Vibration Analysis and Control of Dynamics Effects of Moving Vehicles over Bridges

by

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ABSTRACT

An extensive review and evaluation of the optimal models to asses the dynamic effects of moving vehicle on a bridge is performed. These models, with increasing grades of complexity, represent the best approximation of the problem under certain assumptions. Case studies are also performed with these models.

In addition to the dynamic analyses, which allow evaluations of the problem and gives valuable hints on the optimal design of bridges subjected to moving vehicles, vibration control devices such Tuned Mass Dampers (TMD) and Multiple Tuned Mass Damper (MTMD) are presented. Results from different authors are included, which allow assessing the applicability and effectiveness of these methods.

Thesis Supervisor: Eduardo A. Kausel

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1. Introduction

The vibrations caused by the passage of vehicles have become an important consideration in the design of bridges. The reasons for this are twofold:

- The stresses increase above those due to a static application of the load.
- An excessive vibration may be noticeable to persons on the bridge. This may have a psychological effect of mistrust on the users of the structure.

With the rapid development of high-speed railways, the dynamic response of railway bridges has received much attention from researchers. To evaluate the complex interaction between the vehicle and the bridge, the vehicle can be modeled as a moving force, as a moving mass or as a moving suspension mass.

In order to control excessive vibrations in bridges under moving loads, it is first necessary to understand how exactly the bridge behaves.

In Part I of this thesis we present various dynamic analysis methods of increasing degree of complexity. While in the literature we can find many different formulations to this problem, here we shall restrict our attention to the simplest ones that suffice to model the problem with engineering accuracy. Through the knowledge of how the various physical parameters affect the dynamic behavior of a bridge, we can maintain its response within acceptable limits.

In Part II, we review various methodologies and devices to control or ameliorate vibrations of the bridge, and present practical examples of their effectiveness.

PART I. DYNAMIC ANALYSIS

2. Vehicles models

According to Au, Cheng, Cheung [1], the interaction problem between moving vehicles and the bridge structures can be approached in four different ways:

- Moving-force model, in which the vehicle is modeled as a force moving along the bridge. The dynamic response of the bridge under the action of a vehicle is captured by this model, although the interaction between the vehicle and the bridge is not considered.
- Moving-mass model, in which the vehicle is represented as a moving mass. This is the most common method, appropriate when the mass of the vehicle is not negligible.
- Moving-vehicle model, in which the vehicle is modeled as a mass with a spring and a damper. It considers the vibrations of the moving mass, which is significant in the presence of road surface irregularities for vehicles running at high speeds.
- Moving-vehicle model considering vehicle's pitching effects presented by Yang, Chang and Yau [2], in which the vehicle is modeled as a rigid beam supported by two spring-dashpot units. It can simulate the pitching effect of the car body on the vehicle and on the response of the bridge. It has been demonstrated that for the case with no track irregularities, the pitching effect can be neglected. However if the dynamic behavior of the vehicle is of major concern, then the pitching effect cannot be generally neglected. Furthermore, the vehicle response will be enormously amplified in the presence of track irregularities. Therefore it is unsafe to disregard this effect in high-speed trains from the point of view of design and also the riding comfort of passengers.

3. Bridge models

A beam model can be adopted in the vast majority of problems to explain the essential characteristics of a bridge.

A plate model may be necessary for slab bridges and in cases in which the movement of the vehicle is not along the centerline. In these cases the modal superposition method may require less computational effort than the commonly used finite element method (FEM).

4. Significant parameters

The most significant parameters affecting the response of the bridge are:

- Natural frequencies of the bridge
- Vehicle velocity
- Relative position of the vehicle on the bridge
- Damping characteristics of the vehicle and the bridge.
- Road irregularities and roughness

5. Analysis of two high-speed trains crossing on a bridge

In cases of bridges with two tracks, it is important to check the response of the vehicle-bridge interaction elicited by trains moving in opposite directions. Lin and Ju [3] used a three-dimensional finite element analysis of a bridge and concluded that the maximum dynamic response caused by two-ways trains is at most twice that of a one way train, and it occurs when both trains travel at the same speed. Finite element results also indicate that the averaged response ratio in the three global directions is about 1.65 when the two-way trains run at the same speed.

6. Modeling a train over a bridge as a beam under a moving load: analytical solution

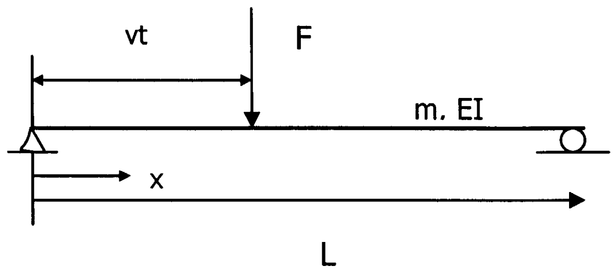


Figure 1 Bridge with a vehicle modeled as a force load.

The analytical solution for the modal response a simple supported beam, subjected to concentrated load, at a point x_F, position of the force in the spam, neglecting damping is:

with

$$\ddot{U}_n + \omega_n^2 \cdot U_n = \frac{F \cdot \phi_n(x_F)}{\int_0^l m \cdot [\phi_n(x)]^2 \, dx} \tag{1}$$

If it is a simply supported beam and prismatic with m = linear mass, the eigenvectors ϕ_n and the eigenvalues ω_n are:

$$\phi_n(x) = \sin\left(\frac{n\pi x}{l}\right) \tag{2}$$

$$\omega_n = \frac{n^2\pi^2}{l^2} \cdot \sqrt{\frac{EI}{m}} \tag{3}$$

For a moving load with constant velocity v, $x_F = vt$ hence, the governing equation of the vertical displacement of the bridge is:

$$\ddot{U}_n + \omega_n^2 \cdot U_n = \frac{2 \cdot F}{m \cdot l} \sin \frac{n\pi \cdot vt}{l} \tag{4}$$

According to Biggs [4], the modal solution for this equation is:

$$U_n = \frac{2 \cdot F}{m \cdot l \cdot \omega_n^2} \cdot (DLF)_n \tag{5}$$

The DLF or Dynamic Load Function is defined as the ratio of the dynamic and static responses.

We see now that this case is the same as that of a single beam without damping, subjected to a sinusoidal forcing function. We solve for the general case:

$$M\ddot{y} + ky = F\sin\Omega t \xrightarrow{Solution} y = C_1 \sin\omega t + C_2 \cos\omega t + \frac{F}{M} \cdot \frac{\sin\Omega t}{\omega^2 - \Omega^2} \tag{6}$$

with initial conditions:

$$y_0 = 0 = C_1 \sin 0 + C_2 \cos 0 + \frac{F}{M} \cdot \frac{\sin 0}{\omega^2 - \Omega^2} \tag{7}$$

$$\dot{y}_0 = 0 = C_1 \omega \cos 0 - C_2 \omega \cos 0 + \frac{F \cdot \Omega}{M} \cdot \frac{\cos 0}{\omega^2 - \Omega^2} \tag{8}$$

Solving C_1 and C_2 and plugging into the general solution, we get the response y:

$$y = \frac{F}{M} \cdot \left(\frac{\sin\Omega t}{\omega^2 - \Omega^2} - \frac{\Omega}{\omega} \frac{\sin\Omega t}{\omega^2 - \Omega^2} \right) \tag{9}$$

From here we get that in our case, the Dynamic Amplification Function is:

$$DLF = \frac{1}{\left(1 - \frac{\Omega_n{}^2}{\omega_n{}^2}\right)} \cdot \left(\sin\Omega_n t - \frac{\Omega_n}{\omega_n} \sin\omega_n t \right) \qquad \text{with} \quad \Omega_{\text{n}} = \frac{n\pi \cdot v}{l} \tag{10}$$

Finally, the response y of any point x of the bridge through time is :

$$y = \sum_{n=1}^{N} U_n \phi_n(x) \tag{11}$$

$$y(x) = \frac{2F}{ml} \cdot \sum_{n=1}^{N} \frac{1}{\left(\omega_n{}^2 - \Omega_n{}^2\right)} \cdot \left(\sin\Omega_n t - \frac{\Omega_n}{\omega_n} \sin\omega_n t \right) \cdot \sin\frac{\text{n}\pi \cdot x}{1} \tag{12}$$

These equations only apply when the force is still on the span. After the force has passed, the response is a free vibration with initial conditions equals to the conditions in the beam at the moment when the force leaves the span.

We can see from the equations that the response under a moving load can be critical if the velocity of the load is such that Ω_n similar to ω_n.

We will have resonance of the loads with the bridge if:

$$\frac{\pi \cdot v}{l} = \omega_n \xrightarrow{\text{velocity of resonance with the structure}} v = \frac{l \cdot \omega_n}{\pi} \tag{13}$$

We also see that the largest participation is generally in the first mode, because the higher modes have larger frequencies ω_n, which would in turn require larger velocities to attain resonance.

Nevertheless, if the velocity of the load approaches the velocity of resonance, we will get unacceptable deflections, which can be critical in design.

The DLF_n difference between static and dynamic response for the case of a simple supported beam can be expressed in each mode as:

$$DLF_n = \frac{1}{\left(\left(\frac{n^2\pi^2}{l^2} \cdot \sqrt{\frac{EI}{m}}\right)^2 - \left(\frac{n\pi \cdot v}{l}\right)^2\right)} \cdot \left(\sin\left(\frac{n\pi \cdot v}{l}\right)t - \frac{\left(\frac{n\pi \cdot v}{l}\right)}{\left(\frac{n^2\pi^2}{l^2} \cdot \sqrt{\frac{EI}{m}}\right)} \sin\left(\frac{n^2\pi^2}{l^2} \cdot \sqrt{\frac{EI}{m}}\right)t\right) \tag{14}$$

We will conduct an analysis of the response of the bridge in each mode, so that we can, from here on, neglect those modes that don't produce a significant response.

6.1. Case study:

If we consider a typical railway bridge [5] with the following properties:

$$L = 50\ m$$
$$v_{vehicle} = 90[m/sg]$$
$$E = 3.303*1010\ [N/m^2]$$
$$m = 43650.864\ [kg/m]$$
$$I = 18.638\ [m4]$$

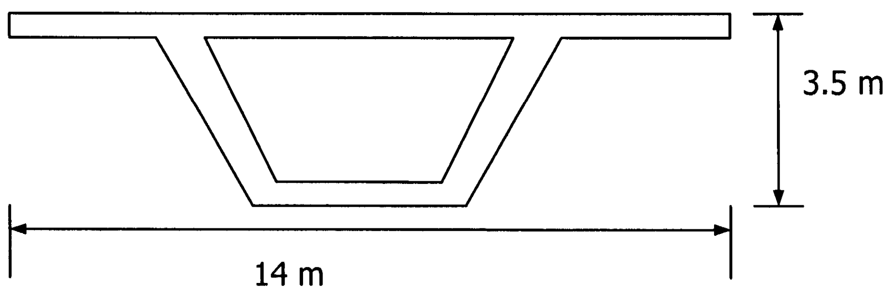
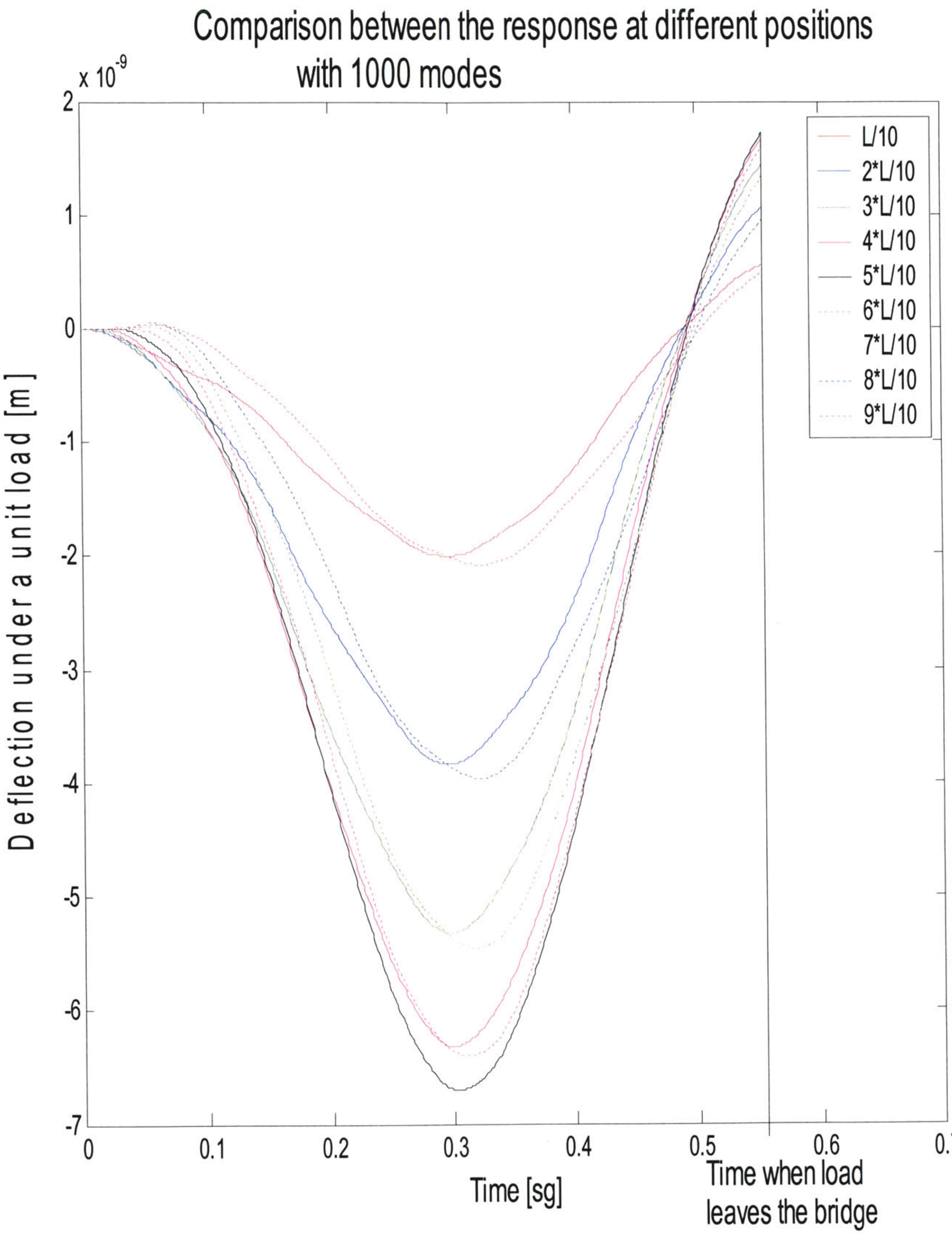


Figure 2. Cross section of the bridge.

With the formulas given previously, a MATLAB program has been developed (see Appendix A) that gives the response of any point of the bridge. With that program, the following figure is drawn, in order to compare the deflection through the bridge in time.

We conclude that the point $x = L/2$ is in where the deflection is largest, so the midspan is the most appropriate location to install a control device to limit the deflection.


