An Efficient Reduced-Order Approach for Nonaffine and Nonlinear Partial Differential Equations

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***Abstract*— In the presence of nonaffine and highly nonlinear terms in parametrized partial differential equations, the standard Galerkin reduced order approach is no longer efficient, because the evaluation of these terms involves high computational complexity. An efficient reduced-order approach is developed to deal with "nonaffineness" and nonlinearity. The efficiency and accuracy of the approach are demonstrated on several test cases, which show significant computational savings relative to classical numerical methods and relative to the standard Galerkin reduced-order approach.**

Keywords: nonaffine equations, nonlinear equations, reduced-order approximation, best points interpolation method.

I. INTRODUCTION

Many systems/processes in engineering and science are described by *parametrized* partial differential equations (PDEs). Typically, the quantities of engineering interests are not the full field variables, but rather certain *outputs*, s, best articulated as functionals of the field variables u. Typical outputs include flowrate, pressure drops, concentration and flux, critical stresses or maximum displacements, and lift and drag forces. These outputs are functions of system parameters, or *inputs*, μ, that serve to identify a particular configuration of the system — geometry, material properties, initial and boundary conditions, and loads. The relevant system behavior is thus described by an implicit input-output relationship, $s(\mu)$, evaluation of which demands solution of the underlying PDE. The design, optimization, control, and characterization of engineering systems often requires repeated, reliable, and real-time prediction of $s(\mu)$.

In this paper, we describe an efficient reduced order approach for the very rapid prediction of functional outputs of *nonaffine* linear elliptic PDEs. The abstract formulation for a nonaffine linear elliptic problem can be stated as follows: given any $\mu \in \mathcal{D} \subset \mathbb{R}^P$, we evaluate

$$s(\mu) = \ell^O(u(\mu); \mu) \ , \tag{1}$$

where $u(\mu) \in X$ is the solution of

$$a(u(\mu), v; \mu) = \ell(v; \mu), \quad \forall v \in X \ . \tag{2}$$

Here $\mathcal{D}$ is the parameter domain in which our P-tuple (input) parameter vector μ resides; $X(\Omega)$ is the appropriate Hilbert space; and Ω is a bounded domain in $\mathbb{R}^{d=2}$ with Lipschitz continuous boundary $\partial\Omega$. Furthermore, a is a continuous parameter-dependent bilinear form as

$$a(w, v; \mu) = \sum_{q=1}^{Q} a^q(w, v, g^q(x; \mu)) \ ; \tag{3}$$

and $\ell(\cdot; \mu), \ell^O(\cdot; \mu)$ are continuous parameter-dependent linear functionals as

$$\ell(v; \mu) = \ell^O(v; \mu) = \int_\Omega v h(x; \mu) \ . \tag{4}$$

(For simplicity we presume that $\ell = \ell^O$). Here the $a^q, 1 \le q \le Q$, are continuous parameter-dependent trilinear forms which take the nonaffine functions $g^q(x; \mu), 1 \le q \le Q$, as the third argument; and the functions $g^q(x; \mu), 1 \le q \le Q$, $h(x; \mu)$ are defined in Ω and dependent on μ.

In our approach, we first construct a reduced set of basis functions for the field variable and apply a standard Galerkin projection of the underlying PDE onto a space spanned by these basis functions. We then develop a collateral reduced-basis expansion for the nonaffine functions. Finally, in the Galerkin representation of the underlying equation, we replace the nonaffine functions with our collateral reduced-basis expansion to obtain a reduced-order model. This reduced-order model is of very low dimension that significant computational savings relative to classical numerical methods and relative to standard Galerkin reduced-order approaches can be achieved. In this paper, we use the *best points interpolation method* introduced earlier in [13] to provide the collateral reduced-basis expansion for nonaffine terms. We shall briefly review the method in Section II.

In addition, we extend the approach to develop the reduced order approximation of *nonlinear* convection-diffusion equations of the form

$$m(\frac{\partial u(\mu, t)}{\partial t}, v) - \int_\Omega \mathbf{f}(u(\mu, t)) \cdot \nabla v + \mu a(u(\mu, t), v) = 0, \tag{5}$$

for all $v \in X, t \in (0, T]$, with initial condition $u(\mu, t) = u_0(x)$; the output of interest $s(\mu, t)$ is related to $u(\mu, t)$ by

$$s(\mu, t) = \ell^O(u(\mu, t)) \ . \tag{6}$$

Here the viscosity parameter μ varies in a parameter space $\mathcal{D} \in \mathbb{R}^1$; and $\mathbf{f}(u(\mu, t)) \equiv (f_1(u(\mu, t)), f_2(u(\mu, t)))$ is the flux vector in which the ith-coordinate flux $f_i(u(\mu, t))$ is a nonlinear function of the field variable $u(\mu, t)$. Furthermore, a and m are symmetric positive-definite bilinear forms; and ℓ^O is a continuous linear functional. For simplicity of exposition, in the nonlinear case these forms and functional are assumed to be *parameter-independent*.

Our approach for the nonlinear convection-diffusion problem (5) will be quite similar to that for the nonaffine linear elliptic problem (2). However, the presence of time parameter

and nonlinearity requires a more sophisticated reduced-order treatment.

A. Previous Work

The reduced-basis methods and proper orthogonal decomposition (POD) reduced-order techniques have been widely used for the rapid solution of linear and nonlinear problems. Success and applications remain largely to affine linear problems [9], [8], [25] and quadratically nonlinear problems such as viscous Burgers-type equations and incompressible Navier-Stokes equations [17], [5], [6], [11], [23], [24], [14], [22]. Recent progress on the development of nonlinear reduced-order model techniques has been made. In particular, linearization methods [1], [19], [18], [12] and polynomial approximation methods [15], [4] have been proposed to treat nonlinear problems quite satisfactorily. However, inefficient representation of the nonlinear terms and fast exponential growth (with the order of nonlinear approximation) of the computational complexity render these methods quite expensive, in particular for strong nonlinearities.

