Dynamic Pricing and Inventory Control with no Backorders under Uncertainty and Competition

by

Elodie Adida

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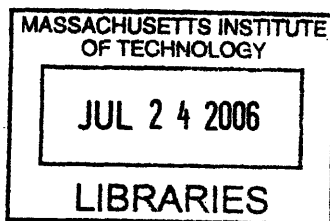MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Author . . . .
Sloan School of Management
May 18, 2006

Certified by. . . . .
Georgia Perakis
J. Spencer Standish Associate Professor of Operations Research
Thesis Supervisor

Accepted by . . . .
James B. Orlin
Edward Pennell Brooks Professor of Operations Research
Codirector, Operations Research Center

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Abstract

Recently, revenue management has become popular in many industries such as the airline, the supply chain, and the transportation industry. Decision makers realize that even small improvements in their operations can have a significant impact on their profits. Nevertheless, determining pricing and inventory optimal policies in more realistic settings may not be a tractable task. Ignoring the potential inaccuracy of parameters may lead to a solution that actually performs poorly, or even that violates some constraints. Finally, competitors impact a supplier's best strategy by influencing her demand, revenues, and field of possible actions. Taking a game theoretic approach and determining the equilibrium of the system can help understand its state in the long run.

This thesis presents a continuous time optimal control model for studying a dynamic pricing and inventory control problem in a make-to-stock manufacturing system. We consider a multi-product capacitated, dynamic setting. We introduce a demand-based model with convex costs. A key part of the model is that no backorders are allowed, as this introduces a constraint on the state variables. We first study the deterministic version of this problem. We introduce and study a solution method that enables to compute the optimal solution on a finite time horizon in a monopoly setting. Our results illustrate the role of capacity and the effects of the dynamic nature of demand. We then introduce an additive model of demand uncertainty. We use a robust optimization approach to protect the solution against data uncertainty in a tractable manner, and without imposing stringent assumptions on available information. We show that the robust formulation is of the same order of complexity as the deterministic problem and demonstrate how to adapt solution method. Finally, we consider a duopoly setting and use a more general model of additive and multiplicative demand uncertainty. We formulate the robust problem as a coupled constraint differential game. Using a quasi-variational inequality reformulation, we prove the existence of Nash equilibria in continuous time and study issues of uniqueness. Finally, we introduce a relaxation-type algorithm and prove its convergence to

a particular Nash equilibrium (normalized Nash equilibrium) in discrete time.

Thesis Supervisor: Georgia Perakis
Title: J. Spencer Standish Associate Professor of Operations Research

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Chapter 1

Introduction

The profitability of operations of a firm is critically affected by decisions regarding its pricing, inventory and production strategy. Static policies usually perform poorly compared to dynamic policies, in which the price and production rate are adjusted over time. Indeed, the demand in particular, and sometimes other components of the system, usually evolve over time. Today, many channels of distribution (such as the internet) allow suppliers to change prices over time when they can benefit from a dynamic strategy. A dynamic strategy requires the price and production rate to be determined at all times in order to maximize the net profit over a time horizon, while obeying some constraints due to stock level limits or production capacity, taking into account production and inventory holding costs. Moreover, when the firm produces simultaneously multiple products, new decisions must be made to allocate the available production capacity among the products.

Practitioners often face the problem of uncertainty when determining model parameters. Ignoring uncertainties may yield a strategy that is useless if the true parameter values differ from their model estimate. Indeed, this strategy may not only be very suboptimal, but may also be infeasible due to constraint violations. It is therefore of great importance to provide a way to incorporate uncertainty in the model, while proposing tractable solutions, and without making unrealistic assumptions.

We consider an oligopoly market with differentiated products, where multiple competitors target the same potential buyers. In such a setting, the price of a product at a given firm affects the demand of all firms for that product. In other words, the demand observed by a given firm for some product depends not only on her own price, but also on the prices applied by her competitors. Therefore, each competitor has to determine her strategy based on her competitors' prices. To understand how competition affects optimal decisions, we place the problem within a game-theoretic setting and study equilibria.