Comparison between the response at different positions
with 1000 modes
x 10^-9
2
1
0
-1
-2
-3
-4
-5
-6
-7
Deflection under a unit load [m]
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
Time [sg]
Time when load
leaves the bridge
L/10
2*L/10
3*L/10
4*L/10
5*L/10
6*L/10
7*L/10
8*L/10
9*L/10

We can see the response in each mode for the midspan position:

N	ωn	Ωn	Max response	Max response/max response of first mode
1	2495.402135	6.981247195	1.66328E-12	100.000 %
2	9981.60854	13.96249439	1.27792E-29	0%
3	22458.61922	20.9437416	2.0489E-14	1.2318%
4	39926.4342	27.9249888	1.5963E-30	0 %
5	62385.0534	34.906236	2.6289E-15	0.1581%
6	89834.4769	41.8874832	4.7047E-31	0%
7	122274.705	48.8687304	6.8424E-16	0.0411%
8	159705.737	55.8499776	1.9937E-31	0%
9	202127.573	62.8312248	2.4178E-16	0.0145%
10	249540.214	69.8124719	3.9243E-31	0%

We notice in the graphs that the first mode response is large compared with the rest and that the total response can be considered, with a very small error to be the response of the first mode.

If a high accuracy is needed and we want to include more modes, we can also note that the third mode response is also larger than the subsequent ones, so only the first and the third modes suffice for calculations, and the rest can be neglected.

In the third graph, the response considering only 1 mode is plotted against the response taking into account the first 1000 modes. We can appreciate the insignificant differences between them.

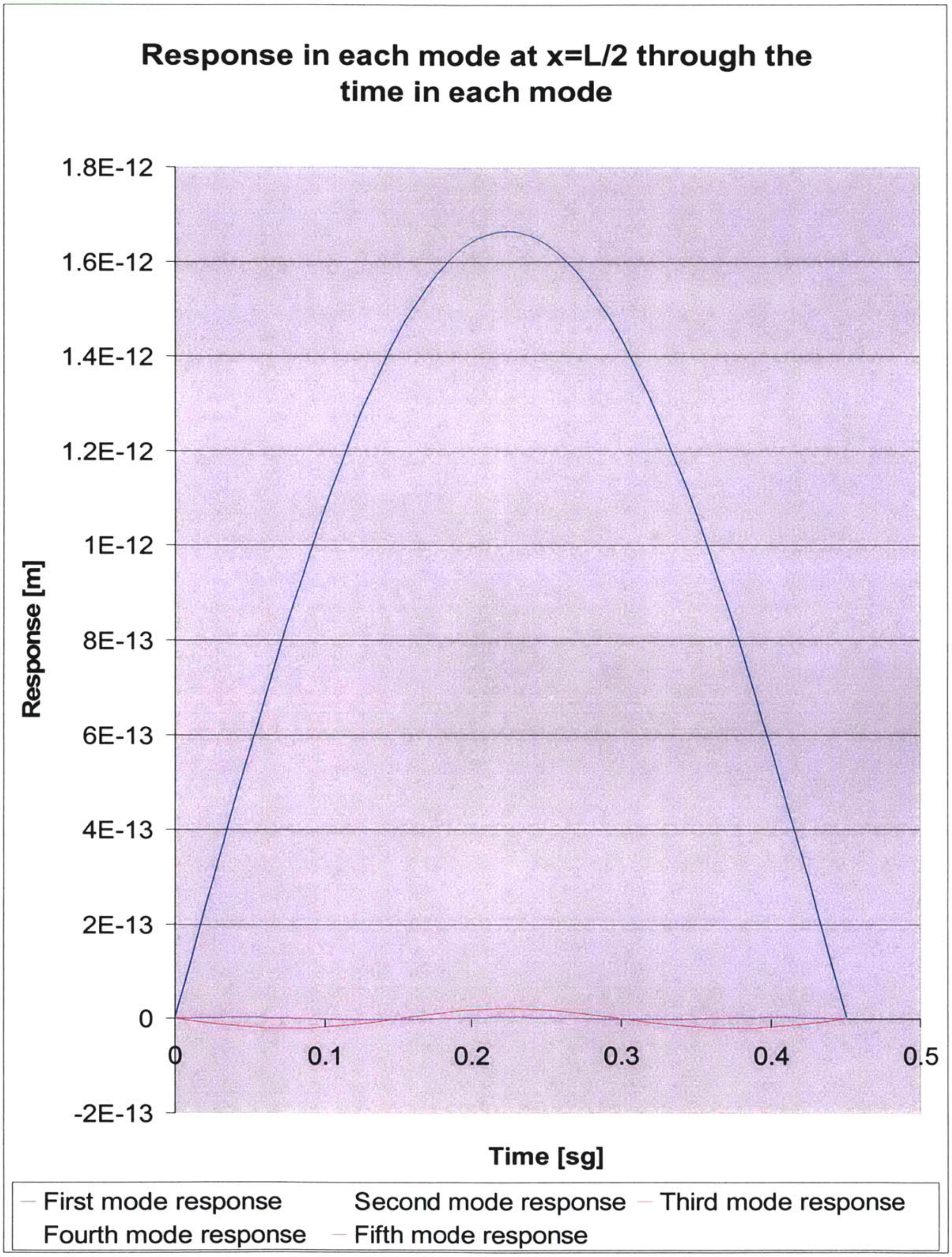


Figure 3. Modal response of the bridge for the given excitation.

If we see only the odd modes larger than the first one, and we plot it in a different scale than in the previous figure:

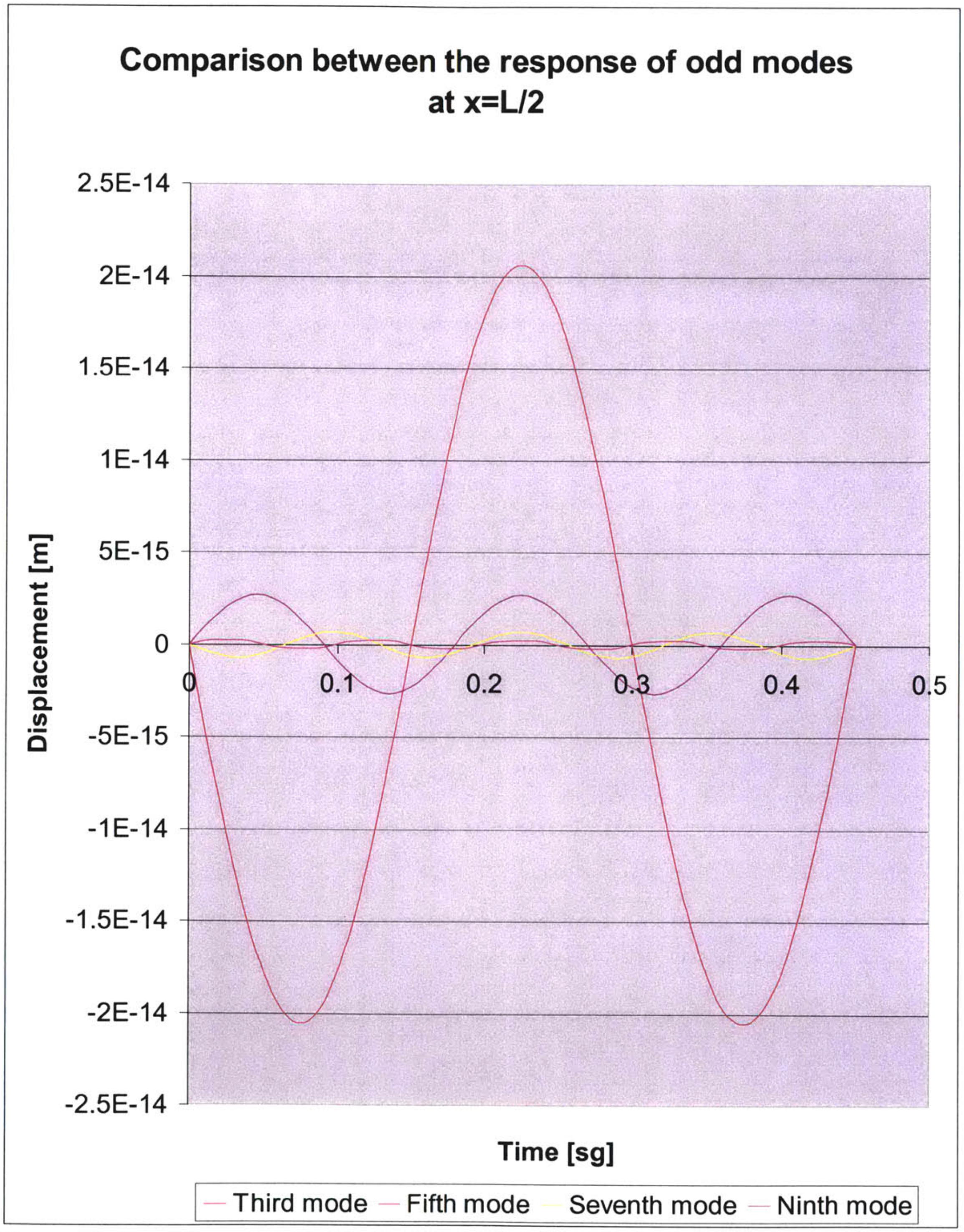


Figure 4. Responses of the bridge in the third, fifth , seventh and ninth mode.

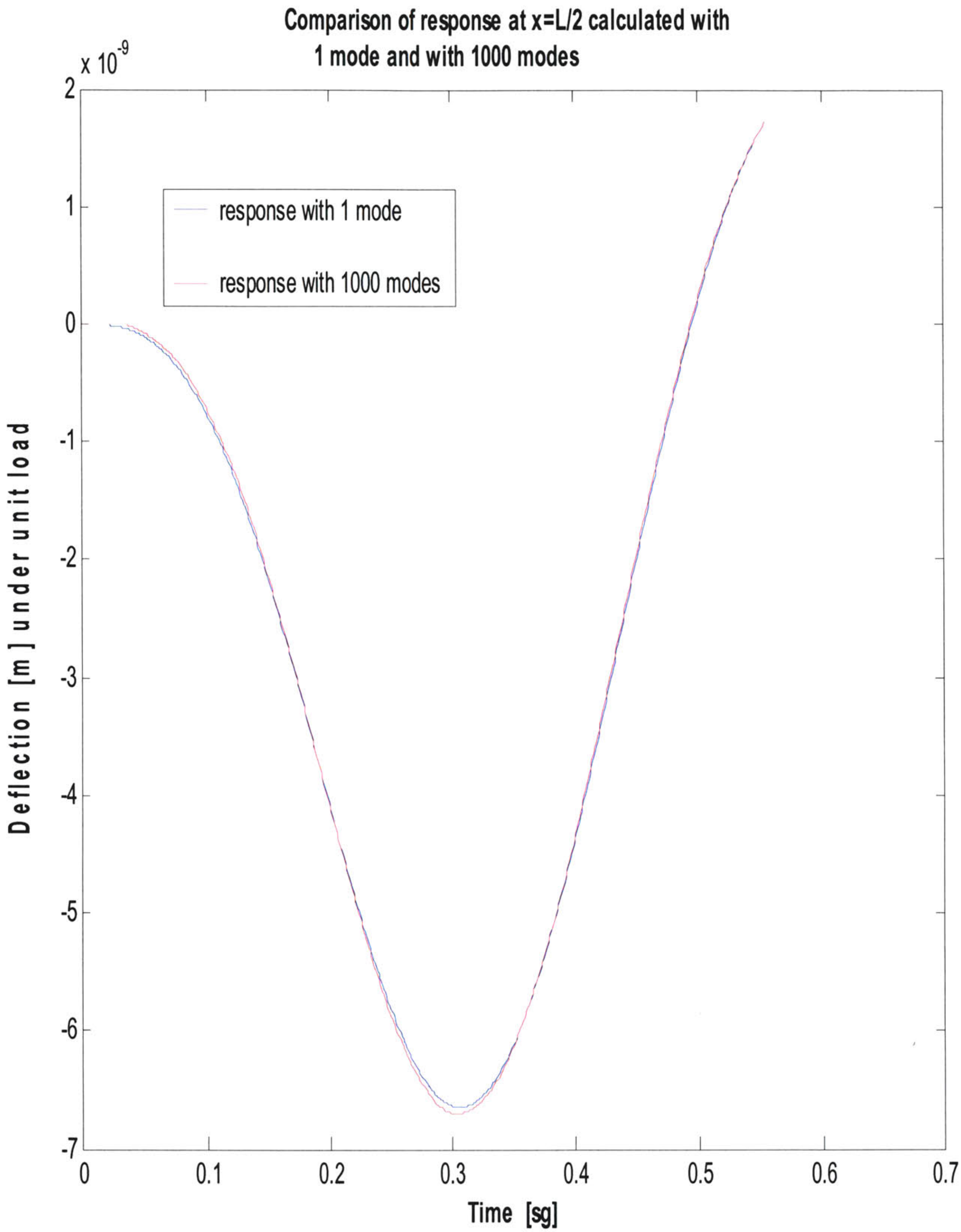


Figure 5. Comparison of the deflection calculated with one and 1000 modes.

It was shown before that, for practical reasons, only the fundamental mode of the bridge will be excited.

We include here an abacus that gives the maximum dynamic amplification factor for any given known parameters E, I, m, L and v, which can be useful for preliminary design. The MATLAB program used to create this abacus is included in appendix B.

If $v/L = 1\ to\ 5$ the dynamic amplification factor is less than 1:

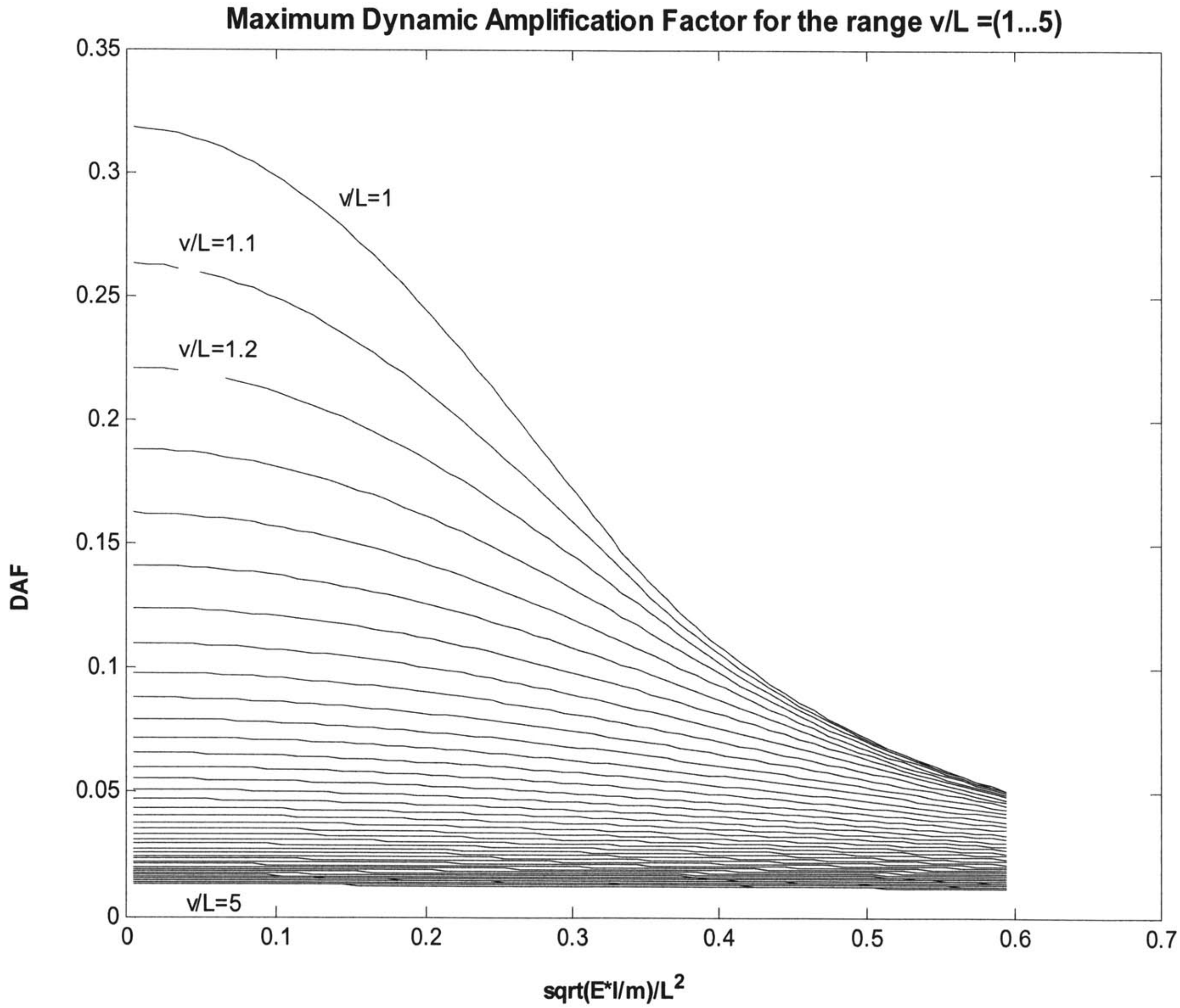


Figure 6. Design Abacus: DLF for a simple supported beam. Overview.

