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A Subdivision Scheme for Continuous-Scale B-Splines and Affine-Invariant Progressive Smoothing

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Abstract

Multiscale representations and smoothing constitute an important topic in different fields as computer vision, CAGD, and image processing. In this work, a multiscale representation of planar shapes is first described. The approach is based on computing classical B-splines of increasing orders, and therefore is automatically affine invariant. The resulting representation satisfies basic scale-space properties at least in a qualitative form, and is simple to implement.

The representation obtained in this way is discrete in scale, since classical B-splines are functions in $\mathbf{C}^{k-2}$, where k is an integer grater or equal than two. We present a subdivision scheme for the computation of B-splines of finite support at continuous scales. With this scheme, B-splines representations in $\mathbf{C}^r$ are obtained for any real r in $[0, \infty)$, and the multiscale representation is extended to continuous scale.

The proposed progressive smoothing receives a discrete set of points as initial shape, while the smoothed curves are represented by continuous (analytical) functions, allowing a straightforward computation of geometric characteristics of the shape.

Keywords: B-spline representations, subdivision schemes, continuous scale, affine invariant, progressive smoothing, computer implementation.

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1 Introduction

Multiscale descriptions of signals have been studied for several years already since Witkin's work [42], and developed in several different frameworks by a number of researchers in the past decade [4, 8, 14, 17, 21, 25, 26, 27, 29, 30, 31, 33, 43]. The idea of scale-space filtering is based on filtering a given initial signal $\Phi_0(\vec{X}) : \mathbb{R}^n \to \mathbb{R}^m$ with a family of filters $\mathcal{K}(\vec{X}, t) : \mathbb{R}^n \to \mathbb{R}^m$, where $t \in \mathbb{R}^+$ represents the scale. In other words, the scale-space is given by $\Phi(\vec{X}, t)$ defined as

$$\Phi(\vec{X}, t) := \Omega_{\mathcal{K}(\vec{X},t)}[\Phi_0(\vec{X})], \tag{1}$$

where $\Omega_{\mathcal{K}(\cdot,t)}[\cdot]$ represents the action of the filter $\mathcal{K}(\cdot, t)$. Larger values of t correspond to signals at coarser resolutions.

A classical example of a scale-space kernel is the Gaussian one. In this case, the scale-space is linear, and the filter in (1) is defined via convolution. The Gaussian kernel is one of the most studied in the theory of scale-spaces [4, 21, 25, 43]. It has some very interesting properties, one of them being that the signal Φ obtained from it, is the solution of the heat equation (with Φ_0 as initial condition) given by

$$\frac{\partial \Phi}{\partial t} = \Delta \Phi.$$

One of the key facts that can be gleaned from the Gaussian example, is that the scale-space can be obtained as the solution of a partial differential equation called an *evolution equation.* This idea was developed in different works [1, 2, 3, 23, 24, 37] for evolution equations different from the classical heat flow.

Let's concentrate now on the family of planar curves

$$\mathcal{C}(u, t) : [a, b] \times [0, \tau) \to \mathbb{R}^2,$$

and define the evolution equation

$$\begin{cases} \frac{\partial \mathcal{C}}{\partial t} = \frac{\partial^2 \mathcal{C}}{\partial p^2}, \\ \mathcal{C}(u, 0) = \mathcal{C}_0(u). \end{cases} \tag{2}$$

If $p \equiv u$, then the classical heat equation is obtained. If $p \equiv v$, where v is the *Euclidean arc-length* [41], the *Euclidean shortening flow,* or *Euclidean geometric heat flow,* is obtained [18, 20]. This equation defines a geometric Euclidean invariant scale-space [2, 24, 31]. If $p \equiv s$, where s is the *affine arc-length* [6], then the *affine shortening flow,* or *affine geometric heat flow,* is obtained [35, 36]. Sapiro and Tannenbaum [35, 36, 38] proved that this equation is the affine analog of the Euclidean shortening flow, and that any simple curve converges to an ellipse when evolving according to it (in the Euclidean case, convergence to a circle is obtained [18, 20]). The affine flow was implemented using an efficient numerical algorithm for curve evolution presented in [32], and based on this, a geometric affine invariant scale-space for planar curves was defined [37].

In general, a number of properties are required from a scale-space [1, 4]. some of the basic properties are:

1. *Completeness.* The original signal should be obtained when $t \to 0$.

2. *Order preserving.* If $\Phi_0 \leq \bar{\Phi}_0$, then $\Phi(t) \leq \bar{\Phi}(t)$ for all $t > 0$.

3. *Causality criterion.* This criterion means that "information" is not added when moving from a finer to a coarser scale. The "information" is in general characterized by zero-crossings, extremum, and so on. The causality criteria is usually also related to the *semi-group property* (specially when the scale-space is derived from evolution equations): The signal $\Phi(\vec{X}, t_2)$ (see equation (1)) can be obtained either from the initial signal $\Phi(\vec{X}, 0)$ via $\Omega(t_2)$, or from an intermediate one $\Phi(\vec{X}, t_1)$ via $\Omega(t_2 - t_1)$ $(t_1 < t_2)$.

In [1], Alvarez *et al.* gave a characterization of the evolution equations for which these and other properties hold. Examples are, of course, both the classical and the geometric heat flows described above [1, 2, 37].

The multiscale representation here developed is not obtained as the solution of an evolution equation. It is based on computing B-spline representations at different orders. We will show that the above properties hold at least in a qualitative form. Since B-splines are affine-invariant, so is the obtained representation.

Classical B-splines are functions in $\mathbf{C}^{k-2}$, where k is an integer number grater or equal than 2. Therefore, the multiscale representation based on classical B-splines of different orders, is discrete in its scale parameter k. Based on subdivision schemes, we present an extension of the classical B-spline basis in order to obtain a finite support basis in $\mathbf{C}^r$, $r \in [0, \infty)$. This way, we also obtain a continuous in scale multiscale B-spline based representation.

The proposed B-spline based progressive smoothing (representation) is easy to implement (see sections 2 and 4). In contrast with frequently used scale-spaces, as the Gaussian one, it is defined directly on an initial discrete set of points, avoiding problems caused by discretization of continuous scale-spaces. (See for example [29] for an analysis of this problem in linear scale-spaces, and a possible solution.) While the initial signal is discrete, the smoothed one is continuous and defined as a linear combination of B-spline basis. This allows a straightforward computation of geometric properties of the smoothed curve, as for example curvature or other invariants [22]. With the continuous scale B-spline basis presented in this paper, the representation is continuous in scale as well. Therefore, the proposed representation is natural for computer shape analysis, since receives as input a digital signal, while keeping a continuous representation which can be helpful for different computations.

The remainder of this paper is organized as follows: Section 2 gives a briefly description of the theory of B-spline approximations. Section 3 presents the affine invariant multiscale representation based on classical B-splines basis. The continuous scale B-spline basis, developed using subdivision schemes, is presented in Section 4. Concluding remarks are given in Section 5.

2 Basic Concepts on B-splines

We briefly describe now the theory of B-spline approximations. For details see for example [5, 9, 39, 40].

Let

$$\mathcal{C}(u) : [a,b] \to \mathbb{R}^2$$

be a planar curve, with Cartesian coordinates $[x(u), y(u)]$. Polynomials are computationally efficient to work with, but it is not always possible to define a satisfactory curve $\mathcal{C}$ using single polynomials for x and y. Then, the curve is divided in segments, each one defined by a given polynomial. The segments are joined together to form a *piecewise polynomial* curve. The joints between the polynomial segments occur at special curve points called *knots*. The sequence u_1, u_2, ... of knots is required to be nondecreasing. The distance between two consecutive knots can be constant or not. Two successive polynomial segments are joined together at a given knot u_j in such a way that the resulting piecewise polynomial has d continuous derivatives. Of course, the order of the polynomials depends on d. In this way, a basis is obtained, and the curve $\mathcal{C}$ is given by a linear combination of it.

Formally, the curve $\mathcal{C}$ is a *B-spline* approximation of the series of points $V_i = [x_i, y_i]$, $1 \leq i \leq n$, called *control vertices*, if it can be written as

$$\mathcal{C}_k(u) = \sum_i^n V_i B_{i,k}, \tag{3}$$

where $B_{i,k} = B(\cdot\,; u_i, u_{i+1}, ..., u_{i+k})$ is the i-th *B-spline basis* of order k for the knot sequence $[u_1, ..., u_{n+k}]$. In particular, $B_{i,k}$ is a piecewise polynomial function of degree $< k$, with breakpoints $u_j, ..., u_{j+k}$.

Several properties can be proven for the basis $B_{i,k}$:

1. $B_{i,k} \geq 0$.

2. $B_{i,k} \equiv 0$ outside the interval $[u_i, u_{i+k}]$. This property shows the locality of the approximation: Changing a given control vertex affects only a corresponding portion of the curve.

3. The basis is normalized:

$$\sum_i B_{i,k}(u) = 1 \text{ on } [u_k \dots u_{n+1}].$$

4. The support of the B-spline basis is minimal among all polynomial splines of order k. This property shows that this representation is optimal in certain sense.

