Essays on the Macroeconomics of the Labor Markets

by

Veronica Guerrieri

B.A., Economics, Bocconi University (2000)
M.A., Economics, Bocconi University (2001)

Submitted to the Department of Economics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Economics

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2006

Signature of Author ....
Department of Economics
21 May 2006

Certified by ....
Daron Acemoglu
Charles P. Kindleberger Professor of Applied Economics
Thesis Supervisor

Certified by ....
Iván Werning
Assistant Professor of Economics
Thesis Supervisor

Accepted by ....
Peter Temin
Chairman, Departmental Committee on Graduate Studies

Essays on the Macroeconomics of the Labor Markets

by

Veronica Guerrieri

Submitted to the Department of Economics
on 21 May 2006, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy in Economics

Abstract

In Chapter 1, I study the efficiency properties of competitive search equilibria in economies with informational asymmetries. Employers and workers are both risk-neutral and ex-ante homogeneous. I characterize an equilibrium where employers post contracts and workers direct their search towards them. When a match is formed, the disutility of labor is drawn randomly and observed privately by the worker. An employment contract is an incentive-compatible mechanism that satisfies a participation constraint on the worker's side. I first show that in a static setting the competitive search equilibrium is constrained efficient, that is, it cannot be Pareto improved by a Social Planner subject to the same informational and participation constraints faced by the decentralized economy. I then show that in a dynamic setting, on the contrary, the equilibrium can be constrained inefficient. The crucial difference between the static and the dynamic environment is that the worker's outside option is exogenously given in the former, while in the latter it is endogenously determined as the equilibrium continuation utility of unemployed workers. Inefficiency arises because the worker's outside option affects the ex-ante cost of information revelation, generating a novel externality which is not internalized by competitive search.

In Chapter 2, I explore whether match-specific heterogeneity, with or without full information, can amplify the responsiveness of unemployment rate and market tightness to productivity shocks. On the contrary, I show that heterogeneity can dampen the response of market tightness to productivity, once one calibrates the model to match two main facts: the finding rate and the finding rate elasticity to market tightness. First, I show a theoretical result for the steady state analysis in the extreme case of no aggegate shock. Then, I report the calibration exercise for alternative specification of the idiosyncratic shocks distribution.

Chapter 3 is the product of joint work with Daron Acemoglu and constructs a model of non-balanced economic growth. The main economic force is the combination of differences in factor proportions and capital deepening. Capital deepening tends to increase the relative output of the sector with a greater capital share (despite the equilibrium reallocation of capital and labor away from that sector). We first illustrate this force using a general two-sector model. We then investigate it further using a class of models with constant elasticity of substitution between two sectors and Cobb-Douglas production functions in each sector. In this class of models, non-balanced growth is shown to be consistent with an asymptotic equilibrium with constant interest rate and capital share in national income. We investigate whether for realistic parameter values, the model generates transitional dynamics that are consistent with both

the more rapid growth of some sectors in the economy and aggregate balanced growth facts. Finally, we construct and analyze a model of "non-balanced endogenous growth," which extends the main results of the paper to an economy with endogenous and directed technical change. This model shows that non-balanced technological progress will generally be an equilibrium phenomenon.

Thesis Supervisor: Daron Acemoglu
Title: Charles P. Kindleberger Professor of Applied Economics

Thesis Supervisor: Iván Werning
Title: Assistant Professor of Economics

Acknowledgments

I am deeply indebted to Daron Acemoglu for his constant guidance, support and advice. His enthusiastic passion for research and for understanding the world have been an invaluable example throughout my Ph.D. experience.

I cannot overstate my debt to Iván Werning, for his precious help and insightful comments. His mentoring greatly improved my work and this thesis.

I owe special thanks to Guido Lorenzoni, for his encouragement during all the stages of this thesis, his insights and priceless suggestions. His genuine curiosity has fueled my love for economics.

I want to thank sincerely Mike Golosov for our endless discussions. He taught me what is Minnesota economics and helped me become a stronger economist.

Many enjoyable conversations with Marios Angeletos, Olivier Blanchard, Ricardo Caballero, Jordi Gali have been of great help for my research. I whish to thank Robert Shimer who inspired my interest in labor markets and has been very helpful in numerous discussions. I am also grateful to him together with Richard Rogerson and Randall Wright for inviting me to present at the NBER Macro Annual the first chapter of this thesis, when it was at a very preliminary stage. I benefited significantly from that presentation and from all the comments of the participants. In particular, I thank Robert Hall and Christopher Pissarides together with the organizers. I am thankful to Aleh Tsyvinski for inviting me to present in his reading group at Harvard, giving me the opportunity of getting the precious comments of himself, Manuel Amador and Pol Antras. I want to thank all the participants to my seminars at MIT, Chicago GSB, Columbia, Cleveland Fed, CREI, IIES, Kellogg MEDS, NYU Stern, NY Fed, Northwestern and Stanford. A particular thank to Ariel Burstein who helped me with the calibrations of the third chapter of the thesis.

I am also very grateful to Francesco Giavazzi and Laura Bottazzi who advised me during my undergraduate studies, who first spurred my interest in economics, and who first encouraged me to pursue graduate studies.

Many other of my friends have contributed to this work. In particular, my officemates David Abrams, Jessica Cohen, Emmanuel Farhi and William Hawkins. My unique friends Gaia and Giusy have supported me in all these years from the other side of the ocean, together with Mattija and Paolo. Alberto, Angelo and Lorenzo have been my family in Cambridge for different periods of my experience.

This work would not have been possible without the support and love of my parents, my aunt and Ilio, my friend Gieky and the memory of my grandmother. I dedicate this thesis to my loved parents.

Alla mia mamma e al mio babbo

Chapter 1

Efficiency of Competitive Search under Asymmetric Information

1.1 Introduction

The extent to which decentralized labor markets achieve efficiency is a central economic question. In labor markets, trade occurs bilaterally and is typically costly. Firms need to post vacancies and workers must spend time searching for jobs. Moreover, employment contracts are commonly characterized by informational imperfections. The surplus produced by a worker-firm match may depend on idiosyncratic features that are private information of the contracting parties. Costly trade and informational imperfections impose a departure from the Walrasian paradigm,[1] but not necessarily from the property of efficiency. Given these frictions, can the price mechanism still achieve a socially optimal allocation of resources?

Search theory typically models labor market imperfections under the heading of *matching frictions,* by assuming an aggregate matching technology. These frictions are meant to capture the general idea that trade is time-consuming and costly both in terms of coordination and of informational incompleteness. The conventional model, built on Diamond (1982), Mortensen (1982a, 1982b) and Pissarides (1984, 1985), combines random matching with a wage determina-

[1] *In his AEA presidential address, Friedman (1968) highlights how, in a Walrasian world, a market economy cannot be kept away from the unemployment level that "...would be ground out by the Walrasian system of general equilibrium equations, provided there is imbedded in them the actual structural characteristics of the labor and commodity markets..."*

tion process based on Nash bargaining. In this context, the equilibrium level of unemployment is generically inefficient. Decentralized markets do not internalize the search externality generated by the matching frictions[2]. However, going back to the Walrasian spirit, a new generation of search models, Shimer (1996), Moen (1997) and Acemoglu and Shimer (1999a), introduces a novel notion of competition in environments with trading frictions, referred to as *competitive search*. In competitive search models firms post wages and workers direct their search towards them. In this environment, decentralized markets internalize the search externality and the resulting equilibrium is efficient. The efficiency property of competitive search has been proven robust in several contexts and sheds light on the power of the price mechanism to induce firms to open the optimal quantity of vacancies.

In this paper, I propose a search model where informational frictions are modeled explicitly. Informational asymmetries seem to be a crucial element of employment relationships. Here, the problem for firms is not only to meet workers, but also to find out the profitability of their match. Employment contracts are designed optimally in order to extract this information and to induce workers to participate in the productive relationship. In this context, a new type of externality arises. The ability of a firm to extract information depends on the worker's outside option. The outside option, in turn, is determined by the contracts offered by other firms in the future. Because of this externality the equilibrium can fail to be constrained efficient. My model retains the Walrasian spirit of competitive search, by allowing firms to post contracts so as to attract workers. Moreover, I allow for general employment contracts. Therefore, the inefficiency does not depend either on the presence of search externalities or on restrictions on the contract space.

I construct a tractable framework for investigating the role of informational asymmetry in search environments. Employers and workers are both risk-neutral and *ex-ante* homogeneous. Employers post contracts and workers direct their search towards them. When a match is formed, the disutility of labor is drawn randomly and observed privately by the worker. An employment contract is an incentive compatible mechanism that satisfies a participation constraint on the worker's side. The participation constraint can be interpreted as the result of lack of commitment. A worker cannot be forced to work, he can always quit and join the ranks

[2]See Hosios (1990).

of the unemployed.[3]

I begin by characterizing the competitive search equilibrium. I show that whenever the *ex-ante* cost of posting a vacancy is positive, *ex-post* inefficiency emerges. In particular, some matches that would produce a positive net surplus are not created. The key source of allocative distortion is that the optimal contract has to offer the same wage to all the types that are hired. This comes straight from incentive compatibility. Moreover, the participation constraint implies that the wage has to be equal to the disutility of the marginal worker. To induce the first-best level of job creation, the wage should be set equal to the firm productivity, driving the firm's profits to zero. This is inconsistent with an equilibrium where firms pay a positive vacancy cost *ex-ante*. A trade-off emerges between the two margins of job creation: efficient creation at the hiring stage has to be sacrificed in order to induce vacancy creation *ex-ante*.

Then, I address my central question: is the competitive search equilibrium *constrained efficient*? I define a social planner who faces *the same frictions* of the competitive economy. The social planner controls the matching process by deciding how many vacancies to post at the beginning of each period and allocates consumption among employed and unemployed workers. He does not observe the match-specific disutility of the workers and has to induce them to reveal it. Moreover, he is subject to the same participation constraint on the workers' side. Workers can always quit and enjoy private utility from leisure, which cannot be transferred. Moreover, workers who quit cannot be distinguished from all the other unemployed workers.

First, I show that in a static setting the competitive search equilibrium is constrained efficient. As in the perfect information benchmark, competition among firms induces them to design contracts in order to attract workers' job applications. This implies that vacancy creation and labor contracts are set to maximize workers' utility, subject to a zero profit condition. This ensures that firms correctly internalize the search externality.

By contrast, I show that in a dynamic setting the competitive search equilibrium is constrained inefficient. The crucial difference between the static and the dynamic environments is that the worker's outside option is exogenously given in the former, while in the latter it is endogenously determined as the continuation utility of unemployed workers. When informational asymmetry is present and workers must be induced to participate, the workers' outside option

[3]This corresponds to the typical *at will* employment contracts enforced in the United States.

affects the *ex-ante* cost of information revelation. This generates a novel externality which is not internalized by dynamic competitive search. Firms who post contracts at time $t+1$ do not take into account the informational cost they impose on contracts designed by other firms at time t, by affecting the workers' outside option. This externality can be the source of constrained inefficiency. The social planner takes into account the impact that the continuation utility of unemployed workers has on current contracts, and can improve upon the equilibrium allocation.

The main result of the paper is that, under asymmetric information and the workers' participation constraint, the competitive search equilibrium is constrained inefficient whenever the economy is away from the steady state. Imagine that the social planner decreases the continuation utility for unemployed workers. An intertemporal trade-off emerges, that is not taken into account by the competitive equilibrium. On one hand, the outside option for a worker who meets a firm today decreases, making it easier for the firm to extract information and increasing job creation today. On the other hand, the social planner has promised to give less utility to the unemployed workers from tomorrow onward. This means that workers will receive smaller informational rents in the future, reducing job creation tomorrow.

When the economy is at the steady state level, the flow of workers out of unemployment, who enjoy the informational gain from a reduction of the outside option, is perfectly offset by the flow of workers into unemployment, who are damaged by a future lower expected utility. When the economy is away from the steady state, the inefficiency depends on the equilibrium dynamics of the unemployment rate. I characterize the direction of the inefficiency and show that it depends on whether the initial unemployment rate is above or below the steady state level. Consider a competitive equilibrium. If the initial unemployment rate is above the steady state level, this means that the mass of potential matches is higher today relative to tomorrow. Hence, the shadow cost of informational extraction is relatively high today and the social planner would like to reduce the continuation utility of unemployed workers in order to achieve higher job creation today. On the contrary, when the unemployment rate is rising, the planner would like to increase the continuation utility of unemployed workers in order to increase job creation tomorrow. The planner can indeed manipulate the continuation utilities by changing the future choices of vacancy creation and hiring margins.

Finally, I explore an alternative environment in which unemployed workers own a transferable endowment which can be seized by the social planner. I show that if these resources are high enough, the social planner can use them to finance the informational rents of employed workers,[4] restoring the full information allocation.[5] Moreover, when these resources are even higher, the full information allocation can be decentralized also by bond posting in private contracts.

Related Literature. My work is related to a vast literature on search theoretic models of the labor market, surveyed by Rogerson, Shimer and Wright (2005). The conventional model builds on Diamond (1982), Mortensen (1982a, 1982b), Pissarides (1984, 1985) and Mortensen and Pissarides (1994).[6] By combining random matching and Nash bargaining, it does not generically achieve efficiency, as shown in Hosios (1990). Departing from this benchmark, more recently, Shimer (1996), Moen (1997), Acemoglu and Shimer (1999a), introduce the equilibrium notion of competitive equilibrium, which combines directed search and wage posting, internalizing the search externality. A series of papers highlights the robustness of the efficiency properties of competitive search.[7] In my model, I use competitive search as equilibrium concept, with the explicit purpose of eliminating the inefficiency coming from the standard search externality and highlighting the novel externality generated by informational frictions.

My work is also related to a growing literature on asymmetric information in search environments. In particular, Shimer and Wright (2004) and Moen and Rosen (2005) analyze, as in my model, labor markets where trading frictions interact with asymmetric information, using competitive search. However, they both explore a static environment, where, as I will show, even in the presence of informational frictions, competitive search keeps its efficiency property. Shimer and Wright (2004) analyze an economy where the employer has some private

[4] An example of feasible policy that I explore in the static analysis is one of subsidizing job creation by taxing lump-sum all the workers.

[5] What is needed to restore the full information allocation is that the economy is able to appropriate enough resources to cover the informational costs without distorting the allocation. In this alternative environment those resources come from home production of unemployed workers. An economy with access to enough external resources could achieve the same outcome.

[6] See Pissarides (2000) for a general treatment.

[7] For example, Acemoglu and Shimer (1999b) show that competitive search is efficient even with ex-ante investments, Mortensen and Wright (2002) generalize results on price determination and show how competitive search achieves efficiency by exploiting all gains from trade. Hawkins (2005) shows that even when a firm can hire more workers, competitive search is efficient when firms post contracts that are general enough.

information about the match and the worker a private effort choice. They show that under mild regularity assumptions, in their environment, contracts take a simple form with at most two wages. In the same spirit, Moen and Rosen (2005) study a competitive search equilibrium with private information on the workers' side. They focus mainly on the impact of asymmetric information on the responsiveness of the unemployment rate to productivity shocks. Moreover, they show that cross subsidization between workers and firms can restore the full information allocation. This result is similar to the one I derive for the case of trasferable endowment.[8]

Another related paper is Faig and Jerez (2004) who propose a theory of commerce, where buyers have private information about their willingness to pay for a product. They also show that the static model is constrained efficient, if the social planner cannot transfer utility across agents. However, they point out that another source of inefficiency can be the non-linearity of the production function. They calibrate a dynamic version of their model embodied in a neoclassical framework where the existence of capital induces a non-linear production function. They show that the welfare losses of competitive search are negligible. In a similar spirit, Wolinsky (2005) analyzes the efficiency properties of a sequential procurement model where a small buyer cannot commit to a mechanism and finds inefficient equilibria. However, in his model the inefficiency arises because of contracting restrictions. The fact that the seller's effort is not contractible distorts the buyer's search intensity. In my paper, private contracts are unrestricted and the equilibrium inefficiency comes from a general equilibrium effect.

My work is also indirectly related to a growing literature more focused on the impact of asymmetric information on the business cycle. Reacting to Shimer (2005) and Hall (2005), they have proceeded in the direction of some form of wage rigidity in order to match the business cycle unemployment volatility.[9]

Finally, from a methodological standpoint my paper is related to the literature on mechanism design with asymmetric information, e.g. Mirrlees (1971), Myerson (1981), Myerson and Satterthwaite (1981), Laffont and Maskin (1980).

[8] See subsection 3.2.

[9] Among others, Kennan (2004) constructs a model with hidden information and bargaining, Menzio (2004) assume employers have private information about productivity and contracts are non-binding, Nagypal (2004) combines workers' heterogeneity, asymmetric information and on-the-job search and Hall and Milgrom (2005) construct an equilibrium characterized by Nash-type bargaining where the threat points are the payoffs of endless delay. None of these papers explore the welfare properties of the models.

The paper is organized as follows. Section 1.2 introduces the static environment of the economy, defines and characterizes the competitive search equilibrium. Section 1.3 analyzes the efficiency properties of the static economy both for the benchmark environment and for alternative settings. Section 1.4 describes the dynamic environment, defines and characterizes the dynamic competitive search equilibrium. Section 1.5 describes the dynamic welfare properties of the model and derives the main result that competitive search, away from the steady state, is constrained inefficient. Finally, Section 2.5 concludes.

1.2 Static Economy

The crucial ingredient of my model is the interaction of informational asymmetry and trading frictions, when there is a participation constraint on the worker's side. This section introduces the static version of a decentralized economy highlighting this interaction. I define and characterize the competitive search equilibrium for this economy.

Environment. The economy is populated by a continuum of measure 1 of workers and a large continuum of potential employers. Both workers and employers are risk-neutral and *ex-ante* homogeneous. Workers can search freely, while employers need to pay an entry cost k to post a vacancy. Each worker wants to match an employer and each employer with an open vacancy wants to match only one worker. When a match is formed, the disutility of labor θ is drawn randomly from the cumulative distribution function $F(.)$, with support $\Theta \equiv [\underline{\theta}, \overline{\theta}]$, and is observed privately by the worker.[10] I assume that the cumulative distribution function $F(.)$ is differentiable, with $f(.)$ being the associated density function, and that it satisfies a monotone hazard rate condition, that is, $d[F(\theta)/f(\theta)]/d\theta > 0$. The net surplus of the match is given by $y - \theta$, where y represents the amount of output generated by a productive match. The value of y is common to all the matches and is exogenously given in the benchmark model.[11]

At the beginning of the period employers can open a vacancy at a cost k which entitles them to post an employment contract $C \in \mathbb{C}$, where $\mathbb{C}$ is the set of *ad interim* incentive compatible

[10]The value θ can also be interpreted as the cost of effort that the worker has to exert to make the match productive, which depends on the specificity of the match.

[11]When the asymmetric information is on the side of the employer, the analysis is similar. The equilibrium allocation still exhibits less trade than in the full information case. However, now firms appropriate ex-post the informational rents required by incentive compatibility.

and individually rational mechanisms. As I describe below, a contract $\mathcal{C} : \Theta \mapsto [0,1] \times \mathbb{R}_+$ specifies the hiring probability and the wage for each matched worker who reports type θ. Therefore the strategy of a firm is a pair $(\sigma, \mathcal{C}) \in \{0,1\} \times \mathbb{C}$ where σ denotes the decision of posting a vacancy and $\mathcal{C}$ is the posted contract. Next, each worker observes all the contracts posted and decides where to apply. He chooses a contract $\mathcal{C} \in \mathbb{C}^P \subset \mathbb{C}$, where $\mathbb{C}^P$ denotes the set of contracts posted by active firms. After workers start to search for a specific contract, matching takes place and for each match the draw θ is realized and is private information of the worker. The worker's behavior is described by a map $(a, s) : \Theta \mapsto \Theta \times \{0,1\}$ that for each type θ specifies a report $\hat{\theta} = s(\theta)$ and a participation decision $a(\theta)$. The worker can either implement the contract, that is choose $a(\theta) = 1$, or walk away, that is choose $a(\theta) = 0$. If he walks away he gets b, a flow of non-transferable utility from leisure. In section 3.2, I will extend the analysis to the case where b is transferable and can be interpreted as home production or unemployment benefit.

Trading frictions in the labor market are modeled through random matching and can be thought of as coordination frictions, as in Burdett, Shi and Wright (2001). Employers and workers know that their matching probabilities will depend on the contract that they respectively post and seek for. Each type of contract $\mathcal{C}$ is associated with a labor submarket, where a mass $v(\mathcal{C})$ of employers posts contracts of type $\mathcal{C}$ and a mass $u(\mathcal{C})$ of unemployed workers applies for jobs at firms offering that type of contract. I assume that each submarket is characterized by a constant returns to scale matching function $m(v(\mathcal{C}), u(\mathcal{C}))$ and by an associated "tightness" $\gamma(\mathcal{C}) = v(\mathcal{C})/u(\mathcal{C})$.[12] Hence, for each contract $\mathcal{C}$, I can define the function $\mu(\gamma) \equiv m(\gamma, 1)$, which represents the probability of a worker applying for $\mathcal{C}$ meeting an employer posting it. On the other hand, the probability of a firm posting $\mathcal{C}$ meeting a worker applying for it is represented by the non-increasing function $\mu(\gamma)/\gamma$.

Assumption A1. The function $\mu(\gamma) : [0, \infty) \longmapsto [0,1]$ satisfies the following conditions:

(i) $\mu(\gamma) \leq \min\{\gamma, 1\}$;[13]

(ii) for any γ such that $\mu(\gamma) < \min\{\gamma, 1\}$, $\mu(\gamma)$ is twice differentiable with $\mu'(\gamma) > 0$ and

[12] In order to simplify the notation, from now on I am going to drop the dependence of u, v and γ on the contract $\mathcal{C}$, whenever it does not cause any confusion.

[13] With discrete time, this condition ensures that both $\mu(\gamma)$ and $\mu(\gamma)/\gamma$ are proper probabilities.

$\mu''(\gamma) < 0.$

This assumption allows me to consider matching functions that either are everywhere differentiable or have one or two kinks. The standard matching functions considered in the literature are covered by one of these two classes. The first category includes the exponential case, while the properly *modified*[14] linear and Cobb Douglas case falls into the second one.

In a decentralized economy the consumption of employed workers is given by the wage. Moreover, the consumption of unemployed workers must be equal to the value of leisure b, where unemployed workers are both unmatched workers and workers who have been matched but have not been hired. Assume that $y > b + \underline{\theta}$ in order to make the problem interesting.[15]

Employment Contracts. Without loss of generality, by invoking the Revelation Principle, I can restrict attention to direct revelation mechanisms, corresponding to a mapping $\mathcal{C} : \Theta \mapsto [0,1] \times \mathbb{R}_{+}$, specifying for each matched worker who reports type θ, the hiring probability $e(\theta) \in [0,1]$ and the wage $\omega(\theta) \in \mathbb{R}_{+}$. The contract must be incentive compatible and individually rational, that is, it has to ensure that the worker reveals truthfully his type and chooses to participate in the employment relationship after the draw has been realized. Individual rationality can be interpreted as a "no-commitment" assumption on the side of the worker. The no-commitment constraint on the worker's side can represent the typical *at will* employment contracts widespread in the United States. Instead, firms can fully commit to the posted contract.

Let $v(\theta, \hat{\theta})$ denote the *ad interim utility* for worker of type θ revealing $\hat{\theta}$, associated with a contract $\mathcal{C}$,[16] with

$$v(\theta, \hat{\theta}) \equiv \omega(\hat{\theta}) - e(\hat{\theta})\theta + [1 - e(\hat{\theta})]b. \tag{1.1}$$

An employment contract is *incentive-compatible* whenever it satisfies

$$v(\theta, \theta) \geq v(\theta, \hat{\theta}) \text{ for all } \theta, \hat{\theta} \in \Theta \tag{IC}$$

[14] *From now on I define the modified version of a function $\hat{\mu}(\gamma)$, the function $\mu(\gamma) = \min\{\hat{\mu}(\gamma), \gamma, 1\}$.*

[15] Notice that if $y < b + \underline{\theta}$ then even with full information the equilibrium would be characterized by zero trade.

[16] In order to simplify the notation, I am going to drop the dependence of $v(.,.)$ on the contract $\mathcal{C}$, since it does not cause any confusion.

and *individually rational* whenever

$$v(\theta, \theta) \geq b \text{ for all } \theta \in \Theta. \tag{IR}$$

I define $\mathbb{C}$ the set of incentive compatible and individually rational direct mechanisms.

Following a standard result in the mechanism design literature,[17] I can reduce the dimensionality of the constraints. In particular, I can state the following lemma.

Lemma 1 *Conditions IC and IR are equivalent to* $e(.)$ *non-increasing and*

$$v(\theta, \theta) = v(\overline{\theta}, \overline{\theta}) + \int_{\theta}^{\overline{\theta}} e(y)\, dy \textit{ for all } \theta \in \Theta, \tag{IC'}$$

$$v(\overline{\theta}, \overline{\theta}) \geq b. \tag{IR'}$$

Proof. The proof that IC is equivalent to IC' and $e(\theta)$ non-increasing is standard and hence omitted.[18] Then, using the monotonicity of $e(\theta)$, the individual rationality constraints IR can be reduced to the one for the worst type $\overline{\theta}$, completing the proof. ∎

This Lemma allows me to separate the problem of finding an optimal allocation from the problem of finding a wage schedule that implements it.

Define $v(\theta, \theta) - v(\overline{\theta}, \overline{\theta})$ as the *informational rent* of a worker of type $\theta \leq \overline{\theta}$, that is, the additional utility that such a worker must receive in order to reveal his own type. Condition IC' ensures that no worker would gain by pretending to have a higher disutility from working than the realized one. Moreover, condition IR' ensures that the worse type does not expect an utility level lower than the one he could get by staying in autarky.

Finally, the large number of potential firms ensures free entry, imposing that the value of an open vacancy must be zero in equilibrium, that is,

$$\frac{\mu(\gamma)}{\gamma} \int_{\underline{\theta}}^{\overline{\theta}} [e(\theta)\, y - \omega(\theta)]\, dF(\theta) = k. \tag{1.2}$$

[17] Among others, Mirlees (1971), Myerson (1981), Myerson and Satterthwaite (1981), Laffont and Maskin (1980).

[18] See Mas-Colell, Winston and Green (1995), Proposition 23.D.2, p. 888.

1.2.1 Static Competitive Search Equilibrium

I now define the concept of competitive search equilibrium in this economy, I prove that it always exists, is unique and I show how to characterize it.

Definition 2 *A static symmetric Competitive Search Equilibrium (CSE) is a set of incentive-compatible and individually rational contracts* $\mathbb{C}^*$ *together with a function* $\Gamma^* : \mathbb{C} \longmapsto \mathbb{R}_+ \cup \infty$ *and a utility level* $U^* \in \mathbb{R}_+$ *satisfying*

(i) employers' ***profit maximization*** *and* ***free-entry****:* $\forall C \equiv [e(\theta), \omega(\theta)]_{\theta \in \Theta}$,

$$\frac{\mu(\Gamma^*(C))}{\Gamma^*(C)} \int_{\underline{\theta}}^{\overline{\theta}} [e(\theta) y - \omega(\theta)] \, dF(\theta) - k \leq 0$$

subject to incentive compatibility IC and individual rationality IR, with equality if $C \in \mathbb{C}^*$*;*

(ii) workers' ***optimal job application****:* $\forall C \equiv [e(\theta), \omega(\theta)]_{\theta \in \Theta}$,

$$U^* \geq \mu(\Gamma^*(C)) \int_{\underline{\theta}}^{\overline{\theta}} [\omega(\theta) - e(\theta)(\theta + b)] \, dF(\theta) + b$$

and $\Gamma^*(C) \geq 0$ *with complementarity slackness, where* U^* *is given by*

$$U^* = \max_{C'} \mu\left(\Gamma^*\left(C'\right)\right) \int_{\underline{\theta}}^{\overline{\theta}} \left[\omega'(\theta) - e'(\theta)(\theta + b)\right] dF(\theta) + b$$

or $U^* = b$ *if* $\mathbb{C}^*$ *is empty.*

In equilibrium, both firms and workers know which market tightness is associated with each contract, that is, they know the function $\Gamma^*(C)$. Given that, profit maximization ensures that firms post the incentive compatible and individually rational contract that maximizes their profits, anticipating the tightness associated even to contracts not offered in equilibrium. This ensures that there are no profitable deviations for the firm and free entry drives profits to zero. Moreover, optimal job application ensures that workers choose which type of contracts to look for, so as to maximize their *ex-ante* utility. In particular, notice that the tightness associated with contracts that are not optimal is zero, since firms will never post those contracts anticipating that they will not be able to attract workers.

It follows that the equilibrium unemployment rate of workers applying to firms posting a contract of type $\mathcal{C}$ is given by

$$u(\mathcal{C}) = 1 - \mu(\Gamma^*(\mathcal{C})) \int_{\underline{\theta}}^{\overline{\theta}} e(\theta)\, dF(\theta)$$

and is affected by both matching and informational frictions. In fact job creation depends not only on the equilibrium matching probability, through $\mu(\Gamma^*(\mathcal{C}))$, but also on the equilibrium hiring decision, once the match is realized, through $\int_{\underline{\theta}}^{\overline{\theta}} e(\theta)\, dF(\theta)$.

Generalizing the standard result in the search literature,[19] I can show that the symmetric competitive search equilibrium is such that the utility of an unemployed worker is maximized subject to the zero profit condition for the employer, the incentive and the participation constraint for the worker.

Proposition 3 *If $\{\mathbb{C}^*, \Gamma^*, U^*\}$ is an equilibrium, then any $\mathcal{C}^* \in \mathbb{C}^*$ and $\gamma^* = \Gamma^*(\mathcal{C}^*)$ solves*

$$U = \max_{e(\theta), \omega(\theta), \gamma} \mu(\gamma) \int_{\underline{\theta}}^{\overline{\theta}} [\omega(\theta) - e(\theta)(\theta + b)]\, dF(\theta) + b \tag{P1}$$

subject to $e(\theta) \in [0,1]$, non-negative consumption, the incentive constraint IC', together with the monotonicity of $e(.)$, the participation constraint IR' and the free-entry condition (1.2). Conversely, if a pair $\{\mathcal{C}^*, \gamma^*\}$ solves the program P1, then there exists an equilibrium $\{\mathbb{C}^*, \Gamma^*, U^*\}$ such that $\mathcal{C}^* \in \mathbb{C}^*$ and $\gamma^* = \Gamma^*(\mathcal{C}^*)$.

Proof. See Appendix. ■

Proposition 3 shows how a CSE must solve Problem P1. The next Proposition shows how it can be equivalently described by a tightness γ and a hiring function $e(\theta)$ solving a simplified program P2 and by an associated wage function $\omega(\theta)$ which can be constructed such that the incentive and the participation constraints are satisfied.

[19] Moen (1997), Acemoglu and Shimer(1999a) analyze a competitive search equilibirum when information is complete. Shimer and Wright (2004) define a competitive search equilibrium with bilateral asymmetric information.

Proposition 4 *Any function* $[e(\theta)]_{\theta\in\Theta}$ *and* γ *which solve Problem P1 solves also*

$$U = \max_{e(.),\gamma} \mu(\gamma) \int_{\underline{\theta}}^{\overline{\theta}} e(\theta)[y-\theta-b]\, dF(\theta) + b - \gamma k \tag{P2}$$

s.t.

$$\mu(\gamma) \int_{\underline{\theta}}^{\overline{\theta}} e(\theta) \left[y-\theta-b-\frac{F(\theta)}{f(\theta)}\right] dF(\theta) \geq \gamma k \tag{1.3}$$

and $e(.)$ *non-increasing.*

Furthermore, for any function $[e(\theta)]_{\theta\in\Theta}$ *and* γ *solving problem P2, there exists a function* $[\omega(\theta)]_{\theta\in\Theta}$ *such that the contract* $C = [e(\theta), \omega(\theta)]_{\theta\in\Theta}$ *and* γ *solve problem P1.*

Proof. See Appendix. ■

Free-entry implies that the entire surplus of the economy accrues to workers. Hence, the competitive search equilibrium maximizes the net surplus of the economy, subject to the constraint that the net output must cover both the *ex-ante* cost of vacancy creation, that is, γk, and the informational rents of all the workers that are hired, that is, $\int_{\underline{\theta}}^{\overline{\theta}} e(\theta) F(\theta)\, d\theta$.[20] Under full information, the maximization problem is unconstrained and the equilibrium coincides immediately with the social optimum.

Finally, the following Proposition establishes the existence and uniqueness of a CSE. The argument relies on the result of Proposition 4, that the existence of a solution to problem P2 is sufficient to prove the existence of a solution for problem P1.

Proposition 5 *A Competitive Search Equilibrium exists and is unique.*

Proof. See Appendix. ■

The proof proceeds by showing the existence of a solution for the relaxed version of problem P2 without assuming that $e(\theta)$ is monotone and, then, by checking that the optimal $e(\theta)$ is effectively monotone, implying that it is also the solution to the original problem.

[20] From condition IC', using integration by parts, it follows that the average information rents are:

$$\int_{\underline{\theta}}^{\overline{\theta}} [v(\theta,\theta) - v(\overline{\theta},\overline{\theta})]\, dF(\theta) = \int_{\underline{\theta}}^{\overline{\theta}} e(\theta) F(\theta)\, d\theta.$$

Equilibrium Characterization. Proposition 4 allows me to characterize the competitive search equilibrium of the static economy in a simple way. Proposition 5 proves that Problem P2 has a unique solution and that the first order conditions are necessary and sufficient to characterize it. The analysis proceeds by focusing on the relaxed problem without the monotonicity assumption on $e(\theta)$. Then, using pointwise maximization for $e(\theta)$, I show that the trading area can be fully described by a cut-off value $\hat{\theta}$ such that

$$e(\theta) = \begin{cases} 1 \text{ if } \theta \leq \hat{\theta} \\ 0 \text{ if } \theta > \hat{\theta} \end{cases}$$

implying that the optimal $e(\theta)$ is in fact monotone. When the constraint is binding[21] and $\mu(\gamma)$ is everywhere differentiable, the equilibrium can be characterized by an array $\hat{\theta}$, γ and λ satisfying the conditions

$$\hat{\theta} = y - b - \lambda \frac{F(\hat{\theta})}{f(\hat{\theta})}, \tag{1.4}$$

$$\mu'(\gamma) \int_{\underline{\theta}}^{\hat{\theta}} \left[y - \theta - b - \lambda \frac{F(\theta)}{f(\theta)} \right] dF(\theta) = k \tag{1.5}$$

and the binding constraint

$$\frac{\mu(\gamma)}{\gamma} \int_{\underline{\theta}}^{\hat{\theta}} \left[y - \theta - b - \frac{F(\theta)}{f(\theta)} \right] dF(\theta) = k. \tag{1.6}$$

The variable λ represents a normalized version of the shadow value of the *informational rents.* I define $\lambda \equiv \hat{\lambda}/(1+\hat{\lambda})$, where $\hat{\lambda}$ is the Lagrangian multiplier attached to the constraint of problem P2. From equation (3.43) it follows that the trading cut-off $\hat{\theta}$ is decreasing in λ, that is, as the constraint is tighter, the shadow value that workers have to receive in order to reveal their information increases and the equilibrium is characterized by less trade. Notice that, when $\mu(\gamma)$ is not differentiable at some point, as I describe in the Appendix, equation (2.6) will be replaced by inequalities involving the left and right derivatives of $\mu(\gamma)$ when the solution will be at the points of non differentiability.

[21]When the constraint is not binding, the informational problem is irrelevant and the competitive search equilibrium is standard.

Notice that when $\lambda = 0$, the constraint (1.3) is slack and γ is simply determined by (2.6). Then, the full information allocation is achieved. This is possible only when the *ex-ante* cost k is zero. As shown in the next Lemma, incentive compatibility would drive employers to zero profits *ex-post*, if the full information allocation would be implemented, contradicting the possibility of an equilibrium where they have to pay a positive cost *ex-ante.*

Lemma 6 *If $k > 0$, then the solution to problem P2 requires $\lambda > 0$, where $\lambda = \hat{\lambda}/(1-\hat{\lambda})$ and $\hat{\lambda}$ is the Lagrangian multiplier attached to the constraint.*

Proof. The proof proceeds by contradiction. Let assume that the solution to problem P2 is an array $\hat{\theta}$, γ and λ with $\lambda = 0$. Then, Proposition 5 implies that $\hat{\theta}$ and γ have to satisfy equations (3.43), (3.44) and (2.6) with $\lambda = 0$. Using (3.43), equation (3.44) can be rewritten as

$$\frac{\mu(\gamma)}{\gamma} \int_{\underline{\theta}}^{\hat{\theta}} \left[\hat{\theta} - \theta - \frac{F(\theta)}{f(\theta)}\right] dF(\theta) = k.$$

Integration by parts implies that

$$\int_{\underline{\theta}}^{\hat{\theta}} \left[\hat{\theta} - \theta\right] dF(\theta) = \int_{\underline{\theta}}^{\hat{\theta}} F(\theta)\, d\theta.$$

It follows that it must be $k = 0$ yielding a contradiction and completing the proof. ■

This argument highlights the main channel driving the misalignment between *ex-ante* and *ex-post* efficiency, which keeps the economy away from the full information allocation. *Ex-post* allocative distortions are necessary to induce employers to open vacancies *ex-ante* and make the economy productive. From now on, I focus on $k > 0$ such that the constraint is binding and the informational problem interesting.

From incentive compatibility it follows that the optimal wage schedule must take the following form:

$$\omega(\theta) = \begin{cases} \omega(\bar{\theta}) + \hat{\theta} + b & \text{if } \theta \leq \hat{\theta} \\ \omega(\bar{\theta}) & \text{if } \theta > \hat{\theta} \end{cases}.$$

Notice that when $k > 0$ and $\lambda > 0$, then $\omega(\bar{\theta}) = 0$. Then, a constant wage is paid only to workers who are effectively hired and is equal to the disutility of the marginal hired worker plus the outside option. In fact, if two hired workers with different types receive different wages, then the worse type would always pretend to be the best in order to get a higher compensation. Moreover, the marginal hired worker would have no incentive to lie if he is indifferent about being unemployed, that is, if he is compensated exactly for his disutility and for the working opportunity cost b. It follows that the wage is increasing in the trading cut-off $\hat{\theta}$. The more trade is generated, the higher the wage must be in order to induce the marginal hired worker to reveal his type.

Remark 7 *The static economy is equivalent to a reduced form economy where firms post a constant wage and workers apply for jobs.*

1.3 Static Efficiency

It is an established result that competitive search correctly internalizes the externality generated by matching frictions. This section investigates the efficiency properties of a static equilibrium when there are both matching frictions and informational imperfections. Does competitive search equilibrium still achieve efficiency?

In the benchmark environment, the workers' outside option b is assumed to be leisure, which is non-transferable and cannot be wastefully destroyed. Consumption must be non negative. The worker's disutility from a match θ is his own private information. Hence, the relevant Social Planning problem must be constrained by incentive compatibility and individual rationality on the workers' side and ends up being very similar to problem P2. The only difference is that the planner could potentially transfer resources to the unemployed workers. However, I show that such a transfer is not desirable and the competitive search equilibrium reaches constrained efficiency in the static setting.

Before turning to the dynamic set-up I also study two variations on the original environment by relaxing the assumption on b: in the first one the social planner can destroy wastefully b, but cannot transfer it, and in the second one b can be freely transferred.

Social Planner. In the static economy, the social planner controls the matching process by deciding how many vacancies to open at the beginning of the period. He does not observe the types of the matched workers and has to induce them to truthfully reveal their match-specific disutility. Moreover, the planning problem is also subject to a participation constraint on the side of the workers, who can decide not to produce and enjoy b. Remember that b is leisure and cannot be destroyed or transferred and that negative consumption is not allowed. Given these environmental constraints, together with the resource constraint of the economy, the social planner decides how to allocate consumption among employed and unemployed workers.

An *allocation* is a pair of functions $[c(\theta), e(\theta)]_{\theta \in \Theta_w}$ representing the consumption and the hiring probability for a matched worker who reports type θ, a value for consumption of unmatched workers C_u, and a value γ denoting the tightness of the market.

As in the case of private contracts, also here, the Revelation Principle allows me to restrict attention to direct revelation mechanisms, without loss of generality. Following the analysis of the previous section, an allocation is *incentive-compatible* when, for all $\theta \in \Theta$, $e(.)$ is non-increasing and

$$v^{SP}(\theta, \theta) = v^{SP}\left(\overline{\theta}, \overline{\theta}\right) + \int_{\theta}^{\overline{\theta}} e(y)\, dy. \tag{1.7}$$

where the *ad-interim* utility for a worker of type θ reporting type $\hat{\theta}$, in the centralized economy, is given by

$$v^{SP}(\theta, \hat{\theta}) = c(\hat{\theta}) - e(\hat{\theta})\theta.$$

Moreover there is a *participation constraint* coming from a lack of commitment on the worker's side together with the assumption that b cannot be destroyed or transferred and that negative consumption is not allowed. It requires that all workers who participate in the society, both if matched and unmatched, consume more than the private utility b that they can appropriate by not participating, that is,

$$C_u \geq b \text{ and } c(\theta) \geq b \text{ for all } \theta \in \Theta. \tag{1.8}$$

Finally, the *resource constraint* for the static economy ensures that aggregate consumption is covered by aggregate net production, that is,

$$\mu(\gamma) \int_{\underline{\theta}}^{\overline{\theta}} c(\theta) dF(\theta) + (1 - \mu(\gamma))\, C_u \leq \mu(\gamma)(y - b) \int_{\underline{\theta}}^{\overline{\theta}} e(\theta)\, dF(\theta) + b - \gamma k. \tag{1.9}$$

I can now define a feasible and a constrained efficient allocation.

Definition 8 *An allocation is feasible iff it (i) is incentive-compatible, that is, satisfies (1.7)* *together with the monotonicity of* $e(.)$*, (ii) satisfies the participation constraint (1.8) and (iii) satisfies the resource constraint (1.9).*

Definition 9 *A constrained efficient allocation maximizes workers' ex-ante utility*

$$\mu(\gamma)\int_{\underline{\theta}}^{\overline{\theta}}[c(\theta)-e(\theta)\theta]\,dF(\theta)+(1-\mu(\gamma))\,C_u$$

subject to feasibility.

The maximization problem defining a constrained efficient allocation can be expressed, after some algebra, as

$$U(b)=\max_{e(\theta),C_u,\gamma}\mu(\gamma)\int_{\underline{\theta}}^{\hat{\theta}}[y-\theta-b]\,dF(\theta)+b-\gamma k \tag{P3}$$

s.t.

$$\int_{\underline{\theta}}^{\overline{\theta}}e(\theta)\left[y-\theta-b-\frac{F(\theta)}{f(\theta)}\right]dF(\theta)\geq\left(\frac{1-\mu(\gamma)}{\mu(\gamma)}\right)(C_u-b)+\frac{\gamma}{\mu(\gamma)}k$$

$$C_u\geq b.$$

The next proposition shows that the competitive search equilibrium is constrained efficient in the static setting

Proposition 10 *Assume that* b *cannot be destroyed or transferred and that negative consumption is not allowed, then a static Competitive Search Equilibrium is constrained efficient.*

Proof. First, notice that C_u does not appear in the objective function so that the social planner cannot do worse by choosing $C_u = b$. Then problem P3 becomes equivalent to problem P2 and competitive search is constrained efficient. ■

It is interesting to notice that, as in the decentralized equilibrium, the social optimum does not reach the full information allocation, that is, $\lambda > 0$. The result is driven by the binding participation constraint for the workers. The proof shows clearly that the Planner could do better by reducing the consumption of unemployed workers, which does not affect the objective function, below the level of leisure b. Hence, in the next two subsections, I explore two variations of the main environment: in the first one the social planner can destroy wastefully b, but cannot transfer it, and in the second one b can be freely transferred.

1.3.1 Money Burning can be Desirable.

I now show how reducing b wastefully, what I refer to as *money burning*, can generate a Pareto improvement. For simplicity, assume that all the workers are unemployed *ex-ante*. Employers get zero profits in expectation due to the free entry assumption. Then, the social welfare coincides with the *ex-ante* value of being unemployed $U(b)$, as defined in problem P3. A Pareto improvement is feasible when $U'(b) < 0$. Considering this alternative environment is useful to highlight, in a simple way, the crucial mechanism of the paper. The source of dynamic constrained inefficiency will come from the fact that it can be socially optimal to reduce *ex-ante* the workers' outside option, which here is exogenously given by b. In the dynamic economy, the workers' outside option is endogenous and corresponds to the continuation value of being unemployed, generating an externality that is not internalized by the competitive search equilibrium. As I will show in section 1.5, the externality comes from the fact that firms who post contracts at time $t+1$ do not internalize the informational cost that they impose on contracts designed by other firms at time t.