More recently, in [2], [7], Maday et al. introduced the empirical interpolation method (EIM) and incorporated the method into the reduced-basis techniques to provide effective reduced-basis treatment of nonaffine and nonlinear parametrized PDEs. This reduced-basis approach has been applied to nonaffine linear and highly nonlinear elliptic and parabolic equations [2], [7] as well as certain quantum chemistry problems [3], and has resulted in quite significant savings relative to traditional solution approaches. We refer the reader to [2], [7] for a detailed discussion of the EIM and the reduced-basis approach presented therein.

B. Structure of the Paper

In Section II, we present a short review of the best points interpolation method introduced in [13]. In Section III, we propose a reduced-order approach for linear elliptic problems with nonaffine parameter dependence based on the BPIM. In Section IV, we extend the methodology developed in Section III to nonlinear convection-diffusion problems. Numerical examples are presented in each sections in order to assess the efficient and accuracy of our approach. Finally, in Section V, we close the paper with concluding remarks and directions for future research.

II. Best Points Interpolation Method

In this section we briefly describe the BPIM to construct a "coefficient-function" approximation of *parameter-dependent* functions. The approximation of such class of functions is of considerable interest in model reduction contexts, since it permits the effective reduced-order treatment of nonaffine and nonlinear PDEs.

A. Interpolation Procedure

We consider the problem of approximating a given μ-dependent function $g(x;\mu) \in L^\infty(\Omega) \cap C^0(\Omega),\ \forall\, \mu \in \mathcal{D}$, by a collateral reduced-basis expansion $g_M(x;\mu)$; here $\mathcal{D} \in \mathbb{R}^P$ is the parameter space in which our P-tuple parameter μ resides. We assume that we are given an approximation space spanned by M orthonormal basis functions, $\Phi_M = \text{span}\{\phi_1, \ldots, \phi_M\}$, with $(\phi_i, \phi_j) = \delta_{ij}, 1 \le i, j \le M$; here δ is the Kronecker symbol and $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product with an induced norm $\|\cdot\| = \sqrt{(\cdot,\cdot)}$. We further assume that we are given an associated set of M interpolation points $\{z_m \in \Omega\}_{m=1}^M$. We then define our coefficient-function approximation as

$$g_M(x;\mu) = \sum_{m=1}^{M} \underline{\beta}_{M\,m}(\mu)\phi_m(x) \tag{7}$$

where the coefficients $\underline{\beta}_{M\,m}(\mu)$ are the solution of

$$\sum_{m=1}^{M} \phi_m(z_i)\underline{\beta}_{M\,m}(\mu) = g(z_i;\mu), \quad i = 1, \ldots, M\ . \tag{8}$$

We observe from (7)-(8) that $g_M(\,\cdot\,;\mu)$ and $g(\,\cdot\,;\mu)$ are equal at the interpolation points $\{z_m\}_{m=1}^M$. Note further that we can express $g_M(x;\mu)$ in terms of the cardinal functions (Lagrange interpolation functions) as

$$g_M(x;\mu) = \sum_{m=1}^{M} g(z_m;\mu)\psi_m(x)\ . \tag{9}$$

Here the cardinal functions $\{\psi_m\}_{m=1}^M$ are defined by $\psi_j(z_i) = \delta_{ij}$, and hence given by

$$\phi_i(x) = \sum_{j=1}^{M} \phi_i(z_j)\,\psi_j(x), \quad 1 \le i \le M\ . \tag{10}$$

Clearly, $\{\psi_m\}_{m=1}^M$ depends on both $\{\phi_m\}_{m=1}^M$ and $\{z_m\}_{m=1}^M$. In order to achieve an accurate approximation, it is important to equip the basis functions $\{\phi_m\}_{m=1}^M$ with good approximation properties. The critical observation is that the manifold $\mathcal{M}^g \equiv \{g(x;\mu)\,|\,\mu \in \mathcal{D}\}$ induced by the parametric dependence is typically low dimensional. This observation is exploited to compute a specific problem-dependent basis set $\{\phi_m\}_{m=1}^M$ which is extremely effective for the approximation of functions in $\mathcal{M}^g$. More specifically, the set $\{\phi_m\}_{m=1}^M$ is constructed from a set of (linearly independent) snapshots

$$\mathcal{G}_K = \{g_k(x) = g(x;\mu_k), \mu_k \in S_K, 1 \le k \le K\}\ ,$$

where $S_K = \{\mu_1, \ldots, \mu_K\}$ is a pre-selected parameter sample set. Typically, S_K is chosen such that any $g(\,\cdot\,;\mu) \in \mathcal{M}^g$ can be approximated very well by a linear combination of the g_k: for any $\mu \in \mathcal{D}$, there exist coefficients $c_k(\mu), 1 \le k \le K$, such that $\sum_{k=1}^K c_k(\mu)g_k(x)$ is very close to $g(x;\mu)$.

In this paper, we employ the POD procedure [21] for the construction of our basis set from the set of snapshots $\mathcal{G}_K$. For an alternative approach of selecting a different basis set with good approximation properties — in particular, the greedy selection process generates maximally independent basis functions — see [2], [7].

Of course, the quality of our coefficient-function approximation depends critically not only on the basis set $\{\phi_m\}_{m=1}^M$ but also the interpolation point set $\{z_m\}_{m=1}^M$. In the following, we describe our approach to determine the interpolation points.

B. Best Interpolation Points

We choose the interpolation points via exploiting the best approximation. The best approximation is defined as

$$g_M^*(\,\cdot\,;\mu) = \arg\min_{w_M \in \Phi_M} \|g(\,\cdot\,;\mu) - w_M\|^2 \ . \qquad (11)$$

It is easily derived from the orthonormality of the ϕ_m that

$$g_M^*(x;\mu) = \sum_{m=1}^{M} \underline{\alpha}_{M\,m}(\mu)\phi_m(x) \ , \qquad (12)$$

where the coefficients are given by

$$\underline{\alpha}_{M\,m}(\mu) = (\phi_m, g(\,\cdot\,;\mu)), \quad m = 1,\ldots,M \ . \qquad (13)$$

We introduce a set of functions $\mathcal{G}_K^* = \{g_M^{*k}(x), 1 \le k \le K\}$, where g_M^{*k} is the best approximation to g_k. It thus follows that

$$g_M^{*k}(x) = \sum_{m=1}^{M} \underline{\alpha}_{M\,m}^k \phi_m(x) \ , \qquad (14)$$

where for $k = 1,\ldots,K$,

$$\underline{\alpha}_{M\,m}^k = (\phi_m, g_k), \quad 1 \le m \le M \ . \qquad (15)$$

