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## **DYNAMIC DERIVATIVE STRATEGIES**

**Jun Liu and Jun Pan**

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# Dynamic Derivative Strategies

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## Abstract

This paper studies the optimal investment strategy of an investor who can access not only the bond and the stock markets, but also the derivatives market. We consider the investment situation where, in addition to the usual diffusive price shocks, the stock market experiences sudden price jumps and stochastic volatility. The dynamic portfolio problem involving derivatives is solved in closed-form. Our results show that derivatives are important in providing access to the risk and return tradeoffs associated with the volatility and jump risks. Moreover, as a vehicle to the volatility risk, derivatives are used by non-myopic investors to exploit the time-varying opportunity set; and as a vehicle to the jump risk, derivatives are used by investors to dis-entangle their simultaneous exposure to the diffusive and jump risks in the stock market. In addition, derivatives investing also affects investors' stock position because of the interaction between the two markets. Finally, calibrating our model to the S&P 500 index and options markets, we find sizable portfolio improvement for taking advantage of derivatives.

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# 1 Introduction

“Derivatives trading is now the world’s biggest business, with an estimated daily turnover of over US\$2.5 trillion and an annual growth rate of around 14%.”<sup>1</sup> Despite increasing usage and growing interest, little is known about the optimal trading strategies with derivatives as part of an investment portfolio. Indeed, academic studies on dynamic asset allocation typically exclude derivatives from the investment portfolios. In a complete market setting, such an exclusion can very well be justified by the fact that derivative securities are redundant [e.g., Black and Scholes (1973) and Cox and Ross (1976)]. When the completeness of the market breaks down — either because of infrequent trading or by the presence of additional sources of uncertainty — it then becomes suboptimal to exclude derivatives.<sup>2</sup>

Building on this economic intuition, this paper goes one step further by asking: What are the optimal dynamic strategies for an investor who can control not just the holdings in the aggregate stock market and a riskless bond, but also derivatives? How much can he benefit from including derivatives?

In this paper, we address these questions by focusing on two specific aspects of market incompleteness that have been well documented in the empirical literature for the aggregate stock market: one arises from stochastic volatility, and the other from jumps.<sup>3</sup> Specifically, we adopt an empirically realistic model for the aggregate stock market that incorporates three types of risk factors: diffusive price shocks, price jumps, and volatility risks. Taking this market condition as given, we solve the dynamic asset allocation problem [Merton (1971)] of a power-utility investor whose investment opportunity includes not only the usual riskless bond and risky stock, but also derivatives on the stock.

The one important aspect of the derivative securities considered in this paper is their non-linear dependence on the risky stock. By itself, the risky stock can only provide a “package deal” of risk exposures: one unit each to the diffusive and jump risks and none to the volatility risk. With the help of derivatives, however, this “package deal” can be broken down into its three individual components. For example, an at-the-money option, being highly sensitive to market volatility, provides a positive exposure to the volatility risk; a deep out-of-the-money put option, being much more sensitive to negative jump risks than diffusive risks, serves to dis-entangle the jump risk from the diffusive risk. From this example, we can see that it is the non-linear nature of derivatives that serves to complete the market

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<sup>1</sup>From *Building the Global Market: A 4000 Year History of Derivatives* by Edward J. Swan.

<sup>2</sup>Among others, the spanning role of derivatives has been studied extensively by Ross (1976), Breeden and Litzenberger (1978), Arditti and John (1980), and Green and Jarrow (1987) in static settings, and, more recently, by Bakshi and Madan (2000) in a dynamic setting. In a buy-and-hold environment, Haugh and Lo (2001) use derivatives to mimic the dynamic trading strategy of the underlying stock. Using historical stock data, Merton, Scholes, and Gladstein (1978, 1982) investigate the return characteristics of various option strategies. Carr, Jin, and Madan (2001) consider the optimal portfolio problem in a pure-jump setting by including as many options as the jump states. In an information context, Brennan and Cao (1996) analyze the role of derivatives in improving trading opportunities. Ahn, Boudoukh, Richardson, and Whitelaw (1999) considers the role of options in a portfolio Value-at-Risk setting.

<sup>3</sup>Both empirical facts have been the object of numerous studies. Among others, Jorion (1989) documented the importance of jumps in the aggregate stock returns. Recent studies documenting the importance of both stochastic volatility and jumps include Andersen, Benzoni, and Lund (2001), Bates (2000), and Bakshi, Cao, and Chen (1997).

with respect to the additional risk factors. Although one can think of derivatives in their most general terms, not all financial contracts can provide such a service. For example, bond derivatives or long-term bonds can only provide access to the risk of the short rate, which is a constant in our setting. Given our focus on the aggregate stock market, individual stocks are unlikely candidates, and their linear nature makes it even more unlikely.

The market incompleteness that makes derivatives valuable in our setting also makes the pricing of such derivatives not unique. In particular, using the risk and return information contained in the underlying risky stock, we are unable to assign the market price of the volatility risk or the relative pricing of the diffusive and jump risks. In other words, when we introduce derivatives to complete the market, say one at-the-money and one out-of-the-money put options, we need to make additional assumptions on the volatility-risk and jump-risk premia implicit in such derivatives. Once such assumptions are made and the derivatives are introduced, the market is complete. Alternatively, we can start with a pricing kernel that supports the risk and return tradeoffs implied by these derivatives and the risky stock. These two approaches are clearly equivalent, and the key element that is important for our analysis is how each of the three risk factors is priced.

To be able to address, for realistic market conditions, the optimal derivative strategies and the quantitative improvement for including derivatives, it is important that we adopt a pricing kernel — or equivalently, a specification for the risk premia in derivatives — that accommodates the empirically documented risk and return tradeoffs implied by options on the aggregate market. Using joint time-series data on the risky stock (the S&P 500 index) and European-style options (the S&P 500 index options), recent studies have documented the importance of the risk premia implicit options, particularly those associated with the volatility and jump risks [Chernov and Ghysels (2000), Pan (2002), Benzoni (1998), and Bakshi and Kapadia (2001)]. Consistent with these findings, Coval and Shumway (2001) report the expected option returns that cannot be explained by the risk and return tradeoff associated with the usual diffusive price shock. Collectively, these empirical studies suggest that the volatility risk and jump risk are indeed priced in options on the aggregate stock market. For this reason, we adopt in this paper a parametric pricing kernel that is capable of separately pricing all three risk factors.

It should be noted that by exogenously introducing a pricing kernel, our analysis is of a partial-equilibrium nature. In fact, this is very much the spirit of the asset allocation problem: a small investor takes the prices (both risks and returns) as given and finds for himself the optimal trading strategy. By the same token, as we later quantify the improvement for including derivatives, we are addressing the improvement in certainty equivalent wealth for this very investor, not the welfare improvement of the society as a whole.<sup>4</sup>

The dynamic asset allocation problem is solved in closed form. Intuitively, this problem can be viewed as being solved in three steps. Given the market completeness with respect to the riskless bond, the risky stock, and the two non-redundant derivative securities, we obtain the investor's optimal wealth dynamics. We then find the optimal exposures to the three risk factors to support this optimal wealth dynamics. Finally, we find the optimal positions on the risky stock and the two derivative securities to achieve the optimal exposures on the risk

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<sup>4</sup>The latter requires an equilibrium treatment. See, for example, the literature on financial innovation [Allen and Gale (1994)].

factors. Closely connected to this last step is the role of derivative securities in completing the market. As discussed earlier, not all financial contracts are capable of achieving this goal, and this statement is formally made in our paper as a non-redundancy condition on the chosen derivatives. Closely connected to the second step is our assumptions on how each of the risk factor is priced. In fact, they are the main economic driving force behind the optimal exposures to the risk factors. As discussed earlier, our specification of the market prices of risks is chosen mainly to accommodate the empirical evidence, particularly that from the option market. As will be shown in two illustrative examples, the optimal derivative strategy and the quantitative improvement for including derivatives change significantly as we change the market prices of such risk factors.

Our first illustrative example is on the role of derivatives as a vehicle to volatility risk. In this setting, the demand for derivatives arises from the need to access the volatility risk. As a result, the optimal portfolio weight on the derivative security depends explicitly on how sensitive the chosen derivative is to the stock volatility. Our result also shows that there are two economically different sources from which the need to access the volatility risk arises. Acting myopically, the investor participates in the derivatives market simply to take advantage of the risk and return tradeoff provided by the volatility risk. For example, if the volatility risk is not priced at all, he would find no “myopic” incentive to take on derivative positions. On the other hand, a negatively priced volatility risk induces him to sell volatility by writing options. Acting non-myopically, the investor holds derivatives to further exploit the time-varying nature of his investment opportunity, which, in our setting, is driven exclusively by the stochastic volatility. As the volatility becomes more persistent, this non-myopic demand for derivatives becomes more prominent, and it also changes sharply around the investment horizon that is close to the half life of the volatility.

To assess the portfolio improvement for participating in the derivatives market, we compare the certainty equivalent wealth of two utility-maximizing investors with and without access to the derivatives market. To further quantify the gain from taking advantage of derivatives, we calibrate the parameters of the stochastic volatility model to those reported by empirical studies on the S&P 500 index and option markets. Our results show that the improvement for including derivatives is driven mostly by the risk and return tradeoff associated with the additional volatility risk. At normal market conditions and with a conservative estimate of the volatility-risk premium, the improvement in certainty equivalent wealth for an investor with relative risk aversion of three is about 14% per year, which becomes higher when the market becomes more volatile.

Our second illustrative example is on the role of derivatives as a vehicle to dis-entangle the jump risk from the diffusive risk. In this setting, the relative attractiveness between the jump risk and diffusive risk is the economic driving force behind our result. If the jump risk is compensated in such a way that the investor finds it to be as attractive as the diffusive risk, then there is no need to dis-entangle the two risk factors. Hence, zero demand for derivatives. Empirically, however, it is generally not true that the two risk factors are rewarded equally. In fact, the empirical evidence from the option market suggests that, for investors with reasonable range of risk aversion, the jump risk is compensated more than the diffusive risk. A recent paper by Bates (2001) addresses this issue in an equilibrium setting by considering an investor with an additional aversion to market crashes.

Apart from the quantitative difference, the jump risk differs from the diffusive risk in an important, qualitative way. Specifically, in the presence of large, negative jumps, the investor is reluctant to hold too much of the jump risk regardless of how much premium is assigned to it. Intuitively, this is because in contrast to the diffusive risks, which can be controlled via continuous trading, the sudden, high-impact nature of jump risks takes away the investor's ability to continuously trade his way out of a leveraged position to avoid negative wealth. As a result, without access to derivatives, the investor avoids taking a too leveraged position on the risky stock [Liu, Longstaff, and Pan (2002)]. The same investor is nevertheless freer to make choices when the worst-case scenarios associated with the jump risk can be taken care of by trading derivatives. In our quantitative example, this is reflected by the optimal trading strategy of taking a larger position on the risky stock and buying deep out-of-the-money put options to hedge out the negative jump risk.

The rest of the paper is organized as follows. Section 2 describes the investment environment including the risky stock and the derivative securities. Section 3 formalizes the investment problem and provides the explicit solutions. Section 4 provides an extensive example on the role of derivatives in the presence of volatility risk, while Section 5 focuses on jump risks. Section 6 concludes the paper. Technical details are provided in the appendices.