I show that for an interesting class of functions $\mu(\gamma)$, there always exists an open set of parameters such that money burning is Pareto improving. Suppose $U(b)$ is differentiable[22] and define

$$g \equiv 1 - \mu(\gamma) F(\hat{\theta}) - \lambda.$$

Notice that g has the same sign of $U'(b)$, which represents the effect of the workers' outside option on welfare. There is a direct positive effect coming from the fact that, as the outside option is higher, workers who end up being unemployed will be better off. This force is summarized by $1 - \mu(\gamma) F(\hat{\theta})$, which represents the *ex-ante* probability of being unemployed at the end of the period. However, there is a negative indirect effect coming from the tightness of the informational constraint, represented by λ. As the outside option increases, the shadow cost of

[22] It is easy to show that when γ, $\hat{\theta}$ and $\hat{\lambda}$ are uniquely defined, then $U(b)$ is differentiable. In fact this is always the case in the rest of the analysis.

revealing information is higher, since workers have a higher opportunity cost of remaining in the employment relationship. When the indirect effect dominates making g negative, a Pareto improvement can be implemented by reducing *ex-ante* the workers' outside option.

Proposition 11 *Whenever $g < 0$, the competitive search equilibrium allocation can be Pareto improved by reducing b.*

Proof. The proof is straightforward. Notice that the last constraint of problem P3 will be always binding and then can be eliminated by substituting for $C_u = b$. Then, from the Envelope condition $dU/db = 1 - (1-\lambda)^{-1}\mu(\gamma) F(\hat{\theta})$ where $\lambda = \hat{\lambda}/(1+\hat{\lambda})$ and $\hat{\lambda}$ is the multiplier attached to the constraint, which implies that $dU/db < 0$ iff $g < 0$. ■

There are two extreme cases that can lead to g being negative: first, if the constraint is extremely tight, that is, $\lambda \to 1$ and second, if the probability of staying unemployed is extremely small, that is, $\mu(\gamma) F(\hat{\theta}(\lambda)) \to 1$. The next two Propositions show that for an interesting class of functions $\mu(\gamma)$, there exists a set of the parameter space (y, k) such that g is negative. My intent is not to fully characterize the set of parameters generating optimal money burning, but to deliver sufficient conditions for this set to exist. Then, I will deliver some insights on its characterization, using the Cobb-Douglas case.

First, I consider two general families of functions $\mu(\gamma)$, which include the specifications commonly used in the search literature: the family of functions everywhere differentiable satisfying a particular restriction on the elasticity stated in assumption A2, and the family of functions reaching 1 with a kink and possibly exhibiting an additional kink when $\mu(\gamma) = \gamma$ at $\gamma > 0$, as specified in assumption A3. Proposition 12 shows that g can be negative for each of these families of functions $\mu(\gamma)$.

The first class of functions $\mu(\gamma)$ includes the exponential case. Functions belonging to this class have to satisfy assumption A1 and A2, where the latter is defined as follows.

Assumption A2. Assume $\mu(\gamma)$ is strictly concave and everywhere differentiable. Furthermore, $\lim_{\gamma\to 0}\mu'(\gamma) = \mu'(0) > 0$ and $\lim_{\gamma\to 0}\eta'(\gamma) > -\mu'(0) f(\bar{\theta}) \int_{\underline{\theta}}^{\bar{\theta}} F(\theta)\, d\theta$ where $\eta(\gamma) = \gamma\mu'(\gamma)/\mu(\gamma)$ denotes the elasticity of $\mu(\gamma)$ and $\eta'(\gamma) \leq 0$.

The second class of functions $\mu(\gamma)$ that I consider includes for example the Leontief specification, $\mu(\gamma) = \min\{\gamma, 1\}$, and the properly modified Cobb-Douglas case, $\mu(\gamma) = \min\{A\gamma^{\alpha}, \gamma, 1\}$ with $\alpha \in (0, 1]$ and $A > 0$. Functions that belong to this class have to satisfy assumptions A1 and A3, where the latter is defined below.

Assumption A3. Assume there exist two cutoffs $\underline{\gamma} \leq \bar{\gamma} < \infty$ such that

$$\begin{aligned} \mu(\gamma) &= \gamma \text{ for } \gamma \leq \bar{\gamma} \\ \mu(\gamma) &= 1 \text{ for } \gamma \geq \underline{\gamma} \end{aligned}$$

with $\lim_{\gamma \searrow \underline{\gamma}} \mu'(\gamma) < 1$ and $\lim_{\gamma \nearrow \bar{\gamma}} \mu'(\gamma) > 0$. Furthermore, assume $\mu(\gamma)$ is strictly concave and exhibits non increasing elasticity for any $\gamma \in (\underline{\gamma}, \bar{\gamma})$.

The following Proposition shows that money burning can Pareto improve the competitive search equilibrium for functions $\mu(\gamma)$ that belong to one of these families of functions, that is, that satisfy A1 and either A2 or A3.

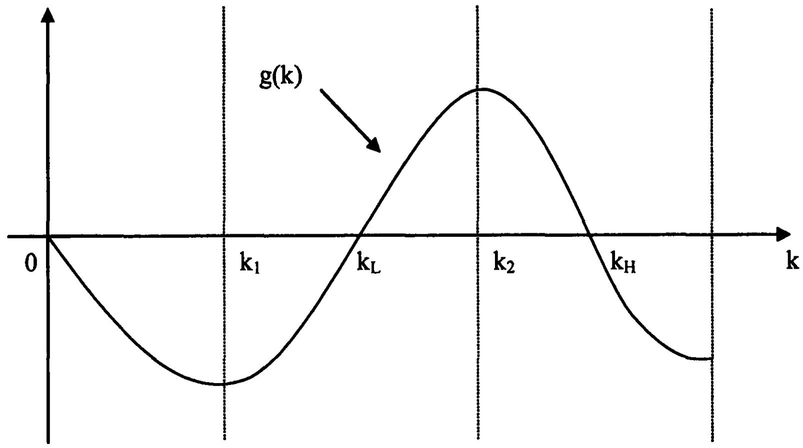
Proposition 12 *Consider any $\mu(.)$ satisfying Assumption A1 and either A2 or A3. For given $F(.)$ and b, there exists an open set of the parameter space (k, y) such that $g < 0$ at the competitive search equilibrium.*

Proof. See the Appendix. ∎

The proof highlights the two forces potentially driving money burning. On one hand, inefficiency can arise when the shadow cost of revealing information is high enough, that is, for $\lambda \to 1$, as I show as being a possibility for the first class of functions and for a subfamily of the second class.[23] On the other hand, inefficiency can arise when $\mu(\gamma) F(\hat{\theta}(\lambda)) \to 1$ and the mass of unemployed workers enjoying b at the end of the period is infinitesimal. This can happen if y is big enough, for the subfamily of functions of the second class for which $\underline{\gamma} < \bar{\gamma}$ and $\mu'(\gamma) < \mu(\gamma)/\gamma$ for any $\gamma \in (\underline{\gamma}, \bar{\gamma})$.

[23] This subfamily of the class of functions satisfying assumptions A1 and A3 reduce to the Leontief case $\mu(\gamma) = \min\{\gamma, 1\}$.

In order to give a flavor of the impact of k on money burning, in the next Proposition, I analyze the *modified* Cobb Douglas case[24] and characterize how g depends on k. Intuition suggests that k has two first-order opposite effects on inefficiency: it reduces the number of vacancies posted and it increases the shadow cost of information revelation, that is, λ. On the top of these, there is an indirect effect of k on λ. This effect is shut down in the Cobb Douglas case, making it analytically tractable. I show that g behaves as in the picture below, highlighting the opposite effects that k can have on money burning. For y big enough, g can be negative both when k is big enough to make the cost of revealing information extremely high, that is, $\lambda \to 1$, and when k is small enough to make the market very tight so that the probability of staying unemployed is extremely small, that is, $\mu(\gamma) F(\hat{\theta}(\lambda)) \to 1$.



Proposition 13 *Assume $\mu(\gamma) = \min\{A\gamma^{\alpha}, \gamma, 1\}$ with $A \in (0,1]$ and $\alpha \in [0,1]$.*[25] *Furthermore, assume $y > b + \bar{\theta} + \tilde{\lambda} F(\bar{\theta})/f(\bar{\theta})$, where $\tilde{\lambda} \equiv \alpha D/[1-\alpha(1-D)]$ with $D \equiv f(\bar{\theta}) \int_{\underline{\theta}}^{\bar{\theta}} F(\theta)\, d\theta$. Then, either g is always negative, or there exist k_L and k_H such that $0 < k_L \leq k_H < \bar{k}$ and at the competitive search equilibrium $g < 0$ for $k \in [0, k_L]$ and $k \in [k_H, \bar{k}]$, where $\bar{k} \equiv \int_{\underline{\theta}}^{\hat{\theta}(1)} \left[y - \theta - b - \frac{F(\theta)}{f(\theta)} \right] dF(\theta)$ with $\hat{\theta}(1) = b + \hat{\theta}(1) + F(\hat{\theta}(1))/f(\hat{\theta}(1))$.*

[24] This is a special case of the functions satisfying Assumption A2, so that the previous Proposition has already shown that there exists a parameter subspace for which $g < 0$.

[25] *Note that the Leontief case is a particular case of this modified Cobb Douglas with $\alpha = 1$.*

Proof. See the Appendix. ■

1.3.2 Transferability can Restore Full Information

Now I explore a second alternative environment in which the worker's outside option b is freely transferable. In this case, I interpret b as home production. If b is high enough, the social planner can achieve the full information allocation, by transferring utility between unemployed and employed workers. In particular, given risk neutrality, the social planner could tax, at no social cost, the unemployed workers in order to finance the informational rents of matched workers. Similarly, the full information allocation could be achieved if the planner had access to enough external resources.

Allowing for the possibility of transferring resources is basically equivalent to reducing the participation constraint to the constraint of non-negative consumption. From problem P3, it appears that, if b is big enough, the Social planner can make a negative net transfer to the unemployed, $C_u - b$, so as to achieve the full information efficient allocation, which is characterized by the cut-off value $\hat{\theta}^{FI}$ such that

$$\hat{\theta}^{FI} = y - b \tag{1.10}$$

and γ^{FI} implicitly defined by

$$\mu'\left(\gamma^{FI}\right) \int_{\underline{\theta}}^{\hat{\theta}^{FI}} [y - \theta - b]\, dF(\theta) = k. \tag{1.11}$$

What allows the planner to achieve the first best is the reduction of the relative value of being unemployed with respect to the one of being employed, that is, the sum of the outside option and the opportunity cost of being employed for the marginal hired worker. In particular, next Proposition shows that subsidizing job creation can restore the full information outcome.

Proposition 14 *Suppose that b is transferable and the following inequality holds*

$$b \geq k\gamma^{FI}.$$

Then the full information allocation can be decentralized by subsidizing job creation with a *lump-sum tax on workers, both employed and unemployed.*

Proof. Assume that the social planner gives a subsidy τ to the firms, by taxing lump-sum employed and unemployed workers. Budget balance imposes that the subsidy τ is covered by a tax equal to $\tau\mu(\gamma)\int_{\underline{\theta}}^{\overline{\theta}} e(\theta)\, dF(\theta)$. Then, the social planning problem maximizes

$$\mu(\gamma)\int_{\underline{\theta}}^{\overline{\theta}} e(\theta)(y-\theta-b)\, dF(\theta)+b-\gamma k$$

subject to

$$\mu(\gamma)\int_{\underline{\theta}}^{\overline{\theta}} e(\theta)\left(y-\theta-b+\tau-\frac{F(\theta)}{f(\theta)}\right) dF(\theta)\geq \gamma k.$$

I guess that $\hat{\theta}^{FI}$, γ^{FI}, τ and $\lambda=0$ are a solution and then I verify it. The social planner can choose the minimal transfer τ which makes the constraint satisfied with equality at the full information allocation, that is, the τ such that

$$\mu\left(\gamma^{FI}\right)\int_{\underline{\theta}}^{\hat{\theta}^{FI}}\left(y-\theta-b+\tau-\frac{F(\theta)}{f(\theta)}\right) dF(\theta)=\gamma^{FI}k$$

yielding $\lambda=0$. Once $\lambda=0$, the first order conditions with respect to $e(.)$ and γ are equivalent to equations (1.10) and (1.11), yielding $\hat{\theta}=\hat{\theta}^{FI}$ and $\gamma=\gamma^{FI}$. Finally, notice that such a τ is feasible only when $b\geq\tau\mu(\gamma)F(\hat{\theta})$, that is, when b is high enough to cover the informational rents. In fact the minimal b sufficient to restore the full information allocation is such that

$$\mu\left(\gamma^{FI}\right)\int_{\underline{\theta}}^{\hat{\theta}^{FI}}\left(y-\theta-\frac{F(\theta)}{f(\theta)}\right) dF(\theta)=\gamma^{FI}k$$

which, using the first order condition with respect to $e(\theta)$ and integration by parts, yields

$$b\geq k\gamma^{FI}$$

completing the proof. ■

The proof of the previous Proposition suggests that the full information allocation can be restored also if there are enough resources to finance both the optimal vacancy creation and the informational rents necessary to sustain the optimal job creation. The next remark follows naturally.

Remark 15 *If the social planner has access to enough external resources, then he achieves the full information allocation.*

These results beg the question wether the competitive search equilibrium can achieve the full information allocation when b is transferable. Indeed, in the next proposition I show that this is the case when b is high enough.

Proposition 16 *Suppose that b is transferable and the following inequality holds*

$$b \geq \frac{k\gamma^{FI}}{\mu\left(\gamma^{FI}\right)},$$

then the competitive search equilibrium achieves the full information allocation.

Proof. The competitive search equilibrium can be now defined exactly as in section 1.2, except for the participation constraints IR which now become

$$v\left(\theta,\theta\right) \geq 0 \text{ for all } \theta \in \Theta.$$

Hence, the constraint in problem P2 can be replaced by

$$\mu\left(\gamma\right)\int_{\underline{\theta}}^{\overline{\theta}} e\left(\theta\right)\left[y-\theta-b-\frac{F\left(\theta\right)}{f\left(\theta\right)}\right]dF\left(\theta\right)+\mu\left(\gamma\right)b \geq \gamma k.$$

Substituting the full information values for γ and $e\left(\theta\right)$ and integrating by parts as in the proof of Lemma 6, it follows that

$$\mu\left(\gamma^{FI}\right)b \geq k\gamma^{FI},$$

completing the proof. ■

Notice that

$$\frac{k\gamma^{FI}}{\mu\left(\gamma^{FI}\right)} \geq k\gamma^{FI}.$$

This implies that the social planner can restore the full information allocation for a larger set of parameters than competitive search can. The difference comes from the fact that firms cannot extract resources from workers who are unmatched, while the social planner can impose an *ex-ante* entry cost for the search market.[26]

When b is transferable, as in the benchmark case, the optimal wage schedule takes the form

$$\omega(\theta) = \begin{cases} \omega(\bar{\theta}) + \hat{\theta} + b & \text{if } \theta \leq \hat{\theta} \\ \omega(\bar{\theta}) & \text{if } \theta > \hat{\theta} \end{cases}.$$

However, now firms can set a negative value for $\omega(\bar{\theta})$ as long as

$$\omega(\bar{\theta}) \geq -b. \tag{1.12}$$

This contract has a natural interpretation as bond posting. The firms ask workers to sign a contingent promise, after the match and before they observe the realization of the shock. Matched workers sign a promise that they will pay an application fee of value $-\omega(\bar{\theta})$. On top of that, if they are hired, they will receive a wage of value $\hat{\theta} + b$. The constraint (1.12) means that workers can credibly promise to pay *ex-post* a value not greater than the value of the home production they obtain when unemployed.

1.4 Dynamic economy.

This section introduces the dynamic environment of this economy leading to the main result of the paper: the dynamic competitive search equilibrium can be constrained inefficient. The crucial difference between the static and the dynamic environment is that the worker's outside option is exogenously given in the former, while in the latter, it is endogenously determined as the equilibrium continuation utility of unemployed workers. Inefficiency arises because the worker's outside option affects the *ex-ante* cost of information revelation, generating a novel externality. The social planner can improve upon the decentralized economy by internalizing this *informational externality.*

[26]Though, allowing for a broader interpretation of competitive search, I could think of market makers who impose an application fee to all the workers who search for a match. This delivers a problem that is isomorphic to the one of the social planner I have described above. In this case, the competitive search equilibrium will be able to restore the full information allocation exactly for the same set of parameters, that is for $b \geq k\gamma^{FI}$.

Environment. Consider an economy with infinite horizon and discrete time. Both workers and employers have linear preferences and discount factor β. The search and production technologies are natural generalizations of the static setting. At the beginning of each period t employers can be either productive or not. Workers can be either employed or unemployed. Non-productive employers can open a vacancy at a cost k which entitles them to post an employment contract $\mathcal{C}_t \in \mathbb{C}$ where $\mathbb{C}$ is the set of *ad interim* incentive compatible and individually rational mechanisms. As I describe below, a contract $\mathcal{C}_t : \Theta \mapsto [0,1] \times \mathbb{R}_+$ specifies the hiring probability and the wage for each matched worker at time t, who reports type θ. Therefore at each time t, a non-productive firm chooses a pair $(\sigma_t, \mathcal{C}_t) \in \{0,1\} \times \mathbb{C}$ where σ_t denotes the decision of posting a vacancy. Next, each unemployed worker observes all the contracts posted and decides where to apply. He chooses a contract $\mathcal{C}_t \in \mathbb{C}_t^P \subset \mathbb{C}$, where $\mathbb{C}_t^P$ denotes the set of contracts posted by active firms at time t. As in the static environment, each contract $\mathcal{C}_t$, is associated to a specific γ_t so that employers and workers know that their matching probabilities will depend on the contract that they respectively post and seek for. After workers start to search for a specific contract, matching takes place and, for each match, the draw θ is realized and is private information of the worker. The behavior of a worker who is matched at time t is described by a map $(a_t, s_t) : \Theta \mapsto \Theta \times \{0,1\}$ that for each type θ specifies a report $\hat{\theta}_t = s(\theta)$ and a participation decision $a_t(\theta)$. After he sees his type, the worker can either implement the contract, that is choose $a_t(\theta) = 1$, or walk away, that is choose $a_t(\theta) = 0$. If he walks away, he stays in autarky for one period, gets a non-transferable utility from leisure b, enters an anonymous pool of unemployed workers and look for another match next period. If the worker is effectively hired, the parties are productive until separation, which happens according to a Poisson process with parameter s. The worker's disutility θ is constant for the duration of the match.

In a decentralized economy, as in the static setting, the consumption of employed workers is given by the contracted wage, which is fixed at the time of the match. Moreover, the consumption of unemployed workers, that is, both unmatched workers and workers who matched but have not been hired, is equal to the value of leisure b.

Employment Contracts and Bellman Values. Invoking the Revelation Principle, without loss of generality, I can again restrict attention to incentive-compatible and individually rational direct revelation mechanisms, corresponding to a mapping $C_t : \Theta \mapsto [0,1] \times \mathbb{R}_+$, specifying for each matched worker at time t who reports type θ, the hiring probability $e_t(\theta) \in [0,1]$ and the wage $\omega_t(\theta) \in \mathbb{R}_+$ which is paid at the beginning of the productive relationship. Notice that I can restrict attention to the set $\mathbb{C}$ of *ad interim* incentive compatible and individually rational mechanisms described above, due to the unemployed anonymity assumption. All the unemployed workers searching for a job cannot be distinguished, so that contracts cannot be conditioned on the past employment history.

Linear preferences, together with the fact that types are fixed over time within a match, imply that the wage profile over the life of the relationship is irrelevant for the analysis. Therefore, I can assume, without loss of generality, that the wage is fully paid at the moment of the match. Analogously I assume that the whole disutility generated over the life of the match, $\alpha\theta$, takes place at the beginning of the relationship. It follows that the continuation value of being employed *net of wages and disutility* at time t, V_t, which from now on I will refer to simply as the continuation utility of employed workers, represents just the discounted expected value of being separated and becoming unemployed, that is,

$$V_t = \beta s U_{t+1} + \beta(1-s)V_{t+1}. \tag{1.13}$$

Moreover the continuation value of an unemployed worker at time t is given by

$$U_t = b + \beta\mu(\gamma_{t+1}) \int_{\underline{\theta}}^{\overline{\theta}} [\omega_{t+1}(\theta) - e_{t+1}(\theta)(\alpha\theta - V_{t+1} + U_{t+1})]\, dF(\theta) + \beta U_{t+1} \tag{1.14}$$

where $\alpha \equiv [1-\beta(1-s)]^{-1}$.

The *ad interim* utility of a worker of type θ, who reports to be of type $\hat{\theta}$ at time t, is given by

$$v_t(\theta,\hat{\theta}) = [\omega_t(\hat{\theta}) - e_t(\hat{\theta})(\alpha\theta - V_t)] + [1-e_t(\hat{\theta})]U_t \tag{1.15}$$

and the expected revenues of the firm, after a match, is given by

$$\int_{\underline{\theta}}^{\overline{\theta}} [e_t(\theta)\alpha y - \omega_t(\theta)]\, dF(\theta).$$

The large number of potential firms ensures free entry and implies that the value of an open vacancy will be zero at each time, that is,

$$\beta \frac{\mu(\gamma_t)}{\gamma_t} \int_{\underline{\theta}}^{\overline{\theta}} [e_t(\theta)\,\alpha y - \omega_t(\theta)]\, dF(\theta) = k. \tag{1.16}$$

A natural generalization of the static analysis, gives that a contract C_t is *incentive-compatible* and *individually rational* whenever $e_t(.)$ is non-increasing and the following conditions hold:

$$v_t(\theta,\theta) = v_t\left(\overline{\theta},\overline{\theta}\right) + \alpha \int_{\theta}^{\overline{\theta}} e_t(y)\, dy \text{ for all } \theta \in \Theta \tag{IC'}$$

and

$$v_t\left(\overline{\theta},\overline{\theta}\right) \geq U_{t-1}. \tag{IR'}$$

Following the static analysis, the informational rents for a worker of type θ who meets a firm at time t are $v_t(\theta,\theta) - v_t\left(\overline{\theta},\overline{\theta}\right)$, as define by equation IC'.

1.4.1 Dynamic Competitive Search Equilibrium

This section defines the dynamic version of the Competitive Search Equilibrium. Generalizing the static definition, a dynamic Competitive Search Equilibrium, in sequential terms, is a sequence of sets of incentive-compatible and individually rational contracts $\{\mathbb{C}_t^*\}$ and a sequence of tightness functions $\{\Gamma_t^*\}$, where $\Gamma_t^* : \mathbb{C}_t^* \longmapsto \mathbb{R}_+ \cup \infty$, such that, at any t employers maximize profits and workers apply optimally for jobs, taking as given the future sequence of sets of contracts, $\{\mathbb{C}_{t+1}\}, \{\mathbb{C}_{t+2}\}, ...$, and tightness functions, $\{\Gamma_t^*\}, \{\Gamma_{t+1}^*\}, ...$

In order to simplify the analytical treatment, we can introduce an equivalent definition of the dynamic competitive search equilibrium in recursive terms.

The first thing to notice is that the pair of continuation utilities for unemployed and employed workers, U and V, are a sufficient statistic for future sets of $\mathbb{C}'s$ and $\Gamma's$. This allows me to describe the dynamic competitive search equilibrium in a recursive way, as shown by the following Definition.

Definition 17 *A dynamic Competitive Search Equilibrium (CSE) is a sequence of sets of incentive-compatible and individually rational contracts* $\{\mathbb{C}_t^*\}$*, a sequence of functions* $\{\Gamma_t^*\}$*,* *where* $\Gamma_t^* : \mathbb{C}_t^* \longmapsto \mathbb{R}_+ \cup \infty$, *and a sequence of pairs of continuation utility levels* $\{U_t^*, V_t^*\}$, *where* $(U_t^*, V_t^*) \in \mathbb{R}_+^2$ *for any* t, *satisfying*

(i) employers' ***profit maximization*** *and* ***free-entry*** *at each time* t: $\forall \mathcal{C}_t \equiv [e_t(\theta), \omega_t(\theta)]_{\theta \in \Theta}$,

$$\frac{\mu(\Gamma_t^*(\mathcal{C}_t))}{\Gamma_t^*(\mathcal{C}_t)} \beta \int_{\underline{\theta}}^{\overline{\theta}} [e_t(\theta)\alpha y - \omega_t(\theta)]\, dF(\theta) - k \leq 0$$

subject to incentive compatibility IC and individual rationality IR, with equality if $\mathcal{C}_t \in \{\mathbb{C}_t^*\}$;

(ii) workers' ***optimal job application*** *at each time* t: $\forall \mathcal{C}_t \equiv [e_t(\theta), \omega_t(\theta)]_{\theta \in \Theta}$, *for given* V_t *and* U_t

$$U_{t-1}^* \geq b + \beta\mu(\Gamma_t^*(\mathcal{C}_t)) \int_{\underline{\theta}}^{\overline{\theta}} [\omega_t(\theta) - e_t(\theta)(\alpha\theta - V_t + U_t)]\, dF(\theta) + \beta U_t$$

and $\Gamma_t^*(\mathcal{C}_t) \geq 0$, *with complementarity slackness, where*

$$U_{t-1}^* = \max_{\mathcal{C}_t'} b + \beta\mu\left(\Gamma_t^*(\mathcal{C}_t')\right) \int_{\underline{\theta}}^{\overline{\theta}} \left[\omega_t'(\theta) - e_t'(\theta)(\alpha\theta - V_t + U_t)\right] dF(\theta) + \beta U_t$$

or $U_{t-1}^* = b + \beta U_t$ *if* $\{\mathbb{C}_t^*\}$ *is empty, and*

$$V_{t-1}^* = \beta s U_t + \beta(1-s)V_t.$$

The definition of the dynamic equilibrium is a natural generalization of the static version. At each point in time employers maximize profits and workers apply optimally for jobs, both taking as given the future values of being employed and unemployed and aware that a market tightness is associated with each contract, even if not offered in equilibrium, according to the function $\Gamma_t^*(\mathcal{C}_t)$. Moreover, profits are driven to zero at each point in time by free entry.

It follows that the unemployment rate of workers applying to firms posting a contract of type $\mathcal{C}_t$ at time t is given by

$$u_{t+1}(\mathcal{C}_{t+1}) = u_t(\mathcal{C}_t)\left[1 - \mu(\Gamma_t^*(\mathcal{C}_t)) \int_{\underline{\theta}}^{\overline{\theta}} e_t(\theta)\, dF(\theta)\right] + (1 - u_t(\mathcal{C}_t))\, s. \qquad (1.17)$$

Generalizing the static result, the next Proposition states a dynamic characterization of a symmetric competitive search equilibrium in recursive terms.

Proposition 18 *If $\{\mathbb{C}_t, \Gamma_t, U_t, V_t\}_{t=0}^{\infty}$ is a Competitive Search Equilibrium, then any pair (C_t^*, γ_t^*) with $C_t^* \in \mathbb{C}_t$ and $\gamma_t^* = \Gamma_t^*(C_t^*)$ satisfy the following*

(i) *for given pair U_t and V_t, for any time t, $C_t = [e_t(\theta), \omega_t(\theta)]_{\theta\in\Theta}$ and γ_t solve*

$$\max_{e_t(.),\omega(.),\gamma_t} b + \beta\mu(\gamma_t)\int_{\underline{\theta}}^{\overline{\theta}} [\omega_t(\theta) - e_t(\theta)(\alpha\theta - V_t + U_t)]\, dF(\theta) + \beta U_t \tag{P4}$$

subject to $e_t(\theta) \in [0,1]$, non-negative consumption, the incentive compatibility constraint IC' together with the monotonicity assumption on $e_t(\theta)$, the individual rationality constraint IR' and the free-entry condition (1.16);

(ii) *for given $\{C_t, \gamma_t\}_{t=0}^{\infty}$, then $\{U_t, V_t\}_{t=0}^{\infty}$ evolve according to (1.13) and (1.14).*

Conversely, if a sequence $\{C_t^*, \gamma_t^*\}_{t=0}^{\infty}$ solves the program P4, then there exists an equilibrium $\{\mathbb{C}_t^*, \Gamma_t^*, U_t^*, V_t^*\}_{t=0}^{\infty}$ such that $C_t^* \in \mathbb{C}_t^*$ and $\gamma_t^* = \Gamma_t^*(C_t^*)$.

In the rest of the analysis I adopt a recursive notation, dropping the t whenever this causes no confusion, and denoting a variable at time $t-1$ with a $-$ sign.

Proposition 18 shows that for given U and V, a (symmetric) equilibrium incentive-compatible and individually-rational contract C and tightness γ must solve Problem P4. The next Proposition shows that the equilibrium can be equivalently described by a hiring function $e(\theta)$ and a tightness γ that solve a simplified program P5. Given $e(\theta)$ and γ, an associated wage function $\omega(\theta)$ can be constructed so that the constraints IC' and IR' are satisfied.

Proposition 19 *For given U and V, any function $[e(\theta)]_{\theta\in\Theta}$ and γ which solve Problem P4, solve also*

$$\begin{aligned} U^{-}(U,V) &= \max_{e(\cdot),\gamma} \beta\mu(\gamma)\int_{\underline{\theta}}^{\overline{\theta}} e(\theta)[\alpha(y-\theta) + V - U]\, dF(\theta) \\ &\quad + b - \gamma k + \beta U \end{aligned} \tag{P5}$$

s.t.

$$\beta\mu(\gamma)\int_{\underline{\theta}}^{\overline{\theta}} e(\theta)\left[\alpha\left(y - \theta - \frac{F(\theta)}{f(\theta)}\right) + V - U\right] dF(\theta) \geq \gamma k. \tag{1.18}$$

Furthermore, for any function $[e(\theta)]_{\theta\in\Theta}$ and γ which solve problem P5, there exists a function $[\omega(\theta)]_{\theta\in\Theta}$ such that the contract $C = [e(\theta), \omega(\theta)]_{\theta\in\Theta}$ and γ solve problem P4.

Proof. The proof is similar to the one of Proposition 4 and is therefore omitted. ■

Equilibrium Characterization. The characterization of the equilibrium allocation for a given continuation utility gap between unemployed and employed workers, $U - V$, is very similar to the static one. In fact, in the dynamic setting, $U - V$ represents the effective outside option of the workers. A generalization of Proposition 5 proves that Problem P5 has a unique solution and that the first order conditions are necessary and sufficient to characterize it. I proceed by studying the relaxed problem without the monotonicity assumption on $e(\theta)$. Then, using pointwise maximization with respect to $e(\theta)$ I show that the trading area can be fully described by a cut-off value $\hat{\theta}$ such that

$$e(\theta) = \begin{cases} 1 \text{ if } \theta \leq \hat{\theta} \\ 0 \text{ if } \theta > \hat{\theta} \end{cases},$$

implying that the optimal $e(\theta)$ is effectively non-increasing. When the constraint (1.18) is binding[27] and $\mu(\gamma)$ is everywhere differentiable the equilibrium can be characterized, for given $U - V$, by an array $\hat{\theta}$, γ and λ satisfying the first order conditions, respectively, for $\hat{\theta}$ and γ

$$\alpha\left(y - \hat{\theta} - \lambda\frac{F(\hat{\theta})}{f(\hat{\theta})}\right) - (U - V) = 0, \tag{1.19}$$

$$\beta\mu'(\gamma)\int_{\underline{\theta}}^{\hat{\theta}}\left[\alpha\left(y - \theta - \lambda\frac{F(\theta)}{f(\theta)}\right) - (U - V)\right]dF(\theta) = k \tag{1.20}$$

and the condition

$$\beta\mu(\gamma)\int_{\underline{\theta}}^{\hat{\theta}}\left[\alpha\left(y - \theta - \frac{F(\theta)}{f(\theta)}\right) - (U - V)\right]dF(\theta) = \gamma k. \tag{1.21}$$

The variable λ represents a normalized version of the shadow value of the *informational rents.*[28] Notice that when $\lambda = 0$, the constraint (1.18) is slack and γ is simply determined by (1.20).

[27] When the constraint is not binding, the informational problem is irrelevant and the competitive search equilibrium is constrained efficient as in the standard result.

[28] Similarly to the static setting, $\lambda \equiv \hat{\lambda}/(1+\hat{\lambda})$, where $\hat{\lambda}$ is the Lagrangian multiplier attached to the constraint (1.18).

Then, the full information allocation is achieved. Clearly, asymmetric information reduces job creation, as the surplus of the economy must cover not only the cost of vacancy creation but also the rents needed to extract information from the workers. As intuition suggests, $\hat{\theta}$ decreases with λ. When $\mu(\gamma)$ is not differentiable at some points, equation (1.20) will be replaced by inequalities involving the left and right derivatives of $\mu(\gamma)$ at the points of non differentiability.

Similarly to the static setting, whenever the cost of posting a vacancy k is positive, then $\lambda > 0$ and the equilibrium is away from the full information allocation.

Lemma 20 *If $k > 0$, then the solution to problem P5 requires $\lambda > 0$, where $\lambda = \hat{\lambda}/(1-\hat{\lambda})$ and $\hat{\lambda}$ is the Lagrangian multiplier attached to constraint (1.18).*

Proof. The proof is analogous to the proof of Lemma 6 and therefore omitted. ■

Proposition 19 shows that at each point in time the competitive search equilibrium γ and $e(\theta)$ are functions only of the expected values of V and U, which evolve according to the law of motions (1.13) and (1.14). Hence, the equilibrium U^{CE} and V^{CE} corresponds to a fixed point, given by

$$U^{CE} = \frac{\alpha\beta\mu\left(\gamma^{CE}\right)\int_{\underline{\theta}}^{\hat{\theta}^{CE}}\left[y-\theta\right]dF(\theta)+b-\gamma^{CE}k}{(1-\beta)\left[1+\alpha\beta\mu\left(\gamma^{CE}\right)F(\hat{\theta}^{CE})\right]} \tag{1.22}$$

$$V^{CE} = \frac{\beta s}{1-\beta(1-s)}U^{CE} \tag{1.23}$$

where, the equilibrium $\hat{\theta}^{CE}$ and γ^{CE} solve Problem P5, for $U = U^{CE}$ and $V = V^{CE}$, that is, satisfy equations (1.19), (1.20) and (1.21). The unemployment rate u, which is the only state variable of the economy, does not affect this problem. This implies that γ and $e(\theta)$ (and $\omega(\theta)$ as well) together with V and U, achieve the steady state values directly in the first period and stay constant over time.[29] The transitional dynamics of the competitive search equilibrium will then be characterized uniquely by the transition of the unemployment rate.

Steady State. Denote the steady state competitive search equilibrium by

$$SS = \left\{\left[e^{CE}(\theta)\right]_{\theta\in\Theta}, \gamma^{CE}, U^{CE}, V^{CE}, u^{SS}\right\}.$$

[29] Note that in equilibrium the continuation utility of the employed turns out to be smaller than the continuation utility of the unemployed. This is natural if one thinks that I define the continuation value of the employed net from wage and disutility.

In steady state not only $e(\theta)$, γ, V and U are constant, but also the unemployment rate u is. The steady state equilibrium is given by equations (1.19), (1.21) and (1.20) with $U = U^{CE}$ and $V = V^{CE}$, (1.22), (1.23) and

$$u^{SS} = s\left[s + \mu\left(\gamma^{CE}\right) F(\hat{\theta}^{CE})\right]^{-1}. \tag{1.24}$$

In the analysis of the static economy, I have shown how decreasing the worker's outside option b can generate a Pareto improvement. In the dynamic environment, the effective worker's outside option corresponds to $U^{CE} - V^{CE}$ and is endogenously determined in equilibrium. In a decentralized economy agents' decisions determine the continuation utility of unemployed workers, without internalizing the cost in terms of efficient creation imposed by the informational rents.

1.5 Dynamic Efficiency

In this section I explore the efficiency properties of the dynamic competitive search equilibrium when matching frictions interact with informational asymmetry.

The static analysis shows that in the benchmark environment, where b cannot be transferred or destroyed, the competitive search equilibrium is constrained efficient. On the contrary, in the analogous dynamic environment, the social planner can improve the decentralized equilibrium allocation, because of an extra instrument to provide incentives. The planner can reward or punish workers reporting a low type, not only through the instantaneous consumption level, but also through continuation utilities. This allows the planner to internalize the externality coming from informational imperfections.

First, I analyze a simple example of inefficiency, driven by the same mechanism described in section 1.3 as *money burning*: changing the worker's outside option can lead to a Pareto Improvement. In the benchmark static environment b is exogenously given and cannot be reduced. However, in a dynamic setting, the social planner can improve the *ex-ante* welfare by affecting the workers' expected continuation utility, through future allocations.

Then, I characterize the social planning problem and show that the competitive equilibrium is constrained inefficient whenever the unemployment rate is away from the steady state level.

Moreover, I show that the steady state competitive search equilibrium, although it satisfies the necessary conditions for constrained efficiency, is not socially optimal according to the utilitarian welfare criterion. In particular, the informational asymmetry makes employed people, who enjoy the informational rents, better off than an utilitarian planner would prescribe.

Finally, I show that in the alternative environment where b is transferable and high enough the social planner can achieve the full information allocation, as it was the case in the static economy.

Social Planner. Analogously to the static setting, the social planner controls the matching process by deciding how many vacancies to open at the beginning of each period. He does not observe the types of the matched workers and has to induce them to truthfully reveal them. Moreover, there is lack of commitment from the side of the workers, who can always decide not to produce and to consume their private non-transferable utility b. I impose on the planner the same anonymity restriction that I impose on the decentralized economy: the pool of unemployed workers is anonymous. Once workers decide to consume b at time t, they can always join back the pool of unemployed at time $t+1$. Given these constraints, together with the resource constraint of the economy, the social planner decides how to allocate intertemporally non-negative consumption among employed and unemployed workers.

An allocation is a sequence of functions $[e_t(\theta)]_{\theta\in\Theta}$ representing the hiring probability for a worker who meets an employer at time t and reports type θ and a sequence of functions $[c_t(\theta, s)]_{\theta\in\Theta s\geq t}$ denoting the consumption at time s of the worker hired at time t reporting type θ, a sequence of consumption values for unmatched workers C_t^U, a sequence of consumption values C_t^V for employed workers who matched at time $\tau \leq t$, and a sequence of tightness values γ_t.[30]

Given that agents have linear utility, the path of consumption does not matter and I can restrict attention to the case in which $c_t(\theta, s) = c_t(\theta)$. Moreover, given that types are fixed over time within a match and there is no commitment problem after the match is implemented, the consumption profile over time is irrelevant for the analysis as in the competitive equilibrium. Thus, without loss of generality, I can characterize the efficient allocation as if the whole con-

[30]Without loss of generality, because of linear preferences, I assume that matched workers who are not hired at time t get the same consumption , C_t^U, of the unmatched ones.

sumption $\alpha c_t(\theta)$ and disutility $\alpha\theta$ took place at the beginning of the relationship. This implies that the *net continuation value* of being employed V, which I will refer to as the employed continuation utility, represents just the value of a current transfer plus the discounted expected value of being separated and becoming unemployed in the future, that is,

$$V_t = C_t^V + \beta s U_{t+1} + \beta(1-s) V_{t+1}.$$

The value of being unemployed at time t is instead

$$U_t = C_t^u + \beta\mu(\gamma_{t+1}) \int_{\underline{\theta}}^{\overline{\theta}} e_{t+1}(\theta) \left[\alpha\left[c_{t+1}(\theta) - \theta\right] + V_{t+1} - U_{t+1}\right] dF(\theta) + \beta U_{t+1}.$$

The *ad interim* utility of a matched worker of type θ, who reports to be of type $\tilde{\theta}$ at time t is given by

$$v_t^{SP}(\theta,\tilde{\theta}) = e_t(\tilde{\theta}) \left[\alpha[c_t(\tilde{\theta}) - \theta] + V_t - U_t\right] + U_t \text{ for all } \tilde{\theta}, \theta \in \Theta.$$

A straightforward generalization of the static analysis gives that an allocation is *incentive-compatible* when $e(.)$ is non-increasing and

$$v_t^{SP}(\theta,\theta) = v_t^{SP}\left(\overline{\theta},\overline{\theta}\right) + \alpha \int_{\theta}^{\overline{\theta}} e_t(y)\, dy \text{ for all } \theta \in \Theta. \tag{IC'}$$

Unemployed workers can choose at any point in time to stay in autarky, enjoy an instantaneous utility from leisure of value b and go back to the anonymous pool of unemployed at the beginning of the following period. Hence, an allocation satisfies the *participation* constraint, whenever

(i) $C_t^U \geq b$

(ii) $v_t^{SP}(\theta,\theta) \geq b - C_t^U + U_t$ for any θ.

Notice that when an allocation is incentive compatible condition *(ii)* reduces to

$$v^{SP}\left(\overline{\theta},\overline{\theta}\right) \geq b - C_t^U + U_t.$$

Hence, next lemma follows.

Lemma 21 *An incentive compatible allocation that satisfies the participation constraint must satisfy* $e(.)$ *non-increasing,* $C_t^U \geq b$ *and*

$$\int_{\underline{\theta}}^{\overline{\theta}} e_t(\theta) \left[\alpha \left(c_t(\theta) - \theta - \frac{F(\theta)}{f(\theta)} \right) + V_t - U_t \right] dF(\theta) + C_t^U - b \geq 0. \tag{1.25}$$

The *intertemporal resource constraint* ensures that aggregate consumption is covered by aggregate output, that is,

$$\begin{aligned} &\sum_{t=0}^{\infty} \beta^t \left\{ u_t \left[\beta\mu(\gamma_{t+1}) \int_{\underline{\theta}}^{\overline{\theta}} e_{t+1}(\theta) c_{t+1}(\theta) dF(\theta) + C_t^U \right] + (1-u_t) C_t^V \right\} \\ \leq\ & \sum_{t=0}^{\infty} \beta^t u_t \left[\mu(\gamma_{t+1}) \int_{\underline{\theta}}^{\overline{\theta}} e_{t+1}(\theta) \alpha y dF(\theta) + b - \gamma_{t+1} k \right], \end{aligned}$$

where u_t follows the law of motion

$$u_{t+1} = u_t \left[1 - \mu(\gamma_t) \int_{\underline{\theta}}^{\overline{\theta}} e_t(\theta) dF(\theta) \right] + (1-u_t) s \tag{1.26}$$

Assume that the social planner can transfer resources intertemporally at the fixed interest rate $r = \beta^{-1} - 1$, by borrowing at the beginning of time t at price β and by paying back at the beginning of time $t+1$. I assume that the economy does not have external resources so that the intertemporal resource constraint has to hold. Define P_t as the net resources of the planner at time t, that is,

$$P_t = \sum_{j=t}^{\infty} \beta^t \left\{ u_t \left[\beta\mu(\gamma_{t+1}) \int_{\underline{\theta}}^{\overline{\theta}} e_{t+1}(\theta) [\alpha y - c_{t+1}(\theta)] dF(\theta) - C_t^U \right] - (1-u_t) C_t^V \right\}.$$

Then, the resource constraint can be written in recursive terms by using the state variable P_t as

$$\begin{aligned} P_t \ \leq\ & u_t \left[\beta\mu(\gamma_{t+1}) \int_{\underline{\theta}}^{\overline{\theta}} e_{t+1}(\theta) \alpha(y - c_{t+1}(\theta)) dF(\theta) + b - \gamma_{t+1} k - C_t^u \right] \\ & - (1-u_t) C_t^V + \beta P_{t+1} \end{aligned} \tag{RC}$$

for any t.

Definition 22 *An allocation is feasible iff it (i) is incentive-compatible, (ii) satisfies the participation constraint and (iii) satisfies the resource constraint for any t.*

1.5.1 An Example of a Feasible Pareto Improvement.

I now show, with a simple example, that in a dynamic economy the competitive search equilibrium can be constrained inefficient. In this example, the mechanism driving inefficiency is the natural generalization of the static *money burning* result described in section 1.3. When all the workers are *ex-ante* unemployed, the economy can achieve a Pareto Improvement by reducing the worker's effective outside option.

Let assume that the economy is at the competitive search equilibrium. From the previous section it means that $e_t(\theta)$, γ_t, U_t and V_t are equal to the steady state competitive equilibrium level, CE, for any $t = 0, 1, \ldots$.

Define the analogous of g for the dynamic economy:

$$g \equiv 1 - \mu(\gamma^{CE})F(\hat{\theta}^{CE}) - \lambda^{CE}. \tag{1.27}$$

The value g represents a monotonic transformation of the effect of the workers' outside option on the *ex-ante* value of being unemployed at the competitive search equilibrium. There is a direct positive effect coming from the probability of being unemployed at the end of the period, $1 - \mu(\gamma^{CE})F(\hat{\theta}^{CE})$, and a negative effect coming from the tightness of the informational constraint, represented by λ^{CE}, since, as the outside option increases, it is more costly to reveal information.

Assume that all the workers start out as unemployed, so that the *ex-ante* welfare coincides with the *ex-ante* value of being unemployed, that is, $W = U_0$. Thus, when g is negative it would be Pareto improving to reduce the workers' outside option.