The multiplicity of the knots governs the smoothness. If a given number τ occurs r times in the knot sequence $[u_i, \ldots, u_{i+k}]$, then the first $k-r-1$ derivatives of $B_{j,k}$ are continuous at the breakpoint τ. Therefore, without knot multiplicity, $\mathcal{C}_k \in \mathbf{C}^{k-2}$.

Computation with B-splines are facilitated by using the following recursive formula [5, 9, 10]:

$$B_{i,k}(u) = \frac{u - u_i}{u_{i+k-1} - u_i} B_{i,k-1} + \frac{u_{i+k} - u}{u_{i+k} - u_{i+1}} B_{i+1,k-1}, \tag{4}$$

together with

$$B_{i,1}(u) = \begin{cases} 1 & u_i \leq u < u_{i+1}, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

The basis $B_{i,k}$ is in fact the repeated convolution of a Haar function with itself, i.e.,

$$B_{0,k} = (*)^k \chi[0,1].$$

3 The B-spline Multiscale Representation

A general *affine* transformation in the plane ($\mathbb{R}^2$) is defined as

$$\tilde{X} = AX + T, \tag{6}$$

where $X \in \mathbb{R}^2$ is a vector, $A \in \mathrm{GL}_2^+(\mathbb{R})$ (the group of invertible real 2×2 matrices with positive determinant) is the affine matrix, and $T \in \mathbb{R}^2$ is a translation vector. It is easy to show that transformations (A, T) of the type (6) form a real algebraic group $\mathcal{A}$, called the *group of proper affine motions.* If $A \in \mathrm{SL}_2(\mathbb{R})$ (i.e., the determinant of A is 1), (6) gives the *group of special affine motions*, $\mathcal{A}_{sp}$.

A quantity Q is called an *invariant* of the group $\mathcal{A}$ if whenever Q transforms into $\tilde{Q}$ by any transformation $(A, T) \in \mathcal{A}$, we obtain $\tilde{Q} = \Psi Q$, where Ψ is a function of (A, T) alone. If $\Psi \equiv 1$ for all $(A, T) \in \mathcal{A}$, Q is called an *absolute invariant* [13].

Observe that from (3), the affine invariant property of the B-spline representation is immediate. If $\{\tilde{V}_i\}_1^n$ is obtained from $\{V_i\}_1^n$ by an affine transformation (A, T), then

$$\tilde{\mathcal{C}}_k(u) = A\mathcal{C}_k(u) + T, \tag{7}$$

where

$$\mathcal{C}_k(u) = \sum_i^n V_i B_{i,k},$$

and

$$\tilde{\mathcal{C}}_k(u) = \sum_i^n \tilde{V}_i B_{i,k}.$$

Based on this, we define now the B-spline affine invariant multiscale shape representation (*BAIM*) of the points $\{V_i\}_1^n$, as the family of curves $\mathcal{C}_k$ obtained from (3) for $k = 2, 3, ...$

Fig. 1 [1] presents the first *BAIM* example. The polygon contains 12 points. In Fig. 1a., the initial polygon is given, together with the corresponding *BAIM* for $k = 2^i$, $i = 1, 2, 4, 6, 7$. In Fig. 1b., the initial polygon is obtained via an affine transformation of the polygon in Fig. 1a. Due to the affine invariant property, the corresponding *BAIM* is related to the one in Fig. 1a by the same affine transformation.

Note that when k increases ($k \to \infty$, see Fig. 1), the B-spline representation converges to the centroid of the initial polygon. The convergence is in such a way that the shape approaches an ellipse. This is obtained from the following Theorem:

Theorem 1 *As k increases ($k \to \infty$), the B-spline representation converges to the centroid of the control points $\{V_i\}_1^n$, its shape becoming elliptical.*

Proof: Let's represent the control points as complex numbers, i.e., $V_i = [x_i + jy_i]$. Then, using the Fourier series expansion of the basis functions $B_{i,k}(u)$ [39, 40], we have that:

$$\begin{aligned}
\mathcal{C}_k(u) &= \sum_{i=1}^{n} V_i B_{i,k} = \\
&= \sum_{i=1}^{n}(x_i + jy_i)\left[\sum_{m=-\infty}^{m=\infty} \exp\{jm\pi i\}\left(\frac{2\sin(m\pi/2)}{m\pi}\right)^k \exp\{jm\pi u\}\right] = \\
&= \sum_{m=-\infty}^{m=\infty}\left(\sum_{i=1}^{n}(x_i + jy_i)\exp\{jm\pi i\}\right)\left(\frac{2\sin(m\pi/2)}{m\pi}\right)^k \exp\{jm\pi u\}.
\end{aligned}$$

We see that when k increases ($k \to \infty$), the B-spline representation converges to the centroid of the initial polygon (only the term for $m = 0$ remains in the sum). Furthermore, the convergence is in such a way that the curve shape approaches an ellipse. This is so since high frequency components of the Fourier transform of the *BAIM* die out much faster than the low frequency ones. Therefore, the limiting curve becomes approximately an ellipse, when only the zero ($m = 0$) and first ($m = \pm 1$) frequency components remain significant. □

As pointed out in the Introduction, a curve evolving according to the affine geometric heat flow, also converges to a point with elliptical shape [35, 36, 37, 38]. In [7], the authors presented a discrete polygonal model of the affine heat flow, and proved, also using Fourier decomposition, that the limiting polygon approaches an ellipse. These results are expected, since all of these three processes are affine smoothing process [7, 37, 40], and the ellipse is the most smooth affine invariant shape [34].

[1] The examples here presented were implemented using the Matlab Spline Toolbox [10].

Before referring to a number of properties of this representation, let's remark basic differences between the *BAIM* and scale-spaces as those presented at the Introduction:

1. Scale-spaces are usually defined over continuous curves, and the *BAIM* is derived from a set $\{V_i\}$ of discrete points. These set of points can be a dense sampling of the curve (even with $n \to \infty$), but it is always a discrete set. The smoothed curves, in contrast, are continuous, given as a linear combination of basis functions. This permits the easy computation of different curve properties.

2. The multiscale representation obtained via scale-spaces is usually continuous ($t \in [0, \tau)$), while the one obtained from the *BAIM* is discrete (k = 2, 3, ...). See Section 4.

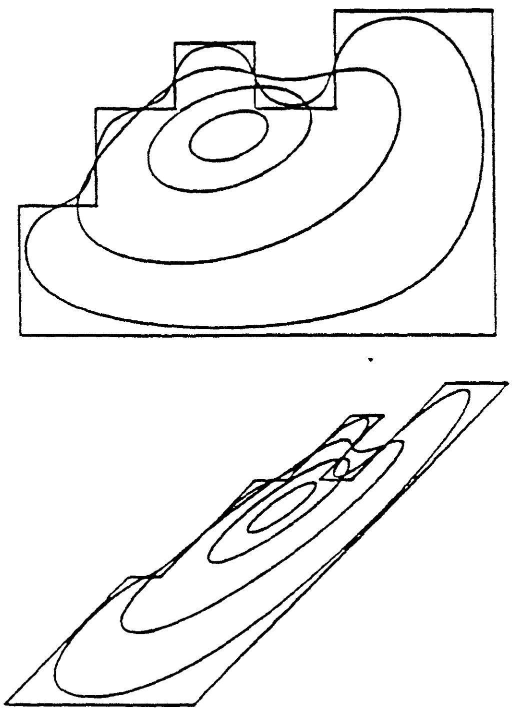

Figure 1. Example of the *BAIM* and its affine invariant property. A 12 points polygon (top) and its correspondent B-spline representations of order $k = 2^i$, i = 1, 2, 4, 6, 7. Note the convergence to the centroid, with an elliptical shape. In the bottom, the original polygon is obtained via an affine transformation of the one in the top. The corresponding *BAIM* is related by the same transformation.

We discuss now, for the *BAIM*, the properties of scale-spaces presented in the Introduction. First note that for $k = 2$, the original set of points, or the polygon defined by them, is obtained. Therefore, the representation is complete.

The curve obtained from the B-spline representation (3) lies in the convex hull of at most k consecutive control points (those which affect the curve at the given point). For other relations between the control points, and the obtained B-spline, see [9]. Then, the order preserving property given in the Introduction, refers to the corresponding convex-hull for the *BAIM*, as well as to the other relations between control points and B-splines given in [9]. Note that when processing isolated shapes, this property is not so crucial in all applications. It becomes important for example when dealing with level sets of images as in [1].

The "causality" criteria holds, since for increasing k, increasingly smoother versions of the initial curve are obtained (see Fig. 1 and 2). Note that when k increases, both the basis functions get smoother, and the curve values are obtained as the weighted mean of more control points. The *BAIM* is a smoothing process with increasing smoothing exponent [40], in the sense that the total difference of any order decreases when k increases (*diminishing property*). The following geometric results can be proven as well, in relation with the smoothing property of B-spline representations. The first result, due to Lane and Riesenfeld [28], refers to the *geometric diminishing property*:

Theorem 2 *Every hyperplane of* $\mathbb{R}^m$ *intersects a B-curve in* $\mathbb{R}^m$ *no more often than the associated control polygon.*

One of the possible requirements related to the causality criteria is that the number of inflection points decreases when the scale increases. The following result, due to Goodman [19], states this.