The overall goal of this research is to provide a model of dynamic pricing and inventory management under demand uncertainty and competition, when no stock-outs are allowed. We are facing contradicting challenges, since we wish to build a model as realistic as possible but at the same time to gain insights from its solution in order to reach a better understanding of how to determine optimal strategies in practice.

1.1 Motivation

Recently, revenue management has become popular in many industries such as the airline, the supply chain, and the transportation industry. Many retail firms have recognized that better revenue management and inventory control may yield significant positive impact. They invest time and effort in an attempt to optimize their prices and inventory policies. One of the critical factors to determine the profitability of a firm is its pricing and production strategy. Many firms hire specialized consultants to help them determine optimal strategies and improve their revenue management. A study by McKinsey and Company on the cost structure of Fortune 1000 companies in the year 2001 shows that pricing is a more powerful lever than variable cost, fixed cost or sales volume improvements. An improvement of 1% in pricing yields an average of 8.6% in operating margin improvement (see [10]). Therefore, companies' ability to survive in a competitive environment depends on the development of efficient pricing

models. Inventory control can also have a major impact on profits. Abernathy et al. note that "Retailers no longer place large seasonal orders for goods in advance instead, they require ongoing replenishment of stock, forcing manufacturers to predict demand and then hold substantial inventories indefinitely. Manufacturers now carry the cost of inventory risk. (...) The inventory demand for stock-keeping units within the same product line can vary significantly. (...) By differentiating inventory policies at the stock-keeping unit level, manufacturers (...) improve the profitability of the entire line." [1]. As a result, dynamic pricing, inventory control and revenue management have become very active research topics and have been extensively studied in the academic literature in Economics, Operations Management, and Marketing (see for example [34], [57], [119], [130]). In such settings, suppliers are maximizing their profits over a time horizon subject to some constraints. Furthermore, data changes over time. Therefore, these problems are typically formulated as constrained dynamic optimization problems. Dynamic Pricing and Inventory Control problems are challenging due to various aspects of the problem:

(a) the intrinsically dynamic aspect;

(b) uncertainty of the demand;

(c) competition among suppliers;

(d) the non separability of the decision variables (prices and production decisions are typically inter-dependent).

1.1.1 Dynamic nature of the problem

In a setting where parameters and decision variables change over time, the state of the system constantly evolves. In particular, the inventory level is variable that represents the current state of the system. It results from the initial state and the entire history of decisions until the current time. In an open-loop framework, the value of data over time is known in advance, and decision makers commit at time zero to their decision over the time horizon, in order to maximize the total profits. In such a

framework, the solution is determined by considering the entire time horizon, rather than by taking a myopic approach that would seek to maximize the instantaneous profits. It is important to realize that the decisions at a given time have an impact on the future state of the system. For example, if the demand is expected to increase over time, and production capacity is small, it may be optimal to choose a high production rate even at the beginning of the horizon even though the low demand at the time does not require it, in order to build up inventory stock. Then it will be possible to satisfy the high demand later on without increasing prices. Otherwise, the low production capacity would not have been sufficient to satisfy the demand, and prices would have to increase in order to slow demand. This example illustrates how optimal decisions are linked over time. Therefore, the dynamic aspect of the problem is essential for determining its optimal solution.

