

STATIC STIFFNESS COEFFICIENTS FOR  
CIRCULAR FOUNDATIONS EMBEDDED  
IN AN ELASTIC MEDIUM

by

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BE, S.U.N.Y. at Stony Brook  
(1973)

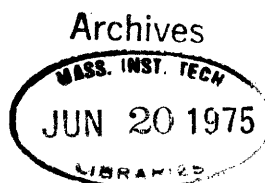
Submitted in partial fulfillment  
of the requirements for the degree of  
Master of Science in  
Civil Engineering

at the  
Massachusetts Institute of Technology  
(May, 1975)

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ABSTRACTSTATIC STIFFNESS COEFFICIENTS FOR  
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Approximate empirical relations for the static stiffness coefficients of circular foundations embedded in (or resting on) a viscoelastic homogeneous stratum (or half-space) are developed. The relations are obtained from a parametric study using a finite element technique. Approximate frequency dependent functions for the stiffness and radiation damping coefficients, and equivalent constant viscous damping ratios have been suggested.

The approximations are used to obtain the response of a rigid cylinder embedded in an elastic stratum which compares well with the response obtained using frequency dependent stiffness functions from a finite element analysis.

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## ACKNOWLEDGMENT

The author would like to thank Prof. R. V. Whitman for his advisement and supervision in writing this thesis. Greatest appreciation is due to Dr. E. Kausel for his most valuable suggestions and time consuming efforts. Special thanks to Prof. J. M. Roesset who suggested the topic.

Last but not least, a great deal is owed to Prof. J. Connor and Mr. C. Reeves for the initiation and support of the Engineering Residence Program with Stone & Webster Eng. Co., without which this work could not have been possible.

To my parents,  
Emile and Nelly

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## List of Symbols

Note: throughout the entire thesis the subscript  
 $x$  denotes swaying and  
 $\phi$  denotes rocking  
 $\phi x$  and  $x\phi$  denote coupling terms

- $a_0$  Dimensionless circular frequency. If a subscript is added, reference is specifically made to the resonant frequency of the appropriate mode.
- $C_s$  Shear wave velocity in the homogeneous soil
- $c_1$  Swaying radiation damping coefficient
- $c_2$  Rocking radiation damping coefficient
- $E$  Embedment depth below grade level
- $f$  Frequency in cps
- $f_0$  Dimensionless frequency in cps. If a subscript is added, reference is specifically made to the resonant frequency of the appropriate mode.
- $H$  Stratum depth below grade level
- $h$  Height of center of stiffness above bottom of footing
- $i$  Imaginary unit
- $I$  Moment of inertia about base
- $G$  Shear modulus of the homogeneous soil
- $K$  Stiffness function
- $K_0$  Static stiffness coefficient
- $\bar{K}_0$  Static stiffness coefficient for a surface footing resting on an elastic half-space
- $k_1$  Swaying stiffness coefficient
- $k_2$  Rocking stiffness coefficient

M	Total mass of structure
R	Radius of circular footing
$\beta_h$	Hysteretic damping ratio
$\beta$	Equivalent viscous damping ratio
$\nu$	Poisson's ratio of the homogeneous soil
$n$	Forcing circular frequency
$\omega$	Circular resonant frequency
$\rho$	Mass density of the homogeneous soil
HNFS	Horizontal natural shear beam frequency of stratum in cps
VNFS	Vertical natural shear beam frequency of stratum in cps

## 1.- Introduction:

### 1.1 Statement of the Problem:

Considerable amount of research has been done in recent years to improve and develop effective solutions for the dynamic response of foundations embedded in a stratum or an elastic half-space. This problem is of special interest in the study of soil-structure interaction as it relates to the seismic analysis and design of structures.

Soil-structure interaction is often accounted for in analyses by a set of springs, one for each degree of freedom, representing the soil stiffnesses. These springs are derived from theoretical solutions of the surface footing problem, for ideally elastic, homogeneous, isotropic half-spaces. Footings of buildings, however, are usually founded beneath the surface of the ground. This has in many cases considerable effect on the dynamic response of such footings in that it increases the resonant frequencies and reduces the resonant amplitudes. The solution of the embedded footing problem is a very difficult one to obtain by rigorous analytical methods, thus only approximate analytical solutions were obtained. Finite element and other techniques were then used to solve the problem for these complicated geometries. A main

problem encountered in numerical solutions for dynamic cases, based on finite element or finite difference techniques, is usage of the proper boundary conditions at the edges of the finite domain which will not result in undesirable reflection of waves into the region of interest. This problem was overcome by Waas (41) for layered media. His solution was then generalized by Kausel (21) and extended for the analysis of axisymmetric systems (e.g. nuclear power plants) subjected to arbitrary non-axisymmetric loads or displacements. A similar procedure was also followed by Liang (25) for the case of a strip footing resting on or embedded in a soil stratum.

Two equivalent general approaches can be used to estimate the dynamic soil-structure interaction effects:

- A- The direct (or complete) approach in which the whole system, soil and structure, is modeled and analyzed together. The excitation or the earthquake motion is specified at some control point in the free field.
- B- The spring(or substructure) method, which consists of three steps. First, the system with a massless structure is subjected to the prescribed ground motion, producing a displacement (acceleration) vector in the structure. Second, the frequency dependent subgrade stiffnesses are determined to

yield the so called "soil springs". Finally the response of the real structure, supported on the frequency dependent soil springs, is computed while the structure is subjected to inertial forces proportional to the acceleration vector obtained in the first step.

If certain simplifying assumptions are made, the second approach is a useful method of solution as it is easily understood by the designer who has to successfully predict the behavior of the real system. As will be explained in sec.1.3, in many cases it is possible to omit the first step and, instead, in the third step subject the structure supported by the soil springs to the control motion applied directly under the foundation. In such an analysis, the frequency dependent stiffnesses are needed, and in many cases only the static ones suffice.

The objective of this study is then to determine a simple method to obtain the soil springs needed in calculating the response of the real structure.

## 1.2 Available Solutions:

This historical background section will be presented in two parts. The first will include the studies made for surface footings, while the second is reserved for embedded footings.

### 1.2.1 Surface Footings:

The dynamic response of foundations resting on an elastic half-space and wave propagation theories are summarized in Richart, Hall and Woods (35). In most of these theories, the foundation is represented as a rigid circular cylinder and the soil on which the foundation rests as an elastic half-space. The elastic half-space representation by Reissner(34), Sung (37), Bycroft (7) and others, idealizes the soil as a homogeneous isotropic, elastic, semi-infinite medium. The rigid foundation has six degrees of freedom: three in translation and three in rotation. In addition to the studies discussed in Richart, Hall and Woods (35), many others have investigated the dynamic response of surface footings, some of which are listed below.

For the elastic, homogeneous, isotropic half-space case, Awojobi and Grootenhuis (4) investigated the response of both rigid circular and very long rectangular bodies for the

vertical, torsional and rocking modes. Awojobi (3) discussed the torsional mode for a rigid circular body, while Grootenhuis (15) discussed the vertical, torsional and rocking modes for a rigid circular or rectangular body (both half-space and elastic stratum cases). Luco and Westmann(27) investigated the vertical, horizontal, torsional, rocking and coupled (rocking and sliding) modes for a rigid circular disc. Veletsos and Wei (40) presented numerical results for flexibility, stiffness and damping coefficients (for various values of Poisson's ratio) over a wide range of dimensionless frequencies for the rigid massless disc. Weissmann (43) presented expressions for the resonant frequency and resonant amplitude of a rigid circular surface footing for the torsional mode, while taking into effect the hysteretic damping of the soil.

Experimental studies on the dynamic behavior of surface foundations have been reported by many investigators. The following comprises some of these studies for the circular surface footing. The vertical mode of vibration has been experimentally investigated by Eastwood (12), Fry (14), Chae (8,9), Drnevich and Hall (11), Grootenhuis and Awojobi (16), Stokoe (36) and Erden (13). Both the torsional and coupled (rocking and sliding) modes have been investigated by Fry (14), Stokoe

(36) and Erden (13).

### 1.2.2 Embedded Footings:

Most of the above mentioned studies apply to ideally elastic, homogeneous, isotropic half-spaces, assumptions which do not apply to many practical cases. Soils are usually non-homogeneous and found in layered strata. Also, most structures are embedded and thus it is necessary to investigate the effect of the embedment on their response.

Approximate analytical solutions of the embedded rigid circular footing on an elastic half-space for the vertical, horizontal and rocking modes were presented by Baranov (5), Novak and Beredugo (32) and Beredugo (6) (he also included embedment in elastic stratum for the vertical mode). Approximate equations of rocking motion are given in terms of input earthquake acceleration, spring constant and damping by Tajimi (38) for a rigid cylindrical body completely embedded in an elastic stratum. Frequency independent stiffness and damping parameters were approximated by Novak (31) for the vertical, torsional and coupled (rocking and sliding) modes of the rigid circular footing on an elastic half-space, while the torsional and coupled (torsional, horizontal and rocking) modes for the same footing are presented by Novak and Sachs (33)



for both half-spaces and elastic strata.

Finite element solutions were obtained for a rigid circular footing embedded in an elastic homogeneous half-space by Kaldjian (19), Kuhlemeyer (24) and Lysmer and Kuhlemeyer (28), for the vertical mode; Kaldjian (20) for the torsional mode; Waas (42) for the vertical and torsional modes (he also included both homogeneous and layered cases); Urlich and Kuhlemeyer (39) for rocking and sliding, Johnson and Christiano (18) for all modes, while Kausel (21) presented a model applicable to all four modes for layered media. Krizek, Gupta and Parmelee (23) presented the rocking and sliding modes for an infinitely long, rigid footing embedded in homogeneous, isotropic and linearly elastic half-space.

Experimental studies on the dynamic behavior of embedded circular footings were made by Anandakrishnan and Krishnaswamy (1), Chae (10), and Gupta (17) for the vertical mode of vibration. Stokoe (36) and Erden (13) presented the vertical, torsional and coupled (rocking and sliding) modes.

The soil springs for complicated geometries (embedment and stratum depths) can be obtained more accurately using finite element techniques, rather than approximate analytical methods.

But such a solution can amount to high computation costs. Thus a parametric study, using a finite element technique, of the embedment and stratum depths for the soil stiffnesses can be quite advantageous. Such a parametric study is the objective of this investigation.

### 1.3 Scope of this Work:

Traditionally, the earthquake motion in the interaction analysis using the spring method has been specified directly under the foundation using the half-space soil springs. A more careful consideration (22) shows, however, that at least three steps are necessary for a rigorous solution of embedded foundations. These steps are (the spring method, see fig 1.1):

A- First, the motion that would occur at the massless foundation is determined. This step is eliminated for surface footings when the motion is specified at the free surface of the soil, and when one dimensional theory (only vertically propagating waves are assumed) is used. In the case of embedded footings, this step is theoretically necessary and both horizontal motion and rocking are expected to occur at the massless foundation.

B- Frequency dependent stiffnesses ("springs") are

then determined to model the soil media. A great amount of research has been devoted to this step. In numerical solutions, the mathematical model of the soil is based on finite elements or finite difference schemes. The system is then analyzed in the frequency domain.

C- Finally, the dynamic analysis of the structure is performed, using the stiffness functions computed in B- to reproduce the soil, and the motion derived in A- as input. This analysis can be performed either in the time or frequency domain. The latter is theoretically necessary due to the frequency dependence of the soil stiffnesses. If the analysis was to be done in the time domain, certain approximations would have to be

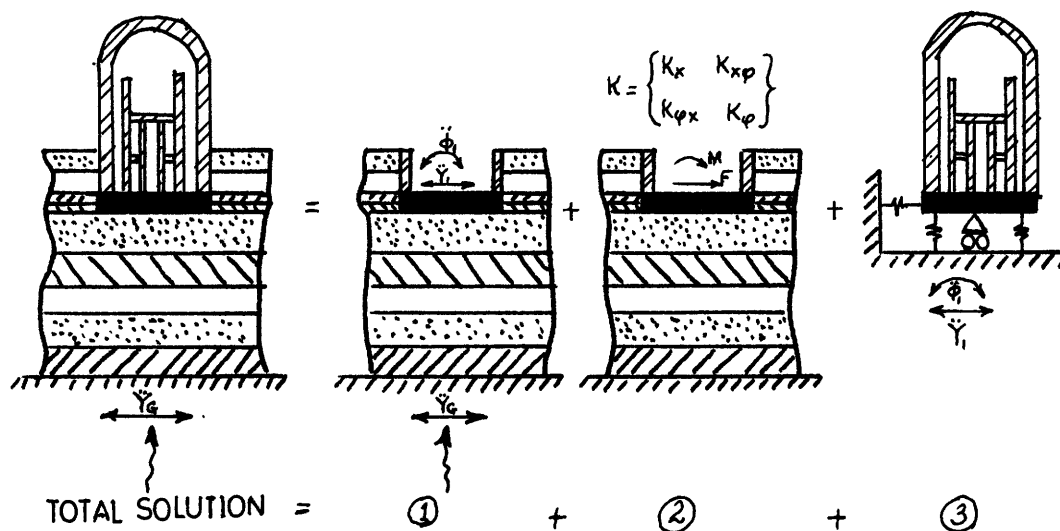


FIG. 1.1: SPRING METHOD

used to account for the frequency dependence of the subgrade stiffnesses (soil springs).

However, the frequency solution is limited to linear problems since it is based on the applicability of the principle of superposition. The non-linear soil behavior must then be simulated using an approximate method which introduces an iterative analysis where modulus and damping are adjusted in each cycle according to the measure of strain resulting from the previous cycle.

The first step becomes quite important for the embedded foundation case. The problem requires the use of finite elements or finite difference techniques, but the structural degrees of freedom are neglected and only an embedded massless rigid footing is considered.

Since only three degrees of freedom are needed to describe the motion of a rigid footing (in a plane), the structural displacements in the first step can be expressed directly in terms of these motion components. As shown in ref.(22), the third step can then be performed subjecting the base of the structure to a prescribed translation (rotation) equal to that of the massless rigid foundation instead of applying the

inertial forces to the structure. It is believed that conservative results can be obtained if the motion obtained from the first step is approximated by the control motion, and the rotational component is neglected. Since the main purpose of this study is to compare true and approximate static stiffnesses, this procedure will be used throughout.

After obtaining the components of motion of the massless foundation, the frequency dependent soil stiffnesses are computed in the second step. The procedure is to subject the base of the foundation, which is assumed to be infinitely rigid and massless, to unit steady state harmonic displacement and rotation, and calculate the corresponding reactions. The following approximations to this step are listed in probable order of decreasing accuracy (22):

- Determine the static stiffnesses of the embedded foundation by finite elements and assume the same frequency variation as for a surface footing on the actual layered medium.
- Apply a scaling factor to the frequency dependent stiffnesses of the surface footing on the layered medium, to account for embedment.
- Apply the static stiffnesses of the actual conditions to the frequency variation of an elastic half-space.

- Use frequency independent stiffness functions obtained by finite elements.
- Use frequency independent stiffness functions obtained by applying the derived scaling factors.
- Use the frequency dependent stiffness functions of the surface footing on an elastic half-space.
- Use the frequency independent stiffnesses of the surface footing on an elastic half-space.

It was suggested in E. Kausel's doctoral dissertation (21), that for an embedded footing the static values are approximately linearly dependent on the embedment and stratum depths. Thus, once these relations are determined, they can be used with some frequency variation obtained from previous studies, resulting in considerable savings in computation costs.

This study aims in determining approximate simple rules (the scaling factors), for the static stiffness coefficients of embedded rigid circular footings in a stratum of soil resting on a rigid half-space. The sidewall is also taken as rigid as the footing. These assumptions apply mostly to the reactor containment structure of nuclear power plants. These structures are usually resting on a rigid circular plate embedded in a stratum which lies on a rigid half-space, i.e. rock. The

sidewalls of the containment, which is embedded, are also considered rigid. The soil is considered homogeneous, isotropic and linearly elastic with hysteretic damping. For simplicity the stratum is considered homogeneous throughout the whole domain. The effect of layering as well as flexibility of the walls will also be discussed later. A computer program written by E. Kausel, hereon referred to as "TRIAX", based on his finite element solution (21) (using the special energy absorbing boundary described in 1.4), will be used in the determination of the simple rules.

The parametric study of embedment and stratum depths is presented in chapter 2. Two typical values for Poisson's ratio will be used ( $\nu = 1/3$  & 0.45). The simple rules for the static stiffness coefficients will be determined graphically. Further study of the effect of weaker backfill, soil layering and flexible sidewalls is also included.

In chapter 3, a comparison of the model used with those of other studies is presented. Comparison of results, of a dynamic analysis of a rigid structure, using the simple rules and the exact stiffnesses obtained from the finite element program for the exact conditions, is included in chapter 4. The comparisons will be in terms of the floor (amplified

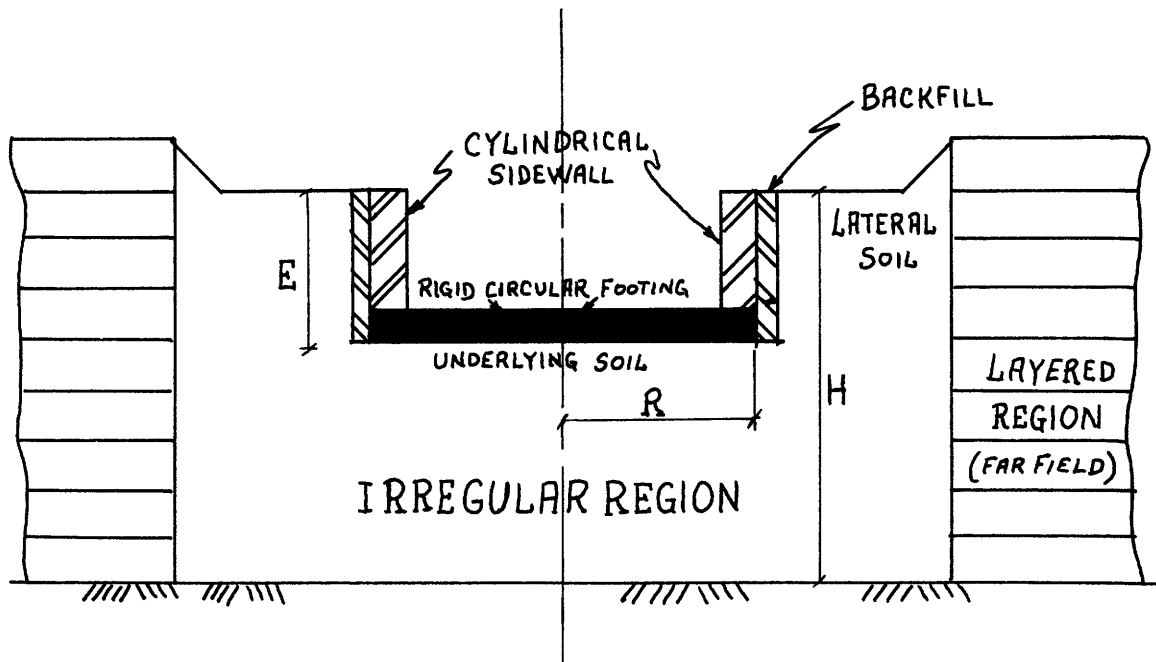


FIG.1.2: DESCRIPTION OF THE MODEL

acceleration) response spectra and transfer functions at the bottom and top of the structure. Conclusions and recommendations are finally presented in chapter 5.

#### 1.4 Description of the Model:

In the finite element formulation, the geometry of the soil is idealized by two regions. The finite irregular region around the footing is connected to the semi-infinite far field as shown in fig 1.2.

Radiation propagates both vertically and laterally. The boundary condition of the irregular region to be used is dictated



by the fact that the radiation of waves away from that region into the far field causes a loss of energy. This boundary condition can be modelled exactly (41,21) when a stratum of soil is resting on a material with higher stiffness properties (e.g. rock), which is the actual case in most practical problems. In such cases, the vertical radiation into the half-space is a small fraction of the lateral radiation in the stratum which occurs only above the fundamental frequency of the stratum. Thus the proper boundary conditions should be of major concern above this fundamental frequency. It is also known that very elongated elements in the horizontal direction will automatically enforce a one dimensional shear type behavior and they will eliminate any lateral radiation.

Such a boundary, representing the actual condition of the finite element columns extending to infinity, has been developed by Waas & Lysmer (30,41) for the two dimensional plane-strain case and the torsional or vertical vibrations of a circular footing. This boundary has the great advantage that it can be located directly at the edge of the footing, with excellent results, leading to an economical and more accurate solution than previous ones by Lysmer and Kuhlemeyer (28,29) or Ang and Newmark ( 2) who developed viscous type boundaries based on a perfect absorption for specific types of waves and incidence angles.

Kausel (21) has recently generalized Waas and Lysmer's boundary to the case of arbitrary Fourier number and expansion order in the finite elements. He has represented the irregular region by means of toroidal finite elements of arbitrary expansion order having three degrees of freedom per nodal ring. In the following study, linear expansions (with four nodes) will be used to minimize computer costs without significant loss of accuracy. The far field is represented by the semi-analytic energy absorbing boundary based on the exact displacement functions in the horizontal direction and expansion in the vertical direction consistent with that used for the finite elements.

## 2.- Parametric Study of Static Stiffness Coefficients:

The effects of embedment on a rigid circular plate welded to a soil stratum, with rigid sidewalls also welded to the lateral soil, subjected to a static displacement and rotation are studied. Approximate correction factors, to account for the embedment effect, are derived and applied to the static stiffnesses of a surface footing resting on a viscoelastic stratum on top of a rigid rock base, or on an elastic half-space. The effects of weaker backfill, soil layering and flexible sidewalls are also briefly investigated.

### 2.1 Method of Solution:

The stiffnesses for the two planesymmetric displacement modes, rocking and swaying, are discussed. The rigid circular plate and sidewalls are idealized by massless finite elements with very high rigidity ( $10^4$  times greater than that of the soil) and the lateral soil is assumed homogeneous throughout the stratum, unless otherwise stated.

The plate is subjected to a unit displacement (or rotation) at the plate-soil interface, and the force (or moment) necessary for equilibrium which by definition is the stiffness,

is computed using TRIAX. The forced horizontal displacement case will be referred to as the swaying mode, while the forced rotation about a horizontal axis as the rocking mode.

The accuracy of the results derived by a finite element analysis is very dependent on the refinement of the mesh used. As one uses finer elements, the theoretical solution is approached. It was concluded from Kausel's dissertation(21), that the rate of convergence of the static stiffness coefficients (for the swaying and rocking modes) to the continuum solution is approximately a linear one. When finite elements with linear expansions are used: "...the relative error in the displacements should be proportional to the square of the typical element size. Stresses and forces are given with less degree of accuracy: as they depend on the partial derivatives of the displacements, the error for them should be approximately linear. The spring constants are determined as reaction forces, and therefore, should converge at an approximate linear rate towards the continuum solution as the mesh is refined." This result suggested a useful linear extrapolation procedure to determine static continuum stiffnesses with the aid of a coarse and a fine mesh. This procedure will be used throughout this study.