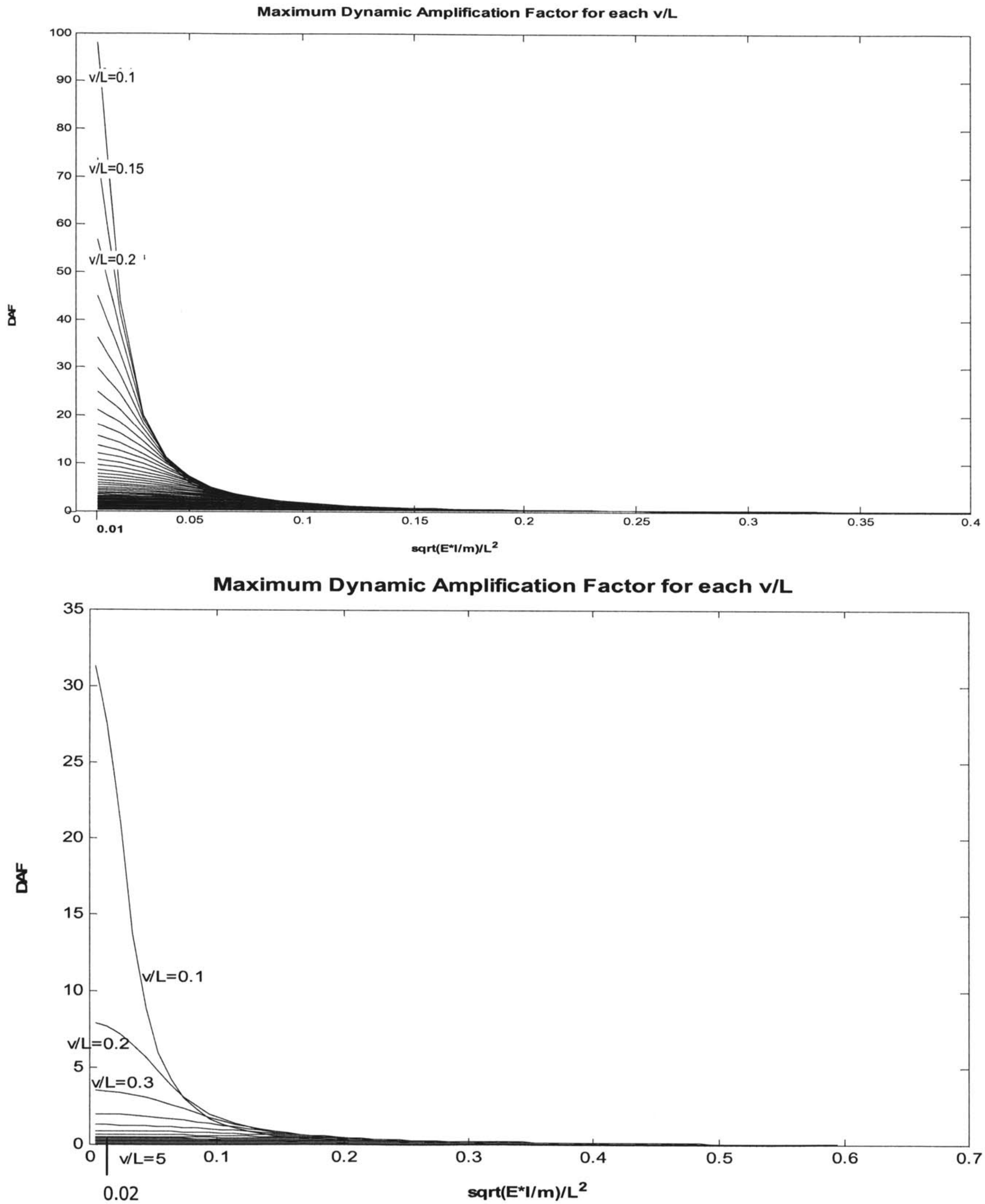
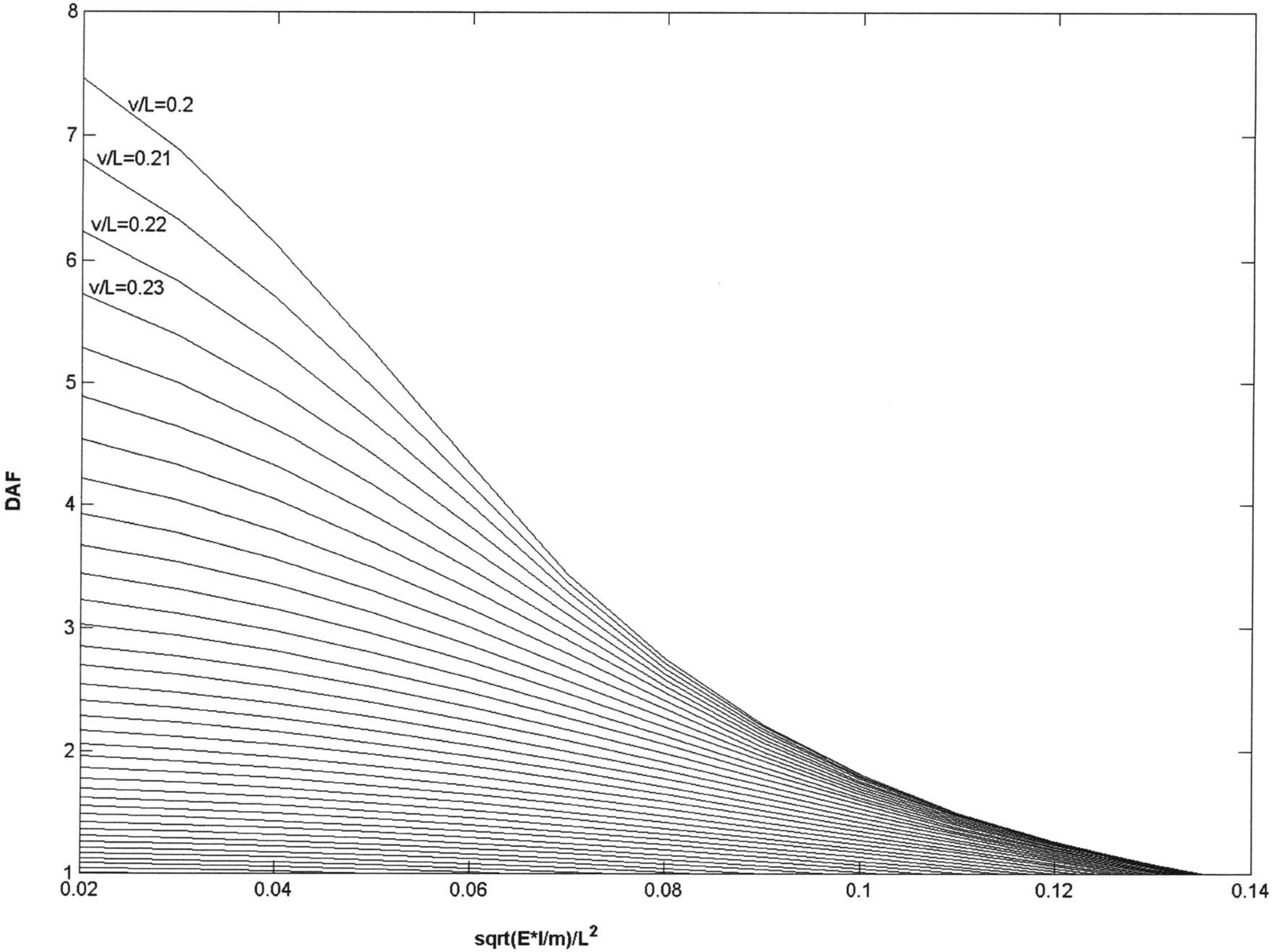


Figure 7. Design Abacus: DLF for a simple supported beam.

Maximum Dynamic Amplification Factor for each v/L



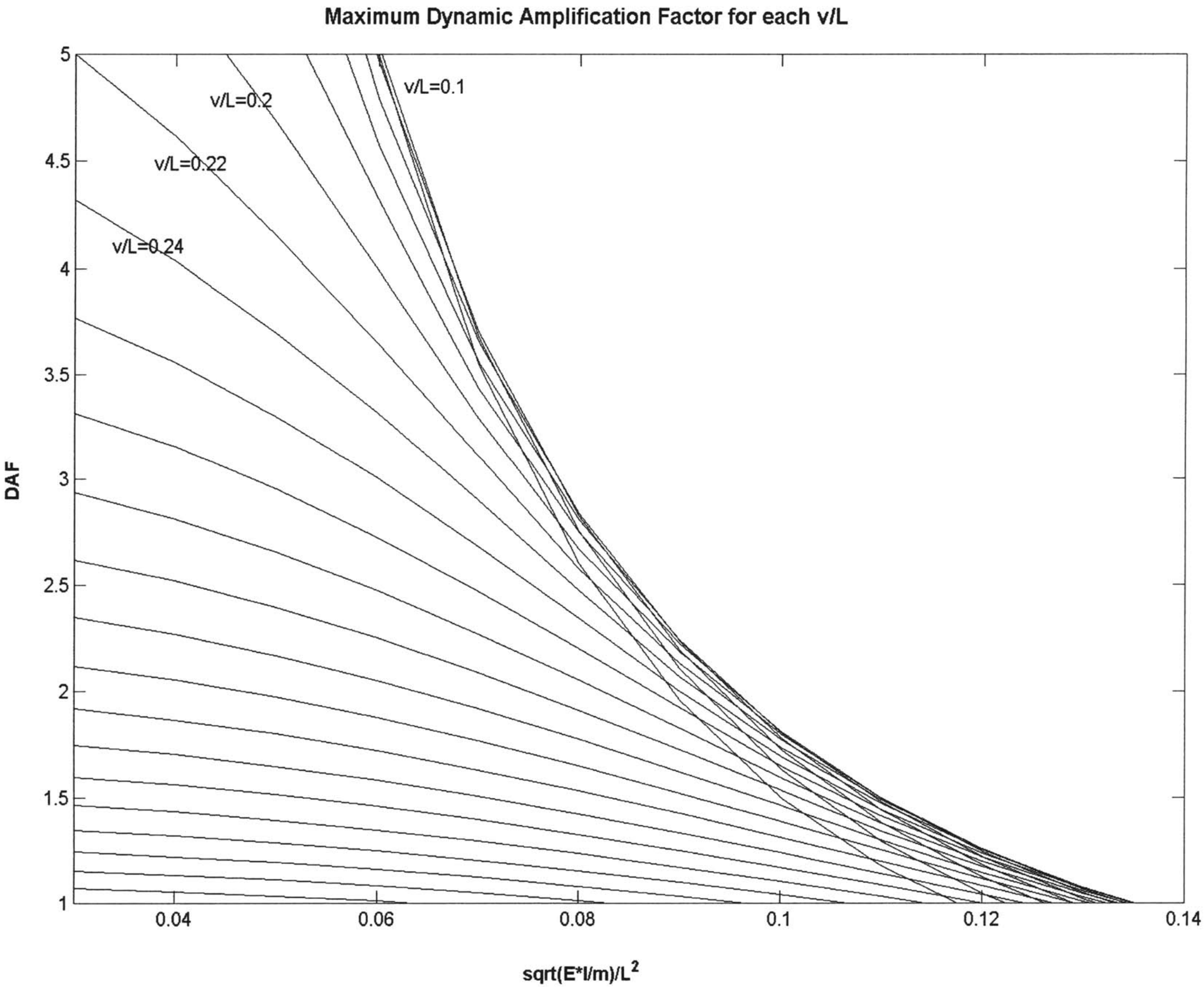


Figure 8 Zoomed Design Abacus: DLF for a simple supported beam.

7. Modeling a train over a bridge as a beam under a rolling mass: analytical solution

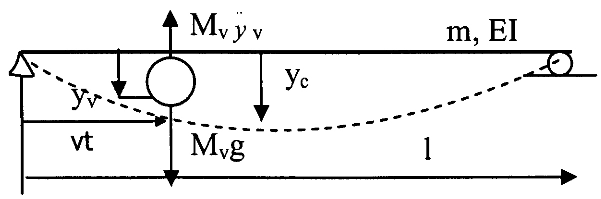


Figure 9. Bridge subjected to a moving mass.

The simplified analysis previously presented is sufficient, if the mass of the train is small compared to the mass of the bridge.

For more accurate solutions, we must consider the effect of the mass of the train on the response of the bridge.

