the tilded matrices $\tilde{Q}$ and $\tilde{\mathcal{Q}}$ are defined through $\tilde{V}$, $\tilde{U}$, $\tilde{X}$

$$\begin{aligned} \tilde{Q}^{\alpha\beta} &= \tilde{U}^{\alpha 3} \frac{1}{1 - \tilde{V}^2} \tilde{V} \tilde{U}^{3\beta} + \tilde{U}^{\alpha\beta}, & \tilde{\mathcal{Q}}^{\alpha\beta} &= \tilde{X}^{\alpha 3} \frac{1}{1 - \tilde{X}^2} \tilde{X} \tilde{X}^{3\beta} + \tilde{X}^{\alpha\beta}, \\ \tilde{Q}_{mn}^{\alpha\alpha'} &= -\left(\tilde{U}^{\alpha 3} \frac{1}{1 - \tilde{V}^2} C \tilde{U}^{3\alpha'} \right)_{mn} + \delta_{0m}\delta_{0n}\tau, & \tilde{\mathcal{Q}}^{\alpha\alpha'} &= -\tilde{X}^{\alpha 3} \frac{1}{1 - \tilde{X}^2} C \tilde{X}^{3\alpha'} \end{aligned} \tag{2.45}$$

with $\alpha, \beta \in 1, 2$; $\alpha', \beta' \in 3, 4$. The matrix $\tilde{U}$ includes zero modes while $\tilde{V}$ does not, so one has to understand $\tilde{U}\tilde{V}$ in (2.45) as a product of $\tilde{U}$, where the first column is dropped, and $\tilde{V}$. Similarly $\tilde{V}\tilde{U}$ is the product of $\tilde{V}$ and $\tilde{U}$ with the first row of $\tilde{U}$ omitted.

The matrices Q^{ij} are not all independent for different i and j. The four-point amplitude is invariant under the twist transformation of either of the two vertices as well as under the interchange of the two (see Figure 2-1). In addition the whole

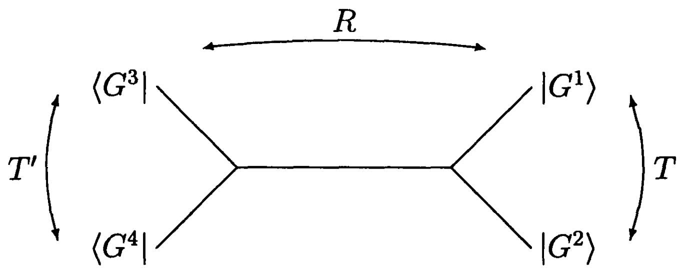

Figure 2-1: Twists T, T' and reflection R are symmetries of the amplitude.

block matrix Q^{ij}_{mn} has been defined in such a way that it is symmetric under the simultaneous exchange of i with j and m with n. Algebraically, we can use properties (A.7a, A.7b, A.7c) of Neumann coefficients to show that the matrices Q^{ij} satisfy

$$\begin{aligned} &(Q^{\alpha\beta})^T = Q^{\beta\alpha}, && CQ^{\alpha\beta}C = Q^{3-\alpha\,3-\beta}, && Q^{\alpha\beta} = Q^{\alpha+2\,\beta+2}, \\ &(Q^{\alpha'\beta'})^T = Q^{\beta'\alpha'}, && CQ^{\alpha'\beta'}C = Q^{7-\alpha'\,7-\beta'}, && Q^{\alpha'\beta'} = Q^{\alpha'-2\,\beta'-2}, \\ &(Q^{\alpha\alpha'})^T = Q^{\alpha'\alpha}, && CQ^{\alpha\alpha'}C = Q^{3-\alpha\,7-\alpha'}, && Q^{\alpha\alpha'} = Q^{\alpha+2\,\alpha'-2}. \end{aligned} \tag{2.46}$$

The analogous relations are satisfied by ghost matrices $\mathcal{Q}$.

Note that we still have some freedom in the definition of the zero modes of the matter matrices Q. Due to the momentum conserving delta function we can add to the exponent in the integrand of (2.42) any expression proportional to $\sum p_i$. To fix this freedom we require that after the addition of such a term the new matrices $\bar{Q}$ satisfy $\bar{Q}^{ii}_{00} = \bar{Q}^{ii}_{0n} = 0$. This gives

$$\bar{Q}^{ij}_{00} = Q^{ij}_{00} - \frac{1}{2}Q^{jj}_{00} - \frac{1}{2}Q^{ii}_{00}, \qquad \bar{Q}^{ij}_{0n} = Q^{ij}_{0n} - Q^{jj}_{0n}. \tag{2.47}$$

and $\bar{Q}^{ij}_{mn} = Q^{ij}_{mn}$ for $m, n > 0$. The addition of any term proportional to $\sum p_i$ corresponds in coordinate space to the addition of a total derivative. In coordinate space we have essentially integrated by parts the terms $\partial_\sigma \partial^\sigma \psi_{\mu_1 \cdots \mu_n}(x)$ and $\partial^{\mu_j} \psi_{\mu_1 \cdots \mu_j \cdots \mu_n}(x)$ thus fixing the freedom of integration by parts.

To summarize, we have rewritten Ω_4 in terms of $\bar{Q}$'s as

$$
\Omega_4 = \frac{\mathcal{N}^2 g^2}{2} \delta(\sum_i p^i) \int_0^\infty d\tau e^\tau \operatorname{Det} \left(\frac{1 - \tilde{X}^2}{(1 - \tilde{V}^2)^{13}} \right)
$$

$$
\times \exp \left(\frac{1}{2} p^i_\mu \bar{Q}^{ij}_{00} p^{j\mu} - p^i_\mu \bar{Q}^{ij}_{0n} J^{j\mu}_n + \frac{1}{2} J^i_{m\mu} \bar{Q}^{ij}_{mn} J^{j\mu}_n - \mathcal{J}^i_{cm} \mathcal{Q}^{ij}_{mn} \mathcal{J}^j_{bn} \right). \quad (2.48)
$$

There are only three independent matrices $\bar{Q}$. For later use we find it convenient to denote the independent $\bar{Q}$'s by $A = \bar{Q}^{12}$, $B = \bar{Q}^{13}$, $C = \bar{Q}^{14}$. Then the matrix $\bar{Q}^{ij}_{mn}$ can be written as

$$
\bar{Q}^{ij}_{mn} = \begin{pmatrix} 0 & A_{mn} & B_{mn} & C_{mn} \\ (-1)^{m+n} A_{mn} & 0 & (-1)^{m+n} C_{mn} & (-1)^{m+n} B_{mn} \\ B_{mn} & C_{mn} & 0 & A_{mn} \\ (-1)^{m+n} C_{mn} & (-1)^{m+n} B_{mn} & (-1)^{m+n} A_{mn} & 0 \end{pmatrix}. \quad (2.49)
$$

In the next section we derive off-shell amplitudes for the massless fields by differentiating Ω_4. The generating function Ω_4 defined in (2.48) and supplemented with the definition of the matrices $\tilde{V}$, $\tilde{X}$, $\bar{Q}$, $\mathcal{Q}$ given in (2.41), (2.43), (2.44), (2.45), (2.47) and (2.49) provides us with all information about the four-point tree-level off-shell amplitudes.

2.3.2 Effective action for massless fields

In this subsection we compute explicit expressions for the general quartic off-shell amplitudes of the massless fields, including derivatives to all orders. Our notation for the massless fields is, as in (2.2),

$$
|\Phi_{\text{massless}}\rangle = \int d^d p \left(A_\mu(p) a^\mu_{-1} - i\alpha(p) b_{-1} c_0 \right) |p\rangle . \quad (2.50)
$$

External states with A_μ and α in the k'th Fock space are inserted using

$$D_\mu^{A,k} = \int dp A_\mu(p) \frac{\partial}{\partial J_{1\mu}^k}\bigg|_{J^k=\mathcal{J}_{b,c}^k=0} \quad \text{and} \quad D^{\alpha,k} = -i\int dp\, \alpha(p) \frac{\partial}{\partial \mathcal{J}_{b1}^k}\frac{\partial}{\partial \mathcal{J}_{c0}^k}\bigg|_{J^k=\mathcal{J}_{b,c}^k=0}. \tag{2.51}$$

We can compute all quartic terms in the effective action $\check{S}[A_\mu, \alpha]$ by computing quartic off-shell amplitudes for the massless fields by acting on Ω_4 with D^A and D^α. First consider the quartic term with four external A's. The relevant off-shell amplitude is given by $\prod_{i=1}^4 D_{\mu_i}^{A,i}\Omega_4$ where Ω_4 is given in (2.48) and $D_{\mu_i}^{A,i}$ is given in (2.51). Performing the differentiations we get

$$\begin{aligned} S_{A^4} = &\frac{1}{2}\mathcal{N}^2 g^2 \int \prod_i dp^i \delta\left(p^1+p^2+p^3+p^4\right) A^{\mu_1}(p_1)A^{\mu_2}(p_2)A^{\mu_3}(p_3)A^{\mu_4}(p_4) \\ &\times \int_0^\infty d\tau e^\tau \operatorname{Det}\left(\frac{1-\tilde{X}^2}{(1-\tilde{V}^2)^{13}}\right)\left(\mathcal{I}_{A^4}^0+\mathcal{I}_{A^4}^2+\mathcal{I}_{A^4}^4\right)\exp\left(\frac{1}{2}p_\mu^i \bar{Q}_{00}^{ij} p^{j\mu}\right). \end{aligned} \tag{2.52}$$

Here $\mathcal{I}_{A^4}^0$, $\mathcal{I}_{A^4}^2$, $\mathcal{I}_{A^4}^4$ are defined by

$$\begin{aligned} \mathcal{I}_{A^4}^0 =&\frac{1}{8}\sum_{i_i\neq i_j} \bar{Q}_{11}^{i_1i_2}\bar{Q}_{11}^{i_3i_4}\eta_{\mu_{i_1}\mu_{i_2}}\eta_{\mu_{i_3}\mu_{i_4}}, \\ \mathcal{I}_{A^4}^2 =&\frac{1}{4}\sum_{i_i\neq i_j} \bar{Q}_{11}^{i_1i_2}\bar{Q}_{10}^{i_3j_1}\bar{Q}_{10}^{i_4j_2}p_{\mu_{i_3}}^{j_1}p_{\mu_{i_4}}^{j_2}\eta_{\mu_{i_1}\mu_{i_2}}, \\ \mathcal{I}_{A^4}^4 =&\bar{Q}_{10}^{1i}\bar{Q}_{10}^{2j}\bar{Q}_{10}^{3k}\bar{Q}_{10}^{4l}p_{\mu_1}^i p_{\mu_2}^j p_{\mu_3}^k p_{\mu_4}^l. \end{aligned} \tag{2.53}$$

Other amplitudes with α's and A's all have the same pattern as (2.52). The amplitude with one α and three A's is obtained by replacing $A_{\mu_{i_1}}(p^{i_1})$ in formula (2.52) with $i\alpha(p^{i_1})$ and the sum of $\mathcal{I}_{A^4}^{0,2,4}$ with the sum of

$$\begin{aligned} \mathcal{I}_{\alpha A^3}^1 &= \frac{1}{2}\sum_{i_i\neq i_j} \mathcal{Q}_{01}^{i_1i_1}\bar{Q}_{11}^{i_2i_3}\bar{Q}_{10}^{i_4k}p_{\mu_{i_4}}^k\eta_{\mu_{i_2}\mu_{i_3}}, \\ \mathcal{I}_{\alpha A^3}^3 &= \frac{1}{6}\sum_{i_i\neq i_j} \mathcal{Q}_{01}^{i_1i_1}\bar{Q}_{10}^{i_2j}\bar{Q}_{10}^{i_3k}\bar{Q}_{10}^{i_4l}p_{\mu_{i_2}}^j p_{\mu_3}^k p_{\mu_4}^l. \end{aligned} \tag{2.54}$$

The amplitude with two A's and two α's is obtained by replacing $A_{\mu_{i_1}}(p^{i_1})A_{\mu_{i_2}}(p^{i_2})$ with $-\alpha(p^{i_1})\alpha(p^{i_2})$ and the sum of $\mathcal{I}^{0,2,4}_{A^4}$ with the sum of

$$\mathcal{I}^0_{\alpha^2 A^2} = \frac{1}{4}\sum_{i_i \neq i_j}(Q^{i_1 i_1}_{01} Q^{i_2 i_2}_{01} - Q^{i_1 i_2}_{01} Q^{i_2 i_1}_{01})\bar{Q}^{i_3 i_4}_{11}\eta_{\mu_{i_3}\mu_{i_4}},$$

$$\mathcal{I}^2_{\alpha^2 A^2} = \frac{1}{4}\sum_{i_i \neq i_j}(Q^{i_1 i_1}_{01} Q^{i_2 i_2}_{01} - Q^{i_1 i_2}_{01} Q^{i_2 i_1}_{01})\bar{Q}^{i_3 k}_{10}\bar{Q}^{i_4 l}_{10} p^k_{\mu_3} p^l_{\mu_4}. \tag{2.55}$$

It is straightforward to write down the analogous expressions for the terms of order $\alpha^3 A$ and α^4. However, as we shall see later, it is possible to extract all the information about the coefficients in the expansion of the effective action for A_μ in powers of field strength up to F^4 from the terms of order $A^4, A^3\alpha$, and $A^2\alpha^2$.

The off-shell amplitudes (2.52), (2.53), (2.54) and (2.55) include contributions from the intermediate gauge field. To compute the quartic terms in the effective action we must subtract, if nonzero, the amplitude with intermediate A_μ. In the case of the abelian theory this amplitude vanishes due to the twist symmetry. In the nonabelian case, however, the amplitude with intermediate A_μ is nonzero. The level truncation method in the next section makes it easy to subtract this contribution at the stage of numerical computation.

As in (2.23), we expand the effective action in powers of p. As an example of a particular term appearing in this expansion, let us consider the space-time independent (zero-derivative) term of (2.52). In the abelian case there is only one such term: $A_\mu A^\mu A_\nu A^\nu$. The coefficient of this term is

$$\gamma = \frac{1}{2}\mathcal{N}^2 g^2 \int_0^\infty d\tau e^\tau \operatorname{Det}\left(\frac{1-\tilde{X}^2}{(1-\tilde{V}^2)^{13}}\right)(A^2_{11} + B^2_{11} + C^2_{11}) \tag{2.56}$$

where the matrices A, B and C are those in (2.49). In the nonabelian case there are two terms, $\operatorname{Tr}(A_\mu A^\mu A_\nu A^\nu)$ and $\operatorname{Tr}(A_\mu A_\nu A^\mu A^\nu)$, which differ in the order of gauge fields. The coefficients of these terms are obtained by keeping $A^2_{11} + C^2_{11}$ and B^2_{11} terms in (2.56) respectively.

2.3.3 Level truncation

Formula (2.56) and analogous formulae for the coefficients of other terms in the effective action contain integrals over complicated functions of infinite-dimensional matrices. Even after truncating the matrices to finite size, these integrals are rather difficult to compute. To get numerical values for the terms in the effective action, we need a good method for approximately evaluating integrals of the form (2.56). In this subsection we describe the method we use to approximate these integrals. For the four-point functions, which are the main focus of the computations in this paper, the method we use is equivalent to truncating the summation over intermediate fields at finite field level. Because the computation is carried out in the oscillator formalism, however, the complexity of the computation only grows polynomially in the field level cutoff.

Tree diagrams with four external fields have a single internal propagator with Schwinger parameter τ. It is convenient to do a change of variables

$$\sigma = e^{-\tau}. \tag{2.57}$$

We then truncate all matrices to size $L \times L$ and expand the integrand in powers of σ up to σ^{M-2}, dropping all terms of higher order in σ. We denote this approximation scheme by $\{L, M\}$. The σ^n term of the series contains the contribution from all intermediate fields at level $k = n+2$, so in this approximation scheme we are keeping all oscillators $a^{\mu}_{k\leq L}$ in the string field expansion, and all intermediate particles in the diagram of mass $m^2 \leq M - 1$. We will use the approximation scheme $\{L, L\}$ throughout this paper. This approximation really imposes only one restriction— the limit on the mass of the intermediate particle. It is perhaps useful to compare the approximation scheme we are using here with those used in previous work on related problems. In [65] analogous integrals were computed by numerical integration. This corresponds to $\{L, \infty\}$ truncation. In earlier papers on level truncation in string field theory, such as [92, 93, 94] and many others, the (L, M) truncation scheme was used, in which fields of mass up to $L - 1$ and interaction vertices with total mass of fields in the vertex up to $M-3$ are kept. Our $\{L, L\}$ truncation scheme is equivalent to the $(L, L+2)$ truncation scheme by that definition.

To explicitly see how the σ expansion works let us write the expansion in σ of a generic integrand and take the integral term by term

$$\int_0^1 \frac{d\sigma}{\sigma^2}\sigma^{p^2}\sum_{n=0}^{\infty} c_n(p^i)\sigma^n = \sum_{n=0}^{\infty}\frac{c_n(p^i)}{p^2+n-1}. \tag{2.58}$$

Here $p = p_1+p_2 = p_3+p_4$ is the intermediate momentum. This is the expansion of the amplitude into poles corresponding to the contributions of (open string) intermediate particles of fixed level. We can clearly see that dropping higher powers of σ in the expansion means dropping the contribution of very massive particles. We also see that to subtract the contribution from the intermediate fields A_μ and α we can simply omit the term $c_1(p)\sigma^{p^2-1}$ in (2.58).

While the Taylor expansion of the integrand might seem difficult, it is in fact quite straightforward. We notice that $\tilde{V}^{rs}$, and $\tilde{X}^{rs}$ are both of order σ. Therefore we can simply expand the integrand in powers of matrices $\tilde{V}$ and $\tilde{X}$. For example, the determinant of the matter Neumann coefficients is

$$\mathrm{Det}(1-\tilde{V}^2)^{-13} = \exp\Big(-13\,\mathrm{Tr}\,\mathrm{Log}(1-\tilde{V}^2)\Big). \tag{2.59}$$

Looking again at (2.52) we notice that the only matrix series' that we will need are $\mathrm{Log}(1-\tilde{V}^2)$ for the determinant (and the analogue for $\tilde{X}$) and $1/(1-\tilde{V}^2)$ for $\bar{Q}^{ij}$. Computation of these series is straightforward.

It is also easy to estimate how computation time grows with L and M. The most time consuming part of the Taylor expansion in σ is the matrix multiplication. Recall that $\tilde{V}$ is an $L\times L$ matrix whose coefficients are proportional to σ^n at leading order. Elements of $\tilde{V}^k$ are polynomials in σ with M terms. To construct a series $a_0 + a_1\tilde{V}+\cdots+a_M\tilde{V}^M+\mathcal{O}(\sigma^{M+1})$ we need M matrix multiplications $\tilde{V}^k\cdot\tilde{V}$. Each matrix multiplication consists of L^3 multiplications of its elements. Each multiplication of the elements has on the average $M/2$ multiplications of monomials. The total complexity therefore grows as L^3M^2.

The method just described allows us to compute approximate coefficients in the effective action at any particular finite level of truncation. In [65], it was found empirically that the level truncation calculation gives approximate results for finite on-shell and off-shell amplitudes with errors which go as a power series in $1/L$. Based on this observation, we can perform a least-squares fit on a finite set of level truncation data for a particular term in the effective action to attain a highly accurate estimate of the coefficient of that term. We use this method to compute coefficients of terms in the effective action which are quartic in A throughout the remainder of this paper.

2.4 The Yang-Mills action

In this section we assemble the Yang-Mills action, picking the appropriate terms from the two, three and four-point Green functions. We write the Yang-Mills action as

$$S_{YM} = \int d^dx \mathrm{Tr}\Big(-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu + ig_{YM}\partial_\mu A_\nu [A^\mu, A^\nu] + \frac{1}{4}g_{YM}^2 [A_\mu, A_\nu][A^\mu, A^\nu]\Big). \quad (2.60)$$

In section 2.4.1 we consider the quadratic terms of the Yang-Mills action. In section 2.4.2 we consider the cubic terms and identify the Yang-Mills coupling constant g_{YM} in terms of the SFT (three tachyon) coupling constant g. This provides us with the expected value for the quartic term. In section 2.4.3 we present the results of a numerical calculation of the (space-time independent) quartic terms and verify that we indeed get the Yang-Mills action.

2.4.1 Quadratic terms

The quadratic term in the action for massless fields, calculated from (2.4), and (2.6) is

$$\check{S}_{A^2} = \int d^dx \,\mathrm{Tr}\Big(-\frac{1}{2}\partial_\mu A_\nu \,\partial^\mu A^\nu - \alpha^2 + \sqrt{2}\alpha \,\partial_\mu A^\mu\Big). \quad (2.61)$$

Completing the square in α and integrating the term $(\partial A)^2$ by parts we obtain

$$\check{S}_{A_2} = \int d^dx \mathrm{Tr}\Big(-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu - B^2\Big) \tag{2.62}$$

where we denote

$$B = \alpha - \frac{1}{\sqrt{2}}\partial_\mu A^\mu . \tag{2.63}$$

Eliminating α using the leading-order equation of motion, $B = 0$, leads to the quadratic terms in (2.60). Subleading terms in the equation of motion for α lead to higher-order terms in the effective action, to which we return in the following sections.

2.4.2 Cubic terms

The cubic terms in the action for the massless fields are obtained by differentiating (2.34). The terms cubic in A are given by

$$\begin{aligned}\check{S}_{A^3} = \frac{\mathcal{N}g}{3}\int \prod_i dp_i \delta(\sum_j p_j)\, \mathrm{Tr}\Big(A_\mu(p_1)A_\nu(p_2)A_\lambda(p_3)\Big)\exp\big(\frac{1}{2}p^r V_{00}^{rr} p^r\big) \\ \times \Big(\big(\eta^{\nu\lambda}p^{r\mu}V_{01}^{r1}V_{11}^{32} + \eta^{\mu\lambda}p^{r\nu}V_{01}^{r2}V_{11}^{13} + \eta^{\mu\nu}p^{r\lambda}V_{01}^{r3}V_{11}^{12}\big) \\ + p^{r\mu}V_{01}^{r1}p^{s\nu}V_{01}^{s2}p^{t\lambda}V_{01}^{t3}\Big).\end{aligned} \tag{2.64}$$

To compare with the Yang-Mills action we perform a Fourier transform and use the properties of the Neumann coefficients to combine similar terms. We then get

$$\begin{aligned}\check{S}_{A^3} = -i\mathcal{N}g\int dx\, \mathrm{Tr}\Big(V_{11}^{12}V_{01}^{12}(\partial_\mu\tilde{A}_\nu[\tilde{A}^\mu,\tilde{A}^\nu]) \\ + \frac{1}{3}(V_{01}^{12})^3(\partial_\lambda\tilde{A}^\mu\partial_\mu\tilde{A}^\nu\partial_\nu\tilde{A}^\lambda - \partial_\nu\tilde{A}^\mu\partial_\lambda\tilde{A}^\nu\partial_\mu\tilde{A}^\lambda) + (V_{01}^{12})^3[\tilde{A}_\nu,\partial^\lambda\tilde{A}_\mu]\partial^\mu\partial^\nu\tilde{A}_\lambda\Big)\end{aligned} \tag{2.65}$$

where, following the notation introduced in (2.36), we have

$$\tilde{A}_\mu = \exp(-\frac{1}{2}V_{00}^{11}\partial^2)A_\mu. \tag{2.66}$$

To reproduce the cubic terms in the Yang-Mills action, we are interested in the terms in (2.65) of order ∂A^3. The remaining terms and the terms coming from the expansion of the exponential of derivatives contribute to higher-order terms in the effective action, which we discuss later. The cubic terms in the action involving the α field are

$$\check{S}_{A\alpha^2} = -i\mathcal{N} g V_{01}^{12} \left(X_{01}^{12}\right)^2 \int dx \, \mathrm{Tr}\left(\tilde{A}^\mu \left[\partial_\mu \tilde{\alpha}, \tilde{\alpha}\right]\right), \tag{2.67}$$
$$\check{S}_{A^2\alpha} = \check{S}_{\alpha^3} = 0.$$

$\check{S}_{A^2\alpha}$ vanishes because $X_{01}^{11} = 0$, and $\check{S}_{\alpha^3}$ is zero because $[\alpha, \alpha] = 0$. After α is eliminated using its equation of motion, (2.67) first contributes terms at order $\partial^3 A^3$.

The first line of (2.65) contributes to the cubic piece of the F^2 term. Substituting the explicit values of the Neumann coefficients:

$$V_{00}^{11} = -\log(27/16), \qquad V_{11}^{12} = 16/27, \tag{2.68}$$
$$V_{01}^{12} = -2\sqrt{2}/3\sqrt{3}, \qquad X_{01}^{12} = 4/(3\sqrt{3}).$$

we write the lowest-derivative term of (2.65) as

$$S_{A^3}^{[1]} = i\frac{g}{\sqrt{2}} \int d^d x \mathrm{Tr}\left(\partial_\mu A_\nu [A^\mu, A^\nu]\right) . \tag{2.69}$$

We can now predict the value of the quartic amplitude at zero momentum. From (2.60) and (2.69) we see that the Yang-Mills constant is related to the SFT coupling constant by

$$g_{YM} = \frac{1}{\sqrt{2}} g. \tag{2.70}$$

This is the same relation between the gauge boson and tachyon couplings as the one given in formula (6.5.14) of Polchinski [91]. We expect the nonderivative part of the quartic term in the effective action to add to the quadratic and cubic terms to form the full Yang-Mills action, so that

$$S_{A_4}^{[0]} = \frac{1}{4} g_{YM}^2 [A_\mu, A_\nu]^2. \tag{2.71}$$

2.4.3 Quartic terms

As we have just seen, to get the full Yang-Mills action the quartic terms in the effective action at $p = 0$ must take the form (2.71). We write the nonderivative part of the SFT quartic effective action as

$$S_{A^4}^{[0]} = g^2 \int dx \Big(\gamma_+ \mathrm{Tr}(A_\mu A^\mu)^2 + \frac{1}{4} \gamma_- \mathrm{Tr}[A_\mu, A^\nu]^2 \Big). \tag{2.72}$$

We can use the method described in section 2.3.3 to numerically approximate the coefficients γ_+ and γ_- in level truncation. In the limit $L \to \infty$ we expect that $\gamma_+ \to 0$ and that $\gamma_- \to g_{YM}^2/g^2 = 1/2$. As follows from formula (2.56) and the comment below it $\gamma_\pm$ are given by:

$$\begin{aligned}
\gamma_+ &= \frac{1}{2} \mathcal{N}^2 \int_0^\infty e^\tau d\tau \, \mathrm{Det} \left(\frac{1 - \tilde{X}^2}{(1 - \tilde{V}^2)^{13}} \right) (A_{11}^2 + B_{11}^2 + C_{11}^2), \\
\gamma_- &= \mathcal{N}^2 \int_0^\infty e^\tau d\tau \, \mathrm{Det} \left(\frac{1 - \tilde{X}^2}{(1 - \tilde{V}^2)^{13}} \right) B_{11}^2.
\end{aligned} \tag{2.73}$$

We have calculated these integrals including contributions from the first 100 levels. We have found that as the level L increases the coefficients γ_+ and γ_- indeed converge to their expected values[2]. The leading term in the deviation decays as $1/L$ as expected. Figure 2-2 shows the graphs of $\gamma_\pm(L)$ vs L.