Notation:

1. $i(\mathcal{C})$ - number of inflections in the curve $\mathcal{C} \in \mathbb{R}^2$.
2. $S(\mathbf{a}) = S(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ - number of strict sign changes in the sequence $\mathbf{a} \in \mathbb{R}^2$.
3. $S(f)$ - number of strict changes in the function $f : (a,b) \to \mathbb{R}$. $S(f) = \sup S(f(t_1), \ldots, f(t_n))$, where the supremum is taken over all the sequences $a < t_1 < \ldots < t_n < b$.

Definition 1 *Suppose* $\mathcal{C} : [a,b] \to \mathbb{R}^2$ *is continuous with piecewise* $\mathbf{C}^1$ *tangent vector*

$$v := \frac{\mathcal{C}'}{|\mathcal{C}'|}.$$

For $t \in (a,b)$, $\kappa(t)$ is defined such that it is positive (negative) if the curve is turning anti-clockwise (clockwise) at $C(t)$. Suppose the curve is constant at $[\alpha,\beta] \subset (a,b)$, but not in any other larger interval. Then, we define

$$\kappa(t) := \begin{cases} \frac{1}{2}\left[v(\alpha^-) \times v'(\alpha^-) + v(\beta^+) \times v'(\beta^+)\right] & \text{if } v(\alpha^-) = v(\beta^+) \\ v(\alpha^-) \times v'(\beta^+) & \text{if } v(\alpha^-) \neq v(\beta^+) \end{cases}$$

Finally,

$$i(C) := S(\kappa).$$

Definition 2 *If the curve C is a polygon, as the control polygon $\{V_i\}_1^n$, we write*

$$I_i := (V_i - V_{i-1}) \times (V_{i+1} - V_i),$$

and

$$i(\{V_i\}_1^n) := S(I_1, ..., I_N).$$

The polygon $\{V\}_i^n$ is regular if the following hold:

1. *It turns to a total angle of magnitude at most π, that is, for some vector $V \in \mathbb{R}^2$, $V \cdot (V_i - V_{i-1}) \geq 0$.*

2. *It does not turn trough an angle of π at any vertex, i.e., $(V_i - V_{i-1}) \neq \lambda(V_i - V_{i+1})$ for any real λ.*

Theorem 3 *If the control polygon $\{V_i\}_1^n$ is regular, then for the B-spline representation C obtained from it via (3), the following relation holds:*

$$i(C) \leq i(\{V_i\}_1^n).$$

Theorem 3 means that if the control polygon does not turn too sharply, then the number of inflection points of the B-spline representation does not exceed the number of inflections points at the control polygon. This result also holds for a large class of basis functions [16].

From the results above, we conclude that when moving from the control polygon to a B-spline representation, no information is added.

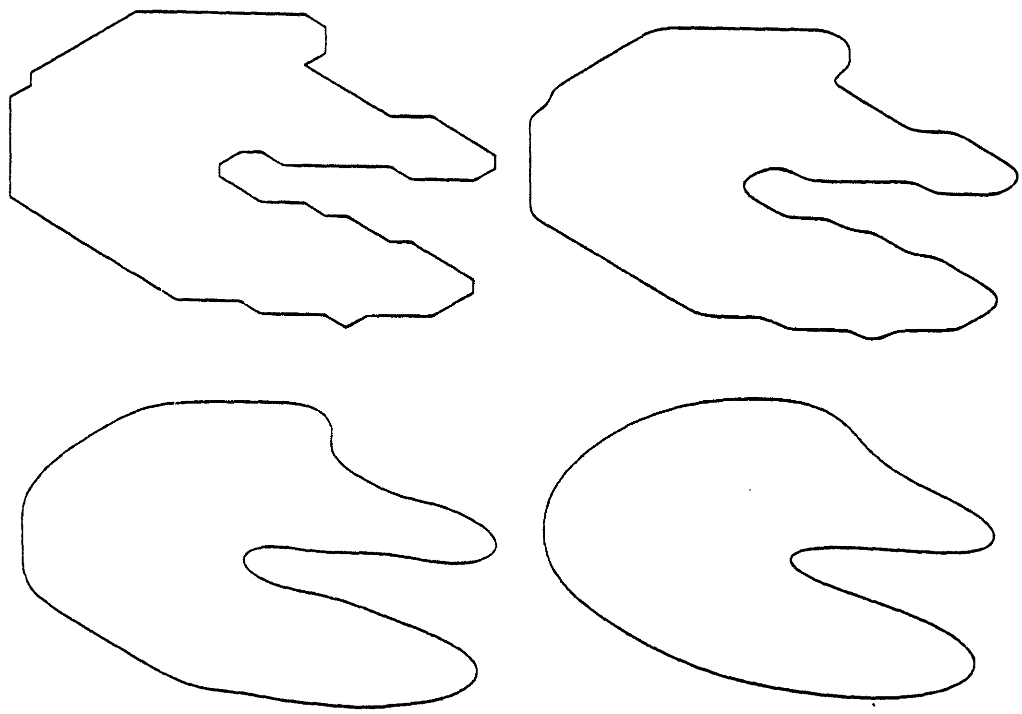

Figure 2. The smoothing property of the *BAIM* is shown in this example (orders 4, 20, and 90). The curve is getting more and more smooth when the order of the B-spline approximation is increased.

As pointed out in the Introduction, when scale-spaces are obtained as solutions of evolution equations, the causality criteria is usually connected to the semi-group property. We want to investigate now this property in a qualitative form. First note that the B-spline basis is obtained via repeated convolution of the Haar function, i.e.,

$$B_k = \chi * B_{k-1},$$

where χ is the indicator function. Therefore, the semi-group property holds for the basis of the representation.

Assume now that given the series of control points $\{V_i\}_1^n$, the corresponding B-spline representations C_{k_1} and C_{k_2} of order k_1 and k_2 respectively are computed ($k_1 < k_2$). Then, in relation to the semi-group property, we ask if we can compute (or at least approach) C_{k_2} from C_{k_1}. From (3) we have

$$C_{k_1}(u) = \sum_i^n V_i B_{i,k_1},$$

$$\mathcal{C}_{k_2}(u) = \sum_i^n V_i B_{i,k_2}.$$

For computing a B-spline representation, we need a discrete set of control points. Therefore, $\mathcal{C}_{k_1}$ is sampled:

$$\{\hat{V}_i\}_1^n := \{\mathcal{C}_{k_1}(u^i)\}_1^n = \{\sum_j^n V_j B_{j,k_1}(u^i)\}_1^n,$$

and the sampling points are used for the computation of a new B-spline approximation of order k_2:

$$\begin{aligned}\hat{\mathcal{C}}_{k_2}(u) &= \sum_i^n \hat{V}_i B_{i,k_2} \\ &= \sum_i^n (\sum_j^n V_j B_{j,k_1}(u^i)) B_{i,k_2}.\end{aligned}$$

The difference between $\mathcal{C}_{k_2}$ and $\hat{\mathcal{C}}_{k_2}$ is given by:

$$\begin{aligned}\| \mathcal{C}_{k_2} - \hat{\mathcal{C}}_{k_2} \| &= \| \sum_i^n V_i B_{i,k_2} - \sum_i^n (\sum_j^n V_j B_{j,k_1}(u^i)) B_{i,k_2} \| \\ &= \| \sum_i^n [V_i - \sum_j^n V_j B_{j,k_1}(u^i)]\, B_{i,k_2} \| \\ &\leq \sum_i^n \| V_i - \sum_j^n V_j B_{j,k_1}(u^i) \| \; B_{i,k_2} \\ &\leq \Upsilon(k_1, \{V_i\}) \sum_i^n B_{i,k_2} \\ &= \Upsilon(k_1, \{V_i\}),\end{aligned}$$

where $\Upsilon(k, \{V_i\})$ is a bound of the distance between the control points and the B-spline of order k. General bounds for the distance between the initial polygon and the corresponding B-spline representation can be found in [9]. Note that since the B-spline basis are normalized, we have

$$\| V_i - (\sum_j^n V_j B_{j,k_1}(u^i)) \| \; B_{i,k_2} = \| \sum_j^n (V_i - V_j) B_{j,k_1}(u^i) \| \; B_{i,k_2},$$

and the error is like a weighted average of the difference between the control points (only those which affects the value at u^i).

Since the bound Υ increases with k [9], so does the error bound. On the other hand, the shape of the spline approximations is similar to the original curve, then it is expected from the shapes $\mathcal{C}_{k_2}$ and $\hat{\mathcal{C}}_{k_2}$ to be similar as well. The error can also be reduced if $\mathcal{C}_{k_1}$ is sampled in such a way that the points $\mathcal{C}_{k_1}(u^i)$ are as close as possible to V_i. This property was experimented and the results are shown in Fig. 3. The original polygon is given in Fig. 3a. The B-spline representations of order 5, 10 , and 15 were computed (Fig. 3b, 3c, 3d, 3e). Then, the B-spline of order $k_1 = 5$ (Fig. 3b) is uniformly sampled, and represented by the same number of points as the original polygon. After that, the B-splines of order (k_2) 10 and 15 were computed, with this polygon as initial data (Fig. 3f and 3g). As expected, the obtained B-spline representations are very similar to the ones obtained with the original polygon in Fig. 3a. (When Fig. 3b and 3f, and Fig. 3c and 3g, are presented in the same draw, almost no distinction between the corresponding curves is obtained.) Small difference were observed when k_1 is increased, but the general (qualitative) shape was always preserved. Therefore, we can conclude that the semi-group property holds in a qualitative form.