Following Biggs [4], the actual force applied to the beam at any instant, assuming that the train to be always in contact with the bridge:

$$Force = M_v g - M_v \ddot{y}$$

The equation of the deflection of the bridge in terms of the modes is:

$$\ddot{U}_n + \omega_n^2 \cdot U_n = \frac{2\cdot M_v}{m\cdot l}\cdot\left(g - \ddot{y}_v\right)\sin\frac{n\pi\cdot vt}{l} \tag{15}$$

y_v is the deflection of the point where the mass is, considering that y_v is:

$$\ddot{y}_v = \sum_{n=1}^{N}\ddot{U}_n\cdot\sin\frac{n\pi\cdot vt}{l} \tag{16}$$

Plugging (16) into the general equation:

$$\ddot{U}_n + \left(\frac{2\cdot M_v}{m\cdot l}\cdot\sin\frac{n\pi\cdot vt}{l}\right)\cdot\left(\sum_{n=1}^{N}\ddot{U}_n\cdot\sin\frac{n\pi\cdot vt}{l}\right) + \omega_n^2\cdot U_n = \frac{2\cdot M_v\cdot g}{m\cdot l}\cdot\sin\frac{n\pi\cdot vt}{l} \tag{17}$$

As was explained before, we can neglect the effect of the modes other than the first one. If we also consider $U_1 = y_c$, (midspan deflection), we get:

$$\ddot{y}_c\cdot\left(1 + \frac{2\cdot M_v}{m\cdot l}\cdot\sin^2\frac{n\pi\cdot vt}{l}\right) + \omega_1^2\cdot y_c = \frac{2\cdot M_v\cdot g}{m\cdot l}\cdot\sin\frac{n\pi\cdot vt}{l} \tag{18}$$

8. Modeling a train over a bridge as a beam subjected to a moving sprung mass: analytical solution

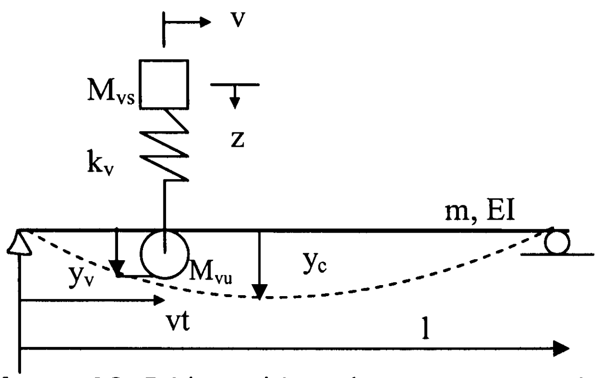


Figure 10. Bridge subjected to a sprung moving mass

This case takes into account the interaction between the vehicle and the bridge. The train is modeled as a mass M_{vu} that always stays in contact with the bridge, a spring k_v and an unsprung mass M_{vs}. According to Biggs [4], the force applied to the beam from the idealized vehicle is:

$$Force = M_{vu}(g - \ddot{y}_v) + [k_v(z - y_v) + M_{vs}g]$$

z is the absolute deflection of the mass M_{vs}, it is measured from the neutral spring position. We substitute this force in the modal equation of the motion:

$$\ddot{U}_n + \omega_n^2 \cdot U_n = \frac{2 \cdot \left(\mathrm{M_{vu}(g - \ddot{y}_v) + [k_v(z - y_v) + M_{vs}g]}\right)}{m \cdot l} \sin\frac{n\pi \cdot vt}{l} \tag{19}$$

We know also, that y_v is:

$$y_v = \sum_{n=1}^{N} U_n \sin\frac{n\pi \cdot vt}{l} \tag{20}$$

$$\ddot{y}_v = \sum_{n=1}^{N} \ddot{U}_n \sin\frac{n\pi \cdot vt}{l} \tag{21}$$

If we rearrange the equation of motion:

$$\frac{ml}{2}\ddot{U}_n + \left(M_{vu}\sin\frac{n\pi\cdot vt}{l}\right)\cdot\left(\sum_{n=1}^{N}\ddot{U}_n\sin\frac{n\pi\cdot vt}{l}\right) + \frac{ml\omega_n^2}{2}\cdot U_n$$
$$= \left[(M_{vu}+M_{vs})\cdot g + k_v\left(z-\sum_{n=1}^{N}U_n\sin\frac{n\pi\cdot vt}{l}\right)\right]\sin\frac{n\pi\cdot vt}{l} \qquad (22)$$

This system has one degree of freedom more than the previous ones, the additional equation, is the dynamic-equilibrium of the mass M_{vs}:

$$M_{vs}\ddot{z} + k_v\left(z-\sum_{n=1}^{N}U_n\sin\frac{n\pi\cdot vt}{l}\right) = 0 \qquad (23)$$

These set of equations cannot be solved analytically, but numerically for the modes of the beam and the sprung mass.

As we have observed before, the first mode dominates the response of the structure. If we want to have a simplified set of equations, if suffices to include only one beam mode. Considering that $y_c = U_1$, we can plug it into (22):

$$\left(\frac{ml}{2}\ddot{U}_n + M_{vu}\sin^2\frac{n\pi\cdot vt}{l}\right)\cdot\ddot{y}_c + \frac{ml\omega_1^2}{2}\cdot y_c = \cdot\left[W_{vt} + k_v\cdot\left(z - y_c\sin\frac{\pi\cdot vt}{l}\right)\right]\sin\frac{\pi\cdot vt}{l}$$

(24)

$$M_{vs}\ddot{z} + k_v\cdot\left(z - y_c\sin\frac{\pi\cdot vt}{l}\right) = 0 \qquad (25)$$

We must keep in mind that to apply these equations to the vibration analysis of a bridge under a heavy vehicle we are making certain assumptions [4]:

- The bridge consists of a floor system and girder may be modeled by a single beam of equivalent rigidity.
- Only the fundamental mode of the bridge need be taken into account (we have already proved the validity of this assumption)
- The vehicle, although having two or more axles and various systems of springs and dampers, may be considered as a one-degree-of-freedom system.
- The entire weight of the vehicle is applied to the bridge at the center of vehicle mass, rather than at the actual wheels.

9. General solution of the vibration of multi-span bridges under N moving sprung masses

This method was presented by Cheung, Au, Zheng and Cheng in [6]. It is a general formulation that allows the calculation of bridges of n spans with N vehicles idealized as two degree of freedom systems over them. It is based on the Lagrangian approach using modified beam vibration functions.

The great advantage of this method is that the total number of unknowns is really small compared with the classical finite element method used by other authors. The convergence is fast and it generally needs less than 20 terms to give good results.

All the formulae can be expressed in matrix form, and therefore is very easy to implement in a code.

A multi-span bridge can be modeled as a continuous linear elastic Bernoulli-Euler bridge with $Q+1$ point supports. The N vehicles that drive through the bridge are modeled as systems of two degree of freedoms with M_{s1}, M_{s2}, c_s, k_s, where $s = 1,2,...,N$. that drive with a velocity $v(t)$ across the bridge.

Each vehicle is composed by a unsprung mass M_{s1} and a sprung mass M_{s2} connected by a spring k_s and a dashpot c_s.

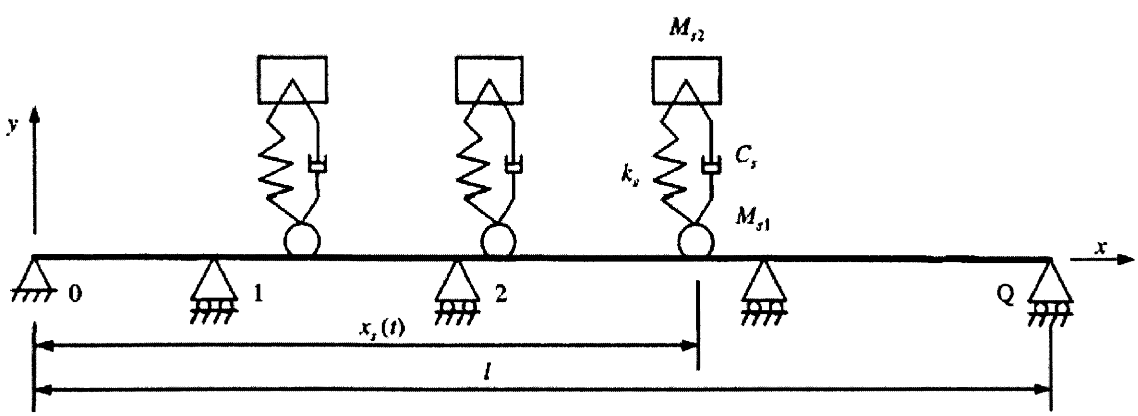


Figure 11. Bridge of Q spans with N vehicles. Capture from [6]

We define the degree of freedoms of the system:

- $w(x,t) =$ deflection of the bridge, > 0 if it is upwards.
- $y_{s1}(t) =$ vertical displacement of the unsprung mass M_{s1}
- $y_{s2}(t) =$ vertical displacement of the sprung mass M_{s2} , both $y_{s1}(t)$ and $y_{s2}(t)$ are measured from their equilibrium position.
- $r(x) =$ roughness of the bridge, vertically upward difference from the mean horizontal profile

We can see that the $y_{s1}(t)$, displacement of the unsprung mass is:

$$y_{s1}(t)=[w(x,t)+r(x)]_{x=x_s(t)} \tag{26}$$

$$\frac{dy_{s1}(t)}{dt}=\left[\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial x}+v\frac{dr}{dx}\right]_{x=x_s(t)} \tag{27}$$

$$\frac{d^2y_{s1}(t)}{dt^2}=\left[\frac{\partial^2 w}{\partial t^2}+2v\frac{\partial^2 w}{\partial x\partial t}+v^2\frac{\partial^2 w}{\partial x^2}+a\frac{\partial w}{\partial x}+v^2\frac{d^2w}{dx^2}+a\frac{dr}{dx}\right]_{x=x_s(t)} \tag{28}$$

The force in the contact between vehicle and bridge $f_{cs}(t)$ can be expressed by:

$$f_{cs}(t)=(M_{s1}+M_{s2})\cdot g+\Delta f_{cs}(t) \tag{29}$$

With g = gravity acceleration and $\Delta f_{cs}(t)$ is the variation of the contact force in time.

The equilibrium of forces in the vertical direction for M_{s1} and M_{s2} are:

$$M_{s1}\frac{d^2y_{s1}(t)}{dt^2}=k_s[y_{s2}(t)-y_{s1}(t)]+c_s\left[\frac{dy_{s2}(t)}{dt}-\frac{dy_{s1}(t)}{dt}\right]_{x=x_s(t)}+\Delta f_{cs}(t) \tag{30}$$

$$M_{s2}\frac{d^2y_{s1}(t)}{dt^2}=-k_s[y_{s2}(t)-y_{s1}(t)]+c_s\left[\frac{dy_{s2}(t)}{dt}-\frac{dy_{s1}(t)}{dt}\right]_{x=x_s(t)} \tag{31}$$

From (29) to (31), the $f_{cs}(t)$ the contact force can be expressed as:

$$f_{cs}(t)=(M_{s1}+M_{s2})\cdot g+M_{s1}\frac{d^2y_{s1}(t)}{dt^2}+M_{s2}\frac{d^2y_{s2}(t)}{dt^2} \tag{32}$$

If we express the vibration of the bridge as the summation of n generalized modes:

$$w(x,t)=\sum_{i=1}^{n} q_i(t)X_i(x) \tag{33}$$

x are the generalized coordinates

$X_i(x)$, $i=1,2,\ldots,n$ are the assumed vibration modes that satisfy the boundary same boundary conditions in all the supports:

$$X_i(x)=\overline{X_i}+\widetilde{X}_i(x) \tag{34}$$

$\overline{X_i}(x)$, $i=1,2,\ldots,n$ are the vibration modes of a beam of total length l of the bridge with the same end supports.

$\widetilde{X}_i(x)$, $i=1,2,\ldots,n$ are cubic spline expressions (or trial functions) that are so chosen that $X_i(x)$ satisfies the boundary conditions at all supports.

We can used the well known Lagrangian equations with the Lagrangian function L, being $Q^*_{is}(t)$ the generalized force we have:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right)-\frac{\partial L}{\partial q_i}=\sum_{s=1}^{N} Q^*_{is}(t), i=1,2,\ldots,h \tag{35}$$

$$Q^*_{is}(t)=-f_{cs}(t)X_i(x)_{x=x_s(t)} \tag{36}$$

Plugging equation (32) into this expression:

$$\begin{aligned}Q^*_{is}(t)=&-(M_{s1}+M_{s1})g\cdot X_i(x_s(t))-M_{s1}X_i(x_s(t))(v^2r''(x_s(t))+ar'(x_s(t)))\\&-M_{s1}X_i(x_s(t))\sum_{j=1}^{n}\{\ddot{q}_j(t)X_j(x_s(t))+2v\dot{q}_j(t)X'_j(x_s(t))+q_j(t)[v^2X''_j(x_s(t))+aX'_j(x_s(t))]\}\\&-M_{s2}\ddot{y}_{s2}X_i(x_s(t))\end{aligned} \tag{37}$$

The Lagrangian function L:

$$L=V-U \tag{38}$$

For this problem V and U are respectively:

$$V = \frac{1}{2}\int_0^l \rho A(x)\left[\frac{\partial w(x,t)}{\partial t}\right]^2 dx \tag{39}$$

$$U = \frac{1}{2}\int_0^l EI(x)\left[\frac{\partial^2 w(x,t)}{\partial x^2}\right]^2 dx \tag{40}$$

If we define m_{ij} and k_{ij} as:

$$m_{ij} = \int_0^l \rho A(x) X_i(x) X_j(x) dx \tag{41}$$

$$k_{ij} = \int_0^l EI(x) X''_i(x) X''_j(x) dx \tag{42}$$

Plugging into equations (39) and (40) the expression given in (33) and with the definitions (41) and (42):

$$V = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\dot{q}_i(t) m_{ij}\ddot{q}_j(t) \tag{43}$$

$$U = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} q_i(t) m_{ij} q_j(t) \tag{44}$$

With these expressions we can express the equation of motion of the bridge and the vehicles by using the Lagrange equation (35):

$$\begin{aligned} &\sum_{j=1}^{n} m^*_{ij}\ddot{q}_j(t) + \sum_{j=1}^{n} c^*_{ij}\dot{q}_j(t) + \sum_{j=1}^{n} k^*_{ij} q_j(t) + \sum_{j=1}^{n} M_{s2} X_i(x_s(t))\ddot{y}_{s2}(t) \\ &= p^*_i(t) \qquad i = 1,2,\ldots,n \end{aligned} \tag{45}$$

This is a compact notation, each term express:

$$m^*_{ij}(t) = m_{ij} + \sum_{s=1}^{N} M_{s1} K_i(x_s(t)) X_j(x_s(t)) \tag{46}$$

$$c^*_{ij}(t) = \sum_{s=1}^{N} 2v \cdot M_{s1} X_i(x_s(t)) X'_j(x_s(t)) \tag{47}$$

$$k^*_{ij}(t) = k_{ij} + \sum_{s=1}^{N} M_{s1} X_i(x_s(t))\left[v^2 X''_j(x_s(t) + aX'_j(x_s(t))\right] \tag{48}$$

$$p^*_i(t) = -\sum_{s=1}^{N}\left[(M_{s1} + M_{s2}) g X_i(x_s(t)) + M_{s1} X_i\left(x_s(t)\right)\left(v^2 r''\left(x_s(t)\right) + a r'\left(x_s(t)\right)\right)\right] \tag{49}$$

These formulae assume that N vehicles are on the bridge; if a vehicle leaves the bridge, it should be excluded from the summation.