The swaying and rocking stiffness functions can be written in the following form (21):

$$K_x = K_{x0} (k_1 + ia_0 c_1) (1 + 2i\beta_{h1}) \quad \text{for swaying (2-1)}$$

$$K_\varphi = K_{\varphi0} (k_2 + ia_0 c_2) (1 + 2i\beta_{h2}) \quad \text{for rocking (2-2)}$$

where:

- $K_{x0}$  and  $K_{\varphi0}$  are the real parts of the static stiffnesses, thus called static spring constants.
- $a_0$  is a dimensionless frequency and is defined by  $a_0 = \frac{\omega R}{C_s}$
- $R$  is the radius of the footing
- $C_s$  is the shear wave velocity
- $k_1, k_2, c_1, c_2$  are the stiffness and radiation damping coefficients, and are functions of the dimensionless frequency  $a_0$ .

The hysteretic damping coefficient  $\beta_h$  is identical to the material damping coefficient  $\beta$  in the soil when the latter is constant throughout the homogeneous stratum. This internal damping in the soil, which is irrelevant for static problems, is assumed to be constant with a value of five percent of critical.

For dynamic problems, it was suggested in (21) that the finite element mesh for swaying, near the plate, should be smaller than 1/6 to 1/8 the shortest shear wave length of interest, and smaller than 1/3 to 1/4 the shortest shear wave length

throughout the finite element region. Because only the static stiffness coefficients are of interest in this parametric study, this rule does not apply. However the meshes will be chosen such that the elements are square with no less than four rows of elements below the footing to insure adequate representation of the subgrade in the region with high strain gradients. Also a correction (as described below) will be applied to account for the discretization error.

## 2.2 Effect of Stratum and Embedment Depths on the Static Stiffness Constants:

Stratum and embedment depths will influence the stiffness functions over the whole frequency range. They are known to increase the static values of the half-space stiffness functions, and affect their frequency dependence. The half-space functions were calculated by Veletsos and Wei (40). Kausel (21) presented the following approximate empirical relations for the static stiffness constants (for a surface footing), which correct the half-space values for the homogeneous stratum depth:

$$K_{x_0} = \bar{K}_{x_0} \left(1 + \frac{1}{2} \frac{R}{H}\right) = \frac{8GR}{2-\nu} \left(1 + \frac{1}{2} \frac{R}{H}\right) \quad (2-3)$$

$$K_{\varphi_0} = \bar{K}_{\varphi_0} \left(1 + \frac{1}{6} \frac{R}{H}\right) = \frac{8GR^3}{3(1-\nu)} \left(1 + \frac{1}{6} \frac{R}{H}\right) \quad (2-4)$$

where:  $-\bar{K}_{x_0}$  and  $\bar{K}_{\phi_0}$  are the half-space static stiffnesses  
 $-H$  is the stratum depth as defined in fig 1.2  
 $-G$  is the shear modulus of the soil and is defined  
 by  $G = \rho C_s^2$   
 $-\rho$  is the mass density of the soil

One can define a point at a distance  $h$  above the plate footing (for either a surface or embedded plate), referred to which the cross coupling term of the stiffnesses is zero. This point is denoted as the center of stiffness and is given by  $h = K_{\phi x_0} / K_{x_0}$ , where  $K_{\phi x_0}$  is the static coupling term. For surface footings,  $h$  is small and results in a small value for the coupling term and thus is usually neglected.

Parametric studies are performed for the following nine cases:

$H/R = 2.$  and  $E/R = .5, 1., 1.5$

$H/R = 3.$  and  $E/R = .5, 1., 1.5$

where  $E$  is the

$H/R = 4.$  and  $E/R = .5, 1., 1.5$

embedment depth

For all models used  $R$ ,  $C_s$  and  $\rho$  are set to unity. The structural properties used are:  $10^{-6}$  for weight density,  $10^6$  for shear wave velocity, and 0.167 for Poisson's ratio ( $\nu$ ).

First, the extrapolation procedure is tested for  $H/R = 2.$

and  $E/R = 0.5$ . Three meshes are tested and the static stiffnesses plotted vs. the inverse of the total number of layers

(fig 2.1). Thus we see that the extrapolation procedure appears

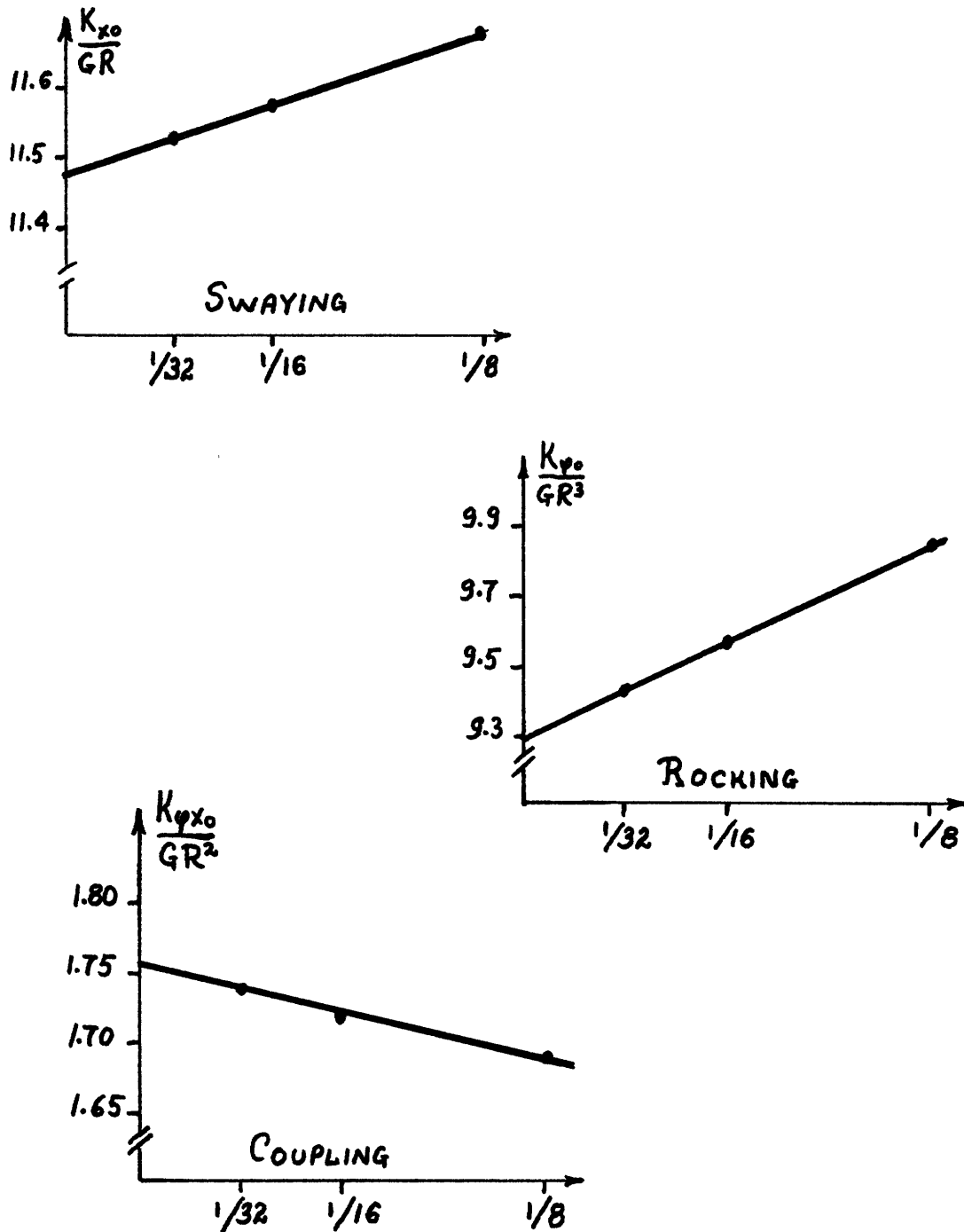


FIG. 2.1 : EXTRAPOLATION PROCEDURE

FOR  $H/R = 2.0$ ;  $E/R = 0.5$ ;  $\nu = 1/3$



Table 2.1: Static swaying and rocking spring constants

		$\nu = 1/3$				$\nu = 0.45$			
E/R	H/R	continuum	coarse	fine	continuum	coarse	fine		
$\frac{K_{xo}}{GR}$	0.0	6.00	--	--	6.45	--	--		
	0.5	11.48	11.682	11.575	12.21	12.525	12.369		
	1.0	16.84	17.095	16.969	18.00	18.372	18.182		
	1.5	25.68	25.885	25.783	27.80	27.896	27.608		
	0.0	5.60	--	--	6.02	--	--		
	0.5	10.08	10.248	10.163	10.74	10.993	10.868		
	1.0	13.75	13.916	13.832	14.68	14.933	14.806		
	1.5	17.72	17.914	17.818	19.00	19.278	19.132		
	0.0	5.35	--	--	5.81	--	--		
	0.5	9.47	9.617	9.541	10.10	10.320	10.208		
	1.0	12.60	12.744	12.673	13.46	13.671	13.564		
	1.5	15.70	15.842	15.768	16.79	17.018	16.904		
$\frac{K_{yo}}{GR}$	0.0	4.30	--	--	5.25	--	--		
	0.5	9.30	9.850	9.564	10.53	11.864	11.198		
	1.0	18.30	19.041	18.673	20.91	22.870	21.892		
	1.5	35.54	36.100	35.822	42.31	44.736	43.523		
	0.0	4.20	--	--	5.12	--	--		
	0.5	8.61	9.110	8.859	9.57	10.658	10.114		
	1.0	16.12	16.696	16.410	17.82	19.092	18.455		
	1.5	28.25	28.921	28.585	31.24	32.784	32.013		
	0.0	4.15	--	--	5.05	--	--		
	0.5	8.41	8.890	8.650	9.29	10.326	9.810		
	1.0	15.51	16.052	15.782	17.00	18.168	17.585		
	1.5	26.70	27.312	27.006	29.22	30.533	29.876		

Note: 1) "continuum" refers to results of extrapolation procedure.  
 2) Values for E/R = 0.0 correspond to Eqs 2-3 & 2-4.

Table 2.2: Static coupling terms

		$\nu = 1/3$				$\nu = 0.45$			
	E/R	H/R	continuum	coarse	fine	continuum	coarse	fine	
	0.5	2	1.75	1.686	1.720	2.46	2.363	2.415	
c	1.0	2	5.79	5.690	5.739	7.15	6.995	7.070	
o	1.5	2	12.49	12.424	12.456	15.22	15.073	15.145	
u	0.5	3	1.39	1.317	1.353	1.96	1.857	1.913	
$K_{\nu} \rho_{\nu} p$	1.0	3	4.64	4.549	4.594	5.63	5.489	5.558	
$GR^2$	1.5	3	9.46	9.368	9.416	11.04	10.885	10.957	
i	0.5	4	1.22	1.154	1.189	1.76	1.638	1.697	
n	1.0	4	4.16	4.068	4.113	5.03	4.891	4.962	
g	1.5	4	8.37	8.275	8.322	9.68	9.538	9.611	

Note: "continuum" refers to results of extrapolation procedure.

valid for the coupling term as well as for swaying and rocking. This procedure, using a fine and a coarse mesh is used in all other eight cases. Tables 2.1 and 2.2 summarize the results for the two sets of runs:  $\nu = 1/3$  and  $\nu = 0.45$ . The values for  $E/R = 0$  (surface footing) are obtained from equations 2-3 and 2-4, where the effect of embedment was not considered but accounting for stratum depth.

### 2.2.1 Swaying and Rocking:

As was mentioned previously, the aim of the parametric study is to determine approximate coefficients, to be applied to equations 2-3 and 2-4 to include the effect of embedment on the static spring constants. These coefficients are determined by evaluating:

$$\frac{K_{x_0}(2-\nu)}{8GR(1+\frac{1}{2}\frac{R}{H})} \quad \text{and} \quad \frac{3 K_{\phi_0}(1-\nu)}{8GR^3(1+\frac{1}{6}\frac{R}{H})}$$

from the static continuum values for all cases (see Table 2.3). For swaying, we see that for all practical purposes, the values of the coefficients for  $\nu = 1/3$  and  $\nu = 0.45$  are the same. Therefore we can conclude that for a homogeneous stratum the coefficients are independent of the soil properties. For rocking, on the other hand, we see a maximum difference of 10% between the two sets of coefficients. This difference can be traced back to two origins:

Table 2.3: Approximate static coefficients

E/R	H/R	s w a y i n g			r o c k i n g		
		$K_{x0}/GR$	$K_{x0}(2-\nu)/8GR(1+\frac{1}{2}\frac{R}{a})$	$K_{\phi_0}/GR^3$	$3K_{\phi_0}(1-\nu)/8GR^3(1+\frac{1}{2}\frac{R}{a})$		
$\nu = 1/3$							
0.0	2	6.00	1.00	4.30	1.00		
0.0	3	5.60	1.00	4.20	1.00		
0.0	4	5.35	1.00	4.15	1.00		
0.5	2	11.48	1.91	9.30	2.15		
0.5	3	10.08	1.80	8.61	2.04		
0.5	4	9.47	1.75	8.41	2.02		
1.0	2	16.84	2.81	18.30	4.22		
1.0	3	13.75	2.45	16.12	3.82		
1.0	4	12.60	2.33	15.51	3.72		
1.5	2	25.68	4.28	35.54	8.20		
1.5	3	17.72	3.16	28.25	6.69		
1.5	4	15.70	2.91	26.70	6.41		
$\nu = 0.45$							
0.0	2	6.45	1.00	5.25	1.00		
0.0	3	6.02	1.00	5.12	1.00		
0.0	4	5.81	1.00	5.05	1.00		
0.5	2	12.21	1.89	10.53	2.00		
0.5	3	10.74	1.78	9.57	1.87		
0.5	4	10.10	1.74	9.29	1.84		
1.0	2	18.00	2.79	20.91	3.98		
1.0	3	14.68	2.44	17.82	3.48		
1.0	4	13.46	2.32	17.00	3.36		
1.5	2	27.80	4.31	42.31	8.06		
1.5	3	19.00	3.16	31.24	6.10		
1.5	4	16.79	2.89	29.22	5.78		

- 1) The extrapolation procedure, even though is theoretically correct, can result in small inaccuracies.
- 2) Even though equations 2-3 and 2-4 are assumed correct for this study, during the derivation of the coefficient  $(1+1/6 R/H)$  for the effect of stratum depth (21), the author showed that in the rocking mode the coefficient was not completely independent of Poisson's ratio. However, this dependence was negligible and the coefficient was derived for the case  $\nu = 1/3$ .

This small difference, between  $\nu = 1/3$  and  $\nu = 0.45$ , can be neglected again as it is a small and negligible one, and the case  $\nu = 1/3$  will be used. Figs 2.2 and 2.3 show the dependence of these coefficients on the stratum depth ratio  $R/H$ .

From figs 2.2 and 2.3, we see that the relationship is a linear one up to approximately  $E/H = 1/2$ . A non-linear relation then exists after that point. It is possible that this limit is larger than  $1/2$ , but this is not evident from the figures. Eventually, when the embedment and the stratum depths are equal, the stiffnesses will reach approximately the half-space solution taking  $G$  and  $\nu$  as the properties of

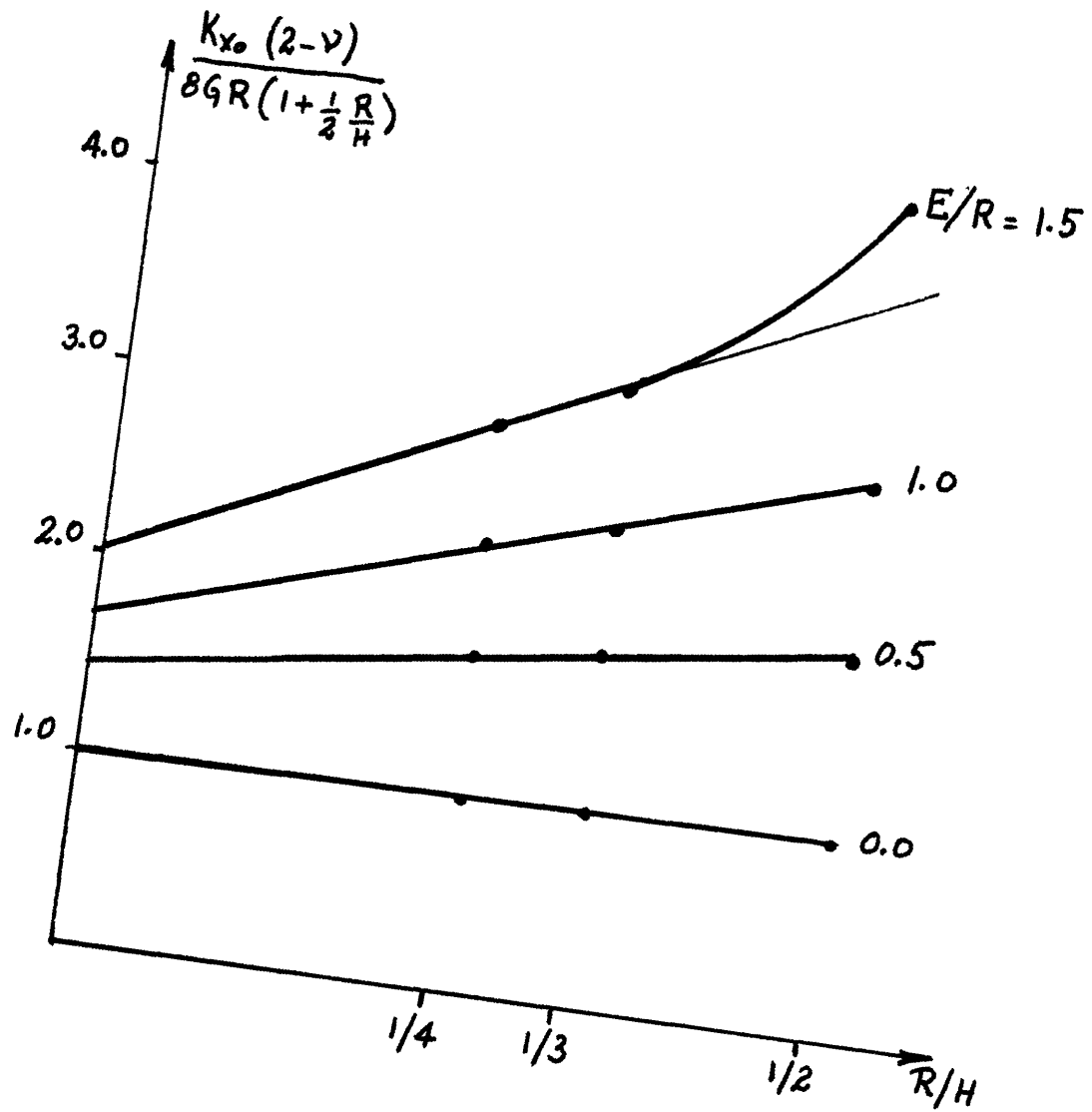


FIG. 2.2: SWAYING STATIC STIFFNESSES  
FOR  $\nu = 1/3$

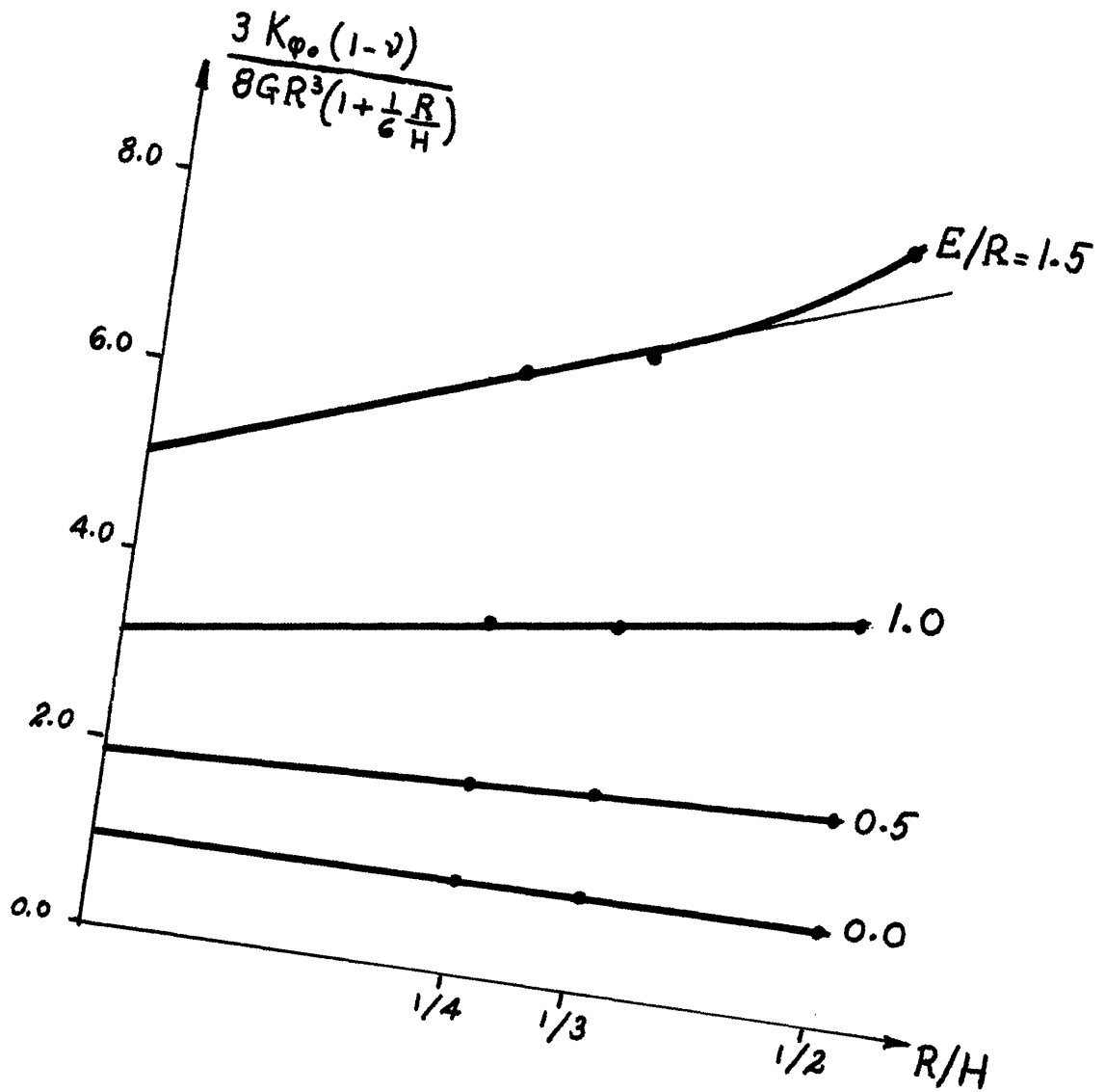


FIG. 2.3 : ROCKING STATIC STIFFNESSES  
FOR  $\nu = 1/3$

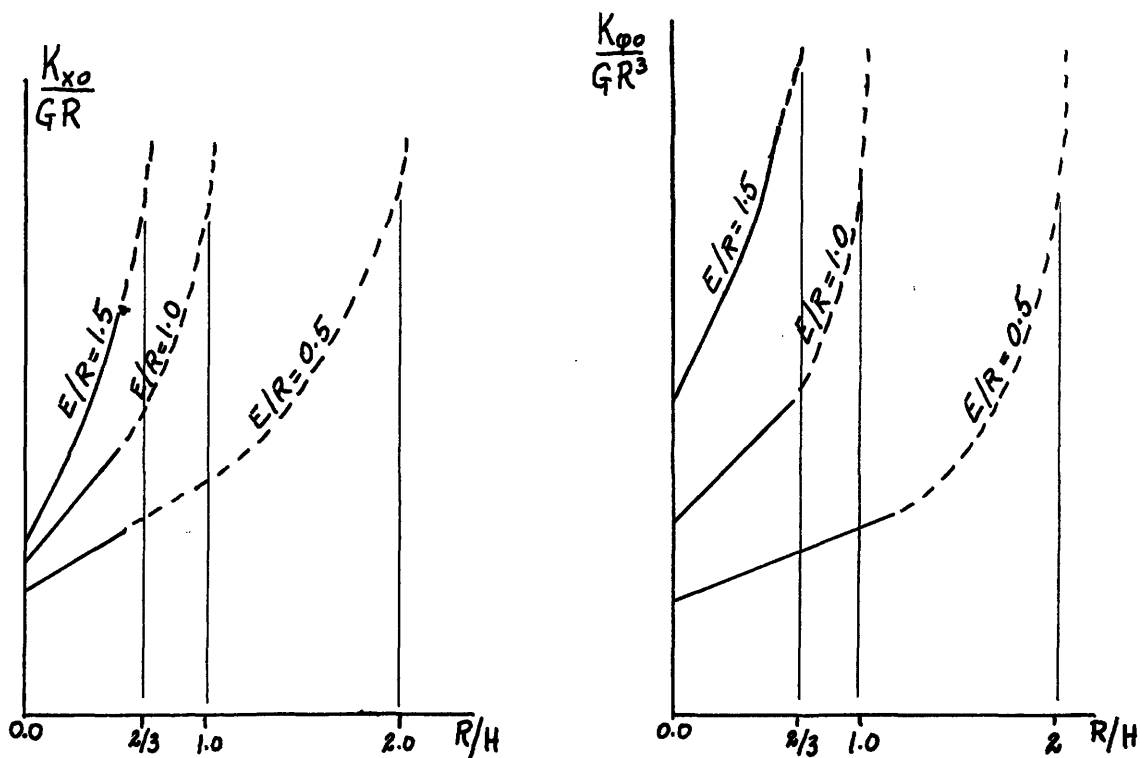


FIG. 2.4: BEHAVIOR OF STATIC STIFFNESS COEFFICIENTS

the rock (see fig 2.4). Actually, the lateral soil will still exist, but will not noticeably affect the stiffnesses.