We then define $\{z_m\}_{m=1}^M$ as the minimizer of the following nonlinear least squares problem

$$\min_{x_1\in\Omega,\ldots,x_M\in\Omega} \sum_{k=1}^{K}\sum_{m=1}^{M} \left(\underline{\alpha}_{M\,m}^k - \underline{\beta}_{M\,m}^k(x_1,\ldots,x_M)\right)^2 \qquad (16)$$

$$\sum_{n=1}^{M} \phi_n(x_m)\underline{\beta}_{M\,n}^k = g_k(x_m), \quad 1 \le m \le M, 1 \le k \le K.$$

The points $\{z_m\}_{m=1}^M$ shall be referred as "best" interpolation points, since they are optimal to the interpolation of the best approximations. The reader is referred to [13] for the solution procedure of the above problem.

In summary, the best interpolation points are found as solution of the minimization problem (16) and the associated cardinal functions are computed via (10). Our coefficient-function approximation $g_M(x;\mu)$ can then be obtained from the interpolation formula (9).

III. Nonaffine Linear Elliptic Equations

In this section we devote to developing the reduced-order approximation of nonaffine linear elliptic equations of the form (2). The main idea is to replace the nonaffine terms with the collateral reduced-basis expansions constructed with using the BPIM. A simple example with geometric variation is provided to illustrate the usefulness of the reduce-order approach for nonaffine problems.

A. Full-Order Approximation

We introduce a suitably fine piecewise-linear finite element approximation space $X_h \subset X$ of *very* large dimension $\mathcal{N}$. The FE discretization of (2) is that: given any $\mu \in \mathcal{D}$, we evaluate

$$s_h(\mu) = \ell^O(u_h(\mu);\mu) \ , \qquad (17)$$

where $u_h(\mu) \in X_h$ satisfies

$$\sum_{q=1}^{Q} a^q(u_h(\mu), v, g^q(x;\mu)) = \int_\Omega h(x;\mu)v, \quad \forall v \in X_h \ . \qquad (18)$$

We shall assume that the discretization is sufficiently rich such that $u_h(\mu)$ and $u(\mu)$ and hence $s_h(\mu)$ and $s(\mu)$ are indistinguishable at the accuracy level of interest. For future reference, we call the FE discretization by "full-order approximation". Of course, for any parameter value μ, the computational cost associated with the full-order approximation is dependent on $\mathcal{N}$ and thus expensive.

In the following, we develop the reduced-order approximation of (2) for the rapid prediction of the output. The reduced-order approximation shall be built upon the full-order approximation, and the reduced-order error will thus be evaluated with respect to $s_h(\mu)$. Our reduced-order approximation must be stable and efficient as $\mathcal{N} \to \infty$.

B. Reduced-Order Approximation

We assume that we are given an approximation space $W_N = \text{span}\{\zeta_1,\ldots,\zeta_N\}$, where the basis functions ζ_n can be constructed using the POD procedure on a set of snapshots; these snapshots are solutions of (18) at selected parameter points. We might seek an approximation $u_N^{\rm SG}(\mu) \in W_N$ that satisfies

$$\sum_{q=1}^{Q} a^q(u_N^{\rm SG}(\mu), v, g^q(x;\mu)) = \int_\Omega h(x;\mu)v, \ \forall v \in W_N, \qquad (19)$$

in terms of which the output approximation is calculated as

$$s_N^{\rm SG}(\mu) = \ell^O(u_N^{\rm SG}(\mu);\mu) \ . \qquad (20)$$

Unfortunately, because of the nonaffine terms $g^q(x,\mu)$ and $h(x,\mu)$, the cost of solving the system (19) will depend on $\mathcal{N}$ — the dimension of the finite element approximation space. Hence, the computational advantage of the standard Galerkin reduced-order approach relative to the full-order approximation is quite modest. See [7] for more detailed discussion about this point.

To obtain an efficient reduced-order approximation, we employ the BPIM to construct the point sets $\{z_m^{g^q}\}_{m=1}^M$, $\{z_m^h\}_{m=1}^M$ and caridinal basis sets $\{\psi_m^{g^q}\}_{m=1}^M$, $\{\psi_m^h\}_{m=1}^M$ for $g^q(x;\mu), 1 \le q \le Q$, and $h(x;\mu)$, respectively, as described in Section II. We define collateral reduced-basis expansions $g_M^q(x,\mu)$ and $h_M(x,\mu)$ by the formula (9). For notational simplification we have assumed that these expansions have the same number of basis functions. We may now replace $g^q(x,\mu), h(x;\mu)$ with $g_M^q(\,\cdot\,,\mu), h_M(x;\mu)$ to obtain: for any $\mu \in \mathcal{D}$, we evaluate

$$s_N(\mu) = \ell^O(u_N(\mu);\mu) \ , \qquad (21)$$

where for all $v \in W_N$, $u_N(\mu) \in W_N$ satisfies

$$\sum_{q=1}^{Q} a^q(u_N(\mu), v, g_M^q(x;\mu)) = \int_\Omega h_M(x;\mu)v \ . \qquad (22)$$

We proceed to derive the fully discrete equations for this reduced-order approximation.