The equation of motion of the masses M_{s2}, that is the motion of the passenger if there are not further suspension devices, can be derived from equation (31):

$$
\begin{aligned}
&-\sum_{j=1}^{N} c_s X_i(x_s(t))\dot{q}_j(t) - \sum_{j=1}^{N}\left[k_s X_j(x_s(t)) + vc_s X'_j(x_s(t))\right]\cdot q_j(t) + M_{s2}\ddot{y}_{s2}(t) \\
&+ c_s\dot{y}_{s2}(t) + k_s y_{s2}(t) = k_s r(x_s(t)) + vc_s r'(x_s(t)) \quad s = 1,2,\ldots,N
\end{aligned}
\tag{50}
$$

In order to implement this formulation in a code, it is best to express it in matrix form:

$$
\begin{aligned}
&\begin{bmatrix} M^* & XM_2 \\ 0 & M_2 \end{bmatrix}\begin{Bmatrix} \ddot{q} \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} C^* & 0 \\ -CX^T & C \end{bmatrix}\begin{Bmatrix} \dot{q} \\ \dot{y}_2 \end{Bmatrix} + \begin{bmatrix} K^* & 0 \\ -KX^T - vCX'^T & K \end{bmatrix}\begin{Bmatrix} q \\ y_2 \end{Bmatrix} \\
&= \begin{Bmatrix} p^* \\ Kr + vCr' \end{Bmatrix}
\end{aligned}
\tag{51}
$$

$$M^* = \left[m_{ij}^*(t)\right] \quad i,j = 1,2,\ldots,n \tag{52}$$

$$C^* = \left[c_{ij}^*(t)\right] \quad i,j = 1,2,\ldots,n \tag{53}$$

$$K^* = \left[k_{ij}^*(t)\right] \quad i,j = 1,2,\ldots,n \tag{54}$$

$$p^* = \left[p_i^*(t)\right] \quad i = 1,2,\ldots,n \tag{55}$$

$$M_2 = diag\left[M_{i2}\right] \quad i = 1,2,\ldots,n \tag{56}$$

$$C = diag\left[c_i\right] \quad i,j = 1,2,\ldots,n \tag{57}$$

$$K = diag\left[k_i\right] \quad i = 1,2,\ldots,n \tag{58}$$

$$X = \left[X_i(x_j(t))\right] \quad i = 1,2,\ldots,n, \quad j = 1,2,\ldots,N \tag{59}$$

The differential equation (51) can be solved with different integration techniques: central differential method, Newmark implicit method, Wilson-θ method...

An important advantage of this approach of the problem is that we don't have many degrees of freedom, hence the size of the matrix is relatively small and the computational effort is not large, regardless of the method used.

PART II VIBRATION CONTROL

10. Introduction

The great expansion of the cities and the automobile industry over the last two centuries has necessitated the development of better transportation structures, such as railways and highways. Because of the cost and scarcity of land, many transportation structures in urban areas have been built with bridges.

The development of stronger but at the same time, lighter materials in the last decades have produced more slender structures, and the bridges are not an exception in this trend

The flexibility of these bridges, combined with the increasing velocities of the new generation of high speed trains, can result in large deformations, potentially dangerous for the bridges and in annoying vibrations for the users of the vehicles.

11. Tuned Mass Dampers (TMD)

A tuned mass damper (TMD) is a device composed by a mass, a spring, and a damper that is attached to a structure in order to reduce its dynamic response.

The tuned mass dampers have turned out to be very effective to control the response of high buildings, and nowadays they have been installed in more than 300 buildings. One of their later applications is to reduce the vibration in bridges under high speed moving loads. The tuned mass damper (TMD) can be installed in a structure more easily than other control devices.

The TMD is a useful device to reduce the response when the external load is in resonance with the structure. However, its utility decreases outside resonant conditions.

As it was explained previously, the larger deflection in a single span bridge takes place in the middle of the span. That is the best location to place the TMD.

The TMD is only tuned to one frequency, however since the bridge has one principal frequency of resonance, its action is very effective if it is well tuned.

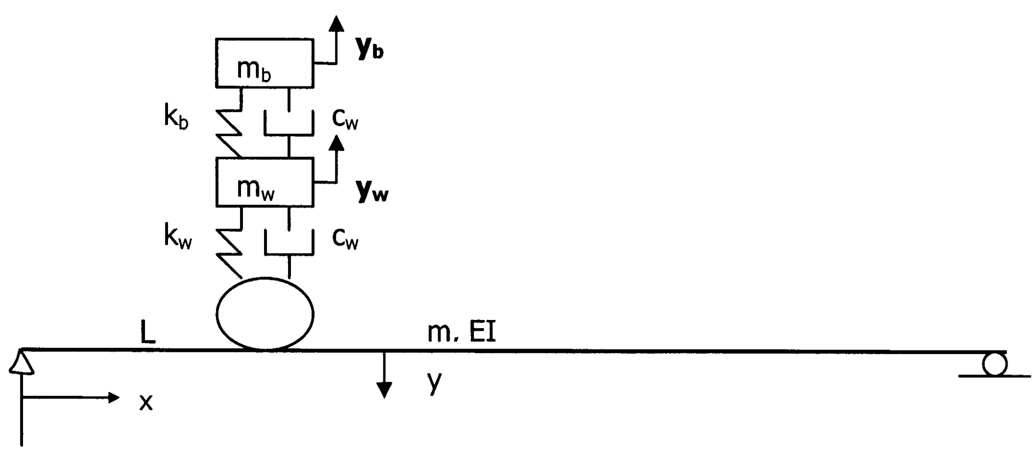


Figure 12. Model of a two spring mass vehicle.

Following the formulation given by [5] :

The equations of motion for the vehicle body y_b and wheel y_w:

$$m_w\ddot{y}_w + m_b\ddot{y}_b + c_w(\dot{y}_w + \dot{y}) + k_w(y_w - y) = 0 \tag{60}$$

$$m_b\ddot{y}_b + c_b(\dot{y}_b - \dot{y}_w) + k_b(y_b - y_w) = 0 \tag{61}$$

The equation of motion of the bridge crossed by vehicle moving at a constant velocity v:

$$EI\frac{\partial^4 y}{\partial x^4}+c\frac{\partial y}{\partial t}+m\frac{\partial^2 y}{\partial t^2}=F(x,t) \tag{62}$$

$$F(x,t)=\{(m_w+m_b)g+m_w\ddot{y}_w+m_b\ddot{y}_b\}\delta(x-vt) \tag{63}$$

Known ζ_n, modal damping ratio, ω_n natural frequency and $\phi_n(x)$ eigenfunction of the bridge, and the frequency and damping ratio of the vehicle body:

$$\omega_b=\sqrt{\frac{k_b}{m_b}},\quad \zeta_b=\frac{c_b}{2m_b\omega_b} \tag{64}$$

$$\omega_w=\sqrt{\frac{k_b}{(m_w+m_b)}},\quad \zeta_w=\frac{c_w}{2(m_w+m_b)\omega_w} \tag{65}$$

Some non-dimensional parameters should be included to have a clearer formulation:

$$\varepsilon=\frac{(m_w+m_b)}{ml},\ \varepsilon_w=\frac{m_w}{ml},\ \varepsilon b=\frac{m_b}{ml},\ \gamma_m=\frac{m_w}{m_b} \tag{66}$$

$$\Omega=\frac{\omega_1}{\omega_w},\quad \gamma_{\mathrm{f}}=\frac{\omega_{\mathrm{b}}}{\omega_{\mathrm{w}}} \tag{67}$$

$$\xi v=\frac{vt}{l},\quad \xi=\frac{x}{l} \tag{68}$$

Also, introducing y_n, static deflection of the bridge:

$$y_m=\frac{C_{static}(m_w+m_b)g}{ml\omega_1^2} \tag{69}$$

$$Y_w=\frac{y_w}{y_m},\ Y_b=\frac{y_b}{y_m},\ Y=\frac{y}{y_m},\ u_n=\frac{q_n}{y_m} \tag{70}$$

The non dimensional equations of motion for the bridge and vehicle can be expressed as:

$$\Omega^2 Y_w+(\varepsilon_b/\varepsilon_w)Y_b+2\zeta_w(\varepsilon/\varepsilon_w)\Omega(Y_w+Y)+(\varepsilon/\varepsilon_w)\Omega(Y_w+Y)=0 \tag{71}$$

$$\Omega^2 Y_b+2\zeta_b\Omega\gamma_f(Y_b-Y_w)+\gamma_f^2(Y_b-Y_w)=0 \tag{72}$$

$$u_n+2\xi_n(\omega_n/\omega_1)u_n+(\omega_n/\omega_1)^2 u_n=\frac{\Delta l}{M_n^2}\left(\frac{1}{C_{static}}+\varepsilon_b Y_b+\varepsilon_w Y_w\right)\varphi_n(\xi_v) \tag{73}$$

With Δ =1.0 for forced vibration and Δ =0.0 for free vibration.

The nondimensional vertical displacement of the bridge Y is:

$$Y(\xi,t)=\sum_{n=1}^{\infty}u_n(\tau)\varphi_n(\xi), \quad \tau=\omega_1 t \tag{74}$$

The response of the bridge with a TMD in its midpoint of the span:

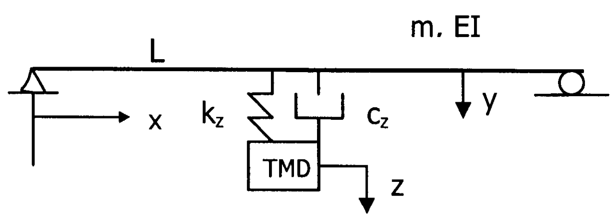


Figure 13 Bridge with Tuned Mass Damper.

The equation of motion of the TMD is given in [5] as:

$$m_z\ddot{z}+c_z(\dot{z}-\dot{y})+k_z(z-y)=0 \tag{75}$$

While the force applied on the bridge by the vehicle acts at the contact point between vehicle and bridge, the TMD inertia force acts at the center of the bridge [5]:

$$F(x,t)=\{(m_w+m_b)g+m_w\ddot{y}_w+m_b y_b\}\delta(x-vt)+(m_z g-m_z\ddot{z})\delta\left(x-\frac{L}{2}\right) \tag{76}$$

Defining some parameters as:

$$Z=\frac{z}{y_m}, \quad \varepsilon_z=\frac{m_z}{ml}, \quad \gamma_z=\frac{\omega_z}{\omega_w} \tag{77}$$

The equation of motion for the TMD is:

$$\Omega^2 Z+2\zeta_z\Omega^2(Z-Y)+\gamma_z^2(Z-Y)=0 \tag{78}$$

The equations of motion for the system in matrix form results is:

$$M\ddot{p}+C\dot{p}+Kp=Q \tag{79}$$

Where:

$$M=\begin{bmatrix} \Omega^2 & (\varepsilon_b/\varepsilon_w)\Omega^2 & 0 & 0 & 0 & \dots \\ 0 & \Omega^2 & 0 & 0 & 0 & \dots \\ 0 & 0 & \Omega^2 & 0 & 0 & \dots \\ -\Delta\varepsilon_w l\varphi_1(\xi v)/M_1^2 & -\Delta\varepsilon_b l\varphi_1(\xi v)/M_1^2 & \varepsilon_z l\varphi_1(L/2)/M_1^2 & 1 & 0 & \dots \\ -\Delta\varepsilon_w l\varphi_2(\xi v)/M_2^2 & -\Delta\varepsilon_b l\varphi_2(\xi v)/M_2^2 & \varepsilon_z l\varphi_2(L/2)/M_2^2 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{80}$$

$$C=\begin{bmatrix} 2\zeta_w(\varepsilon/\varepsilon_w)\Omega & 0 & 0 & 2\zeta_w(\varepsilon/\varepsilon_w)\Omega\varphi_1(\xi_v) & 2\zeta_w(\varepsilon/\varepsilon_w)\Omega\varphi_2(\xi_v) & \dots \\ -2\xi_b\Omega\gamma_f & 2\zeta_b\Omega\gamma_f & 0 & 0 & 0 & \dots \\ 0 & 0 & 2\zeta_z\Omega^2 & -2\zeta_z\Omega^2\varphi_1(L/2) & -2\zeta_z\Omega^2\varphi_2(L/2) & \dots \\ 0 & 0 & 0 & 2\zeta_n(\omega_1/\omega_1) & 0 & \dots \\ 0 & 0 & 0 & 0 & 2\zeta_n(\omega_2/\omega_1) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{81}$$

$$K=\begin{bmatrix} (\varepsilon/\varepsilon_w) & 0 & 0 & (\varepsilon/\varepsilon_w)\varphi_1(\zeta_v) & (\varepsilon/\varepsilon_w)\varphi_2(\zeta_v) & \dots \\ -\gamma_f^2 & \gamma_f^2 & 0 & 0 & 0 & \dots \\ 0 & 0 & \gamma_z^2 & -\gamma_z^2\varphi_1(L/2) & -\gamma_z^2\varphi_2(L/2) & \dots \\ 0 & 0 & 0 & (\omega_1/\omega_1)^2 & 0 & \dots \\ 0 & 0 & 0 & 0 & (\omega_2/\omega_1)^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{82}$$

$$Q=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{l}{C_{static}M_1^2}\left(\Delta\varphi_1(\xi_v)+\dfrac{\varepsilon_z\varphi_1(L/2)}{\varepsilon}\right) \\ \dfrac{l}{C_{static}M_2^2}\left(\Delta\varphi_2(\xi_v)+\dfrac{\varepsilon_z\varphi_2(L/2)}{\varepsilon}\right) \\ \vdots \end{bmatrix} \tag{83}$$

$$p=\begin{bmatrix} Y_w \\ Y_b \\ Z \\ u_1 \\ u_2 \\ \vdots \end{bmatrix} \tag{84}$$

11.1. Tuning condition of the TMD:

There are many tuning conditions proposed, however, the most commonly used [7] is Den Hartog's optimum tuning condition [8]:

$$\varepsilon_z = \frac{m_z}{ml}, \quad \omega_z = \frac{\omega_n}{1+\varepsilon_z} \tag{85}$$

$$\left(\frac{c_z}{c_c}\right)^2 = \frac{3\varepsilon_z}{8(1+\varepsilon_z)^3} \quad c_c = 2m_z\omega_n \tag{86}$$

TMD damping affects the dynamic response of the structures more extensively than the damping values of the structures [5]. The critical damping proposed by Tsai [9] keep away from response increase due to inadequate damper tuning and beating phenomenon:

$$\zeta_z = \zeta_n + \sqrt{\varepsilon_z} \tag{87}$$

The mass ratios ε_Z ∈ (0.01-0.04) are generally recommended [5].