Therefore, the relationships which are to be derived will apply only when  $E/H \leq 1/2$ . The majority of containment structures for nuclear power plants fall within this range.

The relations of interest proposed here are of the following



forms (see figs 2.2 and 2.3):

$$K_{x_0} = \frac{8GR}{2-\nu} \left(1 + \frac{1}{2} \frac{R}{H}\right) \left(\gamma + \alpha \frac{R}{H}\right) = \frac{8GR}{2-\nu} \left(1 + \frac{1}{2} \frac{R}{H}\right) \left(1 + \frac{\alpha}{\gamma} \frac{R}{H}\right) \gamma \quad (2-5)$$

$$K_{\varphi_0} = \frac{8GR^3}{3(1-\nu)} \left(1 + \frac{1}{6} \frac{R}{H}\right) \left(\beta + C \frac{R}{H}\right) = \frac{8GR^3}{3(1-\nu)} \left(1 + \frac{1}{6} \frac{R}{H}\right) \left(1 + \frac{C}{\beta} \frac{R}{H}\right) \beta \quad (2-6)$$

$\alpha$  and  $C$  are the slopes of the straight lines, and  $\gamma$  and  $\beta$  are the intercepts of the straight lines. Notice that all four parameters are functions of  $E/R$ . Calculating  $\gamma$ ,  $\frac{\alpha}{\gamma}$ ,  $\beta$  and  $\frac{C}{\beta}$  and plotting them as functions of  $E/R$  we obtain the results tabulated in table 2.4 and plotted in figs 2.5 and 2.6.

Table 2.4:  $\gamma$ ,  $\alpha/\gamma$ ,  $\beta$  and  $C/\beta$  as functions of  $E/R$

E/R	swaying			rocking		
	$\gamma$	$\alpha$	$\alpha/\gamma$	$\beta$	$C$	$C/\beta$
0.	1.	0.	0.	1.	0.	0.
.5	1.45	1.05	.72	1.9	.6	.31
1.	1.7	2.2	1.29	3.1	2.2	.71
1.5	2.	3.6	1.8	5.	5.2	1.04

Figs 2.5 and 2.6 exhibit a linear dependence of  $\gamma$ ,  $\alpha/\gamma$  and  $C/\beta$  on  $E/R$ . However,  $\beta$  is linear only up to  $E/R = 1.$ , beyond that point divergence from linearity seems to exist. This divergence from linearity can be neglected, for practical purposes, and will be approximated as a straight line. It follows then, that the static springs can be obtained from

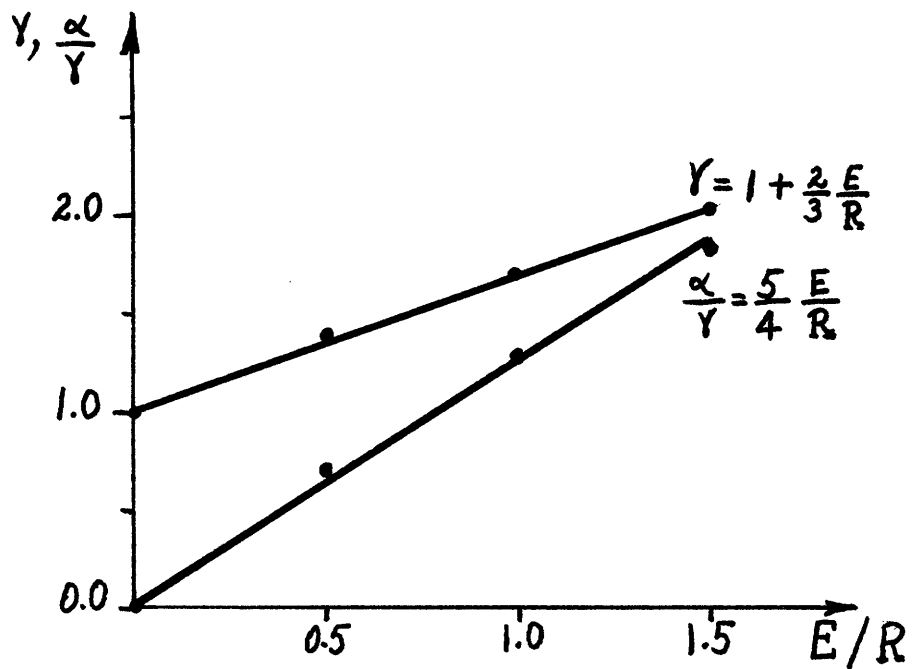


FIG. 2.5: CONTROLLING PARAMETERS OF THE  
STATIC SWAYING STIFFNESS

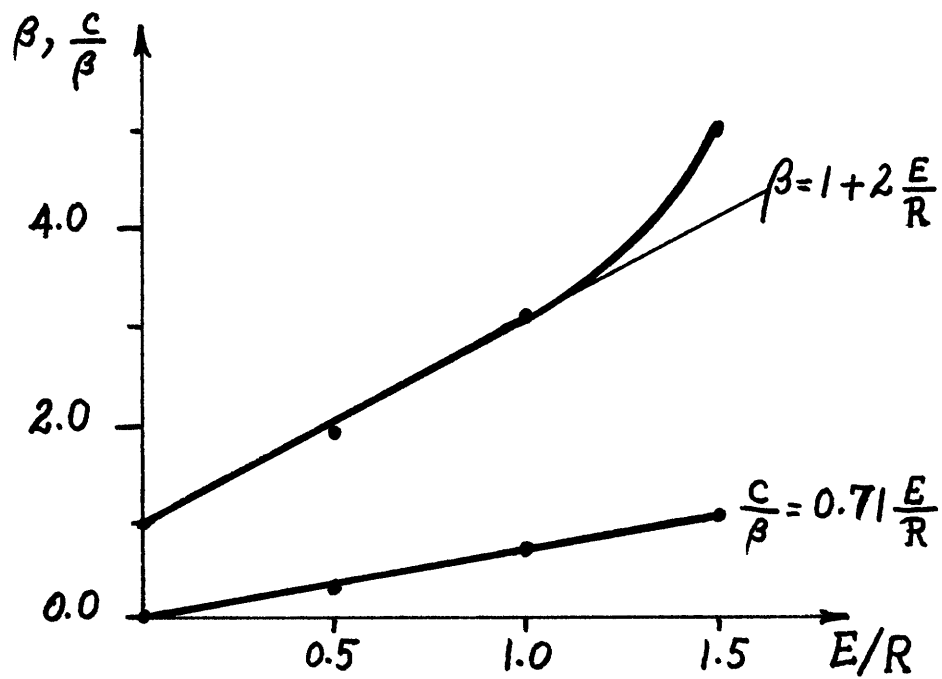


FIG. 2.6: CONTROLLING PARAMETERS OF THE  
STATIC ROCKING STIFFNESS

the approximate empirical relations:

$$K_{x_0} = \bar{K}_{x_0} \left(1 + \frac{1}{2} \frac{R}{H}\right) \left(1 + \frac{2}{3} \frac{E}{R}\right) \left(1 + \frac{1}{4} \frac{E}{H}\right) \quad \text{for } E/H \leq 1/2 \quad (2-7)$$

$$K_{\varphi_0} = \bar{K}_{\varphi_0} \left(1 + \frac{1}{6} \frac{R}{H}\right) \left(1 + 2 \frac{E}{R}\right) \left(1 + 0.71 \frac{E}{H}\right) \quad \text{for } E/H \leq 1/2 \quad (2-8)$$

where  $\bar{K}_{x_0}$  and  $\bar{K}_{\varphi_0}$  are the half-space static spring constants for surface footings:

$$\bar{K}_{x_0} = \frac{8GR}{2-\nu} \quad \text{and} \quad \bar{K}_{\varphi_0} = \frac{8GR^3}{3(1-\nu)}$$

It is believed that for very shallow strata, the flexibility of the base rock should be included in the analysis. This may be of particular importance for deeply embedded footings in shallow strata and is demonstrated using a study by Luco (26). He presented a rigorous mathematical solution for a surface footing sitting on two layered media. The first is a soil stratum while the second is a stiffer half-space representing the rock medium. He presented stiffness functions for three different ratios of  $G_{\text{ROCK}}/G_{\text{SOIL}}$ . Table 2.5 lists the properties for the three cases and the ratios of the static stiffnesses to the half-space values (derived from his study for  $a_0 = 0$ ). These ratios are plotted in fig 2.7.

The corresponding approximate ratios for the surface footing are obtained from equations 2-7 and 2-8 by setting  $E = 0$ .

We obtain:

$$\frac{K_{x_0}}{\bar{K}_{x_0}} = 1 + \frac{1}{2} \frac{R}{H} \quad \text{and} \quad \frac{K_{\varphi_0}}{\bar{K}_{\varphi_0}} = 1 + \frac{1}{6} \frac{R}{H} \quad (2.9)$$

Table 2.5: Static stiffness ratios for surface footings derived from Luco's study.

R/H	swaying			rocking			properties			
	case 1	case 2	case 3	case 1	case 2	case 3	case 1	case 2	case 3	
0.0	1	1	1	1	1	1	$C_{s1}/C_{s2}$	0.8	0.4	0.2
1/5	--	1.08	1.08	--	1	1	$R/\rho_2$	0.85	0.85	0.85
1/3	1.08	--	--	1	--	--	$\gamma_1$	0.25	0.30	0.35
1/2	--	1.2	--	--	1.05	1.05	$\gamma_2$	0.25	0.25	0.25
1	1.18	1.4	1.5	1.1	1.22	1.26	$G_2/G_1$	1.84	7.35	29.4
2	1.28	1.8	2.07	1.25	1.65	1.90	subscript 1 refers to soil			
3	1.46	--	--	1.48	--	--	subscript 2 refers to rock			

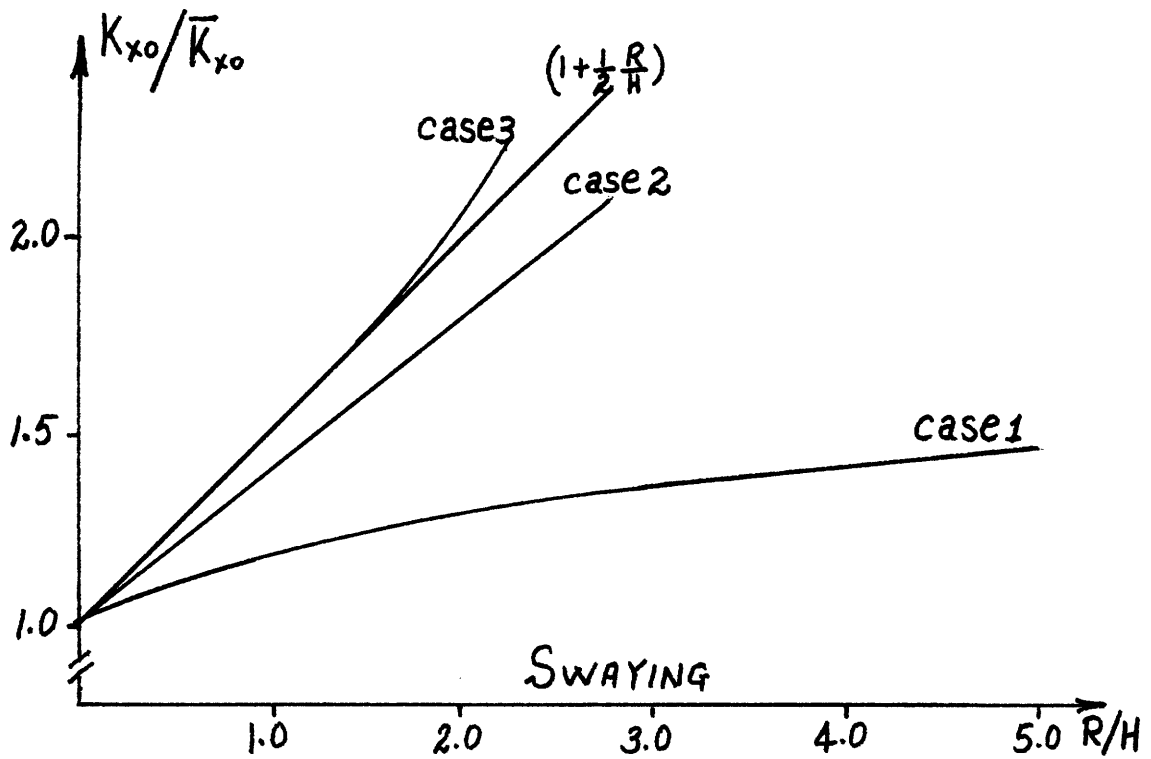
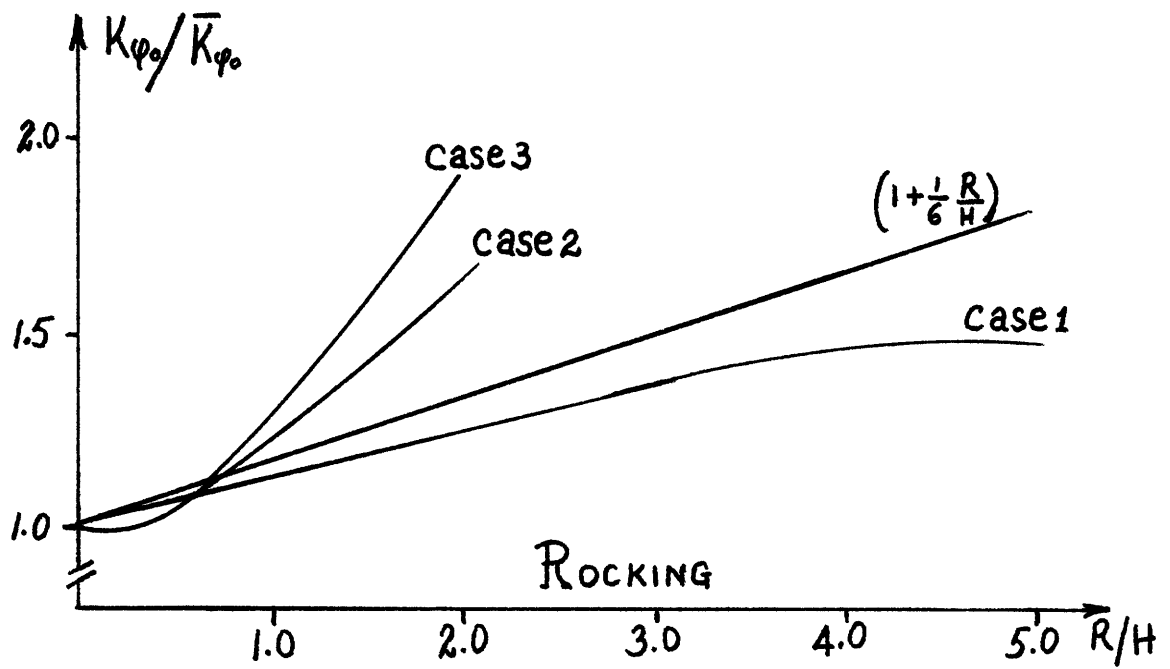


FIG. 2-7: STATIC RATIOS DERIVED FROM LUCO'S STUDY

These ratios are also included in fig 2.7. For the rocking mode the approximate ratio agrees well with Luco's values for all three cases in the region  $R/H < 1$ . On the other hand, the approximate ratio for the swaying mode, agrees perfectly with Luco's case 3 (rigid rock). This comparison demonstrates that the rock flexibility cannot be disregarded for very shallow strata, say  $H/R < 1$ .

### 2.2.2 Coupling Term:

When a plate, whether at the surface or embedded, is rotated at the base, a certain amount of horizontal force will develop. Thus a coupling term will always exist. For an embedded footing it is more pronounced than for a surface footing. Table 2.2 lists the coupling terms obtained from TRIAX for the coarse and fine meshes. The extrapolation procedure is also used to obtain the continuum values. The height of center of stiffness ( $h = K_{\varphi x_0} / K_{x_0}$ ) is then calculated and listed in table 2.6. Fig 2.8 shows the relationship between  $h/R$  and  $E/R$ . We see that the static coupling term is also linear (with respect to  $E/R$ ) in the region where  $E/H \leq 1/2$ . Again when  $E/R = H/R$ , we reach approximately the case of a surface footing on a half-space (rock). For such condition  $h$  is very small, thus we expect  $h/R$  to decrease as  $E/R$  approaches  $H/R$  (see fig 2.9).

Table 2.6: Height of center of stiffness

H/R	E/R	$\nu = 1/3$				$\nu = 0.45$				
		$K_{\varphi x_0}/GR^2$	$K_{x_0}/GR$	$\frac{h}{R} = \frac{K_{\varphi x_0}}{RK_{x_0}}$	$K_{\varphi x_0}/GR^2$	$K_{x_0}/GR$	$\frac{h}{R} = \frac{K_{\varphi x_0}}{RK_{x_0}}$	$K_{\varphi x_0}/GR^2$	$K_{x_0}/GR$	$\frac{h}{R} = \frac{K_{\varphi x_0}}{RK_{x_0}}$
2	0.5	1.75	11.48	0.152	2.46	12.21	0.201			
2	1.0	5.79	16.84	0.344	7.15	18.00	0.397			
2	1.5	12.49	25.68	0.486	15.22	27.80	0.547			
3	0.5	1.39	10.08	0.138	1.96	10.74	0.182			
3	1.0	4.64	13.75	0.337	5.63	14.68	0.384			
3	1.5	9.46	17.72	0.534	11.04	19.00	0.581			
4	0.5	1.22	9.47	0.129	1.76	10.10	0.174			
4	1.0	4.16	12.60	0.330	5.03	13.46	0.374			
4	1.5	8.37	15.70	0.533	9.68	16.79	0.577			