## 2 The Model

### 2.1 The Stock Price Dynamics

The fundamental securities in this economy are a riskless bond that pays a constant rate of interest  $r$ , and a risky stock that represents the aggregate equity market. To capture the empirical features that are important in the time-series data on the aggregate stock market, we assume the following dynamics for the price process  $S$  of the risky stock,

$$dS_t = (r + \eta V_t + \mu(\lambda - \lambda^Q) V_t) S_t dt + \sqrt{V_t} S_t dB_t + \mu S_{t-} (dN_t - \lambda V_t dt) \quad (1)$$

$$dV_t = \kappa(\bar{v} - V_t) dt + \sigma \sqrt{V_t} \left( \rho dB_t + \sqrt{1 - \rho^2} dZ_t \right), \quad (2)$$

where  $B$  and  $Z$  are standard Brownian motions, and  $N$  is a pure-jump process. All three random shocks  $B$ ,  $Z$ , and  $N$  are assumed to be independent.

This model incorporates, in addition to the usual diffusive price shock  $B$ , two risk factors that are important in characterizing the aggregate stock market: stochastic volatility and price jumps. Specifically, the instantaneous variance process  $V$  is a stochastic process with long-run mean  $\bar{v} > 0$ , mean-reversion rate  $\kappa > 0$ , and volatility coefficient  $\sigma \geq 0$ . This formulation of stochastic volatility, due to Heston (1993), allows the diffusive price shock  $B$  to enter the volatility dynamics via the constant coefficient  $\rho \in (-1, 1)$ , introducing correlations between the price and volatility shocks — a feature that is important in the data.

The random arrival of jump events is dictated by the pure-jump process  $N$  with stochastic arrival intensity  $\{\lambda V_t : t \geq 0\}$  for constant  $\lambda \geq 0$ . Intuitively, the conditional probability at time  $t$  of another jump before  $t + \Delta t$  is, for some small  $\Delta t$ , approximately  $\lambda V_t \Delta t$ . This formulation, due to Bates (2000), has the intuitive interpretation that jumps are more likely

to occur during volatile markets. Following Cox and Ross (1976), we adopt deterministic jump amplitudes. That is, conditional on a jump arrival, the stock price jumps by a constant multiple of  $\mu > -1$ , with the limiting case of  $-1$  representing the situation of total ruin. As it becomes clear later, this specification of deterministic jump amplitude simplifies our analysis in the sense that only one additional derivative security is needed to complete the market with respect to the jump component.<sup>5</sup> This formulation, though simple, is capable of capturing the sudden and high-impact nature of jumps that cannot be produced by diffusions.

Finally,  $\eta$  and  $\lambda^Q$  are constant coefficients capturing the two components of the equity premium: one for the diffusive risk  $B$ , the other for the jump risk  $N$ . More detailed discussions on these two parameters will be provided in the next section as we introduce the pricing kernel for this economy.

## 2.2 The Derivative Securities and the Pricing Kernel

In addition to investing in the risky stock and the riskless bond, the investor is also given the chance to include derivatives in his portfolio. Clearly, the relevant derivative securities are those that serve to expand the dimension of risk and return tradeoffs for the investor. More specifically for our setting, such derivatives are those that provide differential exposures to the three fundamental risk factors in the economy.

For concreteness, we consider the class of derivatives whose time- $t$  price  $O_t$  depends on the underlying stock price  $S_t$  and the stock volatility  $V_t$  through  $O_t = g(S_t, V_t)$ , for some function  $g$ . Although more complicated derivatives can be adopted in our setting, this class of derivatives provides the cleanest intuition possible. Letting  $\tau$  be its time to expiration, this particular derivative is defined by its payoff structure at the time of expiration. For example, a derivative with a linear payoff structure  $g(S_\tau, V_\tau) = S_\tau$  is clearly the stock itself, and it must be that  $g(S_t, V_t) = S_t$  at any time  $t < \tau$ . On the other hand, for some strike price  $K > 0$ , a derivative with the non-linear payoff structure  $g(S_\tau, V_\tau) = (S_\tau - K)^+$  is a European-style call option, while that with  $g(S_\tau, V_\tau) = (K - S_\tau)^+$  is a European-style put option. Unlike our earlier example for the linear contract, the pricing relation  $g(S_t, V_t)$  at  $t < \tau$  is not uniquely defined in these two cases by using the information contained in the risky stock only. In other words, by including multiple sources of risks in a non-trivial way, the market is incomplete with respect to the risky stock and riskless bond.

Clearly, the market can be completed once we introduce enough non-redundant derivatives  $O_t^{(i)} = g^{(i)}(S_t, V_t)$  for  $i = 1, 2, \dots, N$ . Alternatively, we can introduce a specific pricing kernel to price all of the risk factors in this economy, and consequently, any derivative securities. These two approaches are indeed equivalent. That is, the particular specification of the  $N$  derivatives that complete the market is linked uniquely to a pricing kernel  $\{\pi_t, 0 \leq t \leq T\}$  such that,

$$O_t^{(i)} = \frac{1}{\pi_t} E_t [\pi_{\tau_i} g^{(i)}(S_{\tau_i}, V_{\tau_i})] , \quad (3)$$

for any  $t \leq \tau_i$ , where  $\tau_i$  is the time to expiration for the  $i$ -th derivative security.

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<sup>5</sup>More generally, one could introduce random jumps with multiple outcomes and use multiple derivatives to help complete the market.

In this paper, we choose the latter approach and start with the following parametric pricing kernel:

$$d\pi_t = -\pi_t \left( r dt + \eta \sqrt{V_t} dB_t + \xi \sqrt{V_t} dZ_t \right) + \left( \frac{\lambda^Q}{\lambda} - 1 \right) \pi_{t-} (dN_t - \lambda V_t dt), \quad (4)$$

where  $\pi_0 = 1$  and the constant coefficients  $\eta$ ,  $\xi$ , and  $\lambda^Q/\lambda$  respectively control the premiums for the diffusive price risk  $B$ , the additional volatility risk  $Z$ , and the jump risk  $N$ . Consistent with this pricing kernel is the following parametric specification of the price dynamics for the  $i$ -th derivative security:

$$\begin{aligned} dO_t^{(i)} = & r O_t^{(i)} dt + \left( g_s^{(i)} S_t + \sigma \rho g_v^{(i)} \right) \left( \eta V_t dt + \sqrt{V_t} dB_t \right) + \sigma \sqrt{1 - \rho^2} g_v^{(i)} \left( \xi V_t dt + \sqrt{V_t} dZ_t \right) \\ & + \Delta g^{(i)} \left( (\lambda - \lambda^Q) V_t dt + dN_t - \lambda V_t dt \right), \end{aligned} \quad (5)$$

where  $g_s^{(i)}$  and  $g_v^{(i)}$  measure the sensitivity of the  $i$ -th derivative price to infinitesimal changes in the stock price and volatility, respectively, and where  $\Delta g^{(i)}$  measures the change in derivative price for each jump in the underlying stock price. Specifically,

$$g_s^{(i)} = \left. \frac{\partial g^{(i)}(s, v)}{\partial s} \right|_{(S_t, V_t)}; \quad g_v^{(i)} = \left. \frac{\partial g^{(i)}(s, v)}{\partial v} \right|_{(S_t, V_t)}; \quad \Delta g^{(i)} = g^{(i)}((1 + \mu) S_t, V_t) - g^{(i)}(S_t, V_t). \quad (6)$$

A derivative with non-zero  $g_s$  provides exposure to the diffusive price shock  $B$ , one with non-zero  $g_v$  provides exposure to the additional volatility risk  $Z$ , and one with non-zero  $\Delta g$  provides exposure to the jump risk  $N$ . To complete the market with respect to these three risk factors, one needs at least three securities. For example, one can start with the risky stock, which provides simultaneous exposure to the diffusive price shock  $B$  and the jump risk  $N$ :  $g_s = \Delta g / \Delta S = 1$ . To separate his exposure to the jump risk from that to the diffusive price shock, the investor can add out-of-the-money put options to his portfolio, which provides more exposure to the jump risk than the diffusive risk:  $|\Delta g / \Delta S| \gg |g_s|$ . Finally, to get himself exposed to the additional volatility risk  $Z$ , he can add at-the-money options, which provides  $g_v > 0$ .

In essence, the role of the derivative securities here is to provide separate exposures to the fundamental risk factors. It is important to point out that not all financial contracts can achieve such a goal. For example, bond derivatives are infeasible because they can only provide exposure to the constant riskfree rate. Other individual stocks are generally infeasible because our risky stock represents the aggregate equity market, which is a linear combination of the individual stocks.<sup>6</sup>

In addition to providing exposures to the risk factors, the derivatives also pick up the associated returns. This risk and return tradeoff is controlled by the specific parametric form of the pricing kernel  $\pi$ , or equivalently, by the particular price dynamic specified for

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<sup>6</sup>Of course, one can think of the extreme case where one group of individual stocks contribute exclusively to the diffusive risk or the jump risk at the aggregate level, but not both. It is even more unlikely that an individual stock that is linear in nature could provide exposure to the volatility risk at the aggregate level.



the derivatives. To be more specific, from either (4) or (5), we can see that the constant  $\eta$  controls the premium for the diffusive price risk  $B$ , the constant  $\xi$  controls that for the additional volatility risk  $Z$ , and the constant ratio  $\lambda^Q/\lambda$  controls that for the jump risk.<sup>7</sup>

Apart from analytical tractability,<sup>8</sup> this specific parametric form has the advantage of having three parameters  $\eta$ ,  $\xi$ , and  $\lambda^Q/\lambda$  to separately price the three risk factors in the economy. This flexibility is in fact supported empirically. Using joint time-series data on the risky stock (the S&P 500 index) and European-style options (the S&P 500 index options), recent studies have documented the importance of the risk premia implicit options, particularly those associated with the volatility and jump risks [Chernov and Ghysels (2000), Pan (2002), Benzoni (1998), and Bakshi and Kapadia (2001)]. Consistent with these findings, Coval and Shumway (2001) report the expected option returns that cannot be explained by the risk and return tradeoff associated with the usual diffusive price shock  $B$ . Collectively, these empirical studies on the options market suggest that the additional risk factors, such as the volatility risk and jump risk, are indeed priced in the option market. Given our focus on the optimal investment decision associated with derivatives, it is all the more important for us to choose a parametric form that accommodates the empirically documented risk and return tradeoff associated with options on the aggregate market.

Although our approach in this paper is partial equilibrium in nature, our choice of pricing kernel can also be related to those derived from equilibrium studies. For the special case of constant volatility, our specific pricing kernel can be mapped to the equilibrium result of Naik and Lee (1990). Letting  $\gamma$  be the relative risk-aversion coefficient of the representative agent, the coefficient for the diffusive-risk premium is  $\eta = \gamma$ , and the coefficient for the jump-risk premium is  $\lambda^Q/\lambda = (1 + \mu)^{-\gamma}$ . In the presence of adverse jump risk ( $\mu < 0$ ), he fears that jumps are more likely to occur ( $\lambda^Q > \lambda$ ), consequently requiring a positive premium for holding the jump risk. It is important to notice that the market prices of both risk factors are controlled by one parameter: the risk-aversion coefficient  $\gamma$  of the representative agent. The empirical evidence from the option market, however, seems to suggest that the jump risk is priced quite differently from the diffusive risk. To accommodate this difference, a recent paper by Bates (2001) introduces a representative agent with an additional crash aversion coefficient  $Y$ . Mapping his equilibrium result to our parametric pricing kernel, we have  $\eta = \gamma$ , and  $\lambda^Q/\lambda = (1 + \mu)^{-\gamma} \exp(Y)$ . The usual risk aversion coefficient  $\gamma$  contributes to the market price of the diffusive risk, while the crash aversion contributes an additional layer to the market price of the jump risk.