To model the dynamics of the system, some researchers choose to view time as discrete. The drawback of such an approach is having to decide the length of the step size. A fine discretization of the time horizon provides more accuracy. Nevertheless, it gives rise to a problem of a huge size, both in terms of number of variables and number of constraints (even though the dimension is finite), which implies significant delays in obtaining good solutions. When viewing time continuously, the inventory may be seen as a continuous flow, and as a result, there is no need to assume that decisions occur only at discrete points on the time axis. The problem can be formulated as a fluid model. Fluid models provide a powerful tool for understanding the behavior of systems where the dynamic aspect plays an important role. They arise in applications as diverse as routing, communication, queueing, supply chain and transportation systems. They have been used in particular as approximations of complex stochastic systems, such as queueing networks. Recent research has proven that an attractive feature of these models in supply chain applications is that they provide good scheduling, production and inventory policies in a variety of settings. Moreover, fluid models allow a continuous time approach instead of having to discretize time. However, since demand depends on prices, these formulations are nonlinear, and as

such, may be very complex to solve. Examples of supply chain industries where fluid models of the type we discuss in this thesis are relevant, include industries with a high volume of throughput and data on costs and demand that change a lot. The hardware as well as the semiconductor industries are such examples. Moreover, we believe that a similar approach can be applied to problems in areas other than dynamic pricing and inventory control, where the dynamic evolution of the system justifies a continuous time approach. We believe that the techniques presented in this thesis may be helpful to those areas as well.

1.1.2 Uncertainty

Most of the time, practitioners and researchers assume that their model represents well reality, and that data are known with certainty. However, uncertainty is inherent in nature and forecasts always involve some degree of randomness since many unexpected events may affect the future. Moreover, the model may inexactly represent the system and the interactions between its components. For instance, modeling the demand as varying linearly with the price, is an approximation of the true relationship between price and demand. Finally, even when the model is exact, its validity relies on the ability to determine the exact values of its parameters, which may be difficult to measure in practice. One way to evaluate those parameters is the use of regression or statistical inference models based on historical data. Nevertheless, these methods provide only an estimate of the parameters. Furthermore, they are based on the assumption that the future can be inferred from the past, and this assumption is not always justified. Ignoring the potential inaccuracy of models parameters may lead to a solution that performs poorly if the actual value differed from the one assumed in the model, or even that is infeasible. In particular, when in the nominal model constraints are binding at this solution, it is likely that even a small perturbation of the data will yield a violation of those constraints.

A common approach to address uncertainty is to assume a probability distribution on the parameters, and use stochastic optimization and dynamic programming.

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However, suppliers may find it difficult in practice to determine such probability distributions. In most applications, it may be unrealistic to know anything other than the first and second moment of the distribution. For reasons mentioned above, even historical data may not be sufficient to have an accurate idea of the probability distribution. Moreover, even if the distribution is known, solving these formulations is often not possible because of what is known as the "curse of dimensionality". As the number of periods increases, the dimension of the problem is so large that the problem becomes intractable.

This idea motivates a robust optimization approach. The main idea is to find the best strategy given some uncertainty model on the data. The field of robust optimization has attracted a lot of research recently by providing an efficient way of finding good solutions that are immune (or *robust*) to data uncertainty. Robust optimization is also easier to use than stochastic optimization and dynamic programming in practice because it does not require to make any assumption on the probability distributions: it only requires a range of variation, which is not very difficult to estimate in most applications. We will consider that the demand parameters are subject to uncertainty, since they are the most difficult to evaluate in practice. The tractability of solving the problem under this approach depends essentially on the structure of the uncertainty set within which the data are allowed to vary. Part of the task in formulating the robust model is to design these sets in such a way that tractability is preserved.

1.1.3 Competition

A third important feature we would like to study is competition. In a non-monopolistic setting, several firms compete. The decisions taken by a firm may affect other firms by influencing her demand, revenues, and field of possible actions. Thus when multiple sellers compete in a market, these mutual interactions motivate a game theoretic approach which makes a key difference in the techniques determining the solution. Such problems have been studied in the literature in economics, revenue management,

and supply chain.

There are many ways of modeling competition based on the number of competitors, whether there is a leader and some followers, whether competitors cooperate, etc. In this thesis, we assume an oligopoly where all suppliers simultaneously seek to optimize their revenues. The question of interest is then to determine if the game has an equilibrium, i.e. a solution for all competitors such that no one has an incentive to unilaterally deviate from it (*Nash equilibrium*). For most settings, it would be unrealistic to assume a monopoly or that competitors cooperate. Oligopolistic settings with no collaboration are thought to protect the customer by guaranteeing fair prices due to the competition between suppliers. Most developed economies have laws that prohibit monopolies and cooperation (US Antitrust law for example). Therefore, in this thesis we will consider a non cooperative oligopolistic competition.