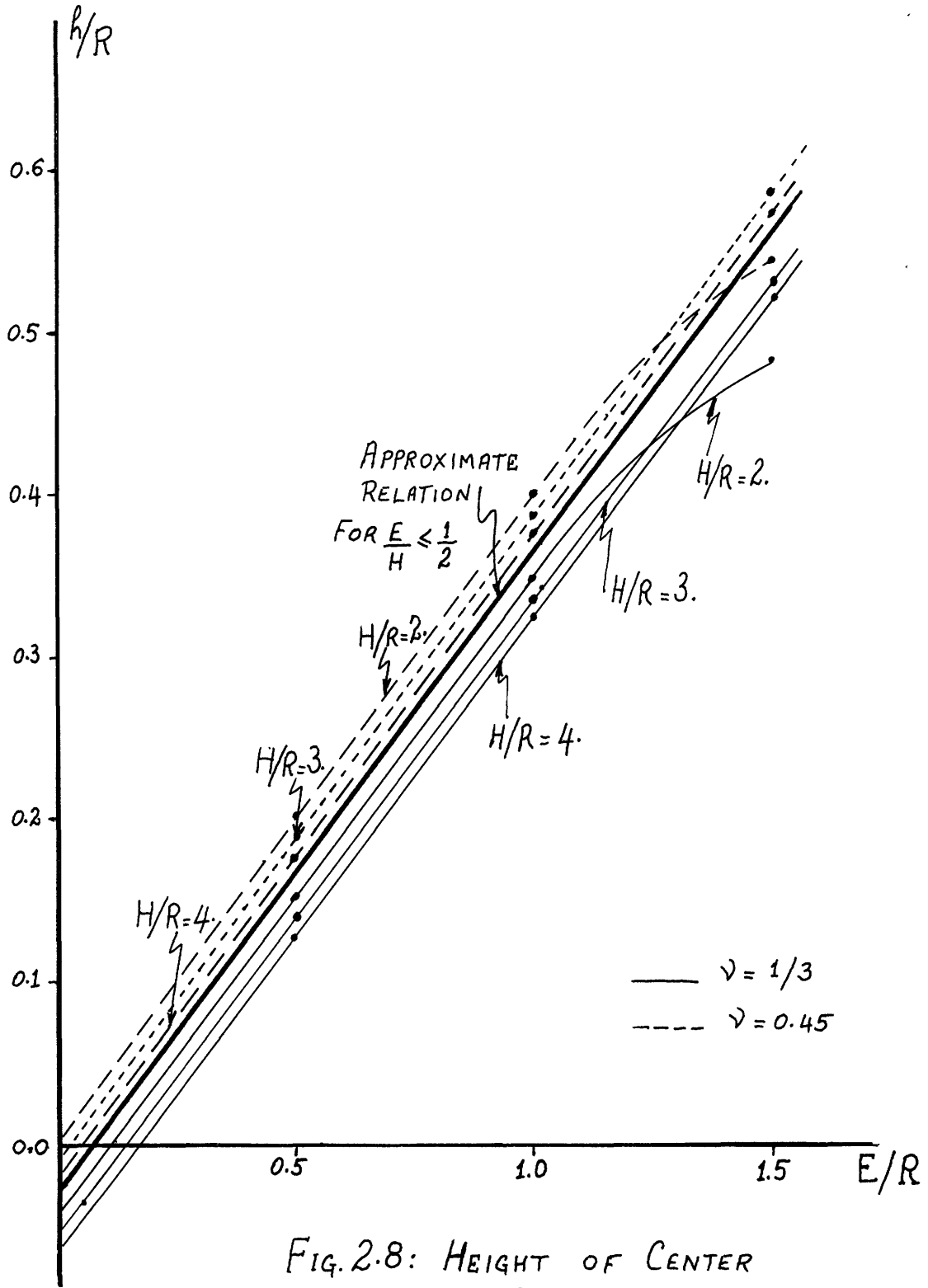


FIG. 2.8: HEIGHT OF CENTER OF STIFFNESS



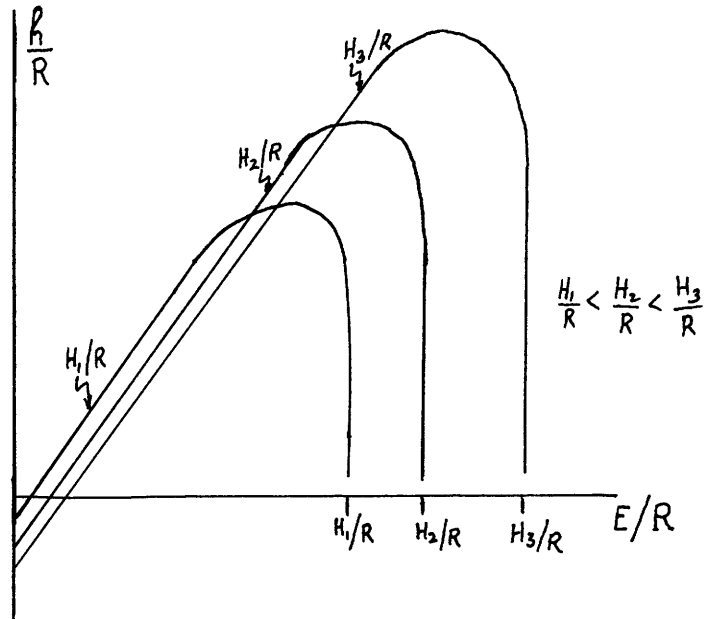


FIG.2.9: BEHAVIOR OF HEIGHT  
OF CENTER OF STIFFNESS

It is also evident that  $h/R$  is dependent on  $\nu$ ,  $H/R$  and  $E/R$ . The dependence on the first two parameters is relatively small and could not accurately be determined. For practical purposes, it is possible to approximate  $h/R$  as independent of  $\nu$  and  $H/R$ , as follows (heavy line in fig 2.8):

$$h/R = 0.4 E/R - 0.03 \quad \text{for } E/H \leq 1/2$$

Thus we can approximate the coupling term as:

$$\begin{aligned} K_{\varphi x_0} &= h K_{x_0} \\ &= (0.4 E - 0.03 R) K_{x_0} \quad \text{for } E/H \leq 1/2 \quad (2-11) \end{aligned}$$

### 2.3 Effect of Weaker Backfill:

One of the assumptions made in the derivation of equations 2-7, 2-8 and 2-11 was that the soil next to the embedded footing has the same properties as beneath it. However, in reality, after the footing and the sidewalls of an embedded structure are poured in place the rest of the excavation is filled in, thus leaving disturbed lateral soil, producing a weaker backfill than the soil beneath the footing. In addition, soils are layered and their stiffness increases with depth.

A new model for the case  $H/R = 2.$ ,  $E/R = 1.$  was developed. The ratio of the thickness of the backfill to the radius of the footing was taken as 0.1, while the shear modulus of the backfill was taken as 0.81 times the shear modulus of the undisturbed soil. The structural properties were unchanged from the previous models. The static stiffnesses were then calculated using TRIAX for the case  $\nu = 1/3$ . The above ratios were used because they represent actual values found in many containment structures. Table 2.7 shows the results, where the continuum values were obtained from the same extrapolation procedure previously used.

Table 2.7: Effect of weaker backfill for  $H/R=2.$ ,  $E/R=1.$ ,  $\nu=1/3$ 

Note:  $G$  is the shear modulus of the underlying soil

mode	coarse	fine	continuum	no backfill
$K_{x_0}/GR$	16.876	16.750	16.62	16.84
$K_{\varphi_0}/GR^3$	18.502	18.167	17.83	18.30
$K_{\varphi x_0}/GR^2$	5.536	5.582	5.63	5.79

It is evident from the table that the static stiffnesses are slightly decreased when the weaker backfill is considered. However, due to the uncertainties of the soil properties, it is possible for all practical purposes, to disregard this slight decrease of the stiffnesses.

#### 2.4 Effect of Flexible Sidewalls:

In the parametric study, the sidewall was assumed to be as rigid as the embedded footing with a ratio of  $10^6$  for the shear modulus of the footing to that of the soil, an assumption needed to compute the soil "springs" in the second step of the spring method discussed in section 1.3. This assumption is not realistic, thus the effect of a flexible sidewall must be evaluated. This effect is known to decrease the subgrade stiffnesses, and it becomes necessary to determine the extent of this decrease for most practical situations.

Table 2.8: Effect of flexible sidewalls for  $E/R = 1.$ ;  $\nu = 1/3$ 

Mode	H/R	Flexible Sidewall			Rigid Sidewall
		coarse	fine	cont.	cont.
s w a y i n g  $\frac{K_{\phi_0}}{GR}$	2.	16.874	16.295	15.72	16.84
	3.	13.754	13.334	12.91	13.75
	4.	12.607	12.248	11.89	12.60
r o c k i n g  $\frac{K_{\phi_0}}{GR^3}$	2.	18.009	16.334	14.66	18.30
	3.	15.784	14.330	12.88	16.12
	4.	15.187	13.807	12.43	15.51
c o u p p l i n g  $\frac{K_{\phi_0}}{GR^2}$	2.	5.238	4.572	3.91	5.79
	3.	4.187	3.651	3.12	4.64
	4.	3.744	3.265	2.79	4.16

Three cases are considered:  $H/R = 2., 3., 4.,$  with  $E/R = 1.$  The soil and structural properties used are the same as in section 2.2, except that the shear modulus of the sidewall is reduced by giving it a value 80 times greater than that of the soil, a factor commonly encountered in practice. The meshes used, coarse and fine, are the same as the previous ones (used in sec. 2.2). Table 2.8 lists the results.

Upon examination of Table 2.8, one notices that the rocking and coupling terms are more sensitive to the sidewall flexibility than the swaying term. Two hypothetical extremes can be considered to explain this effect (see fig 2.10). In one extreme, the foundation has no sidewalls (or infinitely flexible sidewalls), while in (b) the wall is infinitely rigid. The actual foundation wall corresponds to an intermediate case between these two extremes. If a unit horizontal displacement is imposed on these two foundation systems, the force required to produce this displacement will not change

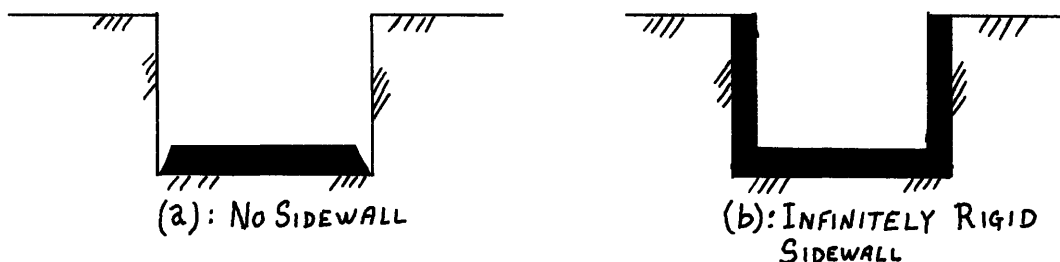


FIG. 2.10: TWO HYPOTHETICAL EXTREMES FOR SIDEWALL FLEXIBILITY.

substantially, since only small tensile/compressive reaction forces act on the rigid sidewall in case (b). Most of the resisting force is offered by shearing stresses below the plate. Rocking, on the other hand, is very sensitive to changes in the stiffnesses of the lateral walls. This results from the fact that the shearing stresses along the sidewalls, which contribute to the rocking stiffness and are magnified by the moment arm, decrease with increasing flexibility of the sidewall. This decrease in the static stiffness coefficients is obviously a function of the extent of the sidewall flexibility assumed. Further analytical and experimental investigations are required to approximate the appropriate decrease to be used.

### 2.5 Effect of Variable Shear Wave Velocity:

Up to this point, the stratum was taken to be homogeneous in all directions. However, in real problems, the soil exists in layers with the shear wave velocity, and thus also the shear modulus, increasing with depth. If one chooses homogeneous properties as existing at the foundation level, the static stiffnesses will be over-estimated. Table 2.9 shows the results obtained for the case  $E/R = 1.$ ,  $H/R = 2.$ ,  $\nu = 1/3$ , with the shear wave velocity varying from 0.5 at the surface

Table 2.9: Effect of variable shear wave velocity for

$$E/R = 1., H/R = 2., \nu = 1/3$$

Note: G is the shear modulus at the level of the footing

mode	constant $C_5$	variable $C_5$
$K_{x_0}/GR$	16.84	15.52
$K_{\varphi_0}/GR^3$	18.30	15.46
$K_{\varphi x_0}/GR^2$	5.79	3.97

to 1. at the foundation level and 1.1 at rock. These values are chosen as they are frequently encountered in nuclear power plants. When the soil shear modulus above the foundation level is decreased, the effect noticed in the previous two cases apply. The sensitivity of the rocking and coupling terms is again evident. Further analytical and experimental studies are needed to determine the approximate reduction of the static stiffnesses.

### 3.- Comparison of Model Used with Other Studies:

In this section, comparisons of the approximate relations derived in sections 2.2.1 and 2.2.2 with results by Urlich and Kuhlemeyer (39) for the influence of embedment in a half-space, and by Johnson and Christiano (18) for the combined effects of embedment and stratum depths are presented. Since the approximate relations provide a good fit to results obtained from TRIAX, this section essentially provides comparisons between the finite element model with the transmitting boundary, used in this study, and other more "conventional" finite element models. As was shown in section 2.2.1, the approximate relations are in good agreement with Luco's theoretical results for surface footings on an elastic stratum underlaid by a rigid base (case 3), and thus provide reasonable validation for the empirical rules derived in this study. Therefore this section will give insight into the magnitude of the errors resulting from standard modeling.

#### 3.1 Embedment in a Half-space:

Urlich and Kuhlemeyer (39) presented a numerical model which was used to solve the problem of steady state coupled rocking and lateral vibrations of footings embedded in an elastic



half-space. They used an analytical model similar to that described by Lysmer and Kuhlemeyer (28) for the steady state vertical vibration of a footing embedded in an elastic half-space. Three dashpots were used oriented perpendicular to each other and located at each boundary nodal point with one being normal to the boundary and two tangential to the boundary. They expressed the coupled rocking and sliding solution for the steady state displacement,  $u$ , and rotation,  $\psi$ , of a surface or embedded rigid, weightless footing of radius  $R$  as:

$$\begin{bmatrix} u \\ \psi R \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} \frac{F}{\bar{K}_{x_0}} e^{i\omega t} \\ \frac{RM}{\bar{K}_{\psi_0}} e^{i\omega t} \end{bmatrix} \quad (3-1a)$$

where  $F_{ij} = f_{ij} + ig_{ij}$  is the dimensionless complex displacement function, ( $f_{ij}$  and  $g_{ij}$  are referred to as flexibility coefficients),  $t$  is time,  $\omega$  is the circular frequency,  $i$  is the imaginary unit,  $F$  and  $M$  are the excitation force and moment amplitudes, respectively, and  $\bar{K}_{x_0}$  and  $\bar{K}_{\psi_0}$  are the static spring constants for surface footings determined from:

$$\bar{K}_{x_0} = \frac{8GR}{2-\nu} \quad \text{and} \quad \bar{K}_{\psi_0} = \frac{8GR}{3(1-\nu)} \quad (3-1b)$$

However, for the static case, which is of interest in this study, the above solution becomes:

$$\begin{bmatrix} u \\ \psi R \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} F/\bar{K}_{x_0} \\ RM/\bar{K}_{\psi_0} \end{bmatrix} \quad (3-2)$$

Taking the inverse of the flexibility to obtain the stiffness matrix  $\underline{K}$ , one obtains:

$$\underline{K} = \frac{1}{D} \begin{bmatrix} f_{22} \bar{K}_{x_0} & -f_{12} \bar{K}_{x_0} R \\ -f_{21} \bar{K}_{\psi_0} / R & f_{11} \bar{K}_{\psi_0} \end{bmatrix} \quad (3-3)$$

where  $D = f_{11} f_{22} - f_{21} f_{12}$ , and  $F_{ij}$  have been replaced by  $f_{ij}$  as  $g_{ij}$  are zero for the static case. From Eq.3-3 we see that the static stiffness coefficients have been expressed as:

$$K_{x_0} = f_{22} \bar{K}_{x_0} / D \quad ; \quad K_{\psi_0} = f_{11} \bar{K}_{\psi_0} / D \quad (3-4)$$

$$K_{x\psi_0} = -f_{12} R \bar{K}_{x_0} / D \quad ; \quad K_{\psi x_0} = -f_{21} \bar{K}_{\psi_0} / RD \quad (3-5)$$

Urlich and Kuhlemeyer showed that a relation exists between  $f_{12}$  and  $f_{21}$ :

$$f_{21} = R^2 \bar{K}_{x_0} f_{12} / \bar{K}_{\psi_0}$$

Thus the expressions in Eq.3-5 become, as expected:

$$K_{x\psi_0} = K_{\psi x_0} = -f_{21} \bar{K}_{\psi_0} / RD$$

or the height of center of stiffness  $h$  (as defined in section 2.2.2) is given by:

$$\frac{h}{R} = \frac{K_{\psi x_0}}{R K_{x_0}} = \frac{-f_{21} \bar{K}_{\psi_0}}{R^2 f_{22} \bar{K}_{x_0}} = \frac{-(2-\nu) f_{21}}{3(1-\nu) f_{22}} \quad (3-6)$$

For the static solution, Urlich and Kuhlemeyer used boundary conditions such that vertical and base boundaries of the mesh were fixed from moving normally and free to move tangentially. The dynamic dashpots were not used for the static solution as

they are properly defined only for the dynamic case. Therefore their static solution models more closely a footing in a finite stratum (specifically  $H/R= 6.$ ). They also used a value of  $\nu=1/3$  for Poisson's ratio and five values for the parameter  $E/R$ . Table 3.1 lists the static flexibility coefficients ( $f_{ij}$ ) which were obtained from plots of their results. Eqs 3-4 and 3-6 are then used to obtain the static stiffness coefficients  $K_{\varphi_0}/\bar{K}_{\varphi_0}$ ,  $K_{x_0}/\bar{K}_{x_0}$  and  $h/R$  which are also plotted in figs 3.1, 3.2, and 3.3, respectively. The corresponding approximate relations for  $K_{\varphi_0}/\bar{K}_{\varphi_0}$  and  $K_{x_0}/\bar{K}_{x_0}$  are obtained from Eqs 2-8 and 2-7 by setting  $1/H = 0.$ (for half-space):

$$K_{\varphi_0}/\bar{K}_{\varphi_0} = 1 + 2E/R \quad \text{and} \quad K_{x_0}/\bar{K}_{x_0} = 1 + 2E/3R \quad (3-7)$$

while  $h/R$  is obtained from fig 2.8 for  $\nu= 1/3$  and  $H/R =4.$

Table 3.1: Static flexibility and stiffness coefficients derived from Ulrich and Kuhlemeyer's study.

$E/R$	$f_{11}$	$f_{22}$	$f_{21}$	$K_{x_0}/\bar{K}_{x_0}$	$K_{\varphi_0}/\bar{K}_{\varphi_0}$	$h/R$
0.0	0.91	0.92	0.08	1.11	1.09	-0.07
0.1	0.77	0.79	0.03	1.31	1.27	-0.03
0.25	0.65	0.64	-.01	1.53	1.55	0.02
0.50	0.55	0.47	-.06	1.80	2.12	0.11
1.00	0.45	0.28	-.11	2.05	3.33	0.33

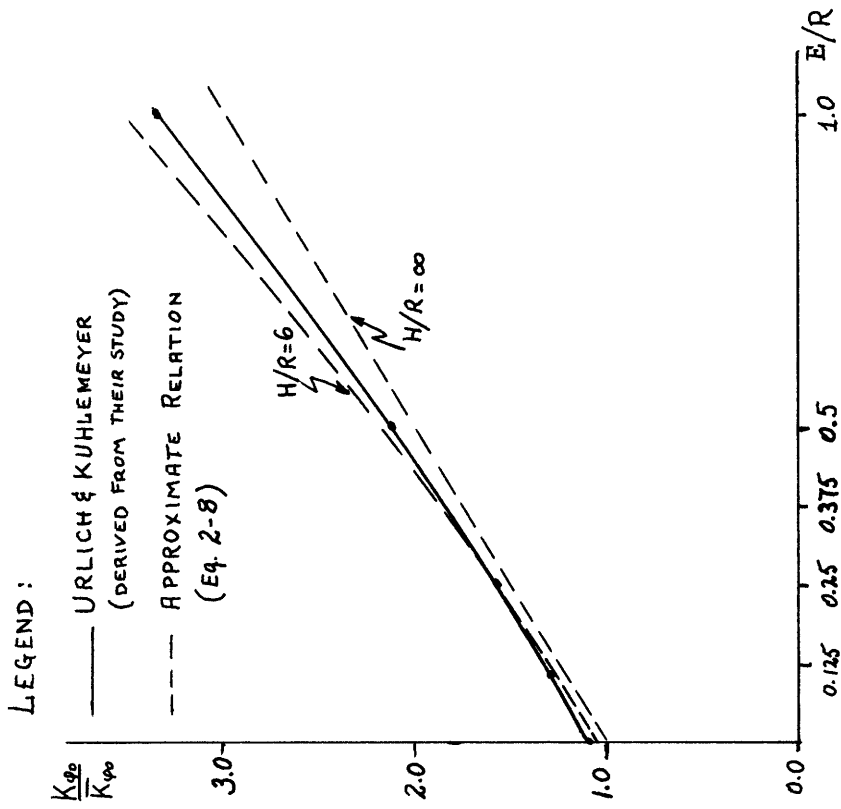


FIG. 3.1: COMPARISON-ROCKING STATIC STIFFNESS COEFFICIENTS OF A CIRCULAR FOOTING EMBEDDED IN AN ELASTIC HALF-SPACE.  $\nu = 1/3$

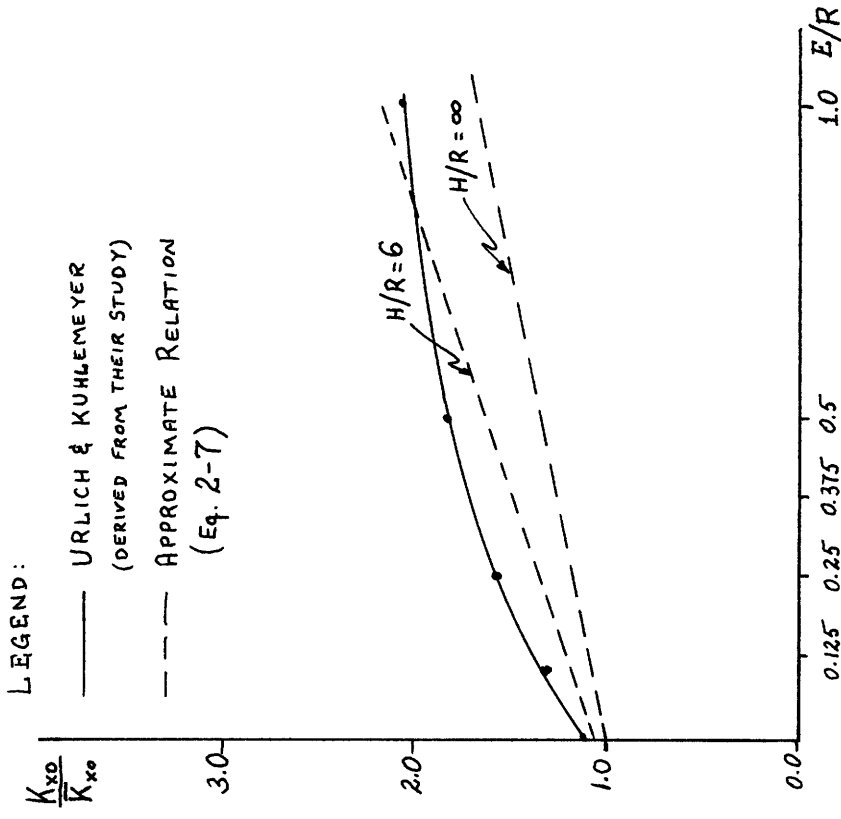


FIG. 3.2: COMPARISON-SWAYING STATIC STIFFNESS COEFFICIENTS OF A CIRCULAR FOOTING EMBEDDED IN AN ELASTIC HALF-SPACE.  $\nu = 1/3$

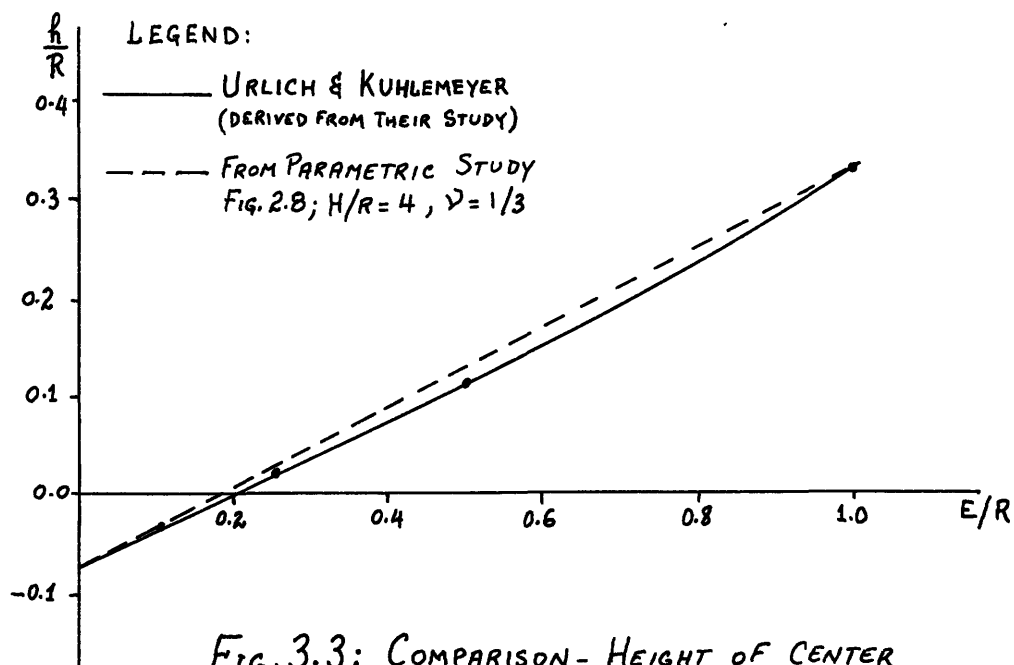


FIG. 3.3: COMPARISON - HEIGHT OF CENTER OF STIFFNESS;  $\nu = 1/3$

These are also plotted in figs 3.1, 3.2 and 3.3 for comparison along with the corresponding values for  $H/R = 6$ . Fig 2.8 is used, for  $h/R$ , rather than Eq. 2-11 due to its greater accuracy, while  $H/R = 4$ . is the closest available value to the half-space condition.