In this respect, we can think of our parametric approach to the pricing kernel as a reduced-form approach. For the purpose of understanding the economic sources of the risk and return, a structural approach such as Naik and Lee (1990) and Bates (2001) is required. For the purpose of obtaining the optimal derivative strategies with given market conditions, however, such a reduced-form approach is in fact sufficient and has been adopted in the asset allocation literature. Finally, to verify that the parametric pricing kernel  $\pi$  is indeed a valid pricing kernel, which rules out arbitrage opportunities involving the riskless bond, the risky

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<sup>7</sup>It should be noted that  $\lambda^Q \geq 0$ , and  $\lambda^Q = 0$  if and only if  $\lambda = 0$ .

<sup>8</sup>For a European-style option with maturity  $\tau_i$  and strike price  $K_i$ , we have  $g^{(i)} = c(S_t, V_t; K_i, \tau_i)$ , where the explicit functional form of  $c$  can be derived via transform analysis. For this specific case, the original solution is given by Bates (2000). See also Heston (1993) and Duffie, Pan, and Singleton (2000).

stock, and any derivative securities, one can apply Ito's lemma and show that  $\pi_t \exp(-rt)$ ,  $\pi_t S_t$ , and  $\pi_t O_t^{(i)}$  are local martingales.<sup>9</sup>

### 3 The Investment Problem and the Solution

The investor starts with a positive wealth  $W_0$ . Given the opportunity to invest in the riskless asset, the risky stock and the derivative securities, he chooses, at each time  $t$ ,  $0 \leq t \leq T$ , to invest a fraction  $\phi_t$  of his wealth in the stock  $S_t$ , and fractions  $\psi_t^{(1)}$  and  $\psi_t^{(2)}$  in the two derivative securities  $O_t^{(1)}$  and  $O_t^{(2)}$ , respectively. The investment objective is to maximize the expected utility of his terminal wealth  $W_T$ ,

$$\max_{\phi_t, \psi_t, 0 \leq t \leq T} E \left( \frac{W_T^{1-\gamma}}{1-\gamma} \right), \quad (7)$$

where  $\gamma > 0$  is the relative risk-aversion coefficient of the investor, and where the wealth process satisfies the self-financing condition

$$\begin{aligned} dW_t = & r W_t dt + \theta_t^B W_t \left( \eta V_t dt + \sqrt{V_t} dB_t \right) + \theta_t^Z W_t \left( \xi V_t dt + \sqrt{V_t} dZ_t \right) \\ & + \theta_{t-}^N W_{t-} \mu \left( (\lambda - \lambda^Q) V_t dt + dN_t - \lambda V_t dt \right), \end{aligned} \quad (8)$$

where  $\theta_t^B$ ,  $\theta_t^Z$ , and  $\theta_t^N$  are defined, for given portfolio weights  $\phi_t$  and  $\psi_t$  on the stock and the derivatives, by

$$\begin{aligned} \theta_t^B &= \phi_t + \sum_{i=1}^2 \psi_t^{(i)} \left( \frac{g_s^{(i)} S_t}{O_t^{(i)}} + \sigma \rho \frac{g_v^{(i)}}{O_t^{(i)}} \right); \quad \theta_t^Z = \sigma \sqrt{1 - \rho^2} \sum_{i=1}^2 \psi_t^{(i)} \frac{g_v^{(i)}}{O_t^{(i)}}; \\ \theta_t^N &= \phi_t + \sum_{i=1}^2 \psi_t^{(i)} \frac{\Delta g^{(i)}}{\mu O_t^{(i)}}. \end{aligned} \quad (9)$$

Effectively, by taking positions  $\phi_t$  and  $\psi_t$  on the risky assets, the investor invests  $\theta^B$  on the diffusive price shock  $B$ ,  $\theta^Z$  on the additional volatility risk  $Z$ , and  $\theta^N$  on the jump risk  $N$ . For example, a portfolio position  $\phi_t$  on the risky stock provides equal exposures to both the diffusive and jump risks in stock prices. Similarly, a portfolio position  $\psi_t$  on the derivative security provides exposure to the volatility risk  $Z$  via a non-zero  $g_v$ , exposure to the diffusive price shock  $B$  via a non-zero  $g_s$ , and exposure to the jump risk via a non-zero  $\Delta g$ .

Except for adding derivative securities in the investor's opportunity set, the investment problem in (7) and (8) is the standard Merton (1971) problem. Before solving for this problem, we should point out that the maturities of the chosen derivatives do not have to match the investment horizon  $T$ . For example, it might be hard for an investor with a 10-year investment horizon to find an option with a matching maturity. He may choose to invest in options with much shorter time to expiration, say LEAPS, which typically expires

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<sup>9</sup>See, for example, Appendix B.2 of Pan (2000).

in one or two years, and switch or roll over to other derivatives in the future. For the purpose of choosing the optimal portfolio weights at time  $t$ , what matters is his choice of derivative securities  $O_t$  at that time, not his future choice of derivatives. This is true as long as, at each point in time in the future, there exist non-redundant derivative securities to complete the market.

We now proceed to solve the investment problem in (7) using the stochastic control approach. Alternatively, our problem can be solved using the Martingale approach of Cox and Huang (1989). Indeed, to further deliver the intuition behind our solution, we will come back and interpret the solution from the angle of the Martingale approach. Following Merton (1971), we define the indirect utility function by

$$J(t, w, v) = \max_{\{\phi_s, \psi_s, t \leq s \leq T\}} E \left( \frac{W_T^{1-\gamma}}{1-\gamma} \middle| W_t = w, V_t = v \right), \quad (10)$$

which, by the principle of optimal stochastic control, satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned} \max_{\phi_t, \psi_t} \left\{ J_t + W_t J_W \left( r_t + \theta^B \eta V_t + \theta_t^Z \xi V_t - \theta^N \mu \lambda^Q V_t \right) + \frac{1}{2} W_t^2 J_{WW} V_t \left( (\theta^B)^2 + (\theta^Z)^2 \right) \right. \\ \left. + \lambda V_t \Delta J + \kappa (\bar{v} - V_t) J_V + \frac{1}{2} \sigma^2 V_t J_{VV} + \sigma V_t W_t J_{WV} \left( \rho \theta^B + \sqrt{1 - \rho^2} \theta_t^Z \right) \right\} = 0, \end{aligned} \quad (11)$$

where  $\Delta J = J(t, W_t(1 + \theta^N \mu), V_t) - J(t, W_t, V_t)$  denotes the jump in the indirect utility function  $J$  for given jumps in the stock price, and where  $J_t$ ,  $J_W$ , and  $J_V$  denote the derivatives of  $J(t, W, V)$  with respect to  $t$ ,  $W$  and  $V$  respectively, and similar notations for higher derivatives.

To solve the HJB equation, we notice that it depends explicitly on the portfolio weights  $\theta^B$ ,  $\theta^Z$ , and  $\theta^N$ , which, as defined in (9), are linear transformations of the portfolio weights  $\phi$  and  $\psi$  on the risky assets. Taking advantage of this structure, we first solve the optimal positions on the risk factors  $B$ ,  $Z$ , and  $N$ , and then transform them back via the linear relation (9) to the optimal positions on the risky assets. This transformation is feasible as long as the chosen derivatives are non-redundant in the following sense.

**Definition:** At any time  $t$ , the derivative securities  $O_t^{(1)}$  and  $O_t^{(2)}$  are non-redundant if

$$\mathcal{D}_t \neq 0 \quad \text{where} \quad \mathcal{D}_t = \left( \frac{\Delta g^{(1)}}{\mu O_t^{(1)}} - \frac{g_s^{(1)} S_t}{O_t^{(1)}} \right) \frac{g_v^{(2)}}{O_t^{(2)}} - \left( \frac{\Delta g^{(2)}}{\mu O_t^{(2)}} - \frac{g_s^{(2)} S_t}{O_t^{(2)}} \right) \frac{g_v^{(1)}}{O_t^{(1)}} \quad (12)$$

Effectively, the non-redundancy condition in (12) guarantees market completeness with respect to the chosen derivative securities, the risky stock, and the riskless bond. Without access to derivatives, linear positions on the risky stock provide equal exposures to the diffusive and jump risks, and none to the volatility risk. To complete the market with respect to the volatility risk, we need to bring in a risky asset that is sensitive to changes in volatility:  $g_v \neq 0$ . To complete the market with respect to the jump risk, we need a

risky asset with different sensitivities to the *infinitesimal* and *large* changes in stock prices:  $g_s S_t / O_t \neq \Delta g / \mu O_t$ . Moreover, (12) also ensures that the two chosen derivative securities are not identical in covering the two risk factors.

**Proposition 1** *Assume that there are non-redundant derivatives available for trade at any time  $t < T$ . Then, for given wealth  $W_t$  and volatility  $V_t$ , the solution to the HJB equation is given by*

$$J(t, W_t, V_t) = \frac{W_t^{1-\gamma}}{1-\gamma} \exp(\gamma h(T-t) + \gamma H(T-t) V_t), \quad (13)$$

where  $h(\cdot)$  and  $H(\cdot)$  are time-dependent coefficients that are independent of the state variables. That is, for any  $0 \leq \tau \leq T$ ,

$$\begin{aligned} h(\tau) &= \frac{2\kappa\bar{v}}{\sigma^2} \ln \left( \frac{2k_2 \exp((k_1 + k_2)\tau/2)}{2k_2 + (k_1 + k_2)(\exp(k_2\tau) - 1)} \right) + \frac{1-\gamma}{\gamma} r\tau \\ H(\tau) &= \frac{\exp(k_2\tau) - 1}{2k_2 + (k_1 + k_2)(\exp(k_2\tau) - 1)} \delta \end{aligned} \quad (14)$$

where

$$\begin{aligned} \delta &= \frac{1-\gamma}{\gamma^2} (\eta^2 + \xi^2) + 2\lambda^Q \left[ \left( \frac{\lambda}{\lambda^Q} \right)^{1/\gamma} + \frac{1}{\gamma} \left( 1 - \frac{\lambda}{\lambda^Q} \right) - 1 \right] \\ k_1 &= \kappa - \frac{1-\gamma}{\gamma} (\eta\rho + \xi\sqrt{1-\rho^2})\sigma; \quad k_2 = \sqrt{k_1^2 - \delta\sigma^2} \end{aligned}$$

The optimal portfolio weights on the risk factors  $B$ ,  $Z$ , and  $N$  are given by

$$\theta_t^{*B} = \frac{\eta}{\gamma} + \sigma\rho H(T-t); \quad \theta_t^{*Z} = \frac{\xi}{\gamma} + \sigma\sqrt{1-\rho^2} H(T-t); \quad \theta_t^{*N} = \frac{1}{\mu} \left( \left( \frac{\lambda}{\lambda^Q} \right)^{1/\gamma} - 1 \right). \quad (15)$$