11.2. Case study:

The formulation here shown, was presented by [5] in the same article we can find a example of the effectiveness of the TMD:

$L = 50\ m$

$E = 3.303*1010\ [N/m^2]$

$m = 3852\ [kg/m3]$

$Damping\ ratio\ (\%) = 0.3$

$v_{vehicle} = 90[m/sg]$

$I = 18.638\ [m^4]$

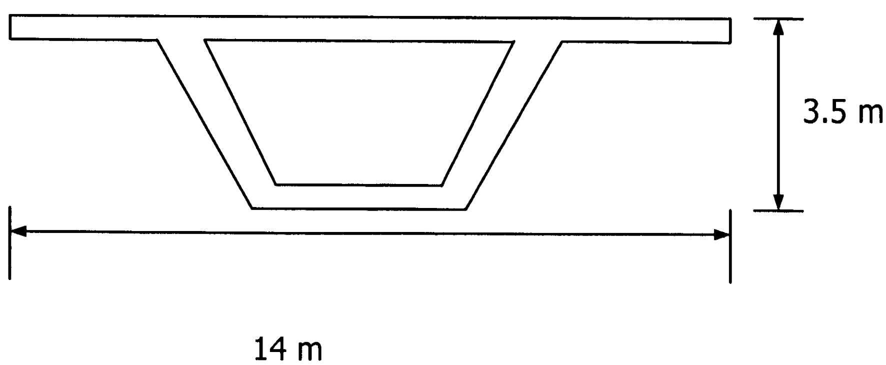
$A = 11.332\ [m^4]$

Wheel TGV:

Locomotive	Semi-passenger	Passenger car
M = 2382 kg	M =2382	M =2383
k =1.0 e7	k =2.35e6	k =1.0e7
c =4e7	c =4e4	c =8e4

Body TGV:

Locomotive	Semi-passenger	Passenger car
M = 13,760 kg	M =17,000	M =17,000
k =5.0 e7	k =2.86e6	k =5.0e7
c =8e4	c =8e4	c =8e4



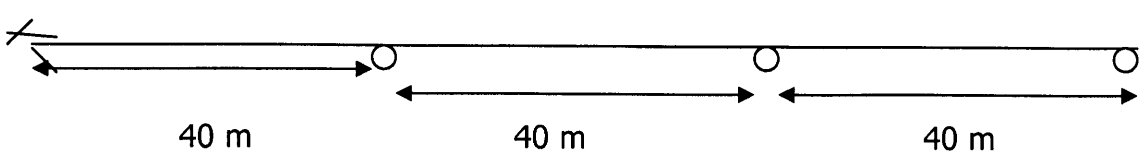


Figure 14. Bridge and vehicle's parameters [5]

Comparison in the response of the bridge under a TGV traveling with different velocities:

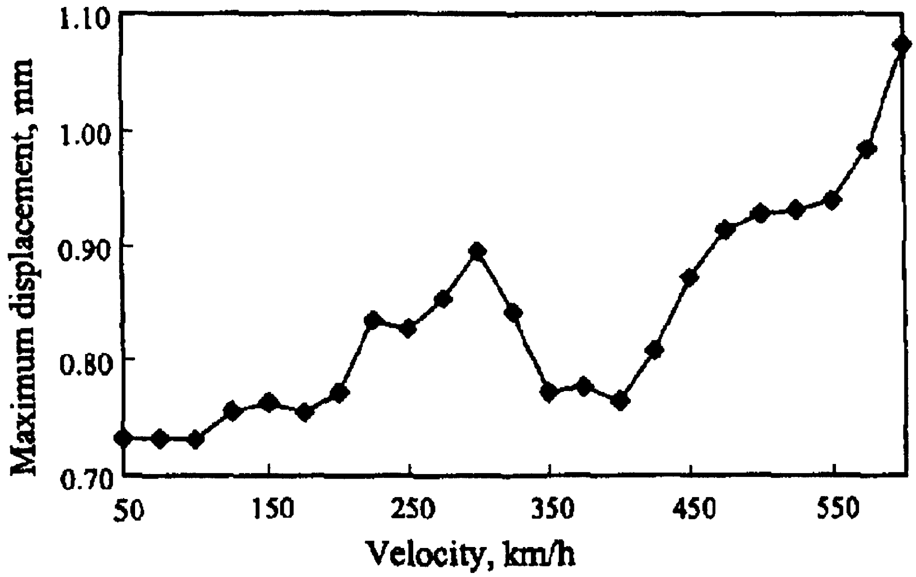


Figure 15. Maximum response at point x=L/2 at different velocities. Capture [5]

It is important to notice that the response with a smaller velocity can be higher. That is because the loading space has its own speed which makes the bridge response maximum. Therefore it is difficult to know a priori which velocity would produce the maximum response, and it is necessary to try with different velocities to find the worst case scenario.

Comparison between the fast Fourier transform (FFT) for the bridge with and without a TMD of mass ratio $\varepsilon_z = 0.01$

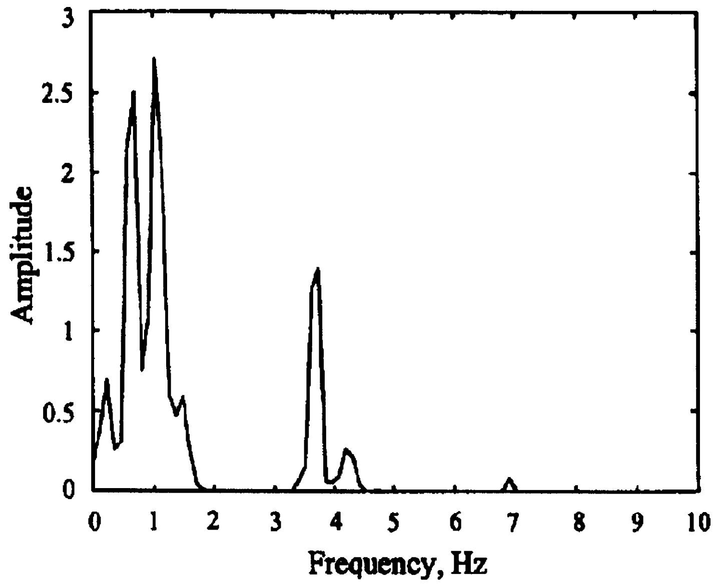


Figure 16. FFT of the displacement x=L/2 without TMD. Capture [5]

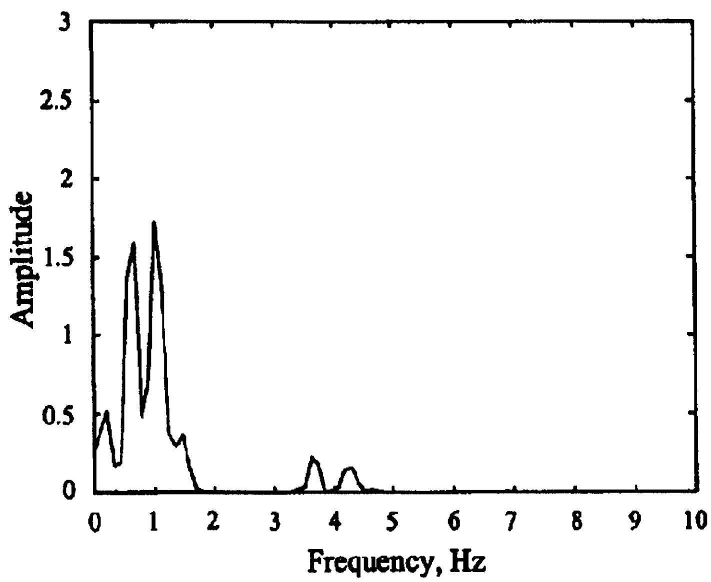


Figure 17. FFT of the displacement x=L/2 witht TMD. . Capture [5]

The acceleration in the vehicles can affect passenger's comfort and cause mistrust in the structure. In this case little acceleration reduction is observed because the time that the vehicle needs to pass across the bridge is too short to increase TMD motion, unlike other cases like earthquake loading.

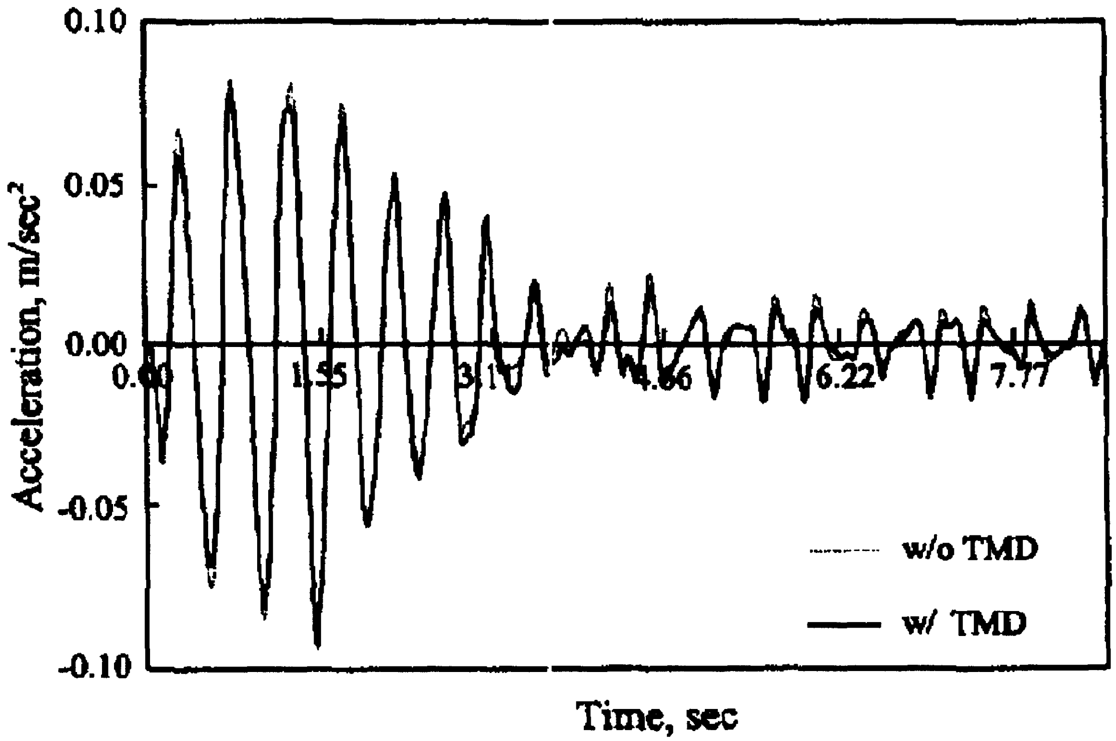


Figure 18. Acceleration of the vehicle body. Capture [5]

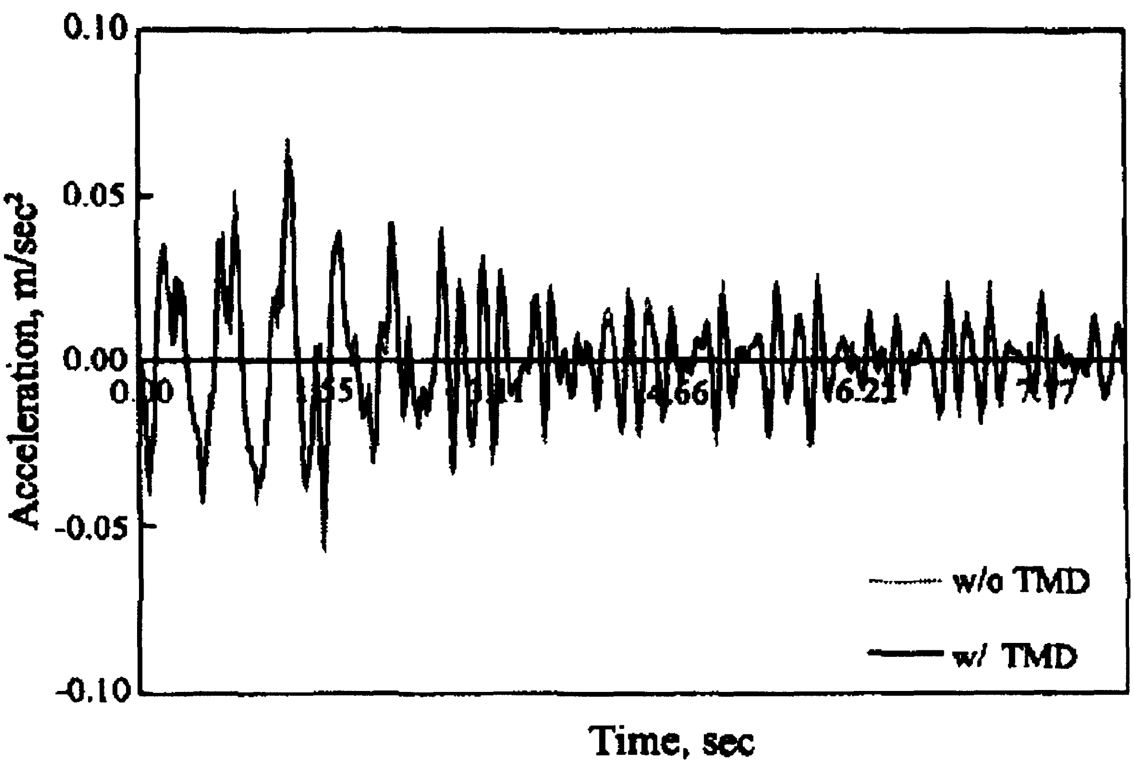


Figure 19. Acceleration of the wheel. Capture [5]

11.3. Conclusions:

We can draw from [5] important conclusions:

- There can be sub-critical speeds within the bridge design speed.
- The bridge impact factors must be changed to adequate levels considering the response of the bridge under a train traveling at sub-critical speeds.
- TMD reduces in a 21% the vertical displacement and free vibration dies out fast when a TGV passes.

12. Multiple Tuned Mass Damper (MTMD)

In [10] Kajikawa et al. concluded that a single TMD couldn't suppress effectively the vibrations caused by traffic, because the dynamic response of a bridge is frequency variant due to vehicle motion. We have seen in the case study [5] in the previous section, the vertical acceleration couldn't be controlled with a TMD.

The control effectiveness of a single TMD comes from the capacity to tune its frequency with the structure. The error in the estimation of the natural frequency of a structure, will affect significantly the design and effectiveness of the TMD. The accuracy in the determination of the natural frequency of a structure is very important. However, because of the uncertainties in the rigidity and mass of the structure as well as its finished details, the frequency of the fundamental mode can be difficult to ascertain with any great accuracy.

A system of multiple tuned mass dampers (MTMD), a multiple parallel placed TMD, can overcome the danger of detuning because it is possible to adjust the natural frequency for each TMD, covering a wider range of frequencies so that the risk that the natural frequency of the structure is not contained by them is reduced drastically.

Xu and Igusa [11] were the first ones to propose an MTMD with uniform distribution of natural frequencies. It has been proved that the MTMD is more reliable and effective in limiting the building vibrations.

Here we present the formulation of a bridge with a MTMD given by Lin et al in [12] and their conclusions in the success of the MTMD in controlling the vibration.

Consider an MTMD with p SDOF TMD installed in parallel on a straight bridge with length L at section $x = x_s$. The l_{th} TMD is away from the centerline at a distance e_{sl}. A train with N_v number of moving loads passes with a velocity v and eccentricity e_v.

If we adopt certain assumptions [12]:

- Bridge modeled as a two dimensional homogeneous, elastic, isotropic beam.
- The train loads are applied in the centerline of the track and it moves with constant speed *v*.