From figs 3.1 and 3.2 one sees that Kuhlemeyer's results compare well with the approximate relations (Eqs 2.8). It is apparent that their swaying and rocking solutions for surface circular footings on an elastic half-space do not perfectly

agree with rigorous analytical solutions, i.e.  $K_{\phi_0} = 8GR/3(1-\nu)$  and  $K_{\chi_0} = 8GR/(2-\nu)$ . These discrepancies do not occur in Eq. 3-7 due to the extrapolation procedure used for the correction of mesh size. Other sources of error in Kuhlemeyer's solution are the boundary conditions and the position of the boundaries relative to the footing used for the half-space representation. These boundary conditions, as mentioned earlier, model more closely an  $H/R = 6$ . stratum rather than a half-space. The swaying term seems to be more sensitive to such modeling errors than both the rocking and the off-diagonal coupling terms. On the other hand, if we apply the approximate relations for  $H/R = 6$ . to account for the fact that Kuhlemeyer's solution does not model the half-space condition, then the agreement improves significantly. The remaining differences are due to Kuhlemeyer's discretization error and his lateral boundary conditions. This shows that great care must be exercised for an accurate determination of the static stiffnesses.

### 3.2 Combined Embedment and Stratum Depths:

Johnson and Christiano (18) dealt with the static load-displacement behavior of circular and strip footings embedded in an elastic stratum which is underlaid by a rigid base.

They reported the results of static finite element analyses of all modes describing the behavior of embedded footings. Their finite element model consists of triangular elements, base boundaries which prevent vertical and horizontal displacements and vertical boundaries which prevent vertical displacements alone. The methods for formulating the governing stiffness equations follow the work of Wilson (45) for axisymmetric systems. They considered a rigid footing with full continuity between the footing and the soil, so that possibility of slip or cleavage, as in Kausel's model used for determining the approximate relations, was precluded. Linear isotropic soil was assumed throughout the stratum with a Poisson's ratio  $\nu = 1/4$ . Their results are presented as a set of plots showing the ratios of the static stiffness coefficients to the corresponding elastic half-space solutions for surface footings (Eq 3-1b).

They concluded that the influence of embedment on the rocking static stiffnesses is greater than for the swaying mode, an effect which can also be predicted from the approximate relations proposed in this study. They also concluded that the influence of the proximity of bedrock is greater for the swaying mode than for the rocking one. Again, this effect is also predicted by the approximate relations.

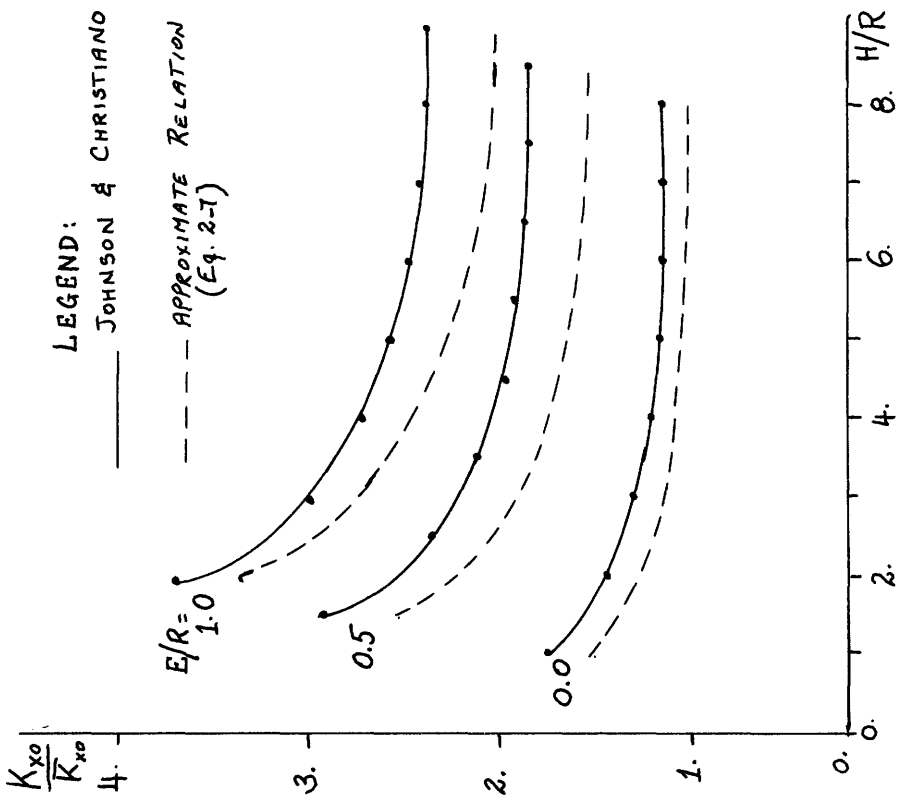


FIG. 3.5: COMPARISON - SWAYING STATIC STIFFNESS COEFFICIENTS OF A CIRCULAR FOOTING EMBEDDED IN AN ELASTIC STRATUM;  $\nu = 1/4$

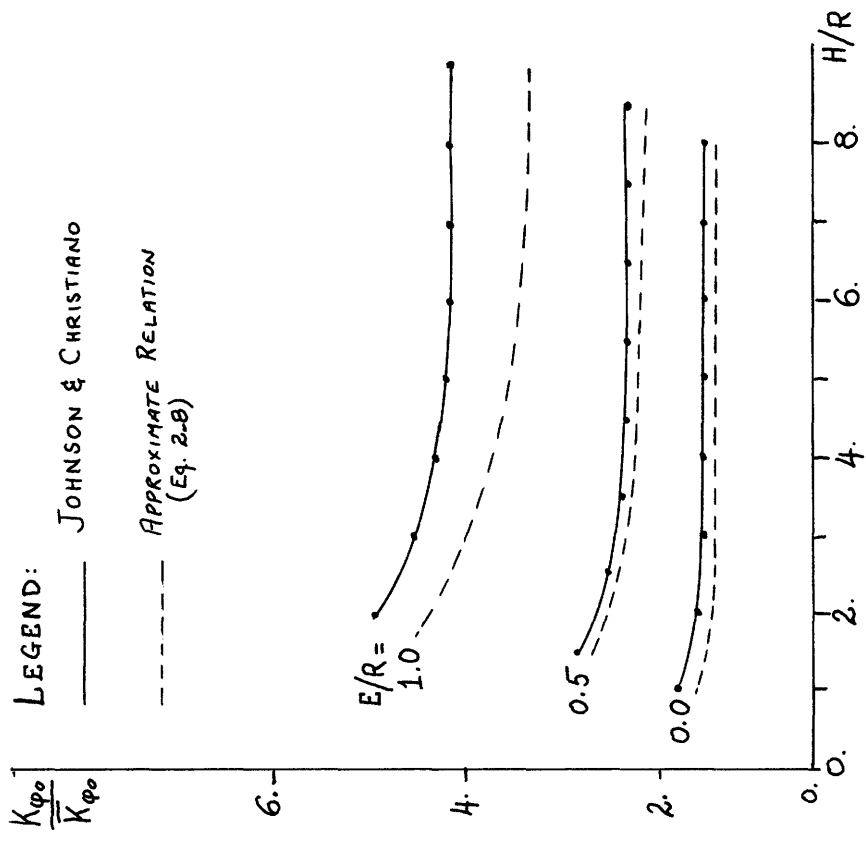


FIG. 3.4: COMPARISON - ROCKING STATIC STIFFNESS COEFFICIENTS OF A CIRCULAR FOOTING EMBEDDED IN AN ELASTIC STRATUM;  $\nu = 1/4$



Figs 3.4 and 3.5 present Johnson and Christiano's static stiffness coefficients for circular footings embedded in an elastic stratum for the rocking and the swaying modes respectively. The corresponding values obtained from the approximate relations (Eqs 2-8 and 2-7) are also included for comparison.

Johnson and Christiano's static stiffness coefficients compare well with the approximate relations. Their values are always larger than those from the approximate relations, and the difference between them appear to be approximately constant for a specific embedment ratio. In the case of the surface footing ( $E/R = 0.0$ ) their values do not tend to the half-space analytical solutions as  $H/R \rightarrow \infty$  while the approximate relations do. However, the general effects of the embedment and stratum depths, evident from their study, seem to agree well with those from the approximate relations. The sources of discrepancies are again attributable to the mesh size used by Johnson and Christiano and the location of the lateral boundary in their finite element mesh.

### 3.3 Concluding Remarks:

The emirical relations derived in this study provide a good approximation for the homogeneous stratum condition and are preferred to a static finite element solution when correction for mesh size is excluded.

However, if accurate values for the static stiffnesses in a layered medium are desired, then the approximate relations should be corrected for this condition, or finite element analyses accounting for mesh size correction be used instead. The correction in this latter alternative is necessary, because the discretization error can be of the same order of magnitude than the effect of layering (see tables 2.1 and 2.9).

#### 4.- Soil-Structure Interaction:

As explained in section 1.1, two general approaches can be used to estimate the dynamic soil-structure interaction effects: the direct (or complete) approach in which the whole system, soil and structure, is modeled and analyzed together; and the spring (or substructure) method which consists of the three steps discussed in section 1.3. Even though numerical methods, such as the finite element method used in the direct approach, are ideally suited for the analysis of complex problems, they result in considerable increases in the cost of computation. In many cases, this cost increase will restrain the analyst from covering a wide range of design parameters to ensure adequate protection against the uncertainties inherent to the nature of soils, thus making the second approach more attractive for many cases.

In performing the seismic analysis of a structure, the stiffness functions given by equations 4-1 and 4-2 are needed to represent the soil under the structure.

$$K_x = K_{x_0} (k_1 + ia_0 c_1) (1 + 2\beta_h i) \quad (4-1)$$

$$K_\psi = K_{\psi_0} (k_2 + ia_0 c_2) (1 + 2\beta_h i) \quad (4-2)$$

The static stiffnesses,  $K_{x_0}$  and  $K_{\psi_0}$ , can be approximated by the relations derived in section 2.2. However, the problem

of approximating the frequency dependent coefficients  $k_1, k_2, c_1,$  and  $c_2$  still remains. In this chapter two methods will be presented to approximate these coefficients, and a seismic analysis is performed for a structure using these approximations. The analysis is also performed using the stiffness functions obtained from TRIAX for the specific case to be studied.

For earthquake loadings, the controversial question as to where the control motion should be specified is often introduced. This question, however, should be resolved in the first step of the spring method. Since our main aim is to test the validity of approximate subgrade stiffness functions, the controversy can be avoided, arbitrarily specifying the horizontal control motion under the foundation as input to the springs. Also, for the same reason it suffices to model the structure as a rigid cylinder embedded in an elastic stratum which is underlain by a rigid base. However, the mass properties and geometry used for the cylinder, and the soil properties of the stratum are typical values for nuclear containment structures. The assumed structural properties are:

$$\text{Radius (R)} = 67.5 \text{ ft.}$$

$$\text{Embedment (E)} = 55.0 \text{ ft.}$$

Height of centroid = 64.7 ft.

Total Mass (M) = 3662. Kslugs

Moment of inertia about base (I) =  $2.98 \times 10^7$  kip-sec<sup>2</sup>-ft

while the stratum (115 ft. deep) consists of layered linear isotropic soil with the properties indicated in table 4.1, and 8% linear hysteretic (material) damping throughout the soil stratum.

Table 4.1: Soils properties of layered stratum used in model.

Note: layer 1 is just below grade level

layer	layer thickness (ft)	weight density (Kcf)	Poisson's ratio	shear wave velocity (ft/sec)
1	5	0.12	0.3	428.
2	5	0.12	0.3	512.
3	5	0.12	0.3	590.
4	5	0.12	0.3	645.
5	5	0.12	0.3	691.
6	7.5	0.12	0.3	738.
7	7.5	0.12	0.3	777.
8	7.5	0.12	0.3	808.
9	7.5	0.12	0.3	848.
10	7.5	0.12	0.3	877.
11	7.5	0.12	0.3	894.
12	10.0	0.13	0.48	880.
13	10.0	0.13	0.48	903.
14	12.5	0.13	0.48	926.
15	12.5	0.13	0.48	934.

In the determination of the approximate stiffness functions, the stratum will be assumed to be homogeneous throughout

the domain with properties equal to those of the layer just below the footing:

$$C_s = 877. \text{ ft/sec}$$

$$G = \rho C_s^2 = 2866. \text{ K/ft}$$

$$\nu = 0.3$$

Using equations 2-7 and 2-8, the approximate static stiffnesses are:

$$K_{x_0} = 2.90 \times 10^6 \text{ K/ft}$$

$$K_{\phi_0} = 1.30 \times 10^{10} \text{ K-ft/rad}$$

and the height of center of stiffness (eq. 2-11) is:

$$h = 19.98 \text{ ft or } K_{\phi x_0} = 57.94 \times 10^6 \text{ K/rad}$$

The finite element analysis using TRIAX with the soil properties in table 4.1 resulted in the following static stiffness values:

$$K_{x_0} = 2.8 \times 10^6 \text{ K/ft}$$

$$K_{\phi_0} = 1.2 \times 10^{10} \text{ K-ft/rad}$$

$$K_{\phi x_0} = 37.3 \times 10^6 \text{ K/rad}$$

The swaying and rocking static stiffnesses agree very well with the approximate values. The coupling term  $K_{\phi x_0}$  has an error of 55%; however as will be seen from the results, this discrepancy will not introduce great errors, because the response is not very sensitive to the coupling term if the center of mass of the structure is well above the center of stiffness. It should be noted that the extrapolation procedure is not used in obtaining the finite element static

stiffnesses and also the effect of soil layering is not included in the approximate values. As was discussed in section 2.5 ("Effect of variable shear wave velocity"), the approximate static stiffnesses will be decreased when the effect of soil layering (layer properties in table 4.1) is considered. In addition, the finite element static swaying and rocking stiffnesses should be reduced by applying the extrapolation procedure (see section 2.2). Since the two effects could decrease the stiffnesses by a comparable amount, these corrections are neglected in this study.

Due to the excellent agreement of the finite element static stiffnesses with the approximate ones, the remainder of the chapter will mainly deal with an evaluation of the importance of the frequency dependence of the stiffness and damping coefficients  $k_1$ ,  $k_2$ ,  $c_1$  and  $c_2$ , on the structural response.

The following resonant swaying and rocking frequencies will be referred to in the sections to follow:

$$\omega_x = \sqrt{\frac{K_{x0}}{M}} ; \quad a_{0x} = \frac{\omega_x R}{C_s} ; \quad f_{0x} = \frac{a_{0x}}{2\pi} \quad (4-3)$$

$$\omega_\varphi = \sqrt{\frac{K_{\varphi 0}}{I}} ; \quad a_{0\varphi} = \frac{\omega_\varphi R}{C_s} ; \quad f_{0\varphi} = \frac{a_{0\varphi}}{2\pi} \quad (4-4)$$

#### 4.1 Approximation of the Frequency Dependent Coefficients:

In the following, a method to account in an approximate manner for the frequency dependence of the coefficients will be analyzed. The discussion will be based on results presented in ref.(21), and the suggested procedures will be evaluated for the rigid cylinder model of a nuclear containment structure described earlier.

##### 4.1.1 Stiffness Coefficients $k_1$ and $k_2$ :

The first interesting feature in the frequency dependence of the stiffness functions is that the real part behaves markedly different for rocking and swaying (see fig. 4.1). The former is a relatively smooth curve, for which the depth of the layer exercises little influence and approaches quite closely the analytical half-space solution for a surface footing after Veletsos and Wei (40). Swaying, however, shows a wavy pattern in which valleys occur close to the natural shear beam frequencies of the soil, an effect which is more pronounced for shallow strata. However, as the stratum depth increases, the amplitudes of the peaks and valleys decrease rapidly, and for  $H/R = 8$   $k_1$  already approaches Veletsos and Wei's half-space solution.



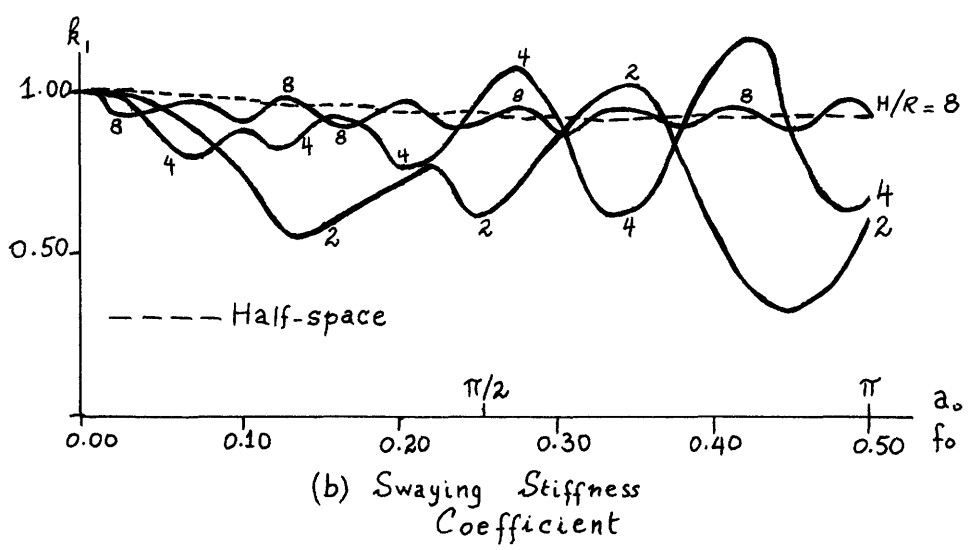
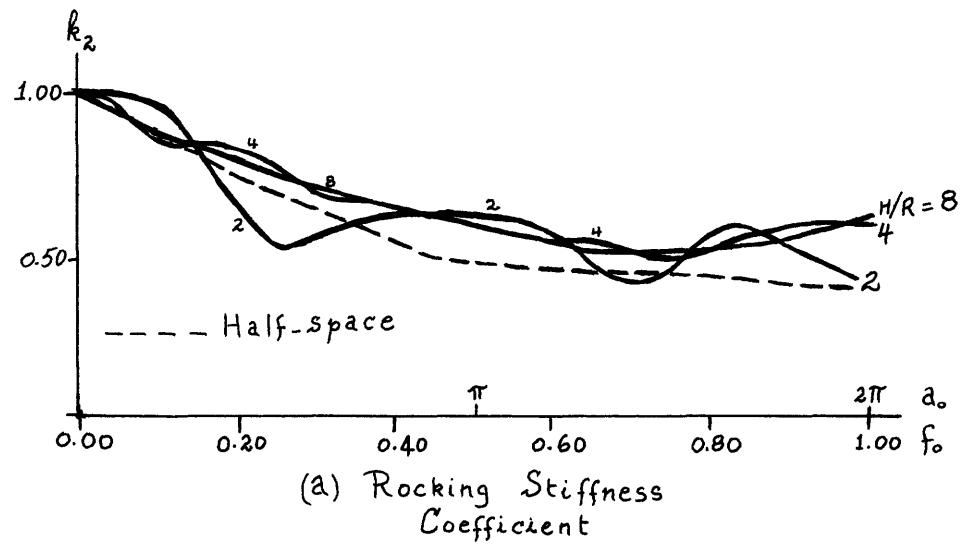


FIG. 4.1: EFFECT OF STRATUM DEPTH ON STIFFNESS COEFFICIENTS FOR SURFACE FOOTINGS.  
 $\nu = 1/3$  ;  $\beta_h = 5\%$ .  
 (OBTAINED FROM REF. 21)

The preceding comparison is done for a hysteretic damping value of 5%. For more typical values of the hysteretic damping ratio ( $5% < \beta_h < 15%$ ) the agreement of the stiffness coefficients with those of the half-space improve even further (ref. 21). A similar effect can be observed by varying Poisson's ratio. On the other hand, embedment does not substantially alter the shape of the stiffness coefficients as was pointed out in said reference.

To better assess the relative importance of the frequency variation of the stiffness coefficients, consider the response of a 1-DOF system supported with a frequency dependent "spring". Its impedance function can be defined as:

$$Z = K_o ( k + 2i \frac{\Omega}{\omega} \beta - (\frac{\Omega}{\omega})^2 )$$

where  $\beta = \beta(\Omega)$  is the equivalent viscous damping given by:

$$2 \frac{\Omega}{\omega} \beta = a_o c, \text{ and } \omega \text{ is some reference frequency. For}$$

the case at hand, this frequency corresponds to the undamped natural resonant frequency of the system computed with the static value of the subgrade stiffness (Eq. 4-3 or 4-4).

In the very low frequency range, ( $\Omega \ll \omega$ ), the stiffness term  $k$  dominates the response. At intermediate frequencies, ( $\Omega/\omega \approx 1$ ), the amplitude of motion is very sensitive to the equivalent viscous damping coefficient  $\beta$ , while in the high frequency range ( $\Omega \gg \omega$ ) the response is totally

controlled by the inertia of the system, and thus the shape of the stiffness function is irrelevant. Therefore, the stiffness coefficients are of major concern only in the low frequency range.

If one limits the depth of the stratum to be greater than the diameter of the foundation ( $H/R > 2$ ), the half-space solution for  $k_1$  and  $k_2$  can be adapted (with reasonable accuracy) to the actual finite depth stratum as a good agreement in the low frequency range, where these coefficients influence the response the most, exists between the two conditions. The relatively moderate differences between the half-space stiffness coefficients and the actual ones will result in slight shifts in the resonant frequencies of the impedance (and thus the transfer) functions. However, since these shifts depend on the square root of the error, they will be small in magnitude.

A more important point in this adaptation is a proper representation of the damping coefficients, as they control the amplitude of the response at resonance. This point is addressed in detail in the following section.