Proposition 23 *Whenever $g < 0$, the competitive search equilibrium allocation is Pareto inefficient.*

Proof. The proof proceeds by construction. Consider the feasible allocation where $\hat{V}_t^* =$

$\hat{V}^{CE}$ for any $t = 1, 2, \ldots$, $\hat{\theta}_t^* = \hat{\theta}^{CE}$ and $U_t^* = U^{CE}$ for any $t = 2, 3, \ldots$; $\gamma_t^* = \gamma^{CE}$ for any $t = 3, 4, \ldots$ and $\gamma_2^* = \gamma^{CE} + \varepsilon$, U_1^* is given by

$$U_1^* = \beta\mu(\gamma_2^*) \int_{\underline{\theta}}^{\hat{\theta}^{CE}} \left[\alpha(y - \theta) + \hat{V}^{CE} - U^{CE}\right] dF(\theta) + b - \gamma_2^* k + \beta U^{CE},$$

$\hat{\theta}_1^*$ and γ_1^* solve problem P2 at time 0 and W^* is given by

$$W^* = \beta\mu(\gamma_1^*) \int_{\underline{\theta}}^{\hat{\theta}_1^*} \left[\alpha(y - \theta) + \hat{V}^{CE} - U_1^*\right] dF(\theta) + b - \gamma_1^* k + \beta U_1^*.$$

First, the Envelope Condition shows that welfare *ex-ante* can be improved by reducing U_1 whenever $g < 0$. It follows that if in period 2, I perturb the competitive equilibrium level of γ, choosing $\gamma_2^* = \gamma + \varepsilon$, leaving everything else at the level of competitive equilibrium, then, by definition, U_1^* will be marginally lower than the competitive equilibrium level, thus increasing *ex-ante* welfare W^*, so that for any $\hat{\theta}_1$ and γ_1 the allocation proposed in the proposition is Pareto improving. Finally, $\hat{\theta}_1^*$ and γ_1^* solve problem P2 for given $\hat{V}^{CE}$ and U_1^*, so that the allocation is also feasible, completing the proof. ■

An equilibrium is constrained inefficient if there exists a feasible allocation that Pareto dominates it. It follows that a dynamic competitive search equilibrium is constrained inefficient whenever g is negative at the equilibrium. I naturally extend the static analysis and show that the dynamic competitive search equilibrium can be constrained inefficient for an interesting class of functions $\mu(.)$. In Section 2, I have shown that for $\mu(.)$ satisfying assumptions A1 and A2 or A3, for a given b there exists a set of the parameter space (k, y) such that the static version of g is negative. Now, I show that varying b for any pair (k, y), I can adjust the equilibrium value of unemployed workers, U^{CE}, so that there exists a set of the parameter space (b, k, y) such that the dynamic version of g is negative. The next proposition is the counterpart of Proposition 12 in the static analysis.

Proposition 24 *For any $\mu(.)$ satisfying Assumption A1 and either A2 or A3 and given $F(.)$, there exists an open set of the parameter space (b, k, y) such that the dynamic competitive search equilibrium is constrained inefficient.*

Proof. See Appendix. ■

1.5.2 The Social Planning Problem

I now characterize the constrained efficient allocation. The social planner, for a given initial rate of unemployment u_0, chooses a Pareto optimal pair U_0 and V_0. The problem can be written in a recursive form, by maximizing at a given t

$$C_t^u + \beta\mu\left(\gamma_{t+1}\right)\int_{\underline{\theta}}^{\hat{\theta}_{t+1}}\left[\alpha(c_{t+1}\left(\theta\right) - \theta) + V_{t+1} - U_{t+1}\right]dF\left(\theta\right) + \beta U_{t+1}$$

subject to feasibility as described in Definition 22, the law of motion of u_t given by (1.26) and the promise-keeping constraint

$$V_t = C_t^V + \beta s U_{t+1} + \beta\left(1-s\right)V_{t+1}.$$

In order to analyze the constrained efficient allocation, it is convenient to approach the social planner problem from a dual perspective. The social planner Bellman equation is a function of three state variables: the promised utility to employed workers, V, the promised utility to unemployed workers, U, and the unemployment rate, u. The planner maximizes the net resources of the economy, subject to two promise-keeping constraints for V and U, the law of motion of u and incentive compatibility and individual rationality, as summarized by Lemma 21. I study a relaxed version of the problem, where I do not impose the monotonicity of $e\left(.\right)$ and then I verify that the result of the optimization gives a monotone $e\left(.\right)$. Indeed, pointwise maximization, as in the competitive equilibrium analysis, implies that there exists a threshold $\hat{\theta}$ such that $e\left(\theta\right) = 1$ iff $\theta \leq \hat{\theta}$. Moreover, when $k > 0$ the constraint (1.25) is binding for any U and V,[31] so that I can solve for the optimal allocation substituting for $c\left(\theta\right)$ and recovering it from the binding constraint at the optimum.

The Bellman equation can be written as:

$$\begin{aligned} P\left(V^-,U^-,u^-\right) &= \max_{\substack{C^u,C^V \\ \hat{\theta},\gamma,u',V',U'}} u^-\beta\mu\left(\gamma\right)\int_{\underline{\theta}}^{\hat{\theta}}\left[\alpha\left(y-\theta-\frac{F\left(\theta\right)}{f\left(\theta\right)}\right)+V-U\right]dF\left(\theta\right) \qquad \text{(P6)} \\ &\quad + u^-\left[\left(1-\beta\mu\left(\gamma\right)\right)\left(b-C^U\right)-\gamma k\right]-\left(1-u^-\right)C^V+\beta P\left(V,U,u\right) \end{aligned}$$

[31] The proof of this statement is very similar to the proof of Lemma 6.

s.t.

$$[\nu] \qquad V^{-} = C^{V} + \beta s U + \beta (1-s) V$$

$$[\eta] \qquad U^{-} = C^{U} + \beta \mu(\gamma) \alpha \int_{\underline{\theta}}^{\hat{\theta}} \frac{F(\theta)}{f(\theta)} dF(\theta) + \beta \mu(\gamma) \left(b - C^{U}\right) + \beta U$$

$$[\pi] \qquad u = u^{-} \left[1 - \mu(\gamma) F(\hat{\theta})\right] + \left(1 - u^{-}\right) s$$

$$[\chi] \qquad C^{u} \geq b$$

I assume that the social planner, as the market economy, does not have access to any external resources. Therefore, for a given u_0, U_0 and V_0 are on the Pareto frontier iff $P(V_0, U_0, u_0) = 0$.

Next, I show the main result of the paper: away from the steady state, a competitive equilibrium is always constrained inefficient. Recall that the competitive equilibrium allocation, $\hat{\theta}^{CE}$, γ^{CE}, V^{CE} and U^{CE}, achieves immediately the steady state level, while the unemployment rate evolves slowly to the steady state, starting at a given u_0.

Proposition 25 *If $u_0 \neq u^{SS}$, then the competitive search equilibrium allocation is constrained inefficient.*

Proof. See Appendix. ■

The proof proceeds by contradiction. I assume that the competitive allocation solves the social planning problem and then I show that the necessary first order conditions are violated. In particular, the optimality condition that is violated is the one determining the level of the promised utility for the unemployed workers. The proof highlights the mechanism driving the inefficiency. The direction of the inefficiency depends on the dynamics of the unemployment rate. Constrained efficiency requires to set the optimal promised utility to unemployed workers, U_{t+1}, so that the marginal benefit of increasing it in terms of social surplus at time t equates the marginal cost at time $t+1$. In particular, the planner needs to satisfy the following Euler equation

$$-P_{U_{t+1}} = \left\{\nu_t (1-u_t) s + \eta_t [1 - \mu(\gamma_t) F(\hat{\theta}_t)] u_t\right\} - (1-\eta_t) u_t \mu(\gamma_t) F(\hat{\theta}_t) \qquad (1.28)$$

where η_t represents the shadow cost of increasing U_t, ν_t the shadow cost of increasing V_t and from the Envelope condition

$$P_{U_{t+1}} = -\eta_{t+1}u_t.$$

The right-hand side of equation (1.28) represents the net benefit at time t of a one unit increase of U_{t+1} for the social planner. The direct effect is to increase by η_t the utility of unemployed workers who will not be hired at the end of the period, $[1-\mu(\gamma_t)F(\hat{\theta}_t)]u_t$, and by ν_t the utility of employed workers who will be separated, $(1-u_t)s$. Moreover there is an indirect effect coming from the fact that an increase of U_{t+1} makes tighter the incentive constraint of the workers hired at time t, that is, $u_t\mu(\gamma_t)F(\hat{\theta}_t)$. The latter effect imposes a cost of $1-\eta_t$ per worker hired.

The left-hand side of the equation represents, instead, the cost of a one unit increase of U_{t+1} in terms of resources at time $t+1$. The cost of giving one additional unit of promised utility to each unemployed worker at time $t+1$ is smaller than one because increasing that utility allows the planner to increase the informational rents of workers who are hired at time $t+1$ and, thus, increase the social surplus in the future.

The Euler condition (1.28), by equating the right and the left-hand sides, implies that the evolution of u_t affects the dynamics of the shadow cost of informational extraction. In particular, when the unemployment rate is decreasing, the social planner wants to reduce the worker's effective outside option because he wants to extract information from the unemployed workers who match today who are relatively more abundant. On the contrary, when the unemployment rate is rising, then there is a gain from increasing the worker's effective outside option because it makes it less costly to extract information from unemployed workers who match tomorrow. The planner can indeed manipulate the continuation utilities by changing the future choices of vacancy creation and hiring margins. In steady state the flows in and out of unemployment are equal and the cost of extracting information is invariant over time. Recall that in competitive equilibrium the allocation $\hat{\theta}$ and γ and the continuation utilities U and V are constant over time. Hence, the equilibrium dynamics are characterized only by the evolution of the unemployment rate. It follows that the competitive equilibrium cannot meet this optimality condition away from the steady state. Denote with $\mathcal{L}$ the Lagrangian associated to P6. Then, we can write:

$$\frac{\partial\mathcal{L}}{\partial U_{t+1}} \gtrless 0 \text{ if } \left(u_{t+1}^{CE}-u_t^{CE}\right)\lambda^{CE} \gtrless 0.$$

The steady state competitive search equilibrium satisfies the necessary conditions of the social planning problem. In fact, when the mass of unemployed workers is constant over time the externality is muted. When the unemployment rate is at the steady state level, the flow of workers out of unemployment, who enjoy the informational gain from a reduction of the outside option, are offset by the flow of workers into unemployment, who are damaged by a future lower expected value.[32]

Proposition 25 generalizes the "money burning" example of Pareto improvement discussed in subsection 5.1. That example relies on the extreme case of all the workers being unemployed *ex ante*, which, by construction, implies that the unemployment rate is decreasing to the steady state level.[33] Hence, it focuses only on one direction of the inefficiency. If we assume by contradiction that the competitive equilibrium is constrained efficient, equation (1.28) could be rewritten as

$$u_{t+1}\left(1-\lambda^{CE}\right)=u_t g+\left(1-u_t\right)s$$

where g is expression (1.27) in subsection 5.1. Recall that g represents the impact of an increase of the promised utility to unemployed workers tomorrow. There are two reasons why the general dynamic analysis differs from the simple "money burning" in the example, where $u_t = 1$ and $g < 0$. First, even though the impact of an increase of the promised utility tomorrow on workers today is overall positive, that is, $g > 0$, the relatively high mass of potential matches today would make the social planner decrease the promised utility U_{t+1}. Second, if there is a mass of employed workers today, that is, $u_t < 1$, then it is possible that the direction of the inefficiency is reversed and in particular it will be the case exactly when $u_t < u_{t+1}$.

1.5.3 Utilitarian welfare

Now I want to consider a particular welfare criterion commonly used in the literature: the utilitarian criterion. It is interesting to notice that the steady state competitive search equilibrium does not maximize the utilitarian welfare function. When utility is perfectly transferable the Pareto frontier is linear and this criterion can be used without loss of generality in order to

[32]Notice that unfortunately problem P6 is not concave so that I cannot state that the first order conditions are also sufficient to characterize the constrained efficient allocation.

[33]The Pareto improvement built on the money burning mechanism works more generally for a high enough ex-ante unemployment rate.

determine the efficient allocation. However, in the context of this environment where b is not transferable, the Pareto frontier is typically not linear. Nevertheless, the utilitarian welfare represents the long run expected welfare of a worker *ex-ante*. For this reason, it is interesting to note that the steady state competitive search equilibrium does not maximize welfare according to this criterion due to the informational asymmetry which makes unemployed people worse off than what the utilitarian planner would prescribe.

Corollary 26 *If P is differentiable, then the competitive search steady state equilibrium does not maximize the utilitarian welfare function.*

Proof. The proof of the previous Proposition shows that the competitive search steady state equilibrium satisfies the first order condition of the social planner problem. Then, either it is not an optimum, since the problem is not concave, and then it cannot maximize the utilitarian welfare function, or it is a point on the Pareto frontier. If this is the case, from the Envelope conditions, given that P is assumed differentiable, then

$$P_U = -u^{CE}(1 - \lambda^{CE})$$

$$P_V = -\left(1 - u^{CE}\right).$$

It follows directly that the allocation does not maximize the total output of the economy, equal to $uU + (1 - u)V$, as long as $\lambda^{CE} > 0$, given that

$$\frac{dU}{dV} = -\frac{1 - u^{CE}}{u^{CE}(1 - \lambda^{CE})} < -\frac{1 - u^{CE}}{u^{CE}}$$

completing the proof. ■

1.5.4 Transferability restores Full Information.

Finally, I consider the alternative environment in which b is freely transferable. As in the static setting, under this relaxed assumption, a net lump-sum transfer from unemployed workers to employed workers, τ_t at time t, makes the competitive equilibrium reach the full information allocation, γ^{FI} and $e^{FI}(.)$, if b is high enough. Budget balance requires that employed workers receive a transfer of $(1-u_t)^{-1}u_t\tau_t$ at time t. Notice that this policy can be interpreted, as in the static section, as a subsidy to job creation financed by a lump-sum tax on workers both employed and unemployed.

Proposition 27 *When b is transferable and high enough, lump-sum transfers from unemployed to employed workers can make the competitive search equilibrium achieve the full information allocation.*

Proof. From the informational constraint, there must exist an H such that

$$\beta\mu\left(\gamma^{FI}\right)\int_{\underline{\theta}}^{\overline{\theta}} e^{FI}(\theta)\left[\alpha(y-\theta)+H-\frac{F(\theta)}{f(\theta)}\right]dF(\theta)=\gamma^{FI}k$$

so that a competitive equilibrium allocation reaching the full information outcome requires $V_t - U_t \geq H\ \forall t$. The equilibrium has to satisfy

$$U_t=\beta\mu\left(\gamma^{FI}\right)\int_{\underline{\theta}}^{\overline{\theta}} e^{FI}(\theta)\left[\alpha(y-\theta)+V_{t+1}-U_{t+1}\right]dF(\theta)+b-\tau_t-\gamma^{FI}k+\beta U_{t+1}$$

$$V_t=\left(\frac{u_t}{1-u_t}\right)\tau_t+\beta sU_{t+1}+\beta(1-s)V_{t+1}$$

$$u_{t+1}=u_t\left[1-\mu\left(\gamma^{FI}\right)\int_{\underline{\theta}}^{\overline{\theta}} e^{FI}(\theta)\,dF(\theta)\right]+(1-u_t)s$$

Then, imposing that $V_t-U_t=H$ for any t and combining the equations above it follows that

$$\begin{aligned}H &= \hat{V}_t-U_t=\left(\frac{1}{1-u_t}\right)\tau_t-\alpha(y-\theta)\beta\mu\left(\gamma^{FI}\right)\int_{\underline{\theta}}^{\overline{\theta}} e^{FI}(\theta)\,dF(\theta)-b+\gamma^{FI}k\\&\quad-\beta sH-\beta\mu\left(\gamma^{FI}\right)\int_{\underline{\theta}}^{\overline{\theta}} e^{FI}(\theta)\,dF(\theta)\,H+\beta H\end{aligned}$$

so that the full information allocation is achievable in competitive equilibrium by imposing the following tax

$$\begin{aligned}\tau_t &= (1-u_t)\left(\alpha(y-\theta)\beta\mu\left(\gamma^{FI}\right)\int_{\underline{\theta}}^{\overline{\theta}} e^{FI}(\theta)\,dF(\theta)+b-\gamma^{FI}k\right)H\\&\quad+(1-u_t)\left(1-\beta+\beta s+\beta\mu\left(\gamma^{FI}\right)\int_{\underline{\theta}}^{\overline{\theta}} e^{FI}(\theta)\,dF(\theta)\right)H\end{aligned}$$

When $b \geq \tau_t$ the policy is feasible completing the proof. ∎

1.6 Conclusions

In this paper, I have modeled the interaction between informational imperfections and matching frictions in labor markets. My focus has been to analyze the ability of labor markets with these features to decentralize the efficient allocation of resources.

The two crucial assumptions of my setup are: the presence of asymmetric information and the fact that workers' always have the option to quit and go back to the unemployed pool. In this setup a new type of externality arises, which can lead to inefficient job creation. In order to highlight the role of this externality, I have used the equilibrium notion of competitive search, which correctly internalizes the search externality generated by matching frictions. I have shown that, under asymmetric information, the efficiency property of competitive search may fail. All along, I have framed the efficiency analysis in terms of constrained efficiency, by defining a social planning problem subject to the same constraints faced by the decentralized economy.

My model shows that the competitive search equilibrium is constrained inefficient whenever the unemployment rate is away from the steady state level. A natural business cycle interpretation would suggest that decentralized economies may react inefficiently to booms and to recessions. In particular, there is insufficient creation in recessions and excessive creation in booms. An interesting area for future research is to add aggregate shocks and to study explicitly the business cycle implications of the model. In particular, it would be interesting to design optimal policies to restore efficiency, for example in terms of optimal subsidies to job creation. The model suggests that countercyclical subsidies may be an optimal response to business cycle shocks.

Appendix

Proof of Proposition 3. The proof follows closely Acemoglu and Shimer (1999a) and proceeds in two steps: step 1 shows that any equilibrium solves problem P1 and step 2 shows that any solution to P1 is part of an equilibrium.

Step 1. Let $\{\mathbb{C}, \Gamma, U\}$ be an equilibrium with $\mathcal{C}^* \in \mathbb{C}$ and $\gamma^* = \Gamma(\mathcal{C}^*)$. I show that $\{\mathcal{C}^*, \gamma^*\}$, where $\mathcal{C}^* = [e^*(\theta), \omega^*(\theta)]_{\theta \in \Theta}$, solves P1. First, profit maximization ensures that $\{\mathcal{C}^*, \gamma^*\}$ solves constraint (1.2) together with IC and IR.

Suppose now that another pair $\{\mathcal{C}, \gamma\}$ satisfies IC, IR and achieves an higher value of the objective, that is,

$$\mu(\gamma) \int_{\underline{\theta}}^{\overline{\theta}} [\omega(\theta) - e(\theta)(\theta + b)] \, dF(\theta) + b > U,$$

I show that it must violates constraint (1.2). Since $\{\mathbb{C}, \Gamma, U\}$ is an equilibrium, optimal job application implies

$$\mu(\Gamma(\mathcal{C})) \int_{\underline{\theta}}^{\overline{\theta}} [\omega(\theta) - e(\theta)(\theta + b)] \, dF(\theta) + b \leq U$$

and given IR this implies that $\mu(\Gamma(\mathcal{C})) < \mu(\gamma)$ and so $\Gamma(\mathcal{C}) > \gamma$. Then, combining this with profit maximization, it follows that

$$\frac{\mu(\Gamma(\mathcal{C}))}{\Gamma(\mathcal{C})} \int_{\underline{\theta}}^{\overline{\theta}} [e(\theta) y - \omega(\theta)] \, dF(\theta) - k < \frac{\mu(\gamma)}{\gamma} \int_{\underline{\theta}}^{\overline{\theta}} [e(\theta) y - \omega(\theta)] \, dF(\theta) - k \leq 0.$$

This implies that $\{\mathcal{C}, \gamma\}$ violates (1.2), completing the proof of the first step.

Step 2. This step shows that for any solution $\{\mathcal{C}^*, \gamma^*\}$ to problem P1, there is an equilibrium $\{\mathbb{C}, \Gamma, U\}$ with $\mathbb{C} = \{\mathcal{C}^*\}$ and $\Gamma(\mathcal{C}^*) = \gamma^*$. Set

$$U = \mu(\gamma^*) \int_{\underline{\theta}}^{\overline{\theta}} [\omega^*(\theta) - e^*(\theta)(\theta + b)] \, dF(\theta) + b$$

and let $\Gamma(\mathcal{C})$ satisfy

$$U = \mu(\Gamma(\mathcal{C})) \int_{\underline{\theta}}^{\overline{\theta}} [\omega(\theta) - e(\theta)(\theta + b)] \, dF(\theta) + b,$$

or $\Gamma(\mathcal{C}) = 0$ if either IC or IR are not satisfied. It follows that $\{\mathbb{C}, \Gamma, U\}$ satisfies the optimal application for jobs.

To complete the proof I now show that it also satisfies firms' profit maximization. Suppose by contradiction that it is violated by a pair $\{C', \gamma'\}$ which satisfies IC and IR, but such that

$$\frac{\mu\left(\Gamma\left(C'\right)\right)}{\Gamma(C')}\int_{\underline{\theta}}^{\overline{\theta}}\left[e'\left(\theta\right)y-\omega'\left(\theta\right)\right]dF\left(\theta\right)-k>0.$$

Then, I can choose $\gamma' > \Gamma\left(C'\right)$ such that

$$\frac{\mu\left(\gamma'\right)}{\gamma'}\int_{\underline{\theta}}^{\overline{\theta}}\left[e'\left(\theta\right)y-\omega'\left(\theta\right)\right]dF\left(\theta\right)-k=0.$$

Then, by the construction of Γ, $\gamma' > \Gamma\left(C'\right)$ and IR imply

$$U<\mu\left(\gamma'\right)\int_{\underline{\theta}}^{\overline{\theta}}\left[\omega'\left(\theta\right)-e'\left(\theta\right)\left(\theta+b\right)\right]dF\left(\theta\right)+b,$$

so that the pair $\{C', \gamma'\}$ satisfies all the constraints, but generates an higher value for the objective function, yielding to a contradiction.

Proof of Proposition 4. The proof proceeds in two steps. First I show that any pair $\{e\left(\theta\right), \gamma\}$ solving problem P1 is also a solution to problem P2 and then I show that for any such a pair, I can construct a wage function $\omega\left(\theta\right)$ which solves problem P1.

Step 1. First, from constraints IC' and IR'

$$\int_{\underline{\theta}}^{\overline{\theta}}v\left(\theta,\theta\right)dF\left(\theta\right)=v\left(\overline{\theta},\overline{\theta}\right)+\int_{\underline{\theta}}^{\overline{\theta}}\left[\int_{\theta}^{\overline{\theta}}e\left(y\right)dy\right]dF\left(\theta\right), \tag{1.29}$$

$$v\left(\overline{\theta},\overline{\theta}\right)\geq b. \tag{1.30}$$

From integration by parts it follows

$$\int_{\underline{\theta}}^{\overline{\theta}}\left[\int_{\theta}^{\overline{\theta}}e\left(y\right)dy\right]dF\left(\theta\right)=\int_{\underline{\theta}}^{\overline{\theta}}e\left(\theta\right)F\left(\theta\right)d\theta,$$

which combined with (2.24) gives

$$\int_{\underline{\theta}}^{\overline{\theta}}\left[\omega\left(\theta\right)-e\left(\theta\right)\left(\theta+b\right)\right]dF\left(\theta\right)+b=\int_{\underline{\theta}}^{\overline{\theta}}e\left(\theta\right)\frac{F\left(\theta\right)}{f\left(\theta\right)}dF\left(\theta\right)+v\left(\overline{\theta},\overline{\theta}\right).$$

Using (2.25) I get

$$\int_{\underline{\theta}}^{\overline{\theta}} \left[\omega(\theta) - e(\theta)\left(\theta + b + \frac{F(\theta)}{f(\theta)}\right)\right] dF(\theta) \geq 0.$$

Then, a relaxed version of problem P1, where I leave to check the monotonicity of $e(\theta)$ for the end, can be rewritten as

$$U = \max_{e(\theta),\omega(\theta),\gamma} \mu(\gamma) \int_{\underline{\theta}}^{\overline{\theta}} \left[\omega(\theta) - e(\theta)(\theta + b)\right] dF(\theta) + b$$

s.t.

$$\mu(\gamma) \int_{\underline{\theta}}^{\overline{\theta}} \left[e(\theta)\, y - \omega(\theta)\right] dF(\theta) = \gamma k$$

$$\int_{\underline{\theta}}^{\overline{\theta}} \left[\omega(\theta) - e(\theta)\left(\theta + b + \frac{F(\theta)}{f(\theta)}\right)\right] dF(\theta) \geq 0$$

where I can eliminate the wage from the program by using the free-entry condition, ending up with problem P2 exactly.

Step 2. Now I show that for any $\{e(\theta), \gamma\}$ solving problem P2, I can construct a wage which, together with the same $\{e(\theta), \gamma\}$, satisfies problem P1. In particular pointwise maximization gives that at the optimum

$$e(\theta) = \begin{cases} 1 & \text{if } \theta \leq \hat{\theta} \\ 0 & \text{if } \theta > \hat{\theta} \end{cases}$$

where $\hat{\theta}$ is implicitly defined by

$$\hat{\theta} = y - b - \lambda \frac{F(\hat{\theta})}{f(\hat{\theta})}.$$

I can construct the wage schedule

$$\omega(\theta) = \begin{cases} \hat{\theta} + b & \text{if } \theta \leq \hat{\theta} \\ 0 & \text{if } \theta > \hat{\theta} \end{cases},$$

which satisfies the IC constraints, completing the proof.

Proof of Proposition 5.

Step 1. Existence.

First, notice that Proposition 4 shows that for any solution $e^*(\theta)$ and γ^* of problem P2 there exists a function $\omega^*(\theta)$ such that $\omega^*(\theta)$, $e^*(\theta)$ and γ^* are a solution of problem P1. Then the existence of a solution of problem P2 is sufficient for the existence of a solution to problem P1. Next, to show existence of a solution to problem P2, I first show that there exists a solution to the relaxed version of P2, without assuming the monotonicity condition on $e(\theta)$, and then I show that $e(\theta)$ is in fact monotone, implying that there exists a solution to the original problem P2.

Pointwise maximization, together with the monotone hazard rate assumption, implies that there exists a threshold $\hat{\theta}$ such that $e(\theta) = 1$ if $\theta < \hat{\theta}$ and $e(\theta) = 0$ otherwise. This shows directly that $e(\theta)$ is in fact non-increasing and that a solution to the relaxed version of problem P2 is also a solution of the original problem. This allows me to reduce the control variables to $\hat{\theta}$ and γ. The relaxed problem can be written as

$$U = \max_{\hat{\theta},\gamma} \mu(\gamma) \int_{\underline{\theta}}^{\hat{\theta}} [y - \theta - b] \, dF(\theta) + b - \gamma k$$

s.t.

$$F(\hat{\theta},\gamma) \equiv \mu(\gamma) \int_{\underline{\theta}}^{\hat{\theta}} \left[y - \theta - b - \frac{F(\theta)}{f(\theta)} \right] dF(\theta) - \gamma k \geq 0.$$

It is straightforward to see that the objective function is continuous in $\hat{\theta}$ and γ and that the constraint set is compact, since $F(\hat{\theta},\gamma)$ is continuous in both its arguments and is not empty, since, for example, $\gamma = 0$ and any $\hat{\theta}$ satisfies it. Existence follows directly.

Step 1. Uniqueness.

Proposition 4 shows that an equilibrium can be characterized by and array $\hat{\theta}$, γ and λ that must satisfy equations (3.43), (2.6) and (3.44). Notice that equation (3.43) defines implicitly $\hat{\theta}$ as a function of λ with $\partial\hat{\theta}/\partial\lambda < 0$ which can be substituted for into equations (2.6) and (3.44). Now there are two equations in two unknown, $\hat{\theta}$ and λ:

$$f_1(\gamma,\lambda) = \mu'(\gamma) \int_{\underline{\theta}}^{\hat{\theta}(\lambda)} \left[y - \theta - b - \lambda \frac{F(\theta)}{f(\theta)} \right] dF(\theta) - k$$

$$f_2(\gamma,\lambda) = \frac{\mu(\gamma)}{\gamma} \int_{\underline{\theta}}^{\hat{\theta}(\lambda)} \left[y - \theta - b - \frac{F(\theta)}{f(\theta)} \right] dF(\theta) - k$$

Notice that $f_1(\gamma,\lambda)$ and $f_2(\gamma,\lambda)$ define implicitly two functions, which I name $\gamma_1(\lambda)$ and $\gamma_2(\lambda)$.

Then, the implicit function theorem implies that

$$\frac{d\gamma_1(\lambda)}{d\lambda}=-\frac{\frac{\partial f_1(\gamma,\lambda)}{\partial\lambda}}{\frac{\partial f_1(\gamma,\lambda)}{\partial\gamma}}>0 \text{ and } \frac{d\gamma_2(\lambda)}{d\lambda}=-\frac{\frac{\partial f_2(\gamma,\lambda)}{\partial\lambda}}{\frac{\partial f_2(\gamma,\lambda)}{\partial\gamma}}>0,$$

since

$$y-\hat{\theta}-b-\frac{F(\hat{\theta})}{f(\hat{\theta})}<y-\hat{\theta}-b-\lambda\frac{F(\hat{\theta})}{f(\hat{\theta})}=0.$$

It follows that the two curves must intersect at most once. Moreover, given that I have proved existence in the previous step, they must intersect exactly at one point, completing the proof.

Proof of Proposition 12. The proof separately analyzes the case of $\mu(\gamma)$ satisfying assumptions A1 and A2 and $\mu(\gamma)$ satisfying A1 and A3.

Case 1. $\mu(\gamma)$ satisfies A1 and A2.

Recall that for any given y and k, the equilibrium $\hat{\theta}$, γ and λ must satisfy equations (3.43), (3.44) and (2.6). Notice that using equation (3.43) and integration by parts yields

$$\int_{\underline{\theta}}^{\hat{\theta}}\left[y-\theta-b-\lambda\frac{F(\theta)}{f(\theta)}\right]dF(\theta)=(1-\lambda)\int_{\underline{\theta}}^{\hat{\theta}}F(\theta)\,d\theta+\lambda\frac{[F(\hat{\theta})]^2}{f(\hat{\theta})}$$

Moreover,

$$\int_{\underline{\theta}}^{\hat{\theta}}\left[y-\theta-b-\frac{F(\theta)}{f(\theta)}\right]dF(\theta)=\lambda\frac{[F(\hat{\theta})]^2}{f(\hat{\theta})}.$$

Then, I can rewrite equations (3.44) and (2.6) as

$$\lambda\frac{\mu(\gamma)}{\gamma}\frac{[F(\hat{\theta})]^2}{f(\hat{\theta})}=k, \tag{1.31}$$

$$\mu'(\gamma)\left[(1-\lambda)\int_{\underline{\theta}}^{\hat{\theta}}F(\theta)\,d\theta+\lambda\frac{[F(\hat{\theta})]^2}{f(\hat{\theta})}\right]=k \tag{1.32}$$

and, when γ is an interior solution, by combining them it follows

$$\frac{\lambda}{1-\lambda}=\left[\frac{\eta(\gamma)}{1-\eta(\gamma)}\right]\frac{f(\hat{\theta})}{[F(\hat{\theta})]^2}\int_{\underline{\theta}}^{\hat{\theta}}F(\theta)\,d\theta, \tag{1.33}$$

where $\eta(\gamma)$ denotes the elasticity of $\mu(\gamma)$, that is,

$$\eta(\gamma) = \frac{\gamma \mu'(\gamma)}{\mu(\gamma)}.$$

Equations (3.43), (1.31) and (1.33) define an equilibrium $\hat{\theta}$, γ and λ.

Consider a family of economies parametrized by a pair (ε, δ) for (ε, δ) belonging to a (one-sided) neighborhood of $(0,0)$ such as $\mathcal{I} \equiv (0, \bar{\varepsilon}) \times (0, \bar{\delta})$ with $\bar{\varepsilon}$ and $\bar{\delta}$ strictly positive. For each pair (ε, δ), set the parameters k and y such that

$$k(\varepsilon, \delta) = \lambda(\varepsilon, \delta) \frac{\mu(\varepsilon)}{\varepsilon} \frac{[F(\bar{\theta} - \delta)]^2}{f(\bar{\theta} - \delta)} \tag{1.34}$$

and

$$y(\varepsilon, \delta) = b + \bar{\theta} - \delta + \lambda(\varepsilon, \delta) \frac{F(\bar{\theta} - \delta)}{f(\bar{\theta} - \delta)}, \tag{1.35}$$

where $\lambda(\varepsilon, \delta)$ solves

$$\frac{\lambda(\varepsilon, \delta)}{1 - \lambda(\varepsilon, \delta)} = \left[\frac{\eta(\varepsilon)}{1 - \eta(\varepsilon)}\right] \frac{f(\bar{\theta} - \delta)}{[F(\hat{\theta})]^2} \int_{\underline{\theta}}^{\bar{\theta} - \delta} F(\theta)\, d\theta$$

and $\eta(\varepsilon) = \varepsilon \mu'(\varepsilon) / \mu(\varepsilon)$. Notice that (1.34) and (1.35) make $\gamma = \varepsilon$ and $\hat{\theta} = \bar{\theta} - \delta$ an equilibrium solution, given that Proposition 5 shows that the first order conditions have a unique solution. Notice that, as we assumed, given that $\varepsilon > 0$, γ is an interior solution.

Next I show that equations (1.34) and (1.35) define a continuous and invertible mapping between the space (ε, δ) and the space (k, y). The determinant of the Jacobian of the bidimensional function $f(\varepsilon, \delta)$, where $f_1 \equiv k(\varepsilon, \delta)$ and $f_2 \equiv y(\varepsilon, \delta)$, is

$$\begin{aligned}
\det J(\varepsilon, \delta) \quad = \quad & -\lambda(\varepsilon, \delta) \frac{1}{\varepsilon} \left[\mu'(\varepsilon) - \frac{\mu(\varepsilon)}{\varepsilon}\right] \frac{[F(\bar{\theta} - \delta)]^2}{f(\bar{\theta} - \delta)} \\
& \left[1 + \lambda(\varepsilon, \delta) \partial \frac{F(\bar{\theta} - \delta)}{f(\bar{\theta} - \delta)} / \partial (\bar{\theta} - \delta) - \frac{\partial \lambda(\varepsilon, \delta)}{\partial \delta} \frac{F(\bar{\theta} - \delta)}{f(\bar{\theta} - \delta)}\right] \\
& - \frac{\mu(\varepsilon)}{\varepsilon} \frac{\partial \lambda(\varepsilon, \delta)}{\partial \varepsilon} \frac{[F(\bar{\theta} - \delta)]^2}{f(\bar{\theta} - \delta)} [1 - \lambda(\varepsilon, \delta)],
\end{aligned}$$

where

$$
\begin{aligned}
\frac{\partial\lambda(\varepsilon,\delta)}{\partial\delta} &= \frac{\eta(\gamma)(1-\eta(\gamma))}{[1-\eta(\gamma)(1-D(\delta))]}\left(\frac{F(\bar{\theta}-\delta)}{f(\bar{\theta}-\delta)}\right)^{-1} \\
&\left[D(\delta)\left(1-\frac{dF(\bar{\theta}-\delta)/f(\bar{\theta}-\delta)}{d(\bar{\theta}-\delta)}\right)+1\right]
\end{aligned}
$$

and

$$
\frac{\partial\lambda(\varepsilon,\delta)}{\partial\varepsilon} = \frac{\eta'(\varepsilon)D(\delta)}{[1-\eta(\varepsilon)(1-D(\delta))]}
$$

since $\eta'(\varepsilon) < 0$, with

$$
D(\delta) \equiv \left(\frac{F(\bar{\theta}-\delta)}{f(\bar{\theta}-\delta)}\right)^{-1} F(\bar{\theta}-\delta)^{-1}\int_{\underline{\theta}}^{\bar{\theta}-\delta} F(\theta)\,d\theta.
$$

Notice that $\det J(\varepsilon,\delta) > 0$ for any $(\varepsilon,\delta) \in \mathcal{I}$, where $\mathcal{I}$ is small enough. This is easy to see. In fact $\partial\lambda(\varepsilon,\delta)/\partial\varepsilon < 0$, given that $\eta'(\varepsilon) < 0$ and $\mu'(\varepsilon) < \mu(\varepsilon)/\varepsilon$ for any $\varepsilon > 0$, given that $\mu(.)$ is strictly concave. Moreover either $\partial\lambda(\varepsilon,\delta)/\partial\delta$ is positive, or, if negative, $\lim_{(\varepsilon,\delta)\to(0,0)}\partial\lambda(\varepsilon,\delta)/\partial\delta = 0$ so that we can choose $\mathcal{I}$ small enough so that $\partial\lambda(\varepsilon,\delta)/\partial\delta$ is sufficiently close to zero for $(\varepsilon,\delta) \in \mathcal{I}$, completing the argument. Notice that this is where the assumption of strict concavity of $\mu(.)$ is required.

Then, think of g as a function of (ε,δ), that is,

$$
g(\varepsilon,\delta) \equiv 1-\mu(\varepsilon)F(\bar{\theta}-\delta)-\lambda(\varepsilon,\delta).
$$

Thus, if for a small enough neighborhood $\mathcal{I}$, $g(\varepsilon,\delta) < 0$ for any $(\varepsilon,\delta) \in \mathcal{I}$, then there exists an open set of the space (k,y) for which $g < 0$.

Next, I show that $\lim_{\varepsilon\to 0} g(\varepsilon,\delta) = 0$ for any $\delta < \bar{\theta}-\underline{\theta}$. Given that $\mu(0) = 0$ and $\mu(\gamma)$ is everywhere differentiable it follows that

$$
\mu'(0) = \lim_{\varepsilon\to 0}\frac{\mu(\varepsilon)}{\varepsilon} \Rightarrow \lim_{\varepsilon\to 0}\eta(\varepsilon) = 1.
$$

Then, equation (1.33) yields

$$
\lim_{\varepsilon\to 0}\lambda(\varepsilon,\delta) = 1\ \forall\delta < \bar{\theta}-\underline{\theta},
$$

given that, by construction, equation (1.35) implies $\hat{\theta} = \bar{\theta} - \delta > \underline{\theta}$ for $\delta < \bar{\theta} - \underline{\theta}$, completing the argument. It follows that

$$\lim_{\varepsilon \to 0} g(\varepsilon, \delta) = 0 \ \forall \delta < \bar{\theta} - \underline{\theta}.$$

Then, to show that $g(\varepsilon, \delta) < 0$ for any $(\varepsilon, \delta) \in \mathcal{I}$ where $\mathcal{I}$ is small enough, it is sufficient to show that $\partial g(\varepsilon, \delta) / \partial \varepsilon < 0$ for any $(\varepsilon, \delta) \in \mathcal{I}$.

Notice that

$$\left. \frac{\partial g(\varepsilon, \delta)}{\partial \varepsilon} \right|_{(\varepsilon,\delta) \in \mathcal{I}} = -\mu'(\varepsilon) F(\bar{\theta} - \delta) - \left. \frac{\partial \lambda(\varepsilon, \delta)}{\partial \varepsilon} \right|_{(\varepsilon,\delta) \in \mathcal{I}}.$$

Equation (1.33) yields

$$\lim_{(\varepsilon,\delta) \to (0,0)} \frac{\partial \lambda(\varepsilon, \delta)}{\partial \varepsilon} = \lim_{\varepsilon \to 0} \frac{\eta'(\varepsilon)}{f(\bar{\theta}) \int_{\underline{\theta}}^{\bar{\theta}} F(\theta)\, d\theta}.$$

Moreover $\lim_{\delta \to 0} F(\bar{\theta} - \delta) = 1$ and, by assumption, $\lim_{\varepsilon \to 0} \mu'(\varepsilon) = \mu'(0) > 0$ and $\lim_{\varepsilon \to 0} \eta'(\varepsilon) > -\mu'(0) f(\bar{\theta}) \int_{\underline{\theta}}^{\bar{\theta}} F(\theta)\, d\theta$. It follows that I can choose $\bar{\varepsilon}$ and $\bar{\delta}$ small enough such that $\partial g(\varepsilon, \delta) / \partial \varepsilon|_{(\varepsilon,\delta) \in \mathcal{I}} < 0$, completing the proof.

Case 2. $\mu(\gamma)$ satisfies A1 and A3.

The proof of this case proceeds in two steps. First I show that there exists a $\bar{k}$ such that either $k \geq \bar{k}$ and $\gamma = 0$ or $k < \bar{k}$ and $\gamma \in [\underline{\gamma}, \bar{\gamma}]$, which is the interesting case. Then, focusing on this case, I divide the family of functions $\mu(\gamma)$, satisfying assumptions A1 and A3 into two subfamilies and I show that in both cases, there exists a parameter set such that g is negative.

Step 1.

In order to analyze the competitive search equilibrium allocation with two points of non-differentiability $\underline{\gamma}$ and $\bar{\gamma}$, first notice that, when the constraint is binding, for given y and k, the equilibrium still has to satisfy equations (3.43) and (3.44).

Define $\tilde{\theta}$ such that

$$y - b - \tilde{\theta} - \frac{F(\tilde{\theta})}{f(\tilde{\theta})} = 0.$$

From equation (3.43), it follows that $\hat{\theta} \geq \tilde{\theta}$. Moreover notice that

$$\tilde{\theta} = \arg\max_{\hat{\theta}} \int_{\underline{\theta}}^{\hat{\theta}} \left[y - \theta - b - \frac{F(\theta)}{f(\theta)} \right] dF(\theta)$$

and I can define $\bar{k}$ the maximum value

$$\bar{k} \equiv \int_{\underline{\theta}}^{\bar{\theta}} \left[y - \theta - b - \frac{F(\theta)}{f(\theta)} \right] dF(\theta) > 0,$$

since $y > b + \underline{\theta}$. For any $k \geq \bar{k}$, given that $\mu(\gamma)/\gamma \leq 1$, the only feasible equilibrium is characterized by zero vacancy, because the total surplus is not enough to cover the cost of information revelation and the cost of creating a vacancy. So, from now on, let $k < \bar{k}$.

It is, then, straightforward to see that the optimal $\gamma \in [\underline{\gamma}, \bar{\gamma}]$. In fact, $\gamma < \underline{\gamma}$ cannot be an equilibrium given that $\mu'(\gamma) = \mu(\gamma)/\gamma = 1$ and

$$\int_{\underline{\theta}}^{\hat{\theta}} \left[y - \theta - b - \lambda \frac{F(\theta)}{f(\theta)} \right] dF(\theta) > \int_{\underline{\theta}}^{\hat{\theta}} \left[y - \theta - b - \frac{F(\theta)}{f(\theta)} \right] dF(\theta) = k.$$

Moreover, $\gamma > \bar{\gamma}$ cannot be an equilibrium since $\mu'(\gamma) = 0$, $\mu(\gamma)/\gamma = 1/\bar{\gamma}$ and

$$0 < \frac{1}{\bar{\gamma}} \int_{\underline{\theta}}^{\hat{\theta}} \left[y - \theta - b - \frac{F(\theta)}{f(\theta)} \right] dF(\theta) = k.$$

It follows that the equilibrium can be characterized by a tuple $\hat{\theta}$, γ and λ solving equations (3.43), (3.44) and

$$\mu'(\gamma) \int_{\underline{\theta}}^{\hat{\theta}} \left[y - \theta - b - \lambda \frac{F(\theta)}{f(\theta)} \right] dF(\theta) \begin{cases} < k & \text{if } \gamma = \underline{\gamma} \\ = k & \text{if } \gamma \in (\underline{\gamma}, \bar{\gamma}) \\ > k & \text{if } \gamma = \bar{\gamma} \end{cases}, \tag{1.36}$$

where $\mu'(\gamma) = \lim_{\gamma \searrow \underline{\gamma}} \mu'(\gamma)$ when $\gamma = \underline{\gamma}$ and $\mu'(\gamma) = \lim_{\gamma \nearrow \bar{\gamma}} \mu'(\gamma)$ when $\gamma = \bar{\gamma}$.

Step 2.

The family of functions $\mu(\gamma)$ satisfying Assumptions A1 and A2 can be divided into two subfamilies: the family of functions for which $\underline{\gamma} = \bar{\gamma}$, that is, the Leontief case $\mu(\gamma) = \min\{\gamma, 1\}$, and the family of function for which $\underline{\gamma} < \bar{\gamma}$ and $\mu'(\gamma) < \mu(\gamma)/\gamma$ for any $\gamma \in (\underline{\gamma}, \bar{\gamma})$.

Subfamily I: $\mu(\gamma) = \min\{\gamma, 1\}$.[34]

[34]Note that $\eta = 1$ represents the extreme case of frictionless labor market. The Proposition can be easily

It follows from the previous analysis that, when $\underline{\gamma} = \bar{\gamma} = 1$ as in the Leontief case and $k < \bar{k}$, then in equilibrium $\gamma = \mu(\gamma) = 1$ and (2.6) is not binding. Then, equations (3.43) and (3.44) complete the equilibrium characterization.