3.1 The BAIM of Continuous Initial Curves and Noisy Polygons

If the original curve is given in a continuous form (by a formula for example), then the curve must be sampled in order to construct the *BAIM*. This is in fact what we did in Fig. 3 for the construction of Fig. 3f and 3g.

Since we are interested in an affine invariant representation, the ideal situation is to do the sampling in an affine invariant form. This can be done for example starting form the point of maximal *affine curvature*, and sampling the curve at constant *affine distance*. This properties are conserved under an affine transformation [6]. In this way, the sampling points (control vertices) are affine invariant. Note also that since the *BAIM* is given as a linear combination of B-spline basis functions, geometric properties of the curve, as affine curvature, can be computed directly, using the classical formulas for derivatives [9]. For instance, it is well known that derivatives of B-splines can be obtained taking finite differences of lower order splines.

If the sampling strategy described above cannot be performed (due to the presence of noise for example), then, a regular sampling can be performed. Since the B-spline basis is normalized, the error of the *BAIM* is controlled by the error in the control vertices coordinates. In other words, if the error in the control vertices coordinates is bounded by ϵ, so is the error in the B-spline representation coordinates. Fig. 4 and 5 show examples of the *BAIM* of noisy initial curves. Note that when k increases, the noise in a given control point extends to bigger segments in the curve (the support of $B_{i,k}$ gets bigger). On the other hand, increasing k, also increases the number of control points which participates in the weighted mean (3) for each point $\mathcal{C}_k(u)$, decreasing the influence of error in a given control point.

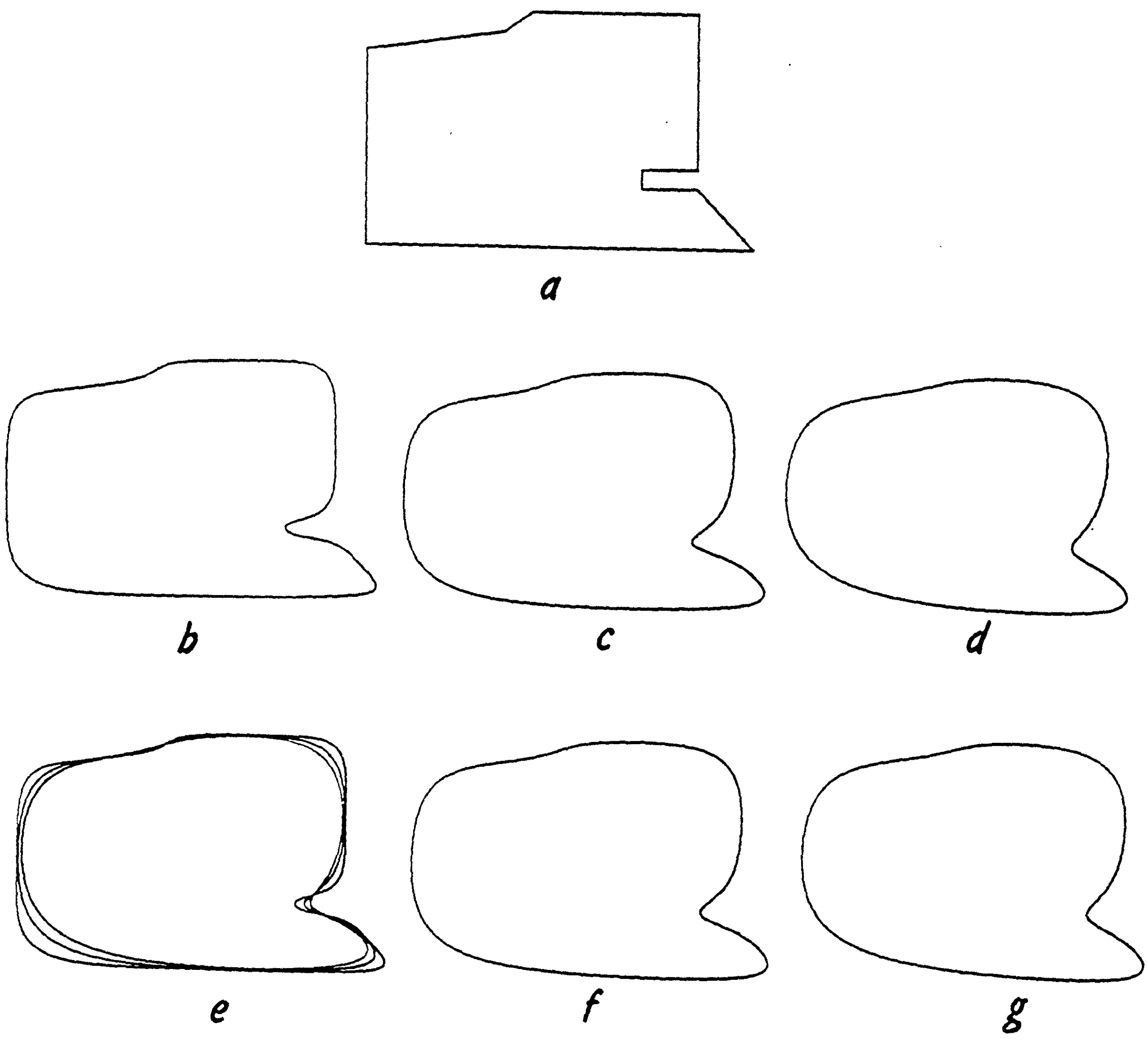

Figure 3. Investigation of the semi-group property for the *BAIM*. (a) The original polygon. (b) B-spline representation of order 5. (c) B-spline representation of order 10. (d) B-spline representation of order 15. (e) B-spline representations of orders 5, 10, and 15. (f) B-spline representation of order 10 obtained from the sampling of the curve in (b). (g) B-spline representation of order 15 obtained from the sampling of the curve in (b).

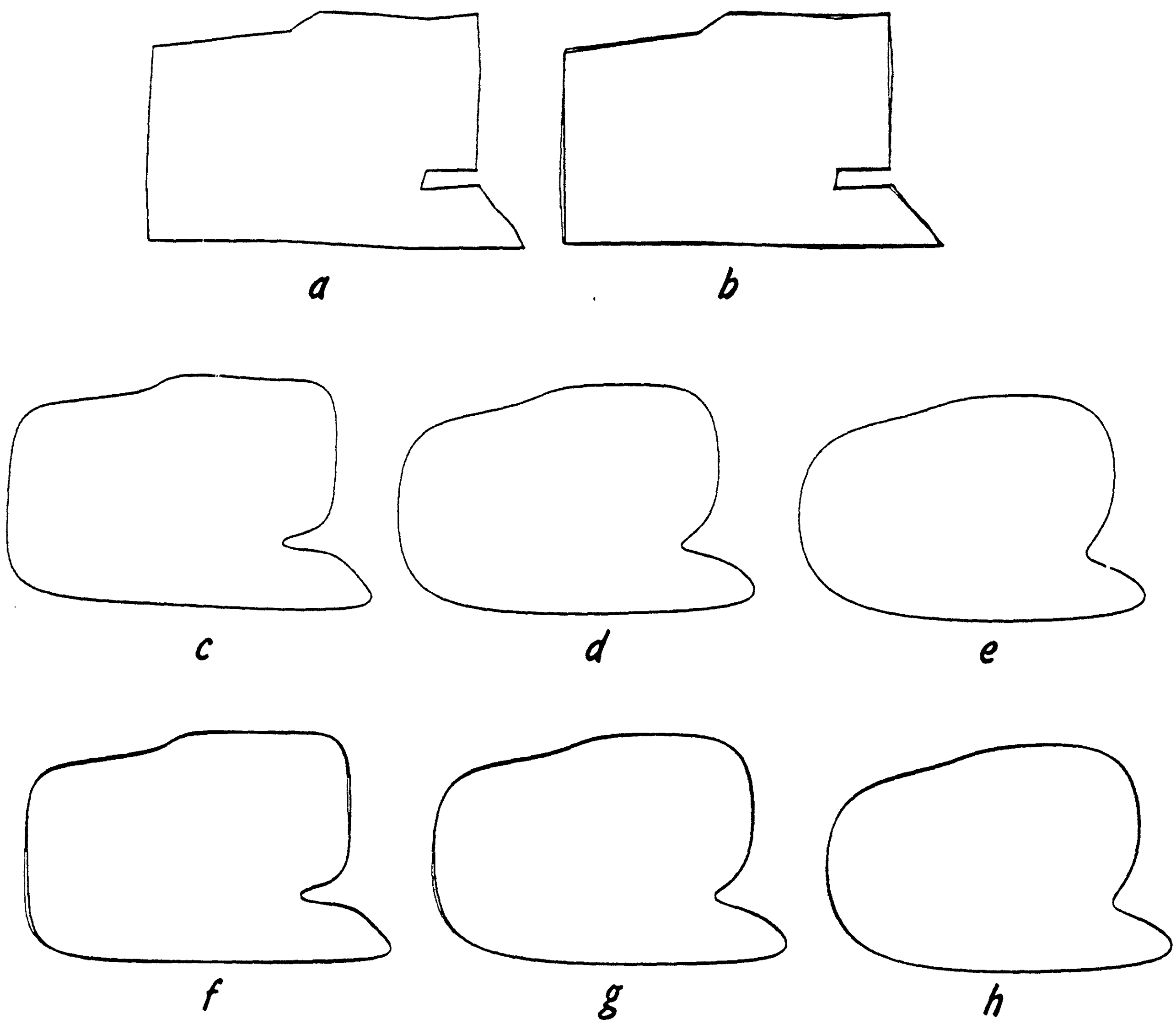

Figure 4. The *BAIM* of a noisy curve. (a) The original polygon. The polygon was obtained from the one in Fig. 3a adding random noise to the coordinates. (b) Original and noisy polygons. (c) B-spline representation of order 5. (d) B-spline representation of order 10. (e) B-spline representation of order 15. (f) B-spline representation of order 5 of the original and noisy polygons. (g) B-spline representation of order 10 of the original and noisy polygons. (h) B-spline representation of order 15 of the original and noisy polygons.