#### 4.1.2 Radiation Damping Coefficients $c_1$ and $c_2$ :

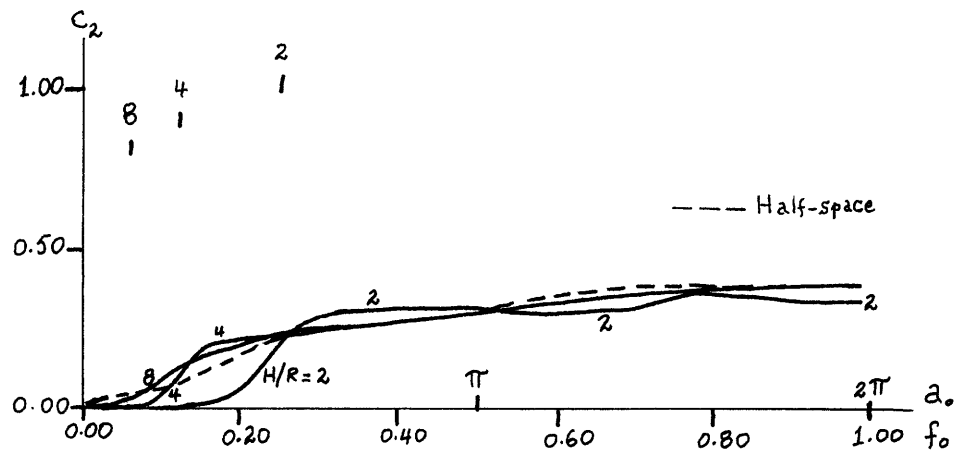
The imaginary part of the stiffness functions, which can be interpreted as an equivalent viscous damping ratio due to radiation of energy away from the footing, is a critical parameter to which the response of the structure is very sensitive.

Referring to fig. 4.2 it can be seen that there are negligible differences of the radiation damping coefficient with respect to the half-space solution for rocking, and except for the wavy nature in shallow strata, the agreement is also remarkably close for swaying. There exists one important difference, however, between the strata and the half-space solutions; namely the former has little radiation damping in the low frequency range, and then rises abruptly at certain critical frequencies. These critical frequencies correspond to the first horizontal natural frequency of the stratum, for swaying, and the first vertical natural frequency of the stratum, for rocking. In the following, these frequencies will be abbreviated by HNFS and VNFS respectively. For a homogeneous stratum, they are given by:

$$\text{HNFS} = \frac{C_s}{4H} \quad (4-5)$$

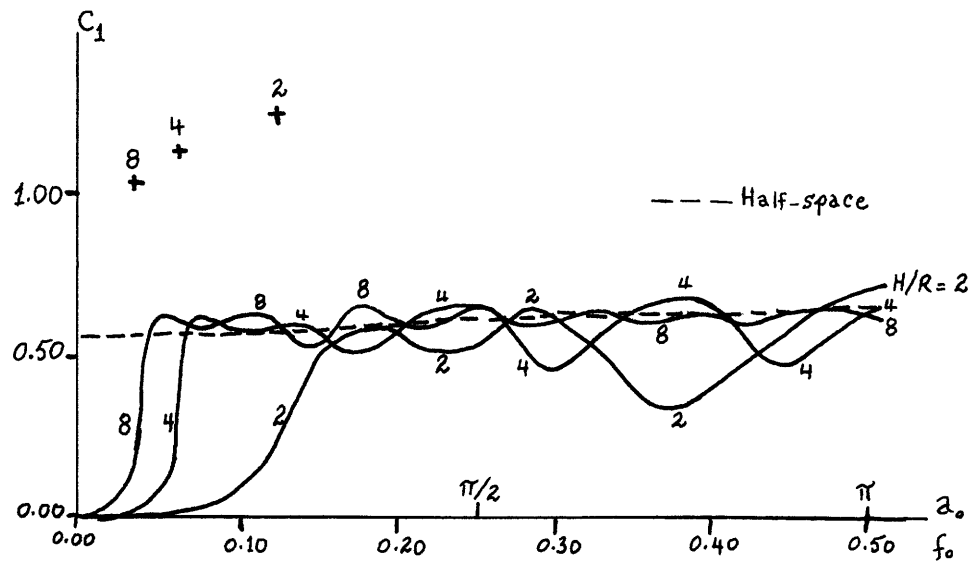
or for dimensionless frequency:  $\frac{R}{4H}$

$$\text{VNFS} = \sqrt{\frac{2(1-\nu)}{1-2\nu}} \text{HNFS} \quad (4-6)$$



(a) Rocking Damping Coefficient

1: indicates VNFS



(b) Swaying Damping Coefficient

+ indicates HNFS

FIG. 4.2: EFFECT OF STRATUM DEPTH ON RADIATION DAMPING COEFFICIENTS FOR SURFACE FOOTINGS.

$$\nu = 1/3; \beta_h = 5\%$$

(OBTAINED FROM REF. 21)

Thus below the stratum natural frequencies, the radiation damping is very low, but it is not zero if there is material damping in the soil. Above the stratum frequencies, the radiation damping is essentially similar to Veletsos and Wei's half-space solution. At first, one would attempt to approximate these coefficients by using zero damping below the stratum natural frequencies (which can be obtained for the actual layered medium) and the half-space solution above them. However, it is quite possible, and it is the case for the rigid cylinder considered here, that the fundamental rocking-swaying frequency of the structure falls below one of the stratum natural frequencies. In such cases the response obtained using the above mentioned approximation for the radiation damping coefficients, would be over-conservative, as one would be using zero radiation damping at the resonant frequency. Thus it is essential to find a better approximation of the radiation damping below the HNFS for swaying and VNFS for rocking.

Kausel (21) presented the following alternate definition of the stiffness functions (Eqs. 4-1 and 4-2), when hysteretic damping is included in the analysis:

$$K_x = K_{x_0} (\bar{k}_1 + i(a_0 \bar{c}_1 + 2\beta_h)) \quad (4-7)$$

and 
$$K_\varphi = K_{\varphi_0} (\bar{k}_2 + i(a_0 \bar{c}_2 + 2\beta_h)) \quad (4-8)$$

Examination of figs. 32 through 35 in ref. (21) reveals that  $\bar{c}_1$  and  $\bar{c}_2$  are essentially zero below the HNFS and VNFS respectively for all hysteretic damping ratios considered (1%, 10%, 20%). This observation may be used to evaluate  $c_1$  and  $c_2$  by the following steps. Since:

$$(k_1 + ia_0 c_1)(1 + 2i\beta_h) = (\bar{k}_1 + i(a_0 \bar{c}_1 + 2\beta_h))$$

$$(k_2 + ia_0 c_2)(1 + 2i\beta_h) = (\bar{k}_2 + i(a_0 \bar{c}_2 + 2\beta_h))$$

it follows that

$$c_1 = \bar{c}_1 + \frac{2\beta_h}{a_0} (1 - k_1)$$

$$c_2 = \bar{c}_2 + \frac{2\beta_h}{a_0} (1 - k_2)$$

or if  $\bar{c}_1$  and  $\bar{c}_2$  are zero:

$$c_1 = \frac{2\beta_h}{a_0} (1 - k_1) \quad \text{if } \frac{\omega}{2\pi} \leq \text{HNFS} \quad (4-9)$$

$$c_2 = \frac{2\beta_h}{a_0} (1 - k_2) \quad \text{if } \frac{\omega}{2\pi} \leq \text{VNFS} \quad (4-10)$$

However, above the fundamental stratum frequencies, the half-space coefficients are a better approximation to the actual coefficients than to the coefficients evaluated using Kausel's  $\bar{c}_1$  and  $\bar{c}_2$ . Thus,  $\bar{c}_1$  and  $\bar{c}_2$  are only used to approximate  $c_1$  and  $c_2$  below the stratum natural frequencies.

In order to evaluate Eqs. 4-9 and 4-10, approximate expressions for  $k_1$ ,  $k_2$  are needed. Limiting  $H/R > 2$  and  $\nu \leq 0.45$ , it is possible to satisfactorily approximate the stiffness coefficients  $k_1$ ,  $k_2$  in the low frequency range by parabolas.

For strata shallower than one plate diameter and/or having a higher Poisson's ratio, the VNFS will be large, and the effect of rock flexibility and radiation into the half-space will be important. Therefore, rock flexibility should be accounted for in such a case (this also applies to the "true" finite element solution).

While the parabolas needed are closely related to the concepts of "effective mass of soil", they will be approximated on the basis of the results presented by Kausel, without consideration of inertia effects in the subgrade. The following sections discuss approximate expressions for  $k_1$  and  $k_2$  to be used in the evaluation of (4-9) and (4-10) below the stratum frequencies.

a) Swaying, c :

Due to the limitation  $H/R \geq 2$  the frequency range of interest, below HNFS, becomes  $f_o < 0.125$ . Fig. 4.3 shows the stiffness coefficient  $k_1$  for different stratum depths in this frequency range; the HNFS for each case is also shown. It is obvious that we do not intend to match the half-space curve, since HNFS is zero for this condition and Veletsos and Wei's damping coefficients can be used for the approximation in the entire frequency domain. From fig. 4.3 we see that the



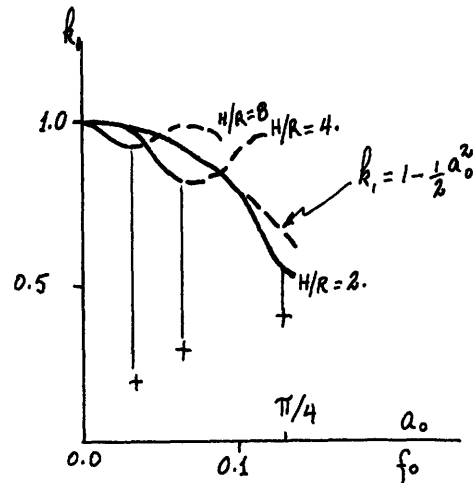


FIG. 4.3: SWAYING STIFFNESS COEFFICIENT FOR SURFACE FOOTINGS.

$$\nu = 1/3 ; \beta_h = 5\%$$

NOTE: + INDICATES HNFS

THIS FIG. IS OBTAINED FROM REF. (21).

curve for  $H/R = 2$  can approximate the others in the frequencies below the HNFS for each condition. This approximation should improve substantially for larger fractions of material damping ( $\beta_h > 0.05$ ). On the other hand, the HNFS is independent of Poisson's ratio, while the portion of  $k_1$  plotted in fig. 4.3 is also essentially independent of Poisson's ratio (ref. 21).

For the rigid cylinder case considered in this study,  $H/R$  is close to 2. Approximating  $k_1$  by a parabola to best fit this stratum depth (for low frequencies) in fig. 4.3, one obtains:

$$k = 1 - 1/2 a_0^2 \quad (4-11)$$

This approximation can also be obtained from a consideration of an effective mass, to be added to the structure, equivalent to a hemisphere of soil below the footing.

Further studies for the usage of Eq. 4-11, or an equivalent expression, for deeper strata are needed. However, an effect to keep in mind is that as the stratum depth increases, the HNFS will decrease. For such cases, the fundamental frequency of most structures will be greater than the HNFS, and the half-space solution will provide a good approximation to the damping coefficients.

Upon substitution of (4-11) into (4-9) one obtains for the swaying damping coefficient:

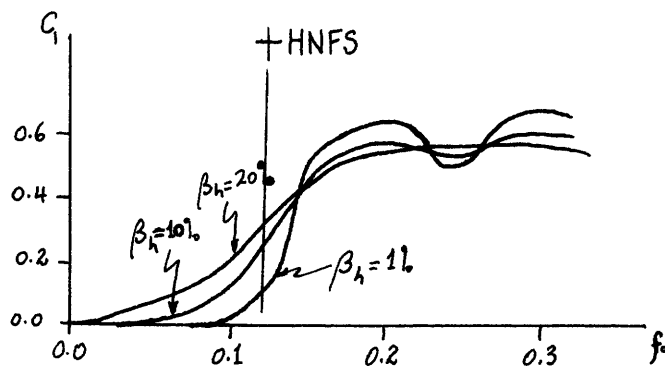


FIG. 4.4 : SWAYING DAMPING COEFFICIENT  
FOR SURFACE FOOTING  
 $H/R = 2.$  ;  $\nu = 1/3$   
(OBTAINED FROM REF. 21)

$$c_1 = 2\pi\beta_h f_o \quad \text{for } f_o \leq \text{HNFS} \quad (4-12)$$

$$= \beta_h a_o$$

This expression for  $c_1$  approximates, with reasonable accuracy, the swaying damping coefficients, in the low frequency range, for various hysteretic dampings presented in ref. (21) and fig. 4.4.

b) Rocking,  $c_2$ :

The rocking damping coefficient presents a more delicate situation, because as we can see from Eq. 4-6 the VNFS is a function of both Poisson's ratio and  $H/R$ . Considering the limits  $H/R > 2$  and  $\nu < 0.45$ , the frequency range of interest

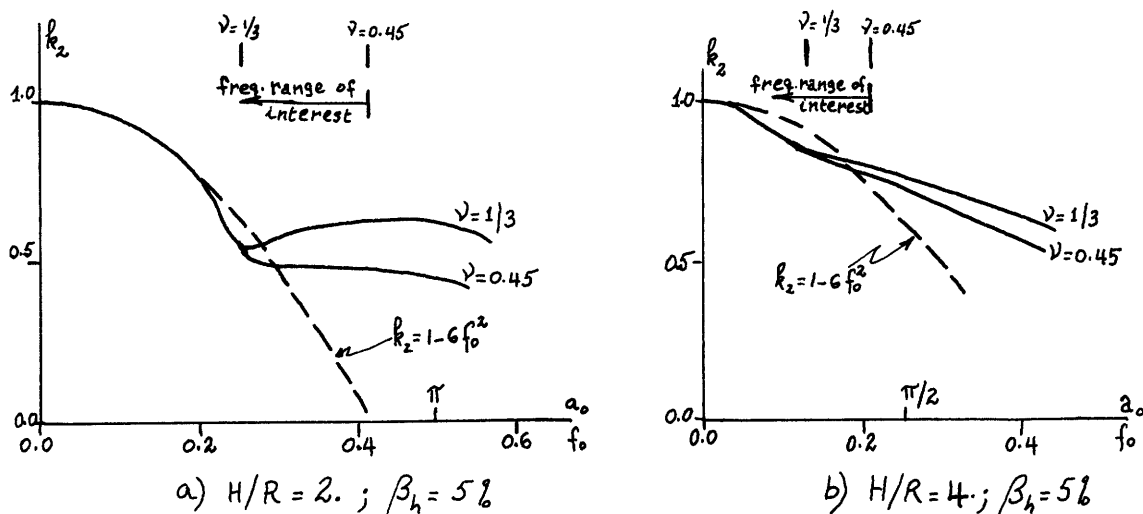


FIG. 4.5: ROCKING STIFFNESS COEFFICIENTS FOR SURFACE FOOTINGS.

| : INDICATES VNFS.  
OBTAINED FROM REF. 21

becomes  $f_0 < 0.41$ . Fig. 4.5a shows the rocking stiffness coefficient,  $k_2$ , for  $H/R = 2$ . One notices that  $k_2$  is almost independent of Poisson's ratio for  $f_0$  less than VNFS for  $\nu = 1/3$ , while in the region between VNFS for  $\nu = 1/3$  and VNFS for  $\nu = 0.45$ ,  $k_2$  is more sensitive to this parameter. As the stratum becomes deeper the VNFS will become smaller and the rocking stiffness coefficient will not depend a great deal on Poisson's ratio in the whole region of interest (see fig. 4.5b for  $H/R = 4$  where the range of interest becomes  $f_0 < 0.21$ ).

Thus approximating the  $k_2$  curve for  $H/R = 2$ , shallow stratum, by a parabola to best fit the low frequency range of the high Poisson's ratio curve, one obtains:

$$k_2 = 1 - 6f_0^2 \quad (4-13)$$

This relation is also plotted in fig. 4.5, and one can see that it approximates the actual curves reasonably well in the regions of interest. Again, this approximation for  $k_2$  can also be obtained from a consideration of an effective soil mass, to be added to the structure, equivalent to  $0.4\rho R^5$  of soil below the footing. This value of the equivalent mass was also presented by Whitman (44). Substituting (4-13) into (4-10), the resulting approximation of the rocking damping coefficient, below the VNFS, is:

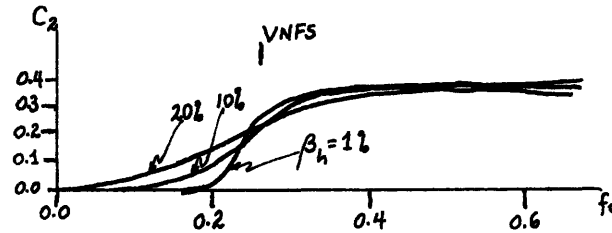


FIG. 4.6: ROCKING DAMPING COEFFICIENT  
FOR SURFACE FOOTING;  
 $H/R=2. ; \nu=1/3$

$$c_2 = 1.9 \beta_h f_o \quad \text{for } f_o \leq \text{VNFS} \quad (4-14)$$

$$= 0.3 \beta_h a_o$$

This relation for  $c_2$  approximates reasonably well the rocking damping coefficients below the VNFS over a range of hysteretic damping values (see fig. 4.6). For very shallow strata, and particularly if Poisson's ratio is large, the rock flexibility and the radiation damping into the half-space should be taken into account both in the approximate and the finite element analyses.

#### 4.1.3 Summary of the Procedure:

In the above a method is presented to approximate the frequency dependent coefficients of the stiffness functions.

The dynamic stiffness functions given by:

$$K_x = K_{x_o} (k_1 + i a_o c_1) (1 + 2i \beta_h)$$

$$K_\varphi = K_{\varphi_o} (k_2 + i a_o c_2) (1 + 2i \beta_h)$$

will be approximated in the following manner:

$$K_{x_0} = \frac{8GR}{2-\nu} \left(1 + \frac{1}{2} \frac{R}{H}\right) \left(1 + \frac{2}{3} \frac{E}{R}\right) \left(1 + \frac{5}{4} \frac{E}{H}\right) \quad \text{for } \frac{E}{H} \leq \frac{1}{2} \quad (4-15)$$

$$K_{\psi_0} = \frac{8GR^3}{3(1-\nu)} \left(1 + \frac{1}{6} \frac{R}{H}\right) \left(1 + 2 \frac{E}{R}\right) \left(1 + 0.71 \frac{E}{H}\right) \quad \text{for } \frac{E}{H} \leq \frac{1}{2} \quad (4-16)$$

$k_1$  and  $k_2$  are obtained from Veletsos and Wei's (40) half-space coefficients. And

$$C_1 = \left\{ \begin{array}{ll} 2\pi \beta_h f_0 = \beta_h a_0 & \text{for } f_0 \leq \text{HNFS} \\ \text{Veletsos \& Wei's coeffs.} & \text{for } f_0 > \text{HNFS} \end{array} \right\}$$

$$C_2 = \left\{ \begin{array}{ll} 1.9 \beta_h f_0 = 0.3 \beta_h a_0 & \text{for } f_0 \leq \text{VNFS} \\ \text{Veletsos \& Wei's coeffs.} & \text{for } f_0 > \text{VNFS} \end{array} \right\}$$

This method will be referred to, from here on, as the "modified half-space" method. The procedure is certainly not rigorous, particularly so for the damping coefficients in the low frequency range. However it constitutes a good engineering approximation to simplify the analysis, as will be demonstrated later with the results of the rigid cylinder case referred to earlier. Figs 4.7 and 4.8 show a comparison of the stiffness and damping coefficients obtained from the modified half-space method to those from a finite element analysis.

In what follows a second method is presented where the coefficients are considered constant over the frequency domain.

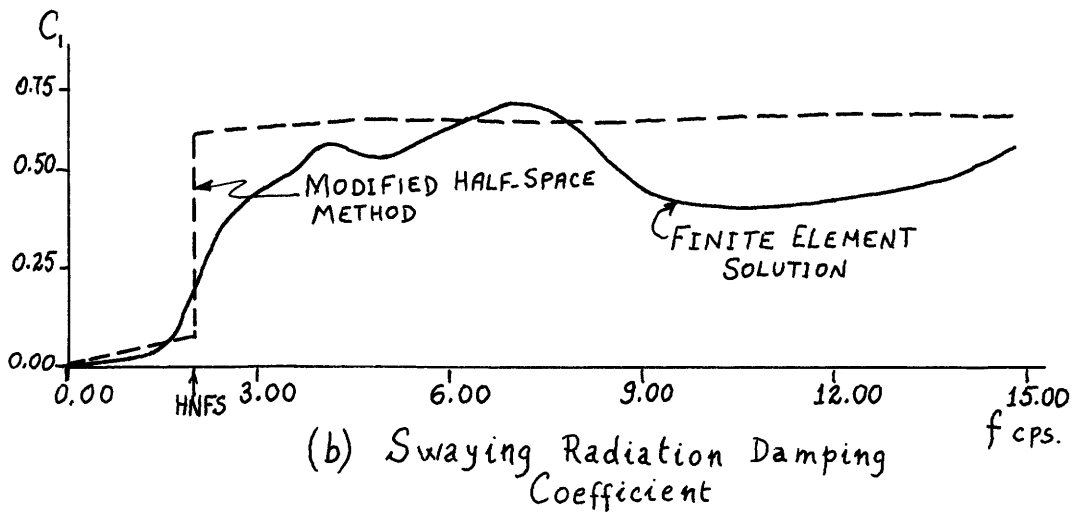
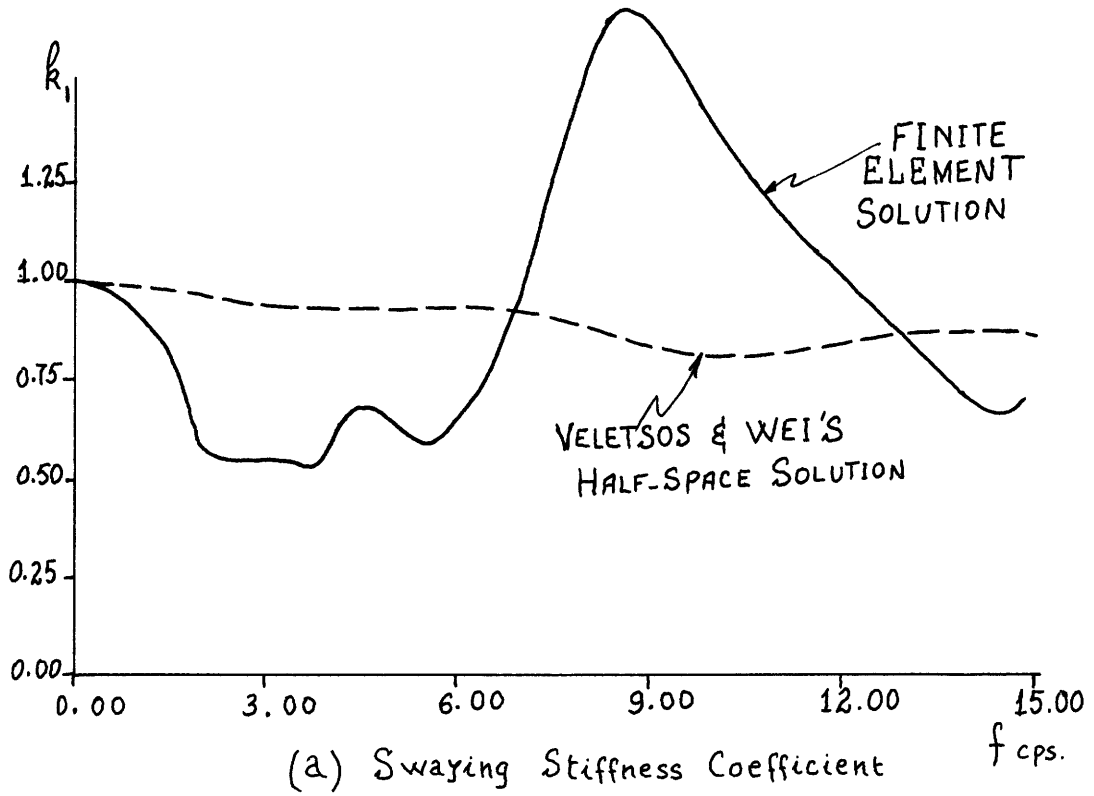