We first express $u_N(\mu) = \sum_{n=1}^{N} \underline{u}_{N\,j}(\mu)\zeta_j$, $g_M^q(x,\mu) = \sum_{m=1}^{M} g(z_m^{g^q};\mu)\psi_m^{g^q}$, $h_M(x,\mu) = \sum_{m=1}^{M} g(z_m^h;\mu)\psi_m^h$, and choose $v = \zeta_i$ to write (22) as

$$\Big(\sum_{q=1}^{Q}\sum_{m=1}^{M} g^q(z_m^{g^q},\mu)\underline{A}_N^{q\,m}\Big)\underline{u}_N(\mu) = \sum_{m=1}^{M} h(z_m^h,\mu)\underline{F}_N^m, \quad (23)$$

where for $1 \le i,j \le N$, $1 \le q \le Q, 1 \le m \le M$, the *parameter-independent* quantities $\underline{A}_N^{q\,m} \in \mathbb{R}^{N\times N}$ and $\underline{F}_N^m \in \mathbb{R}^N$ are given by

$$\underline{A}_{N\,i,j}^{q\,m} = a^q(\zeta_i,\zeta_j,\psi_m^{g^q}), \quad (24)$$

$$\underline{F}_{N\,i}^m = \ell(\psi_m^h,\zeta_i)\ . \quad (25)$$

The output approximation can then be calculated as

$$s_N(\mu) = \Big(\sum_{m=1}^{M} h(z_m^h,\mu)\underline{F}_N^m\Big)^T \underline{u}_N(\mu)\ . \quad (26)$$

Finally, we develop the offline-online procedure [16], [20], [14] for the very rapid evaluation of the output (26) as follows. In the offline stage, we compute and save the quantities $\underline{A}_N^{q\,m}$, $1 \le q \le Q, 1 \le m \le M$, and $\underline{F}_N^m$, $1 \le m \le M$. The offline computation is expensive. However, in the online stage — for each new parameter value μ — we perform the sum in the parentheses and the right-hand side of (23) at cost $\mathcal{O}(QMN^2)$, invert the linear system at cost $\mathcal{O}(N^3)$, and evaluate the output $s_N(\mu)$ at cost $\mathcal{O}(MN)$. In summary, the operation count for the online stage is only $\mathcal{O}(QMN^2+N^3)$. Since $N,\ M \ll \mathcal{N}$, we expect significant computational savings in the online stage relative to the full-order approximation (18) and relative to the standard Galerkin reduced-order model built upon (19).

C. Example 1: Geometric Variation

We consider the Poisson equation $\nabla^2 u = 1$ on a *parameter-dependent* domain $\hat{\Omega}(R,\kappa)$ with homogeneous Dirichlet condition. The physical domain $\hat{\Omega}(R,\kappa)$ shown in Figure 1 depends on the parameter vector $\mu = (\mu^1,\mu^2) \equiv (R,\kappa) \in \mathcal{D}$ in which R controls the size of the domain and κ controls the shape of the domain; here $\mathcal{D} \equiv [1,10]\times[1,10]$. Due to symmetry, the problem can be reformulated as $\nabla^2 u = 1$ on $\Omega^{\rm o}(\mu)$ with homogeneous Neumann condition on $\Gamma_{\rm N}^{\rm o} \equiv \Gamma_1^{\rm o} \cup \Gamma_3^{\rm o}$ and homogeneous Dirichlet condition on $\Gamma_{\rm D}^{\rm o} \equiv \Gamma_2^{\rm o} \cup \Gamma_4^{\rm o}$; here $\Omega^{\rm o}(\mu)$ is a "cut" domain as shown in Figure 2(a). The output of interest is the average of the field variable over the physical domain $s(\mu) = \int_{\Omega^{\rm o}(\mu)} u(\mu)$.

We treat the geometric variation by using an geometric transformation that maps the parameter-dependent cut domain $\Omega^{\rm o}(\mu)$ to a fixed reference domain $\Omega \equiv]0,1[\times]0,1[$ shown in Figure 2(b). It can be shown that in the reference domain the weak formulation has the form (2)-(4) in which the trilinear

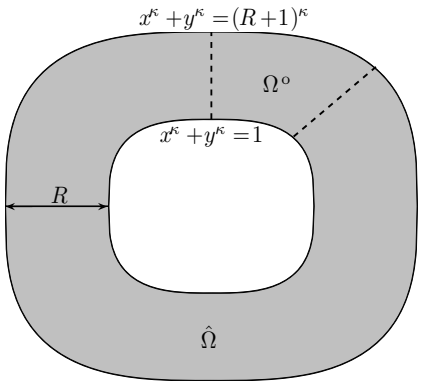


Fig. 1: The physical domain $\hat{\Omega}$ varies with geometric parameters R and κ. The "cut" domain $\Omega^{\rm o}$ is formed by the dash lines and the boundary of the physical domain.

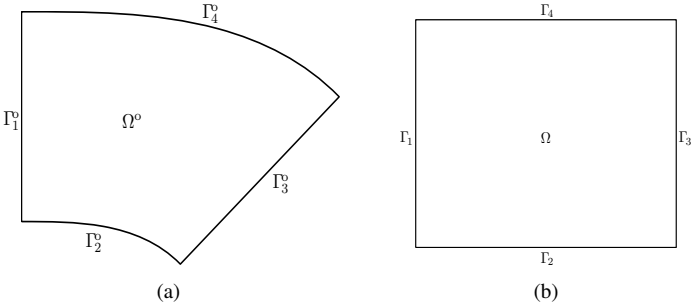


Fig. 2: (a) Original cut domain and (b) Reference domain.

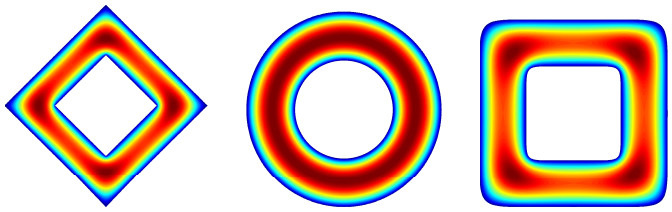
forms, linear functionals, and nonaffine functions are

$$\begin{aligned}
a^1(w,v,g^1(x;\mu)) &= \int_\Omega g^1(x;\mu)\frac{\partial w}{\partial x^1}\frac{\partial v}{\partial x^1},\\
a^2(w,v,g^2(x;\mu)) &= \int_\Omega g^2(x;\mu)\frac{\partial w}{\partial x^2}\frac{\partial v}{\partial x^2},\\
a^3(w,v,g^3(x;\mu)) &= \int_\Omega g^3(x;\mu)\left(\frac{\partial w}{\partial x^1}\frac{\partial v}{\partial x^2}+\frac{\partial w}{\partial x^2}\frac{\partial v}{\partial x^1}\right),\\
\ell(v;\mu) &= \int_\Omega h(x;\mu)v,\\
g^1(x;\mu) &= \frac{R(1+(x^1)^2)}{Rx^2+1},\\
g^2(x;\mu) &= \frac{R(x^1)^{2\kappa-2}(x^2)+Rx^2+(x^1)^{2\kappa-2}+1}{R((x^1)^\kappa+1)^2},\\
g^3(x;\mu) &= \frac{(x^1)^{\kappa-1}-x}{(x^1)^\kappa+1},\\
h(x;\mu) &= R(1+(x^1)^\kappa)^{-2/\kappa}(Rx^2+1).
\end{aligned}$$

The output is evaluated as $s(\mu) = \ell^O(u(\mu);\mu) = \ell(u(\mu);\mu)$.