Think of g as a function of λ

$$g(\lambda) \equiv 1 - \mu(\gamma) F(\hat{\theta}(\lambda)) - \lambda,$$

where, for this first case, I think of $\hat{\theta}$ as a function of λ implicitly defined by equation (3.43), that is,

$$\hat{\theta}(\lambda) = y - b - \lambda \frac{F(\hat{\theta}(\lambda))}{f(\hat{\theta}(\lambda))}. \tag{1.37}$$

Notice that

$$g(1) = -F(\hat{\theta}(1)) < 0,$$

since $y > b + \underline{\theta}$ implies $\hat{\theta}(1) > \underline{\theta}$. Given that $g(\lambda)$ is a continuous function, then there exists a set of $\lambda \leq 1$ such that $g(\lambda)$ is negative. For each λ in this set, I can set the parameter k such that

$$k = \int_{\underline{\theta}}^{\hat{\theta}(\lambda)} \left[y - \theta - b - \frac{F(\theta)}{f(\theta)} \right] dF(\theta). \tag{1.38}$$

Observe that

$$\frac{dk}{d\lambda} = \left[y - \hat{\theta}(\lambda) - b - \frac{F(\hat{\theta}(\lambda))}{f(\hat{\theta}(\lambda))} \right] f(\hat{\theta}(\lambda)) \frac{d\hat{\theta}(\lambda)}{d\lambda} > 0,$$

since from equation (1.37) it follows that

$$y - \hat{\theta}(\lambda) - b - \frac{F(\hat{\theta}(\lambda))}{f(\hat{\theta}(\lambda))} \leq y - \hat{\theta}(\lambda) - b - \lambda \left[\frac{F(\hat{\theta}(\lambda))}{f(\hat{\theta}(\lambda))} \right] = 0$$

and, given the monotone hazard rate assumption, that $d\hat{\theta}(\lambda)/d\lambda < 0$. Thus, for any $y > b + \underline{\theta}$, there exists a $\hat{k} < \bar{k}$ such that $g < 0$ for any $k \in (\hat{k}, \bar{k}]$, completing the proof for the Leontief case.[35]

---

extended to the more general Leontief case where $\mu(\gamma) = \min\{\eta\gamma, 1\}$ with $\eta \in (0, 1]$ and the labor market is frictional for any $\eta < 1$.

[35] In particular, $g(0) > 0$ and $g'(.) < 0$ so that, for any $y > \tilde{y}$, there exists a $\hat{k}$ such that $g < 0$ for any $k \in (\hat{k}, \bar{k}]$ and $g > 0$ for any $k \in [0, \hat{k}]$. The Leontief case is a special case of Proposition 13 that I discuss below.

Subfamily II: $\underline{\gamma} < \bar{\gamma}$ and $\mu'(\gamma) < \mu(\gamma)/\gamma \; \forall \gamma \in (\underline{\gamma}, \bar{\gamma})$.

Assume that the equilibrium is characterized by an interior solution for $\gamma \in (\underline{\gamma}, \bar{\gamma})$. Then, as derived in the proof of Proposition 12, the equilibrium γ, $\hat{\theta}$ and λ must satisfy

$$\hat{\theta} = y - b - \lambda \frac{F(\hat{\theta})}{f(\hat{\theta})}, \tag{1.39}$$

$$\lambda \frac{[F(\hat{\theta})]^2}{f(\hat{\theta})} \frac{\mu(\gamma)}{\gamma} = k, \tag{1.40}$$

$$\frac{\lambda}{1-\lambda} = \left[\frac{\eta(\gamma)}{1-\eta(\gamma)}\right] \frac{f(\hat{\theta})}{[F(\hat{\theta})]^2} \int_{\underline{\theta}}^{\hat{\theta}} F(\theta)\, d\theta, \tag{1.41}$$

where $\eta(\gamma)$ denotes the elasticity of $\mu(\gamma)$, that is,

$$\eta(\gamma) = \frac{\gamma \mu'(\gamma)}{\mu(\gamma)}.$$

Consider a family of economies parametrized by a pair (ε, δ) for (ε, δ) belonging to a (one-sided) neighborhood of $(0,0)$ such as $\mathcal{I} \equiv (0, \bar{\varepsilon}) \times (0, \bar{\delta})$ with $\bar{\varepsilon}$ and $\bar{\delta}$ strictly positive.

Think of λ as a function of ε and δ, $\lambda(\varepsilon, \delta)$, defined implicitly by equation (1.41) with $\gamma = \bar{\gamma} - \varepsilon$ and $\hat{\theta} = \bar{\theta} - \delta$. Notice that $\lambda(\varepsilon, \delta)$ is increasing in ε, given the monotone elasticity assumption. The Proposition assumes that there exists a $\bar{\gamma} < \infty$ such that $\mu(\bar{\gamma}) = 1$ and $\lim_{\gamma \nearrow \bar{\gamma}} \eta(\gamma) > 0$, which implies $\lambda(\varepsilon, \delta) > 0$ for any $\varepsilon > 0$, since $\lim_{\varepsilon \to 0} \lambda(\varepsilon, \delta) > 0$ and $\partial \lambda(\varepsilon, \delta)/\partial \varepsilon > 0$ for any $\delta < \bar{\theta} - \underline{\theta}$.

For each pair (ε, δ), set the parameters k and y such that

$$k = \lambda(\varepsilon, \delta) \frac{\mu(\bar{\gamma} - \varepsilon)}{\bar{\gamma} - \varepsilon} \frac{[F(\bar{\theta} - \delta)]^2}{f(\bar{\theta} - \delta)} \tag{1.42}$$

and

$$y = b + \bar{\theta} - \delta + \lambda(\varepsilon, \delta) \frac{F(\bar{\theta} - \delta)}{f(\bar{\theta} - \delta)}, \tag{1.43}$$

which make $\gamma = \bar{\gamma} - \varepsilon$ and $\hat{\theta} = \bar{\theta} - \delta$ an equilibrium solution. Notice that, as we have assumed, γ is an interior solution given that $\varepsilon > 0$.

Then, think of g as a function of (ε, δ), that is,

$$g(\varepsilon, \delta) \equiv 1 - \mu(\bar{\gamma} - \varepsilon) F(\bar{\theta} - \delta) - \lambda(\varepsilon, \delta).$$

Since equations (1.42) and (1.43) define a continuous and invertible[36] mapping between the space (ε, δ) and the space (k, y), if for a small enough neighborhood $\mathcal{I}$, $g(\varepsilon, \delta) < 0$ for any $(\varepsilon, \delta) \in \mathcal{I}$, then there exists an open set of the space (k, y) for which $g < 0$.

I can choose $\bar{\varepsilon}$ and $\bar{\delta}$ small enough such that $g(\varepsilon, \delta) < 0$ for any $(\varepsilon, \delta) \in \mathcal{I}$, given that

$$\lim_{(\varepsilon,\delta)\to(0,0)} g(\varepsilon, \delta) = -\lambda(\varepsilon, \delta) < 0,$$

completing the proof.

Proof of Proposition 13. The proof proceeds in two steps. First I characterize the equilibrium solution. Then, I describe a set of the parameter space such that g is negative.

Step 1.

First of all, notice that

$$\mu(\gamma) = \begin{cases} \gamma & \text{if } \gamma \leq \underline{\gamma} \\ A\gamma^{\alpha} & \text{if } \underline{\gamma} < \gamma \leq \bar{\gamma} \\ 1 & \text{if } \gamma > \bar{\gamma} \end{cases},$$

where

$$\underline{\gamma} \equiv A^{\frac{1}{1-\alpha}} \text{ and } \bar{\gamma} \equiv A^{-\frac{1}{\alpha}}.$$

Moreover, as described in the proof of Proposition 12, given that $y > b + \underline{\theta}$, there exists a $\bar{k} > 0$ defined as

$$\bar{k} \equiv \int_{\underline{\theta}}^{\tilde{\theta}} \left[y - \theta - b - \frac{F(\theta)}{f(\theta)} \right] dF(\theta),$$

where $\tilde{\theta}$ solves $y - b - \tilde{\theta} - F(\tilde{\theta})/f(\tilde{\theta}) = 0$. I assume $k < \bar{k}$[37], so that the equilibrium allocation must satisfy equations (3.43), (3.44) and (1.36) for $\gamma \in [\underline{\gamma}, \bar{\gamma}]$.

[36] The proof of invertibility is analogous to the one in the proof of Proposition 12 and thus requires strict concavity of $\mu(.)$ and non-increasing elasticity.

[37] Excluding equilibria with zero vacancy, that is such that $k \geq \bar{k}$, ensures that $\gamma = 0$ is never optimal.

Think of γ as a function of k. Next, I show that there exist two cut-off values k_1 and k_2 such that $\gamma(k) = \bar{\gamma}$ for any $k < k_1$, $\gamma(k) = \underline{\gamma}$ for any $k > k_2$ and $\gamma(k) \in (\underline{\gamma}, \bar{\gamma})$ for any $k \in (k_1, k_2)$. When $k \in (k_1, k_2)$, there is an interior solution with $\gamma \in (\underline{\gamma}, \bar{\gamma})$ such that, as derived in the proof of Proposition 12, the equilibrium $\hat{\theta}$, γ and λ must satisfy

$$\hat{\theta} = y - b - \lambda \frac{F(\hat{\theta})}{f(\hat{\theta})}, \tag{1.44}$$

$$\lambda \frac{[F(\hat{\theta})]^2}{f(\hat{\theta})} \frac{\mu(\gamma)}{\gamma} = k, \tag{1.45}$$

$$\frac{\lambda}{1-\lambda} = \left[\frac{\alpha}{1-\alpha}\right] \frac{f(\hat{\theta})}{[F(\hat{\theta})]^2} \int_{\underline{\theta}}^{\hat{\theta}} F(\theta)\, d\theta, \tag{1.46}$$

since the elasticity of $\mu(.)$, $\eta(\gamma(k)) = \gamma\mu'(\gamma)/\mu(\gamma) = \alpha$ for any $\gamma \in (\underline{\gamma}, \bar{\gamma})$.

Notice that, from equations (1.44)-(1.46) it follows that when $k \in (k_1, k_2)$ γ depends on k, while λ and $\hat{\theta}$ are constant. More precisely, using the implicit function theorem, equation (1.44) defines $\hat{\theta}$ as a continuous increasing function of λ and, using that, equation (1.46) defines implicitly $\lambda = \tilde{\lambda}$ for any $k \in (k_1, k_2)$. Then, $\tilde{\lambda}$ is implicitly defined by

$$\tilde{\lambda} = \frac{\alpha D(\hat{\theta}(\tilde{\lambda}))}{1 - \alpha(1 - D(\hat{\theta}(\tilde{\lambda})))},$$

where $D(\hat{\theta}(\tilde{\lambda})) \equiv f(\hat{\theta}(\tilde{\lambda})) \int_{\underline{\theta}}^{\hat{\theta}(\tilde{\lambda})} F(\theta)\, d\theta / [F(\hat{\theta}(\tilde{\lambda}))]^2$. Notice that the monotone hazard rate assumption implies

$$E\left[F(\theta)/f(\theta) \mid \theta < \hat{\theta}\right] < F(\hat{\theta})/f(\hat{\theta}).$$

so that $D(\hat{\theta}(\tilde{\lambda})) < 1$ and $\tilde{\lambda} < 1$ as well. Thus, using $\lambda = \tilde{\lambda}$ and $\hat{\theta}(\tilde{\lambda})$, equation (1.45) defines implicitly $\gamma(k)$.

Now, define

$$k_1 \equiv \frac{1}{\bar{\gamma}} \int_{\underline{\theta}}^{\hat{\theta}(\tilde{\lambda})} \left[y - \theta - b - \frac{F(\theta)}{f(\theta)}\right] dF(\theta)$$

and

$$k_2 \equiv \int_{\underline{\theta}}^{\hat{\theta}(\tilde{\lambda})} \left[y - \theta - b - \frac{F(\theta)}{f(\theta)}\right] dF(\theta).$$

From equation (3.43), (3.44) and (1.36), it follows that an equilibrium for any $k \leq k_1$ is characterized by $\gamma = \bar{\gamma}$ and $\lambda(k)$ implicitly defined by

$$\int_{\underline{\theta}}^{\hat{\theta}(\lambda(k))} \left[y - \theta - b - \frac{F(\theta)}{f(\theta)} \right] dF(\theta) = \bar{\gamma} k,$$

so that by definition $\lambda(k_1) = \tilde{\lambda}$. Instead, an equilibrium for any $k \geq k_2$ is characterized by $\gamma = \underline{\gamma}$ and $\lambda(k)$ implicitly defined by

$$\int_{\underline{\theta}}^{\hat{\theta}(\lambda(k))} \left[y - \theta - b - \frac{F(\theta)}{f(\theta)} \right] dF(\theta) = k,$$

so that by definition $\lambda(k_2) = \tilde{\lambda}$.

Notice that for any $k \leq k_1$

$$\frac{d\lambda(k)}{dk} = \left[\frac{d(\hat{\theta}(\lambda))}{d\lambda} \left[y - \hat{\theta}(\lambda) - b - \frac{F(\hat{\theta}(\lambda))}{f(\hat{\theta}(\lambda))} \right] f(\hat{\theta}(\lambda)) \right]^{-1} > 0 \tag{1.47}$$

since equation (3.43) implies

$$y - \hat{\theta}(\lambda) - b - \frac{F(\hat{\theta}(\lambda))}{f(\hat{\theta}(\lambda))} < y - \hat{\theta}(\lambda) - b - \lambda \frac{F(\hat{\theta}(\lambda))}{f(\hat{\theta}(\lambda))} = 0$$

and using the implicit function theorem and the monotone hazard rate assumption, it follows directly that $d(\hat{\theta}(\lambda))/d\lambda < 0$. An analogous argument shows that $d\lambda(k)/dk > 0$ for any $k \geq k_2$. Notice that the assumption $y > b + \underline{\theta}$ yields $\bar{k} > k_2$.

It follows that we can think of $\lambda(k)$, as a non decreasing function of k with $\lambda(k) = 0$ for $k = 0$ and $\lambda(k) = 1$ for $k = \bar{k}$. Moreover it is strictly increasing for $k < k_1$ and $k > k_2$ and is constant at $\tilde{\lambda}$ for $k_1 \leq k \leq k_2$.

Step 2.

Think of g as a function of k, that is,

$$g(k) \equiv 1 - \mu(\gamma(k)) F(\hat{\theta}(\lambda(k))) - \lambda(k).$$

Then,

$$g(0) = 1 - F(\hat{\theta}(0)) = 0 \text{ and } g(\bar{k}) = -\underline{\gamma} F(\hat{\theta}(1)) < 0,$$

where $g(0) = 0$ since $\hat{\theta}(0) = \bar{\theta}$, given that by assumption $y > b + \bar{\theta} + \tilde{\lambda}F(\bar{\theta})/f(\bar{\theta}) > b + \bar{\theta}$. Moreover $g'(k) < 0$ for $0 < k < k_1$, since $\gamma = \bar{\gamma}$, $\mu(\bar{\gamma}) = 1$ and

$$g'(k) = -\left[1 + f(\hat{\theta}(\lambda(k)))\frac{d\hat{\theta}(\lambda(k))}{d\lambda(k)}\right]\frac{d\lambda(k)}{dk} < 0,$$

where $d\hat{\theta}(\lambda(k))/d\lambda(k) < 0$ and $d\lambda(k)/dk > 0$ by equation (1.47). With a similar argument we can show that $g'(k) < 0$ for $k_2 < k < \bar{k}$. Instead, for $k_1 < k < k_2$, $d\lambda(k)/dk = 0$ but γ is interior and strict concavity of $\mu(\gamma)$ implies

$$g'(k) = -\mu'(\gamma(k))\,\mu''(\gamma(k))\,F(\hat{\theta}(\lambda(k))\int_{\underline{\theta}}^{\hat{\theta}(\lambda(k))} F(\theta)\,d\theta > 0.$$

Then, $g(k)$ has a minimum in k_1 and a maximum in k_2. Notice that $g(k_1) = 1 - F(\hat{\theta}(\tilde{\lambda})) - \tilde{\lambda}$, so that

$$g(k_1) < 0 \Leftrightarrow F(\hat{\theta}(\tilde{\lambda})) > 1 - \tilde{\lambda}.$$

Thus, $y > b + \bar{\theta} + \tilde{\lambda}/f(\bar{\theta})$ implies $\hat{\theta}(\tilde{\lambda}) = \bar{\theta}$, so that $F(\hat{\theta}(\tilde{\lambda})) = 1 > 1 - \tilde{\lambda}$ given that $\tilde{\lambda} < 1$. Moreover $g(k_2) = 1 - \underline{\gamma}F(\hat{\theta}(\tilde{\lambda})) - \tilde{\lambda}$ yielding

$$g(k_2) < 0 \Leftrightarrow \frac{1 - \tilde{\lambda}}{\underline{\gamma}} < 1.$$

Thus, if $\underline{\gamma} > 1 - \tilde{\lambda}$ then $g(k_2) < 0$ leading $g(k) < 0$ for any $k < \bar{k}$. If, otherwise, $\underline{\gamma} < 1 - \tilde{\lambda}$, then $g(k_2) > 0$. In the latter case, it follows, by a continuity argument, that there exist k_L and k_H such that $k_1 < k_L < k_2 < k_H < \bar{k}$ so that in equilibrium $g < 0$ for $k \in [0, k_L]$ and $k \in [k_H, \bar{k}]$, completing the proof.

Proof of Proposition24. The proof analyzes separately the two cases of functions $\mu(\gamma)$ satisfying assumptions A1 and, respectively, A2 and A3.

Case 1. $\mu(\gamma)$ satisfies A1 and A2.
First, notice that at the competitive equilibrium, (1.23) implies that $\hat{V}^{CE} - U^{CE} = -(1-\beta)\,\alpha U^{CE}$. For now, fix $(1-\beta)\,\alpha U^{CE} = \hat{b}$. Then, equations (1.19), (1.21) and (1.20) define an equilibrium $\hat{\theta}^{CE}$, γ^{CE} and λ^{CE} for $\hat{b}$ given, and, for $\alpha = 1$, are equivalent to equations (3.43), (3.44) and

(2.6) that define a static equilibrium where $b = \hat{b}$. Thus, an argument analogous to the case 1 of the proof of Proposition 12, adjusted for $\alpha < 1$, shows that for each $\hat{b}$ and a given $F(.)$ and $\mu(.)$ satisfying Assumption A1 and A2, there exists an open set of the parameter space (k, y) such that $g = 1 - \mu(\gamma^{CE})F(\hat{\theta}^{CE}) - \lambda^{CE} < 0$. In particular such a set of (k, y) is characterized by $\gamma^{CE} \to 0$ and $\hat{\theta}^{CE} \to \bar{\theta}$. Notice that, from equation (1.22), there exists a continuous and invertible mapping between $\hat{b}$ and b such that $(1-\beta)\,\alpha U^{CE} = \hat{b}$, so that for any open set of $\hat{b}$, there exists a correspondent open set of b. The last thing left to check is that $b > 0$, which is the case whenever

$$\hat{b} > \alpha^2 \beta \mu\left(\gamma^{CE}\right) \int_{\underline{\theta}}^{\hat{\theta}^{CE}} [y - \theta]\, dF(\theta) - \gamma^{CE} k. \tag{1.48}$$

Notice that when $\gamma^{CE} \to 0$ and $\hat{\theta}^{CE} \to \bar{\theta}$ the right-hand side of the equation is zero, competing the proof.

Case 2. $\mu(\gamma)$ satisfies A1 and A3.

An analogous argument to the proof of the previous case, using the case 2 of Proposition 12, shows that for given $\hat{b}$, $F(.)$ and $\mu(.)$ satisfying Assumption A1 and A3, there exists an open set of the parameter space (k, y) such that $g < 0$. Now such a set of (k, y) is characterized by $\gamma^{CE} \to \bar{\gamma}$ and $\hat{\theta}^{CE} \to \bar{\theta}$. Moreover, given the expressions for (k, y) in the proof of Proposition 12, I can choose an open set of $\hat{b}$ such that

$$\hat{b} > \frac{\alpha^2\beta}{1-\alpha^2\beta}\left[\bar{\theta} - \int_{\underline{\theta}}^{\bar{\theta}} \theta dF(\theta)\right] - \frac{1}{f(\bar{\theta})} \lim_{(\varepsilon,\delta)\to(0,0)} \lambda(\varepsilon, \delta),$$

where $\lambda(\varepsilon, \delta)$ does not depend on $\hat{b}$, competing the proof.

Proof of Proposition 25. The Proof proceeds by contradiction: assume that the competitive equilibrium is constrained efficient and then show that it is impossible. Note that if the competitive equilibrium allocation $\hat{\theta}^{CE}$, γ^{CE}, V^{CE}, U^{CE} is constrained efficient for a given initial value of unemployment rate u_0, then it must solve the social planner problem, that is, it must satisfy the first order conditions:

$$\alpha(y - \hat{\theta}) + V' - U' - \alpha(1-\eta)\frac{F(\hat{\theta})}{f(\hat{\theta})} - \pi = 0, \tag{1.49}$$

$$\beta\mu'(\gamma)\int_{\underline{\theta}}^{\hat{\theta}}\left[\alpha(y-\theta)+V'-U'-\alpha(1-\eta)\frac{F(\theta)}{f(\theta)}-\pi\right]dF(\theta) \tag{1.50}$$
$$= k+\beta\mu'(\gamma)(1-\eta)\left(b-C^U\right),$$

$$\chi = u(1-\eta)(1-\beta\mu(\gamma)), \tag{1.51}$$

$$\nu \leq 1 \text{ when } C^V \geq 0, \tag{1.52}$$

$$\beta P_V' + u\beta\mu(\gamma)F(\hat{\theta}) + \nu(1-u)\beta(1-s) = 0, \tag{1.53}$$

$$\beta P_U' - u\beta\mu(\gamma)F(\hat{\theta}) + \nu(1-u)\beta s + \eta u\beta = 0, \tag{1.54}$$

$$P_u' = \pi, \tag{1.55}$$

where the multipliers η, π, χ, ν must be such that

$$V = C^V + \beta s U' + \beta(1-s)V', \tag{1.56}$$

$$U = C^U + \beta\mu(\gamma)\int_{\underline{\theta}}^{\hat{\theta}}\frac{F(\theta)}{f(\theta)}dF(\theta) + \beta U', \tag{1.57}$$

$$u' = u\left[1-\mu(\gamma)F(\hat{\theta})\right] + (1-u)s, \tag{1.58}$$

$$C^U \geq b. \tag{1.59}$$

Moreover, recall that when $k > 0$ the informational constraint is binding, that is, it must be true that

$$\int_{\underline{\theta}}^{\overline{\theta}} e(\theta)\left[\alpha\left(c(\theta)-\theta-\frac{F(\theta)}{f(\theta)}\right)+V'-U'\right]dF(\theta)+C^U-b=0 \tag{1.60}$$

and at the optimum

$$P = u\left[\beta\mu(\gamma)\int_{\underline{\theta}}^{\hat{\theta}}\alpha(y-c(\theta))\,dF(\theta)+b-\gamma_t k-C^U\right]-(1-u)C^V+\beta P'. \tag{1.61}$$

Finally, the Envelope conditions are

$$P_U = -u\eta, \tag{1.62}$$

$$P_V = -(1-u)\nu, \tag{1.63}$$

$$
\begin{aligned}
P_u &= \beta\mu(\gamma)\int_{\underline{\theta}}^{\hat{\theta}}\left[\alpha(y-\theta)+V'-U'-\frac{F(\theta)}{f(\theta)}\right]dF(\theta) \\
&\quad+\left[(1-\beta\mu(\gamma))\left(b-C^{U}\right)-\gamma k\right]+C^{V}+\beta\pi\left[1-\mu(\gamma)F(\hat{\theta})-s\right].
\end{aligned}
\tag{1.64}
$$

Now guess that the steady state competitive search equilibrium $\hat{\theta}^{CE}$, γ^{CE}, u^{CE}, V^{CE}, U^{CE}, $C^U=b$, $C^V=0$ and $B=0$ is a solution to this problem.
Moreover, recall that $\hat{\theta}^{CE}$, γ^{CE}, U^{CE}, V^{CE}, u_t^{CE} and the normalized multiplier λ^{CE} must satisfy $C^U=b$, $C^V=P=0$ and solve the system of equations

$$
\alpha(y-\hat{\theta}^{CE})+V^{CE}-U^{CE}-\alpha\lambda^{CE}\frac{F(\hat{\theta}^{CE})}{f(\hat{\theta}^{CE})}=0, \tag{1.65}
$$

$$
\beta\mu'\left(\gamma^{CE}\right)\int_{\underline{\theta}}^{\hat{\theta}}\left[\alpha(y-\theta-(1-\beta)U^{CE})-\alpha\lambda^{CE}\frac{F(\theta)}{f(\theta)}\right]dF(\theta)=k, \tag{1.66}
$$

$$
\beta\mu\left(\gamma^{CE}\right)\int_{\underline{\theta}}^{\hat{\theta}}\left[\alpha(y-\theta-(1-\beta)U^{CE})-\alpha\frac{F(\theta)}{f(\theta)}\right]dF(\theta)=\gamma^{CE}k, \tag{1.67}
$$

$$
U^{CE}=\frac{\alpha\beta\mu\left(\gamma^{CE}\right)\int_{\underline{\theta}}^{\hat{\theta}^{CE}}[y-\theta]\,dF(\theta)+b-\gamma^{CE}k}{(1-\beta)\left[1+\alpha\beta\mu\left(\gamma^{CE}\right)F(\hat{\theta}^{CE})\right]}, \tag{1.68}
$$

$$
V^{CE}=\frac{\beta s}{1-\beta(1-s)}U^{CE}, \tag{1.69}
$$

and

$$
u_{t+1}=u_t\left[1-\mu\left(\gamma^{CE}\right)F(\hat{\theta}^{CE})\right]+(1-u_t)\,s.
$$

First, combining equations (1.49) and (1.50) with equations (1.65) and (1.66), it follows

$$
\left\{
\begin{array}{c}
\alpha\left(1-\eta-\lambda^{CE}\right)\frac{F(\hat{\theta}^{CE})}{f(\hat{\theta}^{CE})}+\pi=0 \\
\alpha\left(1-\eta-\lambda^{CE}\right)\int_{\underline{\theta}}^{\hat{\theta}}\frac{F(\theta)}{f(\theta)}dF(\theta)+\pi\int_{\underline{\theta}}^{\hat{\theta}^{CE}}dF(\theta)=0
\end{array}
\right. .
$$

Given that

$$
\frac{F(\hat{\theta}^{CE})}{f(\hat{\theta}^{CE})}\left[\frac{\int_{\underline{\theta}}^{\hat{\theta}}\frac{F(\theta)}{f(\theta)}dF(\theta)}{\int_{\underline{\theta}}^{\hat{\theta}^{CE}}dF(\theta)}\right]^{-1}>1,
$$

it must be that

$$\eta = 1 - \lambda^{CE} \text{ and } \pi = 0.$$

It follows that $\chi > 0$ and $C^U = b$ as we assumed.

Notice that, using the Envelope condition (1.62), I can rewrite equation (1.53) as

$$\nu_{t+1} (1 - u_{t+1}) = u_t \mu (\gamma) F(\hat{\theta}) + \nu_t (1 - u_t) (1 - s),$$

which for $\nu_{t+1} = \nu_t = \nu^{CE}$ and $\gamma = \gamma^{CE}$, $\hat{\theta} = \hat{\theta}^{CE}$, using the law of motion for u, yields

$$\left(1 - \nu^{CE}\right) [u_{t+1} - u_t - (1 - u_t) s] = 0,$$

implying $\nu^{CE} = 1$, as long as job creation is different from zero. Moreover, using the Envelope condition (1.63), I can rewrite equation (1.54) as

$$\left(1 - \eta_{t+1}\right) u_{t+1} = (1 - \eta_t) u_t + (1 - \nu_t) (1 - u_t) s,$$

which, when $\nu^{CE} = 1$, $\gamma = \gamma^{CE}$, $\hat{\theta} = \hat{\theta}^{CE}$, $1 - \eta_{t+1} = 1 - \eta_t = \lambda^{CE}$, using the law of motion of u, yields

$$\left(u_t^{CE} - u_{t+1}^{CE}\right) \lambda^{CE} = 0.$$

A contradiction follows immediately as long as the unemployment rate is not at the steady state value and λ^{CE} is different from zero, which is the case whenever $k > 0$, as we assumed. This implies that, away from the steady state, the competitive search equilibrium is constrained inefficient completing the proof.

Bibliography

[1] Acemoglu, Daron and Robert Shimer, "Efficient Unemployment Insurance", *Journal of Political Economy*, 1999a, 107, 893-928.

[2] Acemoglu, Daron and Robert Shimer, "Holdups and Efficiency with Search Frictions", *International Economic Review*, 1999b, 40, 827-850.

[3] Burdett, Kenneth, Shouyong Shi and Randall Wright, "Pricing and Matching with Frictions", *Journal of Political Economy*, 2001, 109, 1060-1085.

[4] Diamond, Peter, "Wage Determination and Efficiency in Search Equilibrium", *Review of Economic Studies*, 1982, 49, 217-227.

[5] Faig, Miquel and Belen Jerez, "A Theory of Commerce", *Journal of Economic Theory*, 2005, 122, 60-99.

[6] Friedman, Milton, "The Role of Monetary Policy", *American Economic Review*, 1968, 58, 1-17.

[7] Hall, Robert and Robert Milgrom, "The Limited Influence of Unemployment on the Wage Bargain", 2005, mimeo.

[8] Hall, Robert, "Job Loss, Job Finding and Unemployment in the U.S. Economy over the Past Fifty Years", 2005, mimeo (prepared for the NBER Macro Annual Conference, April 2005).

[9] Hawkins, William, "Competitive Search and Efficiency of Hiring", 2005, MIT mimeo.

[10] Hosios, Arthur J., "On the Efficiency of Matching and Related Models of Search and Unemployment", *Review of Economic Studies*, (1990) 57, 279-298.

[11] Kennan, John, "Private Information, Wage Bargaining, and Employment Fluctuations," May 2005, University of Wisonsin, mimeo.

[12] Laffont, Jean-Jacques and Eric Maskin, "A Differential Approach to Dominant Strategy Mechanisms", *Econometrica*, 1980, 48, 1507-1520.

[13] Mas-Colell, Andreu, Michael D. Whinston and Jerry R. Green, *Microeconomic Theory*, Oxford University Press, New York, Oxford, 1995.

[14] Menzio, Guido, "High-Frequency Wage Rigidity", 2004, Northwestern University, mimeo.

[15] Mirrlees, James, "An Exploration in the Theory of Optimum Income Taxation", *Review of Economic Studies*, 1971, 38, 175-208.

[16] Moen, Espen, "Competitive Search Equilibrium", *Journal of Political Economy*, 1997, 105, 385-411.

[17] Moen, Espen and Asa Rosen, "Incentives in Competitive Search Equilibrium and Wage Rigidity", 2005, Northwestern University, mimeo.

[18] Myerson, Roger B., "Optimal Auction Design", *Mathematics of Operations Research*, 1981, 6, 58-73.

[19] Myerson, Roger B. and Mark A. Satterthwaite, "Efficient Mechanisms for Bilateral Trading", *Journal of Economic Theory*, 1981, 29, 265-281.

[20] Mortensen, Dale, "The Matching Process as a Noncooperative Bargaining Game", in *The Economics of Information and Uncertainty*, ed. John J. McCall, Chicago: University of Chicago Press for the NBER, 1982a.

[21] Mortensen, Dale, "Property Rights and Efficiency in Mating, Racing, and Related Games", *American Economic Review*, 1982b, 72, 968-979.

[22] Mortensen, Dale and Christopher Pissarides, "Job Creation and Job Destruction in the Theory of Unemployment", *Review of Economic Studies*, 1994, 61, 397-415.

[23] Mortensen, Dale and Randall Wright, "Competitive Pricing and Efficiency in Search Equilibrium", *International Economic Review*, 2002, 43, 1-20.

[24] Nagypál, Eva, "Worker Reallocation over the Business Cycle: The Importance of Job-to-Job Transitions, Part 1: Evidence," July 2004, Northwestern University, mimeo.

[25] Pissarides, Christopher, "Search Intensity, Job Advertising, and Efficiency", *Journal of Labor Economics*, 1984, 2, 128-143.

[26] Pissarides, Christopher, "Short-run Equilibrium Dynamics of Unemployment, Vacancies and Real Wages", *American Economic Review*, 1985, 75, 676-90.

[27] Pissarides, Christopher, *Equilibrium Unemployment Theory*, second ed., Cambridge, MA, MIT Press, 2000.

[28] Rogerson, Richard, Shimer, Robert and Randall Wright, "Search-Theoretic Models of the Labor Market: A Survey", 2005 forthcoming *Journal of Economic Literature.*

[29] Shimer, Robert, "Contracts in Frictional Labor Markets", 1996, MIT, mimeo.

[30] Shimer, Robert and Randall Wright, "Competitive Search Equilibrium with Asymmetric Information", 2004, mimeo.

[31] Shimer, Robert "The Cyclical Behavior of Unemployment and Vacancies", *American Economic Review,* 2005, 95, 25-49.

[32] Wolinsky, Asher, "Procurement via Sequential Search", *Journal of Political Economy,* 2005, 113, 785-810.

Chapter 2

Heterogeneity and Unemployment Volatility

2.1 Introduction

Whether search models can match the cyclical behavior of unemployment and vacancies in the U.S. economy is an issue that has recently received new attention. Shimer (2005) has opened the debate by showing that the conventional search model, built on Diamond (1982), Mortensen (1982a, 1982b) and Pissarides (1984, 1985), which combines random matching and Nash bargaining, cannot account for the high responsiveness of unemployment to productivity shocks. The main idea is that wages absorb the shocks reducing the responsiveness of firms' profits and hence of job creation.

In this paper, I explore a search model with ex-post heterogeneous workers with both full and asymmetric information. The main difference with the standard model is that the finding rate is now coming from the combination of the standard matching margin and of the endogenous hiring margin generated by heterogeneity. Matched workers are hit by a match-specific training cost that they have to sustain to become productive. When the productivity shock hits the economy, both the expected cost for a matched worker and the expected value of being unemployed increase and wages with them. However, the hiring margin becomes looser and firms' profits can increase because of that. When the information is asymmetric, in the sense that the workers know privately their type, there is an additional potential source of volatility. The firm has to pay the workers some rents in order to give them the incentive to reveal their information. When there is an high productivity shock, then there is more surplus to cover the rents and the distortion becomes smaller, generating a boost in job creation. On the other side, when the hiring margin increases, the rents that need to be paid to the workers are higher and asymmetric information could dampen the responsiveness of vacancies.

I construct a competitive search model, where workers are hit by match-specific idiosyncratic shocks, which can be interpreted as the training costs a worker has to pay in order to be productive in the match. The model is a generalization of the one described in Chapter 1 to a stochastic environment with an aggregate productivity shock.[1] Employers and workers are both risk-neutral and *ex-ante* homogeneous. Employers post contracts and workers direct their search towards them. When a match is formed, the training cost is drawn randomly. I consider both the case of this cost being public information and the case of it being privately observed by the worker. An employment contract is a mechanism that is incentive compatible, when the information is private, and always satisfies a participation constraint on the worker's side. A worker cannot be forced to work, he can always quit and join the ranks of the unemployed.

First, I introduce the model with full information and I show the main analytical result: in steady state heterogeneity does not amplify the responsiveness of market tightness to productivity. Once one tries to calibrate the model to match two main facts, the finding rate and the finding rate elasticity to market tightness, then, on the opposite, heterogeneity dampens the response of market tightness to productivity. Next, I go to the model with asymmetric information and show that in steady state there is an additional effect on the response of market tightness to productivity shocks. This additional effect comes from the binding incentive compatibility constraint. However, the sign of this effect is ambiguous.

Then, I show the results of the main calibrations. I describe how I parametrize the model to match the U.S. labor market facts, documented in Shimer (2005), in order to investigate the role of heterogeneity and asymmetric information in explaining the U.S. unemployment and vacancies volatility. I explore two family of distributions for the idiosyncratic shocks: Pareto

[1] As I explain in Section 2 the only difference in the basic environment described in Chapter 1 is that the worker's type is now represented by a sunk cost that the worker has to pay at the moment of hiring, such as a training or a mobility cost, instead of a flow cost, such as work effort. This change makes the model with aggregate shocks more tractable becuase avoids endogenous separation.

and Uniform. For each distribution I compare the limit case of the degenerate distribution, which reduces to the standard model studied by Shimer (2005), to the model with proper heterogeneity and full information and to the model with asymmetric information. I show that neither heterogeneity nor asymmetric information help to increase the response of market tightness and unemployment to productivity. In fact, I show that with the proper calibration, the model with full information can dampen the volatility of market tightness as the steady state analysis suggests.

Related Literature. The most related paper to my work is Shimer (2005), who shows that the standard search model with random matching and Nash Bargaining fails to match U.S. data in terms of unemployment and vacancies volatility in response to productivity shocks. Shimer (2005) and Hall (2005a) highlight that Nash Bargaining makes wages very responsive to productivity shocks, reducing the volatility of firms profits and hence of job creation.

A growing group of papers react to Shimer (2005) by introducing some form of wage rigidity into the picture. Among others, Hall (2005b) constructs a model exhibiting wage rigidity, Menzio (2004) assume employers have private information about productivity and contracts are not-binding, Nagypal (2004) combines workers' heterogeneity, asymmetric information and on-the-job search.

Another paper related to my work is Brugemann and Moscarini (2005). They investigate the properties of wage determination in search models which limit the responsiveness of unemployment and vacancies to productivity shocks. Moreover they show that asymmetric information does not typically help in violating those properties and does not allow search models to generate reasonable high level of unemployment volatility. My paper put more structure on the model and focus more on the role of heterogeneity, with both full and asymmetric information.

On a different ground, Hagedorn and Manovskii (2005) show a calibration where the value of non-market activity is much higher than Shimer[2], generating an higher elasticity of market tightness to productivity. In my calibration I am careful to keep the value of non-market activity being less than half than labor income, in order to shut down that explanation and highlight the role of heterogeneity and information.

[2]They calibrate the worker's bargaining power and the non-market activity to match elasticity of wages and average firms' profits.

My model is also very related to Shimer and Wright (2004) and Moen and Rosen (2004) who both model asymmetric information in competitive search model. Their environment is very close to mine, once I introduce asymmetric information. However, Shimer and Wright (2004) study a static environment and Moen and Rosen (2004) focus on the steady state analysis. The model presented in this paper is the stochastic version of the dynamic model in Chapter 1. In fact, Chapter 1 explores the dynamic efficiency properties of asymmetric information in competitive search models, without modeling explicitly the aggregate productivity shock.

This Chapter is organized as follows. Section 2.2 introduces the model with ex-post heterogeneous workers and full information, defines and characterizes the competitive search equilibrium, illustrates some comparative statics result in the extreme case of no aggregate shock. Section 2.3 extends the model to the case of asymmetric information, defines and characterizes the competitive search equilibrium. Section 2.4 describes the calibration exercises and show the results of the main specifications. Finally, Section 2.5 concludes.

2.2 The model with Full Information

I first consider the full information version of the dynamic model in the first Chapter augmented with an aggregate shock to productivity. I show that when the distribution of the idiosyncratic shocks is degenerate the model boils down to the discrete version of Shimer (2005), where the Hosios condition is satisfied. When, instead, the distribution of idiosyncratic shocks is non-degenerate, then the reaction of market tightness to productivity shocks cannot be amplified.

Environment. Consider an economy with infinite horizon and discrete time, populated by a continuum of measure 1 of workers and a large continuum of potential employers. Both workers and employers have linear preferences with discount factor β and are *ex-ante* homogeneous. Workers can search freely, while employers need to pay an entry cost k to post a vacancy. When a match is formed, the worker has to face some training cost θ, where θ is drawn randomly from the cumulative distribution function $F(.)$, with support $\Theta \equiv [\underline{\theta}, \overline{\theta}]$. I assume that the cumulative distribution function $F(.)$ is differentiable, with $f(.)$ being the associated density function, and that it satisfies a monotone hazard rate condition, that is, $d[F(\theta)/f(\theta)]/d\theta > 0$. Any worker-employer match produces y_t units of output at any time t in which it is productive.

Productivity y_t[3] follows a first order Markov process in discrete time, according to some distribution $G(y', y) = \Pr(y_{t+1} \leq y' | y_t = y)$ with finite support $Y \equiv [y_1, ..., y_N]$. The net surplus of the match is given by $y_t - \theta$ at time t, when the match is created, and y_s at any future time $s \geq t$ in which the match is still productive. Both θ and $y's$ are common knowledge.

At the beginning of each period t employers can be either productive or not. Workers can be either employed or unemployed. Non-productive employers can open a vacancy at a cost k which entitles them to post an employment contract $\mathcal{C}_t \in \mathbb{C}$ where $\mathbb{C}$ is the set of individually rational mechanisms. As I describe below, a contract $\mathcal{C}_t : \Theta \times Y \mapsto [0,1] \times \mathbb{R}_+$ specifies the hiring probability and the expected net present value of wages for each worker of type θ matched at time t when productivity is y_t. Therefore at each time t, a non-productive firm chooses a pair $(\sigma_t, \mathcal{C}_t) \in \{0,1\} \times \mathbb{C}$ where σ_t denotes the decision of posting a vacancy. Next, each unemployed worker observes all the contracts posted and decides where to apply. He chooses a contract $\mathcal{C}_t \in \mathbb{C}_t^P \subset \mathbb{C}$, where $\mathbb{C}_t^P$ denotes the set of contracts posted by active firms at time t. Each contract $\mathcal{C}_t$, is associated to a specific γ_t so that employers and workers know that their matching probabilities will depend on the contract that they respectively post and seek for. After workers start to search for a specific contract, matching takes place and, for each match, the draw θ is realized. At the same time the aggregate productivity shock y_t is realized. The behavior of a worker who is matched at time t is described by a function $a_t : \Theta \times Y \mapsto \{0,1\}$ that for each type θ and productivity y_t, specifies participation decision $a_t(\theta, y_t)$. The worker can either implement the contract, that is choose $a_t(\theta, y_t) = 1$, or walk away, that is choose $a_t(\theta, y_t) = 0$. If he walks away, he is unemployed for one period, gets a non-transferable utility from leisure b and looks for another match next period. If the worker is hired, the parties are productive until separation, which happens according to a Poisson process with parameter s. Note that at any point in time $\tau > t$, it is never optimal to end the match created at time t, as long as $y_\tau > 0$. The main difference with the environment of the model presented in the first chapter is about the idiosyncratic shock θ. In this chapter, θ is a sunk cost that the worker suffers at the moment of the match, which I interpret as cost of training. In the previous chapter it is a disutility that the worker suffers each period in which he is productive,

[3]Note that productivity y_t is common to all the matches existing at time t even though created at different times $t - s$, for $s = 0, 1, .., t$.

which I interpret as work effort. This change is crucial to make the model tractable once I introduce the aggregate shock. If θ was was per-period work effort, when there are bad times the firm would like to fire workers with an high enough θ, generating an additional endogenous separation mechanism.

Trading frictions in the labor market are modeled through random matching. Employers and workers know that their matching probabilities will depend on the contract that they respectively post and seek for. Each type of contract $\mathcal{C}$ is associated with a labor submarket, where a mass $v(\mathcal{C})$ of employers posts contracts of type $\mathcal{C}$ and a mass $u(\mathcal{C})$ of unemployed workers applies for jobs at firms offering that type of contract. I assume that each submarket is characterized by a constant returns to scale matching function $m(v(\mathcal{C}), u(\mathcal{C}))$ and by an associated "tightness" $\gamma(\mathcal{C}) = v(\mathcal{C})/u(\mathcal{C})$.[4] Hence, for each contract $\mathcal{C}$, I can define the function $\mu(\gamma) \equiv m(\gamma, 1)$, which represents the probability of a worker applying for $\mathcal{C}$ meeting an employer posting it. On the other hand, the probability of a firm posting $\mathcal{C}$ meeting a worker applying for it is represented by the non-increasing function $\mu(\gamma)/\gamma$.

Assumption A1. The function $\mu(\gamma) : [0, \infty) \longmapsto [0, 1]$ satisfies the following conditions:

(i) $\mu(\gamma) \leq \min\{\gamma, 1\}$;[5]

(ii) for any γ such that $\mu(\gamma) < \min\{\gamma, 1\}$, $\mu(\gamma)$ is twice differentiable with $\mu'(\gamma) > 0$ and $\mu''(\gamma) < 0$.

Employment Contracts and Bellman Values. A contract $\mathcal{C}_t : \Theta \times Y \mapsto [0,1] \times \mathbb{R}_+$, specifies for each worker of type θ matched at time t when productivity is y_t, the hiring probability $e_t(\theta, y_t) \in [0,1]$ and the expected net present value of wages $\omega_t(\theta, y_t) \in \mathbb{R}_+$. Notice that linear preferences, together with the fact that types are fixed over time within a match, imply that the wage profile over the life of the relationship is irrelevant for the analysis. Therefore, I can define the continuation value of being employed *net of wages and training cost* at time t, V_t, which

[4]In order to simplify the notation, from now on I am going to drop the dependence of u, v and γ on the contract $\mathcal{C}$, whenever it does not cause any confusion.