Table 2.1 explicitly lists the results from the first 10 levels. At level 100 we get $\gamma_+ = 0.0037$, $\gamma_- = 0.4992$ which is within 0.5% of the expected values. One can improve precision even more by doing a least-squares fit of $\gamma_\pm(L)$ with an expansion in powers of $1/L$ with indeterminate coefficients. The contributions to $\gamma_\pm$ from the even and odd level fields are oscillatory. Thus, the fit for only even or only odd levels

[2]In [95] there is an analytic proof of this result

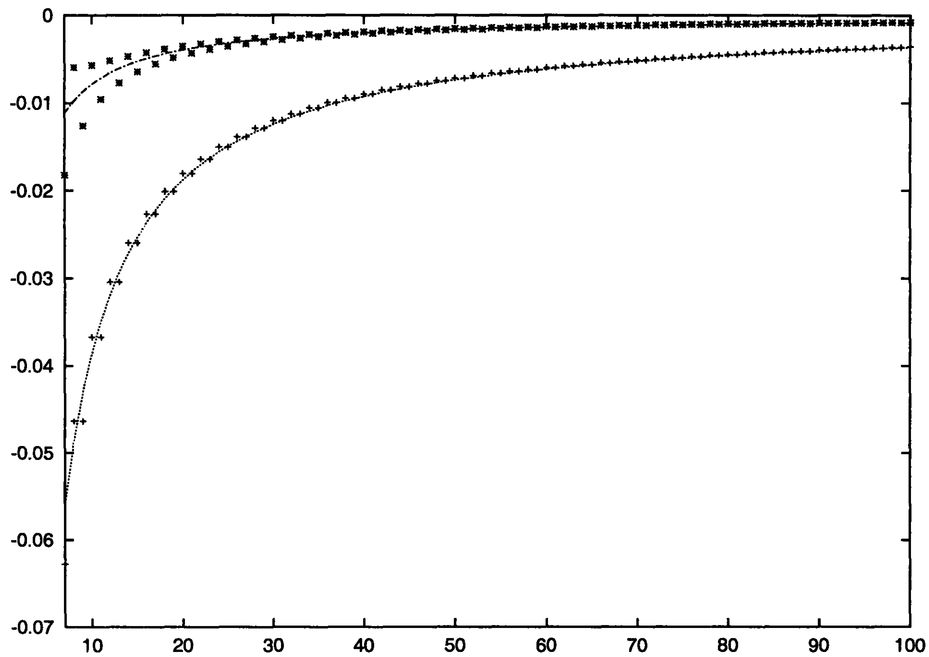

Figure 2-2: Deviation of the coefficients of quartic terms in the effective action from the expected values, as a function of the level of truncation L. The coefficient γ_+ is shown with crosses and $\gamma_- - 1/2$ is shown with stars. The curves given by fitting with a power series in $1/L$ are graphed in both cases.

Level	$\gamma_+(n)$	$\gamma_-(n)$	$\gamma_-(n) - \frac{1}{2}$
0	-0.844	0	-0.500
2	-0.200	0.592	0.092
3	-0.200	0.417	-0.083
4	-0.097	0.504	0.004
5	-0.097	0.468	-0.032
6	-0.063	0.495	-0.005
7	-0.063	0.483	-0.017
8	-0.047	0.494	-0.006
9	-0.047	0.487	-0.013
10	-0.037	0.494	-0.006

Table 2.1: Coefficients of the constant quartic terms in the action for the first 10 levels.

works much better. The least-squares fit for the last 25 even levels gives

$$\begin{aligned}\gamma_+(L) &\approx -5\cdot 10^{-8} - \frac{0.35807}{L} - \frac{0.0091}{L^2} - \frac{1.6}{L^3} + \frac{15}{L^4} + \cdots \\ \gamma_-(L) &\approx \frac{1}{2} - 2\cdot 10^{-8} - \frac{0.0795838}{L} + \frac{0.1212}{L^2} + \frac{1.02}{L^3} - \frac{1.24}{L^4} + \cdots . \end{aligned} \tag{2.74}$$

We see that when $L \to \infty$ the fitted values of $\gamma_\pm$ are in agreement with the Yang-Mills quartic term to 7 digits of precision[3].

The calculations we have described so far provide convincing evidence that the SFT effective action for A_μ reproduces the nonabelian Yang-Mills action. This is encouraging in several respects. First, it shows that our method of computing Feynman diagrams in SFT is working well. Second, the agreement with on-shell calculations is another direct confirmation that cubic SFT provides a correct off-shell generalization of bosonic string theory. Third, it encourages us to extend these calculations further to get more information about the full effective action of A_μ.

2.5 The abelian Born-Infeld action

In this section we consider the abelian theory, and compute terms in the effective action which go beyond the leading Yang-Mills action computed in the previous section. As discussed in Section 2.3, we expect that the effective vector field theory computed from string field theory should be equivalent under a field redefinition to a theory whose leading terms at each order in A take the Born-Infeld form (2.26). In this section we give evidence that this is indeed the case. In the abelian theory, the terms at order A^3 vanish identically, so the quartic terms are the first ones of interest beyond the quadratic Yang-Mills action. In subsection 2.5.1 we use our results on the general quartic term from 2.3.2 to explicitly compute the terms in the effective action at order $\partial^2 A^4$. We find that these terms are nonvanishing. We find, however,

[3]Note that in [96], an earlier attempt was made to calculate the coefficients $\gamma_\pm$ from SFT. The results in that paper are incorrect; the error made there was that odd-level fields, which do not contribute in the abelian action due to twist symmetry, were neglected. As these fields do contribute in the nonabelian theory, the result for γ_- obtained in [96] had the wrong numerical value. Our calculation here automatically includes odd-level fields, and reproduces correctly the expected value.

that the gauge invariance of the effective action constrains the terms at this order to live on a one-parameter family of terms related through field redefinitions, and that the terms we find are generated from the Yang-Mills terms $\hat{F}^2$ with an appropriate field redefinition. We discuss general issues of field redefinition and gauge invariance in subsection 2.5.2; this discussion gives us a framework with which to analyze more complicated terms in the effective action. In subsection 2.5.3 we analyze terms of the form $\partial^4 A^4$, and show that these terms indeed take the form predicted by the Born-Infeld action after the appropriate field redefinition. In subsection 2.5.4 we consider higher-order terms with no derivatives, and give evidence that terms of order $(A \cdot A)^n$ vanish up to $n = 5$ in the string field theory effective action.

2.5.1 Terms of the form $\partial^2 A^4$

In the abelian theory, all terms in the Born-Infeld action have the same number of fields and derivatives. If we assume that the effective action for A_μ calculated in SFT directly matches the Born-Infeld action (plus higher-order derivative corrections) we would expect the $\partial^2 A^4$ terms in the expansion of the effective action to vanish. The most general form of the quartic terms with two derivatives is parameterized as [4]

$$S^2_{A^4} = g^2 \int d^{26}x \big(c_1 A_\mu A^\mu \partial_\sigma A_\nu \partial^\sigma A^\nu + c_2 A_\mu A_\nu \partial_\sigma A^\mu \partial^\sigma A^\nu + c_3 A_\mu A_\nu \partial^\mu A_\sigma \partial^\nu A^\sigma \\ + c_4 A_\mu A_\nu \partial^\nu A_\sigma \partial^\sigma A^\mu + c_5 A_\mu A_\nu A_\sigma \partial^\mu \partial^\nu A^\sigma + c_6 A_\mu A^\mu \partial_\sigma A^\nu \partial_\nu A^\sigma \big). \quad (2.75)$$

When α is eliminated from the massless effective action $\check{S}$ using the equation of motion, we might then expect that all coefficients c_n in the resulting action (2.75) should vanish. Let us now compute these terms explicitly. From (2.62) and (2.67) we see that the equation of motion for α in the effective theory of the massless fields

[4] Recall that in section 2.3.1 we fixed the integration by parts freedom by integrating by parts all terms with $\partial^2 A_\lambda$ and $\partial \cdot A$. Formula (2.75) gives the most general combination of terms with four A's and two derivatives that do not have $\partial^2 A_\lambda$ and $\partial \cdot A$.

reads (in the abelian theory)

$$\alpha = \frac{1}{\sqrt{2}}\partial^\mu A_\mu + \mathcal{O}((A,\alpha)^3)\,. \tag{2.76}$$

The coefficients $c_1, \ldots, c_6$ thus get contributions from the two-derivative term of (2.52), the one-derivative term of (2.54) and the zero-derivative term of (2.55). We first consider the contribution from the four-gauge boson amplitude (2.52). All the expressions for these contributions, which we denote $(\delta c_i)_{A^4}$, are of the form

$$(\delta c_i)_{A^4} = \frac{1}{2}\mathcal{N}^2 \int_0^\infty d\tau e^\tau \,\mathrm{Det}\left(\frac{1-\tilde{X}^2}{(1-\tilde{V}^2)^{13}}\right) P_{\partial^2 A^4,i}(A,B,C). \tag{2.77}$$

Here $P_{\partial^2 A^4,i}$ are polynomials in the elements of the matrices A, B and C which were defined in (2.49). It is straightforward to derive expressions for the polynomials $P_{\partial^2 A^4,i}$ from (2.52) and (2.53), so we just give the result here

$$\begin{aligned}
P_{\partial^2 A^4,1} &= -2\big(A_{11}^2 A_{00} + B_{11}^2 B_{00} + C_{11}^2 C_{00}\big),\\
P_{\partial^2 A^4,2} &= -2\big(A_{11}^2(B_{00}+C_{00}) + B_{11}^2(A_{00}+C_{00}) + C_{11}^2(A_{00}+B_{00})\big),\\
P_{\partial^2 A^4,3} &= 2\big(A_{11}(B_{10}^2+C_{10}^2) - B_{11}(A_{10}^2+C_{10}^2) + C_{11}(A_{10}^2+B_{10}^2)\big),\\
P_{\partial^2 A^4,4} &= 4\big(A_{11}A_{10}(B_{10}+C_{10}) - B_{11}B_{10}(A_{10}+C_{10}) + C_{11}C_{10}(A_{10}+B_{10})\big),\\
P_{\partial^2 A^4,5} &= 4\big(A_{11}B_{10}C_{10} - B_{11}A_{10}C_{10} + C_{11}A_{10}B_{10}\big),\\
P_{\partial^2 A^4,6} &= 2\big(A_{11}A_{10}^2 - B_{11}B_{10}^2 + C_{11}C_{10}^2\big).
\end{aligned} \tag{2.78}$$

The terms in the effective action $\check{S}$ which contain α's and contribute to $S[A]$ at order $\partial^2 A^4$ can similarly be computed from (2.54) and (2.55) and are given by

$$\check{S}^{[1]}_{\alpha A^3} + \check{S}^{[0]}_{\alpha^2 A^2} = g^2 \int d^{26}x\big(\sigma_1 \alpha A_\mu A_\nu \partial^\mu A^\nu + \sigma_2 \partial^\mu \alpha A_\mu A_\nu A^\nu + \sigma_3 \alpha^2 A_\mu A^\mu\big) \tag{2.79}$$

where the coefficients σ_i are given by

$$
\begin{aligned}
P_{\partial\alpha A^3,1} &= 4\mathcal{Q}^{11}_{01}\big(A_{11}(B_{10}+C_{10}) - B_{11}(A_{10}+C_{10}) + C_{11}(B_{10}+A_{10})\big),\\
P_{\partial\alpha A^3,2} &= 4\mathcal{Q}^{11}_{01}\big(A_{11}A_{10} - B_{11}B_{10} + C_{11}C_{10}\big), \qquad (2.80)\\
P_{\alpha^2 A^2} &= 2\big((\mathcal{Q}^{11}_{01})^2 - (\mathcal{Q}^{12}_{01})^2\big)A_{11} - \big((\mathcal{Q}^{11}_{01})^2 - (\mathcal{Q}^{13}_{01})^2\big)B_{11} + \big((\mathcal{Q}^{11}_{01})^2 - (\mathcal{Q}^{14}_{01})^2\big)C_{11}.
\end{aligned}
$$

Computation of the integrals up to level 100 and using a least-squares fit gives us

$$
\begin{array}{lll}
(\delta c_1)_{A^4} \approx -2.1513026, & (\delta c_4)_{A^4} \approx \ 0.9132288, & \sigma_1 \approx -0.4673613,\\
(\delta c_2)_{A^4} \approx -4.3026050, & (\delta c_5)_{A^4} \approx -2.0134501, & \sigma_2 \approx \ 0.2171165,\\
(\delta c_3)_{A^4} \approx -2.0134501, & (\delta c_6)_{A^4} \approx \ 1.4633393, & \sigma_3 \approx \ 1.6829758.
\end{array} \qquad (2.81)
$$

Elimination of α with (2.76) gives

$$
\begin{array}{ll}
c_1 \approx -2.1513026, & c_4 \approx 4.302605,\\
c_2 \approx -4.302605, & c_5 \approx 0,\\
c_3 \approx 0, & c_6 \approx 2.1513026.
\end{array} \qquad (2.82)
$$

These coefficients are not zero, so that the SFT effective action does not reproduce the abelian Born-Infeld action in a straightforward manner. Thus, we need to consider a field redefinition to put the effective action into the usual Born-Infeld form. To understand how this field redefinition works, it is useful to study the gauge transformation in the effective theory. Without directly computing this gauge transformation, we can write the general form that the transformation must take; the leading terms can be parameterized as

$$
\begin{aligned}
\delta A_\mu = \partial_\mu\lambda + g_{YM}^2(&\varsigma_1 A^2\partial_\mu\lambda + \varsigma_2 A_\nu\partial_\mu A^\nu\lambda\\
&+ \varsigma_3 A_\nu\partial^\nu A_\mu\lambda + \varsigma_4 A_\mu\partial\cdot A\lambda + \varsigma_5 A_\mu A_\nu\partial^\nu\lambda) + \mathcal{O}(\partial^3 A^2\lambda). \qquad (2.83)
\end{aligned}
$$

The action (2.75) must be invariant under this gauge transformation. This gauge invariance imposes a number of *a priori* restrictions on the coefficients c_i, ς_i. When we vary the F^2 term in the effective action (2.60) the nonlinear part of (2.83) generates $\partial^3 A^3 \lambda$ terms. Gauge invariance requires that these terms cancel the terms arising from the linear gauge transformation of the $\partial^2 A^4$ terms in (2.75). This cancellation gives homogeneous linear equations for the parameters c_i and ς_i. The general solution of these equations depends on one free parameter γ:

$$\begin{aligned} c_1 &= -c_6 = -\gamma, & \varsigma_1 &= -\gamma, \\ c_2 &= -c_4 = -2\gamma, & \varsigma_5 &= -2\gamma, \\ c_3 &= c_5 = 0, & \varsigma_2 &= \varsigma_3 = \varsigma_4 = 0. \end{aligned} \tag{2.84}$$

The coefficients c_i calculated above satisfy these relations to 7 digits of precision. From the numerical values of the c_i's, we find

$$\gamma \approx 2.1513026 \pm 0.0000005. \tag{2.85}$$

We have thus found that the $\partial^2 A^4$ terms in the effective vector field action derived from SFT lie on a one-parameter family of possible combinations of terms which have a gauge invariance of the desired form. We can identify the degree of freedom associated with this parameter as arising from the existence of a family of field transformations with nontrivial terms at order A^3

$$\begin{aligned} \hat{A}_\mu &= A_\mu + g^2 \gamma A^2 A_\mu, \\ \hat{\lambda} &= \lambda. \end{aligned} \tag{2.86}$$

We can use this field redefinition to relate a field $\hat{A}$ with the standard gauge transformation $\delta \hat{A}_\mu = \partial_\mu \hat{\lambda}$ to a field A transforming under (2.83) with ς_i and γ satisfying

(2.84). Indeed, plugging this change of variables into

$$\delta \hat{A}_\mu = \partial \hat{\lambda}, \qquad (2.87)$$
$$S_{BI} = -\frac{1}{4}\int dx \hat{F}^2 + \mathcal{O}(\hat{F}^3).$$

gives (2.83) and (2.75) with c_i, ς_i satisfying (2.84).

We have thus found that nonvanishing $\partial^2 A^4$ terms arise in the vector field effective action derived from SFT, but that these terms can be removed by a field redefinition. We would like to emphasize that the logic of this subsection relies upon using the fact that the effective vector field theory has a gauge invariance. The existence of this invariance constrains the action sufficiently that we can identify a field redefinition that puts the gauge transformation into standard form, without knowing in advance the explicit form of the gauge invariance in the effective theory. Knowing the field redefinition, however, in turn allows us to identify this gauge invariance explicitly. This interplay between field redefinitions and gauge invariance plays a key role in understanding higher-order terms in the effective action, which we explore further in the following subsection.

2.5.2 Gauge invariance and field redefinitions

In this subsection we discuss some aspects of the ideas of gauge invariance and field redefinitions in more detail. In the previous subsection, we determined a piece of the field redefinition relating the vector field A in the effective action derived from string field theory to the gauge field $\hat{A}$ in the Born-Infeld action by using the existence of a gauge invariance in the effective theory. The rationale for the existence of the field transformation from A to $\hat{A}$ can be understood based on the general theorem of the rigidity of the Yang-Mills gauge transformation [82, 97]. This theorem states that any deformation of the Yang-Mills gauge invariance can be mapped to the standard gauge invariance through a field redefinition. At the classical level this field redefinition can be expressed as

$$\hat{A}_\mu = \hat{A}_\mu(A),$$
$$\hat{\lambda} = \hat{\lambda}(A, \lambda). \tag{2.88}$$

This theorem explains, for example, why noncommutative Yang-Mills theory, which has a complicated gauge invariance involving the noncommutative star product, can be mapped through the Seiberg-Witten map (field redefinition) to a gauge theory written in terms of a gauge field with standard transformation rules [12, 98]. Since in string field theory the parameter α' (which we have set to unity) parameterizes the deformation of the standard gauge transformation of A_μ, the theorem states that some field redefinition exists which takes the effective vector field theory arising from SFT to a theory which can be written in terms of the field strength $\hat{F}_{\mu\nu}$ and covariant derivative $\hat{D}_\mu$ of a gauge field $\hat{A}_\mu$ with the standard transformation rule[5].

There are two ways in which we can make use of this theorem. Given the explicit expression for the effective action from SFT, one can assume that such a transformation exists, write the most general covariant action at the order of interest, and find a field redefinition which takes this to the effective action computed in SFT. Applying this approach, for example, to the $\partial^2 A^4$ terms discussed in the previous subsection, we would start with the covariant action $\hat{F}^2$, multiplied by an unknown overall coefficient ζ, write the field redefinition (2.86) in terms of the unknown γ, plug in the field redefinition, and match with the effective action (2.75), which would allow us to fix γ and $\zeta = -1/4$.

A more direct approach can be used when we have an explicit expression for the gauge invariance of the effective theory. In this case we can simply try to construct a field redefinition which relates this invariance to the usual Yang-Mills gauge invariance. When finding the field redefinition relating the deformed and undeformed theories, however, a further subtlety arises, which was previously encountered in related situations [14, 15]. Namely, there exists for any theory a class of trivial gauge

[5]In odd dimensions there would also be a possibility of Chern-Simons terms

invariances. Consider a theory with fields ϕ_i and action $S(\phi_i)$. This theory has trivial gauge transformations of the form

$$\delta\phi_i = \mu_{ij}\frac{\delta S}{\delta\phi_j} \tag{2.89}$$

where $\mu_{ij} = -\mu_{ji}$. Indeed, the variation of the action under this transformation is $\delta S = \mu_{ij}\frac{\delta S}{\delta\phi_i}\frac{\delta S}{\delta\phi_j} = 0$. These transformations are called trivial because they do not correspond to a constraint in the Hamiltonian picture. The conserved charges associated with trivial transformations are identically zero. In comparing the gauge invariance of the effective action $S[A]$ to that of the Born-Infeld action, we need to keep in mind the possibility that the gauge invariances are not necessarily simply related by a field redefinition, but that the invariance of the effective theory may include additional terms of the form (2.89). In considering this possibility, we can make use of a theorem (theorem 3.1 of [99]), which states that under suitable regularity assumptions on the functions $\frac{\delta S}{\delta\phi_i}$ any gauge transformation that vanishes on shell can be written in the form (2.89). Thus, when identifying the field redefinition transforming the effective vector field A to the gauge field $\hat{A}$, we allow for the possible addition of trivial terms.

The benefit of the first method described above for determining the field redefinition is that we do not need to know the explicit form of the gauge transformation. Once the field redefinition is known we can find the gauge transformation law in the effective theory of A_μ up to trivial terms by plugging the field redefinition into the standard gauge transformation law of $\hat{A}_\mu$. In the explicit example of $\partial^2 A^4$ terms considered in the previous subsection we determined that the gauge transformation of the vector field A_μ is given by

$$\delta A_\mu = \partial_\mu\lambda - g_{\rm YM}^2\gamma(A^2\partial_\mu\lambda - 2A_\mu A_\nu\partial^\nu\lambda) \tag{2.90}$$

plus possible trivial terms which we did not consider. We have found the numerical value of γ in (2.85). If we had been able to directly compute this gauge transformation law, finding the field redefinition (2.86) would have been trivial. Unfortunately, as we shall see in a moment, the procedure for computing the higher-order terms in the gauge invariance of the effective theory is complicated to implement, which makes the second method less practical in general for determining the field redefinition. We can, however, at least compute the terms in the gauge invariance which are of order $A\lambda$ directly from the definition (2.5). Thus, for these terms the second method just outlined for computing the field redefinition can be used. We use this method in section 2.6.1 to compute the field redefinition including terms at order ∂A^2 and $\partial^2 A$ in the nonabelian theory.

Let us note that the field redefinition that makes the gauge transformation standard is not unique. There is a class of field redefinitions that preserves the gauge structure and mass-shell condition

$$\begin{aligned} \hat{A}'_\mu &= \hat{A}_\mu + T_\mu(\hat{F}) + \hat{D}_\mu \xi(\hat{A}), \\ \hat{\lambda}' &= \hat{\lambda} + \delta_{\hat{\lambda}} \xi(\hat{A}_\mu) . \end{aligned} \tag{2.91}$$

In this field redefinition $T_\mu(\hat{F})$ depends on $\hat{A}_\mu$ only through the covariant field strength and its covariant derivatives. The term ξ is a trivial (pure gauge) field redefinition, which is essentially a gauge transformation with parameter $\xi(A)$. The resulting ambiguity in the effective Lagrangian has a field theory interpretation based on the equivalence theorem [100]. According to this theorem, different Lagrangians give the same S-matrix elements if they are related by a change of variables in which both fields have the same gauge variation and satisfy the same mass-shell condition.

Let us now describe briefly how the different forms of gauge invariance arise in the world-sheet and string field theory approaches to computing the vector field action. We primarily carry out this discussion in the context of the abelian theory, although similar arguments can be made in the nonabelian case. In a world-sheet sigma model calculation one introduces the boundary interaction term

$$\oint_\gamma A_\mu \frac{dX^\mu}{d\tau} d\tau. \tag{2.92}$$

This term is explicitly invariant under

$$A_\mu \to A_\mu + \partial_\mu \lambda. \tag{2.93}$$

Provided that one can find a systematic method of calculation that respects this gauge invariance, the resulting effective action will possess this gauge invariance as well. This is the reason calculations such as those in [11, 16] give an effective action with the usual gauge invariance.

In the cubic SFT calculation, on the other hand, the gauge invariance is much more complicated. The original theory has an infinite number of gauge invariances, given by (2.5). We have fixed all but one of these gauge symmetries; the remaining symmetry comes from a gauge transformation that may change the field α, but which keeps all other auxiliary fields at zero. A direct construction of this gauge transformation in the effective theory of A_μ is rather complicated, but can be described perturbatively in three steps:

1. Make an SFT gauge transformation (in the full theory with an infinite number of fields) with the parameter

   $$|\Lambda'\rangle = \frac{i}{\sqrt{2}}\lambda(x) b_{-1} |0\rangle \,. \tag{2.94}$$

   This gauge transformation transforms α and A_μ as

   $$\begin{aligned} \delta A_\mu &= \partial_\mu \lambda + i g_{YM}(\cdots), \\ \delta\alpha &= \sqrt{2}\,\partial^2\lambda + i g_{YM}(\cdots), \end{aligned} \tag{2.95}$$

   and transforms all fields in the theory in a computable fashion.

2. The gauge transformation $|\Lambda'\rangle$ takes us away from the gauge slice we have fixed by generating fields associated with states containing c_0 at all higher levels. We now have to make a second gauge transformation with a parameter $|\Lambda''(\lambda)\rangle$ that will restore our gauge of choice. The order of magnitude of the auxiliary

fields we have generated at higher levels is $\mathcal{O}(g\lambda\Phi)$. Therefore $|\Lambda''(\lambda)\rangle$ is of order $g\lambda\Phi$. Since we already used the gauge parameter at level zero, we will choose $|\Lambda''\rangle$ to have nonvanishing components only for massive modes. Then this gauge transformation does not change the massless fields linearly, so the contribution to the gauge transformation at the massless level will be of order $\mathcal{O}(g^2\lambda\Phi^2)$. The gauge transformation generated by $|\Lambda''(\lambda)\rangle$ can be computed as a perturbative expansion in g. Combining this with our original gauge transformation generated by $|\Lambda'\rangle$ gives us a new gauge transformation which transforms the massless fields linearly according to (2.95), but which also keeps us in our chosen gauge slice.

3. In the third step we eliminate all the fields besides A_μ using the classical equations of motion. The SFT equations of motion are

$$Q_B\,|\Phi\rangle = -g\langle\Phi,\Phi\,|V_3\rangle\,. \tag{2.96}$$

The BRST operator preserves the level of fields; therefore, the solutions for massive fields and α in terms of A_μ will be of the form

$$\psi_{\mu_1,\ldots,\mu_n} = \mathcal{O}(gA^2), \tag{2.97}$$

$$\alpha = \frac{1}{\sqrt{2}}\partial\cdot A + \mathcal{O}(gA^2) \tag{2.98}$$

where $\psi_{\mu_1,\ldots,\mu_n}$ is a generic massive field. Using these EOM to eliminate the massive fields and α in the gauge transformation of A_μ will give terms of order $\mathcal{O}(g^2A^2)$.

To summarize, the gauge transformation in the effective theory for A_μ is of the form

$$\delta A_\mu = \partial_\mu\lambda + R_\mu(A,\lambda), \tag{2.99}$$

where R_μ is a specific function of A and λ at order $g^2A^2\lambda$, which can in principle be computed using the method just described. In the nonabelian theory, there will also be terms at order $gA\lambda$ arising directly from the original gauge transformation $|\Lambda\rangle$; these terms are less complicated and can be computed directly from the cubic string field vertex.

In this subsection, we have discussed two approaches to computing the field redefinition which takes us from the effective action $S[A]$ to a covariant action written in terms of the gauge field $\hat{A}$, which should have the form of the Born-Infeld action plus derivative corrections. In the following sections we use these two approaches to check that various higher-order terms in the SFT effective action indeed agree with known terms in the Born-Infeld action, in both the abelian and nonabelian theories.

2.5.3 Terms of the form $\partial^4 A^4$

The goal of this subsection is to verify that after an appropriate field redefinition the $\partial^4 A^4$ terms in the abelian effective action derived from SFT transform into the $\hat{F}^4 - \frac{1}{4}(\hat{F}^2)^2$ terms of the Born-Infeld action (including the correct constant factor of $(2\pi g_{YM})^2/8$). To demonstrate this, we use the first method discussed in the previous subsection. Since the total number of $\partial^4 A^4$ terms is large we restrict attention to a subset of terms: namely those terms where indices on derivatives are all contracted together. These terms are independent from other terms at the same order in the effective action. By virtue of the equations of motion (2.76) the diagrams with α do not contribute to these terms. This significant simplification is the reason why we choose to concentrate on these terms. Although we only compute a subset of the possible terms in the effective action, however, we find that these terms are sufficient to fix both coefficients in the Born-Infeld action at order F^4.