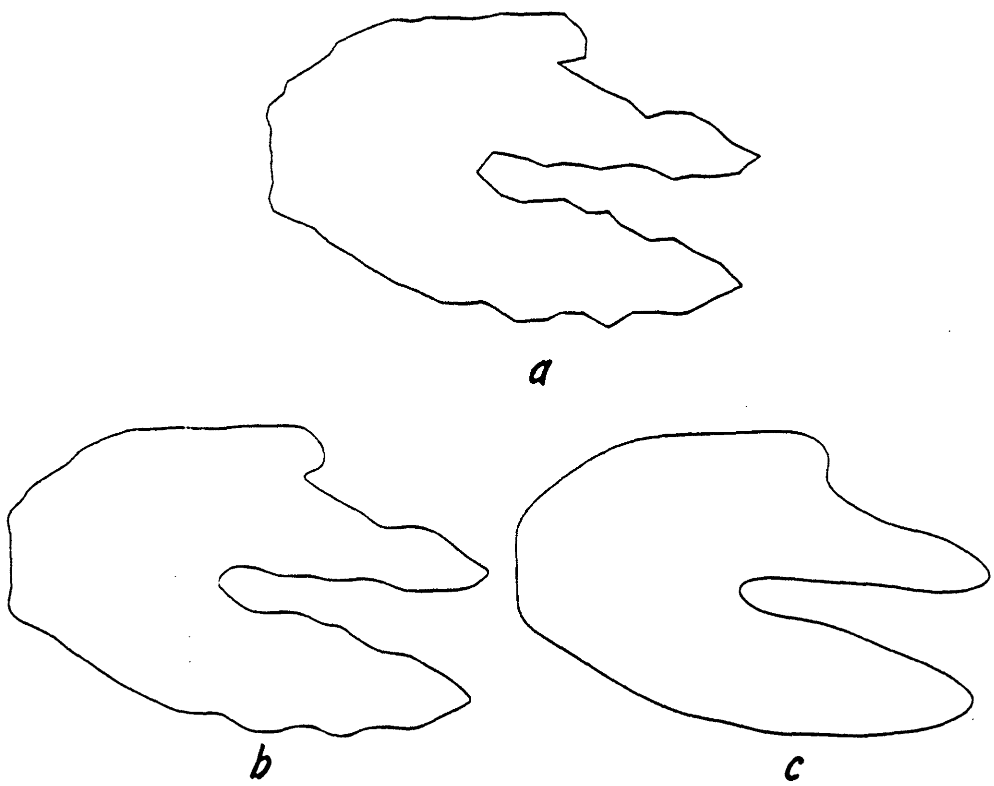

Figure 5. A second example of the *BAIM* of a noisy curve (compare with Figure 2). (a) The noisy curve. (b) B-spline representation of order 4. (c) B-spline representation of order 20.

4 Continuous-Scale B-Splines

The multiscale representation described so far is discrete in the sense that the B-splines $B_{0,k} = (*)^k \chi_{[0,1]}$ that are used to generate it are indexed by a discrete parameter $k \in \mathbb{N} - \{0\}$. In order to obtain a continuous multiscale representation, one needs to extend these generators to a family of compactly supported functions $B_{0,r}$, $r \in [0, +\infty)$, that coincides with the previous one on the integer values of r.

The semi-group property of the multiscale representation is expressed in the integer case by the relation $B_{0,k} = B_{0,k-1} * \chi_{[0,1]}$ indicating that $B_{0,k}$ is obtained by smoothing $B_{0,k-1}$. To preserve this causality, we would like that for $r_1 > r_2$, B_{0,r_1} is obtained by as smoothing operation applied to B_{0,r_2}. Recall that if f is in $\mathbf{C}^n$ but not in $\mathbf{C}^{n+1}$, then its (global) Hölder exponent is given by $\mu(f) = n + \nu$ with

$$\nu = \inf_x \left(\liminf_{|t| \to 0} \frac{\log |f^{(n)}(x+t) - f^{(n)}(x)|}{\log |t|} \right), \tag{8}$$

where $f^{(n)}$ is the n-th derivative of the function f. If we use the Hölder exponent to measure the regularity of our generators, we would like to have

$$r_1 > r_2 \Rightarrow \mu(B_{0,r_1}) > \mu(B_{0,r_2}). \tag{9}$$

A straightforward technique to extend the B-spline family is to define for $r = k+s$, $k \in \mathbb{N} - \{0\}$, $s \in [0,1]$,

$$B_{0,r} = (1-s)B_{0,k} + sB_{0,k+1}. \tag{10}$$

With such a definition, it is clear however that the property (9) will not be satisfied since we have, for all $0 \leq s < 1$, $\mu(B_{0,k+s}) = \mu(B_{0,k}) = k-1$.

We shall thus use a more sophisticated extension of the B-spline family. This extension is based on "subdivision schemes" that are frequently used in computer-aided geometric design.

Subdivision schemes constitute a useful tool for the fast generation of smooth curves and surfaces from a set of control points by means of iterative refinements. In the most often considered binary unidimensional case, one starts from a sequence $s_0(k)$ and obtains at step j a sequence $s_j(2^{-j}k)$, generated from the previous one by linear rules :

$$\begin{array}{l} s_j(2^{-j}2k) = \sum_n a_{j,k}(n)s_{j-1}(2^{-j+1}(k-n)), \\ s_j(2^{-j}(2k+1)) = \sum_n b_{j,k}(n)s_{j-1}(2^{-j+1}(k-n)). \end{array} \tag{11}$$

The masks $a_{j,k}(n)$ and $b_{j,k}(n)$ are in general finite sequences, a property that is clearly useful for the practical implementation of (11).

A natural problem is then to study the convergence of such an algorithm to a limit function. In particular, the scheme is said to be strongly convergent if and only if there exists a continuous function $f(x)$ such that $\lim_{j\to+\infty}(\sup_k |s_j(2^{-j}k) - f(2^{-j}k)|) = 0$. One can study more general type of convergence with the use of a smooth function g that is well localized in space (for example compactly supported) and satisfies the interpolation property $g(k) = \delta_k$. One can then define $f_j(x) = \sum_k s_j(2^{-j}k)g(2^jx - k)$ and study the convergence in a functional sense of f_j to f.

A subdivision scheme is said to be stationary when the masks a and b are independent of the parameters j and k. In that case, the limit $f(x)$ is given by

$$f(x) = \sum_{k\in\mathbb{Z}} s_0(k)\varphi(x-k), \tag{12}$$

where φ is the limit function obtained from the Dirac sequence $s_0(k) = \delta_{0,k}$. For this reason φ is often called the "limit function" of the stationary subdivision. One can also rewrite (11) as

$$s_j(2^{-j}k) = \sum_n c(k-2n)s_{j-1}(2^{-j+1}n), \tag{13}$$

with $c(2k) = a(k)$ and $c(2k+1) = b(k)$. Note that (11) is equivalent to fill the sequence s_{j-1} with zeros at the intermediate points $2^{-j}(2k+1)$ and apply a discrete convolution with the sequence $c(k)$. As a consequence, the function φ is the result of an infinite number of discrete convolutions at finer and finer scales. It can also be expressed in the Fourier domain by the infinite product

$$\hat{\varphi}(\omega) = \prod_{j=1}^{+\infty} m(2^{-j}\omega), \tag{14}$$

where $m(\omega) = \frac{1}{2}\sum_n c(n)e^{-in\omega}$ is 2π-periodic function. Note that if $c(n) = 0$ for $n < a$ and $N > b$, i.e. $m(\omega)$ is a finite Fourier series, then φ is compactly supported in $[a, b]$.

Detailed reviews of stationary subdivision and their possible generalizations have been done by Cavaretta, Dahmen and Micchelli in [11] and Dyn in [15].

The B-spline $B_{0,k}$ can be viewed as the limit function of a stationary subdivision scheme associated to the trigonometric polynomial

$$m_k(\omega) = \left(\frac{1+e^{-i\omega}}{2}\right)^k, \tag{15}$$

since we have indeed

$$\hat{B}_{0,k}(\omega) = \left(\frac{1+e^{-i\omega}}{\omega}\right)^k = \prod_{j=1}^{+\infty} m_k(2^{-j}\omega). \tag{16}$$

Note that the coefficients of the subdivision are given by $c_k(n) = 2^{-k+1}\binom{k}{n}$ for $n = 0 \cdots k$. It is thus possible to use the above described subdivision algorithm to generate the B-spline discrete representation in a very fast way.