1.2   Literature review

In this section, we provide an overview of the literature on topics closely related to the thesis. We first discuss some related papers on revenue management, dynamic pricing and optimal control in a monopoly setting. Then we present references that are relevant to modeling demand uncertainty. Finally, we mention literature relative to oligopolistic competition.

1.2.1   Dynamic pricing, inventory control, revenue management, supply chain, and optimal control

There is a huge literature on inventory control as well as pricing and revenue management. For example, the book by Porteus [108] and the book by Zipkin [130] review inventory management techniques, while the book by Talluri and van Ryzin [119] provides an overview of the revenue management and pricing literature. In this section,

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we provide details and references on advances in areas related to different aspects of the thesis.

In a setting where the problem has a dynamic aspect, such as traffic control, queueing networks, supply chain, or transportation, there is a connection with fluid models. These models can be viewed (when there is no stochasticity) as continuous time optimal control models. A large part of the literature on continuous-time optimal control models has focused on the solution of linear formulations (see for example Anderson [4], [5], Pullan [109]). This part of the literature shows existence of an optimal solution with piecewise constant controls. Pullan in particular showed strong duality and designed a class of algorithms solution convergent. For the solution of linear fluid models, Bertsimas and Luo [93] construct an algorithm solving state constrained separated continuous linear programs under some assumptions. Fluid models also connect with semi-infinite programming problems. Tunçel and Todd [121] study the asymptotic behavior of interior point methods for semi-infinite programming by finding the limits of search directions, potential functions and central paths as the number of variables becomes infinite.

However, when fluid models are nonlinear, the dynamic together with the nonlinear aspect of the problem make them harder to analyze. Nonlinear fluid models are particularly useful for dynamic pricing and inventory management applications, as we explained above. A variety of models have been proposed in the literature for such applications (see references below). These models typically differ due to the production cost, inventory cost, and demand functions considered. More theoretically, many papers study general continuous time optimal control models. [7], [73], [85] and [115] give formulations of the Maximum Principle under state variable constraints. Clarke ([43], [44], [45], [46] with others) and Devdariani and Ledyaev [51] provide theoretical results on global optimality conditions.

The literature on dynamic pricing is growing fast. Elmaghraby and Keskinocak in [57] and the references therein provide a comprehensive literature review of dynamic

pricing models while Bitran and Caldentey [34] provide an overview of research on dynamic pricing and its relation to Revenue Management. Furthermore, Zipkin [130] and the references therein provide a thorough review of recent advances in inventory control theory and its relation to supply chain. Chan, Shen, Simchi-Levi and Swann [39] review research on coordination of pricing and inventory decisions. Finally, Yano and Gilbert [126] and the references therein provide a review of pricing and production/procurement decisions.

A large volume of literature studies a demand model for the single-product case. For example Pekelman [105] solves the dynamic pricing and production policy problem for a single product optimizing over a finite time horizon. He models the demand as a linear function of the price with time-varying coefficients. The model uses linear inventory cost with a constant coefficient, and a general strictly convex production cost. The model does not allow backorders (negative inventory levels). Feichtinger and Hartl [60] extend this model by considering a general nonlinear demand function and allowing backorders, with both piecewise linear and strictly convex inventory costs. They obtain phase diagrams for the equilibrium and transient behavior of the optimal solution with a finite or infinite time horizon. Another extension is introduced by Thompson, Sethi and Teng in [120], where the production rate and the level of inventory are bounded, and the production cost is either linear or strictly convex. Gaimon [64] considers additional controls by allowing decisions on the maximal production rate as well as price and production output, where the change in maximal production rate has an effect on the production cost. [56], [78] and [89] consider the case of centralized or decentralized decisions between a distributor and a manufacturer in an industrial channel of distribution. Locke Anderson [91] considers production decisions when the production of a final good requires as input the production of an intermediate good. In the single-product model, Jørgensen [79] uses a continuous time optimal control model to study demand learning effects while Laurent-Varin [90] introduces an interior-point solution algorithm.