Transforming the  $\theta^*$ 's to the optimal portfolio weights on the risky assets,  $\phi_t^*$  for the stock and  $\psi_t^*$  for derivatives, we have

$$\begin{aligned} \phi_t^* &= \theta_t^{*B} - \sum_{i=1}^2 \psi_t^{*(i)} \left( \frac{g_s^{(i)} S_t}{O_t^{(i)}} + \sigma\rho \frac{g_v^{(i)}}{O_t^{(i)}} \right) \\ \psi_t^{*(1)} &= \frac{1}{\mathcal{D}_t} \left[ \frac{g_v^{(2)}}{O_t^{(2)}} \left( \theta_t^{*N} - \theta_t^{*B} - \frac{\theta_t^{*Z} \rho}{\sqrt{1-\rho^2}} \right) - \left( \frac{\Delta g^{(2)}}{\mu O_t^{(2)}} - \frac{g_s^{(2)} S_t}{O_t^{(2)}} \right) \frac{\theta_t^{*Z}}{\sigma\sqrt{1-\rho^2}} \right] \\ \psi_t^{*(2)} &= \frac{1}{\mathcal{D}_t} \left[ \left( \frac{\Delta g^{(1)}}{\mu O_t^{(1)}} - \frac{g_s^{(1)} S_t}{O_t^{(1)}} \right) \frac{\theta_t^{*Z}}{\sigma\sqrt{1-\rho^2}} - \frac{g_v^{(1)}}{O_t^{(1)}} \left( \theta_t^{*N} - \theta_t^{*B} - \frac{\theta_t^{*Z} \rho}{\sqrt{1-\rho^2}} \right) \right]. \end{aligned} \quad (16)$$

**Proof:** See Appendix.

To further deliver the intuition behind the result in Proposition 1, we can examine our result from the angle of the Martingale approach. Given that the market is complete after

introducing the derivative securities (or equivalently the pricing kernel  $\pi$ ), the terminal wealth  $W_T^*$  associated with the optimal portfolio strategy can be solved directly from,

$$\max_{W_T} E_0 \left( \frac{W_T^{1-\gamma}}{1-\gamma} \right) \quad \text{subject to} \quad E_0 (\pi_T W_T) = W_0. \quad (17)$$

Solving this constrained optimization problem explicitly, and using the fact that, at any time  $t < T$ ,  $W_t^* = E(\pi_T W_T^*) / \pi_t$ , we can show that the optimal wealth dynamics  $\{W_t^*, 0 \leq t \leq T\}$  indeed follows that specified in (8), with  $\theta^B$ ,  $\theta^Z$  and  $\theta^N$  replaced by the optimal solution given by (15) in Proposition 1.

From this perspective, our results can be interpreted as three steps. First, solve for the optimal wealth dynamics. Second, find the optimal exposures  $\theta^{*B}$ ,  $\theta^{*Z}$ , and  $\theta^{*N}$  to the fundamental risk factors to support this optimal wealth dynamics. Finally, find the optimal positions  $\phi^*$ ,  $\psi^{*(1)}$ , and  $\psi^{*(2)}$  on the risky stock and the two derivative securities to achieve the optimal exposures on the risk factors. Clearly, the mapping in this last step is only feasible when the market is indeed incomplete with respect to the three securities  $S$ ,  $O^{(1)}$  and  $O^{(2)}$ . That is, when the non-redundancy condition (12) is satisfied.

To further illustrate our results, we consider two examples in the next two sections, one on volatility risks and the other on jump risks.

## 4 Example I: Derivatives and Volatility Risk

This section focuses on the role of derivative securities as a vehicle to stochastic volatility. For this, we specialize in an economy with volatility risk but no jump risk. Specifically, we turn off the jump component in (1) and (2) by letting  $\mu = 0$  and  $\lambda = \lambda^Q = 0$ .

In such a setting, only one derivative security with non-zero sensitivity to volatility risk is needed to help complete the market. Denoting this derivative security by  $O_t$ , we can readily use the result of Proposition 1 to derive the optimal portfolio weights:

$$\phi_t^* = \frac{\eta}{\gamma} - \frac{\xi \rho}{\gamma \sqrt{1-\rho^2}} - \psi_t^* \frac{g_s S_t}{O_t} \quad (18)$$

$$\psi_t^* = \left( \frac{\xi}{\gamma \sigma \sqrt{1-\rho^2}} + H(T-t) \right) \frac{O_t}{g_v}, \quad (19)$$

where  $\phi_t^*$  and  $\psi_t^*$  denote the optimal positions on the risky stock and the derivative security, respectively, and where  $H$  is as defined in (14) with the simplifying restriction of no jumps.

### 4.1 The Demand for Derivatives

The optimal derivative position  $\psi^*$  in (19) is inversely proportional to  $g_v/O_t$ , which measures the volatility exposure for each dollar invested in the derivative security. Intuitively, the demand for derivatives arises in this setting from the need to access the volatility risk. The more “volatility exposure per dollar” a derivative security provides, the more effective it is as a vehicle to the volatility risk. Hence a smaller portion of the investor’s wealth needs to be

invested in this derivative security. By contrast, financial contracts with lower sensitivities to the aggregate market volatility are less effective for the same purpose. Of course, the extreme case will be those linear securities (e.g. individual stocks) that provide zero exposure to the volatility risk.

The demand for derivatives — or the need for volatility exposures — arises for two economically different reasons. First, a myopic investor finds the derivative security attractive because, as a vehicle to the volatility risk, it could potentially expand his dimension of risk and return tradeoffs. This myopic demand for derivatives is reflected in the first term of  $\psi_t^*$ . For example, a negatively priced volatility risk ( $\xi < 0$ ) makes short positions on volatility attractive, inducing investors to sell derivatives with positive “volatility exposure per dollar.” Similarly, a positive volatility-risk premium ( $\xi > 0$ ) induces opposite trading strategies. Moreover, the less risk-averse investor is more aggressive in taking advantage of the risk and return tradeoff through investing in derivatives.

Second, for an investor who acts non-myopically, there is a benefit in derivative investments even when the myopic demand diminishes with a zero volatility risk-premium ( $\xi = 0$ ). This non-myopic demand for derivatives is reflected in the second term of  $\psi_t^*$ . Without any loss of generality, let’s consider an option whose volatility exposure is positive ( $g_v > 0$ ). In our setting, the Sharpe ratio of the option return is driven exclusively by the stochastic volatility. In fact, it is proportional to the volatility. This implies that a higher realized option return at one instant is associated with a higher Sharpe ratio (better risk-return tradeoff) for the next-instant option return. In other words, a good outcome is more likely to be followed by another good outcome. By the same token, a bad outcome in the option return predicts a sequence of less attractive future risk-return tradeoffs. An investor with relative risk aversion  $\gamma > 1$  is particularly averse to sequences of negative outcomes because his utility is unbounded from below. On the other hand, an investor with  $\gamma < 1$  benefits from sequences of positive outcomes because his utility is unbounded from above. As a result, they act quite differently in response to this temporal uncertainty. The one with  $\gamma > 1$  takes a short position on volatility so as to hedge against the temporal uncertainty, while the one with  $\gamma < 1$  takes a long position on volatility so as to speculate on the temporal uncertainty. Indeed, it is easy to verify that  $H(T - t)$ , which is the driving force of this nonmyopic term, is strictly positive for investors with  $\gamma < 1$ , and strictly negative for investor with  $\gamma > 1$ , and zero for the log-utility investor.<sup>10</sup>

## 4.2 The Demand for Stock

Given that the volatility risk exposure is taken care of by the derivative holding, the “net” demand for stock should simply be linked to the risk and return tradeoff associated with the price risk. Focusing on the first term of  $\phi_t^*$  in (18), this is indeed true. Specifically, it is proportional to the attractiveness of the stock and inversely proportional to the investor’s risk aversion.

The interaction between the derivative security and its underlying stock, however, com-

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<sup>10</sup>One way to show this is by taking advantage of the ordinary differential equation (A.1) for  $H(\cdot)$  with the additional constraints of no jumps. Given the initial condition  $H(0) = 0$ , it is easy to see that the driving force for the sign of  $H$  is the constant term which has the same sign as  $1 - \gamma$ .

plicates the optimal demand for stocks. For example, by holding a call option, one effectively invests a fraction  $g_s$  — typically referred to as the “delta” of the option — on the underlying stock. The last term in  $\phi^*$  is there to correct this “delta” effect. In addition, there is also a “correlation” effect that originates from the negative correlation between the volatility and price shocks (typically referred to as the leverage effect [Black (1976)]). Specifically, a short position on the volatility automatically involves long positions on the price shock, and equivalently the underlying stock. The second term in  $\phi^*$  is there to correct this “correlation” effect.

### 4.3 Empirical Properties of the Optimal Strategies

To examine the empirical properties of our results, we fix a set of base-case parameters for our current model, using the results from the existing empirical studies.<sup>11</sup> Specifically, for the one-factor volatility risk, we set its long-run mean at  $\bar{v} = (0.13)^2$ , its rate of mean-reversion at  $\kappa = 5$ , and its volatility coefficient at  $\sigma = 0.25$ . The correlation between the price and volatility risks is set at  $\rho = -0.40$ .

Important for our analysis is how the risk factors are priced. Given the well established empirical property of the equity risk premium, calibrating the market price of the Brownian shocks  $B$  is straightforward. Specifically, setting  $\eta = 4$  and coupling it with the base-case value of  $\bar{v} = (0.13)^2$  for the long-run mean of volatility, we have an average equity risk premium of 6.76% per year.

The properties of the market price of the volatility risk, however, are not as well established. In part because that volatility is not a directly tradeable asset, there is less consensus on reasonable values for market prices of volatility risk. Empirically, however, there is strong support that volatility risk is indeed priced. For example, using the joint time-series data on the S&P 500 index and options, Chernov and Ghysels (2000), Pan (2002), Benzoni (1998), and Bakshi and Kapadia (2001) report that volatility risks are negatively priced. That is, short positions on volatility are compensated with a positive premium. Similarly, Coval and Shumway (2001) report large negative returns generated by positions that are long on volatility.

Given that the volatility risk at the aggregate level is generally related to the economic activity [Officer (1973); Schwert (1989)], it is quite plausible that it is priced. At an intuitive level, the negative volatility risk premium could be supported by the fact that the aggregate market volatility is typically high during recessions. A short position on volatility, which loses value when volatility becomes high during recessions, is therefore relatively more risky than a long position on volatility, requiring an additional risk premium.

Instead of calibrating the volatility-risk premium coefficient  $\xi$  to the existing empirical results, however, we will allow this coefficient to vary in our analysis so as to get a better

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<sup>11</sup>The empirical properties of the Heston (1993) model have been extensively examined using either the time-series data on the S&P 500 index alone [Andersen, Benzoni, and Lund (2001); Eraker, Johannes, and Polson (2000)], or the joint time-series data on the S&P 500 index and options [Chernov and Ghysels (2000); Pan (2002)]. Because of different sample periods and/or empirical approaches in these studies, the exact model estimates may differ from one paper to another. Our chosen model parameters are in the generally agreed region, with the exception of those reported by Chernov and Ghysels (2000).

understanding of how different levels and signs of the volatility risk premium could affect the optimal investment decision.