The terms we are interested in have the general form

$$S^4_{(\partial\cdot\partial)A^4} = g^2 \int d^{26}x \big(d_1(\partial_\mu A_\lambda \partial^\mu A^\lambda)^2 + d_2 \partial_\mu A_\lambda \partial_\nu A^\lambda \partial^\mu A_\sigma \partial^\nu A^\sigma \\ + d_3 A_\lambda \partial_\nu A^\lambda \partial_\mu A_\sigma \partial^\mu \partial^\nu A^\sigma + d_4 \partial_\mu A_\lambda \partial_\nu A^\lambda A_\sigma \partial^\mu \partial^\nu A^\sigma \\ + d_5 A_\lambda A^\lambda \partial_\mu \partial_\nu A_\sigma \partial^\mu \partial^\nu A^\sigma + d_6 A_\lambda \partial_\mu \partial_\nu A^\lambda A_\sigma \partial^\mu \partial^\nu A^\sigma\big). \quad (2.100)$$

The coefficients for these terms in the effective action are given by

$$d_i = \frac{1}{2}\mathcal{N}^2 \int_0^\infty d\tau e^\tau \operatorname{Det}\left(\frac{1-\tilde{X}^2}{(1-\tilde{V}^2)^{13}}\right) P_i^{(4)}(A,B,C) \tag{2.101}$$

with

$$\begin{aligned}
P_1^{(4)} &= P_5^{(4)} = A_{11}^2 A_{00}^2 + B_{11}^2 B_{00}^2 + C_{11}^2 C_{00}^2, \\
P_2^{(4)} &= P_6^{(4)} = A_{11}^2\left(B_{00}^2 + C_{00}^2\right) + B_{11}^2\left(A_{00}^2 + C_{00}^2\right) + C_{11}^2\left(A_{00}^2 + B_{00}^2\right), \\
P_3^{(4)} &= 4A_{11}^2 A_{00}\left(B_{00} + C_{00}\right) + 4B_{11}^2 B_{00}\left(A_{00} + C_{00}\right) + 4C_{11}^2 C_{00}\left(A_{00} + B_{00}\right), \\
P_4^{(4)} &= 4A_{11}^2 B_{00} C_{00} + 4B_{11}^2 A_{00} C_{00} + 4C_{11}^2 A_{00} B_{00}.
\end{aligned} \tag{2.102}$$

Computation of the integrals gives us

$$\begin{aligned}
d_1 &= d_5 \approx 3.14707539, \qquad & d_3 &\approx 18.51562023, \\
d_2 &= d_6 \approx 2.96365920, \qquad & d_4 &\approx 0.99251621.
\end{aligned} \tag{2.103}$$

To match these coefficients with the BI action we need to write the general field redefinition to order $\partial^2 A^3$ (again, keeping only terms with all derivatives contracted)

$$\begin{aligned}
\hat{A}_\mu = A_\mu + g^2\big(\gamma A^2 A_\mu &+ \alpha_1 A_\mu A_\sigma \partial^2 A^\sigma + \alpha_2 A^2 \partial^2 A_\mu \\
&+ \alpha_3 A_\mu \partial_\lambda A_\sigma \partial^\lambda A^\sigma + \alpha_4 A_\sigma \partial_\lambda A_\mu \partial^\lambda A^\sigma\big).
\end{aligned} \tag{2.104}$$

Using the general theorem quoted in the previous subsection, we know that there is a field redefinition relating the action containing the terms (2.100) to a covariant action written in terms of a conventional field strength $\hat{F}$. The coefficients of $\hat{F}^2$ and $\hat{F}^3$ are already fixed, so the most generic action up to $\hat{F}^4$ is

$$\operatorname{Tr}\int dx\Big(-\frac{1}{4}\hat{F}^2 + g^2\left(a\hat{F}^4 + b(\hat{F}^2)^2\right) + O(\hat{F}^6)\Big). \tag{2.105}$$

We plug the change of variables (2.104) into this equation and collect $\partial^4 A^4$ terms with derivatives contracted together:

$$
\begin{aligned}
g^2 \int d^{26}x \Big(&(\alpha_1 - \alpha_3 + 4b)(\partial_\mu A_\lambda \partial^\mu A^\lambda)^2 \\
&\qquad + (\alpha_1 + 2\alpha_2 - \alpha_4 + 2a)\partial_\mu A_\lambda \partial_\nu A^\lambda \partial^\mu A_\sigma \partial^\nu A^\sigma \\
&\qquad + (4\alpha_1 + 4\alpha_2 - 2\alpha_3 - \alpha_4) A_\lambda \partial_\nu A^\lambda \partial_\mu A_\sigma \partial^\mu \partial^\nu A^\sigma \\
&\qquad\quad + (2\alpha_1 + 2\alpha_2 - \alpha_4)\partial_\mu A_\lambda \partial_\nu A^\lambda A_\sigma \partial^\mu \partial^\nu A^\sigma \\
&\qquad\quad + \alpha_2 A_\lambda A^\lambda \partial_\mu \partial_\nu A_\sigma \partial^\mu \partial^\nu A^\sigma + \alpha_1 A_\lambda \partial_\mu \partial_\nu A^\lambda A_\sigma \partial^\mu \partial^\nu A^\sigma \Big).
\end{aligned}
\tag{2.106}
$$

The assumption that (2.100) can be written as (2.106) translates into a system of linear equations for a, b and $\alpha_1, \ldots \alpha_4$ with the right hand side given by $d_1, \ldots d_6$. This system is non-degenerate and has a unique solution

$$
\begin{aligned}
\alpha_1 &= d_6 \approx 2.9636592, \\
\alpha_2 &= d_5 \approx 3.1470754, \\
\alpha_3 &= \frac{1}{2}(-d_3 + d_4 + 2d_5 + 2d_6) \approx -2.6508174, \\
\alpha_4 &= -d_4 + 2d_5 + 2d_6 \approx 11.2289530, \\
a &= \frac{1}{2}(d_2 - d_4 + d_6) \approx 2.4674011, \\
b &= \frac{1}{8}(2d_1 - d_3 + d_4 + 2d_5) \approx -0.6168503.
\end{aligned}
\tag{2.107}
$$

This determines the coefficients a and b in the effective action (2.105) to 8 digits of precision. These values agree precisely with those that we expect from the Born-Infeld action, which are given by

$$
\begin{aligned}
a &= \frac{\pi^2}{4} \approx 2.4674011, \\
b &= -\frac{\pi^2}{16} \approx -0.6168502.
\end{aligned}
\tag{2.108}
$$

Thus, we see that after a field redefinition, the effective vector theory derived from string field theory agrees with Born-Infeld to order F^4, and correctly fixes the coefficients of both terms at that order. This calculation could in principle be continued to compute higher-derivative corrections to the Born-Infeld action of the form $\partial^6 A^4$ and higher, but we do not pursue such calculations further here.

Note that, assuming we know that the Born-Infeld action takes the form

$$S_{BI} = -T \int dx \sqrt{-\det\left(\eta^{\mu\nu} + T^{-\frac{1}{2}} F^{\mu\nu}\right)}. \tag{2.109}$$

with undetermined D-brane tension, we can fix $T = 1/(2\pi\alpha' g_{YM})^2$ from the coefficients at F^2 and F^4. We may thus think of the calculations done so far as providing another way to determine the D-brane tension from SFT.

2.5.4 Terms of the form A^{2n}

In the preceding discussion we have focused on terms in the effective action which are at most quartic in the vector field A_μ. It is clearly of interest to extend this discussion to terms of higher order in A. A complete analysis of higher-order terms, including all momentum dependence, involves considerable additional computation. We have initiated analysis of higher-order terms by considering the simplest class of such terms: those with no momentum dependence. As for the quartic terms of the form $(A^\mu A_\mu)^2$ discussed in Section 4.2, we expect that all terms in the effective action of the form

$$(A^\mu A_\mu)^n \tag{2.110}$$

should vanish identically when all diagrams are considered. In this subsection we consider terms of the form (2.110). We find good numerical evidence that these terms indeed vanish, up to terms of the form A^{10}.

In Section 4.2 we found strong numerical evidence that the term (2.110) vanishes for $n = 2$ by showing that the coefficient γ_+ in (2.72) approaches 0 in the level-truncation approximation. This A^4 term involves only one possible diagram. As n increases, the number of diagrams involved in computing A^{2n} increases exponentially, and the complexity of each diagram also increases, so that the primary method used in this paper becomes difficult to implement. To study the terms (2.110) we have used a somewhat different method, in which we directly truncate the theory by only including fields up to a fixed total oscillator level, and then computing the cubic terms for each of the fields below the desired level. This was the original method of level truncation used in [92] to compute the tachyon 4-point function, and in later work [93, 94] on level truncation on the problem of tachyon condensation. As discussed in Section 3.3, the method we are using for explicitly calculating the quartic terms in the action involves truncating on the level of the intermediate state in the 4-point diagram, so that the two methods give the same answers. While level truncation on oscillators is very efficient for computing low-order diagrams at high level, however, level truncation on fields is more efficient for computing high-order diagrams at low level.

In [94], a recursive approach was used to calculate coefficients of ϕ^n in the effective tachyon potential from string field theory using level truncation on fields. Given a cubic potential

$$V = \sum_{i,j} d_{ij}\, \psi_i \psi_j + \sum_{i,j,k} g t_{ijk}\, \psi_i \psi_j \psi_k \tag{2.111}$$

for a finite number of fields $\psi_i, i = 1, \ldots, N$ at $p = 0$, the effective action for $a = \psi_1$ when all other fields are integrated out is given by

$$V_{\text{eff}}(a) = \sum_{n=2}^{\infty} \frac{1}{n} v^1_{n-1} a^n g^n \tag{2.112}$$

where v^i_n represents the summation over all graphs with n external a edges and a single external ψ^i, with no internal a's. The $v's$ satisfy the recursion relations

$$\begin{aligned} v^i_1 &= \delta^i_1 \\ v^i_n &= \frac{3}{2} \sum_{m=1}^{n-1} d^{ij}\, t_{jkl}\, \hat{v}^k_m \hat{v}^l_{n-m} \end{aligned} \tag{2.113}$$

where d^{ij} is the inverse matrix to d_{ij} and

$$\hat{v}^i_n = \begin{cases} 0, & i = 1 \text{ and } n > 1 \\ v^i_n, & \text{otherwise} \end{cases} \tag{2.114}$$

has been defined to project out internal a edges.

We have used these relations to compute the effective action for A_μ at $p = 0$. We computed all quadratic and cubic interactions between fields up to level 8 associated with states which are scalars in 25 of the space-time dimensions and which include an arbitrary number of matter oscillators a^{25}_{-n}. Plugging the resulting quadratic and cubic coefficients into the recursion relations (2.113) allows us to compute the coefficients $c_{2n} = v^1_{2n-1}/2n$ in the effective action for the gauge field A_μ

$$\sum_{n=1}^{\infty} -c_{2n} g^n (A^\mu A_\mu)^n \tag{2.115}$$

for small values of n . We have computed these coefficients up to $n = 7$ at different levels of field truncation up to $L = 8$. The results of this computation are given in Table 2.2 up to $n = 5$, including the predicted value at $L = \infty$ from a $1/L$ fit to the data at levels 2, 4, 6 and 8. The results in Table 2.2 indicate that, as expected, all

Level	c_4	c_6	c_8	c_{10}
2	0.200	1.883	6.954	28.65
4	0.097	1.029	6.542	37.49
6	0.063	0.689	5.287	37.62
8	0.046	0.517	4.325	34.18
∞	0.001	0.014	-0.229	1.959

Table 2.2: Coefficients of A^{2n} at various levels of truncation

coefficients c_{2n} will vanish when the level is taken to infinity. The initial contribution at level 2 is canceled to within 0.6% for terms A^4, within 0.8% for terms A^6, within 4% for terms A^8, and within 7% for terms A^{10}. It is an impressive success of the level-truncation method that for c_{10}, the cancellation predicted by the $1/L$ expansion is so good, given that the coefficients computed in level truncation *increase* until level

$L = 8$. We have also computed the coefficients for larger values of n, but for $n > 5$ the numerics are less compelling. Indeed, the approximations to the coefficients c_{12} and beyond continue to grow up to level 8. We expect that a good prediction of the cancellation of these higher-order terms would require going to higher level.

The results found here indicate that the method of level truncation in string field theory seems robust enough to correctly compute higher-order terms in the vector field effective action. Computing terms with derivatives at order A^6 and beyond would require some additional work, but it seems that a reasonably efficient computer program should be able to do quite well at computing these terms, even to fairly high powers of A.

2.6 The nonabelian Born-Infeld action

We now consider the theory with a nonabelian gauge group. As we discussed in section 2.2.3, the first term beyond the Yang-Mills action in the nonabelian analogue of the Born-Infeld action has the form $\mathrm{Tr}\, \hat{F}^3$. As in the previous section, we expect that a field redefinition is necessary to get this term from the effective nonabelian vector field theory derived from SFT. In this section we compute the terms in the effective vector field theory to orders $\partial^3 A^3$ and $\partial^2 A^4$, and we verify that after a field redefinition these terms reproduce the corresponding pieces of the $\hat{F}^3$ term, with the correct coefficients. In section 2.6.1 we analyze $\partial^3 A^3$ terms, and in subsection 2.6.2 we consider the $\partial^2 A^4$ terms.

2.6.1 $\partial^3 A^3$ terms

In section 2.4.2 we showed that the terms of the form ∂A^3 in the nonabelian SFT effective action for A contribute to the $\hat{F}^2$ term after a field redefinition. We now consider terms at order $\partial^3 A^3$. Recall from (2.65) and (2.67) that the full effective action for α and A at this order is given by

$$\check{S}_{(A+\alpha)^3}[A,\alpha] = ig_{YM}\int dx\,\mathrm{Tr}\Big(\frac{1}{6}(\partial_\lambda\tilde{A}^\mu\partial_\mu\tilde{A}^\nu\partial_\nu\tilde{A}^\lambda - \partial_\nu\tilde{A}^\mu\partial_\lambda\tilde{A}^\nu\partial_\mu\tilde{A}^\lambda)$$
$$- \partial_\mu\tilde{A}_\nu[\tilde{A}^\mu,\tilde{A}^\nu] + \frac{1}{2}[\tilde{A}_\nu,\partial^\lambda\tilde{A}_\mu]\partial^\mu\partial^\nu\tilde{A}_\lambda + \tilde{A}^\mu\,[\partial_\mu\tilde{\alpha};\tilde{\alpha}]\Big) \quad (2.116)$$

where $\tilde{A}_\mu = \exp(-\frac{1}{2}V_{00}^{11}\partial^2)A_\mu$, and similarly for $\tilde{\alpha}$. After eliminating α from (2.116) using the equation of motion obtained from (2.61) and integrating by parts to remove terms containing ∂A, we find that the complete set of terms at order ∂^3A^3 is given by

$$S^{[3]}_{A^3}[A] = ig_{YM}\int dx\,\mathrm{Tr}\Big(\frac{2}{3}(\partial_\lambda A^\mu\partial_\mu A^\nu\partial_\nu A^\lambda - \partial_\nu A^\mu\partial_\lambda A^\nu\partial_\mu A^\lambda)$$
$$+\frac{1}{2}V_{00}^{11}\left(\partial_\mu\partial^2A_\nu[A^\mu,A^\nu] + \partial_\mu A_\nu[\partial^2A^\mu,A^\nu] + \partial_\mu A_\nu[A^\mu,\partial^2A^\nu]\right)\Big). \quad (2.117)$$

Note that unlike the quartic terms in A, our expressions for these terms are exact.

Let us now consider the possible terms that we can get after the field redefinition to the field $\hat{A}$ with standard gauge transformation rules. Following the analysis of [90], we write the most general covariant action to order $\hat{F}^3$ (keeping D at order $F^{1/2}$ as discussed above)

$$-\frac{1}{4}\hat{F}^2 + ig_{YM}a\hat{F}^3 + \chi\hat{D}_\sigma\hat{F}^{\sigma\mu}\hat{D}^\nu\hat{F}_{\nu\mu} + \mathcal{O}(\hat{F}^4), \quad (2.118)$$

where

$$\hat{D}_\mu = \partial_\mu - ig_{YM}[\hat{A}_\mu,\cdot\,]. \quad (2.119)$$

The action (2.118) is not invariant under field redefinitions which keep the gauge invariance unchanged. Under the field redefinition

$$\hat{A}'_\mu = \hat{A}_\mu + v\hat{D}_\sigma\hat{F}^\sigma_{\ \mu}. \quad (2.120)$$

we have

$$
\begin{aligned}
a' &= a, \\
\chi' &= \chi - v. \qquad (2.121)
\end{aligned}
$$

Thus, the coefficient a is defined unambiguously, while χ can be set to any chosen value by a field redefinition.

Just as we have an exact formula for the cubic terms in the SFT action, we can also compute the gauge transformation rule exactly to quadratic order in A using (2.5). After some calculation, we find that the gauge variation for A_μ to order $A^2\lambda$ is given by (before integrating out α)

$$
\begin{aligned}
\delta A_\mu = \partial_\mu \lambda - i g_{YM} \Big([A_\mu, \lambda]_\star - [\partial_\mu A_\nu, \partial^\nu \lambda]_\star + [A^\nu, \partial_\mu \partial_\nu \lambda]_\star + \\
\frac{1}{\sqrt{2}} [\partial_\mu B, \lambda]_\star - \frac{1}{\sqrt{2}} [B, \partial_\mu \lambda]_\star \Big). \qquad (2.122)
\end{aligned}
$$

where $B = \alpha - \frac{1}{\sqrt{2}} \partial_\mu A^\mu$ as in section 2.4.1. The commutators are taken with respect to the product

$$
f(x) \star g(x) = f(x) e^{-V_{00}^{11} (\overleftarrow{\partial}^2 + \overleftarrow{\partial} \cdot \overrightarrow{\partial} + \overrightarrow{\partial}^2)} g(x). \qquad (2.123)
$$

The equation of motion for α at leading order is simply $B = 0$. Eliminating α we therefore have

$$
\delta A_\mu = \partial_\mu \lambda - i g_{YM} \Big([A_\mu, \lambda]_\star + [\partial^\nu \lambda, \partial_\mu A_\nu]_\star + [A^\nu, \partial_\mu \partial_\nu \lambda]_\star \Big). \qquad (2.124)
$$

We are interested in considering the terms at order $\partial^2 A \lambda$ in this gauge variation. Recall that in section 2.5.2 we observed that the gauge transformation may include extra trivial terms which vanish on shell. Since the leading term in the equation of motion for A arises at order $\partial^2 A$, it is possible that (2.124) may contain a term of the form

$$
\delta A_\mu = \rho \, [\lambda, \partial^2 A_\mu - \partial_\mu \partial \cdot A] + \mathcal{O}(\lambda A^2) \qquad (2.125)
$$

in addition to a part which can be transformed into the standard nonabelian gauge variation through a field redefinition. Thus, we wish to consider the one-parameter family of gauge transformations

$$
\begin{aligned}
&\delta A = \partial_\mu \lambda - i g_{YM}\big([A_\mu, \lambda] - V_{00}^{11}[\partial^2 A_\mu, \lambda] \\
&- V_{00}^{11}[\partial_\nu A_\mu, \partial^\nu \lambda] - V_{00}^{11}[A_\mu, \partial^2 \lambda] + \rho\, [\lambda, \partial^2 A_\mu - \partial_\mu \partial \cdot A] + \mathcal{O}(\lambda A^2, \lambda \partial^4 A)\big), \quad (2.126)
\end{aligned}
$$

where ρ is an as-yet undetermined constant. We now need to show, following the second method discussed in subsection 2.5.2, that there exists a field redefinition which takes a field A with action (2.117) and a gauge transformation of the form (2.126) to a gauge field $\hat{A}$ with an action of the form (2.118) and the standard nonabelian gauge transformation rule.

The leading terms in the field redefinition can be parameterized as

$$
\begin{aligned}
&\hat{A}_\mu = A_\mu + v_1 \partial_\mu \partial \cdot A + v_2 \partial^2 A_\mu + i g_{YM}\big(v_3 [A_\sigma, \partial_\mu A^\sigma] + v_4 [A_\mu, \partial \cdot A] + v_5 [\partial_\sigma A_\mu, A^\sigma]\big), \\
&\hat{\lambda} = \lambda + v_6 \partial^2 \lambda + i g_{YM}\big(v_7 [\partial \cdot A, \lambda] + v_8 [A_\sigma, \partial^\sigma \lambda]\big). \quad (2.127)
\end{aligned}
$$

The coefficient v_1 can be chosen arbitrarily through a gauge transformation, so we simply choose $v_1 = -v_2$. The requirement that the RHS of (2.127) varied with (2.126) and rewritten in terms of $\hat{A}$, $\hat{\lambda}$ gives the standard transformation law for $\hat{A}$, $\hat{\lambda}$ up to terms of order $\mathcal{O}(\hat{\lambda}\hat{A}^2)$ gives a system of linear equations with solutions depending on one free parameter v.

$$
\begin{aligned}
v_2 &= -v_1 = v, & \rho &= V_{00}^{11}, \\
v_3 &= 1 - \frac{1}{2} V_{00}^{11} + v, & v_6 &= 0, \\
v_4 &= -V_{00}^{11} + v, & v_7 &= V_{00}^{11}, \qquad (2.128)\\
v_5 &= -V_{00}^{11} + 2v, & v_8 &= \frac{1}{2} V_{00}^{11}.
\end{aligned}
$$

It is easy to see that the parameter v generates the field redefinition (2.120). For simplicity, we set $v = 0$. The field redefinition is then given by

$$\hat{A}_\mu = A_\mu - ig_{YM}\Big((\frac{1}{2}V_{00}^{11} - 1)[A_\sigma, \partial_\mu A^\sigma] + V_{00}^{11}[A_\mu, \partial \cdot A] + V_{00}^{11}[\partial_\sigma A_\mu, A^\sigma] \Big) . \quad (2.129)$$

We can now plug in the field redefinition (2.129) into the action (2.118) and compare with the $\partial^3 A^3$ term in the SFT effective action (2.117). We find agreement when the coefficients in (2.118) are given by

$$a = \frac{2}{3}, \qquad \chi = 0. \quad (2.130)$$

Thus, we have shown that the terms of order $\partial^3 A^3$ in the effective nonabelian vector field action derived from SFT are in complete agreement with the first nontrivial term in the nonabelian analogue of the Born-Infeld theory, including the overall constant. Note that while the coefficient of a agrees with that in (2.28), the condition $\chi = 0$ followed directly from our choice $v = 0$; other choices of v would lead to other values of χ, which would be equivalent under the field redefinition (2.120).

2.6.2 $\partial^2 A^4$ terms

In the abelian theory, the $\partial^2 A^4$ terms disappear after the field redefinition. In the nonabelian case, however, the term proportional to $\hat{F}^3$ contains terms of the form $\partial^2 \hat{A}^4$. In this subsection, we show that these terms are correctly reproduced by string field theory after the appropriate field redefinition. Just as in section 2.5.3, for simplicity we shall concentrate on the $\partial^2 A^4$ terms where the Lorentz indices on derivatives are contracted together.

The terms of interest in the effective nonabelian vector field action can be written in the form

$$\begin{aligned} S_{A^4}^{[2]} = g_{YM}^2 \int d^{26}x \Big(& f_1 \partial_\sigma A_\mu A^\mu \partial^\sigma A_\nu A^\nu + f_2 \partial_\sigma A_\mu A^\mu A^\nu \partial^\sigma A_\nu + f_3 A^\mu \partial_\sigma A_\mu A_\nu \partial^\sigma A^\nu \\ & + f_4 \partial_\sigma A_\mu \partial^\sigma A^\mu A_\nu A^\nu + f_5 \partial_\sigma A_\mu \partial^\sigma A_\nu A^\mu A^\nu + f_6 \partial_\sigma A_\mu A^\nu \partial^\sigma A_\mu A^\nu \Big) \end{aligned} \quad (2.131)$$

where the coefficients f_i will be determined below. The coefficients of the terms in the field redefinition which are linear and quadratic in A were fixed in the previous subsection. The relevant terms in the field redefinition for computing the terms we are interested in here are generic terms of order A^3 with no derivatives, as well as those from (2.129) that do not have ∂_μ's contracted with A_μ's. Keeping only these terms we can parametrize the field redefinition as

$$\hat{A}_\mu = A_\mu + ig_{YM}(1 - \frac{V_{00}^{11}}{2})[A_\sigma, \partial_\mu A^\sigma] + g_{YM}^2\big(\rho_1 A_\sigma A_\mu A^\sigma + \rho_2 A^2 A_\mu + \rho_3 A_\mu A^2\big). \quad (2.132)$$

In the abelian case this formula reduces to (2.86) with $\rho_1 + \rho_2 + \rho_3 = 2\gamma$. Plugging this field redefinition into the action

$$\hat{S}[\hat{A}_\mu] = \int \mathrm{Tr}\left(-\frac{1}{4}\hat{F}^2 + \frac{2i}{3}g_{YM}\hat{F}^3 + \mathcal{O}(\hat{F}^4)\right). \quad (2.133)$$

and collecting $\partial^2 A^4$ terms with indices on derivatives contracted together we get

$$\begin{aligned} g_{YM}^2 \int dx \Big[&(\frac{1}{2}V_{00}^{11} - 1 - \rho_3)\partial_\sigma A_\mu A^\mu \partial^\sigma A_\nu A^\nu - (\rho_2 + \rho_3 + V_{00}^{11})\partial_\sigma A_\mu A^\mu A_\nu \partial^\sigma A^\nu \\ &+ (\frac{1}{2}V_{00}^{11} - 1 - \rho_2)A_\mu \partial^\sigma A^\mu A_\nu \partial^\sigma A^\nu - (\rho_2 + \rho_3)\partial_\sigma A_\mu \partial^\sigma A^\mu A_\nu A^\nu \\ &+ (2 - 2\rho_1)\partial_\sigma A_\mu \partial^\sigma A_\nu A^\mu A^\nu - \rho_1 \partial_\sigma A_\mu A_\nu \partial^\sigma A^\mu A_\nu \Big]. \end{aligned} \quad (2.134)$$

Comparing (2.134) and (2.131) we can write the unknown coefficients in the field redefinition in terms of the f_i's through

$$\rho_1 = -f_6, \qquad \rho_2 = \rho_3 = -\frac{1}{2}f_4. \quad (2.135)$$

We also find a set of constraints on the f_i's which we expect the values computed from the SFT calculation to satisfy, namely

$$f_1 - \frac{1}{2}f_4 = -1 + \frac{1}{2}V_{00}^{11}, \qquad f_2 - f_4 = -V_{00}^{11}, \qquad f_5 - 2f_6 = 2. \quad (2.136)$$

On the string field theory side the coefficients f_i are given by

$$f_i = \frac{1}{2}\mathcal{N}^2 \int_0^\infty d\tau e^\tau \,\mathrm{Det}\left(\frac{1-\tilde{X}^2}{(1-\tilde{V}^2)^{13}}\right) P_{\partial^2 A^4,i}(A,B,C) \tag{2.137}$$

where, in complete analogy with the previous examples, the polynomials $P_{\partial^2 A^4,i}$ derived from (2.52) and (2.53) have the form

$$\begin{aligned}
P_{\partial^2 A^4,1} &= -2\big(A_{11}^2 B_{00} + C_{11}^2 B_{00}\big), & P_{\partial^2 A^4,4} &= -4\big(A_{11}^2 A_{00} + C_{11}^2 C_{00}\big),\\
P_{\partial^2 A^4,2} &= -4\big(A_{11}^2 C_{00} + C_{11}^2 A_{00}\big), & P_{\partial^2 A^4,5} &= -4B_{11}^2\big(A_{00} + C_{00}\big),\\
P_{\partial^2 A^4,3} &= -2\big(A_{11}^2 B_{00} + C_{11}^2 B_{00}\big), & P_{\partial^2 A^4,6} &= -4B_{11}^2 B_{00}.
\end{aligned} \tag{2.138}$$

Numerical computation of the integrals gives

$$\begin{aligned}
f_1 &\approx -2.2827697, & f_4 &\approx -2.0422916,\\
f_2 &\approx -1.5190433, & f_5 &\approx -2.5206270,\\
f_3 &\approx -2.2827697, & f_6 &\approx -2.2603135.
\end{aligned} \tag{2.139}$$

As one can easily check, the relations (2.136) are satisfied with high accuracy. This verifies that the $\partial^2 A^4$ terms we have computed in the effective vector field action are in agreement with the $\hat{F}^3$ term in the nonabelian analogue of the Born-Infeld action.

2.7 Summary

In this section we have computed the effective action for the massless open string vector field by integrating out all other fields in Witten's cubic open bosonic string field theory. We have calculated the leading terms in the off-shell action $S[A]$ for the massless vector field A_μ, which we have transformed using a field redefinition into an action $\hat{S}[\hat{A}]$ for a gauge field $\hat{A}$ which transforms under the standard gauge transformation rules. For the abelian theory, we have shown that the resulting action agrees with the Born-Infeld action to order $\hat{F}^4$, and that zero-momentum terms vanish to order A^{10}. For the nonabelian theory, we have shown agreement with the nonabelian effective vector field action previously computed by world-sheet methods to order $\hat{F}^3$. These results demonstrate that string field theory provides a systematic approach to computing the effective action for massless string fields. In principle, the calculation in this paper could be continued to determine higher-derivative corrections to the abelian Born-Infeld action and higher-order terms in the nonabelian theory.

As we have seen in this section, comparing the string field theory effective action to the effective gauge theory action computed using world-sheet methods is complicated by the fact that the fields defined in SFT are related through a nontrivial field redefinition to the fields defined through world-sheet methods. In particular, the massless vector field in SFT has a nonstandard gauge invariance, which is only related to the usual Yang-Mills gauge invariance through a complicated field redefinition. This is a similar situation to that encountered in noncommutative gauge theories, where the gauge field in the noncommutative theory—whose gauge transformation rule is nonstandard and involves the noncommutative star product—is related to a gauge field with conventional transformation rules through the Seiberg-Witten map. In the case of noncommutative Yang-Mills theories, the structure of the field redefinition is closely related to the structure of the gauge-invariant observables of the theory, which in that case are given by open Wilson lines [101]. A related construction appeared in [102], where a field redefinition was used to construct matrix objects transforming naturally under the D4-brane gauge field in a matrix theory of D0-branes and D4-branes. An important outstanding problem in string field theory is to attain a better understanding of the observables of the theory (some progress in this direction was made in [103, 104]). It seems likely that the problem of finding the field redefinition between SFT and world-sheet fields is related to the problem of understanding the proper observables for open string field theory.

Chapter 3

Scattering amplitudes and effective actions from the tachyon non-linear β-function

In this chapter, following the work with V. Forini, G. Grignani, M. Orselli and G. Nardelli [105], we compute the non-linear tachyon β-function of the open bosonic string theory at tree-level. We construct the Witten-Shatashvili (WS) space-time effective action S and prove that it has a very simple universal form in terms of the renormalized tachyon field.