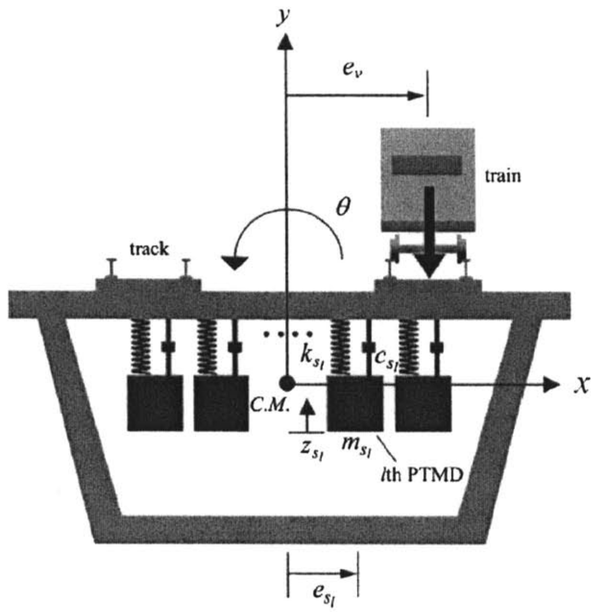


Figure 20. Bridge with TMD at section $x = x_s$ Capture [12]

The flexural and torsional motions of the bridge at section x are [12]:

$$\overline{m}(x)\frac{\partial^2 y(x,t)}{\partial t^2} + C_y(x)\frac{\partial y(x,t)}{\partial t} + \frac{\partial^2 y(x,t)}{\partial x^2}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right] = F_y(x,t) \quad (88)$$

$$\bar{J}_m(x)\frac{\partial^2 \theta(x,t)}{\partial t^2} + C_\theta(x)\frac{\partial \theta(x,t)}{\partial t} - \frac{\partial}{\partial x}\left(GJ_T(x)\frac{\partial \theta(x,t)}{\partial x} - EC_w(x)\frac{\partial^3 \theta(x,t)}{\partial x^3}\right) = F_\theta(x,t) \quad (89)$$

$\overline{m}(x)$, $C_y(x)$, and $EI(x)$ represent mass per unit length, the damping coefficient of the flexural motion, and the rigidity of the flexural motion of the bridge at section x.

$\bar{J}_m(x)$, $C_\theta(x)$, $GJ_T(x)$, and $EC_w(x)$ are the polar mass moment of inertia per unit length, the damping coefficient of the torsional motion, the rigidity in pure torsion , and the rigidity of warping torsion of the bridge at section x.

Yang et. al [13] proposed the following equation for the train load, where p_k is the magnitude of the k_{th} load and t_k denotes the time when the k_{th} load reaches the bridge.

$$F_y(x,t) = -\sum_{k=1}^{Nv} p_k \delta[x - v(t - t_k)]H(t - t_k)$$
$$+\sum_{i=1}^{p} \{k_{sl}[z_{sl} - y(x_s,t) - e_{sl}\theta(x_s,t)] + c_{sl}[\dot{z}_{sl} - \dot{y}(x_s,t) - e_{sl}\dot{\theta}(x_s,t)]\}\delta(x - x_s) \tag{90}$$

The second term on the right hand side determines the location of the k_{th} load on the bridge, the third term determines whether the k_{th} load is on the bridge or not

The F_θ proposed in [12]:

$$F_\theta(x,t) = -\sum_{k=1}^{Nv} e_v p_k \delta[x - v(t - t_k)]H(t - t_k)$$
$$+\sum_{i=1}^{p} e_{sl}\{k_{sl}[z_{sl} - y(x_s,t) - e_{sl}\theta(x_s,t)] + c_{sl}[\dot{z}_{sl} - \dot{y}(x_s,t) - e_{sl}\dot{\theta}(x_s,t)]\}\delta(x - x_s) \tag{91}$$

$H(t,t_k) = U(t\text{-}t_k)\text{-}U[t\text{-}(t_k + v/L)]$, $U(t)$ and $\delta(t)$ are the step function and the Dirac delta function:

$$\int_{-\infty}^{\infty} \delta(x)dx = \begin{Bmatrix} 1, & x = 0 \\ 0, & x \neq 0 \end{Bmatrix} \tag{92}$$

$$U(t) = \begin{Bmatrix} 1, & t \geq 0 \\ 0, & t < 0 \end{Bmatrix} \tag{93}$$

The vertical motion of the l_{th} TMD is:

$$\begin{aligned} &m_{sl}\ddot{z}_{sl}(t) + c_{sl}[\dot{z}_{sl} - \dot{y}(x_s,t) - e_{sl}\dot{\theta}(x_s,t)] + k_{sl}[z_{sl} - y(x_s,t) - e_{sl}\theta(x_s,t)] = 0 \\ &l = 1,2,\ldots,p \end{aligned} \tag{94}$$

m_{sl}, c_{sl}, and k_{sl} are the mass, damping coefficient, and stiffness coefficient of the l_{th} TMD, $y(x,t)$, $\theta(x,t)$ and $z_{sl}(t)$ indicate the vertical displacement of the bridge, the torsional angle of the bridge, and the vertical displacement of the l_{th} TMD, respectively.

The p_k depend on the train model. Depending on the level of complexity we want the problem to have, the train can be considered as:

- Force model:

$$p_k = m_{vk} g \tag{95}$$

- Mass model:

$$p_k = m_{vk}\{g + \ddot{z}_v(t)\} \tag{96}$$

- Moving suspension mass model where the wheel masses are neglected:

$$p_k = m_{vk}[g + \ddot{z}_v(t)] \tag{97}$$

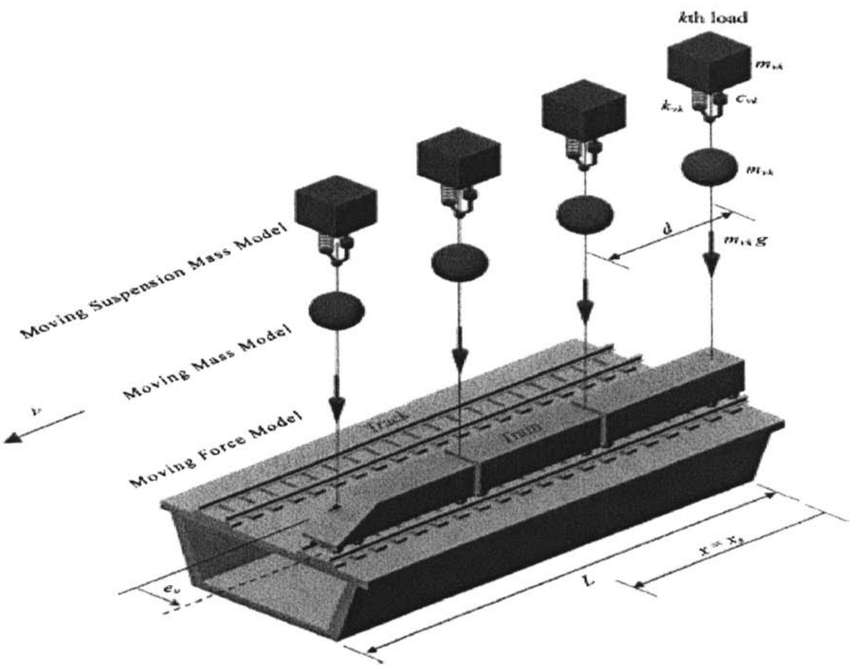


Figure 21. Bridge with the three models of loads: force, moving mass and sprung moving mass. Capture [12]

m_{vk} ad $z_v(t)$ are the mass and the vertical displacement of the k_{th} train.

Modal analysis is employed to separate the governing parameters [12]:

$$y(x,t) = \sum_{j=1}^{N} \phi_j(x)\eta_j(t) = \Phi^T(x)\eta(t) \tag{98}$$

$$\theta(x,t) = \sum_{j=1}^{N} \psi_j(x)\gamma_j(t) = \psi^T(x)\gamma(t) \tag{99}$$

N = number of nodes to be considered

$\Phi^T(x) = \{\phi_1(\mathrm{x}), \phi_2(\mathrm{x}) \ldots \phi_N(\mathrm{x})\}$, and $\Psi^T(x) = \{\psi_1(\mathrm{x}), \psi_2(\mathrm{x}) \ldots \psi_N(\mathrm{x})\}$ mode shape vectors.

$\eta(t)$ and $\gamma(t)$ are the modal-response vectors.

$v_{sl}(t) = z_s l(t)-y(x_s,t)-e_{sl}\theta(x_s,t)$ is the stroke, the displacement of the l_{th} TMD relative to the bridge where the l_{th} TMD is located.

If we plug equations (90), (91), (98) and (99) in equations (88) and (89) and we premultiplying $\Phi(x)$, $\Psi(x)$, we get in matrix form:

$$M_y\ddot{\eta}(t) + C_y\dot{\eta}(t) + K_y\eta(t) = F_{yv}(t) + F_{y,MTMD}(t) \tag{100}$$

$$M_\theta\ddot{\eta}(t) + C_\theta\dot{\eta}(t) + K_\theta\eta(t) = F_{\theta v}(t) + F_{\theta,MTMD}(t) \tag{101}$$

Where M_y, M_θ, C_y, C_θ and K_y, K_θ are $N*N$ matrices, representing modal mass, damping and stiffness of the bridge flexural motion and of the bridge torsional motion respectively. After the application of $\Phi(x)$, $\Psi(x)$, we get N uncoupled equations:

$$\ddot{\eta}_j(t) + 2\xi_{y_j}\omega_{y_j}\dot{\eta}_j(t) + \omega_{yj}^2\eta_j(t) = F_{yj}^v(t) + F_{yj}^{MTMD}(t) \tag{102}$$

$$\ddot{\gamma}_j(t) + 2\xi_{\theta_j}\omega_{\theta_j}\dot{\gamma}_j(t) + \omega_{\theta j}^2\gamma_j(t) = F_{\theta j}^v(t) + F_{\theta j}^{MTMD}(t) \tag{103}$$

$\omega_{yj} = \sqrt{k_{yj}/m_{yj}}$ and $\xi_{yj} = c_{yj}/2m_{yj}\omega_{yj}$ are the j_{th} flexural modal frequency and damping ratio of the bridge

$\omega_{\theta j} = \sqrt{k_{\theta j}/m_{\theta j}}$ and $\xi_{\theta j} = c_{\theta j}/2m_{\theta j}\omega_{\theta j}$ are the j_{th} torsional modal frequency and damping ratio of the bridge and:

$$m_{yj} = \int_0^L \overline{m}(x)\phi_j^2(x)dx,\ \ m_{\theta j} = \int_0^L \overline{J}_m(x)\psi_j^2(x)dx, \tag{104}$$

$$c_{yj} = \int_0^L C_y(x)\phi_j^2(x)dx,\ \ c_{\theta j} = \int_0^L C_\theta(x)\psi_j^2(x)dx, \tag{105}$$

$$k_{yj} = \int_0^L EI(x)\left[\phi_j''(x)\right]^2,\ \ k_{\theta j} = \int_0^L \left\{GJ_T(x)\left[\psi_j'(x)\right]^2 + EC_w(x)\left[\psi_j''(x)\right]^2\right\}dx, \tag{106}$$

The j_{th} flexural modal force F_{yj}^v, and torsional modal force $F_{\theta j}^v$:

$$F_{yj}^v = -\sum_{k=1}^{Nv}\frac{p_k}{m_{yj}}\phi_j(vt - vt_k)H(t,t_k) \tag{107}$$

$$F_{\theta j}^{v} = -e_v \sum_{k=1}^{Nv} \frac{p_k}{m_{\theta j}} \psi_j(vt - vt_k) H(t, t_k) \tag{108}$$

Also:

$$F_{yj}^{MTMD} = \sum_{l=1}^{p} \left(2\xi_{sl}\omega_{sl}\dot{v}_{sl} + \omega_{sl}^2 v_{sl}\right)\mu_{sl\,yj} \tag{109}$$

$$F_{\theta j}^{MTMD} = \sum_{l=1}^{p} \left(2\xi_{sl}\omega_{sl}\dot{v}_{sl} + \omega_{sl}^2 v_{sl}\right)\mu_{sl\,\theta j} \tag{110}$$

Where $\mu_{slyj} = \phi_j(x_s) m_{sl}/m_{yj}$ and $\mu_{sl\theta j} = \phi_j(x_s) m_{sl}/m_{\theta j}$ are ratios of the l_{th} TMD mass to the j_{th} flexural and torsional modal mass of the bridge

The equation (94) can be rearranged into the displacement of the l_{th} TMD:

$$\left[\ddot{v}_{sl}(t) + \Phi^T(xs)\ddot{\eta}(t) + e_{sl}\Psi^T(x_s)\ddot{\gamma}(t)\right] + 2\xi_{sl}\omega_{sl}\dot{v}_{sl}(t) + \omega^2 v_{sl}(t) = 0 \tag{111}$$

If we express the equations of motion in matrix form:

$$\begin{bmatrix} M_{yy} & 0 & 0 \\ 0 & M_{\theta\theta} & 0 \\ M_{sy} & M_{s\theta} & M_s \end{bmatrix} \begin{Bmatrix} \ddot{\eta}(t) \\ \ddot{\gamma}(t) \\ \ddot{v}_s(t) \end{Bmatrix} + \begin{bmatrix} C_{yy} & 0 & C_{ys} \\ 0 & C_{\theta\theta} & C_{\theta s} \\ 0 & 0 & C_s \end{bmatrix} \begin{Bmatrix} \dot{\eta}(t) \\ \dot{\gamma}(t) \\ \dot{v}_s(t) \end{Bmatrix}$$
$$+ \begin{bmatrix} K_{yy} & 0 & K_{ys} \\ 0 & K_{\theta\theta} & K_{\theta s} \\ 0 & 0 & K_s \end{bmatrix} \begin{Bmatrix} \eta(t) \\ \gamma(t) \\ v_s(t) \end{Bmatrix} = \begin{Bmatrix} F_{yv}(t) \\ F_{\theta v}(t) \\ 0 \end{Bmatrix} \tag{112}$$

where 0 is the null matrix and :

$$M_{sy}^T = \begin{bmatrix} \phi_1(x_s) & \phi_1(x_s) & \cdots & \phi_1(x_s) \\ \phi_2(x_s) & \phi_2(x_s) & \cdots & \phi_2(x_s) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N(x_s) & \phi_N(x_s) & \cdots & \phi_N(x_s) \end{bmatrix} \tag{113}$$

$$M_{s\theta}^T = \begin{bmatrix} \psi_1(x_s)e_{s1} & \psi_1(x_s)e_{s2} & \cdots & \psi_1(x_s)e_{sp} \\ \psi_2(x_s)e_{s1} & \psi_2(x_s)e_{s2} & \cdots & \psi_2(x_s)e_{sp} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_N(x_s)e_{s1} & \psi_N(x_s)e_{s2} & \cdots & \psi_N(x_s)e_{sp} \end{bmatrix} \tag{114}$$

$$C_{ys} = \begin{bmatrix} -2\xi_{s1}\omega_{s1}\mu_{s1}y_1 & -2\xi_{s2}\omega_{s2}\mu_{s2}y_1 & \cdots & -2\xi_{sp}\omega_{sp}\mu_{sp}y_1 \\ -2\xi_{s1}\omega_{s1}\mu_{s1}y_2 & -2\xi_{s2}\omega_{s2}\mu_{s2}y_2 & \cdots & -2\xi_{sp}\omega_{sp}\mu_{sp}y_2 \\ \vdots & \vdots & \ddots & \vdots \\ -2\xi_{s1}\omega_{s1}\mu_{s1}y_N & -2\xi_{s2}\omega_{s2}\mu_{s2}y_N & \cdots & -2\xi_{sp}\omega_{sp}\mu_{sp}y_N \end{bmatrix} \tag{115}$$

$$C_{\theta s} = \begin{bmatrix} -2\xi_{s1}\omega_{s1}\mu_{s1}\theta_1 & -2\xi_{s2}\omega_{s2}\mu_{s2}\theta_1 & \cdots & -2\xi_{sp}\omega_{sp}\mu_{sp}\theta_1 \\ -2\xi_{s1}\omega_{s1}\mu_{s1}\theta_2 & -2\xi_{s2}\omega_{s2}\mu_{s2}\theta_2 & \cdots & -2\xi_{sp}\omega_{sp}\mu_{sp}\theta_2 \\ \vdots & \vdots & \ddots & \vdots \\ -2\xi_{s1}\omega_{s1}\mu_{s1}\theta_N & -2\xi_{s2}\omega_{s2}\mu_{s2}\theta_N & \cdots & -2\xi_{sp}\omega_{sp}\mu_{sp}\theta_N \end{bmatrix} \tag{116}$$

$$K_{ys} = \begin{bmatrix} -\omega_{s1}^2\mu_{s1}y_1 & -\omega_{s2}^2\mu_{s2}y_1 & \cdots & -\omega_{sp}^2\mu_{sp}y_1 \\ -\omega_{s1}^2\mu_{s1}y_2 & -\omega_{s2}^2\mu_{s2}y_2 & \cdots & -\omega_{sp}^2\mu_{sp}y_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\omega_{s1}^2\mu_{s1}y_N & -\omega_{s2}^2\mu_{s2}y_N & \cdots & -\omega_{sp}^2\mu_{sp}y_N \end{bmatrix} \tag{117}$$

$$K_{\theta s} = \begin{bmatrix} -\omega_{s1}^2\mu_{s1}\theta_1 & -\omega_{s2}^2\mu_{s2}\theta_1 & \cdots & -\omega_{sp}^2\mu_{sp}\theta_1 \\ -\omega_{s1}^2\mu_{s1}\theta_2 & -\omega_{s2}^2\mu_{s2}\theta_2 & \cdots & -\omega_{sp}^2\mu_{sp}\theta_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\omega_{s1}^2\mu_{s1}\theta_N & -\omega_{s2}^2\mu_{s2}\theta_N & \cdots & -\omega_{sp}^2\mu_{sp}\theta_N \end{bmatrix} \tag{118}$$

12.1. Dynamic of the load

A train acts on a bridge like a series of similar repetitive loads. This loading is like a steady impact on the bridge if the train is moving with constant velocity [12], this will have a different response that a bridge under a single vehicle load that only acts a short time.