FIG. 4.7: SWAYING COEFFICIENTS FOR RIGID CYLINDER PROBLEM.  
 $\beta_h = 8\%$

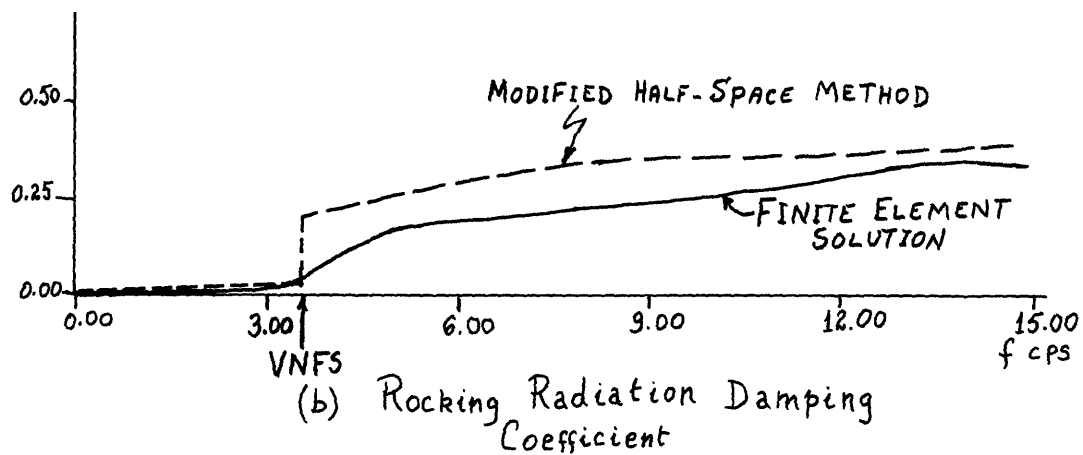
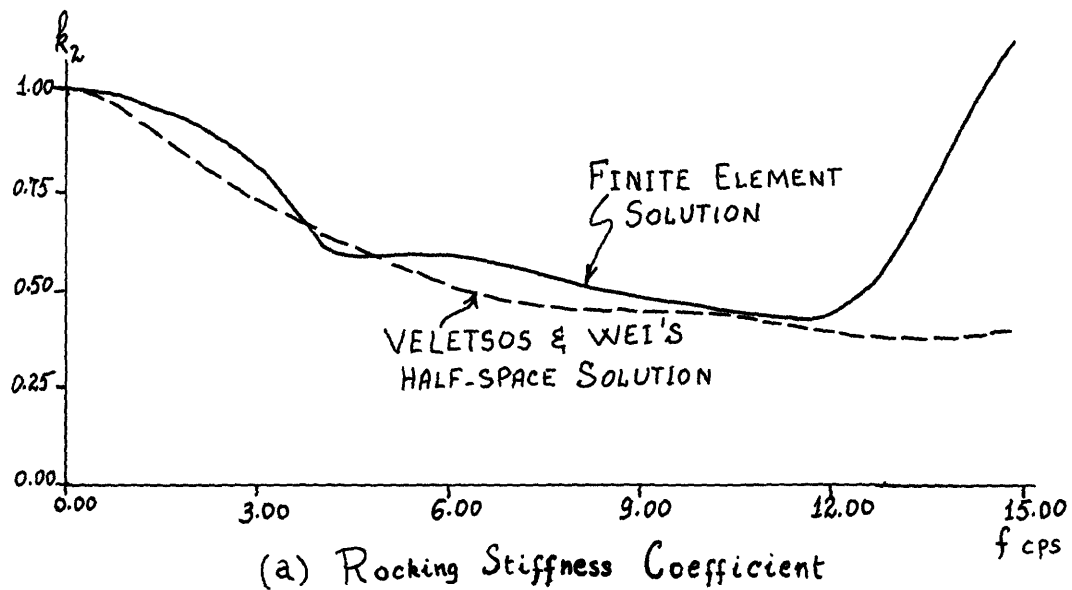


FIG. 4.8: ROCKING COEFFICIENTS FOR RIGID CYLINDER PROBLEM.  
 $\beta_h = 8\%$



#### 4.2 Approximate Constant Stiffnesses:

In many cases, it is desirable to choose frequency independent stiffnesses to represent the soil in the seismic analysis. If one can obtain such stiffnesses, a preliminary analysis could be performed in the conventional time domain. Due to the low costs of such a method, the effect of many variables could be tested for the structure on hand. The procedure described in this section shall be referred to as the modified constant stiffness method.

As in the modified half-space method, the static value of the stiffness function is evaluated first, using the approximate relations (Eqs. 4-15 and 4-16). Then, the stiffness coefficients,  $k_1$  and  $k_2$ , in Eqs. 4-1 and 4-2 could be either approximated to unity, or evaluated with the half-space curves at the resonant swaying and rocking frequencies respectively and taken independent of the frequency  $\Omega$  (or  $f$ ). On the other hand, the equivalent viscous damping ratios  $\beta_x, \beta_\varphi$  may follow from an evaluation of the radiation damping coefficients  $c_1, c_2$  at the resonant swaying and rocking frequencies respectively. The relationship between the damping ratios and the damping coefficients is:

$$c_1 = \frac{2\beta_x}{\omega_x} \frac{c_s}{R} \quad \text{or} \quad \beta_x = \frac{\omega_x R}{2c_s} c_1$$

$$c_2 = \frac{2\beta_\varphi}{\omega_\varphi} \frac{c_s}{R} \quad \text{or} \quad \beta_\varphi = \frac{\omega_\varphi R}{2c_s} c_2$$

where the reference frequencies are:

$$\omega_x = \sqrt{\frac{K_{x0}}{M}} \quad \text{and} \quad \omega_\varphi = \sqrt{\frac{K_{\varphi 0}}{I}}$$

In general  $\beta_x$  and  $\beta_\varphi$  are functions of the driving frequency  $\Omega$ . To estimate these damping ratios, the following procedure based on the modified half-space method can be used:

- If the resonant frequency is below the stratum frequency, then the damping ratio can be evaluated using the approximations derived in section 4.1.2 for  $c_1$  and  $c_2$ , and evaluated at the resonant frequency  $\omega_x$  or  $\omega_\varphi$ . The result is:

$$\begin{aligned} \beta_x &= (1/2) \beta_h a_{ox}^2 && \text{if } f_x < \text{HNFS} \\ &= 20 \beta_h f_{ox}^2 && \text{where } \beta_h \text{ is the hysteretic} \\ &&& \text{damping ratio} \end{aligned}$$

$$\begin{aligned} \text{also, } \beta_\varphi &= 0.15 \beta_h a_{o\varphi}^2 && \text{if } f_\varphi < \text{VNFS} \\ &= 6 \beta_h f_{o\varphi}^2 \end{aligned}$$

- If the resonant frequency is above the stratum frequency, then the damping ratio is evaluated using the half-space values for  $c_1$  and  $c_2$ , at the resonant frequency.

Frequently, the half-space solution is not available, or it is desired to get a closed form expression without using tabulated half-space values. In such cases, the

following expressions suggested by Richart et al (35) give excellent approximations:

$$\beta_x = 0.288 a_{ox}^2 \quad \text{if } f_x \geq \text{HNFS}$$

$$\beta_\varphi = \frac{0.15 a_{o\varphi}}{1 + \left(\frac{1}{a_{o\varphi}}\right)^2} \quad \text{if } f_\varphi \geq \text{VNFS}$$

The damping coefficients  $c_1$  and  $c_2$  corresponding to these approximations are:

$$c_1 = 0.576$$

$$c_2 = \frac{0.3}{1 + \left(\frac{1}{a_{o\varphi}}\right)^2}$$

If these expressions are plotted as a function of frequency, one will notice an excellent agreement with Veletsos and Wei's half-space solution.

#### 4.3 Solution of the Rigid Cylinder Case:

The rigid cylinder, described previously, is now analyzed using the "true" finite element solution, the modified half-space method and the modified constant stiffness method. In the latter method, the coefficients  $k_1$ ,  $k_2$  were equated to unity, while the equivalent viscous damping ratios above the stratum frequencies were evaluated using Richart's approximations.

A preliminary estimation of how close the approximate methods will come to the finite element solution, can be obtained plotting the absolute value of the impedance function as a function of frequency. The impedance is given by:  $(k + ia_0c) - (\frac{n}{\omega})^2$ , which is the reciprocal of the transfer function for a pure (uncoupled) mode. The functions, using the three methods described, are plotted in fig. 4.9 for the rocking mode of the rigid cylinder. This mode is chosen, since it is the critical one for the case under consideration. From the given properties one calculates the following frequencies:

$$\text{VNFS} = 3.57 \text{ cps}$$

$$\text{HNFS} = 1.91 \text{ cps}$$

using the modified constant stiffness functions, the uncoupled natural frequencies of the system are:

$$\text{Rocking: } 3.25 \text{ cps}$$

$$\text{Swaying: } 4.48 \text{ cps}$$

and the coupled rocking-swaying frequencies are:

$$\text{First mode: } 2.97 \text{ cps}$$

$$\text{Second mode: } 5.7 \text{ cps}$$

Thus we see that the fundamental natural frequency is below the VNFS. Fig 4.9 shows a good agreement between the three methods, except for a small shift of the resonant frequency according to the constant stiffness method as compared to the other methods. This shift could be eliminated to a

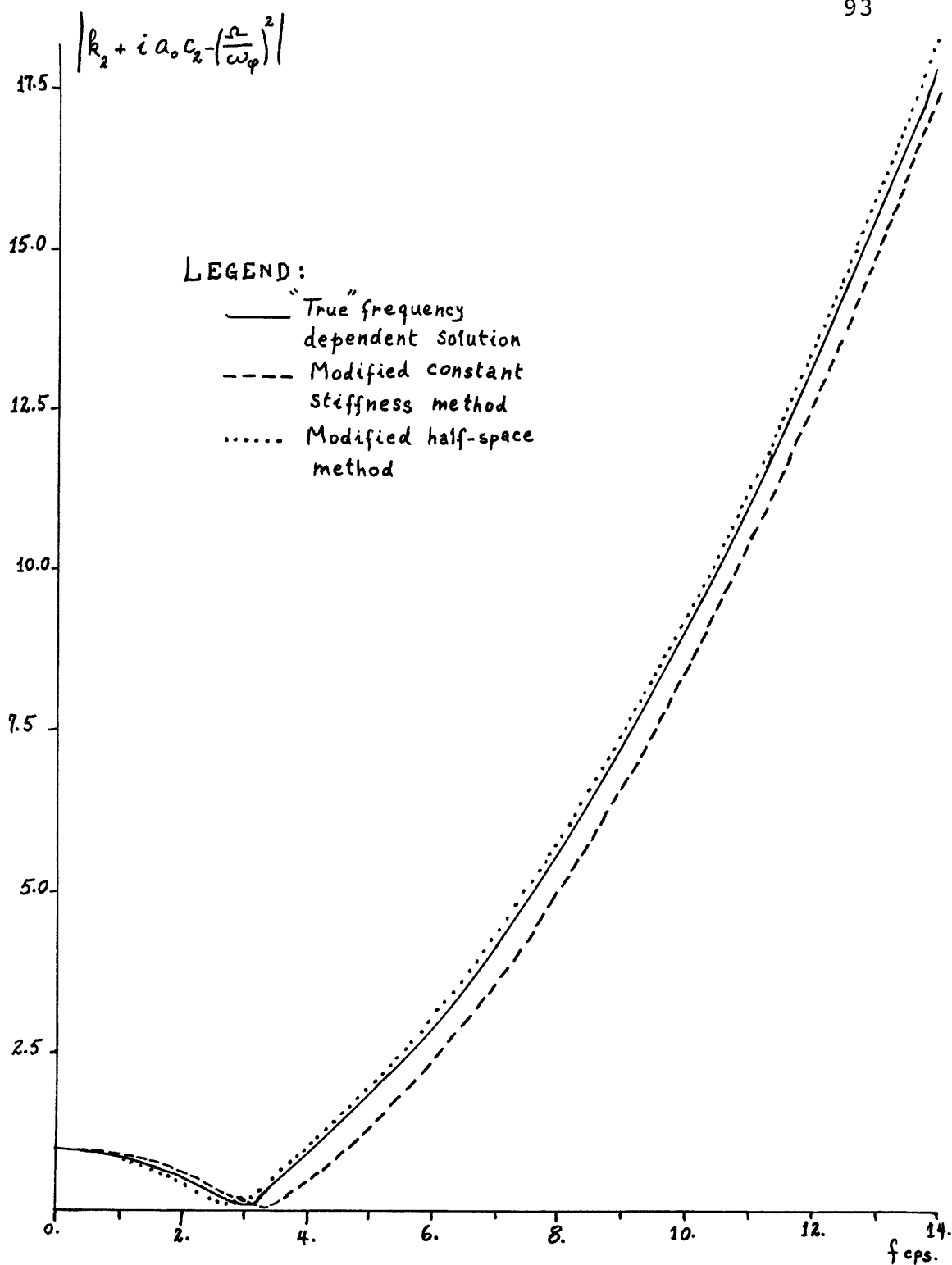
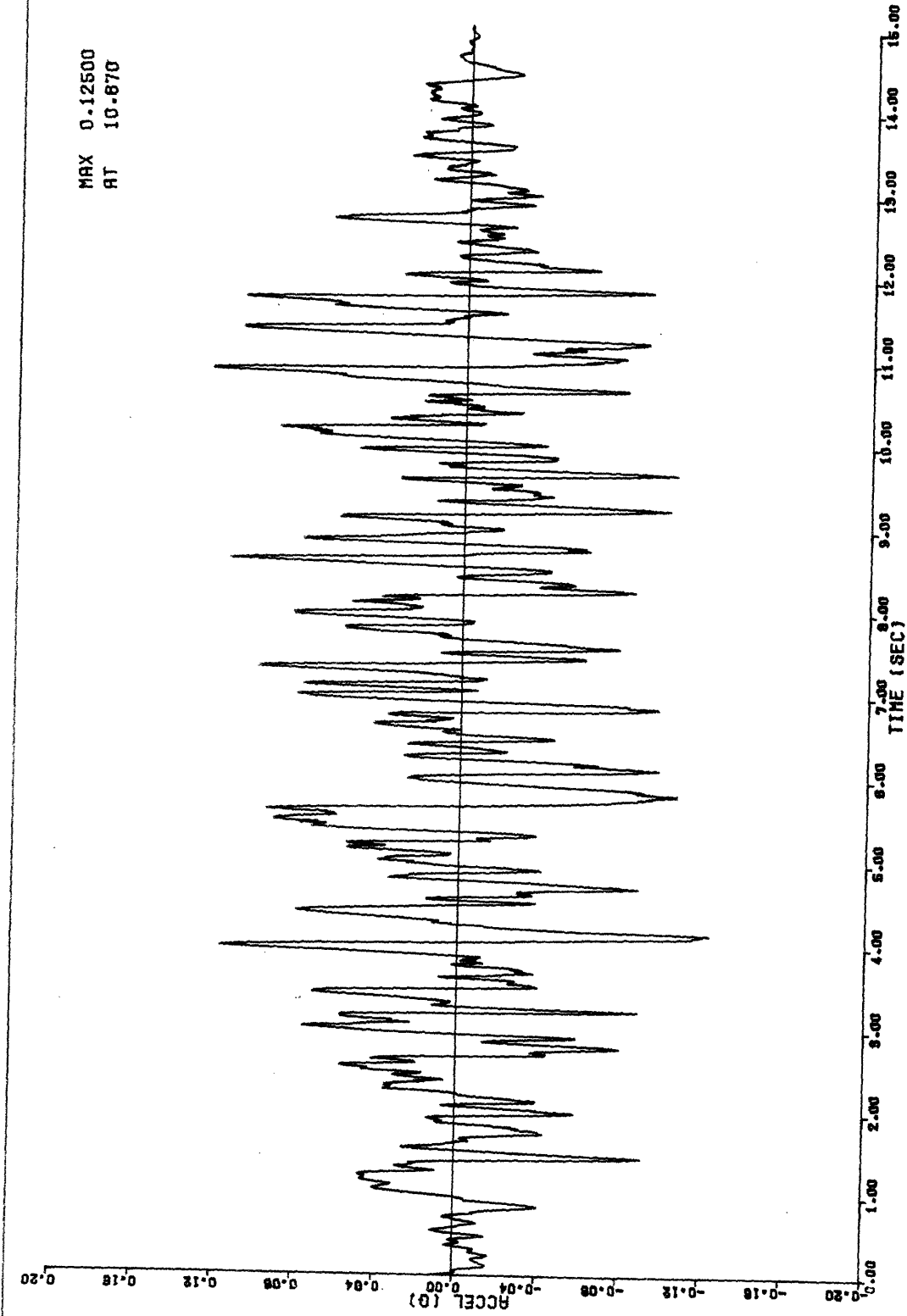


FIG. 4.9: ABSOLUTE VALUE OF THE ROCKING IMPEDANCE FOR THE RIGID CYLINDER CASE.

MAX 0.12500  
AT 10.870



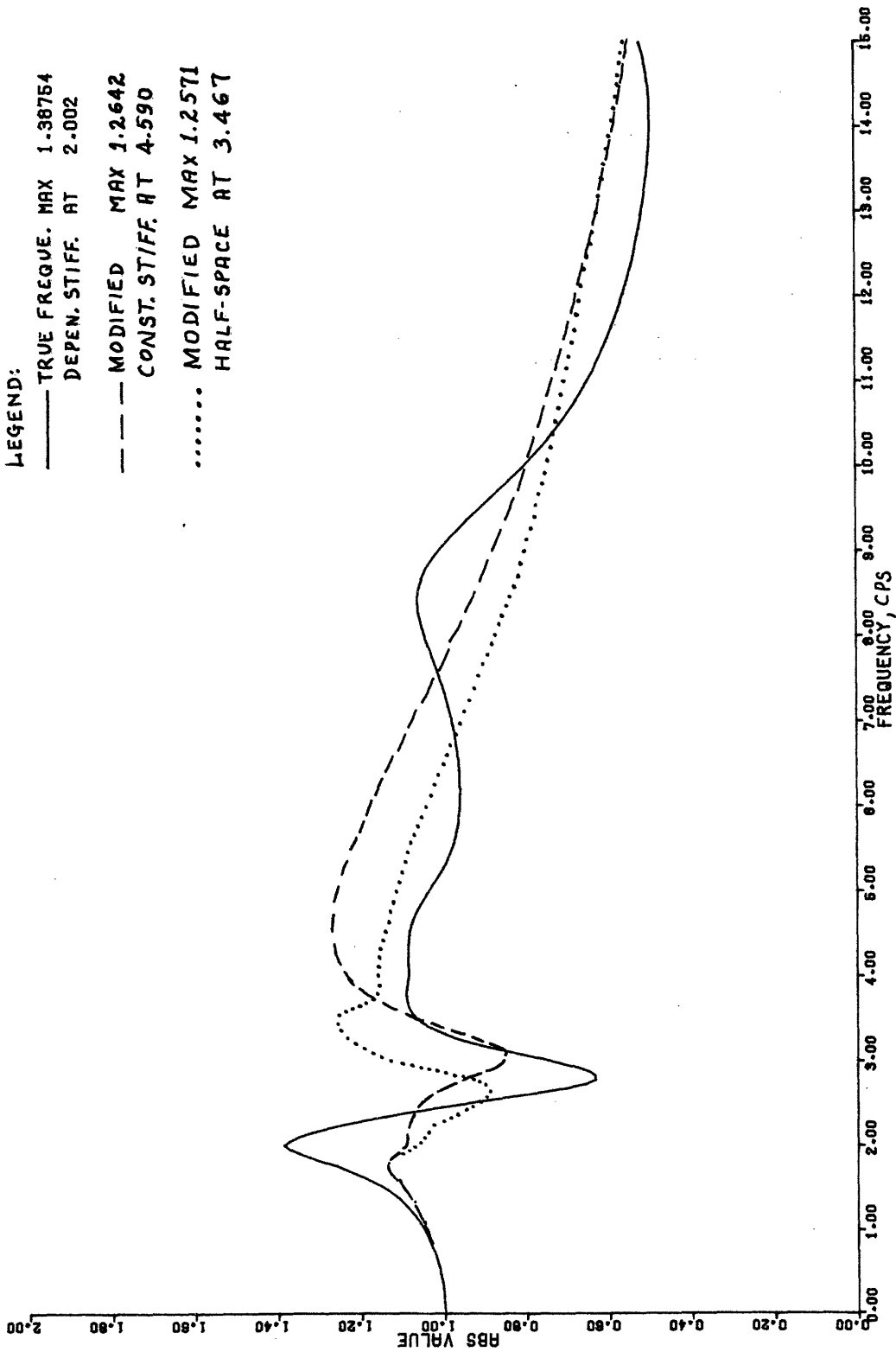
INPUT EARTHQUAKE. TIME HISTORY HORIZ EXCIT. AT FREE FOUND. AMAX= .125 G  
APP. CONSTANT STIFF., MODIFIED RAD. DAMPING BELOW S.F., 8% HYST. DAMPING

FIG. 4.10

great extent, if one would evaluate  $k_2$  at the resonant frequency rather than assigning to it the constant value 1.

Fig. 4.10 shows the time history of the 0.125g peak acceleration artificial earthquake used for the horizontal excitation.