3.1 Introduction

Considering the two-dimensional field theory on the world-sheet , by classical power-counting the tachyon field $T(X)$ has dimension one and is a relevant operator. If $T(X)$ is the only interaction, the field theory is perturbatively super-renormalizable. If $T(X)$ and the other fields are adjusted so that the sigma model that they define is at an infrared fixed point of the renormalization group (RG), these background fields are a solution of the classical equations of motion of string theory. The effective action for a generic coupling constant g^i (which is identified to any field corresponding to on of the open bosonic string excitations) is related to the renormalized partition function of open string theory on the disk, $Z(g^i)$, through

$$S = \left(1 - \beta^i \frac{\delta}{\delta g^i}\right) Z(g^i) \quad , \tag{3.1}$$

where β^i is the beta-function[1] of the coupling g^i. Note that (3.1) fixes the additive ambiguity in S by requiring that at RG fixed points g^*, in which $\beta^i(g^*) = 0$,

$$S(g^*) = Z(g^*) \quad . \tag{3.2}$$

The derivative of the action S with respect to the coupling constant g^i must be related to the β-function through a metric according to

$$\frac{\partial S}{\partial g^i} = -\beta^j G_{ij}(g) \; . \tag{3.3}$$

G_{ij} should be a non-degenerate metric, otherwise there would be an extra zero which could not be interpreted as a conformal field theory on the world sheet. Eq.(3.3) indicates that the RG flow is actually a gradient flow. The prescription (3.1) provides a definition of the metric G_{ij} in the space of couplings.

The β-functions appearing in (3.1) are in general non-linear functions of the couplings g^i. When the linear parts of the β^i (*i.e* the anomalous dimensions λ_i of the corresponding coupling) satisfy a so called "resonant condition", the non linear parts of the β-function cannot be removed by a coordinate redefinition in the space of couplings [22]. Such resonant condition is nothing but the mass-shell condition so that, near the mass-shell, the β-functions are necessarily non-linear.

However, when the resonant condition does not hold, a possible choice of coordinates on the space of string fields is one in which the β-functions are exactly linear. This choice can always be made locally [24] and is well suited to studying processes which are far off-shell, such as tachyon condensation. These coordinates, however, become singular when the components of the string field (*e.g.* $T(X)$, $A_\mu(X)$ *etc.*)

[1] In this paper the β function is positive for relevant perturbations. In some other papers on the subject[24] the opposite conventions are used.

go on-shell. These coordinates can be used to construct, for example, the tachyon effective potential, but become singular when one tries to derive an effective action which reproduces the on-shell amplitudes. In particular, if the Veneziano amplitude needs to emerge from the tachyon effective action it is necessary to consider the whole non-linear β-function in (3.1). A complete renormalization of the theory in fact makes the β-function non-linear in $T(X)$ [106] so that, since the vanishing of the β-function is the field equation for T, these nonlinear terms describe tachyon scattering. One of the goal of this paper is to construct non-linear expressions for the β-functions which are valid away from the RG fixed point. With these expressions for the non-linear tachyon β-function we shall construct the Witten-Shatashvili (WS) space-time action (3.1). We shall prove that (3.1) has the following very simple form in the coupling space coordinates in which the tachyon β-function is non-linear

$$S = K \int d^{26}X \left[1 - T_R(X) + \beta^T(X)\right] \quad , \tag{3.4}$$

where T_R is the renormalized tachyon field and K is a constant related to the D25-brane tension. This formula is universal as it does not depend on how many couplings are switched on. Eq. (3.4) arises from the expression that links the renormalized tachyon field to the partition function that appears in (3.1), namely $Z = K \int d^{26}X(1-T_R)$. T_R is then a non-linear function of the bare coupling T and in these coordinates the β-function is non-linear. When couplings other than the tachyon are introduced in Z, β^T will depend on them so that S will provide the space-time effective action also for these couplings.

With this prescription we shall compute the non-linear β-function β^T for the tachyon field up to the third order in powers of the field and to any order in derivatives of the field. From this we shall show that the solutions of the RG fixed point equations generate the three and four-point open bosonic string scattering amplitudes involving tachyons. Then, with the same renormalization prescription, we shall compute β^T to the leading orders in derivatives but to any power of the tachyon field and we shall show that S coincides with the one-found in [23, 24, 25]. Obviously, S up to the first three powers of T and expanded to the leading order in powers of derivatives can be obtained from both calculations and the results coincide.

In the case of profiles $T_R(k)$ that have support near the on-shell momentum $k^2 \simeq 1$ the equation $\beta^T(k) = 0$ can be derived as the equation of motion of an action. We shall show that this action coincides with the tachyon effective action computed, for the almost on-shell profiles, form the cubic string field theory up to the fourth power of the tachyon field.

The knowledge of the non-linear tachyon β-function is very important also for another reason. The solutions of the equation $\beta^T = 0$ give the conformal fixed points, the backgrounds that are consistent with the string dynamics. In the case of slowly varying tachyon profiles, we shall show that the equations of motion for the WS action can be made identical to the RG fixed point equation $\beta^T = 0$. We shall find solutions of this equation to which correspond a finite value of the WS action. Being solutions of the RG equations, these solitons are lower dimensional D-branes for which the finite value of S provides a quite accurate prediction of the D-brane tension.

We shall also show that the WS action constructed in terms of a linear β-function [107] is related to the action (3.4) by a field redefinition, and that this field redefinition becomes singular on-shell. This is in agreement with the Poincaré-Dulac theorem [108] used in [22] to prove that when the resonant condition holds, namely near the on-shellness, the β-function has to be non-linear.

3.2 Boundary string field theory

In Witten's construction of open boundary string field theory [19] the space of all two dimensional worldsheet field theories on the unit disk, which are conformal in the interior of the disk but have arbitrary boundary interactions, is described by the world-sheet action

$$\mathcal{S} = \mathcal{S}_0 + \int_0^{2\pi} \frac{d\tau}{2\pi} \mathcal{V} \tag{3.5}$$

where $\mathcal{S}_0$ is a free action describing an open plus closed conformal background and $\mathcal{V}$ is a general perturbation defined on the disk boundary. We will discuss the twenty six dimensional bosonic string, for which (3.5) can be expressed in terms of a derivative expansion (or level expansion) of the form

$$\mathcal{V} = T(X) + A_\mu(X)\partial_\tau X^\mu + B_{\mu\nu}(X)\partial_\tau X^\mu \partial_\tau X^\nu + C_\mu(X)\partial_\tau^2 X^\mu + \cdots \tag{3.6}$$

Without the perturbation $\mathcal{V}$ the boundary conditions on X are $\partial_r X^\mu|_{r=1} = 0$, where r is the radial variable on the disk.

$\mathcal{V}$ is a ghost number zero operator and it is useful to introduce a ghost number one operator $\mathcal{O}$ via

$$\mathcal{V} = b_{-1}\mathcal{O}. \tag{3.7}$$

We shall consider the simplest case in which ghosts decouple from matter so that, as in (3.6), $\mathcal{V}$ is constructed out of matter fields alone

$$\mathcal{O} = c\mathcal{V}\ . \tag{3.8}$$

The space-time string field theory action S is defined through its derivative dS which is a two point function computed with the worldsheet action (3.5). More generally one can introduce some basis elements $\mathcal{V}_i$ for operators of ghost number 0 so that the space of boundary perturbations $\mathcal{V}$ can be parametrized as

$$\mathcal{V} = \sum_i g^i \mathcal{V}_i \tag{3.9}$$

where the coefficients g^i are couplings on the world-sheet theory, which are regarded as fields from the space-time point of view, and $\mathcal{O} = \sum_i g^i \mathcal{O}_i$. In this parametrization the space-time action is defined through its derivatives with respect to the couplings and has the form

$$\frac{\partial S}{\partial g^i} = \frac{K}{2}\int_0^{2\pi}\frac{d\tau}{2\pi}\int_0^{2\pi}\frac{d\tau'}{2\pi}\langle \mathcal{O}_i(\tau)\{Q, \mathcal{O}(\tau')\}\rangle_g\ , \tag{3.10}$$

where Q is the BRST charge and the correlator is evaluated with the full perturbed worldsheet action $\mathcal{S}$.

If $\mathcal{V}_i$ is a conformal primary field of dimension Δ_i, for $\mathcal{O}$'s of the form (3.8), one has

$$\{Q, c\mathcal{V}_i\} = (1 - \Delta_i)c\partial_\tau c\mathcal{V}_i \ , \tag{3.11}$$

so that from (3.10) one gets

$$\frac{\partial S}{\partial g^i} = -(1 - \Delta_j)g^j G_{ij}(g) \ , \tag{3.12}$$

where

$$G_{ij} = 2K \int_0^{2\pi} \frac{d\tau}{2\pi} \int_0^{2\pi} \frac{d\tau'}{2\pi} \sin^2(\frac{\tau - \tau'}{2}) \langle \mathcal{V}_i(\tau)\mathcal{V}_j(\tau') \rangle_g \ . \tag{3.13}$$

Eq.(3.12) cannot be true in general, since it does not transform covariantly under reparametrizations of the space of theories, $g^j \to f^j(g^i)$. Indeed, $\partial_i S$ and G_{ij} transform as tensors, (the latter is the metric on the space of worldsheet theories), but g^i does not.

The correct covariant generalization of (3.12) was given in [21, 22]. The worldsheet RG defines a natural vector field on the space of theories: the β-function $\beta^i(g)$, which transforms as a covariant vector under reparametrizations of g^i. The covariant form of (3.12) is thus (3.3). If we assume that total derivatives inside the correlation function decouple and that there are no contact terms, it turns out that the β-function in (3.1) is the linear β-function. This implies that the equations of motion derived from the action (3.1) are just linear. However, as shown by Shatashvili [21, 22], contact terms show up in the computation on the world-sheet and cannot be ignored. The point is that the operator Q, which is constructed out of the BRST operator in the bulk and should be independent on the couplings because the perturbation is on the boundary, actually depends on the couplings when the contour integral approaches the boundary of the disk. A way to fix the structure of the contact terms is to consider that, since dS is a one-form, the derivative of dS should be zero independently of the choice of the contact terms that one makes in the computation. This leads to the following formula for the vector field in equation (3.1)

$$\beta^i = (1 - \Delta_i)g^i + \alpha^i_{jk} g^j g^k + \gamma^i_{jkl} g^j g^k g^l + \cdots \tag{3.14}$$

This is an expression for the β-function with all the non-linear terms. According to the Poincare-Dulac Theorem about vector fields (whose relevance to the β-function related issues was stressed many times by Zamolodchikov [108]) every vector field can be linearized by an appropriate redefinition of the coordinates up to the resonant term. In the second order of equation (3.14) the resonance condition is given by

$$\Delta_j + \Delta_k - \Delta_i = 1 \ . \tag{3.15}$$

The resonance condition means that the β-function cannot be linearized by a coordinate transformation and that all the non-linear terms cannot be removed from the β-function equation (3.14). When g^i is the tachyon field $T(k)$, the resonant condition (3.15) corresponds to the mass-shell conditions for three tachyons. We shall prove in what follows that the WS action S up to the third order in the tachyon fields, constructed in terms of the linear β-function [107], is related to the S made of a non-linear β-function by a field redefinition, but that this field redefinition becomes singular on-shell.

3.3 Integration over the bulk variables

Let us now restrict ourselves to the specific example of open strings propagating in a tachyon background. The partition function reads

$$Z = \int [dX^\mu(\sigma,\tau)] \exp\left(-S[X]\right) \ , \tag{3.16}$$

where the action is

$$S[X] = \int d\sigma d\tau \frac{1}{4\pi} \partial_a X(\sigma,\tau) \cdot \partial_a X(\sigma,\tau) + \int_0^{2\pi} \frac{d\tau}{2\pi} T(X(\tau)) \ . \tag{3.17}$$

Here, the first term in (3.17) is the bulk action and is integrated over the volume of the unit disk. The second term in (3.17) is integrated on the circle which is the boundary of the unit disk and describes the interactions. The scalar fields X^μ have D components with $\mu = 1, ..., D$ and we shall assume $D = 26$ in what follows for a critical string. We are working in a system of units where $\alpha' = 1$.

We begin with the observation that the bulk excitations can be integrated out of (3.16) to get an effective non-local field theory which lives on the boundary [109]. To do this we write the field in the bulk as [110]

$$X = X_{\rm cl} + X_{\rm qu} \ ,$$

where

$$\partial^2 X_{\rm cl} = 0$$

and $X_{\rm cl}$ approaches the fixed (for now) boundary value of X,

$$X_{\rm cl} \to X_{\rm bdry} \text{ and } X_{\rm qu} \to 0 \ .$$

Then, in the bulk, the functional measure is $dX = dX_{\rm qu}$ and

$$S = \int \frac{d^2\sigma}{4\pi} \partial X_{\rm qu} \cdot \partial X_{\rm qu} + \int \frac{d\tau}{2\pi} \left\{ \frac{1}{2} X^\mu |i\partial_\tau| X^\mu + T(X) \right\} \ , \tag{3.18}$$

where we omitted the cl index in the last integral. Then, the integration of $X_{\rm qu}$ produces a multiplicative constant in the partition function - the partition function of the Dirichlet string, which we shall denote K. The kinetic term in the boundary action is non-local. The absolute value of the derivative operator is defined by the Fourier transform,

$$|i\partial_\tau| \delta(\tau - \tau') = \sum_n \frac{|n|}{2\pi} e^{in(\tau - \tau')} \ .$$

The partition function of the boundary theory is then

$$Z(J) = K \int [dX_\mu] e^{-\int_0^{2\pi} \frac{d\tau}{2\pi} \left(\frac{1}{2} X^\mu |i\partial| X^\mu + T(X) - J \cdot X \right)} \ , \tag{3.19}$$

where we have added a source $J^\mu(\tau)$ so that the path integral can be used as a generating functional for correlators of the fields X^μ restricted to the boundary. In particular, this source will allow us to compute the correlation functions of vertex operators of open string degrees of freedom. The remaining path integral over the boundary $X^\mu(\tau)$ defines a one-dimensional field theory with non-local kinetic term. If the tachyon field were absent ($T = 0$), the further integration over $X^\mu(\tau)$ would give a factor which converts the Dirichlet string partition function to the Neumann string partition function.

3.4 Partition function on the disk and the renormalized tachyon field

When only the tachyon field is considered as a boundary perturbation, the Witten-Shatashvili action is given by

$$S = \left(1 - \int \beta^T \frac{\delta}{\delta T}\right) Z \ , \tag{3.20}$$

where Z is the partition function of the boundary theory on the disk and β^T is the tachyon β-function. It is useful to introduce a constant source term k for the zero mode of the X field, the integral over the zero mode variable will just provide the energy-momentum conservation δ-function. The partition function (3.19) in the presence of this constant source reads

$$Z(k) = K \int [dX_\mu] e^{-\int_0^{2\pi} \frac{d\tau}{2\pi}\left(\frac{1}{2}X^\mu|i\partial_\tau|X^\mu + T(X) - ik\cdot\hat{X}\right)} \ , \tag{3.21}$$

where $\hat{X}$ is the zero mode which is defined by

$$\hat{X}^\mu = \int_0^{2\pi} \frac{d\tau}{2\pi} X^\mu(\tau) \ . \tag{3.22}$$

In this section we shall expand the exponential in eq.(3.21) in powers of $T(X)$. The first non-trivial term is

$$Z^{(1)}(k) = -K \int [dX_\mu] \int dk_1 \int_0^{2\pi} \frac{d\tau_1}{2\pi} T(k_1) e^{-\int_0^{2\pi} \frac{d\tau}{2\pi} \left(\frac{1}{2} X^\mu |i\partial_\tau| X^\mu\right) - ik\hat{X} + ik_1 X(\tau_1)} \ . \quad (3.23)$$

The functional integral over the non-zero modes of $X(\tau)$ gives

$$Z^{(1)}(k) = -K \int d\hat{X}_\mu \int dk_1 T(k_1) e^{-\frac{k_1^2}{2} G(0) + i(k_1 - k)\hat{X}} \ , \quad (3.24)$$

where $G(\tau)$ is the Green function of the operator $|i\partial_\tau|$

$$G(\tau_1 - \tau_2) = 2 \sum_{n=1}^{\infty} e^{\epsilon n} \frac{\cos n\,(\tau_1 - \tau_2)}{n} = -\log\left[1 - 2e^{-\epsilon} \cos(\tau_1 - \tau_2) + e^{-2\epsilon}\right] \quad (3.25)$$

and ϵ is an ultraviolet cut-off. In all the calculations we shall use the following prescription for $G(\tau)$

$$G(\tau) = \begin{cases} -\log\left[c \sin^2\left(\frac{\tau}{2}\right)\right] & \tau \neq 0 \\ -2\log\epsilon & \tau = 0. \end{cases} \quad (3.26)$$

The coefficient c reflects the ambiguity involved in subtracting the divergent terms. Its value is scheme dependent and should be fixed by some renormalization prescription. We choose the value $c = 4$ for later convenience. This arbitrariness was discussed in [25, 110]. The integrals over the zero-modes in eq.(3.24) give a 26-dimensional δ-function so that

$$-Z^{(1)}(k) = KT(k)\epsilon^{k^2-1} \quad (3.27)$$

and we can identify

$$T_R(k) \equiv T(k)\epsilon^{k^2-1} = -\frac{Z^{(1)}(k)}{K} \ . \quad (3.28)$$

This equation provides the renormalized coupling T_R in terms of the bare coupling T to the lowest order in perturbation theory. $1 - k^2$ is the anomalous dimension of the tachyon field. The second order term in T is given by

$$Z^{(2)}(k) = K \int_0^{2\pi} \frac{d\tau_1}{4\pi} \frac{d\tau_2}{2\pi} \int dk_1 dk_2 T(k_1) T(k_2) \left\langle e^{ik_1 X(\tau_1)} e^{ik_2 X(\tau_2)} e^{-ik\hat{X}} \right\rangle . \tag{3.29}$$

Again in (3.29) the integral over the zero modes $\hat{X}^\mu$ gives just a 26-dimensional δ-function, $\delta(k - k_1 - k_2)$, and we can perform the integral over the non-zero modes of $X(\tau)$ to get

$$\begin{aligned} Z^{(2)}(k) = \quad & K \int_0^{2\pi} \frac{d\tau_1}{4\pi} \frac{d\tau_2}{2\pi} \int dk_1 dk_2 (2\pi)^D \delta(k - k_1 - k_2) T(k_1) T(k_2) \\ & \exp\left[-\frac{1}{2} \left(k_1^2 + k_2^2 \right) G(0) - k_1 k_2 G(\tau_1 - \tau_2) \right] . \end{aligned} \tag{3.30}$$

The integral in (3.30) becomes

$$\begin{aligned} Z^{(2)}(k) = \quad & K \int dk_1 dk_2 (2\pi)^D \delta(k - k_1 - k_2) \epsilon^{k_1^2 + k_2^2 - 2} T(k_1) T(k_2) \\ & \int_0^{2\pi} \frac{d\tau_1}{4\pi} \frac{d\tau_2}{2\pi} \left[4 \sin^2 \left(\frac{\tau_1 - \tau_2}{2} \right) \right]^{k_1 k_2} . \end{aligned} \tag{3.31}$$

The integral over the relative variable $x = (\tau_1 - \tau_2)/2$ does not need regularization, it converges when $1 + 2k_1 k_2 > 0$, providing the result

$$Z^{(2)}(k) = \frac{K}{2} \int dk_1 dk_2 (2\pi)^D \delta(k - k_1 - k_2) \epsilon^{k_1^2 + k_2^2 - 2} T(k_1) T(k_2) \frac{\Gamma(1 + 2k_1 k_2)}{\Gamma^2(1 + k_1 k_2)} . \tag{3.32}$$

The integrand in (3.32) can be analytically continued also to the region where $1 + 2k_1 k_2 < 0$, so that the integral can be performed.

To the second order in perturbation theory the renormalized coupling in terms of the bare coupling reads

$$\begin{aligned} T_R(k) &= -\frac{Z^{(1)}(k) + Z^{(2)}(k)}{K} \\ &= \epsilon^{k^2 - 1} \left[T(k) - \frac{1}{2} \int dk_1 dk_2 (2\pi)^D \delta(k - k_1 - k_2) T(k_1) T(k_2) \epsilon^{-(1 + 2k_1 k_2)} \frac{\Gamma(1 + 2k_1 k_2)}{\Gamma^2(1 + k_1 k_2)} \right] . \end{aligned} \tag{3.33}$$

The third order contribution to the partition function is given by

$$Z^{(3)}(k) = -\frac{K}{3!} \int dk_1 dk_2 dk_3 (2\pi)^D \delta\left(k - \sum_{i=1}^{3} k_i\right) \epsilon^{\sum_{i=1}^{3} k_i^2 - 3} T(k_1) T(k_2) T(k_3) I(k_1, k_2, k_3)\,, \tag{3.34}$$

where $I(k_1, k_2, k_3)$ is the integral

$$\begin{aligned} I(k_1, k_2, k_3) = \; & \frac{2^{2k_1k_2+2k_2k_3+2k_1k_3}}{(2\pi)^3} \int_0^{2\pi} d\tau_1 d\tau_2 d\tau_3 \left[\sin^2\left(\frac{\tau_1 - \tau_2}{2}\right)\right]^{k_1k_2} \\ & \left[\sin^2\left(\frac{\tau_2 - \tau_3}{2}\right)\right]^{k_2k_3} \left[\sin^2\left(\frac{\tau_1 - \tau_3}{2}\right)\right]^{k_1k_3} . \end{aligned} \tag{3.35}$$

The complete computation of $I(k_1, k_2, k_3)$ will be given in Appendix B. The result is given by the completely symmetric formula

$$I(a_1, a_2, a_3) = \frac{\Gamma(1 + a_1 + a_2 + a_3)\Gamma(1 + 2a_1)\Gamma(1 + 2a_2)\Gamma(1 + 2a_3)}{\Gamma(1 + a_1)\Gamma(1 + a_2)\Gamma(1 + a_3)\Gamma(1 + a_1 + a_2)\Gamma(1 + a_2 + a_3)\Gamma(1 + a_1 + a_3)}\,, \tag{3.36}$$

where we have set $a_1 = k_1k_2$, $a_2 = k_2k_3$ and $a_3 = k_1k_3$. The integral (3.35) converges when $1+a_1+a_2+a_3 > 0$, but its result (3.36) can be analytically continued also outside this convergence region. The result (3.36) is in agreement with the one obtained, with a different procedure, in [107] but does not coincide with the one provided in the appendix of ref. [24]. Up to the third order in powers of T and to all orders in k_i the relation between the bare and the renormalized couplings reads

$$\begin{aligned} T_R(k) &= -\frac{Z^{(1)}(k) + Z^{(2)}(k) + Z^{(3)}(k)}{K} \\ &= \epsilon^{k^2-1} \Bigg[T(k) - \frac{1}{2} \int dk_1 dk_2 (2\pi)^D \delta\left(k - \sum_{i=1}^{3} k_i\right) T(k_1) T(k_2) \epsilon^{-(1+2k_1k_2)} \frac{\Gamma\left(1 + 2k_1k_2\right)}{\Gamma^2\left(1 + k_1k_2\right)} \\ &+ \int dk_1 dk_2 dk_3 \frac{(2\pi)^D}{3!} \delta\left(k - \sum_{i=1}^{3} k_i\right) T(k_1) T(k_2) T(k_3) \epsilon^{-2(1+\sum_{i<j} k_ik_j)} I(k_1, k_2, k_3) \Bigg] . \end{aligned} \tag{3.37}$$

In section 6 we shall use this expression to construct the non-linear β-function.

The renormalized tachyon field can be constructed to all powers of the bare tachyon field in the case in which the tachyon profile appearing in (3.21) is a slowly varying function of X^μ. In this case one can consider an expansion of (3.21) in powers of derivatives of T. To this purpose consider the n-th term in the expansion of (3.21) in powers of $T(X(\tau))$, $Z^{(n)}(k)$. Taking the Fourier transform of the tachyon field and performing all the contractions of the $X(\tau_i)$ fields, for $Z^{(n)}(k)$ we get

$$Z^{(n)}(k) = K\frac{(-1)^n}{n!}\epsilon^{-n}\int\prod_{i=1}^{n} dk_i T(k_i)\int_0^{2\pi}\prod_{i=1}^{n}\left(\frac{d\tau_i}{2\pi}\right)$$
$$e^{-\sum_{i=1}^{n}\frac{k_i^2}{2}G(0)-\sum_{i<j}^{n}k_ik_jG(\tau_i-\tau_j)}\delta\left(k-\sum_{i=1}^{n}k_i\right) . \tag{3.38}$$

Note that with our regularization prescription the dependence on the cut-off in (3.38) comes only from the zero distance propagator $G(0)$ and from the explicit scale dependence of the tachyon field. If the tachyon profile is a slowly varying function of X^μ we can expand inside the integrand of (3.38) in powers of the momenta k_i. The leading and next to leading terms in this expansion read

$$\begin{aligned} Z^{(n)}(k) = \quad & K\frac{(-1)^n}{n!}\prod_{i=1}^{n}\int dk_i\delta\left(k-\sum_{i=1}^{n}k_i\right)\epsilon^{-n}\prod_{i=1}^{n}T(k_i) \\ & \left(1+\sum_{i=1}^{n}k_i^2\log\epsilon+\sum_{i<j}^{n}k_ik_j\log\frac{c}{4}\right) , \end{aligned} \tag{3.39}$$

where the last term comes from the integral over a couple of τ variables of the propagator $G(\tau_i-\tau_j)$, the other integrations over τ_k $k\neq i,j$ being trivial. Here we have kept explicit the ambiguity c appearing in the propagator (3.26) to show that the result greatly simplifies with the choice $c=4$. Unless otherwise stated, we shall adopt this choice throughout the paper. As before, the renormalized tachyon field $T_R(k)$ can be obtained from (3.39) by summing over n from 1 to ∞, changing sign and dividing by K. Taking the Fourier transform of $T_R(k)$ with $c=4$, to all orders in the bare tachyon field and to the leading order in derivatives, we get the renormalized tachyon field $T_R(X)$

$$T_R(X) = 1-\exp\left\{-\frac{1}{\epsilon}\left[T(X)-\Delta T(X)\log\epsilon\right]\right\} \tag{3.40}$$

where Δ is the Laplacian. Again in section 6 we shall use this expression to compute the non-linear tachyon β-function.

From eqs.(3.28,3.33,3.37,3.40) it is clear that the general relation between the renormalized tachyon field $T_R(X)$ and the partition function $Z \equiv Z(k=0)$ is simply

$$Z = K \int d^{26}X \, [1 - T_R(X)] \ . \tag{3.41}$$

This expression is true also when other couplings are present. T_R in this case would be a non linear function also of the other bare couplings but its relation with the partition function of the theory would always be given by (3.41). We shall prove eq.(3.41) in the next section.