If the bridge is simply supported, the vertical vibration $\Phi(x)$ is like we stated in previous sections:

$$\Phi(x)=\left\{\sin\frac{\pi x}{L},\sin\frac{2\pi x}{L},\ldots\sin\frac{N\pi x}{L}\right\} \tag{119}$$

If we do the Fourier transform of $F_{yj}^{v}(\omega)$ the j_{th} flexural modal train load [12]:

$$\begin{aligned}F_{yj}^{v}(\omega)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}F_{yj}^{v}(t)e^{-i\omega t}dt\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}-\sum_{k=1}^{N_v}\left\{\frac{p_k}{m_{yj}}\phi_j(vt-vt_k)H(t,t_k)\right\}dt\\&=-\frac{1}{2\pi}\sum_{k=1}^{N_v}\left\{\int_{-\infty}^{\infty}\frac{p_k}{m_{yj}}\left[\sin\frac{j\pi v(t-t_k)}{L}\right]H(t,t_k)dt\right\}\end{aligned} \tag{120}$$

If each moving load p_k has the same magnitude and the same spacing d :

$$F_{yj}^{v}(x)=\frac{\dfrac{p_0\pi vj}{m_{yj}\omega^2 L}}{1-\left(\dfrac{j\pi v}{\omega L}\right)^2}\left[\frac{1}{2}+\frac{\sin\left(\dfrac{\omega d}{2v}N_v-\dfrac{\omega d}{2v}\right)}{\sin\left(\dfrac{\omega d}{2v}\right)}\right]\left[(-1)^j e^{-i\omega L/v}-1\right] \tag{121}$$

$F_{yj}^{v}(\omega)$ would be large as:

- $\sin(\omega d/2v)\approx 0$
- or $\omega=2\pi nv/d$ $(n=1,2,\ldots)$ where v/d is the impact frequency of the wheels

The bridge will have resonant speeds at:

$$v=\frac{\omega_{yj}d}{2n\pi} \tag{122}$$

The resonance does not only occur when the train travels at high speeds, also:

$$v=\frac{\omega_{yj}L}{n\pi} \tag{123}$$

The major critical condition $(n=1)$ only occur when the train speed is several times the first resonant speeds; this is almost impossible for general high-speed railway bridges [12], [13]. A single car passing through a bridge will not produce any resonant response.

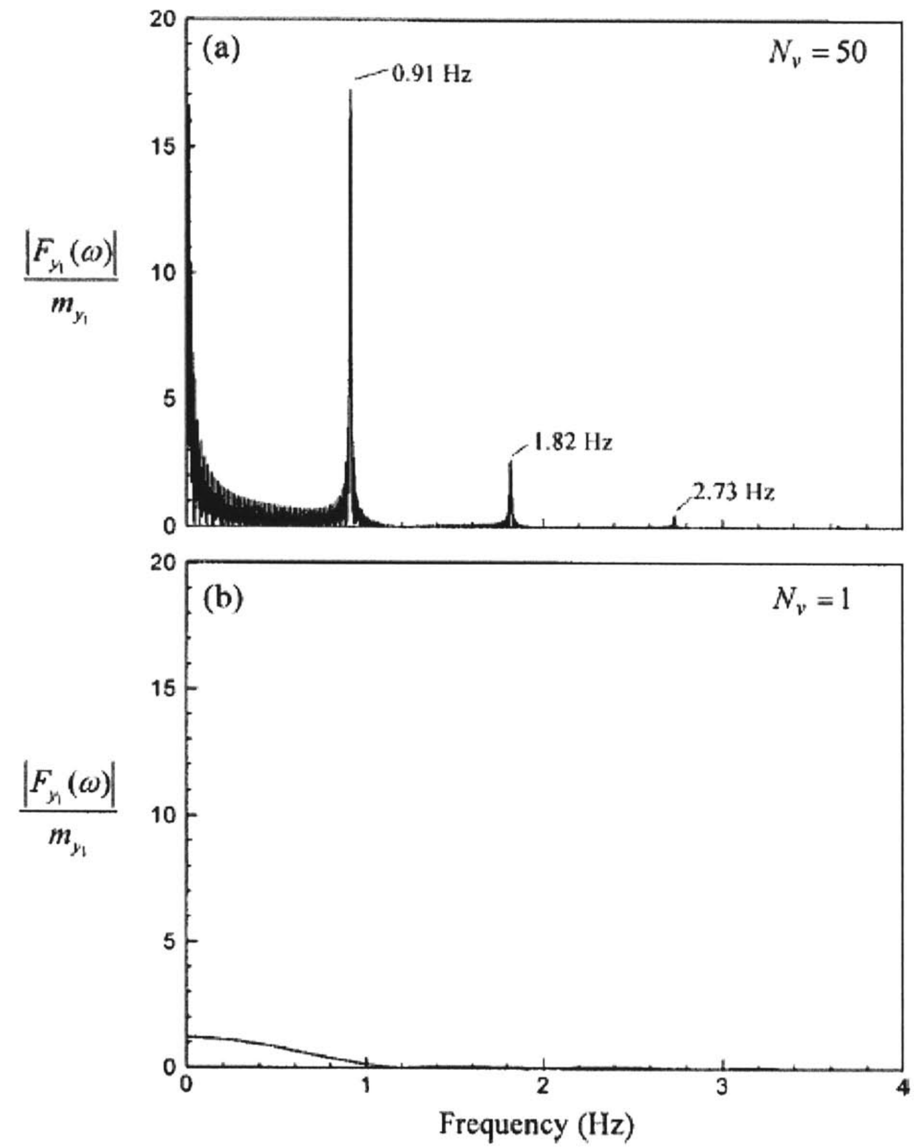


Figure 22. First modal Fourier transform with 1) 50 loads and 2) One load. Capture [12]

12.2. Case study:

In [12] we can find in addition to the formulation of the MTMD, numerical examples of the effectiveness of the MTMD in a Taiwan High-Speed Railway Type for different train types.

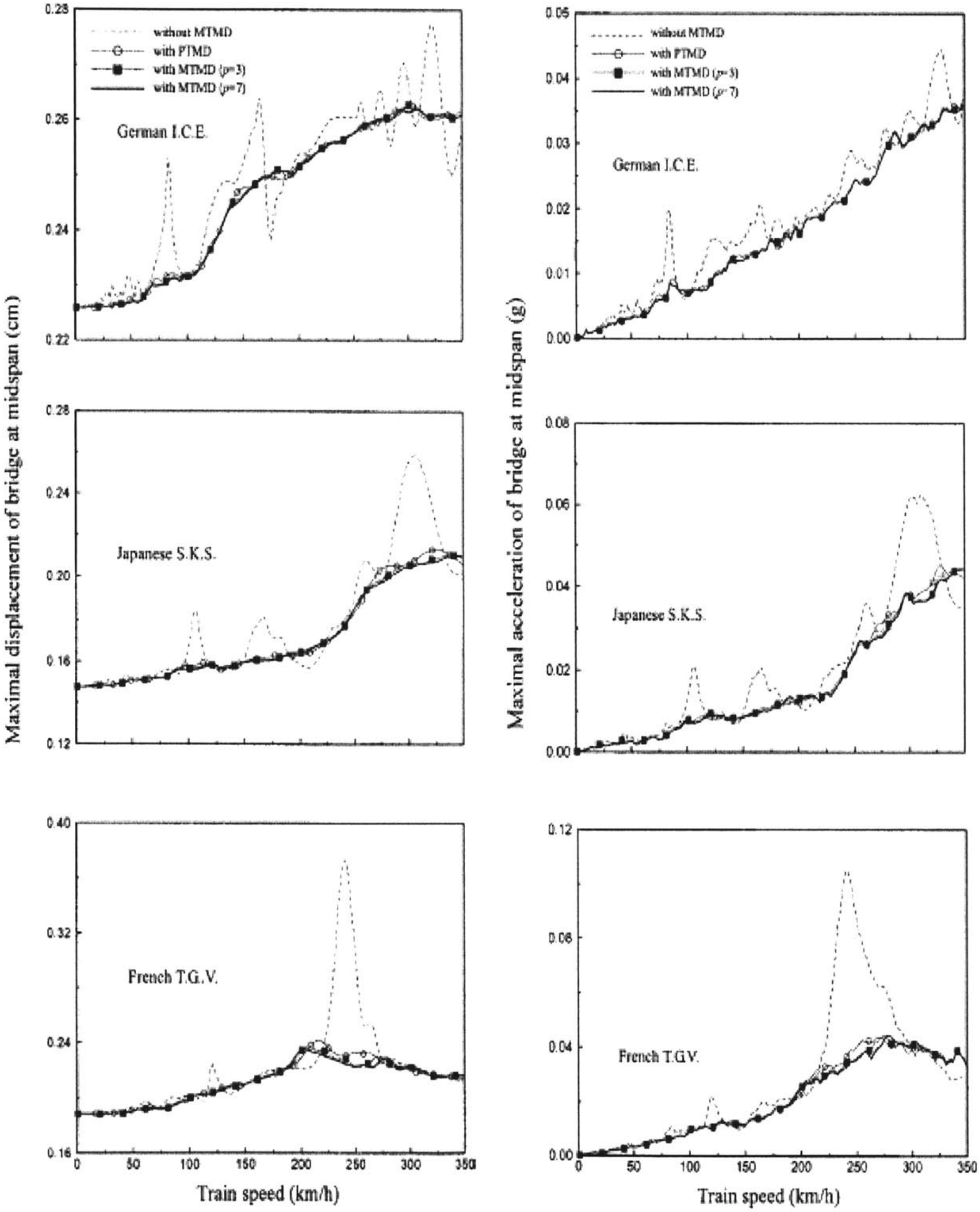


Figure 23. Maximal displacement and accelerations for a Taiwan High-Speed Train with and without MTMD. Capture [12]

12.3. Conclusions:

Important conclusions for the design of bridges can be derived from [12]:

- If the natural frequencies of the bridge are multiples of the impact frequency of the of the train, the resonant effect will take place, although the train doesn't travel at high speeds.

- The MTMD can control effectively the dynamic response of the bridge and train only if are dominated by the resonant response within the design train speed.

- The error in the estimation of the bridge frequencies and the bridge-train interaction will affect the control effectiveness of a single TMD. However a MTMD system with the same mass but a wider range of frequencies, is less affected by the detuning effect, being more reliable and robust than a single TMD.

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APPENDIX A:

MATLAB CODE TO GET THE ANALYTICAL DEFORMATION AT ANY GIVEN POINT UNDER A MOVING LOAD

```
L=50
n=1000                              %number of nodes
xcontrol =5*L/10  ;                 %Point of beam where deflection is analysed
v= 90                               %velocity  of the train in m/sg
F= -1                               %Weight of the force
m=43651.935                         %[kg/m] Linear density
E=3.303e10                          %Young Modulus  [N/m2]
I=18.638                            %Moment of inertia [m4]
timestep = 0.001
T=[0:timestep:L/v] ;                %Range of t in which I see the response
y=zeros(size(T))';                  %Column vector
for i=1:n  ;
    Wn=i*pi*v/L;
    wn=i^2*pi^2/L^2*(E*I/m)^0.5;
    deltay=2*F/(m*L)/(wn^2-Wn^2)*(sin(Wn*T') ...
          -Wn/wn*sin(wn*T'))*sin(i*pi*xcontrol/L);
    y=y+deltay;
    end
plot(T,y,'r-')
hold on
end
```

APPENDIX B:

MATLAB CODE TO GENERATE A DYNAMIC AMPLIFICATION FACTOR

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% Generation of dynamic amplification factor curve %%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
hold off

n=1        % Mode in which I am looking the dynamic amplification factor
%  Number of curves that I get with the parameter v/L
curve0 = 0.2                      % Initial curve
curvefin = 1                      % Final curve
curvestep = 0.01                  % Interval of the curves
curves= [curve0:curvestep:curvefin]   % Vector that stores the vL parameters

% The x axis:
x0= 0.02                                    % Where x starts
minunitaxis = 0.01 ;                         % Minimum unit of x
xfin = 0.4                                  % Where x finishes
% IMPORTANT!! xfin = x0 + q* minutaxis with q integer
x_axis = [x0:minunitaxis:xfin] ; % Axis vector adimensional (E*I/m)^0.5/L^2

% The y axis stored for each x
DLF = zeros(size(curves),size(x_axis));   % Matrix in which the dynamic
                 % amplification factors will be stored
                 % Each row is the DLF for a given relation of v/L relation
timestep= 0.001
for p = 1:(curvefin-curve0)/curvestep +1    % We do a cycle for each curve
    v_L = curves(p)
    for j=1:(xfin-x0)/minunitaxis+1
        timestep = 1/(100*v_L);
        T=[0:timestep:1/v_L] ;       %Range of t in which I see the response
        ck = x_axis(j)
        Wn=n*pi*v_L;
        wn=n^2*pi^2*ck;

        DLF_T=1/(wn^2-Wn^2)*(sin(Wn*T)-Wn/wn*sin(wn*T));
        DLF (p,j)= max(abs(DLF_T));
```

```
    end
end
for q=1:(curvefin-curve0)/curvestep +1
    cur = DLF(q,:)
figure(8);
    plot(x_axis,cur,'k-')
   hold on
end

end
```