[5]With discrete time, this condition ensures that both $\mu(\gamma)$ and $\mu(\gamma)/\gamma$ are proper probabilities.

from now on I will refer to simply as the continuation utility of employed workers, represents just the discounted expected value of being separated and becoming unemployed, that is,

$$V_t(y_t) = \beta E_t\left[sU_{t+1}(y_{t+1}) + (1-s)V_{t+1}(y_{t+1})\,|y_t\right].$$

Moreover the continuation value of an unemployed worker at time t is given by

$$\begin{aligned} U_t(y_t) \quad &= \quad \beta E_t\left[\mu\left(\gamma_{t+1}(y_t)\right)\int_{\underline{\theta}}^{\overline{\theta}}\left[\omega_{t+1}(\theta, y_{t+1}) - e_{t+1}(\theta, y_{t+1})\left[\theta - V_{t+1}(y_{t+1})\right]\right]dF(\theta)\,|y_t\right] \\ &\quad +b+\beta E\left[\left(1-\int_{\underline{\theta}}^{\overline{\theta}} e_{t+1}(\theta, y_{t+1})\,dF(\theta)\right)U_{t+1}(y_{t+1})\,|y_t\right]. \end{aligned}$$

The utility of a worker of type θ who meets an employer at time t when productivity is y_t, is given by

$$v_t(\theta, y_t) = \left[\omega_t(\theta, y_t) - e_t(\theta, y_t)\left(\theta - V_t(y_t)\right)\right] + \left[1 - e_t(\theta, y_t)\right]U_t(y_t)$$

and the expected revenues of the firm, after a match, is given by

$$\int_{\underline{\theta}}^{\overline{\theta}}\left[e_t(\theta, y_t)\,\alpha y - \omega_t(\theta, y_t)\right]dF(\theta),$$

where $\alpha \equiv \left[1 - \beta(1-s)\right]^{-1}$.

The large number of potential firms ensures free entry and implies that the value of an open vacancy will be zero at each time, that is,

$$\beta\frac{\mu(\gamma_t)}{\gamma_t}\int_{\underline{\theta}}^{\overline{\theta}}\left[e_t(\theta, y_t)\,\alpha y_t - \omega_t(\theta, y_t)\right]dF(\theta) = k. \tag{2.1}$$

In order to make every type θ to participate, it must be that

$$v_t(\theta, y_t) \geq U_t(y_t) \quad \text{for any } \theta, y_t. \tag{2.2}$$

2.2.1 Competitive Search Equilibrium

This section generalizes the standard definition of the Competitive Search Equilibrium to an environment with aggregate shock. In order to simplify the analytical treatment, we adopt a definition in recursive terms, following Chapter 1.

It is possible to show that a recursive Competitive Search Equilibrium takes the following simple form. It is a set of contracts contingent on the aggregate shock only $\mathbb{C}^*(y)$, a tightness function $\Gamma^*(y)$, where $\Gamma^*(y) : \mathbb{C}^*(y) \longmapsto \mathbb{R}_+ \cup \infty$ and a pair of continuation utility functions $\{U^*(y), V^*(y)\}$ such that, given y, employers maximize profits and workers apply optimally for jobs, taking as given the future values of being employed $V(y)$ and unemployed $U(y)$ and aware that a market tightness is associated with each contract, even if not offered in equilibrium. Moreover, profits are driven to zero by free entry. I define more formally a recursive CSE in Appendix A.

Following similar steps Next Proposition states the characterization of a stochastic symmetric competitive search equilibrium in recursive terms. In the rest of the analysis I adopt a recursive notation, dropping the t whenever this causes no confusion, and denoting a variable at time $t+1$ with a prime.

Proposition 28 *If $\{\mathbb{C}(y), \Gamma(y), U(y), V(y)\}$ is a Recursive Competitive Search Equilibrium, then any pair $(C^*(y), \gamma^*(y))$ with $C^*(y) \in \mathbb{C}(y)$ and $\gamma^*(y) = \Gamma^*(C^*(y))$ satisfy the following*

(i) *for given pair of functions $\{U(y), V(y)\}$, $C(y) = [e(\theta, y), \omega(\theta, y)]_{\theta\in\Theta, y\in Y}$ and $\gamma(y)$ solve*

$$\begin{aligned} W(U(y), V(y), y) \quad &= \max_{\substack{e(\theta,y),\omega(\theta,y) \\ \gamma(y)}} \beta\mu(\gamma(y)) \int_{\underline{\theta}}^{\overline{\theta}} [\omega(\theta, y) - e(\theta, y)(\alpha\theta - V(y))] \, dF(\theta) \\ &+\beta \left[1 - \mu(\gamma(y)) \int_{\underline{\theta}}^{\overline{\theta}} e(\theta, y) \, dF(\theta)\right] U(y) \qquad \text{(P1)} \end{aligned}$$

subject to $e(\theta, y) \in [0, 1]$, the individual rationality constraints (2.2) and the free-entry condition (2.1);

(ii) *for given pair $\{C(y), \gamma(y)\}$, then $\{U(y), V(y)\}$ evolve according to*

$$U(y) = b + \int W(U(y'), V(y'), y') dG(y'|y)$$

and

$$V(y) = \beta \int \left[sU(y') + (1-s)V(y')\right] dG(y'|y).$$

Conversely, if a pair of functions $\{C^*(y), \gamma^*(y)\}$ *solves the program P4, then there exists an equilibrium* $\{\mathbb{C}^*(y), \Gamma^*(y), U^*(y), V^*(y)\}$ *such that* $C^*(y) \in \mathbb{C}^*(y)$ *and* $\gamma^*(y) = \Gamma^*(C^*(y))$.

It follows that the unemployment rate of workers applying to firms posting a contract of type $C(y)$ is given by

$$u'(C(y')) = u(C(y))\left[1 - \mu(\Gamma^*(C(y)))\int_{\underline{\theta}}^{\overline{\theta}} e(\theta, y)\, dF(\theta)\right] + (1 - u(C(y)))\, s. \tag{2.3}$$

Proof. Similar to the proof of Proposition 10 in Chapter 1. ■

Equilibrium Characterization. Using pointwise maximization with respect to $e(\theta)$ I show that the trading area can be fully described by a cut-off value $\hat{\theta}(y)$ such that

$$e(\theta, y) = \begin{cases} 1 \text{ if } \theta \leq \hat{\theta}(y) \\ 0 \text{ if } \theta > \hat{\theta}(y) \end{cases}.$$

Then, when $\mu(\gamma)$ is everywhere differentiable the equilibrium can be characterized for given y and given continuation utilities $U(y)$ and $V(y)$, by a pair of functions $\hat{\theta}(y)$ and $\gamma(y)$ satisfying the first order conditions of the maximization problem P1, that is,

$$\alpha y - \hat{\theta}(y) - [U(y) - V(y)] = 0 \tag{2.4}$$

$$\beta\mu'(\gamma(y)) \int_0^{\hat{\theta}(y)} [\alpha y - \theta - [U(y) - V(y)]]\, dF(\theta) = k. \tag{2.5}$$

2.2.2 Comparative Statics

In this section I look at some comparative statics to get a sense of the volatility of job creation implied by the model. Following Shimer (2005), I derive the elasticity of market tightness with respect to productivity, when there are no aggregate shocks. I first consider the case of a degenerate distribution for the idiosyncratic shocks and I show that in this case my model is isomorphic to the one analyzed by Shimer (2005) with random matching and Nash bargaining, where the Hosios condition is met. Next, I show that when the distribution of idiosyncratic shocks becomes non-degenerate, the effective finding rate depends not only on the equilibrium matching probability $\mu(\gamma(y))$, but also on the equilibrium hiring decision $F(\hat{\theta}(y))$ which may amplify the impact of productivity shocks on market tightness and unemployment rate. However, I show that, in the deterministic case, the response of market tightness to productivity does not change, while the hiring margin magnifies the impact of productivity on unemployment rate.

Let assume that the distribution of the productivity shock is degenerate and y is fixed. Then, from the fixed point problem described in the second part of Proposition (28) we get the equilibrium values of $U(y)$ and $V(y)$

$$U(y) = \frac{\beta\mu(\gamma(y)) \int_{\underline{\theta}}^{\hat{\theta}(y)} [\alpha y - \theta]\, dF(\theta) + b - \gamma(y)\, k}{(1-\beta)\left[1 + \alpha\beta\mu(\gamma(y))\, F(\hat{\theta}(y))\right]}, \tag{2.6}$$

$$V(y) = \frac{\beta s}{1-\beta(1-s)} U(y), \tag{2.7}$$

where the equilibrium $\hat{\theta}(y)$ and $\gamma(y)$ solve Problem P1. Combining equations (2.6) and (3.46) with the first order conditions (3.43) and (3.44) and after some algebra, the equilibrium can be characterized by the following two conditions

$$\hat{\theta}(y) = \alpha(y-b) - \alpha\left(\frac{1-\delta}{\delta}\right)\gamma(y)\, k, \tag{2.8}$$

$$\frac{\int_0^{\hat{\theta}(y)} [y - \theta/\alpha - b]\, dF(\theta)}{F(\hat{\theta})} = \left[\frac{1}{\alpha\beta\delta\mu(\gamma)\, F(\hat{\theta})} + \frac{1-\delta}{\delta}\right] k\gamma. \tag{2.9}$$

First, notice that if we assume that the distribution of the idiosyncratic shocks is also degenerate, then for θ low enough, matches become always productive and the equilibrium can be characterized simply by

$$y - \theta/\alpha - b = \left[\frac{1}{\alpha\beta\delta\mu(\gamma)} + \frac{1-\delta}{\delta}\right] k\gamma$$

which is exactly the discrete time version of the equilibrium equation of the model described in Shimer (2005) and gives elasticity of market tightness with respect to productivity $\varepsilon_{\gamma y}$ given by the following expression

$$\varepsilon_{\gamma y}=\frac{y}{y-\theta/\alpha-b}\frac{(1-\delta)\,\beta\mu\,(\gamma)+(1-\beta\,(1-s))}{(1-\delta)\,[\beta\mu\,(\gamma)+(1-\beta\,(1-s))]}.$$

Next, I show that when the distribution of the idiosyncratic shocks becomes non-degenerate, then there is an extra term in the expression for $\varepsilon_{\gamma y}$ coming from the responsiveness of the hiring margin to productivity which does not help in making market tightness more elastic to productivity.

Let start by keeping $\hat{\theta}$ fixed. Notice that the finding rate and the unemployment are identical to an economy with degenerate distribution of idiosyncratic shocks (denoted by D) with θ equal to 0, as in Shimer (2005), with the same y, β, s and with parameters:

$$\eta_D = \eta F(\hat{\theta}) \tag{2.10}$$

$$b_D = E\left[\theta/\alpha+b|\theta\leq\hat{\theta}\right] \tag{2.11}$$

$$\delta_D = \delta \tag{2.12}$$

$$k_D = k \tag{2.13}$$

where the response of the market tightness to productivity, $\gamma_D'(y)$, satisfies

$$\gamma_D'(y)=\frac{\gamma}{y-b_D}\frac{1-\beta\,(1-s)+\beta\mu_D\,(\gamma)\,(1-\delta)}{[1-\beta\,(1-s)-\beta\mu_D\,(\gamma)]\,(1-\delta)}.$$

Then, notice that we can write equation (2.9) as the following implicit function

$$h(\hat{\theta}\,(y)\,,\gamma\,(y)\,,y)\equiv\frac{\int_0^{\hat{\theta}(y)}[y-\theta/\alpha-b]\,dF\,(\theta)}{F(\hat{\theta})}-\left[\frac{1-\beta\,(1-s)}{\beta\delta\mu\,(\gamma)\,F(\hat{\theta})}+\frac{1-\delta}{\delta}\right]k\gamma=0.$$

In appendix A I show that $\gamma'(y)$ satisfies

$$\frac{\gamma'(y)}{\gamma'(y)_D}=1+h_{\hat{\theta}}(\hat{\theta}\,(y)\,,\gamma\,(y)\,,y)\hat{\theta}'\,(y)\,.$$

Next Lemma shows that $h_{\hat{\theta}}(\hat{\theta}(y), \gamma(y), y) = 0$ and the derivative of the market tightness with respect to productivity in the model with idiosyncratic shocks coincides with the degenerate case.

Lemma 29 *Suppose conditions (2.10)-(2.13) are satisfied, then* $\gamma'(y) = \gamma'_D(y)$.

Proof. See Appendix A. ■

The Lemma shows that keeping constant the elasticity of the matching function, the response of the hiring margin to productivity does not change the response of the market tightness. On the one hand, when the hiring margin increases, firms have an incentive to post more vacancies because the return for vacancy posted is higher. On the other hand, firms have an higher cost for vacancy posted because of two effects: a direct one, that is, the expected learning cost for a matched worker has increased, and and indirect one, that is, the outside option of the matched worker has increased as well. The Lemma shows that in equilibrium the increase of the benefit of a posted vacancy coincides exactly with the increase of its cost. This implies that the response of the market tightness to productivity is not affected by the response of the hiring margin.

Next, I show that when the distribution of the idiosyncratic shocks θ is degenerate and there is no aggregate shock, then the response to productivity of the cut-off $\hat{\theta}(y)$, that is, the hiring margin, is always positive. By differentiating condition (3.43), I obtain

$$\hat{\theta}'(y) = \alpha y - \alpha \left(\frac{1-\delta}{\delta}\right) k\gamma'(y),$$

and plugging that together with $\gamma'(y)$, given conditions (2.10)-(2.13), the previous Lemma gives

$$\hat{\theta}'(y) = \left[1 - \beta(1-s) + \beta\mu(\gamma) F(\hat{\theta})\right]^{-1} > 0. \tag{2.14}$$

This immediately implies that the model with idiosyncratic shocks generate a bigger impact of productivity on the job creation rate and, hence, on the unemployment rate. Let define the job creation rate $j(y) = \mu(\gamma(y)) F(\hat{\theta}(y))$. In the degenerate version of the model simply

$j_D(y) = \mu_D(\gamma(y))$. Notice that we can write the steady state equation for unemployment as

$$u(y) = \frac{s}{s + j(y)}. \tag{2.15}$$

The next Lemma shows that the response of unemployment to productivity is always higher in the model with heterogeneity.

Lemma 30 *Suppose conditions (2.10)-(2.13) are satisfied, then* $u'(y) > u'_D(y)$.

Proof. From equation (2.14) and the definition of $j(y)$ and $j_D(y)$, it follows that

$$j'(y) = j'_D(y) + \eta\gamma(y)^{\delta} f(\hat{\theta}(y))\hat{\theta}'(y) > j'_D(y).$$

Using that and differentiating equation (2.15) gives

$$u'(y) = j'(y)\frac{y}{s + j(y)} > j'_D(y)\frac{y}{s + j_D(y)} = u'_D(y),$$

completing the proof. ■

Notice that the parametrization of the previous analysis allows both the models to match the finding rate coming from the data, but not the finding rate elasticity to market tightness. As Shimer (2005) shows, there exists a loglinear relationship between market tightness and finding rate which can be estimated in the data. In the degenerate model the finding rate elasticity to market tightness is represented by δ which can be estimated in the data, while in the model with a non-degenerate distribution for idiosyncratic shocks where $j = \eta\gamma^{\delta}F(\hat{\theta})$, then

$$\frac{d\log j}{d\log y} = \frac{f(\hat{\theta})}{F(\hat{\theta})}\frac{d\log\hat{\theta}}{d\log y} + \delta\frac{d\log\gamma}{d\log y}$$

so that the object that can be estimated corresponds to

$$\frac{d\log j}{d\log\gamma} = \frac{f(\hat{\theta})}{F(\hat{\theta})}\frac{\frac{d\log\theta}{d\log y}}{\frac{d\log\gamma}{d\log y}} + \delta.$$

Hence, consider the alternative parametrization

$$\eta_D \gamma^{\delta_D} = \eta \gamma^{\delta} F(\hat{\theta}) \tag{2.16}$$

$$b_D = E\left[\theta/\alpha + b | \theta \leq \hat{\theta}\right] \tag{2.17}$$

$$\delta_D = \frac{f\left(\hat{\theta}\right) \frac{d\log\theta}{d\log y}}{F\left(\hat{\theta}\right) \frac{d\log\gamma}{d\log y}} + \delta \tag{2.18}$$

$$k_D = \frac{\frac{1-\beta(1-s)}{\beta\delta\mu(\gamma)F(\hat{\theta})} + \frac{1-\delta}{\delta}}{\frac{1-\beta(1-s)}{\beta\delta_D \eta_D \gamma^{\delta_D} F(\hat{\theta})} + \frac{1-\delta_D}{\delta_D}} k \tag{2.19}$$

where again the finding rate and the unemployment of a non-degenerate economy are identical to the ones of an economy with degenerate distribution of θ equal to 0, with those parameters and the same y, β, s.

Proposition 31 *Suppose conditions (2.16)-(2.19) are satisfied, then* $\gamma'(y) < \gamma'_D(y)$.

Proof. *See Appendix A.* ■

I will show in Chapter 2.4 that the dynamics of the stochastic model follow closely these analytical steady state results. In fact, when I calibrate the model with Pareto distribution of idiosyncratic shocks, then $\delta \simeq \delta_D$ and the first parametrization exercise is approximately correct, implying that the market tightness volatility in the model with heterogeneity is approximately equal to the one in the standard model with a degenerate distribution of idiosyncratic shocks. When, instead, I calibrate the model with Uniform distribution, I show that it must be that $\delta < \delta_D$ and in fact the market tightness volatility results even lower than the degenerate case.

2.3 Asymmetric information

In this chapter, I introduce asymmetric information, following tightly the model of Chapter 1 augmented with an aggregate shock on productivity, together with the difference that θ is now a sunk cost (such as learning cost) instead of a flow cost (such as work effort) as explained in the full information case. In chapter 2.4, I show that asymmetric information can generate further amplification or dampening in the volatility of the unemployment rate, depending on the distribution of the idiosyncratic shocks.

Environment. The economy is exactly the same described in section 2.2, with the only difference that the training cost θ is now private information of the worker. An employer who opens a vacancy, pays a cost k and post a contract $\mathcal{C}_t \in \mathbb{C}$ where $\mathbb{C}$ is the set of incentive compatible and individually rational mechanisms. As I describe below, a contract $\mathcal{C}_t : \Theta \times Y \mapsto [0,1] \times \mathbb{R}_+$ specifies the hiring probability and the expected net present value of wages for each worker matched at time t when productivity is y_t, reporting type θ. Therefore at each time t, a non-productive firm chooses a pair $(\sigma_t, \mathcal{C}_t) \in \{0,1\} \times \mathbb{C}$ where σ_t denotes the decision of posting a vacancy. Next, each unemployed worker observes all the contracts posted and decides where to apply. He chooses a contract $\mathcal{C}_t \in \mathbb{C}_t^P \subset \mathbb{C}$, where $\mathbb{C}_t^P$ denotes the set of contracts posted by active firms at time t. After workers start to search for a specific contract, matching takes place and, for each match, the draw θ is realized. At the same time the aggregate productivity shock y_t is realized. The behavior of a worker who is matched at time t is described by a map $(a_t, s_t) : \Theta \times Y \mapsto \Theta \times \{0,1\}$ that for each type θ and productivity y_t specifies a report $\hat{\theta}_t(y_t) = s(\theta, y)$ and a participation decision $a_t(\theta, y_t)$. After he sees his type, the worker can either implement the contract, that is choose $a_t(\theta, y_t) = 1$, or walk away, that is choose $a_t(\theta, y_t) = 0$. As in the case of full information, if he walks away, he is unemployed for one period, gets a non-transferable utility from leisure b and looks for another match next period. If the worker is hired, the parties are productive until separation, which happens according to a Poisson process with parameter s.

Employment Contracts and Bellman Values. Invoking the Revelation Principle, without loss of generality, I can again restrict attention to incentive-compatible and individually rational direct revelation mechanisms, corresponding to a mapping $\mathcal{C}_t : \Theta \times Y \mapsto [0,1] \times \mathbb{R}_+$, specifying for each matched worker at time t who reports type θ, the hiring probability $e_t(\theta, y_t) \in [0,1]$ and the expected net present value of wages $\omega_t(\theta, y_t) \in \mathbb{R}_+$. Notice that I can restrict attention to the set $\mathbb{C}$ of *ad interim* incentive compatible and individually rational mechanisms described above, due to the unemployed anonymity assumption. All the unemployed workers searching for a job cannot be distinguished, so that contracts cannot be conditioned on the past employment history. Moreover, as in the full information case, linear preferences, together with the fact that types are fixed over time within a match, imply that the wage profile over the life of the relationship is irrelevant for the analysis. Therefore, I can define the continuation value of being employed *net of wages and training cost* at time t conditional on the productivity shock is y_t as

$$V_t(y_t) = \beta E_t \left[s U_{t+1}(y_{t+1}) + (1-s) V_{t+1}(y_{t+1}) \, | y_t \right].$$

and the continuation value of an unemployed worker at time t conditional on y_t as

$$\begin{aligned} U_t(y_t) &= \beta E_t \left[\mu\left(\gamma_{t+1}(y_t)\right) \int_{\underline{\theta}}^{\overline{\theta}} \left[\omega_{t+1}(\theta, y_{t+1}) - e_{t+1}(\theta, y_{t+1}) \left[\theta - V_{t+1}(y_{t+1}) + U_{t+1}(y_{t+1}) \right] \right] dF(\theta) \, | y_t \right] \\ &\quad + b + \beta E \left[U_{t+1}(y_{t+1}) \, | y_t \right]. \end{aligned}$$

The expected utility of a worker of type θ reporting type $\hat{\theta}$ at time t, is given by

$$v_t(\theta, \hat{\theta}, y_t) = [\omega_t(\hat{\theta}, y_t) - e_t(\hat{\theta}, y_t)(\theta - V_t(y_t))] + [1 - e_t(\hat{\theta}, y_t)] U_t(y_t).$$

An employment contract is *incentive-compatible (IC)* whenever it satisfies

$$v(\theta, \theta, y) \geq v(\theta, \hat{\theta}, y) \text{ for all } \theta, \hat{\theta} \in \Theta \tag{IC}$$

and *individually rational (IR)* whenever

$$v(\theta, \theta, y) \geq U_t(y_t) \text{ for all } \theta \in \Theta. \tag{IR}$$

I define $\mathbb{C}$ the set of incentive compatible and individually rational direct mechanisms. Following Myerson (1981), as stated in Lemma 1 of Chapter 1, I can reduce IC and IR to a monotonicity condition $e_\theta(\theta, y) < 0$ for all θ, the IR binding for the worse type

$$v\left(\overline{\theta}, \overline{\theta}, y\right) \geq U_t(y_t) \tag{2.20}$$

and the following condition

$$v(\theta,\theta,y)=v\left(\overline{\theta},\overline{\theta},y\right)+\int_{\theta}^{\overline{\theta}}e(x,y)\,dx\ \forall\theta\in\Theta. \tag{2.21}$$

The expected revenues of the firm, after a match, are given by

$$\int_{\underline{\theta}}^{\overline{\theta}}\left[e_t(\theta,y_t)\,\alpha y-\omega_t(\theta,y_t)\right]dF(\theta).$$

The large number of potential firms ensures free entry and implies that the value of an open vacancy will be zero at each time, that is,

$$\beta\frac{\mu(\gamma_t)}{\gamma_t}\int_{\underline{\theta}}^{\overline{\theta}}\left[e_t(\theta,y_t)\,\alpha y_t-\omega_t(\theta,y_t)\right]dF(\theta)=k. \tag{2.22}$$

2.3.1 Competitive Search Equilibrium

This section generalizes the definition of the Competitive Search Equilibrium introduced in section 2.2.1 to an environment with asymmetric information.

As in the case of full information, it is possible to show that a recursive Competitive Search Equilibrium takes the following simple form. It is a set of incentive-compatible and individually-rational contracts contingent on the aggregate shock only $\mathbb{C}^*(y)$, a tightness function $\Gamma^*(y)$, where $\Gamma^*(y):\mathbb{C}^*(y)\longmapsto\mathbb{R}_+\cup\infty$ and a pair of continuation utility functions $\{U^*(y),V^*(y)\}$ such that, given y, employers maximize profits and workers apply optimally for jobs, taking as given the future values of being employed $V(y)$ and unemployed $U(y)$ and aware that a market tightness is associated with each contract, even if not offered in equilibrium. Moreover, profits are driven to zero by free entry.

Next Proposition is the analogous of Proposition 28 and states the characterization of a stochastic symmetric competitive search equilibrium under asymmetric information in recursive terms.

Proposition 32 *If $\{\mathbb{C}(y),\Gamma(y),U(y),V(y)\}$ is a Recursive Competitive Search Equilibrium, then any pair $(C^*(y),\gamma^*(y))$ with $C^*(y)\in\mathbb{C}(y)$ and $\gamma^*(y)=\Gamma^*(C^*(y))$ satisfy the following*

(i) *for given pair of functions* $\{U(y), V(y)\}$, $C(y) = [e(\theta, y), \omega(\theta, y)]_{\theta \in \Theta, y \in Y}$ *and* $\gamma(y)$ *solve*

$$
\begin{aligned}
W(U(y), V(y), y) &= \max_{\substack{e(\theta,y),\omega(\theta,y) \\ \gamma(y)}} \beta\mu(\gamma(y)) \int_{\underline{\theta}}^{\overline{\theta}} [\omega(\theta, y) - e(\theta, y)(\alpha\theta - V(y))]\, dF(\theta) \\
&\quad + \beta \left[1 - \mu(\gamma(y)) \int_{\underline{\theta}}^{\overline{\theta}} e(\theta, y)\, dF(\theta)\right] U(y) \qquad \text{(P2)}
\end{aligned}
$$

subject to $e(\theta, y) \in [0, 1]$, *the free-entry condition (2.22), the constraints IC and IR reduced to the conditions (2.20), (2.21) and* $e_\theta(\theta, y) < 0$;

(ii) *for given pair* $\{C(y), \gamma(y)\}$, *then* $\{U(y), V(y)\}$ *evolve according to*

$$U(y) = b + \int W(U(y'), V(y'), y')dG(y'|y)$$

and

$$V(y) = \beta \int [sU(y') + (1 - s)V(y')]\, dG(y'|y).$$

Conversely, if a pair of functions $\{C^*(y), \gamma^*(y)\}$ *solves the program P4, then there exists an equilibrium* $\{\mathbb{C}^*(y), \Gamma^*(y), U^*(y), V^*(y)\}$ *such that* $C^*(y) \in \mathbb{C}^*(y)$ *and* $\gamma^*(y) = \Gamma^*(C^*(y))$.

Proof. *Similar to the proof of Proposition 10 in Chapter 1.* ■

Proposition 32 shows that for given $U(y)$ and $V(y)$, a recursive symmetric equilibrium incentive-compatible and individually-rational contract $C(y)$ and tightness $\gamma(y)$ must solve Problem P2. The next Proposition shows that the equilibrium can be equivalently described by a hiring function $e(\theta, y)$ and a tightness $\gamma(y)$ that solve a simplified program P3. Given $e(\theta, y)$ and $\gamma(y)$, an associated wage function $\omega(\theta, y)$ can be constructed so that the constraints IC and IR are satisfied.

Proposition 33 *For given* $U(y)$ *and* $V(y)$, *any couple of functions* $e(\theta, y)$ *and* $\gamma(y)$ *which solve Problem P2, solve also*

$$
\begin{aligned}
W(U(y), V(y), y) &= \max_{e(\theta,y),\gamma(y)} \beta\mu(\gamma(y)) \int_{\underline{\theta}}^{\overline{\theta}} e(\theta, y)[\alpha y - \alpha\theta + V(y) - U(y)]\, dF(\theta) \text{(P3)} \\
&\quad + \beta U(y) - \gamma(y)k
\end{aligned}
$$

s.t.

$$\beta\mu\left(\gamma\left(y\right)\right)\int_{\underline{\theta}}^{\overline{\theta}}e\left(\theta,y\right)\left[\alpha y-\theta-\frac{F\left(\theta\right)}{f\left(\theta\right)}+V\left(y\right)-U\left(y\right)\right]dF\left(\theta\right)\geq\gamma\left(y\right)k$$

Furthermore, for any pair of functions $e(\theta, y)$ and $\gamma(y)$ which solve problem P3, there exists a function $\omega(\theta, y)$ such that the contract $C(y) = [e(\theta, y), \omega(\theta, y)]_{\theta\in\Theta, y\in Y}$ and $\gamma(y)$ solve problem P2.

Proof. Similar to Proposition 11 in Chapter 1. ■

Equilibrium Characterization. I proceed by studying the relaxed problem without the monotonicity assumption on $e(\theta, y)$. Then, using pointwise maximization with respect to $e(\theta, y)$ I show that the trading area can be fully described by a cut-off value $\hat{\theta}(y)$ such that

$$e\left(\theta,y\right)=\begin{cases}1\text{ if }\theta\leq\hat{\theta}\left(y\right)\\0\text{ if }\theta>\hat{\theta}\left(y\right)\end{cases},$$

implying that the optimal $e(\theta)$ is effectively non-increasing. When the constraint of problem P3 is binding[6] and $\mu(\gamma)$ is everywhere differentiable[7] the equilibrium can be characterized, for given $U(y) - V(y)$, by a set of functions $\hat{\theta}(y)$, $\gamma(y)$ and $\lambda(y)$ satisfying the first order conditions, respectively, for $\hat{\theta}(y)$ and $\gamma(y)$

$$\alpha y-\hat{\theta}\left(y\right)-\lambda\frac{F(\hat{\theta}\left(y\right))}{f(\hat{\theta}\left(y\right))}-\left(U\left(y\right)-V\left(y\right)\right)=0, \tag{2.23}$$

$$\beta\mu'\left(\gamma\left(y\right)\right)\int_{\underline{\theta}}^{\hat{\theta}(y)}\left[\alpha y-\theta-\lambda\left(y\right)\frac{F\left(\theta\right)}{f\left(\theta\right)}-\left(U\left(y\right)-V\left(y\right)\right)\right]dF\left(\theta\right)=k \tag{2.24}$$

and the constraint

$$\beta\mu\left(\gamma\left(y\right)\right)\int_{\underline{\theta}}^{\hat{\theta}(y)}\left[\alpha y-\theta-\frac{F\left(\theta\right)}{f\left(\theta\right)}-\left(U\left(y\right)-V\left(y\right)\right)\right]dF\left(\theta\right)=\gamma\left(y\right)k. \tag{2.25}$$

The variable $\lambda(y)$ represents a normalized version[8] of the shadow value of the *informational*

[6] A generalization of Lemma 2 in Chapter 1 gives that the constraint is binding if and only if the cost of posting a vacancy k is strictly positive.

[7] When $\mu(\gamma(y))$ is not differentiable at some points, equation (2.25) will be replaced by inequalities involving the left and right derivatives of $\mu(\gamma(y))$ at the points of non differentiability.

[8] Similarly to Chapter 1, $\lambda \equiv \hat{\lambda}/(1+\hat{\lambda})$, where $\hat{\lambda}$ is the Lagrangian multiplier attached to the constraint of

rents.[9] Notice that when $\lambda(y) = 0$, the constraint is slack and the full information allocation described in section (2.2) is achieved. Clearly, asymmetric information reduces job creation, as the surplus of the economy must cover not only the cost of vacancy creation but also the rents needed to extract information from the workers. As intuition suggests, $\hat{\theta}(y)$ decreases with $\lambda(y)$.

2.3.2 Steady state with no aggregate shocks

In order to give some intuition of the role of asymmetric information, let consider the case of a degenerate distribution of the productivity shock, that is, keeping y fixed. Then, from the fixed point problem described in the second part of Proposition (32) we get the steady state equilibrium values of $U(y)$ and $V(y)$

$$U(y) = \frac{\beta\mu(\gamma(y)) \int_{\underline{\theta}}^{\hat{\theta}(y)} [\alpha y - \theta]\, dF(\theta) + b - \gamma(y)\, k}{(1-\beta)\left[1 + \alpha\beta\mu(\gamma(y))\, F(\hat{\theta}(y))\right]}, \tag{2.26}$$

$$V(y) = \frac{\beta s}{1 - \beta(1-s)} U(y), \tag{2.27}$$

where the equilibrium $\hat{\theta}(y)$ and $\gamma(y)$ solve Problem P3. Combining these equations with the first order conditions (2.23)-(2.25) and after some algebra, the equilibrium can be characterized by the following three conditions

$$y - \frac{\hat{\theta}}{\alpha} - \frac{\lambda}{\alpha}\frac{F(\hat{\theta})}{f(\hat{\theta})} - b + \gamma k = \beta\mu(\gamma) \int_{\underline{\theta}}^{\hat{\theta}} \left(\hat{\theta} - \theta + \lambda\frac{F(\hat{\theta})}{f(\hat{\theta})}\right) dF(\theta),$$

$$\beta\delta\mu(\gamma) \int_{\underline{\theta}}^{\hat{\theta}} \left[\hat{\theta} - \theta + \lambda\left(\frac{F(\hat{\theta})}{f(\hat{\theta})} - \frac{F(\theta)}{f(\theta)}\right)\right] dF(\theta) = \gamma k,$$

$$\lambda = 1 - \left[\delta\beta\frac{\mu(\gamma)}{\gamma} \int_{\underline{\theta}}^{\hat{\theta}} \frac{F(\theta)}{f(\theta)} dF(\theta)\right]^{-1} (1-\delta)\, k.$$

---

problem P3.

[9] As in Chapter 1, define $v(\theta, \theta, y) - v(\overline{\theta}, \overline{\theta}, y)$ as the *informational rent* of a worker of type $\theta \leq \overline{\theta}$, that is, the additional utility that such a worker must receive in order to reveal his own type.

Notice that, when $\lambda = 0$, they boil down to (3.43) and (3.44). The third equation shows that at a first order approximation λ decreases with γ, since the more vacancies are posted the higher becomes the expected surplus created per vacancy which cover the workers' rents, and increases with $\hat{\theta}$, because as the hiring margin increases more rents need to be paid to the workers and the distortion get worse. On the other hand, the first two equations show that as λ changes, market tightness and hiring margin are changing endogenously. The responsiveness of market tightness to productivity seems ambiguous. The calibration in Section 2.4 show that in fact asymmetric information can either amplify or dampen the market tightness volatility.

2.4 Unemployment volatility

In this section I analyze the cyclical behavior of unemployment and market tightness generated by the model described, with and without full information. I also compare these results to the benchmark model with no idiosyncratic shocks. First I describe how I calibrate the different versions of the model. Then I show two sets of results, using two different family of distribution for the idiosyncratic shocks: Pareto and Uniform.

2.4.1 Calibration

I now describe how I parametrize the model to match the U.S. facts, documented in Shimer (2005), in order to investigate the role of heterogeneity and asymmetric information in explaining the U.S. unemployment rate volatility.

I normalize a time period to be one quarter and then I set the discount factor β to .988, consistent with an annual discount factor of .953.

I choose a separation rate s of .034, consistent with jobs lasting about 2.5 years on average, as in Shimer (2005).

I set the instantaneous utility of unemployed workers b such that the $b + E\left[\theta/\alpha|\theta \leq \hat{\theta}\,(y)\right]$ is approximately equal to .4 of the labor income. In my model the effective opportunity cost of being employed for a matched worker of type θ is $b + \theta/\alpha$ so that to calibrate b equal to .4 of the labor income, would be equivalent to choose an effective higher value of leisure, in the direction of Hagerdon and Manovskii (2005). I want to shut down their mechanism to highlight the role of heterogeneity and information.

For the job-finding rate, j_t, I use the series constructed by Shimer (2005)[10]. He constructs the series for the job-finding rate from 1951 to 2003, using the series for the behavior of unemployment level and the short-term unemployment level constructed by the BLS from the CPS. In particular, j_t is such that the number of unemployed next month is equal to the number of unemployed last month who did not find a job, that is, $(1 - j_t)\, u_t$, and workers who have lost the job between the beginning of this month and the end of it, denoted by u^s_{t+1}[11], that is,

$$j_t = 1 - \frac{u_{t+1} - u^s_{t+1}}{u_t}.$$

Using this series the monthly finding rate is .45, corresponding to a quarterly finding rate of .8336. As explained in section 2.2, in the model with idiosyncratic shocks θ, the finding rate corresponds to $\mu\left(\gamma\left(y\right)\right) F(\hat{\theta}\left(y\right))$ because unemployed workers in order to get a job have both to match an employer and being hired from him. The hiring margin is trivial in the standard model, because it is optimal to hire any matched worker and the finding rate reduces simply to $\mu\left(\gamma\right)$. Hence, I am going to match

$$E\left[\mu\left(\gamma\left(y\right)\right) F(\hat{\theta}\left(y\right))\right] = .8336. \tag{2.28}$$

I use the standard Cobb-Douglas matching function $\mu\left(\gamma\right) = \eta\gamma^{\delta}$. Shimer (2005) estimates the elasticity parameter $\delta = .28$ with a first-order autoregressive residual using detrended data on the job-finding rate and the v/u ratio for the U.S. between 1951 and 2003. In fact, when the distribution of the idiosyncratic shocks is degenerate, the finding rate coincides with the matching function. However, when the distribution of the idiosyncratic shocks is not degenerate,

[10] I thank Rob Shimer for making available the series.

[11] Notice that u^s_{t+1} represents the number of workers unemployed for less than a month.

the finding rate coincides with $j_t = \mu(\gamma_t) F(\hat{\theta}_t)$ and[12]

$$\frac{Cov(\log j_t, \log \gamma_t)}{Var(\log \gamma_t)} = .28. \tag{2.29}$$

I am going to perform two different types of exercises: *quasi-calibration* and *full-calibration*. In the quasi-calibration I am keeping the elasticity of the finding rate to the market tightness γ equal to .28, and calibrate η in order to match the expression (2.28). In the full calibration, instead, I am calibrating δ and η in order to match simultaneously expressions (2.28) and (2.29).

Moreover, in both the exercises, I fix $k = .213$, as in Shimer (2005), adjusting the normalization for the equilibrium γ, which is meaningless in the model.[13]

I choose a simple stochastic process for labor productivity as a two-state Markov process with the transition matrix estimated by Hamilton (1981), that is $\pi_{hh} = .9049$ and $\pi_{ll} = .7550$, where π_{hh} is the probability of staying in the high productivity state once you are there and π_{ll} is the equivalent for the low productivity state. Moreover I choose the gap in the state levels in order to have a standard deviation approximately equal to .02, as in Shimer (2005).

Finally, I analyze two different distributions for the idiosyncratic shocks. First I use a Pareto with lower bound $\underline{\theta}$ equal to .4 and parameter φ equal to 3 and then a Uniform distribution with range between 1.25 and 2 as I will explain below.

[12] Consider the following loglinear approximation:

$$\begin{aligned} \log j_t &= \beta \log \theta_t + \delta \log \gamma_t \\ \log \theta_t &= \tau \log \gamma_t \\ \log \gamma_t &= \alpha_\gamma \log y_t \end{aligned}$$

Given that $\log j_t = (\beta\tau + \delta) \alpha_\gamma \log y_t$, the estimate of the finding rate elasticity to market tightness corresponds to

$$\frac{Cov(\log j_t, \log \gamma_t)}{Var(\log \gamma_t)} = \frac{(\beta\tau + \delta) \alpha_\gamma^2 Var[\log y_t]}{\alpha_\gamma^2 Var[\log y_t]} = \beta\tau + \delta \approx .28,$$

since

$$\log j_t = (\beta\tau + \delta) \alpha_\gamma \log y_t.$$

Notice that

$$\frac{Var[\log j_t]}{Var[\log \gamma_t]} = (\beta\tau + \delta)^2$$

so that the ratio of the volatility of job creation with respect to market tightness is in fact pinned down from the data.

[13] As in Shimer (2005), it is the case that if I double k and multiply η by 2^δ, then γ is reduced by 1/2 and the rate at which firms find workers is doubled, but the worker's finding rate is not affected.

2.4.2 Results

In this section, I report the main results coming from my calibration exercises. I use the calibrated parameter values to simulate the model and create artificial time series of unemployment and vacancy rates, market tightness and finding rate. Then, I compare the volatility of the simulated series across the three main variants of the model: the case of degenerate distribution of the idiosyncratic shocks (D), that is, the benchmark model, the case of heterogenous workers with full information (FI) and the case of heterogeneous workers with asymmetric information (AI). I focus on two family of distributions for the idiosyncratic shocks: Pareto and Uniform. In both cases I analyze the results from the quasi-calibration exercise and from the full-calibration one and I find that, when I consider the proper calibration, heterogeneity does not help much in amplifying the volatility of unemployment and vacancy rates in comparison with the benchmark model, even with asymmetric information.

I start by analyzing the case of the Pareto distribution. Table 1 and 2 report the standard deviation for unemployment, vacancy, market tightness and finding rate in logs for respectively the quasi-calibration exercise and the full-calibration one.

	u	v	γ	j
D	.0416	.1264	.1678	.0470
FI	.0507	.1170	.1674	.0569
AI	.0553	.1193	.1741	.0629

Table 2.1: Quasi-Calibration with Pareto Distribution and $\delta = .28$.

As table 1 shows, the quasi-calibration exercise is interesting because it highlight that, mechanically, the introduction of heterogeneity generates more volatility in the unemployment rate due to the movements in the endogenous hiring margin, as Lemma 30 suggests. Moreover, since we keep δ constant, Lemma 29 applies locally at a first approximation. As one can notice, the market tightness volatility is approximately the same in the degenerate model and in the model with full information. When asymmetric information jumps into the picture, there is an extra source of volatility both for unemployment rate and market tightness. This amplification effect comes from the fact that the distortion generated by the informational rents that the employers have to pay to the workers is also responsive to productivity. However, the increase in volatility is minuscule. Moreover, once one considers the proper calibration exercise the increase in volatility virtually disappears. Moreover, as Lemma 3 suggests, the market tightness volatility reduces slightly. Table 2 shows the results of the full-calibration exercise, where δ is properly calibrated.

	u	v	γ	j
D	.0416	.1264	.1678	.0470
FI	.0414	.1254	.1665	.0468
AI	.0417	.1281	.1694	.0475

Table 2.2: Full-Calibration with Pareto Distribution and $\delta = .28$.

Next, let analyze the case of the Uniform distribution. Following the Pareto case, I report the results from both quasi-calibration exercises and full-calibration ones. First, it is interesting to notice that a proper calibration exercise imposes some restrictions on the parameters of the Uniform distribution. In this version of the paper, the distribution of the idiosyncratic shocks is not pinned down by the data. However this example shows that not all the distributions are consistent with the model. In fact, the range of the Uniform imposes a lower bound on the response of the finding rate to the business cycle fluctuations, that is, on the matching parameter δ^D for the degenerate distribution. Given that the data impose to fix $\delta^D = .28$, the set of the possible distributions consistent with the model is restricted. For the model with full information, a proper calibration imposes a lower bound on the range of 1.5244. Once asymmetric information is introduced, the range needs to be at least 1.25. I am reporting separately the results for the full information case with $A = 1.5244$ and for the asymmetric information case with $A = 1.25$, in order to make interesting the comparison[14].

Tables 3 and 4 show the comparison of the results for the model with a degenerate distribution of the idiosyncratic shocks and the model with a Uniform distribution with range $A = 1.5244$, respectively for the quasi-calibration and the full-calibration.

[14]The uniform distribution with range $A = 1.25$ is consistent also with the full information case, but would generate a corner solution (the hiring margin would be irrelevant) making the results coincide with the ones for the degenerate case.

	u	v	γ	j
D	.0386	.1291	.1514	.0424
FI	.0386	.1291	.1514	.0424

Table 2.3: Quasi-Calibration with Uniform Distribution, $A = 1.5244$ and $\delta^D = .28$.

	u	v	γ	j
D	.0386	.1291	.1514	.0424
FI	.0392	.1309	.1536	.0430

Table 2.4: Full Calibration with Uniform Distribution, $A = 1.5244$ and $\delta^D = .28$.

Tables 5 and 6 show the comparison of the results for the model with a degenerate distribution of the idiosyncratic shocks and the model with a Uniform distribution with range $A = 1.25$ under asymmetric information, respectively for the quasi-calibration and the full-calibration.

	u	v	γ	j
D	.0386	.1291	.1514	.0424
AI	.0618	.1316	.1642	.0672

Table 2.5: Quasi-Calibration with Uniform Distribution, $A = 1.25$ and $\delta^D = .28$.

As in the case of the Pareto, from the quasi-calibrations, it seems that heterogeneity boosts the volatility of unemployment rate. Asymmetric information seems to exacerbate this amplification. On the other hand, once one matches the finding rate elasticity as in the full-calibrations, both unemployment rates and market tightness volatilities do not move significantly.