3.5 Background-field method

The partition function of the boundary theory on the disk in general is given by

$$Z = K \int [dX_\mu] e^{-\left(S_0[X] + \int_0^{2\pi} \frac{d\tau}{2\pi} \mathcal{V}[X(\tau)]\right)} \ , \tag{3.42}$$

where $S_0 = \int d\tau X^\mu |i\partial_\tau| X^\mu$ and $\mathcal{V}[X(\tau)]$ is given in (3.6). Our goal is to determine the relationship between the renormalized and the bare couplings of the one-dimensional field theory. To this purpose we shall make use of the background field method [106]. We expand the fields X^μ around a classical background X_0^μ which satisfies the equations of motion and which varies slowly compared to the cut-off scale,

$$X^\mu = X_0^\mu + Y^\mu \ .$$

The effective action is $S_{\text{eff}}[X_0] = -\log Z[X_0]$ and the aim of the renormalization process is to rewrite the local terms of $S_{\text{eff}}[X_0]$ in terms of renormalized couplings in such a way that $S_{\text{eff}}[X_0]$ has the same form of the original action

$$S_{\text{eff}}[X_0]\Big|_{\text{local}} = S_0[X_0] + \int_0^{2\pi} \frac{d\tau}{2\pi} \mathcal{V}_R[X_0(\tau)] \ . \tag{3.43}$$

$Z[X_0]$ can be conveniently calculated in powers of the boundary interaction $\mathcal{V}$. The first order for example reads, up to the multiplicative constant K,

$$-\int_0^{2\pi}\frac{d\tau}{2\pi}\int dk e^{ikX_0} \quad \langle [T(k)+A_\mu(k)\partial_\tau(X_0^\mu+Y^\mu)+B_{\mu\nu}(k)\partial_\tau(X_0^\mu+Y^\mu)\partial_\tau(X_0^\nu+Y^\nu) \\ +C_\mu(k)\partial_\tau^2(X_0^\mu+Y^\mu)+\cdots]\, e^{ikY}\rangle \ . \tag{3.44}$$

The renormalized couplings $T_R(k)$ will be given by the opposite of the coefficient of the term in (3.44) that does not contain X_0 derivatives. Analogously, the renormalized $A_\mu^R(k)$ will be determined by the coefficient of $\partial_\tau X_0^\mu$, $B_{\mu\nu}^R(k)$ by the coefficient of $\partial_\tau X_0^\mu \partial_\tau X_0^\nu$ and so on. The second order term in the expansion of $Z[X_0]$ is

$$\int_0^{2\pi}\frac{d\tau_1}{2\pi}\int_0^{2\pi}\frac{d\tau_2}{4\pi}\int dk_1 dk_2 e^{ik_1X_0(\tau_1)+ik_2X_0(\tau_2)}\left\langle e^{ik_1Y(\tau_1)+ik_2Y(\tau_2)}\right.$$

$$[T(k_1)+A_\mu(k_1)\partial_{\tau_1}(X_0^\mu+Y^\mu)+\ldots]\,[T(k_2)+A_\nu(k_2)\partial_{\tau_2}(X_0^\nu+Y^\nu)+\ldots]\rangle \tag{3.45}$$

An expansion of the background field X_0 in powers of its derivatives is required to determine the coefficients of 1, $\partial_\tau X_0^\mu$, $\partial_\tau X_0^\mu \partial_\tau X_0^\nu$, $\ldots$,

$$X_0(\tau_2)=X_0(\tau_1)+(\tau_2-\tau_1)\partial_{\tau_1}X_0(\tau_1)+\ldots \ . \tag{3.46}$$

If we are interested in renormalization of couplings of the form $\exp[ikX_0]$, namely in the renormalized tachyon field $T_R(k)$, we can disregard the terms in (3.46,3.45) involving derivatives acting on X_0. For example, at the second order, the only non-vanishing terms in T and A_μ contributing to T_R are

$$T_R(k)=-\quad\int dk_1\int dk_2\delta(k-k_1-k_2)\int_0^{2\pi}\frac{d\tau_2}{4\pi}\left\langle e^{ik_1Y(\tau_1)+ik_2Y(\tau_2)}\right.$$

$$\left.[T(k_1)T(k_2)+A_\mu(k_1)A_\nu(k_2)\partial_{\tau_1}Y^\mu\partial_{\tau_2}Y^\nu+\ldots]\right\rangle \ , \tag{3.47}$$

where the correlator does not depend on τ_1 since the propagator (3.26) of $X(\tau)$ and its derivatives are periodic functions on the unit circle. It is not difficult to see that

$T_R(k)$ in (3.47) coincides with the opposite of the second order term in the expansion of the partition function

$$Z(k) = \int [dY_\mu] e^{-\left(S_0[Y]+\int_0^{2\pi}\frac{d\tau}{2\pi}\mathcal{V}[Y(\tau)]\right)-ik\hat{Y}} \tag{3.48}$$

in powers of the couplings. Here k is a constant source for the zero mode of the Y^μ field, $\hat{Y}^\mu$ (3.22). Such a constant source will just provide the δ-function in (3.47) that imposes the energy-momentum conservation. This will be true at any order in the expansion in powers of the coupling fields. Therefore, to all orders in whatever coupling, the expression for the renormalized tachyon field $T_R(X)$ is related to the partition function $Z = Z(k=0)$ precisely by (3.41), which is the relation that we wanted to prove. Note that T_R depends not only on the bare tachyon field but also on the other coupling fields (in particular T_R will exists also if one starts from a boundary interaction that does not contain the bare tachyon). As a consequence, the tachyon β function will contain for example also the gauge field [111], and this is as it should be, since the solution of the equation $\beta^T = 0$ will then describe the scattering of a tachyon by other excitations (*e.g.* from (3.47) by two vector fields).

3.6 β-function

In this section we shall perform a calculation of the non-linear tachyon β-function. The resulting expression will then be used to derive the Witten-Shatashvili action (3.4,3.20). Following [106], the most general RG equations for a set of couplings g^i can be written as

$$\beta^i \equiv \frac{dg^i}{dt} = \lambda_i g^i + \alpha^i_{jk} g^j g^k + \gamma^i_{jkl} g^j g^k g^l + \cdots \quad , \tag{3.49}$$

where the scale t is $t = -\log\epsilon$, λ_i are the anomalous dimensions corresponding to the couplings g^i and there is no summation in the first term on the right-hand side.

This equation has the solution

$$g^i(t) = e^{\lambda_i t} g^i(0) + \left[e^{(\lambda_j+\lambda_k)t} - e^{\lambda_i t}\right] \frac{\alpha^i_{jk}}{\lambda_j + \lambda_k - \lambda_i} g^j(0) g^k(0) + b^i_{jkl}(t) g^j(0) g^k(0) g^l(0) + \cdots \ , \tag{3.50}$$

where $g^i(0)$ are the bare couplings and

$$\begin{aligned} b^i_{jkl}(t) g^j(0) g^k(0) g^l(0) = &\left[\left(\frac{2\alpha^i_{jm}\alpha^m_{kl}}{\lambda_j + \lambda_m - \lambda_i} - \gamma^i_{jkl}\right) \frac{e^{\lambda_i t}}{\lambda_j + \lambda_k + \lambda_l - \lambda_i}\right. \\ &+ \left(\frac{2\alpha^i_{jm}\alpha^m_{kl}}{\lambda_k + \lambda_l - \lambda_m} + \gamma^i_{jkl}\right) \frac{e^{(\lambda_j+\lambda_k+\lambda_l)t}}{\lambda_j + \lambda_k + \lambda_l - \lambda_i} \\ &\left. - \frac{2\alpha^i_{jm}\alpha^m_{kl}}{(\lambda_j + \lambda_m - \lambda_i)(\lambda_k + \lambda_l - \lambda_m)} e^{(\lambda_j+\lambda_k)t}\right] g^j(0) g^k(0) g^l(0) \ . \end{aligned} \tag{3.51}$$

Let us now consider the case of interest for this paper: open strings propagating in a tachyon background. In this case the coupling g^i is the tachyon field $T(k)$. Then $\lambda_i = 1 - k^2$ and $\lambda_j = 1 - k_j^2$. Comparing the general solution (3.50) with eq.(3.37) derived in the previous section, we will be able to identify the renormalized tachyon field in terms of the bare field up to the third order in powers of the field and to all orders in its derivatives. In the second order term of (3.37) the coefficient proportional to $e^{\lambda_i t} = \epsilon^{1-k^2}$ appearing in (3.49) is absent. This is due to the fact that the convergence condition for the integral (3.31), $1 + 2k_1 k_2 > 0$, implies that $\lambda_j + \lambda_k > \lambda_i$ so that in the limit $t \to \infty$ the dominant contribution comes from $e^{(\lambda_j+\lambda_k)t}$. From similar arguments, the first and the second terms of the right-hand side of (3.51) are negligible compared to the second term, due to the convergence conditions for the integral $I(k_1, k_2, k_3)$ computed in the previous section. This is a general feature of our renormalization procedure. At the n-th order in the bare coupling in the expansion (3.50), the renormalized coupling will contain only the term of the form

$$e^{t \sum_{k=1}^{n} \lambda_k} \ . \tag{3.52}$$

This is due to the fact that the integrals over the τ's do not need an explicit regulator, rather they can be evaluated in a specific region of the k_i variables and then analytically continued. Therefore the only dependence on the cut-off does not come from such integrals but from the propagators (3.26) evaluated at zero distance.

Comparing our result for the renormalized tachyon field (3.37) with the general expressions (3.50,3.51), for the coefficients in the expansion (3.49) we find

$$
\begin{aligned}
\alpha^i_{jk} &= -\frac{1}{2}\frac{\Gamma(2+2k_jk_k)}{\Gamma^2(1+k_jk_k)}\delta(k-k_j-k_k) \\
\gamma^i_{jkl} &= \frac{1}{3!}\int dk_jdk_kdk_l\delta(k-k_j-k_k-k_l)\,[2(1+k_jk_k+k_jk_l+k_kk_l)I(k_j,k_k,k_l) \\
&- \left(\frac{\Gamma(2+2k_jk_k+2k_jk_l)\Gamma(1+2k_kk_l)}{\Gamma^2(1+k_jk_k+k_jk_l)\Gamma^2(1+k_kk_l)}+\text{cycl.}\right)\Bigg] \ ,
\end{aligned}
\tag{3.53}
$$

where $I(k_j,k_k,k_l)$ is given in equation (3.36). The perturbative expression for the β-function up to the third order in the tachyon field obtained using this procedure therefore is

$$
\begin{aligned}
\beta^T(k) &= (1-k^2)T_R(k) - \frac{1}{2}\int dk_1dk_2(2\pi)^D\delta(k-k_1-k_2)T_R(k_1)T_R(k_2)\frac{\Gamma(2+2k_1k_2)}{\Gamma^2(1+k_1k_2)} \\
&+\frac{1}{3!}\int dk_1dk_2dk_3(2\pi)^D\delta(k-k_1-k_2-k_3)T_R(k_1)T_R(k_2)T_R(k_3) \\
&\left[2(1+k_1k_2+k_1k_3+k_2k_3)I(k_1,k_2,k_3) - \left(\frac{\Gamma(2+2k_1k_2+2k_1k_3)\Gamma(1+2k_2k_3)}{\Gamma^2(1+k_1k_2+k_1k_3)\Gamma^2(1+k_2k_3)}+\text{cycl.}\right)\right].
\end{aligned}
\tag{3.54}
$$

We have thus succeeded in deriving a β-function for tachyon backgrounds which do not satisfy the linearized on-shell condition. Exactly the same result can be obtained by taking the derivative of (3.37) (or of the opposite of $Z(k)$) with respect to the logarithm of the cut-off $-\log\epsilon$. The result obtained in this way must then be expressed in terms of the renormalized field by inverting (3.37) and it coincides with (3.54).

It is interesting to note that all the known conformal tachyon profiles, like e^{iX^0} or $\cos X^i$ where i is a space index, are solutions of the equation $\beta^T(X)=0$, where $\beta^T(X)$ is the Fourier transform of (3.54). These solutions and perturbations around them have been recently used to construct tachyon effective actions around the on-shellness [112, 113, 114, 40, 115] and to study the problem of the rolling tachyon [28, 29, 30, 31, 32, 42, 35].

That the non-linear β-function (3.54) is the correct one can be shown by solving the $\beta^T(k) = 0$ equation perturbatively. The solution of this equation will generate the correct scattering amplitudes of open string theory [106]. This in turn will show the validity of the general formula (3.41). To the lowest order the equation is $(1 - k^2)T_0(k) = 0$, so that the solution $T_0(k)$ satisfies the linearized on-shell condition. By writing $T(k) = T_0(k) + T_1(k)$ and substituting into the equation $\beta_T(k) = 0$, to the next order we find

$$T_1(k) = \frac{1}{2}\int dk_1 dk_2 (2\pi)^D \delta(k - k_1 - k_2) T_0(k_1) T_0(k_2) \frac{\Gamma(k^2)}{(1-k^2)\,\Gamma^2(k^2/2)} \,. \tag{3.55}$$

The presence of the couplings T_0 in (3.55) sets two of the three k_i on-shell. To pick out the propagator pole corresponding to the third k we set it on-shell too. The scattering amplitude for three on-shell tachyons is given by the residue of the pole and is $1/2\pi$ with our normalization.

The calculation at the next order proceed in a similar fashion. One sets $T(k) = T_0(k) + T_1(k) + T_2(k)$ and finds

$$\begin{aligned} T_2(k) = &-\frac{(2\pi)^D}{3!(1-k^2)} \int dk_1 dk_2 dk_3 \delta(k - k_1 - k_2 - k_3) T_0(k_1) T_0(k_2) T_0(k_3) I(k_1,k_2,k_3) \\ &\left\{ 2\left(1 + \sum_{i<j} k_i k_j\right) I(k_1,k_2,k_3) - \left[\frac{\Gamma(2+2k_1k_2+2k_1k_3)\Gamma(1+2k_2k_3)}{\Gamma^2(1+k_1k_2+k_1k_3)\Gamma^2(1+k_2k_3)} + \text{cycl.}\right] \right. \\ &\left. - \left[\frac{\Gamma(2+2k_1k_2+2k_1k_3)\Gamma(2+2k_2k_3)}{\Gamma^2(1+k_1k_2+k_1k_3)\Gamma^2(1+k_2k_3)[1-(k_2+k_3)^2]} + \text{cycl.}\right] \right\} \,. \end{aligned} \tag{3.56}$$

When all the tachyons are on-shell, the last two terms on eq. (3.56) cancel and, as it should be for consistency, the residue of the pole in k is the scattering amplitude of four on-shell tachyons. It is given by

$$\frac{\Gamma(1+2k_1k_2)\Gamma(1+2k_2k_3)\Gamma(1+2k_1k_3)}{\Gamma(1+k_1k_2)\Gamma(1+k_2k_3)\Gamma(1+k_1k_3)\Gamma(1+k_1k_2+k_2k_3)\Gamma(1+k_2k_3+k_1k_3)\Gamma(1+k_1k_2+k_1k_3)} \,, \tag{3.57}$$

where the on-shell condition is $1 + k_1k_2 + k_2k_3 + k_1k_3 = 0$. By means of the on-shell condition, from the above expression, we recover, up to a normalization constant, the Veneziano amplitude, the scattering amplitude of four on-shell tachyons. Eq.(3.57) in fact becomes

$$\frac{1}{\pi^3}\Gamma(1+2k_1k_2)\Gamma(1+2k_2k_3)\Gamma(1+2k_1k_3)\sin(\pi k_1)\sin(\pi k_2)\sin(\pi k_3)$$
$$= \frac{1}{(2\pi)^2}\left[B\left(1+2k_1k_2, 1+2k_2k_3\right)+\text{cycl.}\right] , \tag{3.58}$$

where $B(x,y)$ is the Euler beta function. The expression between square brackets is just the Veneziano amplitude. The ambiguity c appearing in the propagator (3.26) could be kept undetermined throughout the calculations of the scattering amplitudes. It is not difficult to see that this would just consistently change the normalization of the on-shell amplitudes.

For tachyon profiles $T_R(k)$ supported over near on-shell momentum $k^2 \simeq 1$, the equation of motion $\beta^T = 0$ with β^T given in (3.54) becomes

$$\begin{aligned}\beta^T(k) &= (1-k^2)T_R(k) - \frac{(2\pi)^D}{2\pi}\int dk_1 dk_2 \delta(k-k_1-k_2)T_R(k_1)T_R(k_2) \\ &+ \frac{(2\pi)^D}{3!(2\pi)^2}\int dk_1 dk_2 dk_3 \delta(k-k_1-k_2-k_3)T_R(k_1)T_R(k_2)T_R(k_3) \\ &\quad \{[B\left(1+2k_1k_2, 1+2k_2k_3\right)+\text{cycl.}] + 2\pi\tan(\pi k_1k_2)\tan(\pi k_1k_3)\tan(\pi k_2k_3)\} = 0 .\end{aligned} \tag{3.59}$$

The coefficients of the quadratic and cubic terms in (3.59) are symmetric with respect to all the k_i and k when these are on the mass-shell. Thus (3.59) can be derived as the equation of motion of an effective action. Such effective action for near on-shell tachyons up to the fourth order in powers of the tachyon fields can be derived from the results of the cubic string field theory. In [65] it was shown that the cubic SFT reproduces the Veneziano amplitude with great accuracy already at level $L = 50$. The tachyon effective action arising from the cubic string field theory for near on shell tachyon profiles $\Phi(k)$ therefore reads

$$\begin{aligned}S_C &= 2\pi^2 T_{25}(2\pi)^D\left\{-\frac{1}{2}\int dk\Phi(k)\Phi(-k)\left(1-k^2\right) + \frac{1}{3}\int\prod_{i=1}^{3}dk_i\Phi(k_i)\delta\left(\sum_{i=1}^{3}k_i\right)\right. \\ &+ \left.\frac{1}{4!}\int\prod_{i=1}^{4}dk_i\Phi(k_i)\delta\left(\sum_{i=1}^{4}k_i\right)\left[B\left(1+2k_1k_2, 1+2k_2k_3\right)+\text{cycl.}\right]\right\} ,\end{aligned}$$

$$ \tag{3.60} $$

where the tachyon momenta in the fourth order term satisfy

$$k_1 = (0, 1, 0, 0, \ldots, 0) \qquad k_2 = (0, \sin\theta, \cos\theta, 0, \ldots, 0) \tag{3.61}$$

$$k_3 = (0, -1, 0, 0, \ldots, 0) \qquad k_4 = (0, -\sin\theta, -\cos\theta, 0, \ldots, 0) \tag{3.62}$$

Since the Veneziano amplitude is completely symmetric in the four momenta k_i, it is not difficult to see that the equation of motion deriving from (3.60) becomes precisely (3.59) once the simple field rescaling $T = 2\pi\Phi$ is performed. Thus the cubic string field theory for almost on-shell tachyons reproduces the non-linear $\beta^T = 0$ equation of motion.

In section 4 we also derived the renormalized tachyon field for the case of a slowly varying tachyon profile, to all orders in the bare field and to the leading order in derivatives, eq.(3.40). From this we can easily compute the corresponding β function. The task in this case is much simpler, as we just need to take the derivative of (3.40) with respect to $-\log\epsilon$

$$\beta(X) = \frac{\partial T_R(X)}{\partial(-\log\epsilon)} = \frac{1}{\epsilon}\exp\left(-\frac{T(X)}{\epsilon}\right)\left\{T(X) + \Delta T(X)\left[1 - \left(1 - \frac{T(X)}{\epsilon}\right)\log\epsilon\right]\right\} . \tag{3.63}$$

Then we have to invert the relation (3.40) between T_R and T. To the leading order in derivatives one has

$$T(X) = -\epsilon\left\{[1 + (\log\epsilon)\Delta]\log(1 - T_R(X))\right\} , \tag{3.64}$$

from which it is clear that the admissible range for T_R is $-\infty \leq T_R \leq 1$. Plugging (3.64) into (3.63) we get the non-linear tachyon β-function to all powers of the renormalized tachyon and to the leading order in its derivatives

$$\beta^T(X) = (1 - T_R(X))\left[-\log(1 - T_R(X)) - \Delta\log(1 - T_R(X))\right] . \tag{3.65}$$

$\beta^T(X) = 0$ is the tachyon equation of motion for a slowly varying tachyon profile.

Since in our calculations of the non-linear β-function we have always used the same coordinates in the space of string fields, the two results (3.65) and (3.54) should coincide when expanded up to the third power of the field and to the leading order in derivatives, respectively. This is indeed the case and the result in both cases reads

$$\beta^T(X) = \Delta T_R + \partial_\mu T_R \partial_\mu T_R + T_R \partial_\mu T_R \partial_\mu T_R \ . \tag{3.66}$$

It is interesting to compute the β-function also in the case in which the ambiguity constant c appearing in (3.26) is kept undetermined. $T_R(k)$ can be easily obtained as before from (3.39) without fixing $c = 4$. By taking the Fourier transform and by differentiating with respect to $-\log\epsilon$, the β-function expressed in terms of the renormalized tachyon field $T_R(X)$ turns out to be

$$\beta^T(X) = (1 - T_R)\left[-\log(1 - T_R) + \frac{\Delta T_R}{1 - T_R} + \left(1 + \frac{1}{2}\log\frac{c}{4}\right)\frac{\partial_\mu T_R \partial_\mu T_R}{(1 - T_R)^2}\right] \ . \tag{3.67}$$

In the next section we shall use also this form of the β-function to construct the Witten-Shatashvili action.

3.7 Witten-Shatashvili action

In this section we shall compute the Witten-Shatashvili action. From the simple expression that relates the partition function to the renormalized tachyon (3.41) it is easy to deduce a simple and universal form for the WS action of the open bosonic string theory

$$S = \left(1 - \int \beta^T \frac{\delta}{\delta T_R}\right) Z[T_R] = K \int d^D X \left[1 - T_R(X) + \beta^T(X)\right] \ . \tag{3.68}$$

This can now be computed in both the cases analyzed in the previous sections. We shall show that the expressions for S that we will obtain are consistent both with the known results on the tachyon potential [24] and with the expected on-shell behavior.

Thus a choice of coordinates in the space of couplings in which the tachyon β-function is non-linear allows one to find not only a simple general formula for the WS action, but provides also a space-time tachyon effective action that describes tachyon physics from the far-off shell to the near on-shell regions.

Let us start with the evaluation of (3.68) up to the third order in the expansion of the tachyon field using the non-linear β-function (3.54). A similar computation was done in [24, 107] by means of the linear β-function, $\beta(k) = (1-k^2)\,T(k)$. We shall later compare the two results. From the renormalized field (3.37) and the β-function (3.54) we arrive at the following expression for the Witten action

$$
\begin{aligned}
S = K\Bigg\{&1 - \frac{1}{2}\int dk(2\pi)^D T_R(k)T_R(-k)\frac{\Gamma(2-2k^2)}{\Gamma^2(1-k^2)} \\
&+\frac{1}{3!}\int dk_1dk_2dk_3(2\pi)^D T_R(k_1)T_R(k_2)T_R(k_3)\delta(k_1+k_2+k_3) \\
&\left[2\left(1+\sum_{i<j}k_ik_j\right)I(k_1,k_2,k_3) - \left(\frac{\Gamma(1+2k_2k_3)\Gamma(2+2k_1k_2+2k_1k_3)}{\Gamma^2(1+k_2k_3)\Gamma^2(1+k_1k_2+k_1k_3)} + \text{cycl.}\right)\right]\Bigg\}.
\end{aligned}
\tag{3.69}
$$

The propagator coming from the quadratic term in (3.69) exhibits the required pole at $k^2 = 1$. There are however also an infinite number of other zeroes and poles. We shall show that these are due to the metric in the coupling space appearing in (3.3). The equations of motion derived from the action (3.69) are

$$
\begin{aligned}
\frac{\delta S}{\delta T_R(-k)} = &-K\frac{\Gamma(2-2k^2)}{\Gamma^2(1-k^2)}(2\pi)^D T(k) \\
&+\frac{K}{2}\int dk_1dk'(2\pi)^D\delta\left(k_1+k'-k\right)T_R(k_1)T_R(k')\cdot \\
&\cdot\Bigg\{2(1-k_1k+k_1k'-kk')I(-k,k_1,k') - \frac{\Gamma(1-2kk_1)\Gamma(2-2kk'+2k_1k')}{\Gamma^2(1-kk_1)\Gamma^2(1-kk'+k'k_1)} \\
&-\frac{\Gamma(1-2kk')\Gamma(2-2kk_1+2k'k_1)}{\Gamma^2(1-kk')\Gamma^2(1-kk_1+k'k_1)} - \frac{\Gamma(1+2k'k_1)\Gamma(2-2kk'-2kk_1)}{\Gamma^2(1+k'k_1)\Gamma^2(1-kk'-kk_1)}\Bigg\}
\end{aligned}
\tag{3.70}
$$

As we did for the equation $\beta^T = 0$ in the previous section, by solving these equations perturbatively it is possible to recover the scattering amplitudes for three on-shell tachyons. To the lowest order the equation is

$$\frac{\Gamma(2-2k^2)}{\Gamma^2(1-k^2)}T_0(k)=0\ . \tag{3.71}$$

At variance with the lowest order solution of $\beta^T = 0$, there are infinite possible solutions of (3.71). We choose the solution for which the tachyon field $T_0(k)$ is on the mass-shell, which corresponds to a consistent string theory background. This choice is also a solution of $\beta^T = 0$ to the lowest order. As we shall show, the other possible zeroes of (3.71) could be interpreted as zeroes of the metric in the space of couplings through eq.(3.3). With such a choice of $T_0(k)$, to the next order we recover the scattering amplitudes for three on-shell tachyons. By writing $T(k) = T_0(k) + T_1(k)$ and substituting it into (3.70) we find

$$\begin{aligned}T_1(k) = \frac{\Gamma^2(1-k^2)}{2\Gamma(2-2k^2)}\int dk_1dk'(2\pi)^D\delta(k-k_1-k')T_0(k_1)T_0(k')\\ \Big\{2(1-k_1k+k_1k'-kk')I(-k,k_1,k') - \frac{\Gamma(1-2kk_1)\Gamma(2-2kk'+2k_1k')}{\Gamma^2(1-kk_1)\Gamma^2(1-kk'+k'k_1)}\\ -\frac{\Gamma(1-2kk')\Gamma(2-2kk_1+2k'k_1)}{\Gamma^2(1-kk')\Gamma^2(1-kk_1+k'k_1)} - \frac{\Gamma(1+2k'k_1)\Gamma(2-2kk'-2kk_1)}{\Gamma^2(1+k'k_1)\Gamma^2(1-kk'-kk_1)}\Big\}\end{aligned} \tag{3.72}$$

Since the two couplings T_0 satisfy the on-shell condition, k_1 and k' are on-shell. To pick out the propagator pole corresponding to the third k we set it on-shell too. The scattering amplitude for three on-shell tachyons is given again by the residue of the pole and with our normalization is $(2\pi)^{-1}$, in precise agreement with the result obtained in the previous section.

The equations (3.70) must be related to the equation $\beta^T = 0$ through a metric $G_{T(k)T(k')}$ as in (3.3), which in this case becomes

$$\frac{\delta S}{\delta T_R(k)} = -\int dk' G_{T(k)T(k')}\beta^{T(k')}\ . \tag{3.73}$$

The Witten-Shatashvili formulation of string field theory provides a prescription for the metric $G_{T(k)T(k')}$ which can then be computed explicitly. To the first two orders in powers of T_R, it is given by

$$
\begin{aligned}
G_{T(k)T(k')} \quad &= K\frac{(2\pi)^D\Gamma(2-2k^2)}{(1-k^2)\Gamma^2(1-k^2)}\delta\left(k+k'\right) - \frac{K}{2}\int dk_1 (2\pi)^D \delta\left(k+k'+k_1\right)\frac{T_R(k_1)}{1-k'^2}\cdot \\
&\cdot\left\{2(1+k_1k+k_1k'+kk')I(k_1,k,k') - \frac{\Gamma(1+2kk_1)\Gamma(2+2kk'+2k_1k')}{\Gamma^2(1+kk_1)\Gamma^2(1+kk'+k'k_1)}\right. \\
&-\frac{\Gamma(1+2kk')\Gamma(2+2kk_1+2k'k_1)}{\Gamma^2(1+kk')\Gamma^2(1+kk_1+k'k_1)} - \frac{\Gamma(1+2k'k_1)\Gamma(2+2kk'+2kk_1)}{\Gamma^2(1+k'k_1)\Gamma^2(1+kk'+kk_1)} \\
&\left.-\frac{\Gamma(2+2k'k_1)\Gamma(2+2kk'+2kk_1)}{\Gamma^2(1+k'k_1)\Gamma^2(1+kk'+kk_1)(1+kk'+kk_1)}\right\}. \qquad (3.74)
\end{aligned}
$$

The first term in this metric coincides with (3.13) for a conformal primary given by the tachyon vertex. From (3.74) it is possible to see that the infinite number of zeroes and poles that the second order term in eq.(3.69) exhibits at $k^2 = 1+n$ and $k^2 = 3/2+n$, respectively, is in fact due to the metric. This is true except for the zero corresponding to the tachyon mass-shell $k^2 = 1$. In fact the metric (3.74) is regular for $k^2 = 1$. This indicates that the kinetic term in eq.(3.69) exhibits the required zero at the tachyon mass-shell and the metric (3.74) can be made responsible for the other extra zeroes and poles. If these zeroes and poles are just an artifact of the expansion in powers of T, it is an open question. It would be interesting to consider for example an expansion around $k^2 = 1+n$ to all orders in T and check if in this case one would still find that the kinetic term exhibits a zero at $k^2 = 1+n$.

Let us turn now to the cubic term in eq.(3.69). If one or two tachyons are on-shell, then the cubic term vanishes. This means that any exchange diagram involving the cubic term vanishes [107]. When all the three tachyons are on-shell, the scattering amplitude for three on-shell tachyons should arise directly as the coefficient of the cubic term. However, the cubic term in (3.69) is ill-defined on shell. Nonetheless, with the most obvious regularization (*i.e.* by going on-shell symmetrically by giving to the three tachyons an identical small mass m, $k_i^2 = 1+m^2$ and then by taking the $m \to 0$ limit) one gets a finite result for the scattering amplitude [107]. Recalling the first of eqs.(3.53) we conclude that this scattering amplitude is $(2\pi)^{-1}$ with our normalization. Also the cubic term in (3.69) has a sequence of poles at finite distances from the tachyon mass-shell. This is related to the fact that the set of couplings that we have taken into account is not complete. If we get far enough from the tachyon mass-shell, we run into the poles due to all the other string states which have not been subtracted.