We now present numerical results for this problem. We first show in Figure 3 the finite element solutions for different parameter values. Here X_h is a piecewise-linear finite element approximation space of dimension $\mathcal{N} = 10,000$.

We next present results obtained with the reduced-order approximation. We choose the test sample $\Xi_{\rm Test}$ as a regular 20×20 grid over $\mathcal{D}$ and define the average relative error in



(a) (b) (c)

Fig. 3: The FE solutions for different parameter values: (a) $\mu = (1, 1)$, (b) $\mu = (1, 2)$, and (c) $\mu = (1, 10)$. Note how the geometry and solution change as μ_2 increases.

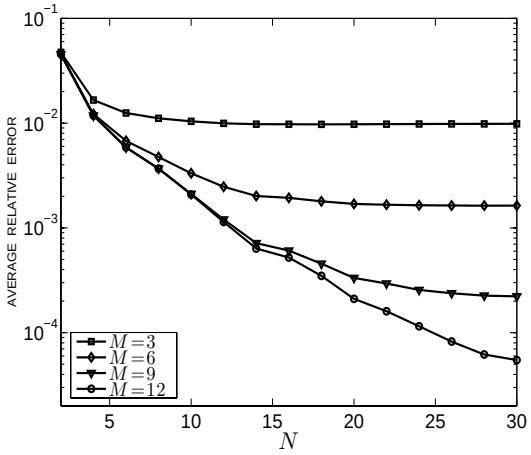


Fig. 4: The average relative error $\epsilon^u_{N,\text{ave,rel}}$ as a function of N and M for the Example 1.

the solution as

$$\epsilon^u_{N,\text{ave,rel}} = \text{mean}_{\mu\in\Xi_{\text{Test}}} \frac{\|u_h(\mu) - u_N(\mu)\|}{\|u_h(\mu)\|}.$$

Figure 4 shows $\epsilon^u_{N,\text{ave,rel}}$ as a function of N and M. We observe that $u_N(\mu)$ converges very rapidly to $u_h(\mu)$. Furthermore, the quality of the reduced-order approximation depends on N and M in a strongly coupled manner: for a fixed value of M the error decreases with N for small N and then levels off for N large enough; when the error does not improve with increasing N, increasing M tends to reduce the error.

Finally, we compare our approach with the standard Galerkin reduced-order approach. For this purpose, we define

$$\begin{aligned} \epsilon^s_{N,\max,\text{rel}} &= \max_{\mu\in\Xi_{\text{Test}}} \frac{s_h(\mu) - s_N(\mu)}{s_h(\mu)} \\ \epsilon^{s,\text{SG}}_{N,\max,\text{rel}} &= \max_{\mu\in\Xi_{\text{Test}}} \frac{s_h(\mu) - s^{\text{SG}}_N(\mu)}{s_h(\mu)} \ , \end{aligned}$$

and introduce $\overline{\text{T}}_N$ and $\overline{\text{T}}^{\text{SG}}_N$ as the normalized computational times for the reduced-order approximation and for the standard Galekin reduced-order model, respectively. Here the times are normalized with respect to the time to solve the full-order approximation. In Table I, we present $\epsilon^s_{N,\max,\text{rel}}$, $\overline{\text{T}}_N$, $\epsilon^{s,\text{SG}}_{N,\max,\text{rel}}$, and $\overline{\text{T}}^{\text{SG}}_N$ as a function of N for a fixed value of $M = 12$. We see that while $\epsilon^s_{N,\max,\text{rel}}$ is very close to $\epsilon^{s,\text{SG}}_{N,\max,\text{rel}}$, $\overline{\text{T}}_N$ is *two orders of magnitude* smaller than $\overline{\text{T}}^{\text{SG}}_N$. This is because the standard Galerkin reduced-order model suffers from the $\mathcal{N}$-dependent cost of evaluating the nonaffine parameter-dependent forms. Consequently, its computational advantage relative to the full-order approximation is quite modest as observed in Table I. In contrast, our reduced-order approximation achieves significant computational savings relative to the full-order approximation and yields a convergence rate similar to that of the standard Galerkin reduced-order model.

N	M	$\epsilon^s_{N,\max,\text{rel}}$	$\overline{\text{T}}_N$	$\epsilon^{s,\text{SG}}_{N,\max,\text{rel}}$	$\overline{\text{T}}^{\text{SG}}_N$
4	12	$4.28\,\text{E}-02$	$2.34\,\text{E}-04$	$4.28\,\text{E}-02$	$1.98\,\text{E}-01$
8	12	$2.14\,\text{E}-02$	$5.75\,\text{E}-04$	$2.14\,\text{E}-02$	$1.98\,\text{E}-01$
12	12	$1.49\,\text{E}-02$	$9.41\,\text{E}-04$	$1.49\,\text{E}-02$	$1.98\,\text{E}-01$
16	12	$5.32\,\text{E}-03$	$1.23\,\text{E}-03$	$5.32\,\text{E}-03$	$1.98\,\text{E}-01$
20	12	$9.76\,\text{E}-04$	$1.59\,\text{E}-03$	$9.75\,\text{E}-04$	$1.98\,\text{E}-01$

TABLE I: Numerical results for the Example 1: $\epsilon^s_{N,\max,\text{rel}}$ and $\overline{\text{T}}_N$ as a function of N and M; $\epsilon^{s,\text{SG}}_{N,\max,\text{rel}}$ and $\overline{\text{T}}^{\text{SG}}_N$ as a function of N. The online computational times are normalized with respect to the time to compute $s_h(\mu)$.