We shall now generate a continuous parameter representation by interpolating between the functions $m_k(\omega)$. We shall thus define for $r = k + s$, $k \in \mathbb{N} - \{0\}$, $s \in [0, 1]$,

$$m_r(\omega) := \left(\frac{1+se^{-i\omega}}{1+s}\right)\left(\frac{1+e^{-i\omega}}{2}\right)^k = \frac{1}{2}\sum_n c_r(n)e^{-in\omega}. \tag{17}$$

Our continuously parametrized B-spline $B_{0,r}$ will then be defined by

$$\hat{B}_{0,r}(\omega) := \prod_{j=1}^{+\infty} m_r(2^{-j}\omega). \tag{18}$$

These functions are compactly supported in $[0, k+1]$ for $k < r \leq k+1$. Their regularity can be studied by several techniques: many contributions have been made to the problem of estimating the regularity of the limit functions of subdivision schemes, due to the diverse possible definitions of regularity. For the Hölder exponent, a method that leads to an exact estimate is described in Daubechies and Lagarias [12]. Applied to our particular limit functions, this method can be summarized as follows: one defines two infinite matrices $(T_\varepsilon)_{i,j} = c_s(2i - j - \varepsilon + 1)$, $\varepsilon = 0, 1$ and study their action on the stable subspace E of the sequences $\{\cdots, 0, 0, s_1, s_2, 0, 0\cdots\}$. The Hölder exponent of $B_{0,r} = B_{0,k+s}$ is then given by

$$\mu(B_{0,r}) = k - \log_2 \rho(T_0, T_1), \tag{19}$$

where $\rho(T^0, T^1)$ is the "joint spectral radius" of T_0 and T_1, defined by

$$\rho(T^0, T^1) := \text{limsup}_{m\to+\infty} \left(\max_{e_j=0 \text{ or } 1} \|T_{e_1}T_{e_2}\cdots T_{e_m}\|_E^{1/m} \right). \tag{20}$$

In E, the operators T_0 and T_1 are given by the two matrices

$$M_0 = \frac{2}{1+s}\begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix} \text{ and } M_1 = \frac{2}{1+s}\begin{pmatrix} s & 1 \\ 0 & 1 \end{pmatrix}. \tag{21}$$

One checks easily that we have in that case $\rho(T_0, T_1) = \|T_0\| = \frac{2}{1+s}$ and we thus have

$$\mu(B_{0,r}) = k - 1 + \log_2(1+s). \tag{22}$$

This formula shows in particular that the smoothing property (9) is satisfied by this construction. Note also, from equation (17) that the defined continuously parametrized B-spline basis coincides with the classical one for integer scales, i.e., for $s = 0, 1$. Other important properties of classical B-splines, as the normalization (see Section 2) can be showed for the extended basis as well. Figure 6 shows the graphs of $B_{0,r}$ for some values of r and an example of a continuously parametrized B-spline representation. The splines orders are given in the graphs.

5 Concluding Remarks

In this paper, an affine invariant multiscale shape representation was described. The representation is obtained via the computation of B-splines of increasing order, and therefore is affine invariant. The representation was first presented using classical B-splines, which are functions in $\mathbf{C}^{k-2}$, obtaining a scale-space which is discrete in scale. We then extended, based on subdivision schemes, this basis to a continuous-scale ones, that is, finite support functions in $\mathbf{C}^{k-2+r}$, where $r \in [0,1]$. When $r = 0$ or $r = 1$, the basis coincides with the classical B-spline one described here. Using this basis, the affine multiscale representation is extended to a continuous scale one as well.

We showed that the basic properties of continuous scale-spaces hold for this representation, at least in a qualitative form. We presented as well a number of geometric properties related to the smoothing behavior of B-spline representations. The proposed B-spline based multiscale representation is easily implemented using the recursive formula for the B-spline basis computation. In contrast with scale-spaces as the Gaussian one, it is defined directly on an initial discrete set of points, avoiding problems caused by discretization of continuous scale-spaces. The smoothed signal is continuous (and analytical), allowing straightforward computation of geometric properties of the smoothed curve, as curvature (this is a main difference with other discrete scale-spaces as the proposed in [29]). Therefore, the proposed representation is natural for computer shape analysis, since receives as input a digital signal, while keeping a continuous representation which can be helpful for different computations.

The same ideas presented in this paper hold for other basis based representations, as well as other subdivision schemes, which keep the basic properties described in this work. We described the basic approach using the classical B-spline, and its extension given in Section 4, because of its attractive properties, as those given in Section 2, and the existence of extensive analysis, which permits to conclude important geometric properties as the theorems presented in this paper.

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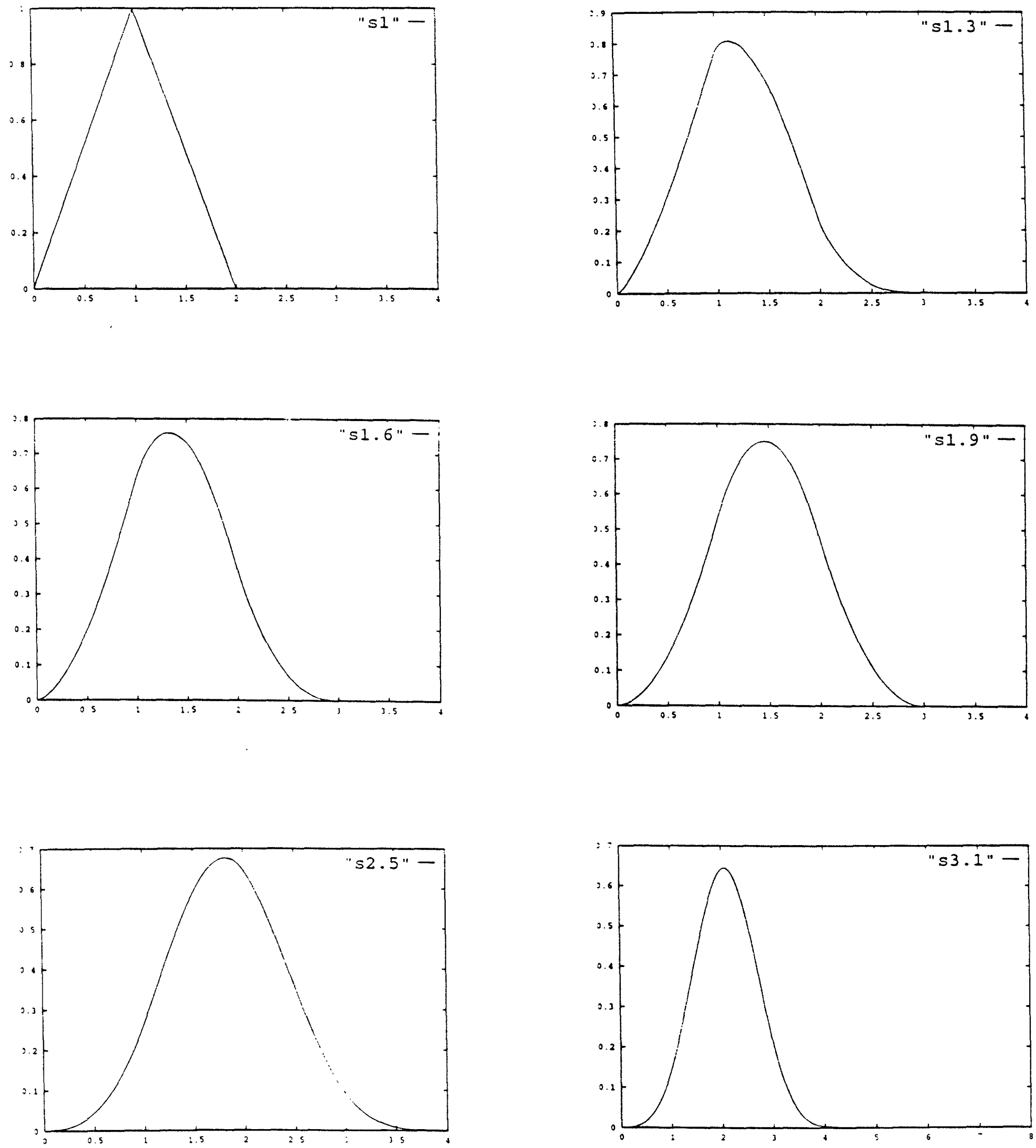
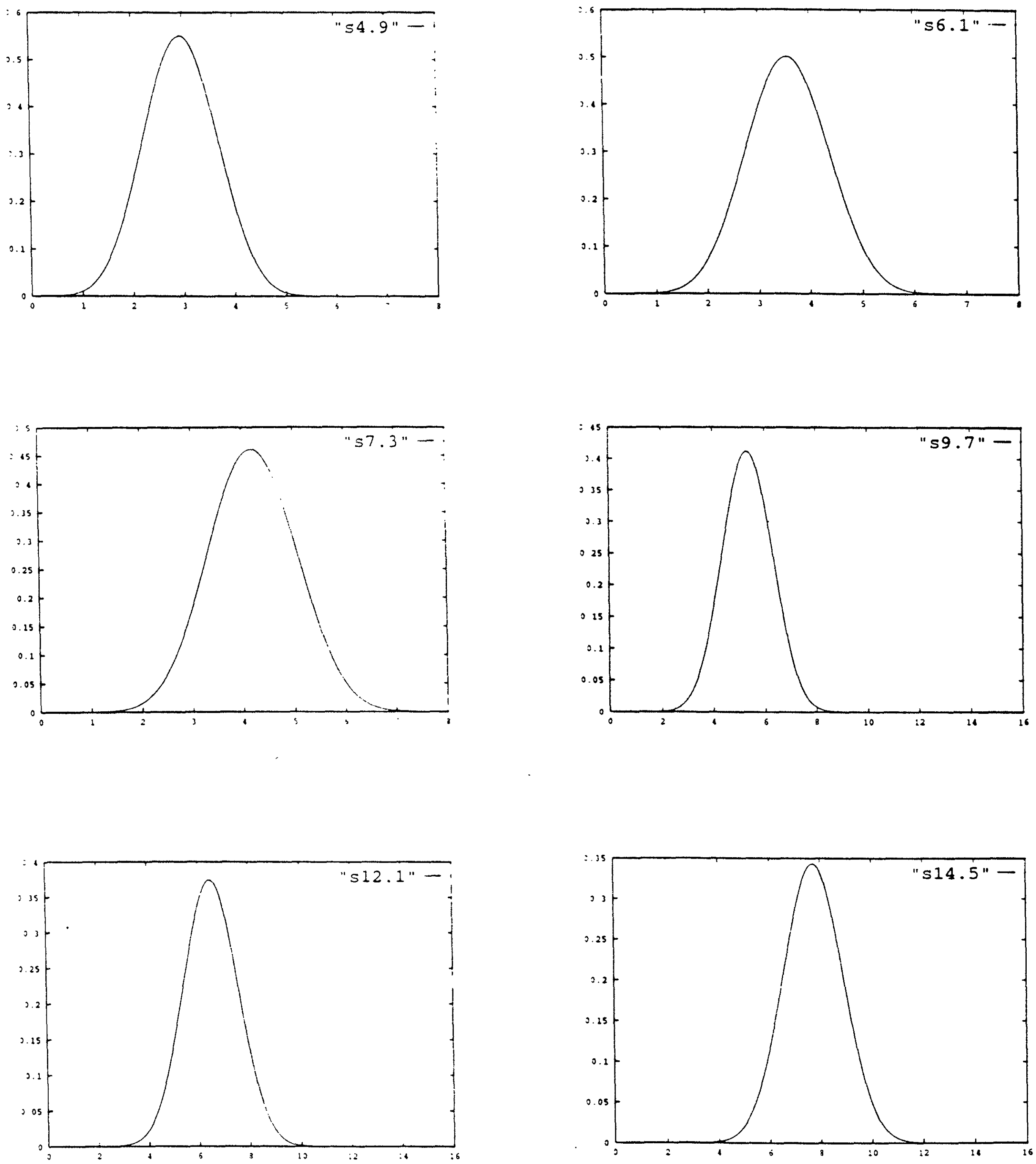