In the next section we shall compare (3.69) with the corresponding action derived from the cubic string field theory. Here we would like to show that, by means of a field redefinition, (3.69) can be rewritten in the form of the WS action obtained from a linear β-function [107], but that this field redefinition becomes singular on-shell. The partition function up to the third order in the bare tachyon field is again given by

$$\begin{aligned} Z(k) &= K\delta(k) - K\epsilon^{k^2-1}\Big[T(k) \\ &-\frac{1}{2}\int dk_1 dk_2 (2\pi)^D \delta\left(k-k_1-k_2\right) \epsilon^{-(1+2k_1k_2)} T(k_1)T(k_2) \frac{\Gamma\left(1+2k_1k_2\right)}{\Gamma^2\left(1+k_1k_2\right)} \\ &+\frac{1}{3!}\int dk_1 dk_2 dk_3 (2\pi)^D \delta\left(k-\sum_{i=1}^{3} k_i\right) \epsilon^{-2(1+\sum_{i<j} k_i k_j)} T(k_1)T(k_2)T(k_3) I(k_1,k_2,k_3)\Big] , \end{aligned} \tag{3.75}$$

where we have used (3.37). If instead of following the general procedure of ref. [106] one renormalizes the theory simply by normal ordering, the β-function turns out to be linear. Thus the renormalized field to all orders in the bare field would just be

$$\phi_R(k) = T(k)\epsilon^{k^2-1} , \tag{3.76}$$

so that $\beta(k) = (1-k^2)\,\phi_R(k)$. The WS action with a linear β-function up to the third order in the tachyon field then reads

$$\begin{aligned} S_L = K\Bigg\{ &1 - \frac{1}{2}\int dk (2\pi)^D \phi_R(k)\phi_R(-k) \frac{\Gamma(2-2k^2)}{\Gamma^2(1-k^2)} \\ &+\frac{1}{3!}\int dk_1 dk_2 dk_3 (2\pi)^D \phi_R(k_1)\phi_R(k_2)\phi_R(k_3) \delta\left(\sum_{i=1}^{3} k_i\right) 2\left(1+\sum_{i<j=2}^{3} k_i k_j\right) I(k_1,k_2,k_3)\Bigg\} \end{aligned} \tag{3.77}$$

in agreement with what found in [107]. If we assume that the fields ϕ_R and T_R are related as follows

$$\phi_R(k) = T_R(k) + \int dk_1 f(k, k_1) T_R(k_1) T_R(k - k_1) + \dots \ , \tag{3.78}$$

by comparing the cubic terms in (3.69) and (3.77) one finds

$$\begin{aligned} & \left[f(k_2 + k_3, k_2) \frac{\Gamma(2 + 2k_1k_2 + 2k_1k_3)}{\Gamma^2(1 + k_1k_2 + k_1k_3)} + \text{cycl.} \right] \\ = \ & \frac{1}{2} \left[\frac{\Gamma(1 + 2k_2k_3)}{\Gamma^2(1 + k_2k_3)} \frac{\Gamma(2 + 2k_1k_2 + 2k_1k_3)}{\Gamma^2(1 + k_1k_2 + k_1k_3)} + \text{cycl.} \right] \ , \end{aligned} \tag{3.79}$$

so that the solution for f is $f(k_1 + k_2, k_1) = \Gamma(1 + 2k_1k_2)/(2\Gamma^2(1 + k_1k_2))$ and the field redefinition becomes

$$\phi_R(k) = T_R(k) + \int dk_1 dk_2 \frac{\Gamma(1 + 2k_1k_2)}{2\Gamma^2(1 + k_1k_2)} T_R(k_1) T_R(k_2) \delta(k - k_1 - k_2) \ . \tag{3.80}$$

It is not difficult to see that if we evaluate this relation when the three tachyon fields are on-shell it becomes singular since $f(k, k_1)$ has a pole. This is in agreement with the Poincaré-Dulac theorem [108] used in [22] to prove that when the resonant condition (3.15) holds, namely near the on-shellness, the β-function has to be non-linear. We showed in fact that the field redefinition that gives from S the WS action constructed in terms of a linear β-function, S_L, becomes singular on-shell.

Let us now turn to the WS action computed in an expansion to the leading order in derivatives and to all orders in the powers of the tachyon fields. If we keep the renormalization ambiguity c undetermined, the β-function is given in (3.67). Using (3.4), S then reads

$$S = K \int dX \, (1 - T_R) \left[1 - \log{(1 - T_R)} + \left(1 + \frac{1}{2} \log \frac{c}{4} \right) \frac{\partial_\mu T_R \partial_\mu T_R}{(1 - T_R)^2} \right] \ , \tag{3.81}$$

where $-\infty \leq T_R \leq 1$. With the field redefinition

$$1 - T_R = e^{-\tilde{T}} \tag{3.82}$$

S becomes

$$S = K \int dX e^{-\tilde{T}} \left[\left(1 + \frac{1}{2} \log \frac{c}{4} \right) \partial_\mu \tilde{T} \partial_\mu \tilde{T} + 1 + \tilde{T} \right] , \tag{3.83}$$

which for $c = 4$ coincides with the space-time tachyon action found in [23, 24]. In particular we shall show in the next section that K coincides with the tension of the D25-brane, $K = T_{25}$, in agreement with the results of ref. [24]. It is not difficult to show that (3.83) can be rewritten, by means of a field redefinition, in the form found in [25] where the renormalization ambiguity was also discussed.

Note that (3.82) is the coordinate transformation in the coupling space that leads form the non-linear β-function (3.65) to the linear beta function $\beta^T = (1 + \Delta)T$. The β-function in fact is a covariant vector in the coupling space and as such it transforms.

We have left the ambiguity c in (3.83) undetermined because we want to show that it is possible to fix c in such a way that the equation of motion deriving from (3.83) coincides with the equation $\beta^T = 0$ with β^T given in (3.67). In fact, in terms of the coordinates (3.82), this equation reads

$$\beta^{\tilde{T}} = \tilde{T} + \Delta \tilde{T} + \frac{1}{2} \log \frac{c}{4} \partial_\mu \tilde{T} \partial_\mu \tilde{T} = 0 \ . \tag{3.84}$$

where we have kept into account that $\beta^{\tilde{T}}$ transforms like a covariant vector in the space of worldsheet theories. Choosing $\log(c/4) = -1$, eq.(3.84) becomes the equation of motion of the action (3.83). This is important because if we find finite action solutions of the equation (3.84), these would be at the same time solutions of the renormalization group equations *and* solitons of the tachyon effective action (3.83). These could then be interpreted as lower dimensional branes. Being solutions of the renormalization group equations they are interpreted as background consistent with the string dynamics, being solitons they must describe branes. The finite action solutions of eq.(3.84) are easy to find

$$\tilde{T}(X) = -n + \frac{1}{2} \sum_{i=1}^{n} (X^i)^2 \ . \tag{3.85}$$

These codimension n solitons can be interpreted as D$(25-n)$-branes. $26-n$ are in fact the number of coordinates on which the profile $\tilde{T}(X)$ does not depend. Substituting the solution (3.85) into the action (3.83) with $\log(c/4) = -1$ we get

$$S = T_{25}(e\sqrt{2\pi})^n V_{26-n} \; . \tag{3.86}$$

Comparing this with the expected result $T_{25-n}V_{26-n}$ we derive the following ratio between the brane tensions

$$R_n = \frac{T_{25-n}}{T_{25}} = (\frac{e}{\sqrt{2\pi}} 2\pi)^n \; . \tag{3.87}$$

With our notation, $\alpha' = 1$, the exact tension ratio should be $R_n = (2\pi)^n$. Thus R_n differs from the one given in (3.87) by a factor $e/\sqrt{2\pi} = 1.084$. It is remarkable that a small derivatives expansion of the WS action truncated just to the second order provides a result with the 93% of accuracy. In particular the result (3.87) is much closer to the exact tension ratio then the one found in [24] with analogous procedure. The solutions of the equations of motion of the WS action considered in [24] were not in fact solutions of the equation $\beta^T = 0$, so that they could not be interpreted as consistent string backgrounds (this was already noticed by the authors of [24] and for this reason the exact tension ratio was obtained with a different procedure). The equations of motion deriving from the WS action are in fact related to the β-function through (3.3) where the metric should in principle be non-degenerate. However, if the metric is computed in some approximation, it could be singular and present solutions that introduce physics beyond that contained in the β-functions. The action (3.83) with $\log(c/4) = -1$ gives an equation of the form (3.3) with the non-degenerate metric $e^{-\tilde{T}}$. The solution of this equation can be at the same time a soliton and a conformal RG fixed point.

In conclusion the general formula (3.4) reproduces all the expected results on tachyon effective actions both in the far off-shell and in the near on-shell regions.

3.8 Summary

In this chapter we have derived some exact results for the non-linear tachyon β-function of the open bosonic string theory. We have shown its relevance in the construction of the Witten-Shatashvili bosonic string field theory. When a non-linear renormalization of the tachyon field is considered [106], the WS action in fact is simply given by (3.4). This formula has a wide range of validity. It can be applied to the case in which the tachyon profile is a slowly varying function of the embedding coordinates of the string to derive the exact tachyon potential and the first derivative terms of the effective action. Eq. (3.4) holds also when the tachyon coupling $T(k)$ is small and has support near the mass-shell. For such tachyon profiles we showed that perturbative solutions of the equation $\beta^T = 0$ provide the expected scattering amplitudes of on-shell tachyons.

The explicit form of the WS action constructed from the tachyon non-linear β-function is in precise agreement with all the conjectures involving tachyon condensation. In particular its normalization can be fixed either by studying the exact tachyon potential or by finding the field redefinition that maps the WS action into the effective tachyon action coming from the cubic string field theory. This field redefinition is non-singular on-shell only if the normalization constant coincides with the tension of the D25-brane.

The knowledge of the non-linear tachyon β-function is very important also for another reason. The solutions of the equation $\beta^T = 0$ give the conformal fixed points, the backgrounds that are consistent with the string dynamics. In the case of slowly varying tachyon profiles, we showed that the equations of motion for the WS action can be made identical to the RG fixed point equation $\beta^T = 0$. This can be done for a particular choice of the renormalization prescription ambiguity. We have found soliton solutions of this equation to which correspond a finite value of the WS action. Being solutions of the RG equations these solitons are lower dimensional D-branes for which the finite value of S provides a very accurate estimate of the D-brane tension.

Chapter 4

Tachyon dynamics at tree level in cubic string field theory

In this chapter, following the work with Ilya Sigalov and Washington Taylor [116], we give evidence that the rolling tachyon in cubic open string field theory has a well-defined but wildly oscillatory time-dependent solution. We show that a field redefinition taking the CSFT effective tachyon action to the analogous boundary string field theory action takes the oscillatory CSFT solution to the pure exponential solution e^t of the BSFT action.

4.1 introduction

An unsolved puzzle in string theory is the fate of unstable D-branes and how to describe their evolution toward stable configurations. It is very important to understand their time evolution and an intriguing conjecture, the rolling tachyon, was proposed by Sen [28]. The rolling tachyon has also been applied to cosmology driven by the tachyon, i.e. the decaying of unstable space filling D-branes describe cosmological solutions [117, 118]. In the decay process as the tachyon approaches the bottom of the potential, the energy is constant and the pressure approaches zero. People think that this form of tachyonic matter may have astrophysical consequences, and they also think that string field theory can confirm its existence.

In CSFT the tachyon dynamics appears to be quite complicated. It looks like that the tachyon rolls past the minimum of the potential, then turns around and begin to oscillate [55, 56]. This behaviour is completely different from the one observed in BSFT where the tachyon seems to approach the stable vacuum. In this chapter we solve this contraddiction carring out a level-truncation analysis for a particular solution. We show that a complicated field redefinition is necessay to map the solution from CSFT to the one in BSFT. This qualitative change in behavior through the field redefinition is possible because the field redefinition relating the tachyon in the two formulations is nonlocal and includes terms with arbitrarily many time derivatives. Such field redefinitions are generically expected to be necessary when relating background-independent string field theory degrees of freedom to variables appropriate for a particular background [14]. A similar field redefinition involving higher derivatives was shown in chapter 2 to be necessary to relate the massless vector field $\hat{A}_\mu$ of CSFT on a D-brane with the usual gauge field A_μ appearing in the Yang-Mills and Born-Infeld actions. Other approaches to the rolling tachyon using CSFT appear in [119]-[122]; related approaches which have been studied include p-adic SFT [123, 124], open-closed SFT [125], and vacuum string field theory [126, 127].

This chapter is organized as follows. Section 4.2 describes the general approach that we use to find the rolling tachyon solution and gives the leading order terms in the solution explicitly. Section 4.3 describes the results of numerically solving the equations of motion in level-truncated CSFT. Section 4.4 is dedicated to finding the leading terms in the field redefinition that relates the effective tachyon actions in Boundary and Cubic String Field Theory. Section 4.5 contains conclusions and a discussion of our results. Some technical details regarding our methods of calculation are relegated to an Appendix.

4.2 Solving the CSFT equations of motion

We are interested in finding a solution to the complete open string field theory equations of motion. The full CSFT action contains an infinite number of fields, coupled through cubic terms which contain exponentials of derivatives. Thus, we have a nonlocal action in which it is difficult to make sense of an initial value problem.

Nonetheless, we can systematically develop a solution valid for all times by assuming that as $t \to -\infty$ the solution approaches the perturbative vacuum at $\phi = 0$. In this limit the equation of motion is the free equation for the tachyon field $\ddot{\phi}(t) = \phi(t)$, with solution $\phi(t) = ce^t$. For $t \ll 0$, we can perform a perturbative expansion in the small parameter e^t. Fixing the string coupling g, we can always choose e^t small enough that this perturbative expansion makes sense. We can then write a solution as a power series in ge^t. We proceed in this fashion and find that this power series indeed seems convergent for all t. A related approach was taken in [55, 56]. In these works, an expansion in cosh$t$ was proposed. This allows a one-parameter family of solutions with $\dot{\phi}(0) = 0$, but is more technically involved due to the more complicated structure of coshnt compared with e^{nt}. We restrict attention here to the simplest case of solutions which can be expanded in e^t, but we expect that a more general class of solutions can be constructed using this approach.

The infinite number of fields of CSFT represents an additional complication. We can, however, systematically integrate out any finite set of fields to arrive at an effective action for the tachyon field which we can then solve using the method just described. We do this using the level-truncation approximation to CSFT including fields up to a fixed level. We find that the resulting trajectory $\phi(t)$ converges well for fixed t as the level of truncation is increased.

We thus compute the solution $\phi(t)$ with the desired behavior e^t as $t \to -\infty$ in two steps. In the first step, described in subsection 4.2.1, we compute the tachyon effective action, eliminating all the other modes using equations of motion. Some technical details of this calculation are relegated to the Appendix. In the second step, described in subsection 4.2.2, we write down the equation of motion for the effective theory. We then solve it perturbatively in powers of g.

4.2.1 Computing the effective action

We are interested in a spatially homogeneous rolling tachyon solution. One way of computing such a solution would be to write the equations of motion for the infinite family of string fields with no spatial tensor indices. Labeling such string fields ψ_i the resulting equations of motion (in the Feynman-Siegel gauge) take the schematic form

$$(\partial_t^2 - m_i^2)\psi_i(t) = g\, e^{V_{00}^{11}(\partial_s^2+\partial_u^2+\partial_s\partial_u)} C_i^{jk}(\partial_u, \partial_s)\ \psi_j(s)\psi_k(u)|_{s=u=t} \tag{4.1}$$

where all possible pairs of fields appear on the RHS. Generally, coefficients C_{ijk} multiplying each term may contain a finite number of derivatives. Plugging in the Ansatz $\phi(t) = \psi_0(t) = e^t + \cdots$ with all other fields vanishing at order e^t it is clear that we can systematically solve the equations for all fields order by order in e^t. This is one way of systematically solving order by order for $\phi(t)$.

We will find it convenient to think of the perturbative solution for $\phi(t)$ in terms of an effective action $S(\phi)$ which arises by integrating out all the massive string fields at tree level. Perturbatively, we can solve the equations of motion (4.1) for all fields except $\phi = \psi_0$ as power series in ϕ, by recursively plugging in the equations of motion for all fields except ϕ on the RHS until all that remains is a perturbative expansion in terms of $\phi(t)$ and its derivatives. We have used two approaches to compute the effective action $S(\phi)$. One approach is to explicitly use the equations (4.1) for all fields up to a fixed level. This approach is useful for generating terms to high powers in g but becomes unwieldy for fields at high levels. The second approach we use is to compute the effective action as a diagrammatic sum using the level truncation on oscillator method developed in [65]. This approach is useful for calculating low-order terms in the effective potential where high-level fields are included. Some details of this are described in Appendix C.

The leading terms in the tachyon action are the quadratic and cubic terms coming directly from the CSFT action

$$S(\phi) = \frac{1}{2}\int \mathrm{d}t\ \phi(t)\left(-\partial_t^2 + 1\right)\phi(t) - \frac{g}{3}\left(e^{V_{00}^{11}(\partial_t^2-1)}\phi(t)\right)^3 + \cdots \tag{4.2}$$

where

$$V_{00}^{11} = -\log\left(\frac{27}{16}\right) \tag{4.3}$$

is the Neumann coefficient for the three tachyon vertex.

Integrating out the massive fields at tree level gives rise to higher-order terms $g^2\phi^3, \ldots$ with even more complicated derivative structures. The resulting effective action can be written in terms of the (temporal) Fourier modes $\phi(w)$ of $\phi(t)$ as

$$S(\phi) = \sum_n \frac{g^{n-2}}{n!} \int \prod_{i=1}^{n} \mathrm{d}w_i \, (2\pi)^n \delta\Big(\sum_i w_i\Big) \Xi_n^{\mathrm{CSFT}}(w_1, \ldots, w_n)\phi(w_1)\ldots\phi(w_n) \tag{4.4}$$

where the functions $\Xi_n^{CSFT}(w_1, \ldots, w_n)$ determine the derivative structure of the terms at order $g^{n-2}\phi^n$. The quadratic and cubic terms are given as above by

$$\Xi_2^{\mathrm{CSFT}}(w_1, w_2) = (1 - w_1 w_2), \tag{4.5}$$

$$\Xi_3^{\mathrm{CSFT}}(w_1, w_2, w_3) = -2\, e^{-\frac{1}{2}V_{00}^{11}(w_1^2+w_2^2+w_3^2+3)}. \tag{4.6}$$

One way to obtain the approximate classical effective action for the tachyon field is to use the equations of motion for a few low level massive fields to eliminate these fields explicitly from the action. The higher level massive fields are set to zero.

Now let us compute explicitly the quartic term in effective action in level 2 truncation. In case of CSFT of a single D-brane the combined level of fields coupled by a cubic interaction must be even. For example, there is no vertex coupling two tachyons (level zero) with the gauge boson (level 1). It follows that there are no tree level Feynman diagrams with all external tachyons and internal fields of odd level. Thus, in calculating the tachyonic effective action we may set odd level fields to 0. Fixing the Feynman-Siegel gauge the only fields involved are the tachyon ϕ and three level 2 massive fields with $m^2 = 1$: β, B_μ and $B_{\mu\nu}$. The terms in the action contributing to four-tachyon term in the effective action are

$$\frac{1}{2}\int dt\, \beta(\partial_t^2+1)\beta - B_{\mu\nu}(\partial_t^2+1)B^{\mu\nu} - B_\mu(\partial_t^2+1)B^\mu +$$
$$g\int dt\, a_1\tilde{\phi}^2\tilde{B}^\mu_\mu + a_2(\tilde{\phi}\partial_t\partial_t\tilde{\phi} - \partial_t\tilde{\phi}\partial_t\tilde{\phi})\tilde{B}^{00} + a_3\tilde{\phi}^2\tilde{\beta} + a_4\tilde{\phi}\partial_t\tilde{\phi}\tilde{B}^0, \quad (4.7)$$

where $\tilde{f} = e^{\frac{1}{2}V_{00}^{11}(\partial_t^2-1)}f$ with $V_{00}^{11} = -\log(27/16)$. Other interaction terms involving level 2 fields, for example β^3 or $B_\mu B^\mu\phi$ would contribute to the effective action at higher powers of ϕ. The coefficients a_1, ... a_4 are real numbers and can be expressed via appropriate matter and ghost Neumann coefficients (see appendix A)

$$a_1 = -\frac{V_{11}^{11}}{\sqrt{2}} \approx 0.130946, \qquad a_2 = \sqrt{2}(V_{01}^{12})^2 \approx 0.419026,$$
$$a_3 = X_{11}^{11} \approx 0.407407, \qquad a_4 = -6V_{02}^{12} \approx 0.628539. \quad (4.8)$$

Using the procedure described above we write down the equations of motion for massive fields, plug them into (4.7) and set $\beta = B_\mu = B_{\mu\nu} = 0$. We then obtain the quartic term in the tachyonic effective action

$$g^2 e^{-3V_{00}^{11}}\int\prod_{i=1}^4(2\pi dw_i)\phi(w_i)\delta(\sum w_i)\frac{\exp\big(-V_{00}^{11}[w_1^2+w_2^2+w_3^2+w_4^2+w_1w_2+w_3w_4]\big)}{1-(w_1+w_2)^2}$$
$$\Big(b_1 + b_2\, w_2(w_2-w_1) + b_3\, w_1w_4(w_2-w_1)(w_4-w_3) + b_4\, w_2w_4\Big) \quad (4.9)$$

where we have denoted

$$b_1 = \frac{1}{2}(13(V_{11}^{11})^2 - (X_{11}^{11})^2), \qquad b_2 = -V_{11}^{11}(V_{01}^{12})^2,$$
$$b_3 = (V_{01}^{12})^4, \qquad b_4 = 18(V_{02}^{12})^2. \quad (4.10)$$

4.2.2 Solving the equations of motion in the effective theory

We now outline the process for solving the equation of motion of the effective theory, and we compute the first perturbative correction to the free solution. We are interested in time-dependent solutions which are uniform in spatial directions. Varying

(4.4) we get an equation of the form

$$(\partial_t^2 - 1)\phi = \sum_{n=2}^{\infty} g^{n-1} K_n(\phi, \dots, \phi) \tag{4.11}$$

where the nonlinear terms of order ϕ^n are denoted by K_n. The specific form of the K_n follow by differentiating (4.4) with respect to $\phi(t)$. The functions Ξ_n appearing in (4.4), and thus the corresponding K_{n-1}'s can in principle be explicitly computed for arbitrary n at any finite level of truncation. In general, K_n will be a complicated momentum-dependent function of its arguments.

The solution of the linearized equations of motion which goes to 0 as $t \to -\infty$ is $\phi(t) = c_1 e^t$. As discussed above, we wish to use perturbation theory to find a rolling solution which is defined by this asymptotic condition as $t \to -\infty$. Note that this asymptotic form places a condition on all derivatives of ϕ in the limit $t \to -\infty$, as appropriate for a solution of an equation with an unbounded number of time derivatives. If we now assume that the full solution can be computed by solving (4.11) using perturbation theory at least in some region $t < t_{\max}$, it can be easily seen that the successive corrections to the asymptotic solution $\phi_1(t) = c_1 e^t$ are of the form $\phi_n(t) = c_n e^{nt}$. In other words, to solve the equations of motion using perturbation theory we expand $\phi(g, t)$ in powers of g

$$\phi(g, t) = \phi_1(t) + g\phi_2(t) + g^2\phi_3(t) + \dots \tag{4.12}$$

where

$$\phi_n(t) = c_n e^{nt}. \tag{4.13}$$

As we will see, our assumption leads to a power series which seems to be convergent for all t and all g. Note that since $g^n e^{nt} = e^{n(t+\log(g))}$, the coupling constant can be set to 1 by translating the time variable and rescaling ϕ, so convergence for fixed g and all t implies convergence for all t and for all g. Plugging (4.12) into (4.11) we find

$$(\partial_t^2 - 1)\phi_n = (n^2-1)c_n e^{nt} = \sum_p \sum_{m_1+m_2+\ldots m_p=n} K_p(\phi_{m_1}, \phi_{m_2}, \phi_{m_p}). \tag{4.14}$$

These equations allow us to solve for $c_{n>1}$ iteratively in n. Having solved the equations for $c_2, \ldots, c_{n-1}$ we can plug them in via (4.13) on the right hand side of (4.14) to determine c_n.

As an example, let us consider the first correction $\phi_2(t) = c_2 e^{2t}$ to the linearized solution $\phi_1(t) = c_1 e^t$. The equation of motion at quadratic order arising from K_2 is

$$(\partial_t^2 - 1)\phi = -e^{\frac{1}{2}V_{00}^{11}(\partial_t^2-1)}(e^{\frac{1}{2}V_{00}^{11}(\partial_t^2-1)}\phi)^2 \tag{4.15}$$

Plugging in $\phi_1 = c_1 e^t$, $\phi_2 = c_2 e^{2t}$ we find

$$c_2(\partial_t^2 - 1)e^{2t} = -c_1^2 e^{\frac{1}{2}V_{00}^{11}(\partial_t^2-1)}(e^{\frac{1}{2}V_{00}^{11}(\partial_t^2-1)}e^t)^2 \tag{4.16}$$

and therefore

$$c_2 = -\frac{1}{3}e^{\frac{3}{2}V_{00}^{11}}c_1^2. \tag{4.17}$$

If we normalize $c_1 = 1$ then the solution to order e^{2t} is

$$\phi(t) = e^t - \frac{64}{243\sqrt{3}}e^{2t} + \ldots. \tag{4.18}$$

The quartic interaction term in the effective action would contribute to coefficients $c_n e^{nt}$ with $n \geq 3$ with the leading order contribution being $c_3 e^{3t}$. From equation (4.14) we have

$$c_3 = \frac{e^{-3t}}{8}\left(2c_2 K_2(e^t, e^{2t}) + K_3(e^t, e^t, e^t)\right). \tag{4.19}$$

where K_3 is obtained by differentiating (4.9) with respect to $\phi(t)$. Then (4.19) gives

$$(\delta c_3)_{\text{cubic}} \simeq 0.0021385, \qquad (\delta c_3)_{\text{quartic}} \simeq 0.0000492826. \tag{4.20}$$

It is quite surprising, that the contribution to c_3 from the quartic term in the effective action is merely 0.2% of the contribution from the cubic term. Adding the contributions we get the rolling solution to second order in perturbation theory in level 2 truncation

$$\phi(t) \simeq e^t - \frac{64\, e^{2\,t}}{243\,\sqrt{3}} + 0.002187\, e^{3\,t} + \ldots \quad . \tag{4.21}$$

4.3 Numerical results

In this section we describe the results of using the level-truncated effective action $S[\phi]$ to compute approximate perturbative solutions to the equation of motion through (4.14). We are testing the convergence of the solution in two respects. In subsection 4.3.1 we check that the solution converges nicely at fixed t when we take into account more and more terms in a perturbative expansion of the effective action while keeping the truncation level fixed at $L = 2$. In subsection 4.3.2 we check that the solution converges well for fixed t when we keep the order of perturbation theory fixed while increasing the truncation level.

4.3.1 Convergence of perturbation theory at $L = 2$

The equation (4.14) allows us to find the successive perturbative contributions to the solution of the equations of motion, given an explicit expression for the terms in the effective action. The solution takes the form

$$\phi(t) = \sum_n c_n e^{nt}. \tag{4.22}$$

Since all the derivatives of e^{nt} are straightforward to compute, as in (4.17), we can replace these derivatives in any operator through $f(\partial_t)e^{nt} \to f(n)e^{nt}$. This manipulation is justified as long as f does not contain a pole at n.