IV. Nonlinear Convection-Diffusion Equations

This section is devoted to the reduced-order approximation of nonlinear convection-diffusion equations of the form (5). A full-order approximation of (5) is first formulated using the FE discretization and the Euler-Backward scheme. A reduced-order approximation of (5) is then developed by using a Galerkin projection and the BPIM. Finally, numerical results for a *non-polynomial* nonlinear example are presented.

A. Full-Order Approximation

We will directly consider the finite element approximation for the spatial discretization and the Euler-Backward scheme for the time integration. We denote the number of time steps by J and let $\Delta t = T/J$. We further introduce a piecewise-linear finite element approximation space X_h of *very* large dimension $\mathcal{N}$. The FE discretization of (5)-(6) is that: given any $\mu \in \mathcal{D}$, for $j = 1, \ldots, J$, we evaluate

$$s_h(\mu, t^j) = \ell^O(u_h(\mu, t^j)) \ , \tag{27}$$

where $u_h(\mu, t^j) \in X_h$ is the solution of

$$\begin{aligned} m(u_h(\mu, t^j), v) - \int_\Omega \mathbf{f}(u_h(\mu, t^j)) \cdot \nabla v + \mu \Delta t \, a(u_h(\mu, t^j), v) \\ = m(\mu, u_h(\mu, t^{j-1}), v), \quad \forall v \in X_h, \end{aligned} \tag{28}$$

and is subject to the initial condition $(u(\mu, 0), v) = (u_0, v), \forall v \in X_h$, for a smooth data $u_0(x)$.

In essence, the full-order approximation (28) is a nonlinear discrete system which can be solved by using a Newton's method. Hence, the computation of the full-order approximation is typically extremely expensive.

B. Reduced-Order Approximation

We assume that we are given an approximation space $W_N = \text{span}\{\zeta_1, \ldots, \zeta_N\}$, where the basis functions ζ_n can be constructed on a set of solutions (snapshots) of (5) at selected parameter points and timesteps. We might then find that $u_N^{\text{SG}}(\mu, t^j) \in W_N, 1 \le j \le J$, satisfies

$$m(u_N^{\text{SG}}(\mu,t^j),v) - \Delta t \int_\Omega f_1(u_N^{\text{SG}}(\mu,t^j))\frac{\partial v}{\partial x^1} - \Delta t \int_\Omega f_2(u_N^{\text{SG}}(\mu,t^j))\frac{\partial v}{\partial x^2} + \mu\Delta t\, a(u_N^{\text{SG}}(\mu,t^j),v) = m(u_N^{\text{SG}}(\mu,t^{j-1}),v), \quad v \in W_N \ . \quad (29)$$

Observe that if f_1 and f_2 are low-order polynomial nonlinearities of u, we can then develop an efficient offline-online procedure by resolving $f_1(u_N^{\text{SG}}(\mu,t^j))$ and $f_2(u_N^{\text{SG}}(\mu,t^j))$ into the sum of products of the basis functions and coefficients [22], [10]. Unfortunately, this strategy can not be applied to high-order polynomial and non-polynomial nonlinearities: the associated flux integrals can only be evaluated by explicitly constructing $f_1(u_N^{\text{SG}}(\mu,t^j))$ and $f_2(u_N^{\text{SG}}(\mu,t^j))$. Consequently, the operation count for the online stage will thus scale as some power of $\mathcal{N}$. Due to this $O(\mathcal{N})$ dependence, it is somewhat disingenuous to interpret (29) as a reduced-order model since the resulting computational advantage relative to the full-order approximation (28) can be modest.

We seek to develop a reduced-order approximation with an online evaluation cost *independent* of $\mathcal{N}$. Towards this goal, we first compute two sets of snapshots

$$\mathcal{F}_L^J = \left\{ \{f_1(u_h(\nu_1,t^j))\}_{j=1}^J, \ldots, \{f_1(u_h(\nu_L,t^j))\}_{j=1}^J \right\}$$

$$\mathcal{H}_L^J = \left\{ \{f_2(u_h(\nu_1,t^j))\}_{j=1}^J, \ldots, \{f_2(u_h(\nu_L,t^j))\}_{j=1}^J \right\}.$$

Upon the two sets of snapshots, we construct $\{z_m^1\}_{m=1}^M$, $\{\psi_m^1\}_{m=1}^M$ and $\{z_m^2\}_{m=1}^M$, $\{\psi_m^2\}_{m=1}^M$ by following the procedure described in Section II. Then for any given $w \in X_h$, we approximate $f_1(w)$ by $f_{1,M}^w = \sum_{m=1}^M f_1(w(z_m))\psi_m$ and $f_2(w)$ by $f_{2,M}^w = \sum_{m=1}^M f_2(w(z_m))\psi_m$.

Hence, we can now replace $f_1(u_N^{\text{SG}}(\mu,t^j))$ and $f_2(u_N^{\text{SG}}(\mu,t^j))$ — as required in the Galerkin projection for $u_N^{\text{SG}}(\mu,t^j)$ — by $f_{1,M}^{u_N}(\mu,t^j)$ and $f_{2,M}^{u_N}(\mu,t^j)$, respectively. Our reduced-order approximation is thus: given any $\mu \in \mathcal{D}$, for $j = 1, \ldots, J$, we evaluate

$$s_N(\mu,t^j) = \ell^O(u_N(\mu,t^j)) \ , \quad (30)$$

where $u_N(\mu,t^j) \in W_N$ satisfies

$$m(u_N(\mu,t^j),v) - \Delta t \int_\Omega f_{1,M}^{u_N}(\mu,t^j)\frac{\partial v}{\partial x^1} - \Delta t \int_\Omega f_{2,M}^{u_N}(\mu,t^j)\frac{\partial v}{\partial x^2} + \mu\Delta t\, a(u_N(\mu,t^j),v) = m(u_N(\mu,t^{j-1}),v), \quad v \in W_N \ . \quad (31)$$