Finally, tables 7 and 8 report the results from the quasi- and the full-calibration for a Uniform distribution with range parameter $A = .5$ where I fix the finding rate elasticity $\delta^D = .55$. In this case the hiring margin does not achieves a corner solution and I can calibrate both the cases of full and asymmetric information with the same distribution of idiosyncratic shocks.

Once again, the results show that the boost in the volatility that asymmetric information seems to generate with the quasi-calibration, is, in fact, more than offset with the proper full-calibration. Notice that once one matches the finding rate elasticity, then heterogeneity

	u	v	γ	j
D	.0386	.1291	.1514	.0424
AI	.0402	.1340	.1572	.0440

Table 2.6: Full-Calibration with Uniform Distribution, $A = 1.25$ and $\delta^D = .28$.

	u	v	γ	j
D	.0808	.1212	.1586	.0872
FI	.0807	.1210	.1584	.0871
AI	.1343	.1380	.1830	.1419

Table 2.7: Quasi-Calibration with Uniform Distribution and $\delta = .55$.

dampens both the responsiveness of market tightness and unemployment to productivity, as the analytical steady state result suggests. In this case, asymmetric information boosts these volatilities in comparison with the full information case, but not enough to offset the degenerate standard model.

2.5 Conclusions

This paper explores the role of heterogeneity and of asymmetric information on the responsiveness of market tightness and unemployment to productivity shocks.

First, I derive some steady state comparative statics. I show that locally the response to productivity shocks is not affected by heterogeneity or can be dampened by it. Then, I show how I choose the parameter values in order to match some statistics for the U.S. economy in the last 50 years. I consider idiosyncratic shocks distributed according to Uniform and Pareto distributions and show that in both cases there is no significant amplification in the volatility of unemployment and vacancies. The analysis of other distribution functions is left for future research.

The main result of the paper is a negative one: match-specific heterogeneity, either with full or asymmetric information, does not amplify significantly the responsiveness of unemployment rate to productivity shocks in comparison with the standard model. This result is very close to Brugemann and Moscarini (2005) who show that asymmetric information does not typically help in violating the properties of wage determination that limit unemployment and vacancies

	u	v	γ	j
D	.0808	.1212	.1586	.0872
FI	.0767	.1152	.1507	.0829
AI	.0782	.1175	.1537	.0845

Table 2.8: Full-Calibration with Uniform Distribution and $\delta = .55$.

volatility. Further investigations will be needed to understand how my model relates to the general formulation of Brugemann and Moscarini (2005).

Appendix

Proof of Lemma 1. First, rewrite equation (2.9) as the following implicit function

$$h(\hat{\theta}(y), \gamma(y), y) \equiv \frac{\int_0^{\hat{\theta}(y)} [y - \theta/\alpha - b]\, dF(\theta)}{F(\hat{\theta})} - \left[\frac{1-\beta(1-s)}{\beta\delta\mu(\gamma)F(\hat{\theta})} + \frac{1-\delta}{\delta}\right] k\gamma = 0.$$

By total differentiating it, I get

$$h_{\hat{\theta}}(\hat{\theta}(y), \gamma(y), y)\hat{\theta}'(y) + h_{\gamma}(\hat{\theta}(y), \gamma(y), y)\gamma'(y) + h_y(\hat{\theta}(y), \gamma(y), y) = 0.$$

Then,

$$\begin{aligned}
\gamma'(y) &= -\frac{h_y(\hat{\theta}(y), \gamma(y), y)}{h_{\gamma}(\hat{\theta}(y), \gamma(y), y)} - \frac{h_{\hat{\theta}}(\hat{\theta}(y), \gamma(y), y)}{h_{\gamma}(\hat{\theta}(y), \gamma(y), y)}\hat{\theta}'(y) \\
&= -\frac{h_y(\hat{\theta}(y), \gamma(y), y)}{h_{\gamma}(\hat{\theta}(y), \gamma(y), y)}\left[1 + \frac{h_{\hat{\theta}}(\hat{\theta}(y), \gamma(y), y)}{h_y(\hat{\theta}(y), \gamma(y), y)}\hat{\theta}'(y)\right],
\end{aligned}$$

where

$$h_{\hat{\theta}}(\hat{\theta}(y), \gamma(y), y) = \left[y - \frac{\hat{\theta}(y)}{\alpha} - b - E\left[y - \frac{\theta}{\alpha} - b|\theta \leq \hat{\theta}\right] + \frac{1-\beta(1-s)}{\beta\delta\mu(\gamma)F(\hat{\theta})}k\gamma\right]\frac{f(\hat{\theta}(y))}{F(\hat{\theta})}.$$

Using equation (2.9) we can rewrite

$$h_{\hat{\theta}}(\hat{\theta}(y), \gamma(y), y) = y - \frac{\hat{\theta}(y)}{\alpha} - b - \left(\frac{1-\delta}{\delta}\right)k\gamma$$

which is equal to zero from the first equilibrium condition equation (2.8), implying that

$$\gamma'(y) = -\frac{h_y(\hat{\theta}(y), \gamma(y), y)}{h_\gamma(\hat{\theta}(y), \gamma(y), y)}.$$

Next, consider the case of a fixed $\hat{\theta} = 0$, then the equilibrium condition becomes

$$h^D(\gamma(y), y) \equiv y - b_D - \left[\frac{1 - \beta(1-s)}{\beta\delta\mu_D(\gamma)} + \frac{1-\delta}{\delta}\right] k\gamma = 0.$$

and by total differentiating it

$$h_\gamma^D(\gamma(y), y)\gamma'(y) + h_y^D(\gamma(y), y) = 0$$

so that

$$\gamma'_D(y) = -\frac{h_y^D(\gamma(y), y)}{h_\gamma^D(\gamma(y), y)}.$$

Conditions (2.10)-(2.13) give that

$$\begin{aligned} h_y(\hat{\theta}(y), \gamma(y), y) &= h_y^D(\gamma(y), y) = 1 \\ h_\gamma(\hat{\theta}(y), \gamma(y), y) &= h_\gamma^D(\gamma(y), y) = -\frac{[1 - \beta(1-s) - \beta\mu_D(\gamma)](1-\delta)}{1 - \beta(1-s) + \beta\mu_D(\gamma)(1-\delta)}(y - b_D) \end{aligned}$$

so that

$$\frac{\gamma'(y)}{\gamma'_D(y)} = 1$$

completing the proof.

Proof of Lemma 31. Following the first steps of the previous proof we can show that

$$\gamma'(y) = -\frac{h_y(\hat{\theta}(y), \gamma(y), y)}{h_\gamma(\hat{\theta}(y), \gamma(y), y)}.$$

and

$$\gamma'_D(y) = -\frac{h_y^D(\gamma(y), y)}{h_\gamma^D(\gamma(y), y)}.$$

Now, conditions (2.16)-(2.19) give that

$$\begin{aligned}
h_y(\hat{\theta}(y), \gamma(y), y) &= h_y^D(\gamma(y), y) = 1 \\
h_\gamma(\hat{\theta}(y), \gamma(y), y) &= \frac{1-\beta(1-s)}{\beta\delta\mu(\gamma) F(\hat{\theta})}\delta\gamma k - \left(\frac{1-\beta(1-s)}{\delta\beta\mu(\gamma) F(\hat{\theta})} + \frac{1-\delta}{\delta}\right) k \\
h_\gamma^D(\gamma(y), y) &= \frac{1-\beta(1-s)}{\beta\delta_D\eta_D\gamma^{\delta_D}}\delta_D\gamma k_D - \left(\frac{1-\beta(1-s)}{\delta_D\beta\eta_D\gamma^{\delta_D}} + \frac{1-\delta_D}{\delta_D}\right) k_D
\end{aligned}$$

Hence, using (2.19), I obtain

$$\frac{1}{\gamma'(y)} - \frac{1}{\gamma_D'(y)} = \frac{1-\beta(1-s)}{\beta\mu(\gamma) F(\hat{\theta})}\gamma(k_S - k) > 0,$$

completing the proof.

Bibliography

[1] Acemoglu, Daron and Robert Shimer, "Efficient Unemployment Insurance", *Journal of Political Economy*, 1999a, 107, 893-928.

[2] Brugemann, Bjorn and Giuseppe Moscarini, "Asymmetric Information and Employment Fluctuations", 2005, Yale University, mimeo.

[3] Diamond, Peter, "Wage Determination and Efficiency in Search Equilibrium", *Review of Economic Studies*, 1982, 49, 217-227.

[4] Hagedorn, Marcus and Iourii Manovskii, "The Cyclical Behavior of Equilibrium Unemployment and Vacancies Revisited", 2005, mimeo.

[5] Hall, Robert, "Job Loss, Job Finding and Unemployment in the U.S. Economy over the Past Fifty Years", 2005a, mimeo (prepared for the NBER Macro Annual Conference, April 2005).

[6] Hall, Robert, "Employment Efficiency and Sticky Wages: Evidence from Flows in the Labor Market", *Review of Economics and Statistics*, 2005b, 87, 397-407.

[7] Hamilton, James, "A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle", *Econometrica*, 1989, 57, 357-384.

[8] Menzio, Guido, "High-Frequency Wage Rigidity", 2004, Northwestern University, mimeo.

[9] Moen, Espen, "Competitive Search Equilibrium", *Journal of Political Economy*, 1997, 105, 385-411.

[10] Moen, Espen and Asa Rosen, "Incentives in Competitive Search Equilibrium and Wage Rigidity", 2005, Northwestern University, mimeo.

[11] Myerson, Roger B., "Optimal Auction Design", *Mathematics of Operations Research*, 1981, 6, 58-73.

[12] Mortensen, Dale, "The Matching Process as a Noncooperative Bargaining Game", in *The Economics of Information and Uncertainty*, ed. John J. McCall, Chicago: University of Chicago Press for the NBER, 1982a.

[13] Mortensen, Dale, "Property Rights and Efficiency in Mating, Racing, and Related Games", *American Economic Review*, 1982b, 72, 968-979.

[14] Nagypál, Eva, "Worker Reallocation over the Business Cycle: The Importance of Job-to-Job Transitions, Part 1: Evidence," July 2004, Northwestern University, mimeo.

[15] Pissarides, Christopher, "Search Intensity, Job Advertising, and Efficiency", *Journal of Labor Economics*, 1984, 2, 128-143.

[16] Pissarides, Christopher, "Short-run Equilibrium Dynamics of Unemployment, Vacancies and Real Wages", *American Economic Review*, 1985, 75, 676-90.

[17] Shimer, Robert, "Contracts in Frictional Labor Markets", 1996, MIT, mimeo.

[18] Shimer, Robert and Randall Wright, "Competitive Search Equilibrium with Asymmetric Information", 2004, mimeo.

[19] Shimer, Robert "The Cyclical Behavior of Unemployment and Vacancies", *American Economic Review*, 2005, 95, 25-49.

Chapter 3

Capital Deepening and Non-Balanced Economic Growth[1]

3.1 Introduction

Most models of economic growth strive to be consistent with the Kaldor facts, i.e., the relative constancy of the growth rate, the capital-output ratio, the share of capital income in GDP and the real interest rate (see Kaldor, 1963, and also Denison, 1974, Barro and Sala-i-Martin, 2004). Beneath this balanced picture, however, are the patterns that Kongsamut, Rebelo and Xie (2001) refer to as the Kuznets facts, which concern the systematic change in the relative importance of various sectors, in particular, agriculture, manufacturing and services (see Kuznets, 1957, 1973, Chenery, 1960, Kongsamut, Rebelo and Xie, 2001). While the Kaldor facts emphasize the balanced nature of economic growth, the Kuznets facts highlight its non-balanced nature.

The Kuznets facts have motivated a small literature, which typically starts by positing non-homothetic preferences consistent with Engel's law. With these preferences, the marginal rate of substitution in consumption changes as an economy grows, directly leading to a pattern of non-balanced growth (e.g., Murphy, Shleifer and Vishny, 1989, Matsuyama, 1992, 2005,

[1]This Chapter is the product of work joint with Daron Acemoglu. Acemoglu acknowledges financial support from the Russell Sage Foundation and the NSF. An early version of this work was circulated under the title "Non-Balanced Endogenous Growth".

Echevarria, 1997, Laitner, 2000, Kongsamut, Rebelo and Xie, 2001, Caselli and Coleman, 2001, Gollin, Parente and Rogerson, 2002). An alternative perspective, proposed by Baumol (1967), emphasizes the potential non-balanced nature of economic growth resulting from differential productivity growth across sectors, but has received less attention in the literature.[2]

This paper has two aims. First, it shows that there is another, and very natural, reason to expect economic growth to be non-balanced. Differences in factor proportions across sectors (i.e., different shares of capital) combined with capital deepening will lead to non-balanced growth. The reason is simple: an increase in capital-labor ratio will raise output more in the sector with greater capital intensity. More specifically, we prove that "balanced technological progress" (in the sense of equal rates of Hicks-neutral technological progress across sectors),[3] capital deepening and differences in factor proportions necessarily cause non-balanced growth. This result holds irrespective of the exact source of economic growth or the process of accumulation.

The second objective of the paper is to present and analyze a tractable two-sector growth model featuring non-balanced growth, and investigate under what circumstances non-balanced growth can be consistent with aggregate Kaldor facts. We do this by constructing a class of economies with constant elasticity of substitution between two sectors and Cobb-Douglas production functions within each sector. We characterize the equilibria in this class of economies with both exogenous and endogenous technological change. We show that equilibrium takes a simple form, and the limiting (asymptotics) equilibrium features constant *but different* growth rates in each sector, constant interest rate and constant share of capital in national income.

Other properties of the limiting equilibrium of this class of economies depend on whether the products of the two sectors are gross substitutes or complements (meaning whether the elasticity of substitution between these products is greater than or less than one). When they are gross substitutes, the sector that is more "capital intensive" (in the sense of having a greater

[2]Two exceptions are the two recent independent papers by Ngai and Pissarides (2006) and Zuleta and Young (2006). Ngai and Pissarides (2006), for example, construct a model of multi-sector economic growth inspired by Baumol. In Ngai and Pissarides's model, there are exogenous TFP differences across sectors, but all sectors have identical Cobb-Douglas production functions. While both of these papers are potentially consistent with the Kuznets and Kaldor facts, they do not contain the main contribution of our paper, non-balanced growth resulting from factor proportion differences and capital deepening.

[3]Hicks-neutral technological progress is both a natural benchmark and also the type of technological progress that the more microfounded models considered later in the paper will generate.

capital share) dominates the economy. The form of the equilibrium is more interesting when the elasticity of substitution between these products is less than one. In this case, the growth rate of the economy is determined by the more slowly growing (less capital-intensive sector). Despite the change in the terms of trade against the faster growing sector, in equilibrium sufficient amounts of capital and labor (and technological progress when this is endogenous) are deployed in this sector to ensure a faster rate of growth.

One interesting feature is that, especially when the elasticity of substitution is less than one,[4] the resulting pattern of economic growth is consistent with the Kuznets facts, without substantially deviating from the Kaldor facts. In particular, even in the limiting equilibrium both sectors grow with positive (and unequal) rates, and more importantly, we show that convergence to this limiting equilibrium may be slow, and along the transition path, growth is non-balanced, while capital share and interest rate vary only by relatively small amounts. Therefore, the equilibrium with an elasticity of substitution less than one may be able to rationalize both the Kuznets and the Kaldor facts.

Naturally, whether or not this is the empirically relevant explanation for the observed Kuznets and Kaldor facts is not answered by this theoretical result. As a preliminary attempt to investigate this question, we undertake a simple calibration of our economy based on US data and look at the medium-run traditional dynamics. This calibration exercise shows that the economy can grow in a non-balanced manner (i.e., with one sector expanding relative to the other) for an extended period of time, while also remaining approximately consistent with the Kaldor facts.

Finally, we present and analyze a model of "non-balanced endogenous growth," which shows

[4] As we will see below, the elasticity of substitution between products will be less than one if and only if the (short-run) elasticity of substitution between labor and capital is less than one. In view of the time-series and cross-industry evidence, a short-run elasticity of substitution between labor and capital less than one appears reasonable.

For example, Hamermesh (1993), Nadiri (1970) and Nerlove (1967) survey a range of early estimates of the elasticity of substitution, which are generally between 0.3 and 0.7. David and Van de Klundert (1965) similarly estimate this elasticity to be in the neighborhood of 0.3. Using the translog production function, Griffin and Gregory (1976) estimate elasticities of substitution for nine OECD economies between 0.06 and 0.52. Berndt (1976), on the other hand, estimates an elasticity of substitution equal to 1, but does not control for a time trend, creating a strong bias towards 1. Using more recent data, and various different specifications, Krusell, Ohanian, Rios-Rull, and Violante (2000) and Antras (2001) also find estimates of the elasticity significantly less than 1. Estimates implied by the response of investment to the user cost of capital also typically imply an elasticity of substitution between capital and labor significantly less than 1 (see, e.g., Chirinko, 1993, Chirinko, Fazzari and Mayer, 1999, or Mairesse, Hall and Mulkay, 1999).

the robustness of our results to endogenous technological progress, and demonstrates how, in the presence of factor proportion differences, the pattern of technological progress itself will be non-balanced. To the best of our knowledge, despite the large literature on endogenous growth, there are no previous studies that combine endogenous technological progress and non-balanced growth.[5]

The rest of the paper is organized as follows. Section 3.2 shows how the combination of factor proportions differences and capital deepening lead to non-balanced growth using a general two-sector growth model. Section 3.3 constructs a more specific model with a constant elasticity of substitution between two sectors and Cobb-Douglas production functions, but exogenous technological progress. It characterizes the full dynamic equilibrium of this economy, and shows how with an elasticity of substitution less than one, the model may generate an equilibrium path that is consistent both with the Kuznets and the Kaldor facts. Section ?? undertakes a simple calibration of the economy outlined in Section 3.3 to investigate whether, for realistic parameter values, the medium-run transitional dynamics generated by the model are consistent with the Kuznets and Kaldor facts. Section 3.4 introduces endogenous technological progress and shows that the results are robust to differential rates of technological progress across sectors. Section 3.5 concludes, and the Appendix contains proofs that are not presented in the text.

3.2 Capital Deepening and Non-Balanced Growth

We first illustrate how differences in factor proportions across sectors combined with capital deepening lead to non-balanced economic growth. To do this, we use a standard two-sector competitive model with constant returns to scale in both sectors, and two factors of production, capital, K, and labor, L. To highlight that the exact nature of the accumulation process is not essential for the results, in this section we take the sequence (process) of capital and labor supplies, $[K(t), L(t)]_{t=0}^{\infty}$, as given (and assume that labor is supplied inelastically). In addition,

[5] See, among others, Romer (1986, 1990), Lucas (1988), Rebelo (1991), Segerstrom, Anant and Dinopoulos (1990), Grossman and Helpman (1991a,b), Aghion and Howitt (1992), Jones (1995), Young (1993). Aghion and Howitt (1998) and Barro and Sala-i-Martin (2004) provide excellent introductions to endogenous growth theory. See also Acemoglu (2002) on models of directed technical change that feature endogenous, but balanced technological progress in different sectors. Acemoglu (2003) presents a model with non-balanced technological progress between two sectors, but in the limiting equilibrium both sectors grow at the same rate.

we omit explicit time dependence when this will cause no confusion.

Final output, Y, is produced as an aggregate of the output of two sectors, Y_1 and Y_2,

$$Y = F(Y_1, Y_2),$$

and we assume that F exhibits constant returns to scale and is continuously differentiable. Output in both sectors is produced with the production functions

$$Y_1 = A_1 G_1 (K_1, L_1) \tag{3.1}$$

and

$$Y_2 = A_2 G_2 (K_2, L_2). \tag{3.2}$$

G_1 and G_2 also exhibit constant returns to scale and are twice differentiable. A_1 and A_2 denote Hicks-neutral technology terms.[6] We also assume that the functions F, G_1 and G_2 satisfy Inada-type conditions, e.g., $\lim_{Y_1 \to 0} \partial F(Y_1, Y_2) / \partial Y_1 = \infty$ for all $Y_2 > 0$, etc. These assumptions ensure interior solutions and simplify the exposition, though they are not necessary for the results presented in this section.

Market clearing implies

$$\begin{aligned} K_1 + K_2 &= K, \\ L_1 + L_2 &= L, \end{aligned} \tag{3.3}$$

where K and L are the (potentially time-varying) supplies of capital and labor, given by the exogenous sequence $[K(t), L(t)]_{t=0}^{\infty}$, which we take to be continuosly differentiable functions of time. Without loss of any generality, we also ignore capital depreciation.

We normalize the price of the final good to 1 in every period, and denote the prices of Y_1 and Y_2 by p_1 and p_2, and wage and rental rate of capital (interest rate) by w and r. We assume

[6] Hicks-neutral technological progress is convenient to work with, and is also relevant since it is the type of technological progress that the models in Sections 3.3 and 3.4 will generate.

that product and factor markets are competitive, so product prices satisfy

$$\frac{p_1}{p_2} = \frac{\partial F(Y_1, Y_2) / \partial Y_1}{\partial F(Y_1, Y_2) / \partial Y_2}, \tag{3.4}$$

and the wage and the interest rate satisfy[7]

$$\begin{aligned} w &= \frac{\partial A_1 G_1 (K_1, L_1)}{\partial L_1} = \frac{\partial A_2 G_2 (K_2, L_2)}{\partial L_2} \\ r &= \frac{\partial A_1 G_1 (K_1, L_1)}{\partial K_1} = \frac{\partial A_2 G_2 (K_2, L_2)}{\partial K_2}. \end{aligned} \tag{3.5}$$

Definition 34 An equilibrium, given factor supply sequences, $[K(t), L(t)]_{t=0}^{\infty}$, is a sequence of product and factor prices, $[p_1(t), p_2(t), w(t), r(t)]_{t=0}^{\infty}$ and factor allocations, $[K_1(t), K_2(t), L_1(t), L_2(t)]_{t=0}^{\infty}$, such that (3.3), (3.4) and (3.5) are satisfied.

Also define the share of capital in the two sectors

$$s_1 \equiv \frac{rK_1}{p_1 Y_1} \text{ and } s_2 \equiv \frac{rK_2}{p_2 Y_2},$$

the capital to labor ratio in the two sectors,

$$k_1 \equiv \frac{K_1}{L_1} \text{ and } k_2 \equiv \frac{K_2}{L_2}$$

and the "per capita production functions" (without the Hicks-neutral technology term),

$$g_1(k_1) \equiv \frac{G_1(K_1, L_1)}{L_1} \text{ and } g_2(k_2) \equiv \frac{G_2(K_2, L_2)}{L_2}.$$

Definition 35 There is *capital deepening* at time t if $\dot{K}(t)/K(t) > \dot{L}(t)/L(t)$.

There are *factor proportion differences* at time t if $s_1(t) \neq s_2(t)$.

In this definition, $s_1(t) \neq s_2(t)$ refers to the equilibrium factor proportions in the two sectors at time t. It does not necessarily mean that these will not be equal at some future date.

[7]Without the Inada-type assumptions, these would have to be written as

$$w \geq \partial A_1 G_1 (K_1, L_1) / \partial L_1 \text{ and } L_1 \geq 0,$$

with complementary slackness, etc.

The next theorem shows that if there is capital deepening and factor proportion differences, then balanced technological progress is not consistent with balanced growth.

Theorem 36 Suppose that at time t, there are factor proportion differences between the two sectors, i.e., $s_1(t) \neq s_2(t)$, technological progress is balanced, i.e., $\dot{A}_1(t)/A_1(t) = \dot{A}_2(t)/A_2(t)$ and there is capital deepening, i.e., $\dot{K}(t)/K(t) > \dot{L}(t)/L(t)$, then growth is not balanced, that is, $\dot{Y}_1(t)/Y_1(t) \neq \dot{Y}_2(t)/Y_2(t)$.

Proof. Differentiating the production function for the two sectors,

$$\frac{\dot{Y}_1}{Y_1} = \frac{\dot{A}_1}{A_1} + s_1\frac{\dot{K}_1}{K_1} + (1-s_1)\frac{L_1}{L_1}$$

and

$$\frac{\dot{Y}_2}{Y_2} = \frac{\dot{A}_2}{A_2} + s_2\frac{\dot{K}_2}{K_2} + (1-s_2)\frac{L_2}{L_2}.$$

Suppose, to obtain a contradiction, that $\dot{Y}_1/Y_1 = \dot{Y}_2/Y_2$. Since $\dot{A}_1/A_1 = \dot{A}_2/A_2$ and $s_1 \neq s_2$, this implies $\dot{k}_1/k_1 \neq \dot{k}_2/k_2$ (otherwise, $\dot{k}_1/k_1 = \dot{k}_2/k_2 > 0$ because of capital deepening and if, for example, $s_1 < s_2$, then $\dot{Y}_1/Y_1 < \dot{Y}_2/Y_2$).

Since F exhibits constant returns to scale, (3.4) implies

$$\frac{\dot{p}_1}{p_1} = \frac{\dot{p}_2}{p_2} = 0. \tag{3.6}$$

Equation (3.5) yields the following interest rate and wage conditions

$$\begin{aligned} r &= p_1 A_1 g_1'(k_1) \\ &= p_2 A_2 g_2'(k_2), \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} w &= p_1 A_1 \left(g_1(k_1) - g_1'(k_1)k_1\right) \\ &= p_2 A_2 \left(g_2(k_2) - g_2'(k_2)k_2\right). \end{aligned} \tag{3.8}$$

Differentiating the interest rate condition, (3.7), with respect to time and using (3.6), we have

$$\frac{\dot{A}_1}{A_1} + \varepsilon_{g_1'} \frac{\dot{k}_1}{k_1} = \frac{\dot{A}_2}{A_2} + \varepsilon_{g_2'} \frac{\dot{k}_2}{k_2}$$

where

$$\varepsilon_{g_1'} \equiv \frac{g_1''(k_1)\, k_1}{g_1'(k_1)} \text{ and } \varepsilon_{g_2'} \equiv \frac{g_2''(k_2)\, k_2}{g_2'(k_2)}.$$

Since $\dot{A}_1/A_1 = \dot{A}_2/A_2$, we must have

$$\varepsilon_{g_1'} \frac{\dot{k}_1}{k_1} = \varepsilon_{g_2'} \frac{\dot{k}_2}{k_2}. \tag{3.9}$$

Differentiating the wage condition, (3.8), with respect to time, using (3.6) and some algebra gives

$$\frac{\dot{A}_1}{A_1} - \frac{s_1}{1-s_1} \varepsilon_{g_1'} \frac{\dot{k}_1}{k_1} = \frac{\dot{A}_2}{A_2} - \frac{s_2}{1-s_2} \varepsilon_{g_2'} \frac{\dot{k}_2}{k_2}.$$

Since $\dot{A}_1/A_1 = \dot{A}_2/A_2$ and $s_1 \neq s_2$, this equation is inconsistent with (3.9), yielding a contradiction and proving the claim. ■

The intuition for this result can be obtained as follows. Suppose there is capital deepening and that both capital and labor are allocated to the two sectors with constant proportions. Because factor proportions differ between the two sectors, say $s_1 < s_2$, such an allocation will generate faster growth in sector 2 than in sector 1 and induce a non-balanced pattern of growth (since there is capital deepening). In equilibrium, the faster growth in sector 2 will naturally change equilibrium prices, and the decline in the relative price of sector 2 will cause some of the labor and capital to be reallocated to sector 1. However, this reallocation cannot entirely offset the greater increase in the output of sector 2, since, if it did, the relative price change that stimulated the reallocation would not take place. Consequently, equilibrium growth will be non-balanced.

It is also useful to relate the results in Theorem 36 to two different strands of the existing literature. First, the pattern of non-balanced growth in Theorem 36 is related to Rybczynski's theorem in international trade (Rybczynski, 1950), which states that for an open economy within the "cone of diversification" (where factor prices do not depend on factor endowments), changes in factor endowments will be absorbed by changes in output mix. Our result can be viewed as a closed-economy equivalent of Rybczynski's theorem; it shows that changes in factor endowments (capital deepening) will be absorbed by faster growth in one sector than the other, even though relative prices of goods and factors will change in response to the change in factor endowments. Second, if the result in Theorem 36 holds also asymptotically (i.e., as $t \to \infty$), it may appear to contradict the celebrated "Turnpike theorems" in the optimal growth literature (e.g., Radner, 1961, Scheinkman, 1976, Bewley, 1982, McKenzie, 1998). These theorems establish convergence to a steady state in multi-sector models with sufficiently high discount factors. The different result in Theorem 36 stems from the presence of "capital deepening." The non-balanced growth result in this theorem would hold asymptotically only if capital deepening continues asymptotically, which is ruled out in the standard Turnpike theorems. The model in the next section will show more explicitly how asymptotic capital deepening can take place.

Finally, the proof of Theorem 36 makes it clear that the two-sector structure is not necessary for this result. In light of this, we also state a generalization for $N \geq 2$ sectors, where aggregate output is given by the constant returns to scale production function

$$Y = F\left(Y_1, Y_2, ..., Y_N\right).$$

The definitions for s, k and g and the other assumptions above naturally generalize to this setting. We have:

Theorem 37 Suppose that at time t, there are factor proportion differences among the N sectors in the sense that there exists i and $j \leq N$ such that $s_i(t) \neq s_j(t)$, technological progress is balanced, i.e., $\dot{A}_i(t)/A_i(t) = \dot{A}_j(t)/A_j(t)$ for all i and $j \leq N$, and there is capital deepening, i.e., $\dot{K}(t)/K(t) > \dot{L}(t)/L(t)$, then growth is not balanced, that is, there exists i and $j \leq N$ such that $\dot{Y}_i(t)/Y_i(t) \neq \dot{Y}_j(t)/Y_j(t)$.

The proof of this theorem parallels that of Theorem 36 and is omitted.

3.3 Two-Sector Growth with Exogenous Technology

The previous section demonstrated that differences in factor proportions across sectors and capital deepening will lead to non-balanced growth. This result was proved for a given (arbitrary) sequence of capital and labor supplies, $[K(t), L(t)]_{t=0}^{\infty}$, but this level of generality does not allow us to fully characterize the equilibrium path and its limiting properties. We now wish to analyze the equilibrium behavior of such an economy fully, which necessitates at least the sequence of capital stocks to be endogenized, and capital deepening to emerge as an equilibrium phenomenon. We will accomplish this by imposing more structure both in terms of specifying preferences and in terms of the production functions. Capital deepening will result from exogenous technological progress, which will in turn be endogenized in Section 3.4.

3.3.1 Demographics, Preferences and Technology

The economy consists of $L(t)$ workers at time t, supplying their labor inelastically. There is exponential population growth,

$$L(t) = \exp(nt) L(0). \tag{3.10}$$

We assume that all households have constant relative risk aversion (CRRA) preferences over total household consumption (rather than per capita consumption), and all population growth takes place within existing households (thus there is no growth in the number of households).[8] This implies that the economy admits a representative agent with CRRA preferences (see, for example, Caselli and Ventura, 2000):

$$\int_0^{\infty} \frac{C(t)^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt,$$

where $C(t)$ is aggregate consumption at time t, ρ is the rate of time preferences and $\theta \geq 0$ is the inverse of the intertemporal elasticity of substitution (or the coefficient of relative risk aversion). We again drop time arguments to simplify the notation whenever this causes no confusion, and continue to assume that there is no depreciation of capital. The flow budget

[8]The alternative would be to specify population growth taking place at the extensive margin, in which case the discount rate of the representative agent would be $\rho - n$ rather than ρ, without any substantive changes in the analysis.

constraint for the representative consumer is:

$$\dot{K} = rK + wL + \Pi - C, \tag{3.11}$$

where K and L denote the total capital stock and the total labor force in the economy, Π is total net corporate profits received by the consumers, w is the equilibrium wage rate and r is the equilibrium interest rate.

The unique final good is produced by combining the output of two sectors with elasticity of substitution $\varepsilon \in [0, \infty)$:

$$Y = \left[\gamma Y_1^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma) Y_2^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}, \tag{3.12}$$

where γ is a distribution parameter which determines the relative importance of the two goods in the aggregate production.

The resource constraint of the economy, in turn, requires that consumption and investment are less than total output, $Y = rK + wL + \Pi$, thus

$$\dot{K} + C \leq Y. \tag{3.13}$$

The two goods Y_1 and Y_2 are produced competitively using constant elasticity of substitution (CES) production functions with elasticity of substitution between intermediates equal to $\nu > 1$:

$$Y_1 = \left(\int_0^{M_1} y_1(i)^{\frac{\nu-1}{\nu}} di\right)^{\frac{\nu}{\nu-1}} \text{ and } Y_2 = \left(\int_0^{M_2} y_2(i)^{\frac{\nu-1}{\nu}} di\right)^{\frac{\nu}{\nu-1}}, \tag{3.14}$$

where $y_1(i)$'s and $y_2(i)$'s denote the intermediates in the sectors that have different capital/labor ratios, and M_1 and M_2 represent the technology terms. In particular M_1 denotes the number of intermediates in sector 1 and M_2 the number of intermediate goods in sector 2.

Intermediate goods are supplied by monopolists that hold the relevant patent,[9] and are

[9] Monopoly power over intermediates is introduced to create continuity with the next section, where monopoly profits will motivate the creation of new intermediates. Since equilibrium markups will be constant, this monopoly power does not have any substantive effect on the form of equilibrium.

produced with the following Cobb-Douglas technologies

$$y_1(i) = l_1(i)^{\alpha_1} k_1(i)^{1-\alpha_1} \text{ and } y_2(i) = l_2(i)^{\alpha_2} k_2(i)^{1-\alpha_2}, \tag{3.15}$$

where $l_1(i)$ and $k_1(i)$ are labor and capital used in the production of good i of sector 1 and $l_2(i)$ and $k_2(i)$ are labor and capital used in the production of good i of sector 2.[10]

The parameters α_1 and α_2 determine which sector is more "capital intensive".[11] When $\alpha_1 > \alpha_2$, sector 1 is less capital intensive, while the converse applies when $\alpha_1 < \alpha_2$. In the rest of the analysis, we assume that

$$\alpha_1 > \alpha_2, \tag{A1}$$

which only rules out the case where $\alpha_1 = \alpha_2$, since the two sectors are otherwise identical and the labeling the sector with the greater capital share is without loss of any generality.

All factor markets are competitive, and market clearing for the two factors imply

$$\int_0^{M_1} l_1(i)di + \int_0^{M_2} l_2(i)\, di \equiv L_1 + L_2 = L, \tag{3.16}$$

and

$$\int_0^{M_1} k_1(i)di + \int_0^{M_2} k_2(i)\, di \equiv K_1 + K_2 = K, \tag{3.17}$$

where the first set of equalities in these equations define K_1, L_1 and K_2, L_2 as the levels of capital and labor used in the two sectors, and the second set of equalities impose market clearing.

The number of intermediate goods in the two sectors evolve at the exogenous rates

$$\frac{\dot{M}_1}{M_1} = m_1 \text{ and } \frac{\dot{M}_2}{M_2} = m_2, \tag{3.18}$$

and each new intermediate is assigned to a monopolist, so that all intermediate goods are owned and produced by monopolists throughout. Since M_1 and M_2 determine productivity in their respective sectors, we will refer to them as "technology".

[10] Strictly speaking, we should have two indices, $i_1 \in [0, M_1]$ and $i_2 \in [0, M_2]$, but we simplify the notation by using a generic i to denote both indices, and let the context determine which index is being referred to.

[11] We use the term "capital intensive" as corresponding to a greater share of capital in value added, i.e., meaning for example that $s_1 > s_2$ in terms of the notation of the previous section. While this share is constant because of the Cobb-Douglas technologies, the equilibrium ratios of capital to labor in the two sectors depend on prices.

3.3.2 Equilibrium

Recall that w and r denote the wage and the capital rental rate, and p_1 and p_2 denote the prices of the Y_1 and Y_2 goods, with the price of the final good normalized to one. Let $[q_1(i)]_{i=1}^{M_1}$ and $[q_2(i)]_{i=1}^{M_2}$ be the prices for labor-intensive and capital-intensive intermediates.

An equilibrium in this economy is given by paths for factor, intermediate and final goods prices r, w, $[q_1(i)]_{i=1}^{M_1}$, $[q_2(i)]_{i=1}^{M_2}$, p_1 and p_2, employment and capital allocation $[l_1(i)]_{i=1}^{M_1}$, $[l_2(i)]_{i=1}^{M_2}$, $[k_1(i)]_{i=1}^{M_1}$, $[k_2(i)]_{i=1}^{M_2}$ such that firms maximize profits and markets clear, and consumption and savings decisions, C and $\dot{K}$, maximize consumer utility.

It is useful to break the characterization of equilibrium into two pieces: static and dynamic. The static part takes the state variables of the economy, which are the capital stock, the labor supply and the technology, K, L, M_1 and M_2, as given, and determines the allocation of capital and labor across sectors and factor and good prices. The dynamic part of the equilibrium determines the evolution of the endogenous state variable, K (the dynamic behavior of L is given by (3.10) and the one of M_1 and M_2 by (3.18)).

First, our choice of numeraire implies that the price of the final good, P, satisfies:

$$1 \equiv P = \left[\gamma^{\varepsilon} p_1^{1-\varepsilon} + (1-\gamma)^{\varepsilon} p_2^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}.$$

Next, since Y_1 and Y_2 are supplied competitively, their prices are equal to the value of their marginal product, thus

$$p_1 = \gamma \left(\frac{Y_1}{Y}\right)^{-\frac{1}{\varepsilon}} \quad \text{and} \quad p_2 = (1-\gamma)\left(\frac{Y_2}{Y}\right)^{-\frac{1}{\varepsilon}}, \tag{3.19}$$

and the demands for intermediates, $y_1(i)$ and $y_2(i)$, are given by the familiar isoelastic demand curves:

$$\frac{q_1(i)}{p_1} = \left(\frac{y_1(i)}{Y_1}\right)^{-\frac{1}{\nu}} \quad \text{and} \quad \frac{q_2(i)}{p_2} = \left(\frac{y_2(i)}{Y_2}\right)^{-\frac{1}{\nu}}. \tag{3.20}$$

The value of the monopolist for intermediate i in the s-intensive sector is given by

$$V_s(i,t) = \int_t^{\infty} \exp\left[-\int_t^{v} r(z)dz\right] \pi_s(i,v)dv, \tag{3.21}$$

for $s = 1, 2$, where $\pi_s(i,t) = (q_s(i,t) - mc_s(i,t))\, y_s(i,t)$ is the flow profits for firm i at time t, with q_s given by the demand curves in (3.20), and mc_s is the marginal cost of production in this sector. Given the production functions in (3.15), the cost functions take the familiar Cobb-Douglas form, $mc_1(i) = \alpha_1^{-\alpha_1}(1-\alpha_1)^{\alpha_1-1} r^{1-\alpha_1} w^{\alpha_1}$, and $mc_2(i) = \alpha_2^{-\alpha_2}(1-\alpha_2)^{\alpha_2-1} r^{1-\alpha_2} w^{\alpha_2}$. In equilibrium, all firms in the same sector will make the same profits, so we have $V_s(i,t) = V_s(t)$, and we use $V_1(t)$ and $V_2(t)$ to denote the value firms in the two sectors at time t. In Section 3.4, these value functions will be used to determine the equilibrium growth rate of the number of intermediate goods, M_1 and M_2.

Each monopolist chooses its price to maximize (3.21). Since prices at time t only influence revenues and costs at that point, profit-maximizing prices will be given by a constant mark-up over marginal cost:

$$q_1(i) = \left(\frac{\nu}{\nu-1}\right) \alpha_1 (1-\alpha_1)^{\alpha_1-1} r^{1-\alpha_1} w^{\alpha_1}, \tag{3.22}$$

$$q_2(i) = \left(\frac{\nu}{\nu-1}\right) \alpha_2 (1-\alpha_2)^{\alpha_2-1} r^{1-\alpha_2} w^{\alpha_2}. \tag{3.23}$$

Equations (3.22) and (3.23) imply that all intermediates in each sector sell at the same price $q_1 = q_1(i)$ for all $i \le M_1$ and $q_2 = q_2(i)$ for all $i \le M_2$. This combined with (3.20) implies that the demand for, and the production of, the same type of intermediate will be the same. Thus:

$$\begin{aligned} y_1(i) &= l_1(i)^{\alpha_1} k_1(i)^{1-\alpha_1} = y_1 = l_1^{\alpha_1} k_1^{1-\alpha_1} \qquad \forall i \le M_1 \\ y_2(i) &= l_2(i)^{\alpha_2} k_2(i)^{1-\alpha_2} = y_2 = l_2^{\alpha_2} k_2^{1-\alpha_2} \qquad \forall i \le M_2, \end{aligned}$$

where l_1 is the level of employment in all intermediates of sector 1, etc.

Market clearing conditions, (3.16) and (3.17), then imply that $l_1 = L_1/M_1$, $k_1 = K_1/M_1$, $l_2 = L_2/M_2$ and $k_2 = K_2/M_2$, so we have the output of each intermediate in the two sectors as

$$y_1 = \frac{L_1^{\alpha_1} K_1^{1-\alpha_1}}{M_1} \quad \text{and} \quad y_2 = \frac{L_2^{\alpha_2} K_2^{1-\alpha_2}}{M_2}. \tag{3.24}$$

Substituting (3.24) into (3.14), we obtain the total supply of labor- and capital-intensive goods as

$$Y_1 = M_1^{\frac{1}{\nu-1}} L_1^{\alpha_1} K_1^{1-\alpha_1} \text{ and } Y_2 = M_2^{\frac{1}{\nu-1}} L_2^{\alpha_2} K_2^{1-\alpha_2}. \tag{3.25}$$

Comparing these (derived) production functions to (3.1) and (3.2) highlights that in this economy, the production functions G_1 and G_2 from the previous section take Cobb-Douglas forms, with one sector always having a higher share of capital than the other sector, and also that $A_1 = M_1^{\frac{1}{\nu-1}}$ and $A_2 = M_2^{\frac{1}{\nu-1}}$.

In addition, combining (3.25) with (3.12) implies that the aggregate output of the economy is:

$$Y = \left[\gamma\left(M_1^{\frac{1}{\nu-1}} L_1^{\alpha_1} K_1^{1-\alpha_1}\right)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)\left(M_2^{\frac{1}{\nu-1}} L_2^{\alpha_2} K_2^{1-\alpha_2}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}. \tag{3.26}$$

Using (3.24) and (3.25) we can rewrite the prices for the labor- and capital-intensive intermediates as $q_1 = \gamma M_1^{\frac{1}{\nu-1}} \left(\frac{Y_1}{Y}\right)^{-\frac{1}{\varepsilon}}$ and $q_2 = (1-\gamma) M_2^{\frac{1}{\nu-1}} \left(\frac{Y_2}{Y}\right)^{-\frac{1}{\varepsilon}}$, and the flow profits from the sale of the labor- and capital-intensive intermediates are:

$$\pi_1 = \frac{\gamma}{\nu}\left(\frac{Y_1}{Y}\right)^{-\frac{1}{\varepsilon}} \frac{Y_1}{M_1} \text{ and } \pi_2 = \frac{1-\gamma}{\nu}\left(\frac{Y_2}{Y}\right)^{-\frac{1}{\varepsilon}} \frac{Y_2}{M_2}. \tag{3.27}$$

Finally, factor prices and the allocation of capital between the two sectors are determined by:[12]

$$w = \left(\frac{\nu-1}{\nu}\right)\gamma\alpha_1\left(\frac{Y}{Y_1}\right)^{\frac{1}{\varepsilon}} \frac{Y_1}{L_1} \tag{3.28}$$

$$w = \left(\frac{\nu-1}{\nu}\right)(1-\gamma)\,\alpha_2\left(\frac{Y}{Y_2}\right)^{\frac{1}{\varepsilon}} \frac{Y_2}{L_2} \tag{3.29}$$

$$r = \left(\frac{\nu-1}{\nu}\right)\gamma(1-\alpha_1)\left(\frac{Y}{Y_1}\right)^{\frac{1}{\varepsilon}} \frac{Y_1}{K_1} \tag{3.30}$$

$$r = \left(\frac{\nu-1}{\nu}\right)(1-\gamma)(1-\alpha_2)\left(\frac{Y}{Y_2}\right)^{\frac{1}{\varepsilon}} \frac{Y_2}{K_2}. \tag{3.31}$$

These factor prices take the familiar form, equal to the marginal product of a factor from (3.26) with a discount, $(\nu-1)/\nu$, due to the the monopoly markup in the intermediate goods.

[12]To obtain these equations, start with the cost functions above, and derive the demand for factors by using Shepherd's Lemma. For example, for the sector 1, these are $l_1 = \left(\frac{\alpha_1}{1-\alpha_1}\frac{r}{w}\right)^{1-\alpha_1} y_1$ and $k_1 = \left(\frac{\alpha_1}{1-\alpha_1}\frac{r}{w}\right)^{-\alpha_1} y_1$. Combine these two equations to derive the equilibrium relationship between r and w. Then using equation (3.22), eliminate r to obtain a relationship between w and q_1. Now combining with the demand curves in (3.20), the market clearing conditions, (3.16) and (3.17), and using (3.25) yields (3.28). The other equations are obtained similarly.