We have computed the functions Ξ_n^{CSFT} and the resulting K_{n-1}'s solving the equation of motion for the level two field up to $n = 7$. We have used these K_n's to compute the resulting approximate coefficients c_n, with $n \leq 6$. To compute the coefficient c_n

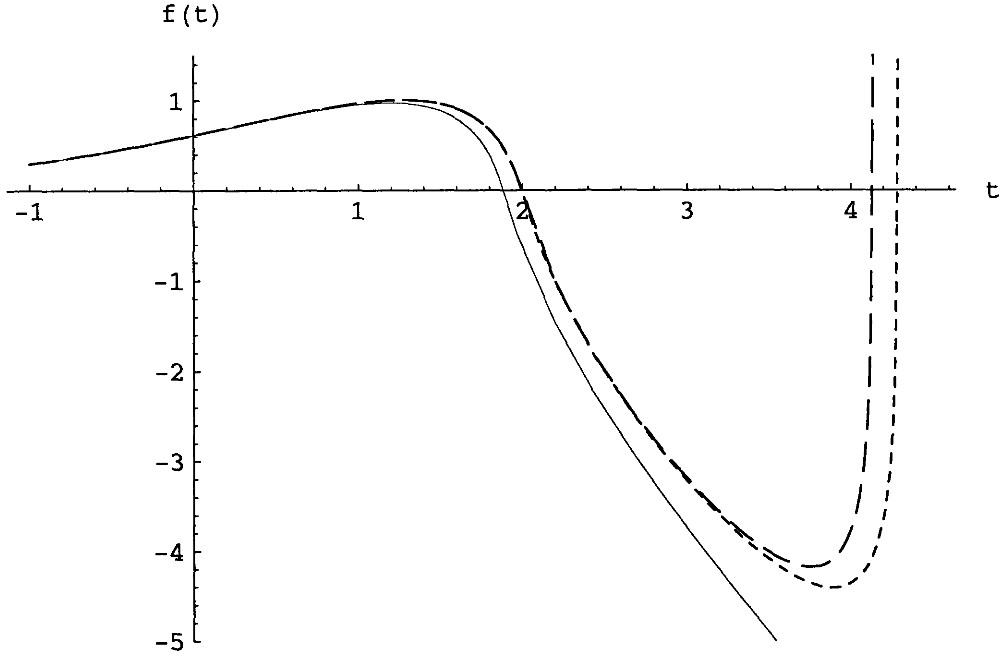

Figure 4-1: Solution $\phi(t)$ including the first two turnaround points, including fields up to level $L = 2$. Solid line graphs approximation $\phi(t) = e^t + c_2 e^{2t}$. Long dashed plot graphs $\phi(t) = e^t + c_2 e^{2t} + c_3 e^{3t}$. The approximate solutions computed up to e^{4t}, e^{5t} and e^{6t} are very close in this range of t and are all represented in short dashed plot. One can see that after going through the first turnaround point with coordinates $\sim$ (1.27, 1.8) the solution goes down, reaching the second turnaround at around $\sim$ (3.9, -81). A function $f(\phi(t)) = \text{sign}(\phi(t)) \log(1 + |\phi(t)|)$ is graphed to show both turnaround points clearly on the same scale.

one needs the effective tachyon action computed to order $n+1$; higher terms in the action contribute only to higher order coefficients.

The $L = 2$ approximation to the solution for the tachyon field is

$$\phi(t) \simeq e^t - \frac{64\, e^{2t}}{243\sqrt{3}} + 0.002187\, e^{3t} - \\ 3.9258\, 10^{-6}\, e^{4t} + 4.9407\, 10^{-10}\, e^{5t} + 6.3227\, 10^{-12}\, e^{6t}. \quad (4.23)$$

Plotting the result we observe that for small enough t the term e^t dominates and the solution decays as e^t at $-\infty$. Then, as t grows, the second term in (4.23) becomes important. The solution turns around and $\phi(t)$ becomes negative, with the major contribution coming from e^{2t}. Then the next mode, e^{3t} becomes dominating and so on. The solution $\phi(t)$ around the first two turnaround points is shown on the figure 4-1.

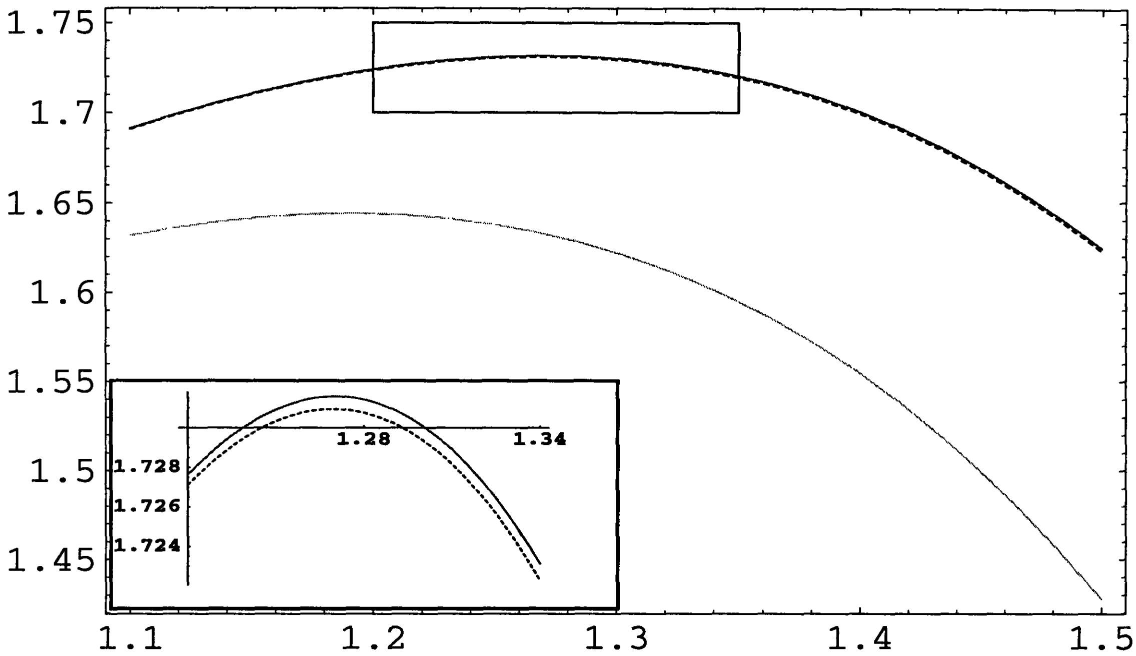

Figure 4-2: First turnaround point for the solution in $L = 2$ truncation scheme. The large plot shows the approximations with ϕ^3 (the gray line) and ϕ^4 (black solid and dashed lines) terms in the action taken into account. The smaller plot zooms in on the approximations with ϕ^4 (the solid line) and ϕ^5 (the dotted line) terms taken into account. The corrections from higher powers of ϕ are very small and the corresponding plots are indistinguishable from the one of the ϕ^5 approximation.

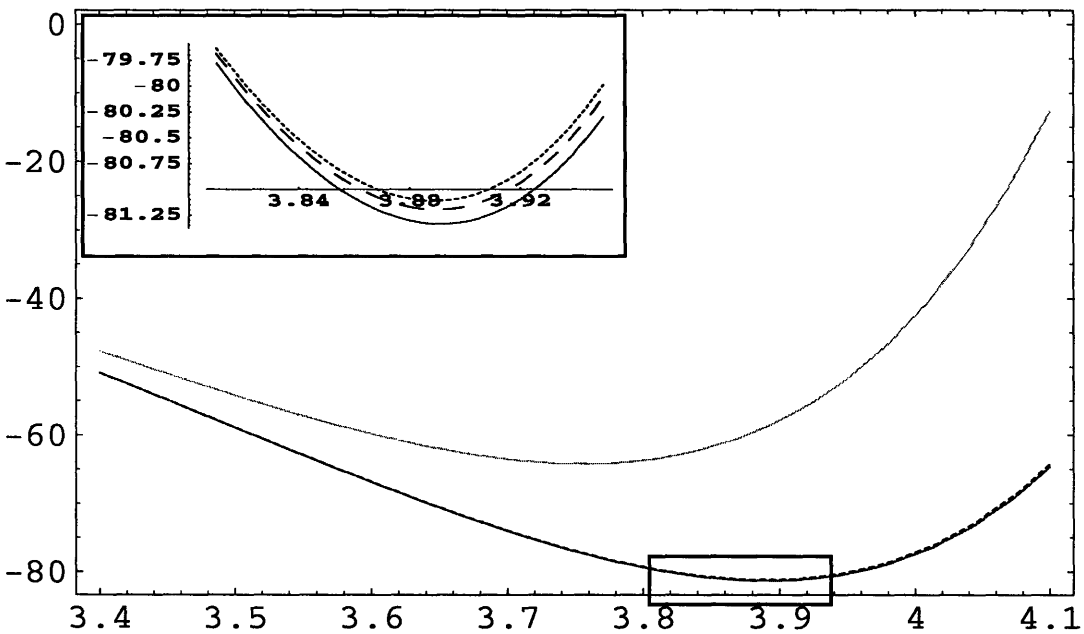

Figure 4-3: Second turnaround point for the solution in $L = 2$ truncation scheme. The gray line on the large plot shows the solution computed with the effective action including terms up to ϕ^4. The black solid and dashed lines represent higher order corrections. On the small plot the solid line includes ϕ^5 corrections, the dotted line includes corrections from ϕ^6 term and the dashed line takes into account the ϕ^7 term.

The positions of the first n turnaround points are quite accurately determined by taking into account the effective action terms up to ϕ^{n+1}. The inclusion of the higher order terms in the action changes the position of the first n turnaround points only slightly. Figures 4-2 and 4-3 illustrate the dependence of the position of the first two turnaround points on the powers of ϕ included in effective action. We interpret these results as strong evidence that, at least for the effective action at truncation level $L = 2$, the solution (4.22) is given by a perturbative series in e^t which converges at least as far as the second turnaround point, and plausibly for all t.

4.3.2 Convergence of level truncation

From the results of the previous subsection, we have confidence that the first two points where the tachyon trajectory turns around are well determined by the ϕ^4 and ϕ^5 terms in the effective action. To check whether these oscillations are truly part of a well-defined trajectory in the full CSFT, we must check to make sure that the turnaround points are stable as our level of truncation is increased and the terms in the effective action are computed more precisely.

We have computed the ϕ^4 term in the effective action at levels of truncation up to $L = 16$. The results of this computation for the approximate trajectory $\phi(t)$ are shown in Figure 4.3.2, which demonstrates the behavior of the first turnaround point as we include higher level fields. This computation shows that the first turnaround point is already determined to within less than 1% by the level $L = 2$ truncation. We take this computation as giving strong evidence that this turnaround point is real. Combining this result with the computation of the previous subsection, we have (to us) convincing evidence that the perturbative expansion e^{nt} for the rolling tachyon solution is valid well past the first turnaround point, and that the level truncation procedure converges to a trajectory containing this turnaround point. Extrapolating the results of this computation, we believe that the qualitative phenomenon of wild oscillations revealed by the level $L = 2$ computation is physically correct, and that more precise calculations at higher level will only shift the positions of the turnaround points mildly, leaving the qualitative behavior intact.

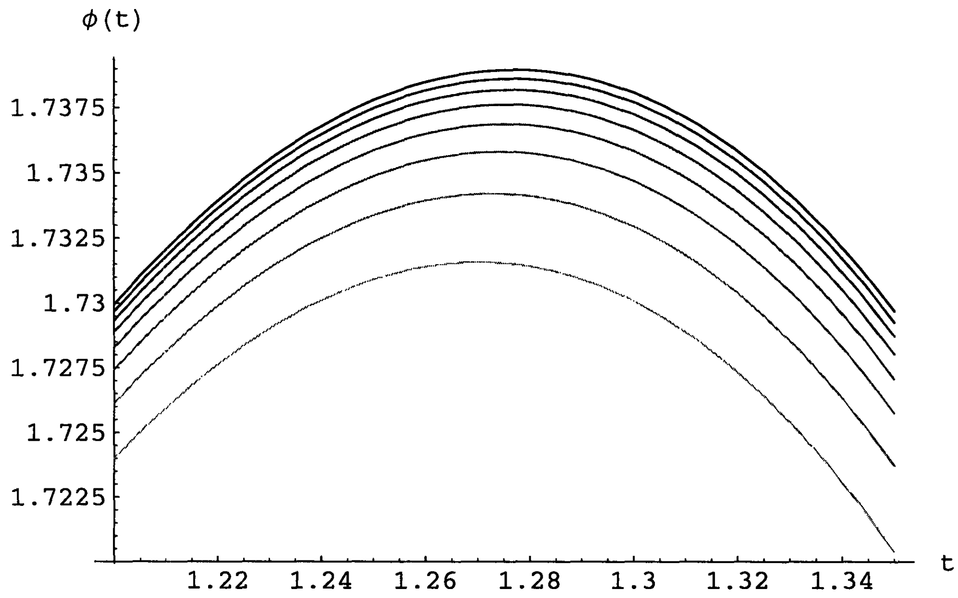

Figure 4-4: The figure shows the convergence of the solution around the first turnaround point as we increase the truncation level. Bottom to top the graphs represent the approximate solutions computed with the effective action computed up to ϕ^4 and truncation level increasing in steps of 2 from $L = 2$ to $L = 16$. We observe that the turnaround point is determined quite precisely already at the level 2. Similar behavior is observed for the second turnaround point.

4.4 Taming the tachyon with a field redefinition

Now that we have confirmed that CSFT gives a well-defined but highly oscillatory time-dependent solution, we want to understand the physics of this solution. Although the oscillations seem quite unnatural from the point of view of familiar theories with only quadratic kinetic terms and a potential, the story is much more subtle in CSFT due to the higher-derivative terms in the action. For example, while the tachyon field rolls immediately into a region with $V(\phi) \gg V(0) = 0$, the energy of the perturbative rolling tachyon solution we have found is conserved, as we have verified by a perturbative calculation of the energy including arbitrary derivative terms, along the lines of similar calculations in [55].

To understand the apparently odd behavior of the rolling tachyon in CSFT, it is useful to consider a related story. In [57] we computed the effective action for the massless vector field on a D-brane in CSFT by integrating out the massive fields. The resulting action did not take the expected form of a Born-Infeld action, but included various extra terms with higher derivatives which appeared because the degrees of freedom natural to CSFT are not the natural degrees of freedom expected for the CFT on a D-brane, but are related to those degrees of freedom by a complicated field redefinition with arbitrary derivative terms. In principle, we expect such a field redefinition to be necessary any time one wishes to compare string field theory (or any other background-independent formulation) with CFT computations in any particular background. The necessity for considering such field redefinitions was previously discussed in [14, 15].

Thus, to compare the complicated time-dependent trajectory we have found for CSFT with the marginal e^t perturbation of the boundary CFT found in [28, 29], we must relate the degrees of freedom of BSFT and CSFT throw a field redefinition which can include arbitrary derivative terms. In this section we compute the leading terms of the field redefinition in the effective field theories for $T(\phi)$, the tachyon field in boundary string field theory and ϕ, the tachyon in cubic string field theory.

The field redefinition can be determined by requiring that the effective actions for the two theories are mapped into each other. We use the effective tachyonic action of BSFT computed up to cubic order in [105]; another approach to computing the BSFT action which may apply more generally was developed in [128]. The BSFT action is determined via partition function for the boundary SFT and the tachyon's beta function. Thus the particular shape of the action depends on the renormalization scheme for the boundary CFT. The BSFT tachyon T is therefore, the renormalized tachyon with renormalization scheme of [105]. We check, that the field redefinition maps the rolling tachyon solution $T(t) = e^t$ to the leading terms in the perturbative solution $\phi(t) = e^t - \frac{64}{243\sqrt{3}}e^{2t}$ which we have computed in the previous section. The fact that the field redefinition is nonsingular at $T = e^t$ is consistent with the anzats $\sum_n c_n e^{nt}$ for the rolling tachyon solution in CSFT.

In parallel with (4.4) we write the action for the boundary tachyon T as

$$S(T) = \sum_n \frac{g^{n-2}}{n!} \int \prod_{i=1}^{n} (2\pi\, dw_i) \delta(\sum_i w_i) \Xi_n^{BSFT}(w_1, \ldots, w_n) T(w_1) \ldots T(w_n) \quad (4.24)$$

where the functions $\Xi_n^{BSFT}(w_1, \ldots, w_n)$ define the derivative structure of the term of

n'th power in T. The kernel for the quadratic terms is

$$\Xi_2^{\mathrm{BSFT}}(w_1, w_2) = \frac{\Gamma(2 - 2w_1w_2)}{\Gamma^2(1 - w_1w_2)}. \tag{4.25}$$

where Γ is the Euler gamma function. Denoting $a_1 = -w_2w_3$, $a_2 = -w_1w_3$, $a_3 = -w_2w_3$ the kernel for the cubic term can be written as

$$\Xi_3^{\mathrm{BSFT}}(w_1, w_2, w_3) = 2(1 + a_1 + a_2 + a_3)I(w_1, w_2, w_3) + J(w_1, w_2, w_3) \tag{4.26}$$

where functions $I(a_1, a_2, a_3)$ and $J(a_1, a_2, a_3)$ are defined by

$$\begin{aligned} I(a_1, a_2, a_3) &= \frac{\Gamma(1 + a_1 + a_2 + a_3)\Gamma(1 + 2a_1)\Gamma(1 + 2a_2)\Gamma(1 + 2a_3)}{\Gamma(1 + a_1)\Gamma(1 + a_2)\Gamma(1 + a_3)\Gamma(1 + a_1 + a_2)\Gamma(1 + a_1 + a_3)\Gamma(1 + a_2 + a_3)}, \\ J(a_1, a_2, a_3) &= -\frac{\Gamma(1 + 2a_1)\Gamma(2 + 2a_2 + 2a_3)}{\Gamma^2(1 + a_1)\Gamma^2(1 + a_2 + a_3)} + \text{cyclic}. \end{aligned} \tag{4.27}$$

We are interested in the field redefinition that relates $S(T)$ with the CSFT action $S(\phi)$ given in (4.4), (4.5), (4.6). A generic field redefinition without explicit time dependence can be written as

$$\begin{aligned} \phi(w_1) = \int dw_2\, \delta(w_1 - w_2) f_1(w_1, w_2) T(w_2) + \\ \int dw_2\, dw_3\, f_2(w_1, w_2, w_3) T(w_2) T(w_3) \delta(w_1 - w_2 - w_3) + \dots . \end{aligned} \tag{4.28}$$

This field redefinition maps the CSFT action to the BSFT action

$$S(\phi(T)) = S(T). \tag{4.29}$$

In order to match the quadratic terms f_1 must satisfy the equation

$$\Xi_2^{\mathrm{BSFT}}(w_1, w_2) - f_1(w_1, w_1) f_1(w_2, w_2) \Xi_2^{CSFT}(w_1, w_2) \approx 0. \tag{4.30}$$

In this equation approximate sign means that the left hand side becomes equal to the right hand side when inserted into $\int dw_1\, dw_2\, \delta(w_1 + w_2)\phi(w_1)\phi(w_2)$ for arbitrary

$\phi(w)$[1]. Solving equation (4.30) we find

$$f_1(w,w) \equiv f_1(w) = \sqrt{\frac{1}{1+w^2}\frac{\Gamma(2+2w^2)}{\Gamma^2(1+w^2)}}. \tag{4.31}$$

The analogous equation for f_2 is

$$\frac{1}{3}\Xi_3^{\mathrm{BSFT}}(w_1,w_2,w_3) \approx \frac{1}{3}f_1(w_1)f_1(w_2)f_1(w_3)\Xi_3^{CSFT}(w_1,w_2,w_3)+ \\ f_1(w_1)f_2(-w_1,w_2,w_3)\Xi_2^{CSFT}(-w_1,w_1). \tag{4.32}$$

The desired field redefinition must also map the mass-shell states correctly, by keeping the mass-shell component of any $\phi(w)$ intact. In other words the mass-shell component of Fourier expansion of $\phi(t)$ can only depend on the same mass-shell component of the Fourier expansion of $T(t)$. This translates to a restriction on f_2

$$f_2(-w_1,w_2,w_3)|_{w_1^2=-1} = 0. \tag{4.33}$$

This constraint is crucial for the field redefinition to correctly relate the on-shell scattering amplitudes for T with those for ϕ. It also ensures that the solution of the classical equations of motion for T maps to the solution of the equations of motion for ϕ.

Equation (4.32) can be simplified be making a substitution

$$f_2(-w_1,w_2,w_3) = \frac{\Xi_3^{BSFT}(w_1,w_2,w_3)/f_1(w_1) - \Xi_3^{\mathrm{CSFT}}(w_1,w_2,w_3)f_1(w_2)f_2(w_3)}{\Xi_2^{CSFT}(-w_1,w_1)}A_2(w_1,w_2,w_3) \tag{4.34}$$

we obtain a simple equation for $A_2(w_1,w_2,w_3)$

$$A_2(w_1,w_2,w_3) \approx \frac{1}{3}. \tag{4.35}$$

[1]When matching the quadratic terms it implies strict equality but in general it is less restrictive. Indeed, considering a discrete analogue it is easy to show that equation $M_{kl}c_kc_l = 0$ is equivalent to $M_{kl} + M_{lk} = 0$. Consequently, equation $M_{n_1,\ldots,n_k}c_{n_1}\cdots c_{n_k} = 0$ is equivalent to $\sum_{\sigma(n_1,\ldots,n_k)} M_{\sigma(n_1),\ldots,\sigma(n_k)} = 0$.

Thus, we need to find a function $A(w_1, w_2, w_3)$ on the momentum conservation hyperplane $-w_1 + w_2 + w_3 = 0$, symmetric under exchange of w_2 and w_3 and satisfying

$$A_2(w_1, w_2, w_3) + A_2(w_2, w_3, w_1) + A_2(w_3, w_1, w_2) = 1, \tag{4.36}$$

with the constraint[2]

$$A_2(w_1, w_2, w_3)|_{w_1^2=-1} = 0. \tag{4.37}$$

It would be sufficient to consider a discrete case, where w_1, w_2, w_3 are integers. Indeed, we can think of the tachyon fields to be obtained by Wick rotation of a periodic field $\phi_{rot}(\tau)$, where $\tau = it$, that live on the D-brane compactified on a circle. It is easy to see, that discretized $A(w_i)$ given by

$$A(w_1, w_2, w_3) = \begin{cases} \frac{1}{3}, & w_{1,2,3} \neq \pm i \\ 0, & w_1 = \pm i \\ \frac{1}{2}, & w_2 = \pm i,\ w_{1,3} \neq \pm i \quad \text{or} \quad w_3 = \pm i,\ w_{1,2} \neq \pm i \\ \frac{1}{3}, & w_1 = -2i,\ w_{2,3} = i \quad \text{or} \quad w_1 = -2i,\ w_{2,3} = i \\ 1 & w_1 = 0,\ w_2 = -w_3 = \pm i \end{cases} \tag{4.38}$$

is a solution to (4.36), (4.37). It is not difficult to generalize the above discussion to the continuum case.

Let us make a few comments on the field redefinition.

- The expression under the square root in the definition of $f_1(w)$ becomes negative for $w^2 < -3/2$. This means that the field redefinition (4.28) is only well defined on the subspace $\phi(w)$ with $w^2 > -3/2$. Within this region $f_1(w)$ is smooth without any zeroes or poles. The mass-shell point, $w^2 = -1$ lies within this region.

[2]In [105] this constraint was overlooked and a simple expression $A(w_1, w_2, w_3) = \frac{1}{3}$ have been used. One can check that this is incorrect on a simple example. Consider mapping the action of harmonic oscillator to the action of harmonic oscillator plus a cubic potential term $-\frac{1}{3}\phi^3$. With the choice of A that preserves the mass-shell modes one gets a field redefinition that correctly maps the solution of harmonic oscillator e^{it} to the perturbative solution of unharmonic oscillator $e^{it} - \frac{1}{3}e^{2it}$.. A naive choice of A gives rise to unwanted additional factor of 1/3.

- The function A_2 represents a universal part of f_2 and is independent of the particular properties of the CSFT and BSFT actions. For example to map the action of harmonic oscillator to the action of unharmonic oscillator we could use the same A.

- The term multiplying $A_2(w_1, w_2, w_3)$ in (4.34) has a number of poles. However it is non-singular in two important cases. The first case was considered in [105] is when the tachyon fields in both frames $T(w_2)$, $T(w_3)$ and $\phi(w_1)$ in (4.28) are on the mass-shell. At this point the two summands in the numerator of (4.34) cancel and there is no pole at this point. The requirement of this cancelation was used in [105] to fix the normalization of BSFT action.

  The second case is the one of the rolling tachyon. In this case $T(w_2)$ and $T(w_3)$ are on them mass-shell: $w_2^2 = w_3^2 = -1$ and $w_1^\mu = w_2^\mu$. A potential singularity is in the term $\Xi_3^{\mathrm{BSFT}}(w_1, w_2, w_3)/f_1(w_1)$ in the numerator. $f_1(w_1)$ has a zero at $w_1^2 = -4$, but the functions I and J in the numerator have a stronger zero resulting in a zero at that point.

Finally, we want to check that the field redefinition maps the rolling tachyon solution of BSFT into the perturbative solution that we have fount in 4.2.2. Plugging the rolling solution $T_{\mathrm{rolling}}(t) = e^t$ into the field redefinition and computing the numerical values we obtain

$$\phi(t) = e^t - \frac{64\, e^{2t}}{243\sqrt{3}} + \dots \tag{4.39}$$

which exactly reproduces the leading order terms in the perturbative CSFT solution found in section 4.2. It seems likely that as we include higher powers of ϕ in the field redefinition we would generate higher powers terms e^{nt} in the perturbative solution.

4.5 Summary

In this chapter we have confirmed and expanded on the earlier results of [55] and [56], which suggested that in CSFT the rolling tachyon oscillates wildly rather than converging to the stable vacuum. We have shown that the oscillatory trajectory is stable when higher-level fields are included and thus correctly represents the dynamics of CSFT. We have further shown that this dynamics is not in conflict with the more physically intuitive e^t dynamics of BSFT by explicitly demonstrating a field redefinition, including arbitrary derivative terms, which (perturbatively) maps the CSFT action to the BSFT action and the oscillatory CSFT solution to the e^t BSFT solution.

This resolves the outstanding puzzle of the apparently different behaviour of the rolling tachyon in these two descriptions of the theory. On the one hand, this serves as further validation of the CSFT framework, which has the added virtue of background-independence, and which has been shown to include disparate vacua at finite points in field space. On the other hand, the results of this paper serve as further confirmation of the complexity of using the degrees of freedom of CSFT to describe even simple physics. As noted in previous work, many phenomena which are very easy to describe with the degrees of freedom natural to CFT, such as marginal deformations [93], and the low-energy Yang-Mills/Born-Infeld dynamics of D-branes [57] are extremely obscure in the variables natural to CSFT. This is in some sense possibly an unavoidable consequence of attempting to work with a background-independent theory: the degrees of freedom natural to any particular background arise in complicated ways from the underlying degrees of freedom of the background-independent theory. This problem becomes even more acute in the known formulations of string field theory, which require a canonical choice of background to expand around, when attempting to describe the physics of a background far from the original canonical background choice, such as when describing the physics of the true vacuum using the CSFT defined around the perturbative vacuum [129, 130]. The complexity of the field redefinitions needed to relate even simple backgrounds such as the rolling tachyon discussed in this paper to the natural CFT variables make it clear that powerful new tools are needed to take string field theory from its current form to a framework in which relevant physics in a variety of backgrounds can be clearly computed and interpreted.

Chapter 5

Conclusions and future directions

OSFT is a powerful tool for studying non-perturbative phenomena and distinct vacua with different geometrical properties. The theory has the virtue of being background-independent and needs a map from the set of field variables from one background's choice to the other. The fact that OSFT's set of variables, which are defined through a complicated algebraic structure, produce different geometrical backgrounds as different solutions of the equations of motion is an important step beyond the perturbative formulation of string theory. It would be ideal if we could have an off-shell formulation of string/M-theory where the various known vacua arise in terms of a single well defined set of degrees of freedom.

The downside of using CSFT approach is that its degrees of freedom are rather complex and to describe simple physics may look complicated. As we have shown in this work, many phenomena clearly described using conformal field theory degrees of freedom, such as the low-energy Yang-Mills/Born-Infeld action and the tachyon dynamics starting at the perturbative vacuum, are unclear in the variable of CSFT. Again, we think that this is a consequence of trying to approach a problem with a theory that is background-independent. For example, if we want to study the effective action for the gauge field from the partition function on the disk, we add a boundary operator that is manifestly gauge invariant by construction. Clearly, the resulting effective action has the usual gauge transformation for A_μ in contrast to the CSFT case where the gauge field is coupled to the other massive field through a totally different gauge symmetry. We remind that situation like this are not unusual, in fact there are other cases where we need a map that relates the standard set of variables to the new one. For example in noncommutative gauge theories the gauge field posses a non-standard gauge transformation which involves a noncommutative product. These guage variables are related to the standard ones by means of a field redefinition, the Seiberg-Witten map.

While we have focused in this work on calculations in the bosonic theory, it would be even more interesting to carry out analogous calculations in the supersymmetric theory. There are currently several candidates for an open superstring field theory, including the Berkovits approach [131] and the (modified) cubic Witten approach [132, 133, 134]. (See [135] for further references and a comparison of these approaches.) For the abelian gauge field's effective action, a superstring calculation should again reproduce the Born-Infeld action, including all higher-derivative terms. In the non-abelian case, it should be possible to compute all the terms in the nonabelian effective action. It would be also interesting to see if in the supersymmetric case the tachyon solution happens to show the wild oscillations that we observe in the bosonic case.