To derive an algebraic discrete system for (31), we first expand

$$\begin{aligned} u_N(\mu,t^j) &= \sum_{n=1}^N \underline{u}_{N\,n}(\mu,t^j)\zeta_n \\ f_{1,M}^{u_N}(\mu,t^j) &= \sum_{m=1}^M f_1(u_N(z_m^1;\mu,t^j))\psi_m^1 \\ f_{2,M}^{u_N}(\mu,t^j) &= \sum_{m=1}^M f_2(u_N(z_m^2;\mu,t^j))\psi_m^2 \ . \end{aligned}$$

Inserting these representations into (31), and choosing $v = \zeta_{n'}$, we immediately obtain the nonlinear algebraic system

$$\underline{M}_N\underline{u}_N(\mu,t^j) - \Delta t\underline{E}_{N,M}^1\underline{F}_M^1(\underline{u}_N(\mu,t^j)) - \Delta t\underline{E}_{N,M}^2\underline{F}_M^2(\underline{u}_N(\mu,t^j)) + \mu\Delta t\underline{A}_N\underline{u}_N(\mu,t^j) = \underline{M}_N\underline{u}_N(\mu,t^{j-1}), \quad j = 1,\ldots,J \ , \quad (32)$$

with the initial solution $\underline{u}_{N\,n}(0) = (u_0,\zeta_n), 1 \le n \le N$. Here $\underline{M}_N \in \mathbb{R}^{N\times N}$ and $\underline{A}_N \in \mathbb{R}^{N\times N}$ are SPD matrices with entries

$$\underline{M}_{N\ i,j} = m(\zeta_j,\zeta_i), \quad \underline{A}_{N\ i,j} = a(\zeta_j,\zeta_i)$$

for $1 \le i,j \le N$; $\underline{E}_{N,M}^1 \in \mathbb{R}^{N\times M}$ and $\underline{E}_{N,M}^2 \in \mathbb{R}^{N\times M}$ are given by

$$\underline{E}_{N,M\,i,j}^1 = \int_\Omega \psi_j^1\frac{\partial\zeta_i}{\partial x^1}, \quad \underline{E}_{N,M\,i,j}^2 = \int_\Omega \psi_j^2\frac{\partial\zeta_i}{\partial x^2}, \quad (33)$$

for $1 \le i \le N$, $1 \le j \le M$; and $\underline{F}_M^1(\underline{u}_N(\mu,t^j)) \in \mathbb{R}^M$ and $\underline{F}_M^2(\underline{u}_N(\mu,t^j)) \in \mathbb{R}^M$ are given by

$$\underline{F}_M^1(\underline{u}_N(\mu,t^j)) = f_1\big(\underline{D}_{M,N}^1\underline{u}_N(\mu,t^j)\big), \quad (34)$$

$$\underline{F}_M^2(\underline{u}_N(\mu,t^j)) = f_2\big(\underline{D}_{M,N}^2\underline{u}_N(\mu,t^j)\big), \quad (35)$$

where $\underline{D}_{M,N}^1 \in \mathbb{R}^{M\times N}$ and $\underline{D}_{M,N}^2 \in \mathbb{R}^{M\times N}$ are matrices with entries

$$\underline{D}_{M,N\ i,j}^1 = \zeta_j(z_i^1), \quad \underline{D}_{M,N\ i,j}^2 = \zeta_j(z_i^2)$$

for $1 \le i \le M, 1 \le j \le N$.

The nonlinear algebraic system (32) can be solved by using a Newton's method for the coefficient vector $\underline{u}_N(\mu,t^j)$. The reduced-order output is thus calculated as

$$s_N(\mu,t^j) = \big(\underline{L}_N^O\big)^T\underline{u}_N(\mu,t^j) \ , \quad (36)$$

where $\underline{L}_N^O \in \mathbb{R}^N$ is given by $\underline{L}_{N\,n}^O = \ell^O(\zeta_n), 1 \le n \le N$. The offline-online procedure for the very rapid computation of (32) may now be developed as follows.

In the offline stage, we compute and store the *parameter-independent* quantities $\underline{u}_N(0)$, $\underline{A}_N$, $\underline{M}_N$, $\underline{E}_{N,M}^1$, $\underline{E}_{N,M}^2$, $\underline{D}_{M,N}^1$, $\underline{D}_{M,N}^2$. In the online stage — for each new parameter value μ — we solve the nonlinear algrebraic system (32) for the coefficient vector $\underline{u}_N(\mu,t^j)$ by using the Newton's method with a computational cost (per Newton iteration per timestep) of only $\mathcal{O}(MN^2 + N^3)$ (see [7] for a detailed complexity analysis). We see that the complexity of the online stage is *independent* of $\mathcal{N}$. Hence, we expect significant computational savings relative to the full-order approximation (28) and relative to standard Galerkin reduced-order approximation built upon (29).

C. Example 2: Buckley-Leverett equation

We consider the two-dimensional Buckley-Leverett equation

$$\frac{\partial u}{\partial t} + \frac{\partial f_1(u)}{\partial x^1} + \frac{\partial f_2(u)}{\partial x^2} - \mu \nabla^2 u = 0, \quad \text{in } \Omega \times (0,T] \quad (37)$$

where $x \in \Omega =]-1.5, 1.5[^2$, $\mu \in \mathcal{D} \equiv [0.05, 0.1]$, $t \in (0,T]$ with $T = 0.5$, and $f_1(u)$ and $f_2(u)$ are two nonlinear *non-polynomial* functions

$$f_1(u) = \frac{u^2}{u^2 + (1-u)^2}, \quad f_2(u) = f_1(u)(1 - 5(1-u)^2);$$

we impose a homogeneous boundary condition on $\partial\Omega$ and an initial condition $u_0(x) = \exp(-16((x^1)^2 + (x^2)^2))$. The output of interest is the average of the field variable over the physical domain.