In a multi-product setting, Bertsimas and Paschalidis [28], Harrison [72] and Meyn [98] study a make-to-stock problem using fluids. Specifically, Bertsimas and Paschalidis in [28] study an inventory control problem with fixed demand rate and capacity rate shared among all classes. Their model allows backorders and computes a production policy by minimizing either a linear or a quadratic inventory cost over successive small intervals. Luo [92] considers a make-to-stock multi-class queueing scheduling problem that minimizes a convex quadratic backorder and holding cost and finds an optimal production policy over the entire time horizon. Kleywegt [86] uses a cutting plane algorithm to solve a multi-class optimal control problem of dynamic pricing with profit linear in terms of selling rate. Fleisher and Sethuraman [62] provide an approximation algorithm to solve the optimal control of fluid queueing networks. Moreover, van Ryzin and McGill [124] designed an adaptive approach within the framework of airline revenue management based on historical observed data. They study an algorithm through stochastic approximation theory. Gallego and van Ryzin [66], [67] consider the problem of dynamically pricing over a finite horizon when demand is stochastic and price sensitive. Finally, Kachani and Perakis [82], [83] take a delay-based approach to determine optimal pricing and production policies, where the price and level of inventory affect the delay (time that a product remains in inventory).

A stream of research has focused on a dynamic programming approach to solve pricing and/or inventory problems (see [2], [26]), by dividing the (possibly infinite) time horizon into time periods and allowing decisions at the beginning of each period, as opposed to the research cited above that takes a continuous approach, including Maglaras and Meissner [95] who approach the pricing problem under fixed capacity by reducing the problem to determining the aggregate rate at which all products jointly consume resource capacity, and defining an efficient frontier.

1.2.2 Demand uncertainty

Economists have used a variety of demand models which have been applied in the revenue management literature (see for example, [6], [125], survey articles [39] and [57]). The problem of demand uncertainty has motivated a significant amount of literature in the field of Revenue Management and Pricing. A number of different approaches have been introduced to model this uncertainty. Zabel [129] considers two models of uncertain demand: a multiplicative model ($d_t = \eta_t u(p_t)$) and an additive model ($d_t = u(p_t) + \eta_t$), where d_t is the demand at time t, p_t is the price at time t, $u(p) = a - \frac{a}{b}p$, i.e. a downward sloping linear demand curve, and η_t is assumed to be either exponentially or uniformly distributed with $E[\eta_t] > 0$. Young [128] and Federgruen and Heching [58] generalize the demand model to be of the form $d_t = \gamma_t(p)\epsilon_t + \delta_t(p)$, where γ and δ have first derivatives non positive and ϵ_t is a random term with a finite mean. Gallego and van Ryzin [66], [67] as well as Bitran and Mondschein [35] assume that demand follows a Poisson process with a deterministic intensity that depends on price and time. Raman and Chatterjee [110] model the stochasticity of the demand by introducing an additive model where the random noise is a continuous time Wiener process.

For additional details and references, see review papers such as [34], [57] and [126].

The Operations Research literature treats the presence of data uncertainty in optimization problem in several ways. The problem is sometimes solved assuming all parameters are deterministic; subsequently sensitivity analysis is performed to study the stability of the nominal solution with respect to small perturbations of the data. Stochastic programming is used when a probability distribution of the underlying uncertain parameters is available, and seeks a solution that performs well and has low probability of constraint violation. Robust optimization is an alternative was to seek an optimal solution of a problem when its data is uncertain.