3.3.3 Static Equilibrium: Comparative Statics

Let us now analyze how changes in the state variables, L, K, M_1 and M_2, impact on equilibrium factor prices and factor shares. As noted in the Introduction, the case with $\varepsilon < 1$ is of greater interest (and empirically more relevant as pointed out in footnote 4), so throughout, we focus on this case (though we give the result for the case in which $\varepsilon > 1$, and we only omit the case with $\varepsilon = 1$, which is standard).

Let us denote the fraction of capital and labor employed in the labor-intensive sector respectively by $\kappa \equiv K_1/K$ and $\lambda \equiv L_1/L$ (clearly $1-\kappa \equiv K_2/K$ and $1-\lambda \equiv L_2/L$). Then Equations (3.28), (3.29), (3.30) and (3.31) imply:

$$\kappa = \left[1 + \left(\frac{1-\alpha_2}{1-\alpha_1}\right)\left(\frac{1-\gamma}{\gamma}\right)\left(\frac{Y_1}{Y_2}\right)^{\frac{1-\varepsilon}{\varepsilon}}\right]^{-1} \tag{3.32}$$

and

$$\lambda = \left[\left(\frac{1-\alpha_1}{1-\alpha_2}\right)\left(\frac{\alpha_2}{\alpha_1}\right)\left(\frac{1-\kappa}{\kappa}\right) + 1\right]^{-1}. \tag{3.33}$$

Equation (3.33) makes it clear that the share of labor in the sector 1, λ, is monotonically increasing in the share of capital in the sector 1, κ. We next determine how these two shares change with capital accumulation and technological change.

Proposition 38 In equilibrium,

1.

$$\frac{d\ln\kappa}{d\ln K} = -\frac{d\ln\kappa}{d\ln L} = \frac{(1-\varepsilon)(\alpha_1-\alpha_2)(1-\kappa)}{1+(1-\varepsilon)(\alpha_1-\alpha_2)(\kappa-\lambda)} > 0 \Leftrightarrow (\alpha_1-\alpha_2)(1-\varepsilon) > 0. \tag{3.34}$$

2.

$$\frac{d\ln\kappa}{d\ln M_2} = -\frac{d\ln\kappa}{d\ln M_1} = \frac{(1-\varepsilon)(1-\kappa)/(\nu-1)}{1+(1-\varepsilon)(\alpha_1-\alpha_2)(\kappa-\lambda)} > 0 \Leftrightarrow \varepsilon < 1. \tag{3.35}$$

The proof of this proposition is straightforward and is omitted.

Equation (3.34), part 1 of the proposition, states that when the elasticity of substitution between sectors, ε, is less than 1, the fraction of capital allocated to the capital-intensive sector declines in the stock of capital (and conversely, when $\varepsilon > 1$, this fraction is increasing in the stock of capital). To obtain the intuition for this comparative static, which is useful for understanding many of the results that will follow, note that if K increases and κ remains constant, then the capital-intensive sector grows by more than the other sector. Equilibrium prices given in (3.19) imply that when $\varepsilon < 1$, the relative price of the capital-intensive sector falls more than proportionately, inducing a greater fraction of capital to be allocated to the sector that is less intensive in capital. The intuition for the converse result when $\varepsilon > 1$ is straightforward.

Moreover, equation (3.35) implies that when the elasticity of substitution, ε, is less than one, an improvement in the technology of a sector causes the share of capital going to that sector to fall. The intuition is again the same: increased production in a sector causes a more than proportional decline in its relative price, inducing a reallocation of capital away from it towards the other sector (again the converse results and intuition apply when $\varepsilon > 1$).

Proposition 38 gives only the comparative statics for κ. Equation (3.33) implies that the same comparative statics applies to λ.

Next, combining (3.28) and (3.30), we also obtain relative factor prices as

$$\frac{w}{r} = \frac{\alpha_1}{1-\alpha_1}\left(\frac{\kappa K}{\lambda L}\right), \tag{3.36}$$

and the capital share in the economy as[13]

$$s_K \equiv 1 - \frac{wL}{Y} = 1 - \left(\frac{v-1}{v}\right)\gamma\alpha_1\left(\frac{Y_1}{Y}\right)^{\frac{\varepsilon-1}{\varepsilon}}\frac{1}{\lambda}. \tag{3.37}$$

Proposition 39 In equilibrium,

1.

$$\frac{d\ln(w/r)}{d\ln K} = -\frac{d\ln(w/r)}{d\ln L} = \frac{1}{1+(1-\varepsilon)(\alpha_1-\alpha_2)(\kappa-\lambda)} > 0.$$

[13]Notice that we define the capital share as one minus the labor share, which makes sure that monopoly profits are included in the capital share. Also s_K refers to the share of capital in national income, and is thus different from the capital shares in the previous section, which were sector specific. Sector-specific capital shares are constant here because of the Cobb-Douglas production functions.

2.

$$\frac{d\ln(w/r)}{d\ln M_2} = -\frac{d\ln(w/r)}{d\ln M_1} = -\frac{(1-\varepsilon)(\kappa-\lambda)/(\nu-1)}{1+(1-\varepsilon)(\alpha_1-\alpha_2)(\kappa-\lambda)} < 0 \Leftrightarrow (\alpha_1-\alpha_2)(1-\varepsilon) > 0.$$

3.

$$\frac{d\ln s_K}{d\ln K} < 0 \Leftrightarrow \varepsilon < 1.$$

4.

$$\frac{d\ln s_K}{d\ln M_2} = -\frac{d\ln s_K}{d\ln M_1} < 0 \Leftrightarrow (\alpha_1-\alpha_2)(1-\varepsilon) > 0.$$

The proof of this proposition is provided in the Appendix.

The most important result in this proposition is part 3, which links the equilibrium relationship between the capital share in national income and the capital stock to the elasticity of substitution. Since a negative relationship between the share of capital in national income and the capital stock is equivalent to capital and labor being gross complements in the aggregate, this result also implies that, as claimed in footnote 4, the elasticity of substitution between capital and labor is less than one if and only if ε is less than one. Intuitively, as in Theorem 36, an increase in the capital stock of the economy causes the output of the more capital-intensive sector to increase relative to the output in the less capital-intensive sector (despite the fact that the share of capital allocated to the less-capital intensive sector increases as shown in equation (3.34)). This then increases the production of the more capital-intensive sector, and when $\varepsilon < 1$, it reduces the relative reward to capital (and the share of capital in national income). The converse result applies when $\varepsilon > 1$.

Moreover, when $\varepsilon < 1$, part 4 implies that an increase in M_1 is "capital biased" and an increase in M_2 is "labor biased". The intuition for why an increase in the productivity of the sector that is intensive in capital is biased toward labor (and vice versa) is once again similar: when the elasticity of substitution between the two sectors, ε, is less than one, an increase in the output of a sector (this time driven by a change in technology) decreases its price more than proportionately, thus reducing the relative compensation of the factor used more intensively in that sector. When $\varepsilon > 1$, we have the converse pattern, and M_2 is "capital biased," while an increase in M_1 is "labor biased"

3.3.4 Dynamic Equilibrium

We now turn to the characterization of the dynamic equilibrium path of this economy. We start with the Euler equation for consumers, which takes the familiar form[14]

$$\frac{\dot{C}}{C} = \frac{1}{\theta}(r - \rho). \tag{3.38}$$

To write the transversality condition, note that the financial wealth of the representative consumer comes from payments to capital and profits, and is given by $W(t) = K(t) + M_1(t) V_1(t) + M_2(t) V_2(t)$, where recall that $V_1(t)$ is the present discounted value of the profits of a firm in the sector 1 at time t and there are $M_1(t)$ such firms, and similarly for $V_2(t)$ and $M_2(t)$. The transversality condition is then:

$$\lim_{t \to \infty} W(t) \exp\left(-\int_0^t r(\tau)\, d\tau\right) = 0, \tag{3.39}$$

which together with the resource constraint given in (3.13) determines the dynamic behavior of consumption and capital stock, C and K. Equations (3.10) and (3.18) give the behavior of L, M_1 and M_2.

We can therefore summarize a dynamic equilibrium as paths of interest rates, labor and capital allocation decisions, r, λ and κ, satisfying (3.28), (3.29), (3.30) and (3.31), and of consumption, capital stock, technology, values of innovation satisfying (3.13), (3.21), (3.38), and (3.39).

Let us also introduce the following notation for growth rates of the key objects in this economy:

$$\frac{\dot{L}_1}{L_1} \equiv n_1, \; \frac{\dot{L}_2}{L_2} \equiv n_2, \; \frac{\dot{L}}{L} \equiv n$$
$$\frac{\dot{K}_1}{K_1} \equiv z_1, \; \frac{\dot{K}_2}{K_2} \equiv z_2, \; \frac{\dot{K}}{K} \equiv z$$
$$\frac{\dot{Y}_1}{Y_1} \equiv g_1, \; \frac{\dot{Y}_2}{Y_2} \equiv g_2, \; \frac{\dot{Y}}{Y} \equiv g,$$

[14] Throughout, we assume that in equilibrium consumption, capital and factor prices are differentiable functions of time, and work with time derivatives, e.g., $\dot{C}$, etc.

so that n_s and z_s denote the growth rate of labor and capital stock, m_s denotes the growth rate of technology, and g_s denotes the growth rate of output in sector s. Moreover, whenever they exist, we denote the corresponding asymptotic growth rates by asterisks, i.e.,

$$n_s^* = \lim_{t\to\infty} n_s \ , \ z_s^* = \lim_{t\to\infty} z_s \text{ and } g_s^* = \lim_{t\to\infty} g_s.$$

Similarly denote the asymptotic capital and labor allocation decisions by asterisks

$$\kappa^* = \lim_{t\to\infty} \kappa \text{ and } \lambda^* = \lim_{t\to\infty} \lambda.$$

We now state and prove two lemmas that will be useful both in this and the next section.

Lemma 40 If $\varepsilon < 1$, then $n_1 \gtreqless n_2 \Leftrightarrow z_1 \gtreqless z_2 \Leftrightarrow g_1 \lesseqgtr g_2$. If $\varepsilon > 1$, then $n_1 \gtreqless n_2 \Leftrightarrow z_1 \gtreqless z_2 \Leftrightarrow g_1 \gtreqless g_2$.

Proof. Differentiating (3.28) and (3.29) with respect to time yields

$$\frac{\dot{w}}{w} + n_1 = \frac{1}{\varepsilon} g + \frac{\varepsilon - 1}{\varepsilon} g_1 \text{ and } \frac{\dot{w}}{w} + n_2 = \frac{1}{\varepsilon} g + \frac{\varepsilon - 1}{\varepsilon} g_2. \tag{3.40}$$

Subtracting the second from the first gives $n_1 - n_2 = (\varepsilon - 1)(g_1 - g_2)/\varepsilon$, and immediately implies the first part of the desired result. Similarly differentiating (3.30) and (3.31) yields

$$\frac{\dot{r}}{r} + z_1 = \frac{1}{\varepsilon} g + \frac{\varepsilon - 1}{\varepsilon} g_1 \text{ and } \frac{\dot{r}}{r} + z_2 = \frac{1}{\varepsilon} g + \frac{\varepsilon - 1}{\varepsilon} g_2. \tag{3.41}$$

Again, subtracting the second from the first gives the second part of the result. ■

This lemma establishes the straightforward, but at first counter-intuitive, result that when the elasticity of substitution between the two sectors is less than one, then the equilibrium growth rate of the capital stock and labor force in the sector that is growing faster must be less than in the other sector. When the elasticity of substitution is greater than one, the converse result obtains. To see the intuition, note that terms of trade (relative prices) shift in favor of the more slowly growing sector. When the elasticity of substitution is less than one, this change in relative prices is more than proportional with the change in quantities, and this encourages more of the factors to be allocated towards the more slowly growing sector.

Lemma 41 Suppose the asymptotic growth rates g_1^* and g_2^* exist. If $\varepsilon < 1$, then $g^* = \min\{g_1^*, g_2^*\}$. If $\varepsilon > 1$, then $g^* = \max\{g_1^*, g_2^*\}$.

Proof. Differentiating the production function for the final good (3.26) we obtain:

$$g = \frac{\left[\gamma Y_1^{\frac{\varepsilon-1}{\varepsilon}} g_1 + (1-\gamma) Y_2^{\frac{\varepsilon-1}{\varepsilon}} g_2\right]}{\left[\gamma Y_1^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma) Y_2^{\frac{\varepsilon-1}{\varepsilon}}\right]} \tag{3.42}$$

which, combined with $\varepsilon < 1$ implies that as $t \to \infty$, $g^* = \min\{g_1^*, g_2^*\}$. Similarly, combined with $\varepsilon > 1$, implies that as $t \to \infty$, $g^* = \max\{g_1^*, g_2^*\}$. ■

Consequently, when the elasticity of substitution is less than 1, the asymptotic growth rate of aggregate output will be determined by the sector that is growing more slowly, and the converse applies when $\varepsilon > 1$.

3.3.5 Constant Growth Paths

We first focus on *asymptotic equilibrium paths*, which are equilibrium paths that the economy tends to as $t \to \infty$. A *constant growth path* (CGP) is defined as an equilibrium path where the asymptotic growth rate of consumption exists and is constant, i.e.,

$$\lim_{t\to\infty} \frac{\dot{C}}{C} = g_C^*.$$

From the Euler equation (3.38), this also implies that the interest rate must be asymptotically constant, i.e., $\lim_{t\to\infty} \dot{r} = 0$.

To establish the existence of a CGP, we impose the following parameter restriction:

$$\max\left\{\frac{m_1}{\alpha_1}, \frac{m_2}{\alpha_2}\right\} \le (\nu - 1)\left(\frac{\rho}{1-\theta} - n\right). \tag{A2}$$

This assumption ensures that the transversality condition (3.39) holds. Terms of the form m_1/α_1 or m_2/α_2 appear naturally in equilibrium, since they capture the "augmented" rate of technological progress. In particular, recall that associated with the technological progress, there will also be equilibrium capital deepening in each sector. The overall effect on labor productivity (and output growth) will depend on the rate of technological progress augmented with the rate of capital deepening. The terms m_1/α_1 or m_2/α_2 capture this, since a lower α_1 or α_2 corresponds to a greater share of capital in the relevant sector, and thus a higher rate of augmented technological progress for a given rate of Hicks-neutral technological change. In this light, Assumption (A2) can be understood as implying that the augmented rate of technological progress should be low enough to satisfy the transversality condition (3.39).

The next theorem is the main result of this part of the paper and characterizes the relatively simple form of the constant growth path (CGP) in the presence of non-balanced growth. Although we characterize a CGP, in the sense that aggregate output grows at a constant rate, it is noteworthy that growth is non-balanced since output, capital and employment in the two sectors grow at *different rates.*

Theorem 42 Suppose Assumptions A1 and A2 hold. Define s and $\sim s$ such that $\frac{m_s}{\alpha_s} = \min\left\{\frac{m_1}{\alpha_1}, \frac{m_2}{\alpha_2}\right\}$ and $\frac{m_{\sim s}}{\alpha_{\sim s}} = \max\left\{\frac{m_1}{\alpha_1}, \frac{m_2}{\alpha_2}\right\}$ when $\varepsilon < 1$, and $\frac{m_s}{\alpha_s} = \max\left\{\frac{m_1}{\alpha_1}, \frac{m_2}{\alpha_2}\right\}$ and $\frac{m_{\sim s}}{\alpha_{\sim s}} = \min\left\{\frac{m_1}{\alpha_1}, \frac{m_2}{\alpha_2}\right\}$ when $\varepsilon > 1$. Then there exists a unique CGP such that

$$g^* = g_C^* = g_s^* = z_s^* = n + \frac{1}{\alpha_s(\nu-1)}m_s \tag{3.43}$$

$$z_{\sim s}^* = n - \frac{(1-\varepsilon)\, m_{\sim s}}{(\nu-1)} + \frac{[1+\alpha_{\sim s}(1-\varepsilon)]\, m_s}{\alpha_s(\nu-1)} < g^* \tag{3.44}$$

$$g_{\sim s}^* = n + \frac{\varepsilon m_{\sim s}}{(\nu-1)} + \frac{[1-\alpha_{\sim s}(1-\varepsilon\alpha_{\sim s}(1-\varepsilon))]\, m_s}{\alpha_s(\nu-1)[1-\alpha_{\sim s}(1-\varepsilon)]} > g^* \tag{3.45}$$

$$n_s^* = n \text{ and } n_{\sim s}^* = n - \frac{(1-\varepsilon)(\alpha_s m_{\sim s} - \alpha_{\sim s} m_s)}{\alpha_s(\nu-1)}. \tag{3.46}$$

Proof. We prove this proposition in three steps.

Step 1: Suppose that $\varepsilon < 1$. Provided that $g_{\sim s}^* \geq g_s^* > 0$, then there exists a unique CGP defined by equations (3.43), (3.44), (3.45) and (3.46) satisfying $g_{\sim s}^* > g_s^* > 0$, where $\frac{m_s}{\alpha_s} = \min\left\{\frac{m_1}{\alpha_1}, \frac{m_2}{\alpha_2}\right\}$ and $\frac{m_{\sim s}}{\alpha_{\sim s}} = \max\left\{\frac{m_1}{\alpha_1}, \frac{m_2}{\alpha_2}\right\}$.

Step 2: Suppose that $\varepsilon > 1$. Provided that $g^*_{\sim s} \leq g^*_s < 0$, then there exists a unique CGP defined by equations (3.43), (3.44), (3.45) and (3.46) satisfying $g^*_{\sim s} < g^*_s < 0$, where $\frac{m_s}{\alpha_s} = \max\left\{\frac{m_1}{\alpha_1}, \frac{m_2}{\alpha_2}\right\}$ and $\frac{m_{\sim s}}{\alpha_{\sim s}} = \min\left\{\frac{m_1}{\alpha_1}, \frac{m_2}{\alpha_2}\right\}$.

Step 3: Any CGP must satisfy $g^*_{\sim s} \geq g^*_s > 0$, when $\varepsilon < 1$ and $g^*_s \geq g^*_{\sim s} > 0$, when $\varepsilon > 1$ with $\frac{m_s}{\alpha_s}$ defined as in the theorem.

The third step then implies that the growth rates characterized in steps 1 and 2 are indeed equilibria and there cannot be any other CGP equilibria, completing the proof.

Proof of Step 1. Let us assume without any loss of generality that $s = 1$, i.e., $\frac{m_s}{\alpha_s} = \frac{m_1}{\alpha_1}$. Given $g^*_2 \geq g^*_1 > 0$, equations (3.32) and (3.33) imply condition that $\lambda^* = \kappa^* = 1$ (in the case where $s = 2$, we would have $\lambda^* = \kappa^* = 0$) and Lemma 41 implies that we must also have $g^* = g^*_1$. This condition together with our system of equations, (3.25) , (3.40) and (3.41), solves uniquely for n^*_1, n^*_2, z^*_1, z^*_2, g^*_1 and g^*_2 as given in equations (3.43), (3.44), (3.45) and (3.46). Note that this solution is consistent with $g^*_2 > g^*_1 > 0$, since Assumptions A1 and A2 imply that $g^*_2 > g^*_1$ and $g^*_1 > 0$. Finally, $C \leq Y$, (3.11) and (3.39) imply that the consumption growth rate, g^*_C, is equal to the growth rate of output, g^* (suppose not, then since $C/Y \to 0$ as $t \to \infty$, the budget constraint (3.11) implies that asymptotically $K(t) = Y(t)$, and integrating the budget constraint gives $K(t) \to \int_0^t Y(s)\,ds$, implying that the capital stock grows more than exponentially, since Y is growing exponentially; violating the transversality condition (3.39)).

Finally, we can verify that an equilibrium with z^*_1, z^*_2, m^*_1, m^*_2, g^*_1 and g^*_2 satisfies the transversality condition (3.39). Note that the transversality condition (3.39) will be satisfied if

$$\lim_{t\to\infty} \frac{\dot{W}(t)}{W(t)} < r^*, \tag{3.47}$$

where r^* is the constant asymptotic interest rate. Since from the Euler equation (3.38) $r^* = \theta g^* + \rho$, (3.47) will be satisfied when $g^*(1-\theta) < \rho$. Assumption A2 ensures that this is the case with $g^* = n + \frac{1}{\alpha_1(\nu-1)}m_1$. A similar argument applies for the case where $\frac{m_s}{\alpha_s} = \frac{m_2}{\alpha_2}$.

Proof of Step 2. The proof of this step is similar to the previous one, and is thus omitted.

Proof of Step 3. We now prove that along all CGPs $g^*_{\sim s} \geq g^*_s > 0$, when $\varepsilon < 1$ and $g^*_s \geq g^*_{\sim s} > 0$, when $\varepsilon > 1$ with $\frac{m_s}{\alpha_s}$ defined as in the theorem. Without any loss of generality, suppose that $\frac{m_s}{\alpha_s} = \frac{m_1}{\alpha_1}$. We now separately derive a contradiction for two configurations, (1)

$g_1^* \geq g_2^*$, or (2) $g_2^* \geq g_1^*$ but $g_1^* \leq 0$.

1. Suppose $g_1^* \geq g_2^*$ and $\varepsilon < 1$. Then, following the same reasoning as in Step 1, the unique solution to the equilibrium conditions (3.25), (3.40) and (3.41), when $\varepsilon < 1$ is:

$$g^* = g_C^* = g_2^* = z_2^* = n + \frac{1}{\alpha_2 (\nu - 1)} m_2 \tag{3.48}$$

$$z_1^* = n - \frac{(1-\varepsilon) m_1}{(\nu - 1)} + \frac{[1 + \alpha_1 (1-\varepsilon)] m_2}{\alpha_2 (\nu - 1)} \tag{3.49}$$

$$g_1^* = n + \frac{\varepsilon m_1}{(\nu - 1)} + \frac{[1 - \alpha_1 (1 - \varepsilon \alpha_1 (1-\varepsilon))] m_2}{\alpha_2 (\nu - 1) [1 - \alpha_1 (1-\varepsilon)]} \tag{3.50}$$

$$n_1^* = n + \frac{(1-\varepsilon)[\alpha_1 m_2 - \alpha_2 m_1]}{\alpha_2 (\nu - 1)} \tag{3.51}$$

   But combining these equations implies that $g_1^* < g_2^*$, which contradicts the hypothesis $g_1^* \geq g_2^* > 0$. The argument for $\varepsilon > 1$ is analogous.

2. Suppose $g_2^* \geq g_1^*$ and $\varepsilon < 1$, then the same steps as above imply that there is a unique solution to equilibrium conditions (3.25), (3.40) and (3.41), which are given by equations (3.43), (3.44), (3.45) and (3.46). But now (3.43) directly contradicts $g_1^* \leq 0$. Finally suppose $g_2^* \geq g_1^*$ and $\varepsilon > 1$, then the unique solution is given by (3.48), (3.49), (3.50) and (3.51), then (3.50) directly contradicts $g_1^* \leq 0$, and this completes the proof.

■

There are a number of important implications of this theorem. First, as long as $m_1/\alpha_1 \neq m_2/\alpha_2$, growth is non-balanced. The intuition for this result is the same as Theorem 1 in the previous section. Differential capital intensities in the two sectors combined with capital deepening in the economy (which itself results from technological progress) ensure faster growth in the more capital-intensive sector. Intuitively, if capital were allocated proportionately to the two sectors, the more capital-intensive sector would grow faster. Because of the changes in prices, capital and labor are reallocated in favor of the less capital-intensive sector, but not enough to fully offset the faster growth in the more capital-intensive sector. This result also highlights that the assumption of balanced technological progress in the previous section (in this context, $m_1 = m_2$) was not necessary for the theorem, but we simply needed to rule out the precise relative rate of technological progress between the two sectors that would ensure balanced growth (in this context, $m_1/\alpha_1 = m_2/\alpha_2$).

Second, while the CGP growth rates looks somewhat complicated because they are written in the general case, they are relatively simple once we restrict attention to parts of the parameter space. For example, when $m_1/\alpha_1 < m_2/\alpha_2$, the capital-intensive sector (sector 2) always grows faster than the labor-intensive one, i.e., $g_1^* < g_2^*$. In addition if $\varepsilon < 1$, the more slowly-growing labor-intensive sector dominates the asymptotic behavior of the economy, and the CGP growth rates are

$$\begin{aligned} g^* &= g_C^* = g_1^* = z_1^* = n + \frac{1}{\alpha_1 (\nu - 1)} m_1, \\ g_2^* &= n + \frac{\varepsilon m_2}{(\nu - 1)} + \frac{[1 - \alpha_2 (1 - \varepsilon\alpha_2 (1 - \varepsilon))] m_1}{\alpha_1 (\nu - 1) [1 - \alpha_2 (1 - \varepsilon)]} > g^*. \end{aligned}$$

In contrast, when $\varepsilon > 1$, the more rapidly-growing capital-intensive sector dominates the asymptotic behavior and

$$\begin{aligned} g^* &= g_C^* = g_2^* = z_2^* = n + \frac{1}{\alpha_2 (\nu - 1)} m_2, \\ g_1^* &= n + \frac{\varepsilon m_1}{(\nu - 1)} + \frac{[1 - \alpha_1 (1 - \varepsilon\alpha_1 (1 - \varepsilon))] m_2}{\alpha_2 (\nu - 1) [1 - \alpha_1 (1 - \varepsilon)]} < g^*. \end{aligned}$$

Third, as the proof makes it clear, in the limit, the share of capital and labor allocated to one of the sector tends to one (e.g., when sector 1 is the asymptotically dominant sector, $\lambda^* = \kappa^* = 1$). Nevertheless, at all points in time both sectors produce positive amounts, so this limit point is never reached. In fact, at all times both sectors grow at rates greater than the rate of population growth in the economy. Moreover, when $\varepsilon < 1$, the sector that is shrinking grows faster than the rest of the economy at all point in time, even asymptotically. Therefore, the rate at which capital and labor are allocated away from this sector is determined in equilibrium to be *exactly* such that this sector still grows faster than the rest of the economy. This is the sense in which non-balanced growth is not a trivial outcome in this economy (with one of the sectors shutting down), but results from the positive but differential growth of the two sectors.

Finally, it can be verified that the share of capital in national income and the interest rate are constant in the CGP. For example, when $m_1/\alpha_1 < m_2/\alpha_2$, $rK/Y = 1 - \alpha_1 (\nu - 1)/\nu$ and when $m_1/\alpha_1 > m_2/\alpha_2$, $rK/Y = 1 - \alpha_2 (\nu - 1)/\nu$. The interest rate, on the other hand, is equal to $r = (1 - \alpha_1)(\nu - 1)\gamma^{\frac{\varepsilon}{\varepsilon - 1}} (\chi^*)^{-\alpha_1}/\nu$ in the first case and $r = (1 - \alpha_2)(\nu - 1)\gamma^{\frac{\varepsilon}{\varepsilon - 1}} (\chi^*)^{-\alpha_2}/\nu$ in the second case, where χ^* is defined below. These results are the basis of the claim in the Introduction that this equilibrium may account for non-balanced growth at the sectoral level, without substantially deviating from the Kaldor facts. In particular, the constant growth path equilibrium matches both the Kaldor facts and generates unequal growth between the two sectors. However, in this constant growth path equilibrium, one of the sectors has already become very small relative to the other. Therefore, this theorem does not answer whether along the equilibrium path (but away from the asymptotic equilibrium), we can have a situation in which both sectors have non-trivial employment levels and the equilibrium capital share in national income and the interest rate are approximately constant. This question and the stability of the constant growth path equilibrium are investigated in the next section.

3.3.6 Dynamics and Stability

The previous section characterized the asymptotic equilibrium, and established the existence of a unique constant growth path. This growth path exhibits non-balanced growth, though asymptotically the economy grows at a constant rate and the share of capital in national income is constant. We now study the equilibrium behavior of this economy away from this asymptotic equilibrium.

The equilibrium behavior away from the asymptotic equilibrium path can be represented by a dynamical system characterizing the behavior of a control variable C and four state variables K, L, M_1 and M_2. The dynamics of aggregate consumption, C, and aggregate capital stock, K, are given by the Euler equation (3.38) and the resource constraint (3.13). Furthemore, the dynamic behavior of L is given by (3.10) and the one of M_1 and M_2 by (3.18).

As noted above, when $\varepsilon > 1$, the sector which grows faster dominates the economy, while when $\varepsilon < 1$, conversely, the slower sector dominates. We want to show that, in both cases, the unique CGP of the previous section is locally stable. Since the case with $\varepsilon < 1$ is more interesting, we emphasize this case in our analysis. Moreover, without loss of generality, we restrict the discussion to the case in which asymptotically the economy is dominated by the labor-intensive sector, sector 1, so that

$$g^* = g_1^* = z_1^* = n + \frac{1}{\alpha_1(\nu-1)} m_1.$$

This means that when we assume $\varepsilon < 1$, the relevant part of the parameter space is where $m_1/\alpha_1 < m_2/\alpha_2$, and, when $\varepsilon > 1$, we must have $m_1/\alpha_1 > m_2/\alpha_2$ (for the rest of the parameter space, it would be sector 2 that dominates the asymptotic behavior).

The equilibrium behavior of this economy can be represented by a system of autonomous non-linear differential equations in three variables,

$$c \equiv \frac{C}{LM_1^{\frac{1}{\alpha_1(\nu-1)}}}, \ \chi \equiv \frac{K}{LM_1^{\frac{1}{\alpha_1(\nu-1)}}} \text{ and } \kappa.$$

Here c is the level of consumption normalized by population and technology (of the sector which will dominate the asymptotic behavior), and is the only control variable; χ is the capital stock normalized by the same denominator, and κ determines the allocation of capital between the two sectors. These two are state variables with given initial conditions χ_0 and κ_0.[15]

The dynamic equilibrium conditions than translate into the following equations:

$$\begin{aligned}
\frac{\dot{c}}{c} &= \frac{1}{\theta}\left[(1-\alpha_1)\left(\frac{\nu-1}{\nu}\right)\gamma\left(\frac{Y}{Y_1}\right)^{\frac{1}{\varepsilon}}\lambda^{\alpha_1}(\kappa\chi)^{-\alpha_1} - \rho\right] - n - \frac{1}{\alpha_1(\nu-1)}m_1, \qquad (3.52)\\
\frac{\dot{\chi}}{\chi} &= \lambda^{\alpha_1}\kappa^{1-\alpha_1}\chi^{-\alpha_1}\left(\frac{Y}{Y_1}\right) - \chi^{-1}c - n - \frac{1}{\alpha_1(\nu-1)}m_1,\\
\frac{\dot{\kappa}}{\kappa} &= \frac{(1-\kappa)\left[(\alpha_1-\alpha_2)\frac{\dot{\chi}}{\chi} + \left(\frac{1}{\nu-1}\right)\left(m_2 - \frac{\alpha_2}{\alpha_1}m_1\right)\right]}{\left(\frac{\varepsilon}{1-\varepsilon}\right) + \alpha_2 + (1-\alpha_1)(1-\kappa) + (1-\alpha_2)\kappa + (\alpha_1-\alpha_2)(1-\lambda)},
\end{aligned}$$

where

$$\left(\frac{Y}{Y_1}\right) = \gamma^{\frac{\varepsilon}{\varepsilon-1}}\left[1 + \left(\frac{1-\alpha_1}{1-\alpha_2}\right)\left(\frac{1}{\kappa}-1\right)\right]^{\frac{\varepsilon}{\varepsilon-1}},$$

and

$$\lambda = \left[\left(\frac{1-\alpha_1}{\alpha_1}\right)\left(\frac{\alpha_2}{1-\alpha_2}\right)\left(\frac{1-\kappa}{\kappa}\right) + 1\right]^{-1}.$$

[15] χ_0 is given by definition, and κ_0 is uniquely pinned down by the static equilibrium allocation of capital at time $t = 0$, given by (3.32).

Clearly, the constant growth path equilibrium characterized above corresponds to a steady-state equilibrium in terms of these three variables, denoted by c^*, χ^* and κ^* (i.e., in the CGP equilibrium, c , χ and κ will be constant). These steady-state values are given by $\kappa^* = 1$,

$$\chi^* = \left[\frac{\nu \left(\theta \left[n + \frac{1}{\alpha_1 (\nu - 1)} m_1 \right] + \rho \right)}{\gamma^{\frac{\varepsilon}{\varepsilon - 1}} (1 - \alpha_1)(\nu - 1)} \right]^{-\frac{1}{\alpha_1}},$$

and

$$c^* = \gamma^{\frac{\varepsilon}{\varepsilon-1}} {\chi^*}^{1-\eta} - \chi^* \left(n + \frac{1}{\alpha_1 (\nu - 1)} m_1 \right).$$

Since there are two state and one control variable, local (saddle-path) stability requires the existence of a (unique) two-dimensional manifold of solutions in the neighborhood of the steady state that converge to c^*, χ^* and κ^*. The next theorem states that this is the case.

Theorem 43 The non-linear system (3.52) is locally (saddle-path) stable, in the sense that in the neighborhood of c^*, χ^* and κ^*, there is a unique two-dimensional manifold of solutions that converge to c^*, χ^* and κ^*.

Proof. Let us rewrite the system (3.52) in a more compact form as

$$\dot{x} = f(x), \tag{3.53}$$

where $x \equiv \begin{pmatrix} c & \chi & \kappa \end{pmatrix}'$. To investigate the dynamics of the system (3.53) in the neighborhood of the steady state, consider the linear system

$$\dot{z} = J(x^*) z,$$

where $z \equiv x - x^*$ and x^* such that $f(x^*) = 0$, where $J(x^*)$ is the Jacobian of $f(x)$ evaluated at x^*. Differentiation and some algebra enable us to write this Jacobian matrix as

$$J(x^*) = \begin{bmatrix} a_{cc} & a_{c\chi} & a_{c\kappa} \\ a_{\chi c} & a_{\chi\chi} & a_{\chi\kappa} \\ a_{\kappa c} & a_{\kappa\chi} & a_{\kappa\kappa} \end{bmatrix},$$

where

$$
\begin{aligned}
a_{cc} &= a_{\kappa c} = a_{\kappa\chi} = 0 \\
a_{c\chi} &= -\gamma^{\frac{\varepsilon}{\varepsilon-1}} \left(\chi^*\right)^{-\alpha_1-1} \left(\frac{\alpha_1 \left(1-\alpha_1\right)}{\theta}\right) \left(\frac{\nu-1}{\nu}\right) \\
a_{c\kappa} &= \gamma^{\frac{\varepsilon}{\varepsilon-1}} \left(\chi^*\right)^{-\alpha_1} \left(\frac{1-\alpha_1}{\theta}\right) \left(\frac{\nu-1}{\nu}\right) \left[\left(\frac{1-\alpha_1}{1-\alpha_2}\right) \left(\frac{1+\alpha_2 \left(1-\varepsilon\right)}{1-\varepsilon}\right) - \alpha_1\right] \\
a_{\chi c} &= -\left(\chi^*\right)^{-1} \\
a_{\chi\chi} &= \gamma^{\frac{\varepsilon}{\varepsilon-1}} \left(\chi^*\right)^{-\alpha_1-1} \left(1-\alpha_1\right) \left[1 - \frac{1}{\theta}\left(\frac{\nu-1}{\nu}\right)\right] + \frac{\left(\chi^*\right)^{-1} \rho}{\theta} \\
a_{\chi\kappa} &= \gamma^{\frac{\varepsilon}{\varepsilon-1}} \left(\chi^*\right)^{-\alpha_1} \left[\left(1-\alpha_1\right) + \left(\frac{1-\alpha_1}{1-\alpha_2}\right) \left(\frac{1+\alpha_2 \left(1-\varepsilon\right)}{1-\varepsilon}\right)\right] \\
a_{\kappa\kappa} &= -\left(\frac{1-\varepsilon}{\nu-1}\right) \left(m_2 - \frac{\alpha_2}{\alpha_1} m_1\right).
\end{aligned}
$$

The determinant of the Jacobian is $\det\left(J\left(x^*\right)\right) = -a_{\kappa\kappa} a_{c\chi} a_{\chi c}$. First, $a_{c\chi}$ and $a_{\chi c}$ are clearly negative. Next, it can be seen that $a_{\kappa\kappa}$ is always negative since we are in the case with $\varepsilon \lesseqgtr 1 \Leftrightarrow m_2/\alpha_2 \gtreqless m_1/\alpha_1$.[16] This implies that $\det\left(J\left(x^*\right)\right) > 0$, so the steady state is hyperbolic. Moreover, either all the eigenvalues are positive or two of them are negative and one positive. To determine which is the case, we look at the characteristic equation given by $\det\left(J\left(x^*\right) - vI\right) = 0$, where v denote the eigenvalues. This equation is

$$
\left(a_{\kappa\kappa} - v\right) \left[v \left(a_{\chi\chi} - v\right) + a_{\chi c} a_{c\chi}\right] = 0,
$$

and shows that one of the eigenvalue is equal to $a_{\kappa\kappa}$ and thus negative, so there must be two negative eigenvalues. Consequently, there exists a unique two-dimensional manifold of solutions in the neighborhood of this steady state, converging to it. This proves local (saddle-path) stability. ■

This result shows that the constant growth path equilibrium is locally stable, and when the initial values of capital, labor and technology are not too far from the constant growth path, the economy will indeed converge to this equilibrium, with non-balanced growth at the sectoral level and constant capital share and interest rate at the aggregate.

[16] As noted above, this is not a parameter restriction. When we have $\varepsilon > 1$ and $m_2/\alpha_2 > m_1/\alpha_1$, for example, then it will be sector 2 that grows more slowly in the limit, and stability will again obtain.

We next investigate the dynamics of some parameterized economies numerically. In particular, we wish to study the speed of convergence to the asymptotic equilibrium, and how the economy behaves along the path of convergence, for example, whether the interest rate and the share of capital change by large amounts along the transition path. Since the economy with $\varepsilon < 1$ is our main focus, we only report simulations for this case.

Recall that our economy is fully characterized by ten parameters, γ, ε, ν, α_1, α_2, ρ, θ, n, m_1, and m_2. We choose a period to correspond to a year, and take a baseline economy with a 1% annual population growth ($n = 0.01$), $\rho = 0.02$ and $\theta = 2$ as in the baseline calibration of the neoclassical growth model in Barro and Sala-i-Martin (2004). We choose the rest of the parameters to be consistent with 3% total growth rate of output ($g = 0.03$), a capital share in national income of approximately 35% ($s_K = 0.35$), and an interest rate of approximately 12% ($r = 0.12$).[17] These choices imply that $\alpha_1 \simeq 0.8$ (essentially to match the share of capital asymptotically), $\varepsilon = 0.5$ and $\nu \simeq 5$ (to match the interest rate) and $m_1 \simeq 0.125$ (to match the growth rate). Finally, in the benchmark simulation, we choose a capital intensity in sector 2 close to that in sector 1, $\alpha_2 = 0.75$ and $m_2 = m_1 \simeq 0.125$ to highlight the non-balanced growth resulting from differential capital intensities in the two sectors. In addition, we choose a low level of γ, $\gamma = 0.15$, to generate a fraction of employment in sector 1 of approximately 40%, and the following initial values: $L(0) = 1$, $K(0) = 1$, $M_L(0) = 1$ and $M_H(0) = 0.1$.

Figure 1 shows the results of the simulations.[18] The four panels depict λ, κ, the interest rate (r) and the capital share (s_K). The solid line is for the benchmark. The first remarkable feature in the simulation is the rate of convergence. The units on the horizontal axis are years, and range from zero to 3000. This shows that it takes a very very long time for the fraction of capital and labor allocated to sector 1 to approach their asymptotic equilibrium value of 1.

[17] Since there is no depreciation in our model, the interest rate also corresponds to the rental rate of capital. We choose 12% as the benchmark value to approximate a reasonable rental rate (e.g., an interest rate of 7% plus 5% depreciation).

[18] To perform the simulations, we first represent the equilibrium as a two-dimensional non-autonomous system in c and χ (rather than the three-dimensional autonomous system analyzed above) since κ can be represented as a function of time only. This two dimensional system has one state and one control variable. Following Judd (1998, ch.10), we then discretize these differential equations using the Euler method, and turn them into a system of first-order difference equations in c_t and χ_t, which can itself be transformed into a second-order non-autonomous system only in χ_t. This second-order equation can be analyzed numerically by reversing time, perturbing χ away from its steady-state value, and then integrating it backward to (χ_0, κ_0) (given the exogenously given sequence of state variables, $\{L_t, M_{1,t}, M_{2,t}\}_{t=0}^{T}$ and boundary conditions $\chi_{T+1} = \chi_T = \chi^*$).

For example, initially, about 40% of employment is in sector 1 and even after 500 years, less than half of employment remains there. This illustrates that even though in the limit one of the sectors employs all of the factors, it takes a very long time for the economy to approach this limit point. Second, in the baseline simulation, the interest rate is essentially constant, and varies only between 0.112 and 0.115 throughout the convergence process. The share of capital in national income is declining visibly, but its range of movement is small (between 0.35 and 0.375). Moreover, in the first 500 years, the capital share essentially moves between 0.37 and 0.375). This is the basis of our claim that this type of model may lead to a pattern of non-balanced sectoral growth (as shown by λ and κ in the top two panels), while generating only small movements in the interest rate and the capital share, thus remaining approximately consistent with the Kaldor facts.

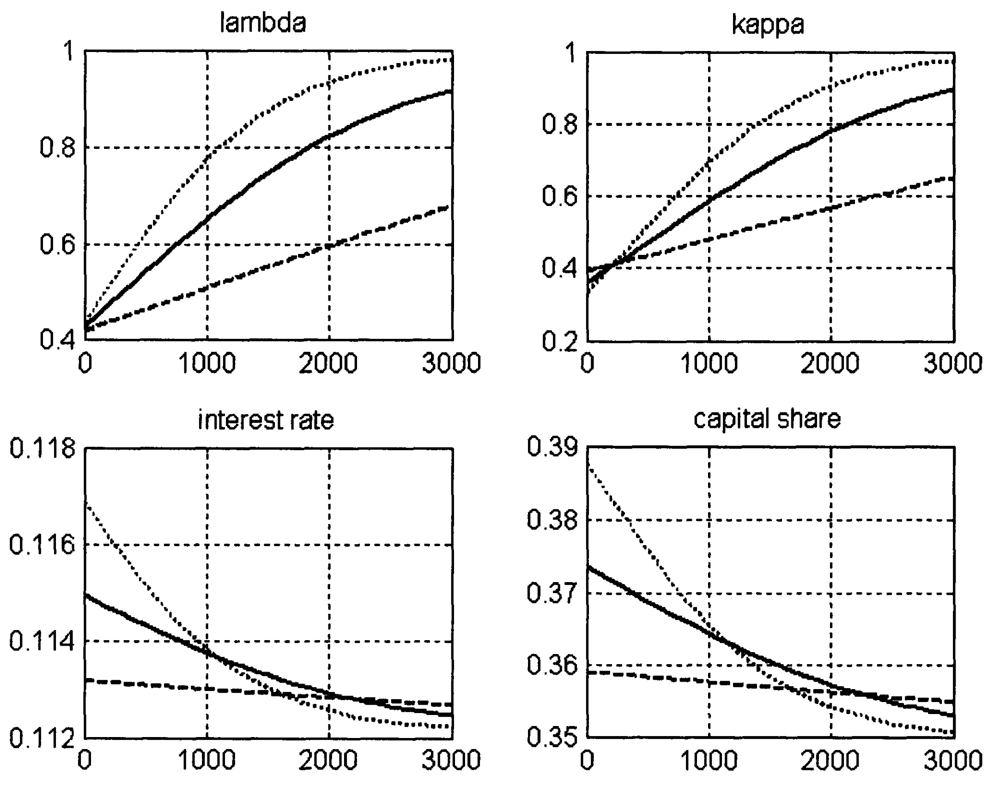


Figure 1. Solid line: $\alpha_2 = .75$. Dotted line: $\alpha_2 = .72$. Dashed line: $\alpha_2 = .78$.

The dashed and dotted lines in Figure 1 show variations with $\alpha_2 = 0.78$ and $\alpha_2 = 0.72$. The dashed line shows that when α_2 is increased even further, convergence is even slower, and now after 3000 years only less than 70% of employment is in sector 1. Moreover, the interest rate and the share of capital in national income are now essentially constant. When α_2 is reduced so that the gap between the capital intensity of the two sectors becomes larger, the speed of convergence is a little faster, but is still very very slow. The range of change of the capital share also becomes larger (between 0.35 and 0.39).

Figure 2 investigates the consequences of varying m_2. The solid line is again for the benchmark simulation, with $m_2 = m_1 \simeq 0.125$, while the dashed and the dotted lines show simulations with $m_2 = 0.12$ and $m_2 = 0.13$. They show that when the TFP growth rate in the capital-intensive sector is reduced, convergence takes much longer. For example, after 3000 years, a little above 60% of employment is in sector 1. The capital share also changes by very little over this time period. In contrast, when the TFP growth rate of the capital-intensive sector is increased, convergence is faster than the benchmark case but still very slow. For example, it takes 2000 years for the share of employment in sector 1 to reach 90%. These simulations therefore show that this class of economies may be able to generate significant non-balanced sectoral growth, without substantially deviating from the Kaldor facts.