The analysis in chapter 2 also has an interesting analogue in the closed string context. Just as the Yang-Mills theory describing a massless gauge field can be extended to a full stringy effective action involving the Born-Infeld action plus derivative corrections, in the closed string context the Einstein theory of gravity becomes extended to a stringy effective action containing higher order terms in the curvature. Some terms in this action have been computed, but they are not yet understood in the same systematic sense as the abelian Born-Infeld theory. A tree-level computation in closed string field theory would give an effective action for the multiplet of massless closed string fields, which should in principle be mapped by a field redefinition to the Einstein action plus higher-curvature terms [14]. Lessons learned about the nonlocal structure of the effective vector field theory discussed in this work may have interesting generalizations to these nonlocal extensions of standard gravity theories.

Another direction in which it would be interesting to extend this work is to carry out an explicit computation of the effective action for the tachyon in an unstable brane background, or for the combined tachyon-vector field effective action. Some progress on the latter problem was made in [15]. Because the mass-shell condition for the tachyon is $p^2 = 1$, it does not seem to make any sense to consider an effective action for the tachyon field, analogous to the Born-Infeld action, where terms of higher order in p are dropped. Indeed, it can be shown that when higher-derivative terms are dropped, any two actions for the tachyon which keep only terms $\partial^k \phi^{m+k}, m \geq 0$, can be made perturbatively equivalent under a field redefinition (which may, however, have a finite radius of convergence in p). Nonetheless, a proposal for an effective tachyon + vector field action of the form

$$S = V(\phi)\sqrt{-\det(\eta_{\mu\nu} + F_{\mu\nu} + \partial_\mu \phi \partial_\nu \phi)} \tag{5.1}$$

was given in [61, 62, 63] (see also [60]). Quite a bit of recent work has focused on this form of the effective action and there seem to be many special properties for this action with particular forms of the potential function $V(\phi)$. It would be very interesting to explicitly construct the tachyon-vector action using the methods outlined in this work. A particularly compelling question related to this action is that of closed string radiation during the tachyon decay process. In order to understand this radiation process, it is necessary to understand back-reaction on the decaying D-brane [136], which in the open string limit corresponds to the computation of loop diagrams. Recent work [64] indicates that for the superstring, SFT loop diagrams on an unstable Dp-brane with $p < 7$ should be finite, so that it should be possible to include loop corrections in the effective tachyon action in such a theory. The resulting effective theory should shed light on the question of closed string radiation from a decaying D-brane.

The tachyon-vector filed effective action can also be computed in the framework of BSFT. When other excited string modes are present, say modes of the vector field A_μ, the form of the WS action should still be given by (3.4) where the renormalized tachyon field depends also on the other string couplings. In particular it would be interesting to apply our renormalization procedure to the other renormalizable case in which the boundary perturbation contains also a vector field. Whether this approach would help in treating also non-renormalizable interactions it is yet not clear.

The decay of unstable systems of D-branes, pictured as a tachyon field rolling down a potential toward a stable minimum, can also be addressed in the context of the boundary string field theory. It involves deforming the world sheet conformal field theory of the unstable D-brane by a conformal, time dependent tachyon profile. It is useful then to construct β-functions which are valid away from the RG fixed point to demonstrate that the renormalization flow exists, to draw the RG-trajectories and to understand where the endpoints of the RG flux are. Thus our approach should reveal important in studying the physics around a conformal fixed points and in particular about the time dependent solutions describing rolling tachyons.

Appendix A

Neumann Coefficients

In this Appendix we give explicit expressions for and properties of the Neumann coefficients that we use throughout this paper. First we define coefficients A_n and B_n by the series expansions

$$\left(\frac{1+iz}{1-iz}\right)^{1/3} = \sum_{n\,\text{even}} A_n z^n + i \sum_{n\,\text{odd}} A_n z^n, \tag{A.1}$$

$$\left(\frac{1+iz}{1-iz}\right)^{2/3} = \sum_{n\,\text{even}} B_n z^n + i \sum_{n\,\text{odd}} B_n z^n\,. \tag{A.2}$$

In terms of A_n and B_n we define the coefficients $N_{mn}^{r,\pm s}$ as follows:

$$\begin{aligned}
N_{nm}^{r,\pm r} &= \frac{1}{3(n \pm m)} \begin{cases} (-1)^n (A_n B_m \pm B_n A_m) & m+n \in 2\mathbb{Z}, \quad m \neq n \\ 0 & m+n \in 2\mathbb{Z}+1 \end{cases}, \\
N_{nm}^{r,\pm(r+1)} &= \frac{1}{6(n \pm m)} \begin{cases} (-1)^{n+1} (A_n B_m \pm B_n A_m) & m+n \in 2\mathbb{Z}, \quad m \neq n \\ \sqrt{3}\,(A_n B_m \mp B_n A_m) & m+n \in 2\mathbb{Z}+1 \end{cases}, \\
N_{nm}^{r,\pm(r-1)} &= \frac{1}{6(n \mp m)} \begin{cases} (-1)^{n+1} (A_n B_m \mp B_n A_m) & m+n \in 2\mathbb{Z}, \quad m \neq n \\ -\sqrt{3}\,(A_n B_m \pm B_n A_m) & m+n \in 2\mathbb{Z}+1 \end{cases}.
\end{aligned} \tag{A.3}$$

The coefficients V^{rs}_{mn} are then given by

$$V^{rs}_{nm} = \sqrt{mn}\,\left(N^{r,s}_{nm} + N^{r,-s}_{nm}\right) \qquad m \neq n,\ m,n > 0\,, \tag{A.4a}$$

$$V^{rr}_{nn} = \frac{1}{3}\Big(2\sum_{k=0}^{n}(-1)^{n-k}A_k^2 - (-1)^n - A_n^2\Big), \qquad n \neq 0\,, \tag{A.4b}$$

$$V^{r(r+1)}_{nn} = V^{r(r+2)}_{nn} = -\frac{1}{2}\big((-1)^n + V^{rr}_{nn}\big) \qquad n \neq 0\,, \tag{A.4c}$$

$$V^{rs}_{0n} = \sqrt{2n}\,\left(N^{r,s}_{0n} + N^{r,-s}_{0n}\right) \qquad n \neq 0\,, \tag{A.4d}$$

$$V^{rr}_{00} = -\ln(27/16)\,. \tag{A.4e}$$

The analogous expressions for the ghost Neumann coefficients are

$$\begin{aligned}
\mathcal{N}^{r,\pm r}_{nm} &= \frac{1}{3(n\pm m)}\begin{cases}(-1)^{n+1}(B_nA_m \pm A_nB_m) & m+n \in 2\mathbb{Z},\quad m\neq n\\ 0 & m+n\in 2\mathbb{Z}+1\end{cases},\\
\mathcal{N}^{r,\pm(r+1)}_{nm} &= \frac{1}{6(n\pm m)}\begin{cases}(-1)^{n}\,(B_nA_m \pm A_nB_m) & m+n \in 2\mathbb{Z},\quad m\neq n\\ -\sqrt{3}\,(B_nA_m \mp A_nB_m) & m+n\in 2\mathbb{Z}+1\end{cases},\\
\mathcal{N}^{r,\pm(r-1)}_{nm} &= \frac{1}{6(n\mp m)}\begin{cases}(-1)^{n}(B_nA_m \mp A_nB_m) & m+n \in 2\mathbb{Z},\quad m\neq n\\ \sqrt{3}\,(B_nA_m \pm A_nB_m) & m+n\in 2\mathbb{Z}+1\end{cases}.
\end{aligned} \tag{A.5}$$

Observe that the ghost formulae (A.5) are related to matter ones (A.4a) by $A_m \to -B_m$, $B_m \to A_m$. The ghost Neumann coefficients are expressed via $\mathcal{N}^{rs}_{nm}$ as

$$X^{rs}_{nm} = m\,(\mathcal{N}^{r,s}_{nm} + \mathcal{N}^{r,-s}_{nm}) \qquad m\neq n,\ m>0\,, \tag{A.6a}$$

$$X^{rr}_{nn} = -\frac{2}{3}(-1)^nA_nB_n + \frac{1}{3}\Big(2\sum_{k=0}^{n}(-1)^{n-k}A_k^2 - (-1)^n - A_n^2\Big) \qquad n\neq 0, \tag{A.6b}$$

$$X^{rs}_{nn} = X^{rs}_{nn} = -\frac{1}{2}\,((-1)^n + X^{rr}_{nn})\,, \qquad r\neq s, n\neq 0, \tag{A.6c}$$

The exponential in the vertex $\langle V_3|$ does not contain X_{n0}, so we have not included an expression for this coefficient; alternatively, we can simply define this coefficient to vanish and include c_0 in the exponential in $\langle V_3|$.

Now we describe some algebraic properties satisfied by V^{rs} and X^{rs}. Define $M^{rs}_{mn} = CV^{rs}$, $\mathcal{M}^{rs}_{mn} = \sqrt{\frac{n}{m}}CX^{rs}_{mn}$. The matrices M and $\mathcal{M}$ satisfy symmetry and cyclicity properties

$$M^{r+1\,s+1} = M^{rs}, \qquad \mathcal{M}^{r+1\,s+1} = \mathcal{M}^{rs}, \tag{A.7a}$$

$$(M^{rs})^T = M^{rs}, \qquad (\mathcal{M}^{rs})^T = \mathcal{M}^{rs}, \tag{A.7b}$$

$$CM^{rs}C = M^{sr}, \qquad C\mathcal{M}^{rs}C = \mathcal{M}^{sr}. \tag{A.7c}$$

This reduces the set of independent matter Neumann matrices to M^{11}, M^{12}, M^{21} and similarly for ghosts. These matrices commute and in addition satisfy

$$M^{11} + M^{12} + M^{21} = -1, \qquad \mathcal{M}^{11} + \mathcal{M}^{12} + \mathcal{M}^{21} = -1, \tag{A.8a}$$

$$M^{12}M^{21} = M^{11}(M^{11} + 1), \qquad \mathcal{M}^{12}\mathcal{M}^{21} = \mathcal{M}^{11}(\mathcal{M}^{11} - 1). \tag{A.8b}$$

These relations imply that there is only one independent Neumann matrix.

Appendix B

Computation of $I(k_1, k_2, k_3)$

In this Appendix we shall compute the integral $I(k_1, k_2, k_3)$ appearing in eq.(3.34)

$$\begin{aligned} I(k_1,k_2,k_3) = \quad & \frac{2^{2k_1k_2+2k_2k_3+2k_1k_3}}{(2\pi)^3} \int_0^{2\pi} d\tau_1 d\tau_2 d\tau_3 \left[\sin^2\left(\frac{\tau_1-\tau_2}{2}\right)\right]^{k_1k_2} \\ & \left[\sin^2\left(\frac{\tau_2-\tau_3}{2}\right)\right]^{k_2k_3} \left[\sin^2\left(\frac{\tau_1-\tau_3}{2}\right)\right]^{k_1k_3} \end{aligned} \tag{B.1}$$

Introducing the variables

$$x = \frac{\tau_1 - \tau_2}{2} \quad , \quad y = \frac{\tau_3 - \tau_1}{2}$$

the integral over τ_1, τ_2 and τ_3 can be written as

$$I = -\frac{4^{k_1k_2+k_2k_3+k_1k_3+1}}{2\pi^3} \int_0^{2\pi} d\tau_1 \int_{\frac{\tau_1}{2}}^{\frac{\tau_1}{2}-\pi} dx \int_{-\frac{\tau_1}{2}}^{\pi-\frac{\tau_1}{2}} dy \left[\sin^2 x\right]^{k_1k_2} \left[\sin^2 y\right]^{k_1k_3} \left[\sin^2(x+y)\right]^{k_2k_3}$$

With a suitable shift of the integration variables we obtain

$$\begin{aligned} I = \quad & \frac{4^{k_1k_2+k_2k_3+k_1k_3}}{\pi^2} \int_0^{\pi} dx \int_0^{\pi} dy \left[\sin x\right]^{2k_1k_2} \left[\sin y\right]^{2k_1k_3} \left[\sin^2(x+y)\right]^{k_2k_3} \\ = \quad & \frac{4^{k_1k_2+k_2k_3}}{\pi^2} \int_0^{\pi} dx \int_0^{\pi} dy \left[\sin x\right]^{2k_1k_2} \left[\sin y\right]^{2k_1k_3} \left[1 - e^{2i(x+y)}\right]^{k_1k_3} \left[1 - e^{-2i(x+y)}\right]^{k_2k_3} \\ = \quad & \frac{4^{k_1k_2+k_1k_3}}{\pi^2} \int_0^{\pi} dx \int_0^{\pi} dy \left[\sin x\right]^{2k_1k_2} \left[\sin y\right]^{2k_1k_3} \sum_{n,m=0}^{\infty} \frac{\Gamma(n-k_2k_3)\Gamma(m-k_2k_3)}{n!m!\Gamma^2(-k_2k_3)} e^{2i(x+y)(n-m)} \end{aligned} \tag{B.2}$$

Integrating over x and y we have

$$I = \sum_{n,m=0}^{\infty} \frac{\Gamma(n-a_3)\Gamma(m-a_3)}{n!m!\Gamma(1+a_1+n-m)\Gamma(1+a_1-n+m)\Gamma(1+a_2+n-m)\Gamma(1+a_2-n+m)} \frac{\Gamma(1+2a_1)\Gamma(1+2a_2)}{\Gamma^2(-a_3)} \tag{B.3}$$

where $a_1 = k_1k_2$, $a_2 = k_2k_3$ and $a_3 = k_1k_3$. Shifting $m \to m-n$ in the sum over m we have

$$\begin{aligned} I &= \sum_{n,m=0}^{\infty} \frac{\Gamma(n-a_2)\Gamma(n+m+a_2)\Gamma(1+2a_1)\Gamma(1+2a_3)}{n!(n+m)!\Gamma^2(-a_2)\Gamma(1+a_1+m)\Gamma(1+a_1-m)\Gamma(1+a_3+m)\Gamma(1+a_3-m)} \\ &+ \sum_{n=0}^{\infty}\sum_{m=-n}^{0} \frac{\Gamma(n-a_2)\Gamma(n+m+a_2)\Gamma(1+2a_1)\Gamma(1+2a_3)}{n!(n+m)!\Gamma^2(-a_2)\Gamma(1+a_1+m)\Gamma(1+a_1-m)\Gamma(1+a_3+m)\Gamma(1+a_3-m)} \\ &- \frac{\Gamma(1+2a_1)\Gamma(1+2a_3)}{\Gamma^2(1+a_1)\Gamma^2(1+a_3)}\ {}_2F_1(-a_2,-a_2;1;1) \end{aligned} \tag{B.4}$$

where ${}_2F_1(\alpha,\beta;\gamma;z)$ is the Hypergeometric function. Changing the sign of the integer m in the second term of the previous equation and noting that

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} = \sum_{n=m}^{\infty}\sum_{m=0}^{\infty}$$

we find

$$\begin{aligned} I &= 2\sum_{m=0}^{\infty} \frac{\Gamma(m-a_2)\Gamma(1+2a_1)\Gamma(1+2a_3)}{\Gamma(-a_2)\Gamma(1+a_1+m)\Gamma(1+a_1-m)\Gamma(1+a_3+m)\Gamma(1+a_3-m)} \\ &\quad {}_2F_1(m-a_2,-a_2;m+1;1) - \frac{\Gamma(1+2a_1)\Gamma(1+2a_3)}{\Gamma^2(1+a_1)\Gamma^2(1+a_3)}\ {}_2F_1(-a_2,-a_2;1;1) \end{aligned} \tag{B.5}$$

It is not difficult to show that the sum over m can be extended to negative values so that we find

$$\begin{aligned} I = &-\frac{\Gamma(1+2a_1)\Gamma(1+2a_2)\Gamma(1+2a_3)}{\Gamma(1-a_1)\Gamma(a_1)\Gamma(1-a_2)\Gamma(a_2)\Gamma(1-a_3)\Gamma(a_3)} \\ &\sum_{m=-\infty}^{\infty} \frac{\Gamma(m-a_1)\Gamma(m-a_2)\Gamma(m-a_3)}{\Gamma(1+m+a_1)\Gamma(1+m+a_2)\Gamma(1+m+a_3)} \end{aligned} \tag{B.6}$$

The series in the second line of the right-hand side of (B.6) is convergent for $1 + a_1 + a_2 + a_3 > 0$. To sum over m we use a standard procedure. Consider the path in Fig.B-1

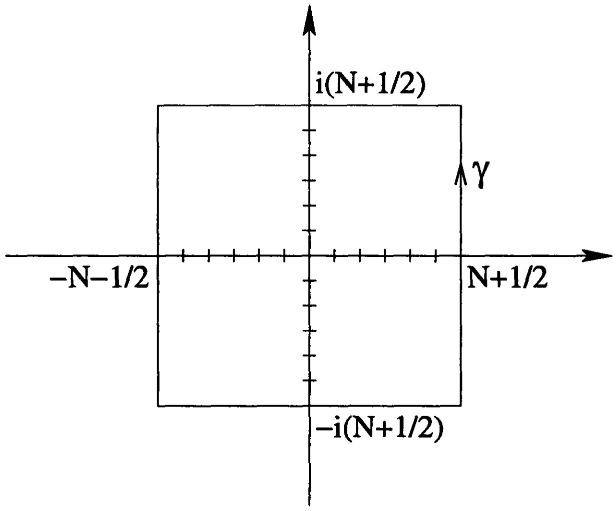

Figure B-1: Path of integration.

Defining

$$S = \sum_{m=-\infty}^{\infty} \frac{\Gamma(m-a_1)\Gamma(m-a_2)\Gamma(m-a_3)}{\Gamma(1+m+a_1)\Gamma(1+m+a_2)\Gamma(1+m+a_3)} \equiv \sum_{m=-\infty}^{\infty} f(m) \tag{B.7}$$

we can write

$$\oint_C \pi \mathrm{cotg} \pi z f(z) dz = \sum_{m=-N}^{N} f(m) + \tilde{S} \tag{B.8}$$

where $\tilde{S}$ is the sum of the residues of $\pi \mathrm{cotg}\pi z f(z)$ evaluated in the poles of $f(z)$. In the limit $N \to \infty$ the left-hand side of the previous equation vanishes reducing S to

$$\begin{aligned} S = & -\frac{\Gamma(1+a_1+a_2+a_3)}{\Gamma(1+a_1)\Gamma(1+a_2)\Gamma(1+a_3)\Gamma(1+a_1+a_2)\Gamma(1+a_1+a_3)\Gamma(1+a_2+a_3)} \\ & \left[\frac{\pi^3 \cos^2 \pi a_1}{\sin \pi a_1 \sin \pi(a_1-a_2) \sin \pi(a_1-a_3)} + \text{cycl.}\right] \end{aligned} \tag{B.9}$$

So that I becomes

$$I = \frac{\Gamma(1+a_1+a_2+a_3)\Gamma(1+2a_1)\Gamma(1+2a_2)\Gamma(1+2a_3)}{\Gamma(1+a_1)\Gamma(1+a_2)\Gamma(1+a_3)\Gamma(1+a_1+a_2)\Gamma(1+a_2+a_3)\Gamma(1+a_1+a_3)} \tag{B.10}$$

Appendix C

Perturbative computation of the effective action for the tachyon field.

We have used two methods to compute the coefficients in the effective action $S[\phi(t)]$. The first method, as described in the main text, consists of solving the equations of motion for each field perturbatively in ϕ. The second method consists of computing the effective action by summing diagrams which can be computed using the method of level truncation on oscillators. This approach is described in subsection C.1

C.1 Effective action from level truncation on oscillators

The classical effective action for the tachyon can be perturbatively computed as a sum over all tree-level connected Feynman diagrams.

A method for computing such diagrams to high levels of truncation in string field theory was presented in [65], and used in [71] to compute the effective action for the massless vector field. Using this method, the contribution of a given Feynman diagram with n vertices, $n-1$ propagators and $n+2$ external fields is given by an

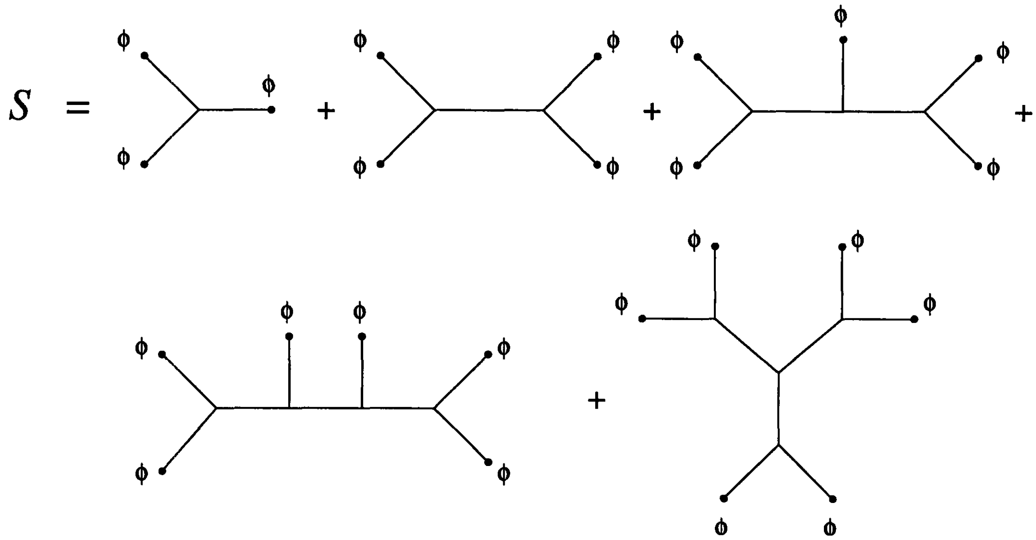

Figure C-1: The first few diagrams contributing to the effective action

integral of the form

$$\delta S = \int \prod_{i=1}^{n} (2\pi d w_i) \phi(w_i) \delta(\sum w_i) \int \prod_{j=1}^{n-3} \frac{d\sigma_j}{\sigma_j^2} \operatorname{Det}\left(\frac{1-\mathcal{X}\mathcal{P}}{(1-\mathcal{V}\mathcal{P})^{13}}\right) \exp\left(-w_i \mathcal{Q}^{ij} w_j\right) \tag{C.1}$$

In this formula $\mathcal{V}$ and $\mathcal{X}$ are $n \times n$ block matrices whose blocks are matter and ghost Neumann coefficients V^{rs} and X^{rs} of the cubic string field theory vertex. More precisely

$$\mathcal{V} = \begin{pmatrix} V^{r_1 s_1} & 0 & \dots & 0 \\ 0 & V^{r_2 s_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & V^{r_n s_n} \end{pmatrix}, \quad \mathcal{X} = \begin{pmatrix} X^{r_1 s_1} & 0 & \dots & 0 \\ 0 & X^{r_2 s_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & X^{r_n s_n} \end{pmatrix}. \tag{C.2}$$

When using level truncation V^{rs} and X^{rs} become $3L \times 3L$ matrices of real numbers. The matrix $\mathcal{P}$ encodes information about propagators, external states and the graph structure of the diagram. We define it as

$$\mathcal{P} = K^T \hat{\mathcal{P}} K. \tag{C.3}$$

Here $\hat{\mathcal{P}}$ is a block-diagonal matrix of the form

$$\hat{\mathcal{P}} = \begin{pmatrix} P(\sigma_1) & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & P(\sigma_2) & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & P(\sigma_{n-3}) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} \tag{C.4}$$

The diagonal blocks $P(\sigma_i)$ correspond to propagators. In the level truncation scheme the block $P(\sigma)$ of $\hat{P}$ is the $2L \times 2L$ matrix

$$P(\sigma) = \begin{pmatrix} 0 & P_{12}(\sigma) \\ P_{21}(\sigma) & 0 \end{pmatrix}. \tag{C.5}$$

where

$$P_{12}(\sigma) = P_{21}(\sigma) = \begin{pmatrix} \sigma & 0 & \dots & 0 \\ 0 & \sigma^2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sigma^4 \end{pmatrix}. \tag{C.6}$$

The last n rows and columns of $\hat{P}$ are filled with zeroes which correspond to external tachyon states. The matrix K is the block permutation matrix that encodes information on the graph structure of the diagram.

As we can see from the figure C-2 in the case of the 4-point diagram a suitable permutation is

$$\kappa = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} \to \begin{pmatrix} 3 & 6 & 1 & 2 & 4 & 5 \end{pmatrix}, \tag{C.7}$$

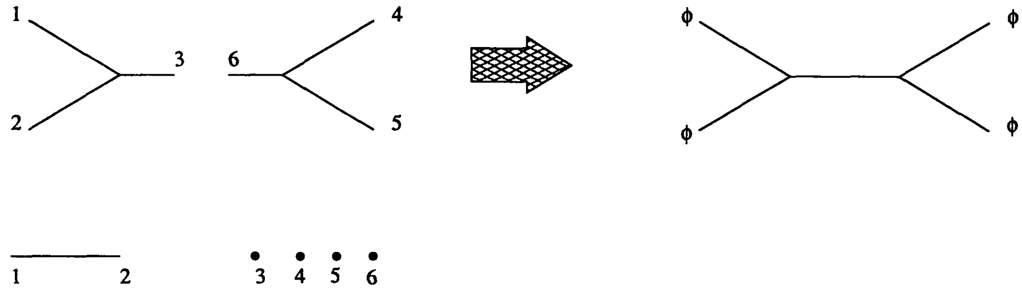

Figure C-2: To construct the 4 point diagram we label consecutively the edges of vertices on one hand and propagators and external states on the other. Matrix K corresponds to a permutation that would glue them in one diagram

which corresponds to

$$K = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{C.8}$$

Multiplying matrices we find

$$\mathcal{V}\mathcal{P} = (\tilde{V}^{11})^2, \qquad\qquad \mathcal{X}\mathcal{P} = (\tilde{X}^{11})^2, \tag{C.9}$$

where

$$\tilde{V}^{11}_{mn} = \sigma^{\frac{m+n}{2}} V^{11}_{mn}, \qquad\qquad \tilde{X}^{11}_{mn} = \sigma^{\frac{m+n}{2}} X^{11}_{mn}. \tag{C.10}$$

The contribution from the Feynman diagram with 4 external tachyons is given by, see subsection 2.3.1,

$$\begin{aligned} &\frac{e^{-3V^{11}_{00}} g^2}{2} \int \prod_{i=1}^{4} (2\pi dw_i)\phi(w_i)\delta(\sum w_i) \\ &\qquad \int \frac{d\sigma}{\sigma^2} \operatorname{Det}\left(\frac{1-(\tilde{X}^{11})^2}{[1-(\tilde{V}^{11})^2]^{13}}\right) \sigma^{-\frac{1}{2}[(w_1+w_2)^2+(w_3+w_4)^2]} \exp\left(-w_i \mathcal{Q}^{ij} w_j\right), \end{aligned} \tag{C.11}$$

with Q^{ij} defined as

$$Q^{ij} = \mathcal{U}^{i3}_{0.}\frac{1}{1-(\tilde{V}^{11})^2}\tilde{V}^{11}\mathcal{U}^{3j}_{.0} + \mathcal{U}^{ij}_{00} \qquad i,j=1,2 \text{ or } i,j=3,4, \tag{C.12}$$

$$Q^{ij} = -\mathcal{U}^{i3}_{0.}\frac{1}{1-(\tilde{V}^{11})^2}C\mathcal{U}^{3j}_{.0} \qquad (i=1,2 \text{ and } j=3,4) \text{ or } (i=3,4 \text{ and } j=1,2), \tag{C.13}$$

$$\mathcal{U}^{ij} = \begin{pmatrix} V^{ij}_{00} - V^{i3}_{00} - V^{3j}_{00} + V^{33}_{00} & \tilde{V}^{ij}_{0n} - \tilde{V}^{3j}_{0n} \\ \tilde{V}^{ij}_{m0} - \tilde{V}^{i3}_{m0} & \tilde{V}^{ij}_{mn} \end{pmatrix}, \tag{C.14}$$

and $C_{mn} = \delta_{mn}(-1)^n$. Considering only the contribution coming from level 2 fields, we have to consider only these Neumann coefficents whose powers and products total oscillator level sum up to 2, i.e. V_{01}, V_{11}, V_{02} and X_{11} [65][1]. Doing so equation (C.11) simplifies a lot and the integral over the modular parameter reduces to

$$\int d\sigma\sigma^{-\frac{1}{2}\left[(w_1+w_2)^2+(w_3+w_4)^2\right]}. \tag{C.15}$$

Performing this integral it is easy to get the same result as in formula (4.9).

[1]If we want to calculate the quartic term in the effective action we have to subtract the contribution from the tachyon in the propagator.

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