Using this set of base-case parameters, particularly the risk-and-return tradeoff implied by the data, we now proceed to provide some quantitative examples of optimal investments in the markets of S&P 500 index and options. To make the intuition as clean as possible, we focus on “delta-neutral” securities. Specifically, we consider the following “delta-neutral” straddle:

$$O_t = g(S_t, V_t; K, \tau) = c(S_t, V_t; K, \tau) + p(S_t, V_t; K, \tau), \quad (20)$$

where  $c$  and  $p$  are pricing formulas for call and put options with the same strike price  $K$  and time to expiration  $\tau$ . The explicit formulation of  $c$  and  $p$  is provided in Appendix B.1. For given stock price  $S_t$ , market volatility  $V_t$ , and time to expiration  $\tau$ , the strike price  $K$  is selected so that the call option has a delta of 0.5, and, by put/call parity, the put option has a delta of  $-0.5$ , making the straddle delta neutral.<sup>12</sup>

Fixing the riskfree rate at 5%, and picking a delta-neutral straddle with 0.1 year to expiration, Figure 1 provides optimal portfolio weights under different scenarios. The top-right panel examines the optimal portfolio allocation with varying volatility-risk premia. Qualitatively, this result is similar to our analysis in Section 4.1. Quantitatively, however, this result indicates that the demand for derivatives is driven mainly by the myopic component. In particular, when the volatility-risk premium is set to zero ( $\xi = 0$ ), the non-myopic demand for straddles is only 2% of the total wealth for an investor with relative risk aversion  $\gamma = 3$  and investment horizon  $T = 5$  years. In contrast, as we set  $\xi = -6$ , which is a conservative estimate for the volatility-risk premium, the optimal portfolio weight in the delta-neutral straddle increases to 54% for the same investor.

The quantitative effect of the non-myopic component can be best seen by varying the investment horizon (bottom left panel), or the volatility persistence (bottom right panel). Consider an investor with  $\gamma = 3$ , who would like to hedge against temporal uncertainty by taking short positions on volatility. The bottom left panel shows that as we increase his investment horizon, this intertemporal hedging demand increases. And, quite intuitively, the change is most noticeable around the region close to the half life of the volatility risk. Similarly, the bottom right panel shows that as we decrease the persistent level of the volatility by increasing the mean-reversion rate  $\kappa$ , there is less benefit in taking advantage of the intertemporal persistence. Hence a reduction in the intertemporal hedging demand.

As the market becomes more volatile, the cost of straddle ( $O_t$ ) increases, but the volatility sensitivity ( $g_v$ ) of such straddles decreases. Effectively the delta-neutral straddles provide less “volatility exposure per dollar” as the market volatility increases. To achieve the optimal volatility exposure, more needs to be invested in the straddle. Hence the increase in  $|\psi^*|$  with the market volatility  $\sqrt{V}$ . As the volatility of the volatility increases, the risk and return tradeoff on the volatility risk becomes less attractive. Hence the decrease in magnitude of the straddle position with increasing “vol of vol”  $\sigma$ . Finally, the optimal strategy with

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<sup>12</sup>Although “delta-neutral” positions can be constructed in numerous ways, we choose the “delta-neutral” straddle mainly because it is made of call and put options which are typically very close to the money. In particular, we intentionally avoid deep out-of-the-money options in our quantitative examples because they are most subject to concerns of option liquidity and jump risks, two important issues that are not accommodated formally in this section.



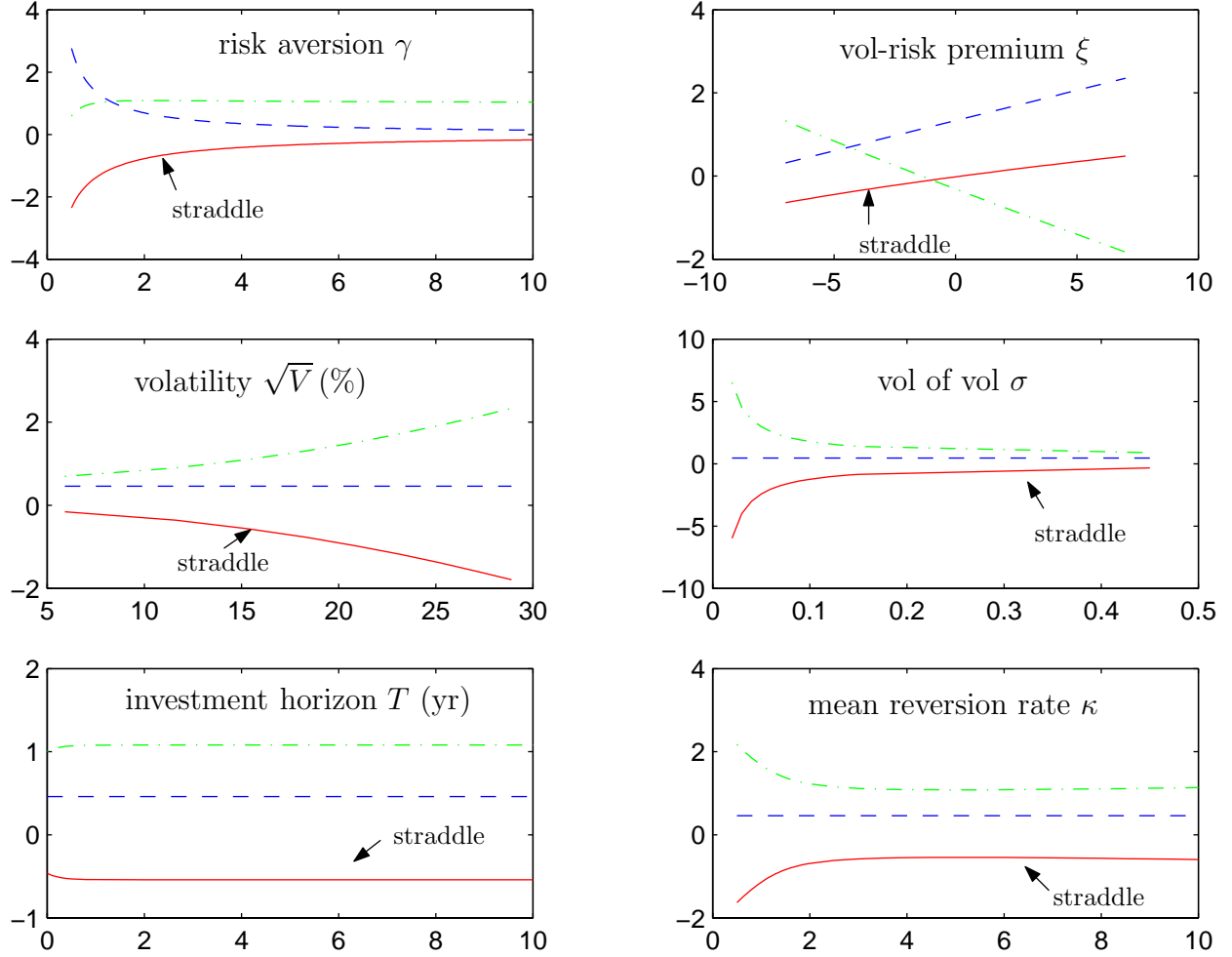


Figure 1: The optimal portfolio weights. The  $y$ -axes are the optimal weight  $\psi^*$  on the “delta-neutral” straddle (solid line),  $\phi^*$  on the risky stock (dashed line), and  $1 - \psi^* - \phi^*$  on the riskfree bank account (dashed-dot line). The base-case parameters are as described in Section 2, and the volatility-risk premium coefficient is fixed at  $\xi = -6$ . The base-case investor is the one with risk aversion  $\gamma = 3$  and investment horizon  $T = 5$  years. The riskfree rate is fixed at  $r = 5\%$ , and the base-case market volatility is fixed at  $\sqrt{V} = 15\%$ .

varying risk aversion  $\gamma$  is as expected: less risk-averse investors are more aggressive in their investment strategies.

#### 4.4 Portfolio Improvement

Consider an investor with an initial wealth of  $W_0$  and an investment horizon of  $T$  years. If he takes advantage of the derivatives market, his optimal expected utility is as provided in Proposition 1 (with the simplifying restriction of no jumps). For a given market volatility of  $V_0$ , his certainty equivalent wealth  $\mathcal{W}^*$  is<sup>13</sup>

$$\mathcal{W}^* = W_0 \exp \left( \frac{\gamma}{1-\gamma} \left[ h(T) + H(T) V_0 \right] \right), \quad (21)$$

where, again, the time-varying coefficients  $h$  and  $H$  are as defined in (14) with the simplifying constraint of no jumps. Alternatively, this investor might choose not to participate in the derivatives market. Let  $\mathcal{W}^{\text{no-op}}$  be the certainty equivalent wealth of such an investor who chooses not to invest in options. To quantify the portfolio improvement for including derivatives, we adopt the following measure<sup>14</sup>

$$\mathcal{R}^{\mathcal{W}} = \frac{\ln \mathcal{W}^* - \ln \mathcal{W}^{\text{no-op}}}{T}. \quad (22)$$

Effectively,  $\mathcal{R}^{\mathcal{W}}$  measures the portfolio improvement in terms of the annualized, continuously compounded return in certainty equivalent wealth. The following Proposition summarizes the results.

**Proposition 2** *For a power-utility investor with risk aversion coefficient  $\gamma > 0$  and investment horizon  $T$ , the improvement for including derivatives is*

$$\mathcal{R}^{\mathcal{W}} = \frac{\gamma}{1-\gamma} \left( \frac{h(T) - h^{\text{no-op}}(T)}{T} + \frac{H(T) - H^{\text{no-op}}(T)}{T} V_0 \right), \quad (23)$$

where  $V_0$  is the initial market volatility, and  $h^{\text{no-op}}$  and  $H^{\text{no-op}}$  are defined in (B.6). For an investor with  $\gamma \neq 1$ , the portfolio improvement for including derivatives is strictly positive. For an investor with log utility, the improvement is strictly positive if  $\xi \neq 0$ , and zero otherwise.

**Proof:** See Appendix B.

The improvement for including derivatives is closely linked to the demand for derivatives. For a myopic investor with log-utility, the demand for derivatives arises from the need to exploit the risk and return tradeoff provided by the volatility risk. When the volatility-risk

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<sup>13</sup>It should be noted that the optimal expected utility is independent of the specific derivative contract chosen by the investor. This is quite intuitive, because, in our setting, the market is complete in the presence of the derivative security.

<sup>14</sup>The indirect utility of the “no-option” investor can be derived using the results from Liu (1998). For the completeness of the paper, it is provided in Appendix B.