Figure 6. Example of the continuous B-spline basis and representation.



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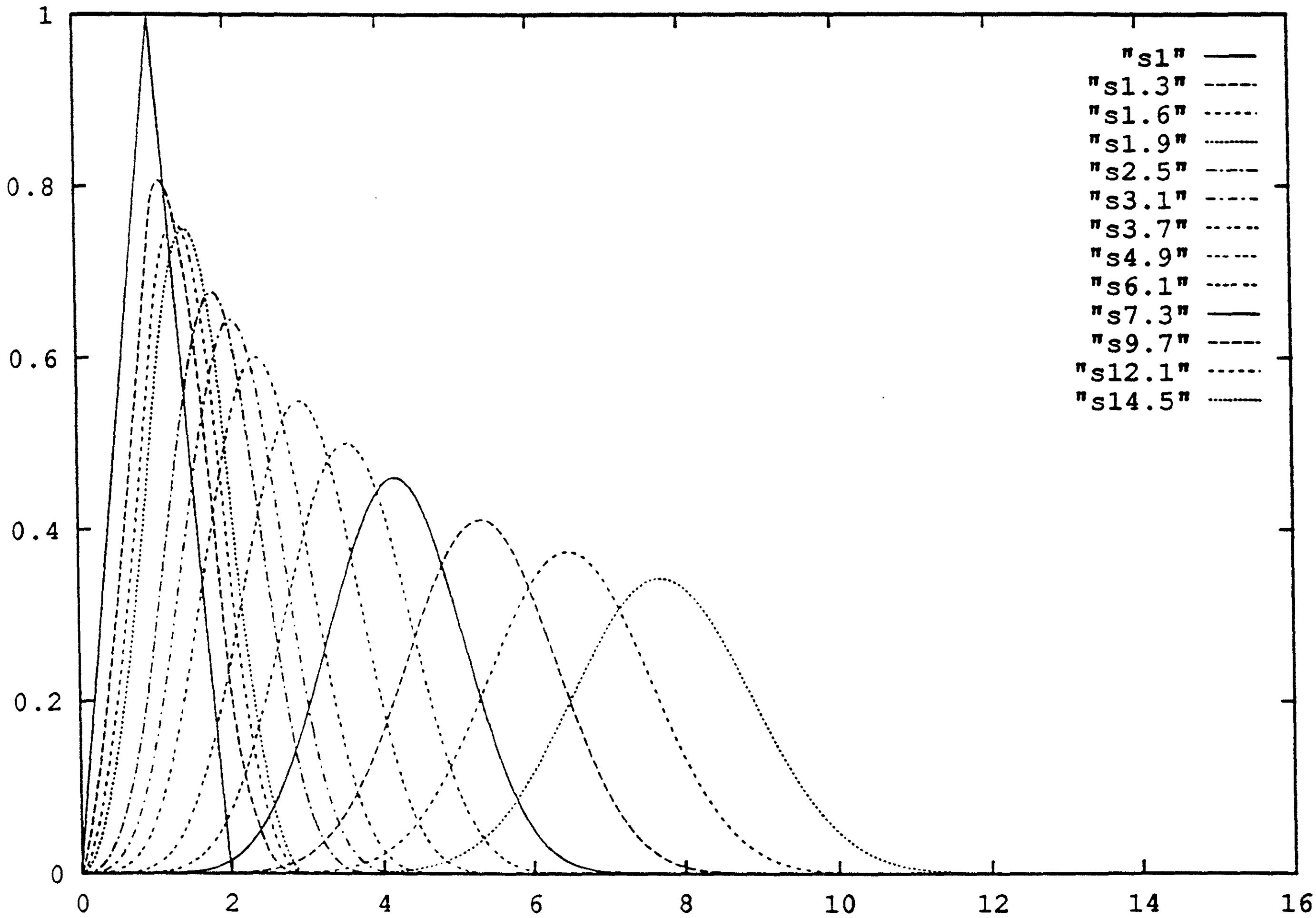

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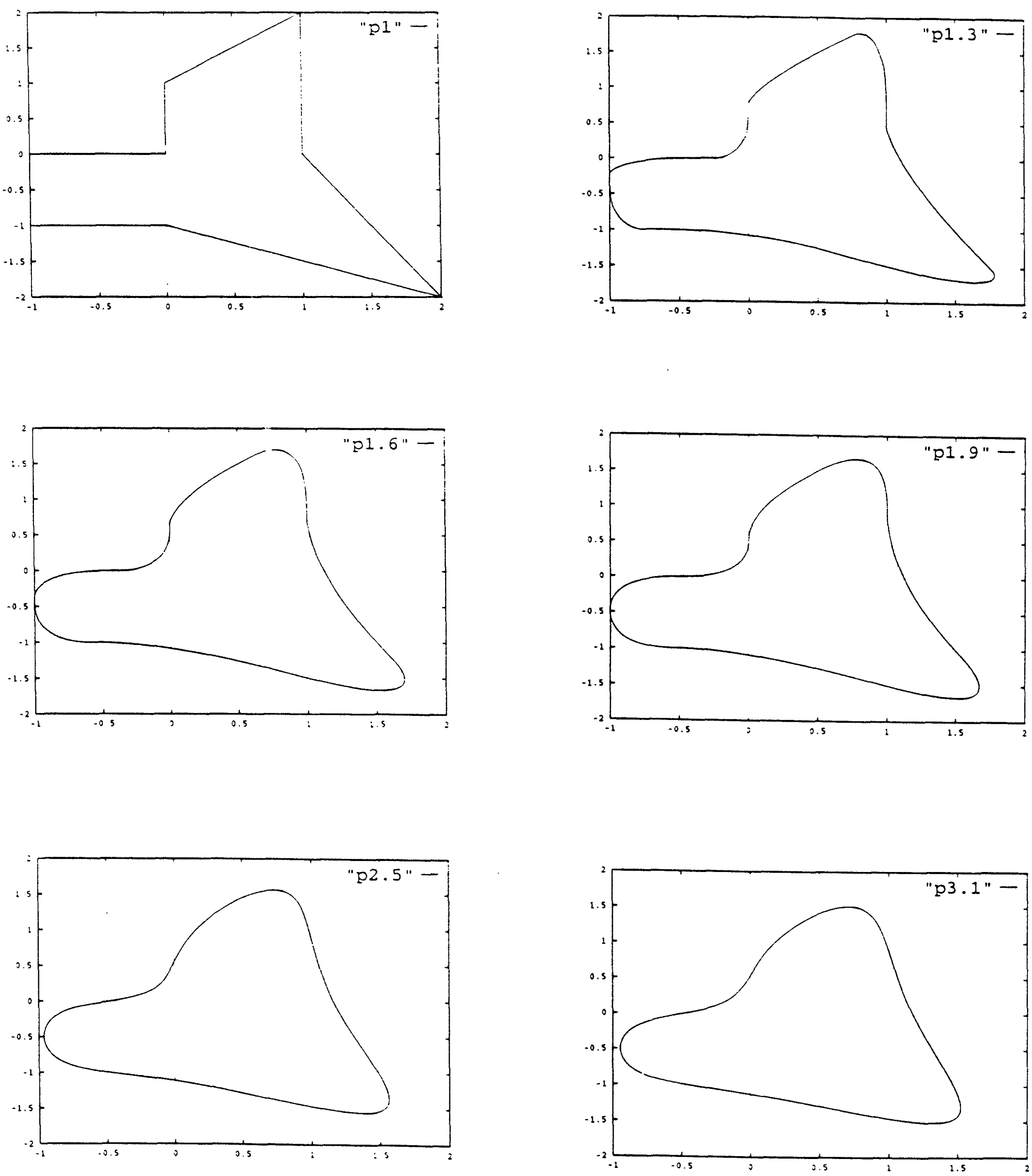