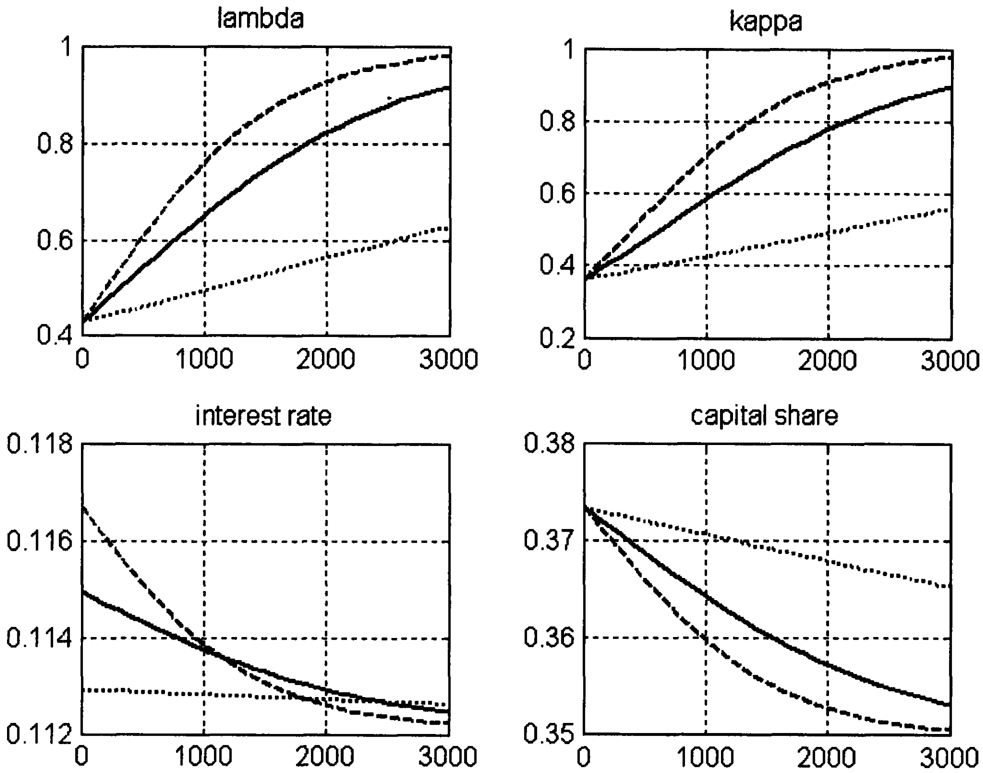


Figure 2. Solid line: $m_2 = .125$. Dotted line: $m_2 = .12$. Dashed line: $m_2 = .13$.

3.4 Non-Balanced Endogenous Growth

In this section we introduce endogenous technological progress. This investigation is motivated by two questions. As already emphasized, non-balanced growth results from capital deepening, and so far, capital deepening was a consequence of exogenous technological change. The first question is whether similar results obtain when technological change itself is endogenous. Second and more important, one may wonder whether endogenous technological change will take place in such a form as to restore balanced growth. The analysis in this section will explicitly show that this is not the case. Finally, endogenizing technological change in this context enables us to derive a model of non-balanced endogenous technological change, which is an important direction for models of endogenous technology given the non-balanced nature of growth in the data.

Demographics, preferences and technology are the same as described in the previous section, except that instead of the exogenous processes for technology given in (3.18), we now need to specify an innovation possibilities frontier, i.e., the technology to transform resources into blueprints for new varieties in the two sectors. In particular, we assume that

$$\dot{M}_1 = b_1 M_1^{-\varphi} X_1 \text{ and } \dot{M}_2 = b_2 M_2^{-\varphi} X_2, \tag{3.54}$$

where $X_1 \geq 0$ and $X_2 \geq 0$ are research expenditures in terms of the final good, b_1 and b_2 are strictly positive constants measuring the technical difficulty of creating new blueprints in the two sectors, and $\varphi \in (-1, \infty)$ measures the degree of spillovers in technology creation.[19]

Finally, we assume that there is free entry into research, and a firm that invents a new intermediate of either sector becomes the monopolist producer with a perpetually enforced patent. Given the value for being the monopolist for intermediate in (3.21), we have two free entry conditions, which will pin down equilibrium technological change:

$$V_1 \leq \frac{M_1^{\varphi}}{b_1} \text{ and } V_2 \leq \frac{M_2^{\varphi}}{b_2}, \tag{3.55}$$

with each condition holding as equality when there is positive R&D expenditure for that sector, i.e., when $X_1 > 0$ or $X_2 > 0$.

The resource constraint of the economy is also modified to incorporate the R&D expenditures,

$$\dot{K} + C + X_1 + X_2 \leq Y. \tag{3.56}$$

Now a dynamic equilibrium is represented by paths of interest wages and rates, w and r, labor and capital allocation decisions, λ and κ, satisfying (3.30) and (3.31), and of consump-

[19] When $\varphi = 0$, there are no spillovers from the current stock of knowledge to future innovations. With $\varphi < 0$, there are positive spillovers and the stock of knowledge in a particular sector makes further innovation in that sector easier. With $\varphi > 0$, there are negative spillovers ("fishing out") and further innovations are more difficult in sectors that are more advanced (see, for example, Jones, 1995, Kortum, 1997). Similar to the results in Jones (1995), Young (1999) and Howitt (1999), there will be endogenous growth for a range of values of φ because of population growth. In the remainder, we will typically think of $\varphi > 0$, so that there are negative spillovers, though this is not important for any of the asymptotic results.

Also, this innovation possibilities frontier assumes that only the final good is used to generate new technologies. The alternative is to have a scarce factor, such as labor or scientists, in which case some amount of positive spillovers would be necessary.

tion, capital stock, technology, values of innovation and research expenditures satisfying (3.21), (3.38), (3.39), (3.54), (3.55) and (3.56). It is also useful to define a path that satisfies all of these equations, possibly except the transversality condition, (3.39), as a *quasi-equilibrium.*

Since the case with $\varepsilon < 1$ is both more interesting, and in view of the discussion in footnote 4, also more realistic, in this section we focus on this case exclusively.

We first note that Propositions 38 and 39 from Section 3.3 still apply and characterize the comparative static responses, and Lemmas 40 and 41 from there determine the behavior of the growth rate of sectoral output, capital and employment. For the analysis of the economy with endogenous technology, we also need an additional result in the next lemma. It shows that provided that (i) $\varepsilon < 1$, (ii) there exists a constant asymptotic interest rate r^* (i.e., $\lim_{t\to\infty} \dot{r} = 0$), and (iii) there is positive population growth, in the asymptotic equilibrium the free entry conditions in (3.55) will both hold as equality:

Lemma 44 Suppose that $\varepsilon < 1$, $n > 0$, and $\lim_{t\to\infty} \dot{r} = 0$, then $\lim_{t\to\infty} (V_1 - M_1^{\varphi}/b_1) = 0$ and $\lim_{t\to\infty} (V_2 - M_2^{\varphi}/b_2) = 0$.

The proof of this lemma is rather long and is provided in the Appendix.

This lemma is an important result for the analysis of non-balanced endogenous growth. It enables us to solve for the asymptotic growth rates from the free entry conditions and obtain a relatively simple characterization of the constant growth path equilibrium.

The economic intuition for the lemma comes from population growth; with population growth, it is always optimal to allocate more capital to each sector, which increases the profitability of intermediate producers in that sector. Consequently, the value of a new blueprint increases asymptotically. This rules out asymptotic equilibrium paths with slack free-entry conditions, because along such paths, the cost of creating a new blueprint would remain constant, ultimately violating the free-entry condition.

To establish the existence of a CGP, we now impose the following parameter restriction, which replaces (A2) in the exogenous technology case:

$$\zeta > \max\left\{\frac{1}{\alpha_1}, \frac{1}{\alpha_1}\left[1 - \frac{(1-\theta)}{\rho}n\right]^{-1}\right\} \tag{A3}$$

where $\zeta \equiv (\nu - 1)(1 + \varphi)$. This assumption ensures that the transversality condition (3.39) holds. Notice that in contrast to Assumption A2 only α_1 features in this assumption, since, given Assumption A1, $\alpha_1 < \alpha_2$ and with endogenous technology, this will make sure that sector 1 is the more slowly growing sector in the asymptotic equilibrium. Assumption A3 also rules out quasi-equilibrium paths where output and consumption grow more than exponentially.

Lemma 45 Suppose Assumption A3 holds and $\varepsilon < 1$, then there exists no quasi-equilibria with $\lim_{t\to\infty} \dot{C}/C = \infty$.

This lemma is proved in the Appendix, where, for completeness, we also show that Assumption A3 is "tight" in the sense that, if first inequality in this assumption, $\zeta > 1/\alpha_1$, did not hold, there always exist quasi-equilibria with more than exponential growth.

Combined Lemmas 44 and 45 imply that the asymptotic equilibrium has to converge either to a limit with constant growth of consumption, or to a limit cycle. Using these results, the next theorem establishes the existence of a unique constant growth path, with non-balanced sectoral growth, but constant share of capital and constant interest rate in the aggregate. It is therefore the analogue of Theorem 42 of the previous section, and is the main result of this section.

Theorem 46 Suppose Assumptions A1 and A3 hold, $\varepsilon < 1$, and $n > 0$, then there exists a unique CGP where

$$g^* = g_C^* = g_1^* = z_1^* = \frac{\alpha_1\zeta}{\alpha_1\zeta - 1}n, \tag{3.57}$$

$$g_2^* = \frac{\alpha_1\zeta\left(1-\varepsilon+\zeta\right)+\varepsilon\zeta\left(\alpha_1-\alpha_2\right)}{\left(\alpha_1\zeta-1\right)\left(1-\varepsilon+\zeta\right)}n > g^*, \tag{3.58}$$

$$z_2^* = \frac{\alpha_1\zeta\left(1-\varepsilon+\zeta\right)-\zeta\left(\alpha_1-\alpha_2\right)\left(1-\varepsilon\right)}{\left(\alpha_1\zeta-1\right)\left(1-\varepsilon+\zeta\right)}n < g^*, \tag{3.59}$$

$$n_1^* = n \quad \text{and} \quad n_2^* = \left[1 - \frac{\zeta\left(1-\varepsilon\right)\left(\alpha_1-\alpha_2\right)}{\left(\alpha_1\zeta-1\right)\left[1-\varepsilon+\zeta\right]}\right]n, \tag{3.60}$$

$$m_1^* = \frac{1}{1+\varphi}z_1^* \quad \text{and} \quad m_2^* = \frac{1}{1+\varphi}z_2^*. \tag{3.61}$$

The proof of this theorem is also given in the Appendix.

There a number of important features worth noting. First, this theorem shows that the equilibrium of this non-balanced endogenous growth economy takes a relatively simple form.

Second, given the equilibrium rates of technological change in the two sectors, m_1^* and m_2^*, the asymptotic growth rates are identical to those in Theorem 42 in the previous section, so that the economy with endogenous technological change gives the same insights as the one with exogenous technology. In particular, there is non-balanced sectoral growth, but in the aggregate, the economy has a limiting equilibrium with constant share of capital in national income and a constant interest rate. Finally, technology is also endogenously non-balanced, and in fact, tries to counteract the non-balanced nature of economic growth. Specifically, there is more technological change towards the sector that is growing more slowly (recall we are focusing on the case where $\varepsilon < 1$), so that the sector with a lower share of capital has an increasing share of capital, employment and technology in the economy. Nevertheless, this sector still grows more slowly than the more capital-intensive sector, and the non-balanced nature of the growth process remains. Therefore, endogenous growth does not restore balance between the two sectors as long as capital intensity (factor proportion) differences between the two sectors remain.[20]

3.5 Conclusion

This paper shows how differences in factor proportions across sectors combined with capital deepening lead to a non-balanced pattern of economic growth. We first illustrated this point using a general two-sector growth model, and then characterized the equilibrium fully using a class of economies with constant elasticity of substitution between sectors and Cobb-Douglas production technologies. This class of economies shows how the pattern of equilibrium may be simultaneously consistent with non-balanced sectoral growth (the so-called Kuznets facts) while also generating asymptotically constant share of capital in national income and interest rate in the aggregate (the Kaldor facts). We also constructed and analyzed a model with endogenous technology featuring non-balanced economic growth.

The main contribution of the paper is theoretical, but it also raises a number of empirical questions. In particular, the analysis suggests that differences in factor proportions across

[20] Since there are two more endogenous state variables and two more control variables now, it is not possible to show local stability analytically. In particular, in addition to c, χ and κ, we need to keep track of the endogenous evolution of M_1, M_2, X_1 and X_2 (or their stationary transformations). Given the size of this system, we are unable to prove local (saddle-path) stability.

sectors will introduce a powerful force towards non-balanced growth, which could be important in accounting for the cross-sectoral patterns of output and employment growth. Whether this is so or not, especially in the context of the differential growth of agriculture, manufacturing and services, appears to be a fruitful area for future research.

3.6 Appendix

3.6.1 Proof of Proposition 39

Parts 1 and 2 follow from differentiating equation (3.36) and Proposition 38. Here we prove parts 3 and 4. Recall the expression for the equilibrium capital share

$$s_K \equiv \frac{rK}{Y} = 1 - \gamma\alpha_1 \left(\frac{Y_1}{Y}\right)^{\frac{\varepsilon-1}{\varepsilon}} \lambda^{-1}$$

with

$$\lambda = \left[\left(\frac{1-\eta}{1-\alpha}\right)\left(\frac{\alpha}{\eta}\right)\left(\frac{1}{\kappa}-1\right)+1\right]^{-1}$$

where combining the production function and the equilibrium capital allocation we get

$$\begin{aligned}\left(\frac{Y_1}{Y}\right)^{\frac{\varepsilon-1}{\varepsilon}} &= \left[\gamma + (1-\gamma)\left(\frac{Y_1}{Y_2}\right)^{\frac{1-\varepsilon}{\varepsilon}}\right]^{-1} \\ &= \gamma^{-1}\left(1+\left(\frac{1-\alpha_1}{1-\alpha_2}\right)\left(\frac{1}{\kappa}-1\right)\right)^{-1}\end{aligned}$$

Then, using the results of Proposition 38, we have

$$\frac{d\ln s_K}{d\ln K} = -S\left(\frac{1-s_K}{s_K}\right)\left(\frac{1-\alpha_1}{1-\alpha_2}\right)\frac{(1-\varepsilon)(\alpha_1-\alpha_2)(1-\kappa)/\kappa}{1+(1-\varepsilon)(\alpha_1-\alpha_2)(\kappa-x)} \tag{3.62}$$

and

$$\frac{d\ln s_K}{d\ln M_2} = -\frac{d\ln s_K}{d\ln M_1} = S\left(\frac{1-s_K}{s_K}\right)\left(\frac{1-\alpha_1}{1-\alpha_2}\right)\frac{(1-\varepsilon)(1-\kappa)/\kappa(\nu-1)}{1+(1-\varepsilon)(\alpha_1-\alpha_2)(\kappa-x)} \tag{3.63}$$

where

$$S \equiv \left[\left(1+\left(\frac{1-\alpha_1}{1-\alpha_2}\right)\left(\frac{1}{\kappa}-1\right)\right)^{-1} - \left(\left(\frac{\alpha_1}{\alpha_2}\right)+\left(\frac{1-\alpha_1}{1-\alpha_2}\right)\left(\frac{1}{\kappa}-1\right)\right)^{-1}\right]$$

with

$$S < 0 \Leftrightarrow \alpha_1 > \alpha_2.$$

This establishes the desired result that

$$\frac{d\ln s_K}{d\ln K} < 0 \Leftrightarrow \varepsilon < 1$$

and

$$\frac{d\ln s_K}{d\ln M_1} = -\frac{d\ln s_K}{d\ln M_2} > 0 \Leftrightarrow (\alpha_1 - \alpha_2)(1-\varepsilon) > 0.$$

■

3.6.2 Proof of Lemma 44

We will prove this lemma in four steps.

Step 1: $m_1^* = m_2^* = 0$ imply $g_2^* = g_1^* = n$.

Step 2: $\limsup_{t\to\infty}(V_1 - M_1^\varphi/b_1) \geq 0$ or $\limsup_{t\to\infty}(V_2 - M_2^\varphi/b_2) \geq 0$.

Step 3: $\limsup_{t\to\infty}(V_1 - M_1^\varphi/b_1) \geq 0$ and $\limsup_{t\to\infty}(V_2 - M_2^\varphi/b_2) \geq 0$.

Step 4: $\lim_{t\to\infty}(V_1 - M_1^\varphi/b_1) = 0$ and $\lim_{t\to\infty}(V_2 - M_2^\varphi/b_2) = 0$.

Proof of Step 1: We first prove that $m_1^* = m_2^* = 0$ imply $g_2^* = g_1^* = n$. To see this, combine equations (3.30) and (3.31) to obtain

$$\left(\frac{Y_2}{Y_1}\right)^{\frac{1-\varepsilon}{\varepsilon}} = \left(\frac{1-\gamma}{\gamma}\right)\left(\frac{1-\alpha_2}{1-\alpha_1}\right)\frac{K_1}{K_2}.$$

Differentiating gives

$$\frac{1-\varepsilon}{\varepsilon}(\alpha_2 n_2 - \alpha_1 n_1) + \frac{1-\varepsilon}{\varepsilon}(1-\alpha_2)z_2 - \frac{1-\varepsilon}{\varepsilon}(1-\alpha_1)z_1 = z_1 - z_2. \tag{3.64}$$

To derive a contradiction, suppose that $g_2^* > g_1^*$. Then, from Lemma 40 $n_2^* < n_1^*$ and $z_2^* < z_1^*$ and from Lemma 41, $g^* = g_1^*$. Next, differentiating (3.25), we obtain

$$g_1 = \alpha_1 n_1 + (1-\alpha_1)z_1 + \frac{1}{\nu-1}m_1 \text{ and } g_2 = \alpha_2 n_2 + (1-\alpha_2)z_2 + \frac{1}{\nu-1}m_2. \tag{3.65}$$

Moreover, since $\lim_{t\to\infty}\dot{r} = 0$, equation (3.41) and the fact that $g^* = g_1^*$ imply that $g_1^* = z_1^*$. This combined with equation (3.65) and $m_1^* = 0$ implies that $z_1^* = n_1^* = n$, which together with

(3.64) implies

$$0 > \frac{1-\varepsilon}{\varepsilon}(1-\alpha_2)(z_2^* - z_1^*) > z_1^* - z_2^*,$$

yielding a contradiction. The argument for the case in which $g_2^* < g_1^*$ is analogous. Since $g_2^* = g_1^* = g^*$ and $m_1^* = m_2^* = 0$, it must also be the case that $g_2^* = g_1^* = n$, completing the proof of step 1.

Proof of Step 2: First note that $\limsup_{t\to\infty}(V_1 - M_1^{\varphi}/b_1) < 0$ and $\limsup_{t\to\infty}(V_2 - M_2^{\varphi}/b_2) < 0$ imply that the free entry conditions, (3.55), are asymptotically slack, so m_1^* and m_2^* exist, and $\lim_{t\to\infty} m_1(t) = m_1^* = 0$ and $\lim_{t\to\infty} m_2(t) = m_2^* = 0$ (since they cannot be negative). In particular, note that if $\limsup_{t\to\infty}(V_1 - M_1^{\varphi}/b_1) < 0$, this implies that there exists no "infinitely-recurring" interval in the limit where the free entry condition holds for sector 1.

Now to derive a contradiction, suppose that $m_2^* = m_1^* = 0$, which, as shown above, implies $g_1^* = g_2^* = g^* = n > 0$. Then, differentiating the second equation in (3.27), we obtain:

$$\lim_{t\to\infty}\frac{\dot{\pi}_2}{\pi_2} = g^* > 0. \tag{3.66}$$

Combining this with $\lim_{t\to\infty}\dot{r} = 0$ and the value function in (3.21) yields: $\lim_{t\to\infty} V_2 = \infty$. Since $m_2^* = 0$ by hypothesis, $M_2^{-\varphi}$ is constant, and we have $\lim_{t\to\infty} V_2 = \infty > \lim_{t\to\infty} M_2^{\varphi}/b_2$, violating the free entry condition (3.55). This proves that m_1^* and m_2^* cannot both equal to 0, and thus $\limsup_{t\to\infty}(V_2 - M_2^{\varphi}/b_2) < 0$ and $\limsup_{t\to\infty}(V_1 - M_1^{\varphi}/b_1) < 0$ is not possible.

Proof of Step 3: We next prove that $\limsup_{t\to\infty}(V_1 - M_1^{\varphi}/b_1) \geq 0$, $\limsup_{t\to\infty}(V_2 - M_2^{\varphi}/b_2) \geq 0$, $\limsup_{t\to\infty} m_1 > 0$ and $\limsup_{t\to\infty} m_2 > 0$.

Suppose, to derive a contradiction, $\limsup_{t\to\infty} m_2 = m_2^* = 0$ and $\limsup_{t\to\infty}(V_2 - M_2^{\varphi}/b_2) < 0$ (the other case is proved analogously). Since, as shown above, $m_1^* = m_2^* = 0$ is not possible, we must have $\limsup_{t\to\infty}(V_1 - M_1^{\varphi}/b_1) \geq 0$ and $\limsup_{t\to\infty} m_1 > 0$.

We start by noting from (3.27), that asymptotically

$$\lim_{t\to\infty}\frac{\dot{\pi}_2}{\pi_2} = \frac{1}{\varepsilon}(g - g_2) + g_2, \tag{3.67}$$

because $m_2^* = 0$. Since

$$\lim_{t\to\infty} \frac{\dot{r}}{r} = \frac{1}{\varepsilon}(g - g_2) + g_2 - z_2 = 0$$

from (3.31), $z_2 > 0$ implies that $\lim_{t\to\infty} \dot{\pi}_2/\pi_2 > 0$. But by the same argument as in Step 2, we have $\lim_{t\to\infty} V_2 = \infty > \lim_{t\to\infty} M_2^\varphi/b_2$, violating the free entry condition (3.55). We therefore only have to show that $z_2 > 0$ (i.e., $\limsup_{t\to\infty} z_2 > 0$). Suppose, to obtain a contradiction, that $z_2^* = 0$. Using (3.40) and (3.41) from the proof of Lemma 40, we have $z_2 - n_2 = z_1 - n_1$, which implies that $n_2^* = 0$ (recall that either $n_1 \geq n$ or $n_2 \geq n$, and since $\lim_{t\to\infty} g \geq n$, either $z_1 \geq n$ or $z_2 \geq n$). But then with $n_2^* = z_2^* = m_2^* = 0$, we have $g_2^* = 0 < g_1^*$, which contradicts $n_2^* < n_1^* = n$ from Lemma 40. A similar argument for the other sector completes the proof that $\limsup_{t\to\infty}(V_1 - M_1^\varphi/b_1) \geq 0$ and $\limsup_{t\to\infty}(V_2 - M_2^\varphi/b_2) \geq 0$.

Proof of Step 4. From the free entry conditions in (3.55), we have that $V_1 - M_1^\varphi/b_1 \leq 0$ and $V_2 - M_2^\varphi/b_2 \leq 0$, thus $\limsup_{t\to\infty}(V_1 - M_1^\varphi/b_1) \leq 0$ and $\limsup_{t\to\infty}(V_2 - M_2^\varphi/b_2) \leq 0$. Combined with Step 3, this implies $\limsup_{t\to\infty}(V_1 - M_1^\varphi/b_1) = 0$ and $\limsup_{t\to\infty}(V_2 - M_2^\varphi/b_2) = 0$. Hence, we only have to prove that $\liminf_{t\to\infty}(V_1 - M_1^\varphi/b_1) \geq 0$ and $\liminf_{t\to\infty}(V_2 - M_2^\varphi/b_2) \geq 0$. We prove the first inequality (the proof of the second is similar).

Suppose, to derive a contradiction, that $\liminf_{t\to\infty}(V_1 - M_1^\varphi/b_1) < 0$. This implies that there exists a (recurring) interval (t_0', t_2') such that $V_1(t) - M_1^\varphi(t)/b_1 < 0$ for all $t \in (t_0', t_2')$. Suppose that (t_0', t_2') is unbounded; this would imply that $\limsup_{t\to\infty} m_1 = m_1^* = 0$, yielding a contradiction with Step 1. Thus (t_0', t_2') must be bounded, so there exists $(t_0, t_2) \supset (t_0', t_2')$ such that for $t \in (t_0, t_2) \setminus (t_0', t_2')$, we have $V_1(t) - M_1^\varphi(t)/b_1 = 0$. Moreover, since $\limsup_{t\to\infty} m_1 > 0$, there also exists an interval $(t_0'', t_2'') \supset (t_0, t_2)$ such that for all $t \in (t_0, t_2) \setminus (t_0'', t_2'')$, $m_1 > 0$.

Next, since $m_1 = 0$ for all $t \in (t_0', t_2')$, we also have $M_1(t_0') = M_1(t_2')$. This implies

$$V_1(t_2') = \frac{M_1^\varphi(t_0')}{b_1} = V_1(t_0'). \tag{3.68}$$

Figure A1 shows this diagrammatically.

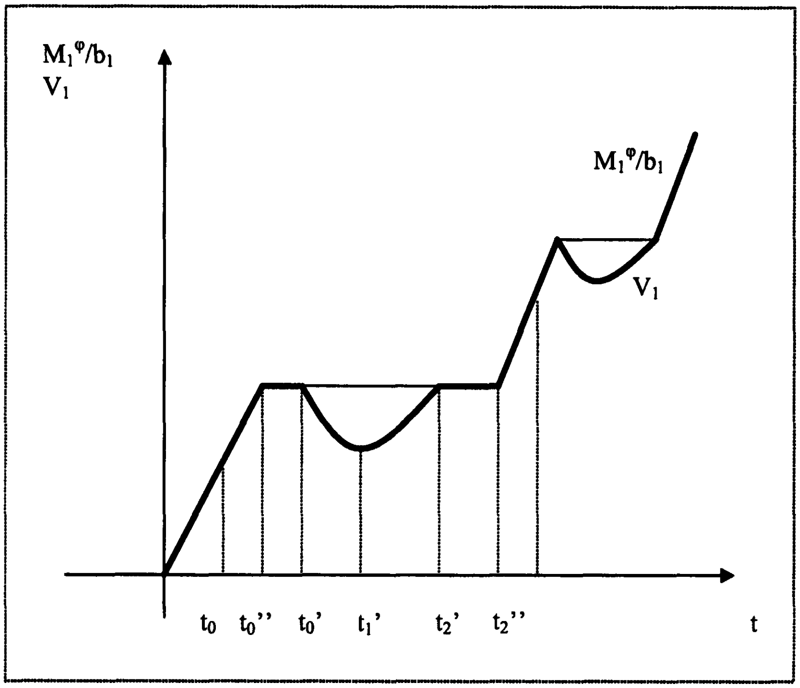


Figure A1: The solid line represents M_1^{φ}/b_1 and the thick line represents V_1.

Let us rewrite (3.21) in the Bellman equation form

$$\frac{\dot{V}_1(t)}{r} = V_1(t) - \frac{\pi_1(t)}{r} \quad \forall t. \tag{3.69}$$

Equation (3.69) also shows that $\dot{V}_1(t)$ is well-defined, so $V_1(t)$ is continuously differetiable in t. Equation (3.68) and the fact that $V_1(t) - M_1^{\varphi}(t_0')/b_1 < 0$ for all $t \in (t_0', t_2')$ imply that $V_1(t)$ reaches a minimum over (t_0', t_2') with $\dot{V}_1(t) = 0$. Let $t_1' < t_2'$ be such that $V_1(t_1')$ is the first local minimum after t_0', which naturally satisfies $V_1(t_1') < V_1(t_0')$. Moreover, using (3.27) and (3.31), we have that for all t

$$\frac{\pi_1(t)}{r^*} = \frac{1}{(\nu-1)(1-\alpha_1)} \frac{K_1(t)}{M_1(t)},$$

where $r^* = \lim_{t\to\infty} r(t)$ is the asymptotic equilibrium interest rate, which exists by hypothesis that $\lim_{t\to\infty} \dot{r}(t) = 0$. Also, using the fact that $\lim_{t\to\infty} \dot{r}(t) = 0$ and the interest rate condition, (3.30), we obtain that since $m_1 = 0$ and $n > 0$, $K_1(t_1') > K_1(t_0')$. In addition, since $M_1(t_1') =$

$M_1(t'_0)$, we can use (3.69) to write

$$\frac{\dot{V}_1(t'_1)}{r^*} = V_1(t'_1) - \frac{1}{(\nu-1)(1-\alpha_1)}\frac{K_1(t'_1)}{M_1(t'_1)} < V_1(t'_0) - \frac{1}{(\nu-1)(1-\alpha_1)}\frac{K_1(t'_0)}{M_1(t'_0)} = \frac{\dot{V}_1(t'_0)}{r^*} < 0$$

which contradicts the fact that $V_1(t'_1)$ is a local minimum, completing the proof of the lemma.■

3.6.3 Proof of Lemma 45 and The Converse Result

Proof of Lemma 45: First, recall that $C \le Y$, (3.11). Hence it is enough to prove that $\lim_{t\to\infty} g = \infty$ will violate the resource constraint. We will prove this separately in two cases, when $g_2^* \ge g_1^*$ and when $g_2^* < g_1^*$

Suppose $g_2^* \ge g_1^*$ and $g^* = \infty$. Then, Lemma 41 implies $g_1^* = g^* = \infty$, and equation (3.41) together with (3.40) and (3.65) yields

$$g = n - \left(\frac{1-\alpha_1}{\alpha_1}\right)\frac{\dot{r}}{r} + \frac{1}{\alpha_1(\nu-1)}m_1.$$

Given $n < \infty$ and $\lim_{t\to\infty} \dot{r}/r > 0$, it must be that asymptotically

$$g^* = \frac{1}{\alpha_1(\nu-1)}m_1^*. \tag{3.70}$$

Combining the technology possibility frontier (3.54) and (3.70) we have

$$\lim_{t\to\infty}\frac{\dot{X}_1}{X_1} = \lim_{t\to\infty}\frac{\dot{m}_1}{m_1} + \alpha_1(1+\varphi)(\nu-1)g^*.$$

Then, the first inequality in Assumption A3, $\zeta > 1/\alpha_1$, implies

$$\lim_{t\to\infty}\frac{\dot{X}_1}{X_1} > g^*,$$

which violates the resource constraint (3.13).

Next suppose that $g_1^* > g_2^*$and $g^* = \infty$. Then, following the steps of above, Lemma 41 implies $g_2^* = g^* = \infty$, and equation (3.41) together with (3.40) and (3.65) yields

$$\left(\frac{1-\alpha_1(1-\varepsilon)}{\varepsilon}\right)g_1 = \alpha_1 n_1 + \left(\frac{1-\alpha_1}{\varepsilon}\right)g - (1-\alpha_1)\frac{\dot{r}}{r} + \left(\frac{1}{\nu-1}\right)m_1.$$

Since $g^* = \infty$, a fortiori $g_1^* = \infty$, and, given $n < \infty$ and $\lim_{t\to\infty} \dot{r}/r > 0$, we have that asymptotically

$$m_1^* = (\nu - 1)\left[\left(\frac{1-\alpha_1(1-\varepsilon)}{\varepsilon}\right) g_1^* - \left(\frac{1-\alpha_1}{\varepsilon}\right) g^*\right]. \tag{3.71}$$

Once again the innovation possibilities frontier (3.18) implies

$$\lim_{t\to\infty} \frac{\dot{X}_1}{X_1} = \lim_{t\to\infty} \frac{\dot{m}_1}{m_1} + \alpha_1 (1-\varphi)\, m_1^*.$$

Then equation (3.70) together with the first inequality in Assumption A3, $\zeta > 1/\alpha_1$, implies

$$\lim_{t\to\infty} \frac{\dot{X}_1}{X_1} > \alpha_1 (1-\varphi)(\nu-1)\, g^* > g^*,$$

which violates the resource constraint (3.13), completing the proof that when Assumption A3 holds any quasi-equilibrium with more than exponential growth violates the resource constraints.■

For completeness, we also prove the converse of Lemma 45, which shows that the use of the first inequality in Assumption A3, $\zeta > 1/\alpha_1$, in this lemma is "tight" in the sense that, if it were relaxed, the converse result would obtain.

Lemma 4': Suppose A1 holds, but $\zeta \equiv (\nu-1)(1-\varphi) \leq \frac{1}{\alpha_1}$, then there exists quasi-equilibria with $\lim_{t\to\infty} g = \infty$.

Proof. This lemma will be proved by showing that in this case

$$g_2^* = g_1^* = g^* = \infty \quad \text{and} \quad z_2^* = z_1^* = z^* \tag{3.72}$$

$$z^* = g^* - \frac{\dot{r}}{r} \tag{3.73}$$

$$\frac{\dot{m}_1}{m_1} = \frac{\dot{m}_2}{m_2} = [1 - \alpha_1 \zeta]\, g \tag{3.74}$$

is a quasi-equilibrium.

From the interest rate conditions in the two sectors (3.30) and (3.31), and (3.72) we obtain

$$z^* = g^* - \lim_{t\to\infty} \frac{\dot{r}}{r}$$

which is exactly condition (3.73). By substituting into (3.65), we obtain

$$g^* = n_1 - \left(\frac{1-\alpha_1}{\alpha_1}\right) \lim_{t\to\infty} \frac{\dot{r}}{r} + \frac{1}{\alpha_1(\nu-1)} m_1^*$$

$$g^* = n_2 - \left(\frac{1-\alpha_2}{\alpha_2}\right) \lim_{t\to\infty} \frac{\dot{r}}{r} + \frac{1}{\alpha_2(\nu-1)} m_2^*$$

which gives $g^* = \frac{1}{\alpha_1(\nu-1)} m_1^*$ and $g^* = \frac{1}{\alpha_2(\nu-1)} m_2^*$, and hence

$$m_1^* = \frac{\alpha_1}{\alpha_2} m_2^*.$$

Differentiating this condition gives equation (3.74).

Finally, we need to check feasibility, i.e., that the R&D expenditures do not grow faster than output. From the technology possibilities frontiers, (3.18), this requires

$$\lim_{t\to\infty} \frac{\dot{X}_1}{X_1} = \frac{\dot{m}_1^*}{m_1^*} + \alpha_1 \zeta g^* \leq g^*$$

$$\lim_{t\to\infty} \frac{\dot{X}_2}{X_2} = \frac{\dot{m}_2^*}{m_2^*} + \alpha_2 \zeta g^* \leq g^*$$

and both these conditions are satisfied given (3.74) and $\alpha_1 > \alpha_2$. ■

3.6.4 Proof of Theorem 46

We prove this proposition in two steps.

Step 1: Provided that $g_2^* \geq g_1^* > 0$, then there exists a unique CGP defined by equations (3.57), (3.58), (3.59), (3.60) and (3.61), satisfying $g_2^* > g_1^* > 0$.

Step 2: All CGP must satisfy $g_2^* \geq g_1^* > 0$.

Proof of Step 1. Lemma 44 establishes that as $t \to \infty$ the free-entry conditions (3.55) must asymptotically hold as equality. Combining (3.55) as equality with (3.69) (and the equivalent for sector 2), we obtain the following conditions that must hold as $t \to \infty$:

$$\frac{\frac{\gamma}{\nu}\left(\frac{Y_1}{Y}\right)^{-\frac{1}{\varepsilon}} Y_1}{r - \varphi m_1} = \frac{M_1^{1+\varphi}}{b_1} \text{ and } \frac{\frac{\gamma}{\nu}\left(\frac{Y_2}{Y}\right)^{-\frac{1}{\varepsilon}} Y_2}{r - \varphi m_2} = \frac{M_2^{1+\varphi}}{b_2} \tag{3.75}$$

Differentiating (3.75) yields:

$$g_1 - \frac{1}{\varepsilon}(g_1 - g) - (1+\varphi)\, m_1 = 0 \text{ and } g_2 - \frac{1}{\varepsilon}(g_2 - g) - (1+\varphi)\, m_2 = 0. \tag{3.76}$$

Then $g_2^* > g_1^* > 0$ and Lemma 41 imply that we must also have $g^* = g_1^*$. This condition together our system of equations, (3.40), (3.41), (3.65), (3.76), solves uniquely for n_1^*, n_2^*,z_1^*, z_2^*, m_1^*, m_2^*, g_1^* and g_2^* (and $\kappa^* = \lambda^* = 1$) as given in equations (3.57), (3.58), (3.59), (3.60) and (3.61). Note that this solution is consistent with $g_2^* > g_1^* > 0$, since Assumptions A1 and A3 immediately imply that $g_2^* > g_1^*$ and $g_1^* > 0$ (which is also consistent with Lemma 44). Finally, $C \leq Y$, (3.11) and (3.39) imply that the consumption growth rate, g_C^*, is equal to the growth rate of output, g^*.

Finally, we can verify that an equilibrium with κ^*, λ^*, n_1^*, n_2^*, z_1^*, z_2^*, m_1^*, m_2^*, g_1^* and g_2^* satisfies the transversality condition (3.39) with a similar argument to the one spelled in the first step of the proof of Theorem 42.

Proof of Step 2. The proof that along all CGPs $g_2^* > g_1^* > 0$ must be true, is analogous to the one of the second step in the proof of Theorem 42.■

Bibliography

[1] Acemoglu, Daron, "Directed Technical Change" *Review of Economic Studies*, LXIX (2002b), 781-810.

[2] Acemoglu, Daron, "Labor- and Capital-Augmenting Technical Change" *Journal of European Economic Association*, I (2003), 1-37.

[3] Aghion, Philippe and Peter Howitt, "A Model of Growth Through Creative Destruction" *Econometrica*, LX (1992), 323-351.

[4] Aghion, Philippe and Peter Howitt, *Endogenous Growth Theory*, Cambridge, MA, MIT Press, 1998.

[5] Antras, Pol, "Is the U.S. Aggregate Production Function Cobb-Douglas? New Estimates of the Elasticity of Substitution" MIT mimeo (2001).

[6] Barro, Robert and Xavier Sala-i-Martin. *Economic Growth*, McGraw Hill, New York (2004, 2nd edition).

[7] Baumol, William J., "Macroeconomics of Unbalanced Growth: The Anatomy of Urban Crisis", *American Economic Review*, LVII (1967), 415-426.

[8] Berndt, Ernst, "Reconciling Alternative Estimates of the Elasticity of Substitution," *Review of Economics and Statistics*, LVIII (1976), 59-68.

[9] Bewley, Truman, "An Integration of Equilibrium Theory and Turnpike Theory" *Journal of Mathematical Economics*, X (1982), 233-267.

[10] Caselli, Francesco and John Coleman, "The U. S. Structural Transformation and Regional Convergence: a Reinterpretation" *Journal of Political Economy* CIX (2001) 584-617.

[11] Chenery, Hollis, "Patterns of Industrial Growth", *American Economic Review,* V (1960), 624-654.

[12] Chirinko, Robert S. "Business Fixed Investment: a Critical Survey of Modeling Strategies, Empirical Results and Policy Implications" *Journal of Economic Literature*, XXXI, (1993), 1875-1911.

[13] Chirinko, Robert S., Steven M. Fazzari and Andrew P. Mayer, "How Responsive Is Business Capital Formation to Its User Cost?" *Journal of Public Economics*, LXXV, (1999), 53-80.

[14] David, Paul, and Th. Van de Klundert, "Biased Efficiency Growth and Capital-Labor Substitution in the U.S., 1899-1960," *American Economic Review,* LV, (1965), 357-393.

[15] Denison, Edward F, "Accounting for United States Economic Growth, 1929-1969" Washington, DC: Brookings Institution (1974).

[16] Echevarria, Cristina, "Changes in Sectoral Composition Associated with Economic Growth", *International Economic Review*, XXXVIII (1997), 431-452.

[17] Gollin, Douglas, Stephen Parente and Richard Rogerson "The Role of Agriculture in Development" *American Economic Review Papers and Proceedings* XCII (2002) 160-164.

[18] Griffin, James M. and Paul R. Gregory, "An Intercountry Translog Model of Energy Substitution Responses", *American Economic Review*, LXVI (1976), 845-857.

[19] Grossman, Gene and Elhanan Helpman, "Quality Ladders in the Theory of Growth" *Review of Economic Studies*, LIIX (1991a), 43-61.

[20] Grossman, Gene and Elhanan Helpman, *Innovation and Growth in the Global Economy,* Cambridge, MA, MIT Press, 1991b.

[21] Hamermesh, David S., *Labor Demand*, Princeton University Press, Princeton 1993.

[22] Howitt, Peter, "Steady Endogenous Growth with Population and R&D Inputs Growing" *Journal of Political Economics*, CVII (1999), 715-730.

[23] Jones, Charles I., "R&D-Based Models of Economic Growth", *Journal of Political Economics*, CIII (1995), 759-784.

[24] Jones, Charles I., "The Shape of Production Functions and the Direction of Technical Change", U.C. Berkeley mimeo (2004).

[25] Judd, Kenneth, *Numerical Methods in Economics*, MIT Press, Cambridge, 1998.

[26] Kaldor, Nicholas , "Capital Accumulation and Economic Growth", in Friedrich A. Lutz and Douglas C. Hague, eds., Proceedings of a Conference Held by the International Economics Association, London, Macmillan.(1963)

[27] Kongsamut, Piyabha , Sergio Rebelo and Danyang Xie, "Beyond Balanced Growth", *Review of Economic Studies*, LXVIII (2001), 869-882.

[28] Kortum, Samuel, "Research, Patenting and Technological Change" *Econometrica*, LXV (1997), 1389-1431.

[29] Krusell, Per; Lee Ohanian and Victor Rios-Rull and Giovanni Violante, "Capital Skill Complementary and Inequality", *Econometrica*, LXIIX (2000), 1029-1053.

[30] Kuznets, Simon, "Quantitative Aspects of the Economic Growth of Nations: II, Industrial Distribution of National Product and Labour Forcce", *Economic Development and Cultural Change*, V Supplement (1957).

[31] Kuznets, Simon, "Modern Economic Growth: Findings and Reflections", *American Economic Review*, LXIII (1973), 829-846.

[32] Laitner, John, "Structural Change and Economic Growth", *Review of Economic Studies*, LXVII (2000), 545-561.

[33] Lucas, Robert E., "On the mechanics of economic development." *Journal of Monetary Economics*, XXII (1988), 3-42.

[34] Matsuyama, Kiminori, "Agricultural Productivity, Comparative Advantage and Economic Growth" *Journal of Economic Theory* LVIII (1992), 317-334.

[35] Matsuyama, Kiminori, "Structural Change" forthcoming *New Pelgrave Dictionary of Economics* (2005).

[36] Mairesse, Jacques, Bronwyn H. Hall and Benoit Mulkay, "Firm-Level Investment in France and the United States: An Exploration over What We Have Returned in Twenty Years" *Annales d'Economie et Statistiques*, LV (1999), 27-67.

[37] McKenzie, Lionel W., "Turnpikes: Richard Ely Lecture" *American Economic Review*, (1998).

[38] Murphy, Kevin, Andrei Shleifer and Robert Vishny, "Income Distribution, Market Size and Industrialization" *Quarterly Journal of Economics* CIV (1989), 537-564.

[39] Nadiri, M. I. "Some Approaches to Theory and Measurement of Total Factor Productivity: a Survey" *Journal of Economic Literature*, VIII, (1970), 1117-77.

[40] Nerlove, Mark, "Recent Empirical Studies of the CES and Related Production Functions" in M. Brown (editor) *The Theory and Empirical Analysis of Production* New York (1967).

[41] Ngai Rachel and Christopher Pissarides (2006), "Structural Change in a Multi-Sector Model of Growth", London School of Economics, mimeo.

[42] Radner, Roy "Paths of Economic Growth That Are Optimal With Regard Only to Final States" *Review of Economic Studies*, XXVIII (1961), 98-104.

[43] Romer, Paul M., "Increasing Returns and Long-Run Growth" *Journal of Political Economy*, XCIV (1986), 1002-1037.

[44] Romer, Paul M., "Endogenous Technological Change" *Journal of Political Economy*, IIC (1990), S71-S102.

[45] Rybczynski, Tadeusz, "Factor Endowments and Relative Commodity Prices" *Economica*, XXII (1955), 336-341.

[46] Scheinkman, José A. "On Optimal Steady States of n-Sector Growth Models When Utility Is Discounted" *Journal of Economic Theory*, XII (1976), 11-20.

[47] Segerstrom, Paul S., T. C. A. Anant, and Elias Dinopoulos, "A Schumpeterian Model of the Product Life Cycle" *American Economic Review* LXXX (1990), 1077-1092.

[48] Young, Alwyn "Invention and Bounded Learning by Doing" *Journal of Political Economy* CI (1993), 443-472.

[49] Young, Alwyn "Growth without Scale Effect" *Journal of Political Economy*, CVI (1998), 41-63.

[50] Zuleta, Hernando and Andrew Young "Labor's Shares—Aggregate and Industry: Accounting for Both in a Model with Induced Innovation" University of Mississippi, mimeo, 2006.