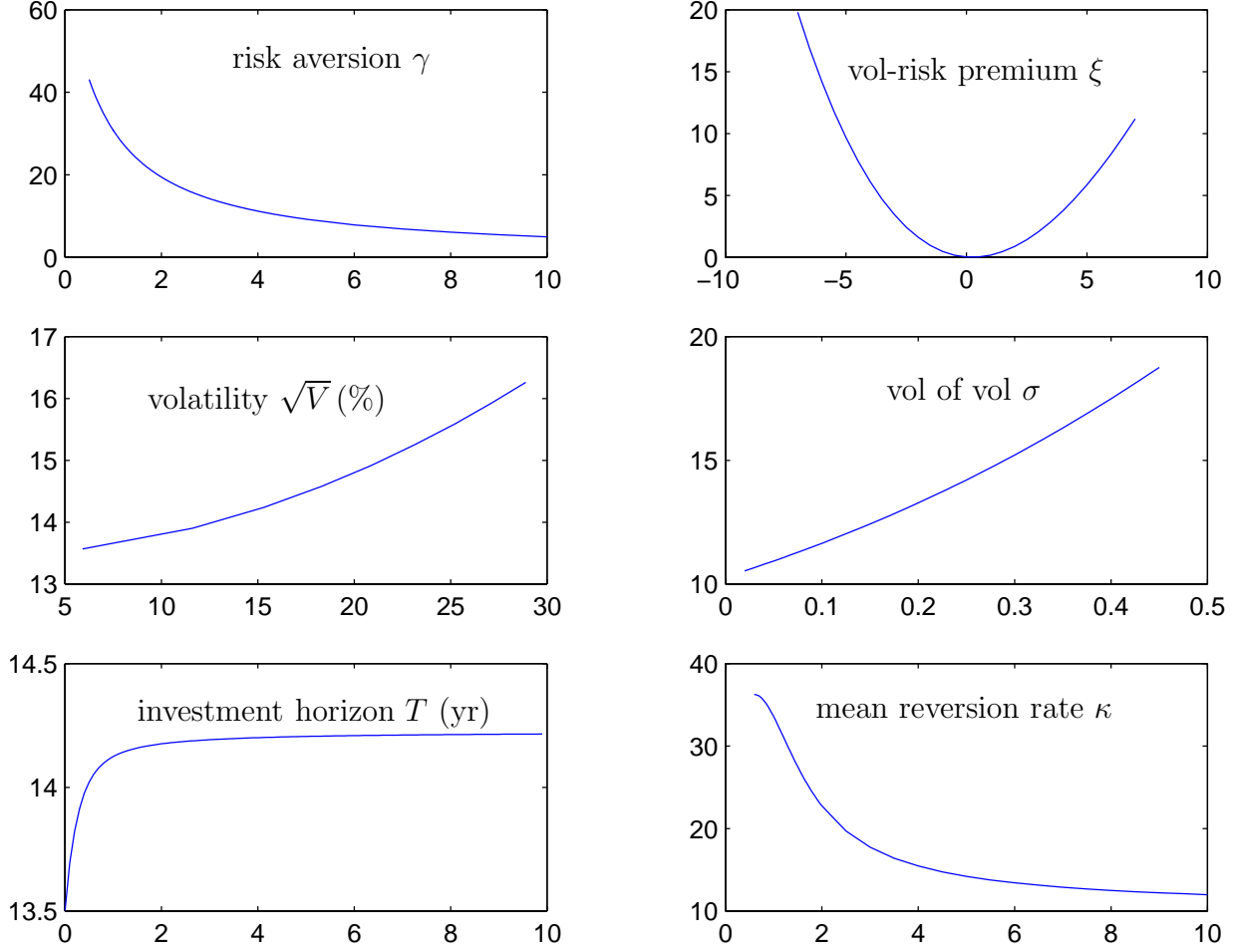


Figure 2: Portfolio improvement for including derivatives. The y-axes are the improvement measure  $\mathcal{R}^W$ , defined by (22) in terms of returns over certainty equivalent wealth. The base-case parameters are as described in Section 2, and the volatility-risk premium coefficient is fixed at  $\xi = -6$ . The base-case investor is the one with risk aversion  $\gamma = 3$  and investment horizon  $T = 5$  years. The riskfree rate is fixed at  $r = 5\%$ , and the base-case market volatility is fixed at  $\sqrt{V} = 15\%$ .

premium is set to zero ( $\xi = 0$ ), there is no myopic demand for derivatives. Consequently, there is no benefit for including derivatives. There are, however, still non-myopic demands for derivatives. Hence the portfolio improvement for a nonmyopic investor is strictly positive regardless of the value of  $\xi$ .

To provide a quantitative assessment of the portfolio improvement, we again use the base-case parameters described in Section 4.3. The results are summarized in Figure 2. Focusing first on the top-right panel, we see that the portfolio improvement is very sensitive to how the volatility risk is priced. At normal market condition with a conservative estimate<sup>15</sup> of the volatility-risk premium  $\xi = -6$ , our results show that the portfolio improvement for including derivatives is about 14.2% per year in certainty equivalent wealth for an investor with risk aversion  $\gamma = 3$ . As the investor becomes less risk averse and more aggressive in taking advantage of the derivatives market, the improvement for including derivatives becomes even higher (top left panel).

We can further evaluate the relative importance of the myopic and nonmyopic components of portfolio improvement by setting  $\xi = 0$ . The portfolio improvement from non-myopic trading of derivatives is as low as 0.02% per year. This is consistent with our earlier result: the demand for derivatives is driven mostly by the myopic component. The non-myopic component of the portfolio improvement is further examined in the bottom panels of Figure 2 as we vary the investment horizon and the persistence of volatility. Quite intuitively, as the investment horizon  $T$  increases, or, as the volatility shock becomes more persistent, the benefit of the derivative security as a hedge against temporal uncertainty becomes more pronounced. Hence there is an increase in portfolio improvement. Finally, from the middle two panels, we can also see that when the market volatility  $\sqrt{V}$  increases, or when the volatility of volatility increases, there is more to be gained from investing in the derivatives market.

## 5 Example II: Derivatives and Jumps

In this section, we examine the role of derivative securities in the presence of jump risks. For this, we specialize in an economy with jump risk but no volatility risk. That is, setting  $V_0 = \bar{v}$  and  $\sigma = 0$ , we have  $V_t = \bar{v}$  at any time  $t$ .

The risky stock is now affected by two types of risk factors: the diffusive price shock with constant volatility  $\sqrt{\bar{v}}$ , and the pure jump with Poisson arrival  $\lambda\bar{v}$  and deterministic jump size  $\mu$ . In the absence of either risk factor, derivative securities are redundant since the market can be completed by dynamic trading of the stock and bond [Black and Scholes (1973) and Cox and Ross (1976)]. In their simultaneous presence, however, one more derivative is needed to complete the market. Applying the result of Proposition 1, the optimal portfolio weights

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<sup>15</sup>For example, Coval and Shumway (2001) report that zero-beta at-the-money straddle positions produce average losses of approximately 3% per week. This number roughly corresponds to  $\xi = -12$ . Using volatility-risk premium to explain the premium implicit in option prices, Pan (2002) reports a total volatility-risk premium that translates to  $\xi = -23$ . This level of volatility-risk premium, however, could be overstated due to the absence of jump and jump-risk premium in the model. In fact, after introducing jumps and estimating jump-risk premium simultaneously with volatility-risk premium, Pan (2002) reports a volatility-risk premium that translates to  $\xi = -10$ .

$\phi$  on the risky stock and  $\psi$  on the derivative are,

$$\phi_t^* = \frac{\eta}{\gamma} - \psi_t^* \frac{g_s S_t}{O_t} \quad (24)$$

$$\psi_t^* = \left( \frac{\Delta g}{\mu O_t} - \frac{g_s S_t}{O_t} \right)^{-1} \left( \frac{1}{\mu} \left[ \left( \frac{\lambda}{\lambda^Q} \right)^{1/\gamma} - 1 \right] - \frac{\eta}{\gamma} \right). \quad (25)$$

## 5.1 The Demand for Derivatives

Evident in our solution is the role of derivative securities in separating the jump risk from the diffusive price risk. Specifically, the optimal demand  $\psi^*$  for the derivative security is inversely proportional to its ability to dis-entangle the two — the more effective it is in providing the separate exposure, the less is needed to be invested in this derivative security. Deep out-of-the-money put options are examples of derivatives with high sensitivities to large price drops but low sensitivities to small price movements. In contrast, if a financial contract is sensitive to infinitesimal price movements in the same way as to large price movements:

$$\frac{\partial g}{\partial S} = \frac{\Delta g}{\Delta S},$$

then it is not effective at all in dis-entangling the two risk factors. Linear financial contracts including the risky stocks are such examples.

Economically, the ultimate driving force for holding derivatives is the risk and return tradeoff involved, which brings us to the second term in the optimal derivative position  $\psi^*$ . A derivative might be able to dis-entangle the two risk factors, but the need for such a dis-entanglement diminishes if the investor finds the two risk factors equally attractive. Recall that the premia for the two risk factors are controlled, respectively, by  $\lambda^Q/\lambda$  and  $\eta$ . Suppose that the relative value of the two coefficients is set so that

$$\frac{\lambda^Q}{\lambda} = \left( 1 + \mu \frac{\eta}{\gamma} \right)^{-\gamma}. \quad (26)$$

From (25), we can see that, under such a constraint, the optimal derivative position  $\psi^*$  is zero for an investor with risk-aversion coefficient  $\gamma$ . In other words, viewing the two risk factors as equally attractive, his desire to dis-entangle the two risk factors diminishes, therefore, so does his demand  $\psi^*$  for the derivative security.

Empirically, however, it is generally not true that the two risk factors are rewarded equally. Specifically, the empirical evidence from the option market suggests that, for a reasonable range of risk aversion  $\gamma$ , the coefficient  $\lambda^Q/\lambda$  is much higher than that implied by (26). If that is the case, then derivatives — with their ability to dis-entangle the two risk factors — can be used by the investor to load more on the jump risk. If the opposite case is true, say the jump risk is not being compensated at all, then derivatives can be used by the investor to carve out his exposure to the jump risk. Later in Section 5.3, we allow the coefficient  $\lambda^Q/\lambda$  for the jump-risk premium to vary, and examine the impact on the optimal derivative position.

Finally, to further emphasize the important role played by derivatives in dis-entangling the two risk factors, let's focus again on the “equally attractive” condition (26). One important observation is that, for given diffusive-risk premium, one cannot always find the appropriate jump-risk premium to make the jump risk equally attractive. In particular, for (26) to hold, it must be that  $1 + \mu\eta/\gamma > 0$ , which can be easily violated when  $\eta/\gamma > 1$  and  $\mu$  is negative and large. This reflects the qualitative difference between the two risk factors: in the presence of large, negative jumps, the investor is reluctant to hold too much of the jump risk regardless of how much premium ( $\lambda^Q/\lambda$ ) is assigned to it. This is because in contrast to the diffusive risks, which can be controlled via continuous trading, the sudden, high-impact nature of jump risks takes away the investor's ability to continuously trade his way out of a leveraged position to avoid negative wealth. As a result, the investor needs to prepare for the worst-case scenario associated with the jump risk so that his wealth stays positive when jump arrives.

## 5.2 The Demand for Stock

To understand how having access to derivatives might change the investor's demand  $\phi^*$  for the risky stock, let's compare our solution for  $\phi^*$  with that for an investor with no access to the derivatives market [Liu, Longstaff, and Pan (2002)]:

$$\phi_t^{*no-op} = \frac{\eta}{\gamma} + \frac{\lambda\mu}{\gamma} \left[ (1 + \mu\phi_t^{*no-op})^{-\gamma} - \frac{\lambda^Q}{\lambda} \right], \quad (27)$$

where  $\phi^{*no-op}$  is the optimal portfolio weight on the risky stock.

By taking a position on the risky stock, an investor is exposed to both the diffusive and jump risks. Without access to derivatives, his optimal stock position is generally a compromise between the two risk factors. This tension is evident in the non-linear equation (27) that gives rise to the optimal stock positions  $\phi^{*no-op}$ . For example, when the diffusive risk becomes more attractive with increasing  $\eta$ , an investor with risk aversion  $\gamma$  would like to increase his position on the diffusive risk via  $\eta/\gamma$ . But the second term in (27) pulls him back, because, at the same time, he is also increasing his exposure to the jump risk. If the jump-risk premium  $\lambda^Q/\lambda$  fails to catch up with the diffusive-risk premium, then tension arises. It is only when the investor finds the two risk factors equally attractive in the sense of (26) does this tension go away.