Fig. 4.11 contains the absolute value of the transfer functions at the base of the cylinder using the true stiffness functions and the two approximate methods. While the three transfer functions follow similar trends, the frequency shift of the modified constant stiffness method, noticed in the absolute value of the rocking impedance (fig. 4.9), is again apparent in the transfer function. It is also evident that the approximate methods result in some unconservative values ( $\sim 25\%$  at 2 cps) close to the first peak in the low frequency range (1.5 to 2.5 cps). This is due to a combination of effects, some of which are: the static coupling term  $K_{\phi \times \phi_0}$  was overestimated by the approximate relation; also the swaying damping coefficients were overestimated in this low frequency range while the rocking damping coefficients were overestimated after the VNFS. On the other hand, this discrepancy does not result in large differences in the acceleration response spectra plots ( $\sim 15\%$  at  $T=0.45$  sec in figs 4.12 and 4.13). In the



TRANSFER FUNCTION FROM FREE FOUND. TO BASE HORIZ. MOTION. ABSOLUTE VALUE  
TRUE FREQUENCY DEPENDENT STIFFNESSES AND APPROXIMATE METHODS

FIG. 4.11



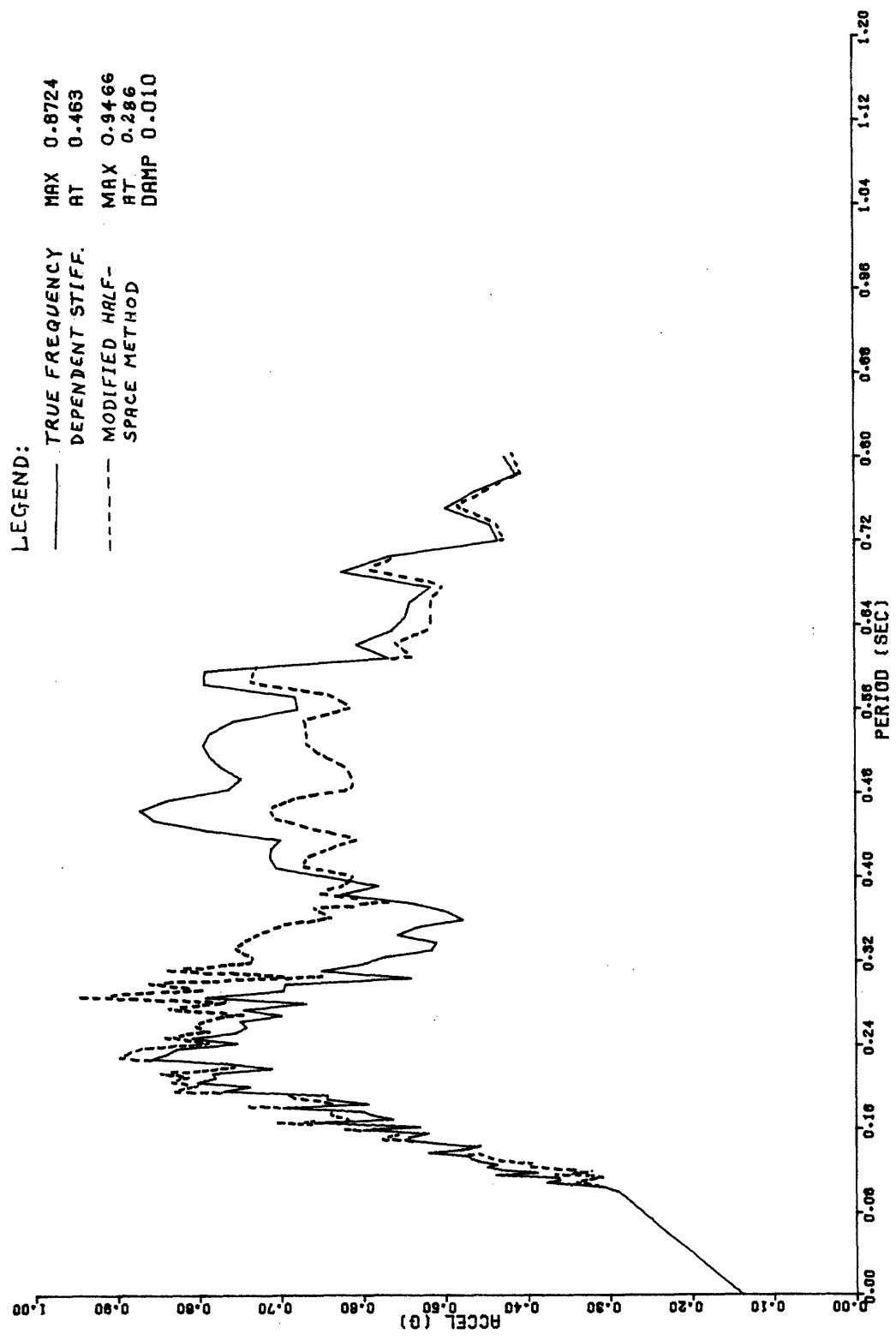
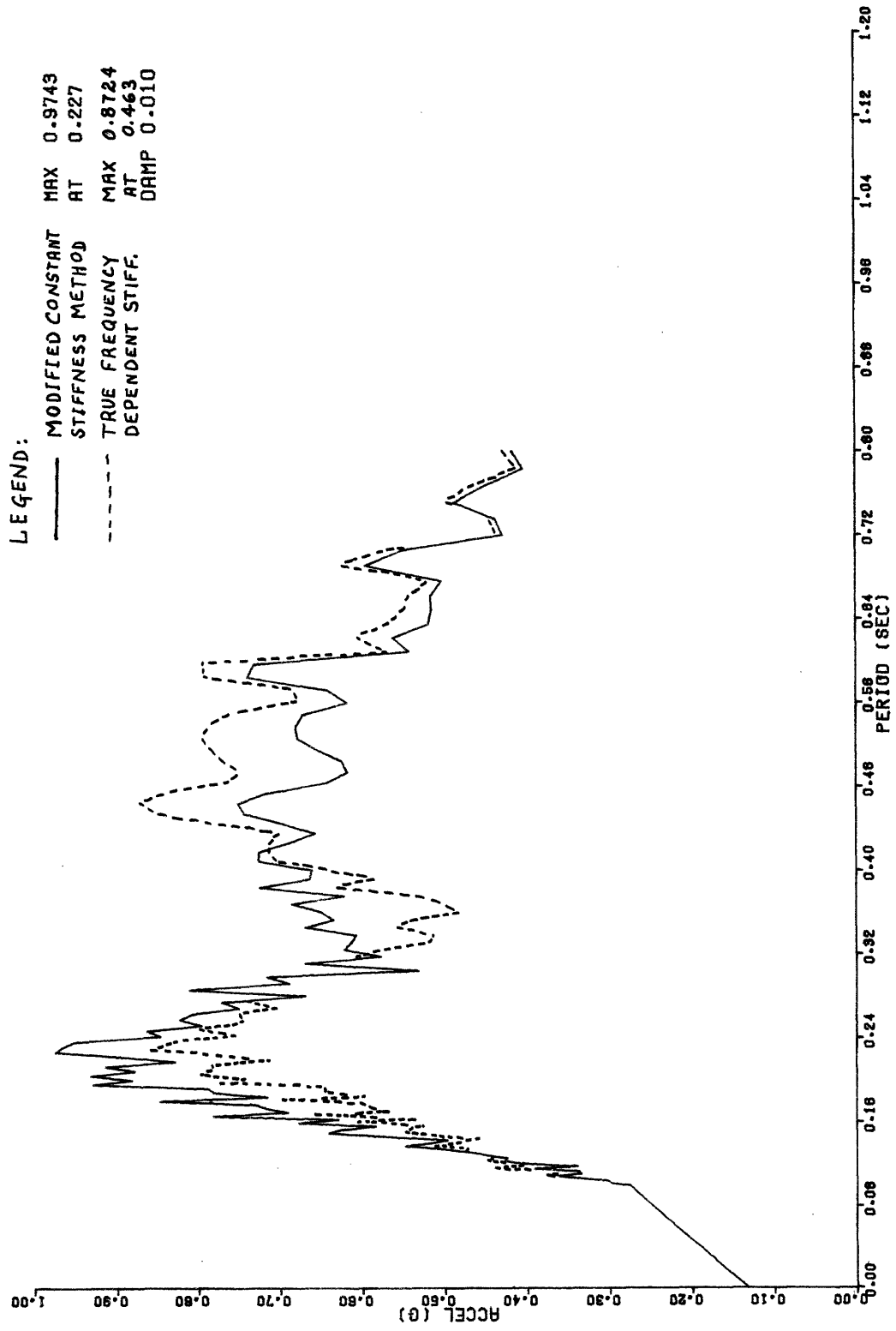


FIG. 4.12

HORIZ. AMPLIFIED RESPONSE SPECTRA OF ACCELERATION AT BASE  
TRUE FREQUENCY DEPENDENT STIFFNESSES AND MODIFIED HALF-SPACE METHOD

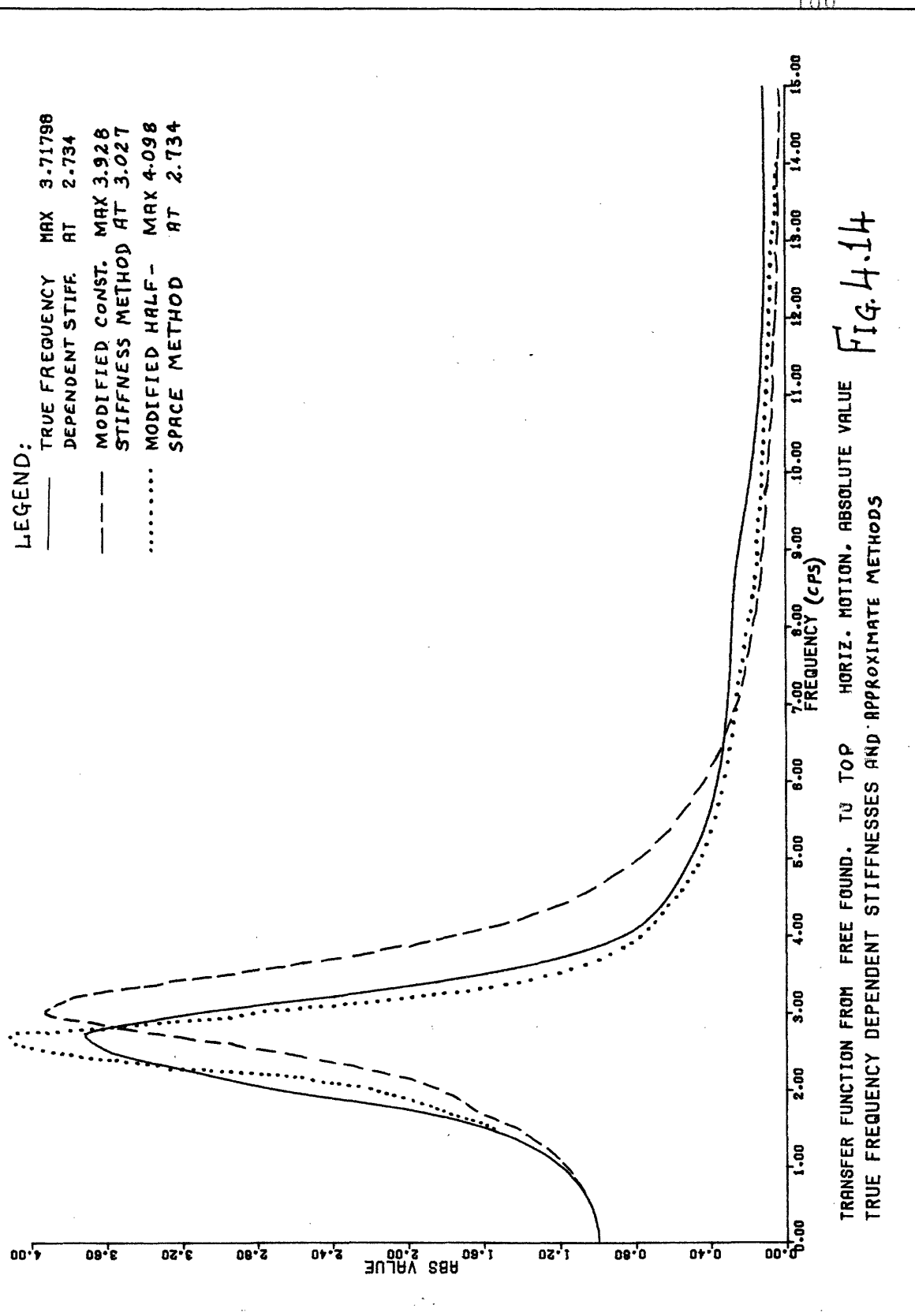


HORIZ. AMPLIFIED RESPONSE SPECTRA OF ACCELERATION AT BASE  
 MODIFIED CONSTANT STIFFNESS METHOD AND TRUE FREQUENCY DEPENDENT STIFF. *Fig. 4.13*

intermediate range (2.5 to 7. cps) the approximate methods result in slightly conservative values in the transfer function. This conservatism however is not reflected in the response spectra plots, keeping the approximate methods accurate in the low period range (figs 4.12 and 4.13).

At the top of the cylinder, on the other hand, the transfer functions and the response spectra obtained from the approximate methods agree much better with the ones obtained from the "true" frequency dependent stiffnesses (figs 4.14 to 4.16). The frequency shift of the modified constant stiffness method is again evident in both the transfer function and the response spectrum (figs 4.14 and 4.16). As previously stated, this shift stems from assuming  $k_2 = 1$  throughout the frequency range.

It should be noted that the rigid cylinder case chosen for this study is testing a very critical limiting situation since it falls slightly below the range for which the approximations in this study were derived. For the rigid cylinder:  $E/H = 0.478$ ;  $H/R = 1.704 < 2$  which is considered a very shallow stratum. While the fundamental frequency is below the VNFS, the approximate methods resulted in good agreements with the true frequency dependent stiffnesses. For deeper strata the approximations are expected to give even better results.



LEGEND:  
 — TRUE FREQUENCY  
 DEPENDENT STIFF.  
 --- MODIFIED HALF-SPACE  
 METHOD.  
 MAX 2.4447  
 AT 0.368  
 MAX 2.6129  
 AT 0.368  
 DAMP 0.010

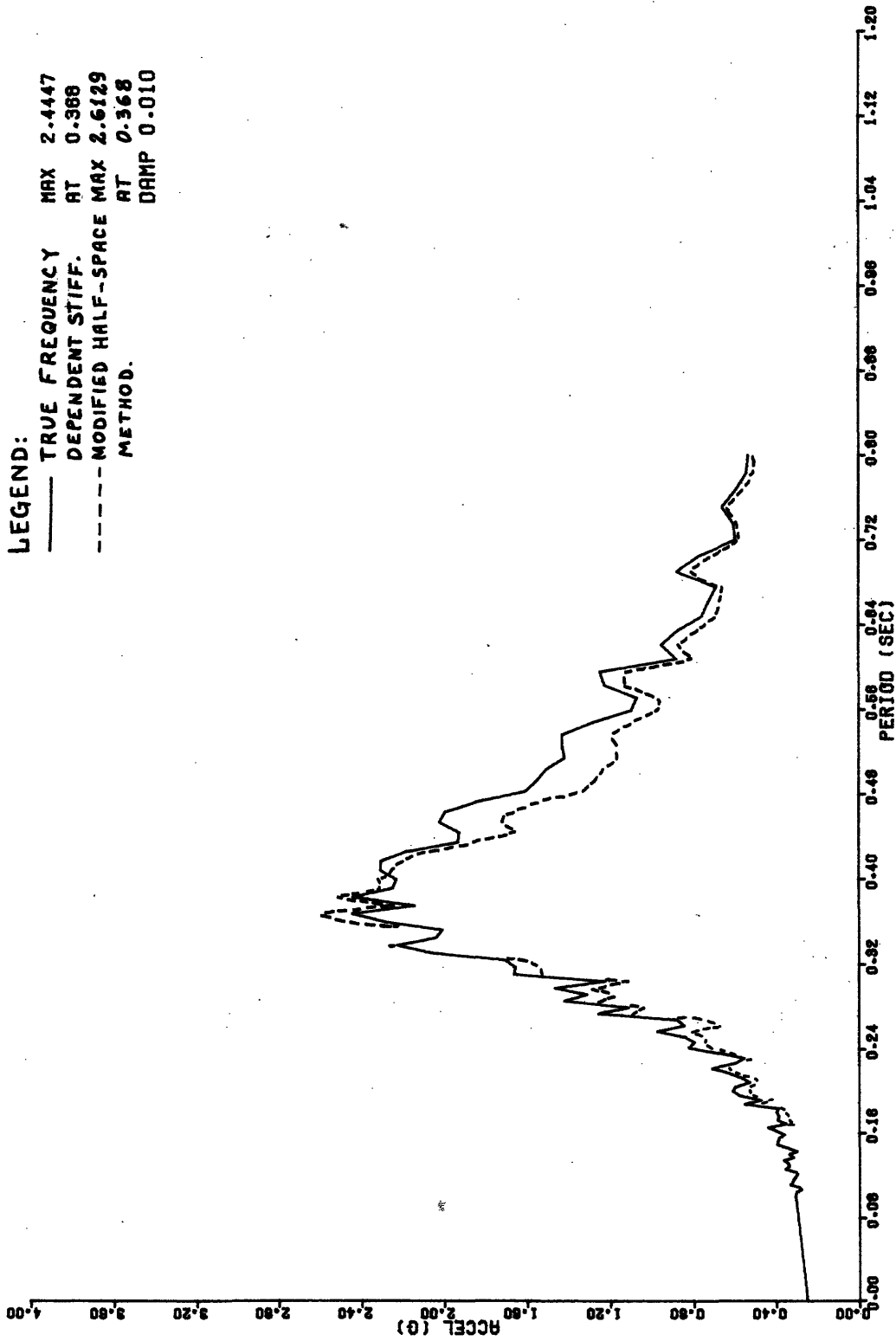
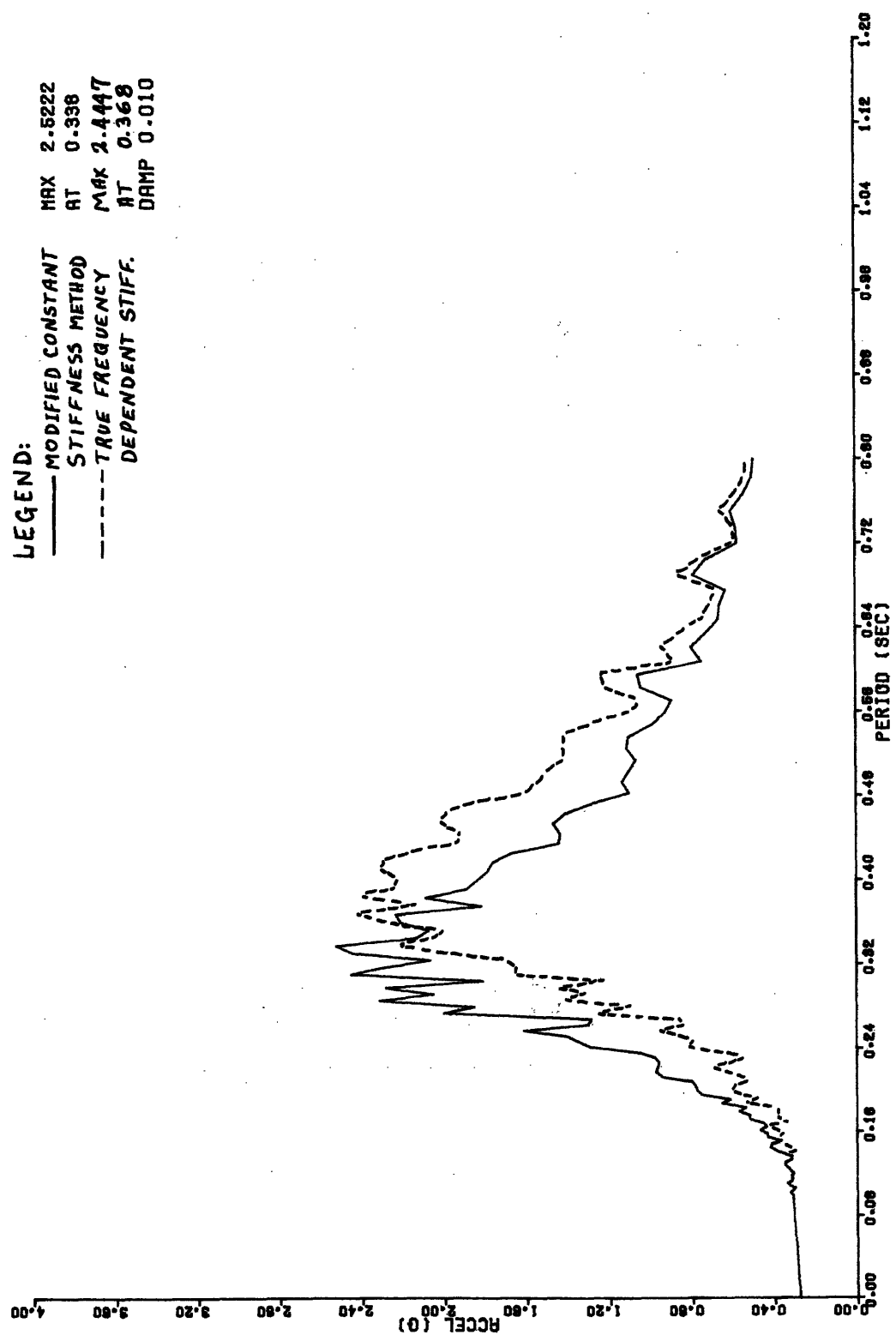


FIG. 4.15

HORIZ. AMPLIFIED RESPONSE SPECTRA OF ACCELERATION AT TOP  
TRUE FREQUENCY DEPENDENT STIFFNESSES AND MODIFIED HALF-SPACE METHOD

**LEGEND:**  
 ——— MODIFIED CONSTANT STIFFNESS METHOD  
 - - - - TRUE FREQUENCY DEPENDENT STIFF.

MAX 2.5222  
 AT 0.338  
 MAX 2.4417  
 AT 0.368  
 DAMP 0.010



HORTZ. AMPLIFIED RESPONSE SPECTRA OF ACCELERATION AT TOP  
 MODIFIED CONSTANT STIFFNESS METHOD AND TRUE FREQUENCY DEPENDENT STIFF. FIG. 4.16

## 5.- Summary and Conclusions:

Approximate empirical relations for the static stiffness coefficients of circular foundations embedded in (or resting on) an elastic homogeneous medium have been developed. Approximate frequency dependent functions for the stiffness and radiation damping coefficients and equivalent constant viscous damping ratios have been suggested. The approximate solutions were used to compute the response of a rigid cylinder embedded in a layered stratum and compared to the response obtained from a finite element analysis of the soil. The approximate static stiffnesses proved to be quite accurate for the rigid cylinder case, while the approximate stiffness and damping coefficients resulted in a good agreement in the response of the cylinder with that obtained using the finite element coefficients. It was found that the modified constant stiffness method gave results which agree well with those using the actual frequency dependent stiffnesses. Thus, such a method can be used to lower the computation costs of a preliminary analysis in the time domain.

During the derivation of the empirical relations and approximate methods, the following restrictions were imposed on the parameters:

$$E/H < 1/2$$

$$H/R \geq 2$$

$$\nu < 0.45$$

If a problem falls outside these ranges, a finite element analysis should be undertaken where the flexibility of the base rock should be taken into account.

Further studies are needed to investigate the effects of soil layering and the flexibility of the sidewall in greater detail, as they could change the results to some extent.

Given all the uncertainties inherent in the determination of the subgrade properties and soil behavior, it is felt that the relations proposed are in excellent agreement with more involved finite element studies. As an engineering tool, approximate methods may prove to be of considerable value, avoiding the use of unnecessarily costly studies, while giving reliable results.



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