A robust optimization formulation was first considered by Soyster [117] in the case of a linear optimization problem where the data were uncertain within a convex set.

He addresses uncertainty by taking a worst-case approach. Nevertheless, such an approach decreases the performance of the solution significantly, and was criticized of being overly conservative. Ben-Tal and Nemirovski ([16],[17]) address the issue of over-conservativeness by considering data uncertainty sets that are ellipsoids, for linear programming (LP) and general convex programming. Numerical examples of their approach for polyhedral uncertainty sets, that allow to reformulate the robust counterpart of an LP as an LP, can be found in [18]. El-Ghaoui et al ([54], [55]) independently developed the approach of robust counterpart applied mostly to uncertain semidefinite programming, by adapting ideas from robust control.

They show that the robust counterpart of many convex optimization problems (LPs, second-order cone problems (SOCPs), semidefinite programming programs (SDPs)) with data within ellipsoidal uncertainty sets can be efficiently solved exactly or approximatively by polynomial-time algorithms. However, for this type of uncertainty sets, complexity increases: the robust counterpart of an LP is reformulated as an SOCP, of an SOCP is reformulated as an SDP, and the robust counterpart of an SDP is NP-hard to solve. In [19], Ben-Tal, Nemirovski and Roos approximate the solution of the NP-hard semidefinite robust counterpart of an SOCP with ellipsoidal uncertainty sets with a single explicit semidefinite program.

Bertsimas and Sim [30] seek a model leading to a robust counterpart problem to an LP that is still a linear optimization problem, by introducing the notion of budget of uncertainty to control the level of conservativeness. They studied the tradeoff between robustness of a solution to a linear programming problem and the sub-optimality of the solution. In [29], they use this approach for discrete optimization and network flow problems. Bertsimas, Pachamanova and Sim [27] propose a framework for robust modeling of LPs using uncertainty sets described by an arbitrary norm. Bertsimas and Sim [31] propose a relaxed robust counterpart for general conic optimization problems that preserves the original structure (robust LPS are LPs, robust SOCPs are SOCPs, robust SDPS are SDPs), and provides probability guarantees on the feasibility of the solution. Bertsimas and Brown [25] construct uncertainty sets for LPs by taking a coherent risk measure as primitive.

The robust optimization methodology has been applied to a number of areas. Bertsimas and Thiele [32] apply robust optimization principles to inventory theory and supply chain management. In [15], Ben-Tal and Nemirovski use robust optimization to formulate a robust truss topology design problem as an SDP. Goldfarb and Iyengar [71] apply robust optimization to portfolio selection problems. They introduce uncertainty sets that allow to reformulate the robust counterpart problem as an SOCP. Ben-Tal et al [14] take a robust optimization approach for multi-period stochastic operations management problems, and in particular the retailer-supplier flexible commitment problem with uncertain demand.

1.2.3 Competition

In economics, many game-theoretic problems in continuous time are modeled as differential games, i.e. games involving a fluid equation. Jørgensen and Zaccour [80] apply a differential game model to control pollution emission. They consider two neighboring countries who make decisions on emissions and investments in abatement technology. The dynamics of capital stocks and the stock of pollution are modeled through a fluid equation. They solve in closed form the cooperative problem and the Nash equilibrium problem, and they design an incentive strategy. Mäler and de Zeeuw [96] use a differential game to model a problem related to acid rains. Neighboring countries face a trade-off between costs of emission reductions and the damage to the soils.