In general, however, the “equally attractive” condition (26) does not hold empirically or theoretically. As mentioned earlier, for some large and negative jumps, no amount of jump-risk premium  $\lambda^Q/\lambda$  can make up for the jump risk. This qualitative difference between the two risk factors also manifests itself in the endogenously determined bound on  $\phi^{*no-op}$ . Specifically, (27) implies that  $1 + \mu\phi^{*no-op} > 0$ . In other words, in the presence of adverse jump risks ( $\mu < 0$ ), the investor cannot afford to take a too leveraged position on the risky stock. The intuition behind this result is the same that makes the “equally attractive” condition impossible to hold for large, negative jumps. That is, when being blindsided by things that he couldn't control, the investor adopts investment strategies that prepare for the worst-case scenarios.

The investor is nevertheless freer to make choices when the worst-case scenarios can be taken care of by trading derivatives. Indeed, for an investor with access to derivatives, the result in (24) indicates that the optimal position on the risky stock is free of the tension between the two risk factors. Specifically, the first term of  $\phi^*$  is to take advantage of the risk and return tradeoff associated with the diffusive risk, while the second term is to correct for the “delta” exposure introduced by the derivative security.

### 5.3 A Quantitative Analysis on Optimal Strategies

For the quantitative analysis, we set the riskfree rate at  $r = 5\%$  and consider three jump cases: 1)  $\mu = -10\%$  jumps once every 10 years; 2)  $\mu = -25\%$  jumps once every 50 years; and 3)  $\mu = -50\%$  jumps once every 200 years. Clearly, these jump cases are designed to capture the infrequent, high-impact nature of large events. For each jump case, we adjust the diffusive component of the market volatility  $\sqrt{v}$  so that the total market volatility is always fixed at 15% a year.

For each jump case, we consider a wide range of jump-risk premia  $\lambda^Q/\lambda$ , starting from the one with zero jump-risk premium:  $\lambda^Q/\lambda = 1$ . For each fixed level of the jump-risk premium, we always adjust the coefficient  $\eta$  for the diffusive-risk premium so that the total equity risk premium is fixed at 8% a year.

Table 1: Optimal Strategies with/without Options

Jump Cases		$\mu = -10\%$ every 10 yrs			$\mu = -25\%$ every 50 yrs			$\mu = -50\%$ every 200 yrs		
$\gamma$	$\lambda^Q/\lambda$	stock only	stock & put		stock only	stock & put		stock only	stock & put	
			$\phi^*$	$\psi^*$		$\phi^*$	$\psi^*$		$\phi^*$	$\psi^*$
0.5	1	6.74	9.34	4.33%	4.00	8.52	2.28%	2.00	8.38	1.85%
	2	6.74	6.25	-0.67%	4.00	7.59	1.40%	2.00	7.94	1.54%
	5	6.74	1.95	-5.63%	4.00	5.88	0.85%	2.00	7.10	1.53%
3	1	1.17	1.56	0.72%	1.12	1.42	0.38%	0.99	1.40	0.31%
	2	1.17	0.82	-0.66%	1.12	1.22	0.12%	0.99	1.31	0.21%
	5	1.17	-0.44	-3.38%	1.12	0.85	-0.34%	0.99	1.13	0.09%
5	1	0.70	0.93	0.43%	0.68	0.85	0.23%	0.62	0.84	0.18%
	2	0.70	0.48	-0.43%	0.68	0.73	0.06%	0.62	0.78	0.13%
	5	0.70	-0.35	-2.28%	0.68	0.50	-0.25%	0.62	0.67	0.04%
10	1	0.35	0.47	0.22%	0.34	0.43	0.11%	0.32	0.42	0.09%
	2	0.35	0.23	-0.23%	0.34	0.36	0.03%	0.32	0.39	0.06%
	5	0.35	-0.21	-1.24%	0.34	0.24	-0.15%	0.32	0.33	0.01%

The quantitative analysis is summarized in Table 1. We choose one-month 5% out-of-the-money (OTM) European-style put options as the derivative security for the investor to include in his portfolio. Known to be highly sensitive to large negative jumps in stock prices, such OTM put options are among the most effective exchange-traded derivatives for

the purpose of dis-entangling the jump risk from the diffusive risk. For an investor with varying degrees of risk aversion  $\gamma$ , Table 1 reports his optimal portfolio weights  $\phi^*$  and  $\psi^*$  on the risky stock and the OTM put option, respectively. For comparison, the optimal portfolio weights for the case of no derivatives (stock only) are also reported.

To put the results in perspective, recall that for all cases considered in Table 1, the total market volatility is always fixed at 15% a year, and the total equity risk premium is always fixed at 8% a year. If there were no jump risks, then options would be redundant and this investor's optimal stock weight would be  $0.08/0.15^2/\gamma$ . This translates to an optimal stock position of 7.11, 1.19, 0.71, and 0.36, respectively, for an investor with  $\gamma = 0.5, 3, 5$ , and 10.

The introduction of the jump component in Table 1 affects the optimal stock positions in important ways. As discussed earlier, the stock-only investor becomes relatively more cautious in the presence of the jump risk.<sup>16</sup> More importantly, because the stock-only investor has no ability to separate his jump exposure from his diffusive exposure, his position is indifferent to how the jump risk is rewarded relative to the diffusive risk: all that matters is the total equity premium, which is fixed at 8% a year.

This, however, is no longer true for the investor who can trade both the risky stock and the put options. In particular, his position depends sensitively on how the jump risk is rewarded. If it is not being compensated ( $\lambda^Q/\lambda = 1$ ), the investor views the exposure to the jump risk as a nuisance. He sees the risky stock simply as an opportunity to achieve his optimal exposure to the diffusive risk. By investing in the risky stock, however, he also exposes himself to the negative jump risk. To carve out this very exposure, he buys put options. In this sense, the put options are playing their traditional hedging role against negative jump risk.

As we increase  $\lambda^Q/\lambda$  in Table 1, the jump-risk premium increases. At some point, there is a switch between the relative attractiveness of the jump and diffusive risks. This is indeed the outcome for some of the cases in Table 1. That is, instead of buying puts, the investor starts writing put options ( $\psi^* < 0$ ) to earn the high premium associated with the jump risk. At the same time, his holding of the risky stock decreases along with the decreasing attractiveness of the diffusive risk.<sup>17</sup>

Finally, it is interesting to notice that, for some of the cases in Table 1, this switch in relative attractiveness never happens, regardless of the magnitude of  $\lambda^Q/\lambda$ . For example, we see that the put option continues to play its hedging role for the last jump case for the investor with  $\gamma = 0.5$ . Using our earlier discussion on the “equally attractive” condition (26), this implies that the jump magnitude in this case is so large that  $1 + \mu\eta/\gamma < 0$  for the given level of  $\eta$  and  $\gamma$ .

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<sup>16</sup>In particular, in the presence of the  $-25\%$  and  $-50\%$  jumps, the endogenously determined portfolio bound kicks in. Specifically, the associated portfolio weights are determined by imposing the constraint that  $1 + \mu\phi \geq 0$ .

<sup>17</sup>It should be noted that part of the reason for this reduction in stock holding is to correct for the “delta” exposure introduced by writing the put. See the last paragraph in Section 5.2.



## 5.4 Portfolio Improvement

In this section, we compare the certainty equivalent wealth of an investor with access to the derivative market with that of a stock-only investor. Suppose that, at time 0, the investor starts with an initial wealth of  $W_0$  and has an investment horizon of  $T$  years. With access to derivatives, his certainty equivalent wealth is<sup>18</sup>

$$\mathcal{W}^* = W_0 \exp \left( rT + \left[ \frac{\gamma}{2} \left( \frac{\eta}{\gamma} \right)^2 + \frac{\gamma}{1-\gamma} \lambda^Q \left( \left( \frac{\lambda}{\lambda^Q} \right)^{1/\gamma} + \frac{1}{\gamma} \left( 1 - \frac{\lambda}{\lambda^Q} \right) - 1 \right) \right] \bar{v}T \right). \quad (28)$$

Without access to derivatives, his certainty equivalent wealth is

$$\mathcal{W}^{*no-op} = W_0 \exp \left( rT + \left[ (\eta - \lambda^Q \mu) \phi^* - \frac{\gamma}{2} \phi^{*2} + \frac{\lambda}{1-\gamma} ((1 + \phi^* \mu)^{1-\gamma} - 1) \right] \bar{v}T \right), \quad (29)$$

where  $\phi^*$ , solved from (27), is the optimal stock position of the stock-only investor [Liu, Longstaff, and Pan (2002)].

Clearly, the investor with access to the derivative security cannot do worse than the stock-only investor. Hence  $\mathcal{W}^* \geq \mathcal{W}^{*no-op}$ . The equality holds if the “equally attractive” condition (26) holds, that is, when the investor has no incentive to dis-entangle his exposures to the two risk factors.

Table 2: Portfolio Improvement for Including Derivatives

Jump Cases		$\mu = -10\%$ every 10 yrs	$\mu = -25\%$ every 50 yrs	$\mu = -50\%$ every 200 yrs
$\gamma$	$\lambda^Q/\lambda$	$\mathcal{R}^W(\%)$	$\mathcal{R}^W(\%)$	$\mathcal{R}^W(\%)$
0.5	1	2.11	8.62	16.74
	2	0.13	5.97	15.14
	5	11.78	1.84	11.28
3	1	0.26	0.43	0.71
	2	0.28	0.06	0.46
	5	7.68	0.46	0.09
5	1	0.15	0.24	0.37
	2	0.19	0.02	0.22
	5	5.12	0.36	0.02
10	1	0.08	0.12	0.17
	2	0.10	0.01	0.09
	5	2.77	0.22	0.003

A quantitative analysis of the portfolio improvement for including derivatives is summarized in Table 2. Adopting the notation developed in Section 4.4, we use  $\mathcal{R}^W$  to measure

<sup>18</sup>The indirect utility for this special case can be solved in a couple of ways. One is by a straightforward derivation similar to that leading to Proposition 1 with the simplifying condition that  $V_t \equiv \bar{v}$ . Alternatively, one can take advantage of our existing solution, particularly the ordinary differential equations (A.1) for  $h$  and  $H$ , and take the limit to the case of constant volatility.

the improvement in terms of the annualized, continuously compounded return in certainty equivalent wealth. The result of Table 2 can be best understood by comparing the related optimal strategies in Table 1. When derivatives are used to hedge the exposure to the jump risk, the more aggressive investor benefits more from having access to derivatives. This is because, in the absence of jump risks, the more aggressive investor typically would like to take larger stock positions. The presence of jump risks restricts them from taking too leveraged positions. With the help of derivatives, however, he is again freer to choose his optimal exposure to the diffusive risk. For the same reason, the improvement for including derivatives decreases when the jump-risk premium increases and the diffusive-risk premium decreases. For example, in the presence of the last jump case, the investor with  $\gamma = 0.5$  buys put options to hedge out his jump-risk exposure. His improvement in certainty equivalent wealth is 16.74% a year when the jump risk is not compensated. When  $\lambda^Q/\lambda$  increases to 5, his improvement in certainty equivalent wealth decreases to 11.28%.

This, however, is not the case when the relative attractiveness of the two risk factors switches, and the investor starts to use derivatives as a way to obtain positive exposure to the jump risk. For example, in the presence of the first jump case, the investor with  $\gamma = 3$  starts writing put options when  $\lambda^Q/\lambda$  increases to 2. His improvement in certainty equivalent wealth is 0.28% a year. When  $\lambda^Q/\lambda$  increases to 5, however, he writes more put options, and his improvement in certainty equivalent wealth increases to 7.68% a year.