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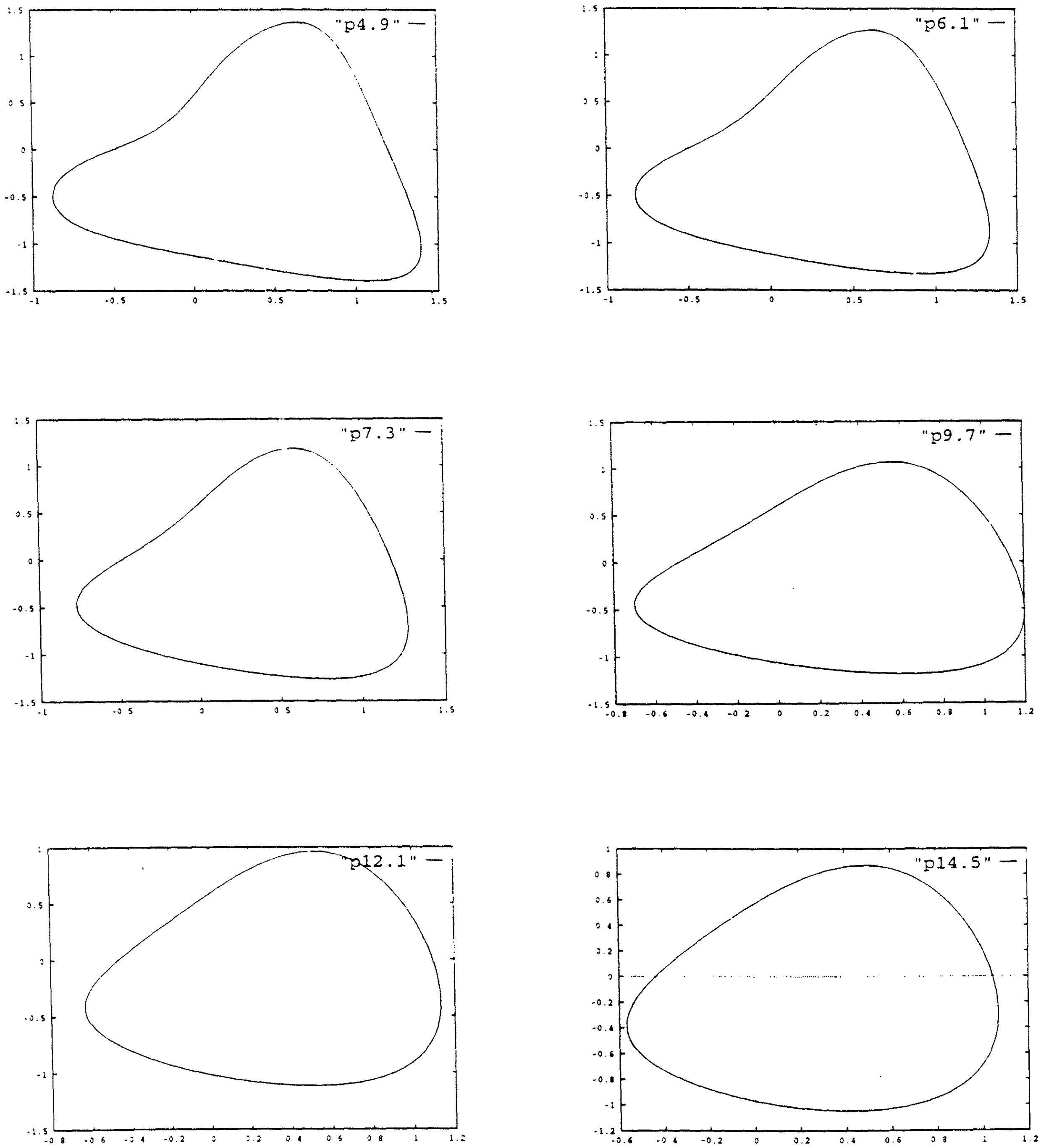

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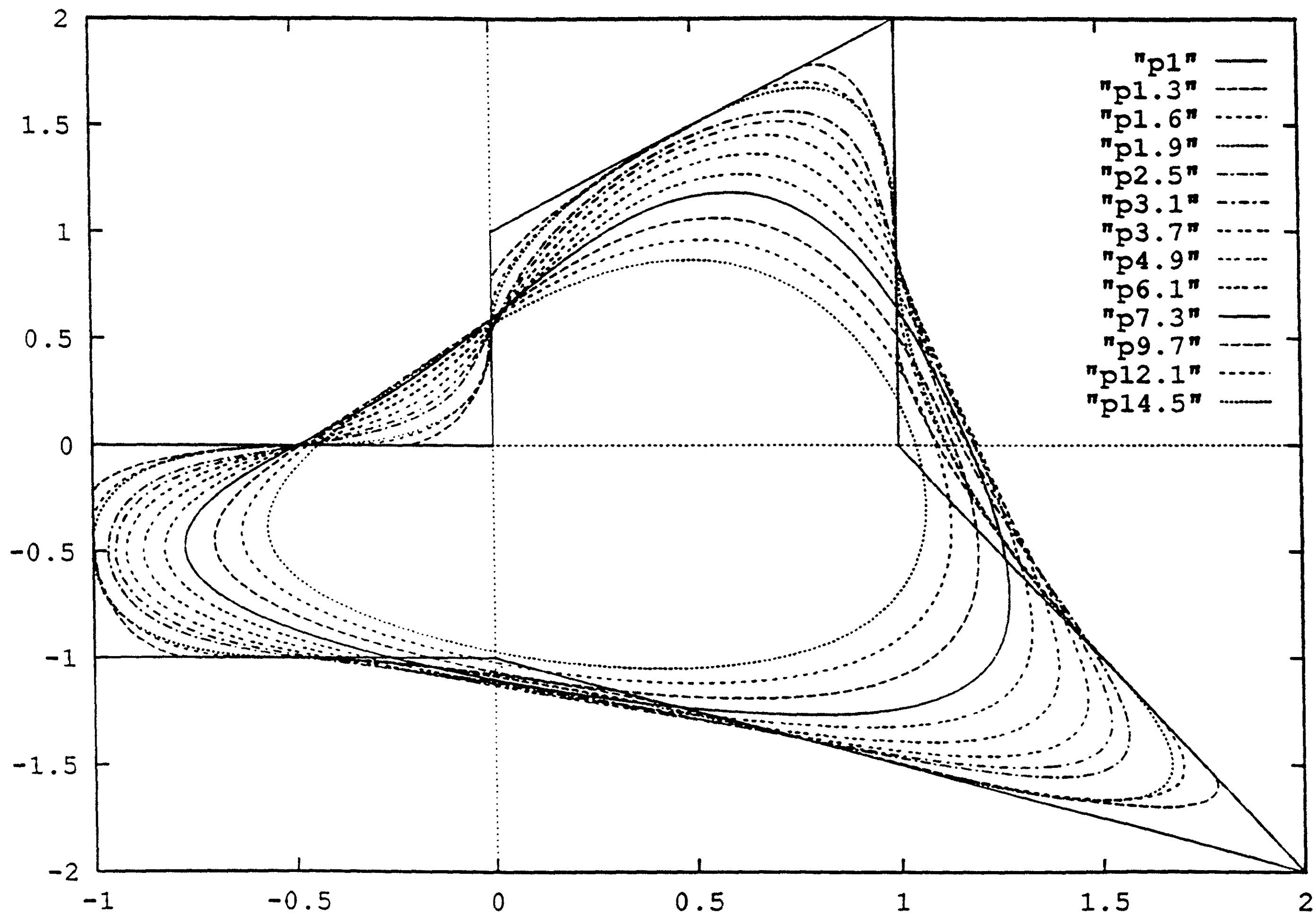

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