The weak formulation is stated as: given $\mu \in \mathcal{D}$, find $s(\mu,t) = \int_\Omega u(\mu,t)$, where $u(\mu,t) \in X = H_0^1(\Omega) \equiv \{v \in H^1(\Omega) \mid v|_{\partial\Omega} = 0\}$ is the solution of

$$\int_\Omega \frac{\partial u}{\partial t} v - \int_\Omega f_1(u) \frac{\partial v}{\partial x^1} - \int_\Omega f_2(u) \frac{\partial v}{\partial x^2} + \mu \int_\Omega \nabla u \cdot \nabla v = 0, \quad \forall v \in X, t \in (0,T] \ . \quad (38)$$

Our abstract statement (5)-(6) then obtains for

$$m(w,v) = \int_\Omega wv, \quad a(w,v) = \int_\omega \nabla w \cdot \nabla v \ , \quad \ell^O(v) = \int_\Omega v.$$

For the full-order approximation (27)-(28), we take $J = 50$ time steps and $X_h \in X$ as a piecewise-linear finite element approximation space of dimension $\mathcal{N} = 10,000$.

We now present numerical results obtained with the reduced-order approximation (32). For this purpose we define the average relative error in the solution as

$$\epsilon^u_{N,\text{ave,rel}} = \text{mean}_{\mu \in \Xi_{\text{Test}}, 1 \le j \le J} \frac{||u_h(\mu,t^j) - u_N(\mu,t^j)||}{||u_h(\mu,t^j)||}.$$

Here $\Xi_{\text{Test}} \subset \mathcal{D}$ is the parameter test sample of size 12. Figure 5 shows $\epsilon^u_{N,\text{ave,rel}}$ as a function of N and M. Again we observe very rapid convergence of the reduced-order approximation to the full-order approximation and the same convergence behavior already seen in the Example 1. This trend of convergence is typical of our reduced-order approximation and basically suggests that optimal combinations of N and M are at the "knees" of the error curves.

Finally, we compare the reduced-basis approximation with the standard Galerkin reduced-order model. For this purpose, we define

$$\epsilon^s_{N,\text{rel,max}} = \max_{\mu \in \Xi_{\text{Test}}, 1 \le j \le J} \frac{|s_h(\mu,t^j) - s_N(\mu,t^j))}{|s_h(\mu,t^j)|}$$

$$\epsilon^{s,\text{SG}}_{N,\text{rel,max}} = \max_{\mu \in \Xi_{\text{Test}}, 1 \le j \le J} \frac{|s_h(\mu,t^j) - s^{\text{SG}}_N(\mu,t^j))|}{|s_h(\mu,t^j)|}$$

and introduce $\overline{\text{T}}_N$ and $\overline{\text{T}}^{\text{SG}}_N$ as the normalized computational times for the reduced-order approximation and for the standard Galekin reduced-order model, respectively. Here the times are normalized with respect to the time to solve the full-order approximation. We present in Table II $\epsilon^s_{N,\text{max,rel}}$ and $\overline{\text{T}}_N$ as a function of N and M, and $\epsilon^{\text{SG}}_{N,\text{rel,max}}$, and $\overline{\text{T}}^{\text{SG}}_N$ as a function of N. We see that for a relative accuracy of less than 0.03 percent ($N = 40$, $M = 50$), the online time to compute $s_N(\mu,t^j)$ is about 1/5000 the time to directly calculate $s_h(\mu,t^j)$. In addition, thanks to the fast convergence of the collateral reduced-basis expansions and the $\mathcal{N}$-*independent* computational cost of the online stage, our reduced-order model achieves *three orders of magnitude* less expensive than the standard Galerkin reduced-order model, while yielding almost the same convergence and accuracy. Meanwhile the standard Galerkin reduced-order model suffers from the $\mathcal{N}$-dependent cost of evaluating the nonlinear terms, thereby having a very modest advantage relative to the full-order approximation.

In summary, the present approach is able to provide efficient and accurate reduced order approximation of highly nonlinear convection-diffusion equations.

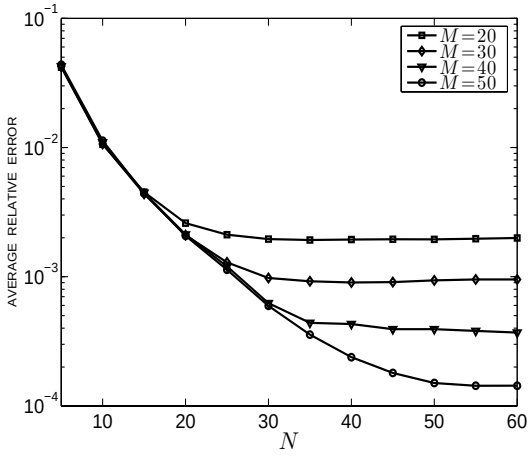


Fig. 5: The average relative error $\epsilon^u_{N,\text{ave,rel}}$ as a function of N and M for the Example 2.

N	M	$\epsilon^s_{N,\text{max,rel}}$	$\overline{\text{T}}_N$	$\epsilon^{s,\text{SG}}_{N,\text{max,rel}}$	$\overline{\text{T}}^{\text{SG}}_N$
10	20	2.19 E−02	4.91 E−05	2.18 E−02	4.15 E−01
20	30	4.78 E−03	7.69 E−05	4.71 E−03	4.15 E−01
30	40	8.13 E−04	1.26 E−04	7.94 E−03	4.15 E−01
40	50	2.56 E−04	1.93 E−04	2.45 E−04	4.15 E−01
50	60	1.19 E−04	2.98 E−04	9.62 E−05	4.15 E−01

TABLE II: Numerical results for the Example 2: $\epsilon^s_{N,\text{max,rel}}$ and $\overline{\text{T}}_N$ as a function of N and M; $\epsilon^{s,\text{SG}}_{N,\text{max,rel}}$ and $\overline{\text{T}}^{\text{SG}}_N$ as a function of N. The online computational times are normalized with respect to the time to compute $s_h(\mu,t^j)$.

V. Conclusions

We have presented a numerical approach for efficient reduced-order approximation of nonaffine and nonlinear partial differential equations. The approach is based on the BPIM developed to provide effective reduced-order treatment of nonaffine and nonlinear terms. Numerical results have shown *very* rapid

convergence of the reduced-order approximation to the full-order approximation. In addition, significant computational savings relative to classical discretization approaches and relative to standard Galerkin reduced-order approach were evident from both the complexity analysis and the numerical examples presented.

We are currently investigating in the application of this approach for compressible Euler and Navier-Stokes equations. Our future research shall also focus on *a posteriori* error estimators and basis adaptivity for the nonaffine and nonlinear problems.

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