## 6 Conclusion

In this paper, we studied the optimal investment strategy of an investor who can access not only the bond and the stock markets, but also the derivatives market. Our results demonstrate the importance of including derivative securities as an integrated part of the optimal portfolio decision. The analytical nature of our solutions also helps establish direct links between the demand for derivatives and their economic sources.

As a vehicle to the additional risk factors such as stochastic volatility and price jumps in the stock market, derivative securities play an important role in expanding the investor's dimension of risk and return tradeoffs. In addition, by providing access to the volatility risk, derivatives are used by non-myopic investors to take advantage of the time-varying nature of their opportunity set. Similarly, by providing access to the jump risk, derivatives are used by investors to dis-entangle their simultaneous exposure to the diffusive and jump risks in the stock market.

Although our analysis focuses on volatility and jump risks, our intuition can be readily extended to other risk factors that are not accessible through linear positions on stocks. The risk factor that gives rise to stochastic predictor is such an example. If, in fact, there are derivatives providing access to such additional risk factors, then demands for the related derivatives will arise from the need to take advantage of the associated risk and return tradeoff, as well as the time-varying investment opportunity provided by such risk factors.

By focusing on the investment opportunity provided by derivative securities, this paper also raised an important question that has yet to be fully examined: What are the reasonable values for the market price of such additional risk factors? While empirically there is strong support indicating that these risk factors are indeed priced in the aggregate market,

our theoretical understanding of this subject is still limited. In particular, while it is easy to include such risk factors in the pricing kernel (as we did in this paper), it remains an open question as to why they are in the pricing kernel,<sup>19</sup> and what types of restrictions are associated with their presence.<sup>20</sup> The importance of these questions naturally arises as we start to treat derivatives as an integrated part of the optimal portfolio decision. Answers to such questions, however, lie outside of the partial-equilibrium approach adopted in this paper and need equilibrium treatment. For example, in an equilibrium setting with an investor with crash aversion, Bates (2001) shows that the jump risk can in fact be priced differently from the diffusive risk.

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<sup>19</sup>For example, in the setting of Campbell and Cochrane (1999), the time-varying risk aversion of an investor gives rise to stochastic volatility, which in turn finds its position in the pricing kernel.

<sup>20</sup>See, for example, Cochrane and Saá-Requejo (2000) and Bernardo and Ledoit (2000) for constraints on the pricing kernel via some intuitive criteria, and their impact on the market prices of the risk factors that affect derivatives pricing.

# Appendices

## A A Proof of Proposition 1

The proof is a standard application of the stochastic control method. Suppose that the indirect utility function  $J$  exists, and is of the conjectured form in (13). Then the first order condition of the HJB Equation (11) implies that the optimal portfolio weights  $\phi^*$  and  $\psi^*$  are indeed as given by (18) and (19), respectively.

Substituting (13), (18), and (19) into the HJB equation (11), one can show that the conjectured form (13) for the indirect utility function  $J$  indeed satisfies the HJB equation (11) if the following ordinary differential equations are satisfied

$$\begin{aligned}\frac{dh(t)}{dt} &= \kappa \bar{v} H(t) + \frac{1-\gamma}{\gamma} r, \\ \frac{dH(t)}{dt} &= \left( -\kappa + \frac{1-\gamma}{\gamma} \left( \eta \rho + \xi \sqrt{1-\rho^2} \right) \sigma \right) H(t) + \frac{\sigma^2}{2} H(t)^2 + \frac{1-\gamma}{2\gamma^2} (\eta^2 + \xi^2) \\ &\quad + \lambda^Q \left[ \left( \frac{\lambda}{\lambda^Q} \right)^{1/\gamma} + \frac{1}{\gamma} \left( 1 - \frac{\lambda}{\lambda^Q} \right) - 1 \right]\end{aligned}\quad (\text{A.1})$$

Using the solutions provided in (14) for  $H$  and  $h$ , it is a straightforward calculation to verify that this is indeed true. ■

## B Appendix to Section 4

### B.1 Option Pricing

Option pricing for the stochastic-volatility model adopted in this paper is well established by Heston (1993). Using the notation established in Section 2, and letting  $\kappa^* = \kappa - \sigma(\rho\eta + \sqrt{1-\rho^2}\xi)$  and  $\bar{v}^* = \kappa\bar{v}/\kappa^*$  be the risk-neutral mean reversion rate and long-run mean, respectively, the time- $t$  prices of European-style call and put options with time  $\tau$  to expiration and striking at  $K$  are

$$C_t = c(S_t, V_t; K, \tau); \quad P_t = p(S_t, V_t; K, \tau), \quad (\text{B.2})$$

where  $S_t$  is the spot price and  $V_t$  is the market volatility at time  $t$ , and where

$$c(S, V; K, \tau) = S \mathcal{P}_1 - e^{-r\tau} K \mathcal{P}_2,$$

and, by put/call parity, the put pricing formula is

$$p(S, V; K, \tau) = e^{-r\tau} K (1 - \mathcal{P}_2) - S (1 - \mathcal{P}_1).$$

Very much like the case of Black and Scholes (1973),  $\mathcal{P}_1$  measures the probability of the call option expiring in the money, while  $\mathcal{P}_2$  is the adjusted probability of the same event.

Specifically,

$$\begin{aligned}\mathcal{P}_1 &= \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{du}{u} \operatorname{Im} \left( e^{A(1-iu)+B(1-iu)V} e^{iu(\ln K - \ln S + r\tau)} \right) \\ \mathcal{P}_2 &= \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{du}{u} \operatorname{Im} \left( e^{A(-iu)+B(-iu)V} e^{iu(\ln K - \ln S + r\tau)} \right)\end{aligned}\tag{B.3}$$

where  $\operatorname{Im}(\cdot)$  denotes the imaginary component of a complex number, and where, for any  $y \in \mathbb{C}$ ,

$$\begin{aligned}B(y) &= -\frac{a(1 - \exp(-qt))}{2q - (q+b)(1 - \exp(-qt))} \\ A(y) &= -\frac{\kappa^* \bar{v}^*}{\sigma^2} \left( (q+b)\tau + 2 \ln \left[ 1 - \frac{q+b}{2q} (1 - e^{-q\tau}) \right] \right)\end{aligned}\tag{B.4}$$

where  $b = \sigma \rho y - \kappa^*$ ,  $a = y(1-y) - 2\lambda^Q (\exp(y)(1+\mu) - 1 - y\mu)$  and  $q = \sqrt{b^2 + a\sigma^2}$ .

Connecting to the notation  $O_t = g(S_t, V_t)$  adopted in Section 2, we can see that for a call option,  $g$  is simply  $c$ , while for a straddle,  $g(S_t, V_t) = c(S_t, V_t; K, \tau) + p(S_t, V_t; K, \tau)$ .

## B.2 The Indirect Utility of a No-Option Investor

A “no-option” investor solves the same investment problem as that in (7) and (8) with the additional constraint that  $\beta_t \equiv 0$ . This problem is solved extensively in Liu (1998). For completeness of the paper, the following summarizes the results that are useful for our analysis of portfolio improvement in Section 4.4.

At any time  $t$ , the indirect utility of a “no-option” investor with a  $T$ -year investment horizon is

$$J^{\text{no-op}}(W_t, V_t, t) = \frac{W_t^{1-\gamma}}{1-\gamma} \exp(\gamma h^{\text{no-op}}(T-t) + \gamma H^{\text{no-op}}(T-t) V_t), \tag{B.5}$$

where  $h^{\text{no-op}}(\cdot)$  and  $H^{\text{no-op}}(\cdot)$  are time-dependent coefficients that are independent of the state variables:

$$\begin{aligned}h^{\text{no-op}}(t) &= \frac{2\kappa\bar{v}}{\sigma^2(\rho^2 + \gamma(1-\rho^2))} \ln \left( \frac{2k_2 \exp((k_1 + k_2)t/2)}{2k_2 + (k_1 + k_2)(\exp(k_2 t) - 1)} \right) + \frac{1-\gamma}{\gamma} r t \\ H^{\text{no-op}}(t) &= \frac{\exp(k_2 t) - 1}{2k_2 + (k_1 + k_2)(\exp(k_2 t) - 1)} \frac{1-\gamma}{\gamma^2} \eta^2\end{aligned}\tag{B.6}$$

where

$$k_1 = \kappa - \frac{1-\gamma}{\gamma} \eta \sigma \rho; \quad k_2 = \sqrt{k_1^2 - \frac{1-\gamma}{\gamma^2} \eta^2 \sigma^2 (\rho^2 + (1-\rho^2)\gamma)}. \tag{B.7}$$

The certainty equivalent wealth of such a “no-option” investor with initial wealth  $W_0$  then becomes

$$\mathcal{W}^{\text{no-op}} = W_0 \exp \left( \frac{\gamma}{1-\gamma} \left[ h^{\text{no-op}}(T) + H^{\text{no-op}}(T) V_0 \right] \right). \tag{B.8}$$

### B.3 Proof of Proposition 2

The indirect utility of an investor with access to derivatives is given in Proposition 1, while that of an investor without access to derivatives is provided in Section B.2. It is then straightforward to verify that the portfolio improvement  $\mathcal{R}^W$  is indeed of the form (23). To show that the improvement is strictly positive for investors with  $\gamma \neq 1$ , let  $DH(t) = H(t) - H^{\text{no-op}}(t)$ , and one can show that

$$DH(t) = \frac{1-\gamma}{2} \exp(-y(t)) \int_t^T \exp(-y(s)) \left( \frac{\xi}{\gamma} - \sqrt{1-\rho^2} \sigma H^{\text{no-op}}(s) \right)^2 ds,$$

where

$$y(t) = \int_t^T \left[ \kappa + \frac{1-\gamma}{\gamma} \left( \eta \rho + \xi \sqrt{1-\rho^2} \sigma \right) + \frac{\sigma^2}{2} \left( H(s) + H^{\text{no-op}}(s) \right) \right] ds$$

is finite for any  $t \leq T$ . Consequently,  $DH(T)/(1-\gamma)$  is strictly positive. Moreover, it is straightforward to show that

$$\frac{Dh(t)}{1-\gamma} = \frac{h(t) - h^{\text{no-op}}(t)}{1-\gamma} = \kappa \bar{v} \int_0^t \frac{DH(s)}{1-\gamma} ds. \quad (\text{B.9})$$

As a result,  $Dh(T)/(1-\gamma)$  is also strictly positive, making  $\mathcal{W}^* > \mathcal{W}^{\text{no-op}}$  for any  $\gamma \neq 1$ .

For the log-utility case, the intertemporal hedging demand is zero. That is,  $H(t) = 0$  and  $H^{\text{no-op}}(t) = 0$  for any  $t$ . One can show that

$$\lim_{\gamma \rightarrow 1} \frac{H^{\text{no-op}}(t)}{1-\gamma} = \frac{1 - \exp(-\kappa t)}{2\kappa} \eta^2; \quad \lim_{\gamma \rightarrow 1} \frac{H(t)}{1-\gamma} = \frac{1 - \exp(-\kappa t)}{2\kappa} (\eta^2 + \xi^2)$$

Moreover, (B.9) also holds for the case of  $\gamma = 1$ , making  $\mathcal{W}^* > \mathcal{W}^{\text{no-op}}$  when  $\xi \neq 0$ , and  $\mathcal{W}^* = \mathcal{W}^{\text{no-op}}$  when  $\xi = 0$ .

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