Literature on oligopolistic competition in the field of pricing and revenue management is emerging fast in recent years. The book by Vives [125] presents a number of pricing models in an oligopoly market. For a survey of joint pricing and production decisions for inventory control in a supply chain setting, the reader can refer to Chan et al. [41] as well as Cachon and Netessine [36]. A wide range of applications of pricing can be found in the literature in economics, marketing and management science (see for example, Jørgensen and Zaccour [81] and Dockner and others [52], and the references therein). Fudenberg and Tirole [63] review a variety of game theoretic models for pricing and capacity decisions. Gaimon [65] studies open and closed

loop Nash equilibria for two firms and a single product setting, where the price and capacity are determined to maximize net profits when the acquisition of new technology reduces the operating cost. For perishable products, Perakis and Sood [106] and Perakis and Nguyen [100] study non cooperative equilibrium policies. Steinberg and Eliashberg [118] study a production and expenditure problem and obtain a dynamic open-loop Nash equilibrium. Feichtinger [59] applies differential games to advertising. He considers two profit-maximizing firms and studies the structure of optimal advertising rates in the Nash equilibrium solution. For a survey of applications of differential games to management science, see [61]. In many cases in these applications, the equilibrium point is obtained by solving the differential form of the Kuhn-Tucker optimality conditions. Nevertheless, it is not always the case that solving these conditions is tractable.

Differential games are useful to model dynamic systems of conflict and cooperation where decisions are made over a time horizon. Başar presents some theoretic results in [11] for non-cooperative dynamic games. Quasi-variational inequality problems were introduced by Bensoussan and Lions [20], [21], [22], motivated by stochastic impulse control problems. Mosco [99] studies topological and order methods to solve this type of problems with implicit constraints and connects these problems with applications to differential games. Quasi-variational inequalities in finite dimension have been studied by several authors such as Yao [127], Pang [101], Pang and Fukushima [102], and Chan and Pang [38]. Cavazzuti and others [37] introduce some relationships between Nash equilibria, variational equilibria and dynamic equilibria for noncooperative games without assuming that the dimension is finite. Cubiotti [48] and Cubiotti and Yen [49] prove the existence of solution for generalized quasi-variational inequalities in infinite-dimensional normed spaces under some conditions. Pang and Stewart [103] recently introduced differential variational inequalities and illustrated this notion in particular for differential games. Using the maximum principle, Schumacher [112] provides an example of a differential variational inequality for a linear quadratic dynamic optimal control problem with state constraints only, and a linear problem

with control constraints.

The degree of interdependency between players' actions impacts directly the difficulty of finding an equilibrium. Rosen [111] extended the literature on finite dimensional n-person non-zero sum games by considering games in which the constraints as well as the payoff function depend on the strategy of every player. He called these games coupled constraint games. He showed the existence of an equilibrium under concavity assumptions, and he introduced the notion of *normalized* Nash equilibrium as a particular Nash equilibrium. He showed uniqueness of a normalized Nash equilibrium under certain assumptions. This type of games have several applications, for example in routing [53], and in environmental economics [75], [76], [88], [87].

Relaxation algorithms provide a powerful tool for finding a Nash Equilibrium when the problem is not tractable enough to solve the necessary conditions. These iterative algorithms rely on averaging the current solution iterate with the solution of the *best response* problem each player solves. Uryas'ev and Rubinstein [123] and Başar [11] study the convergence of such algorithms in finite dimensions for finding the equilibria of a non cooperative game for some payoff functions on a closed, compact, subset of $\mathbb{R}^m$. Berridge and Krawczyk [24] apply these techniques to games with non linear payoff functions and coupled constraints arising in economics. Krawczyk and Uryas'ev [88] consider similar problems and use steepest-descent step-size control. Contreras et al. [47] illustrates this method for finding equilibria in electricity markets.

1.3 Overall goal and structure of the thesis

The overall goal of this thesis is to introduce and study an optimization problem for dynamic pricing and inventory control with no backorders for a make-to-stock manufacturing system. We proceed as follows:

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Deterministic problem in a monopoly setting: We first present the basic model in a monopoly setting with no uncertainty in the data.

Robust formulation: We then introduce uncertainty in the demand parameters and take a robust optimization approach.

Robust formulation under competition: We finally add the aspect of competition to the model and address jointly uncertainty of demand in a duopoly competitive setting. Our goal is to study the Nash equilibria of the system.

Given the complexity of the problems we consider, it is clearly impossible to obtain closed form solutions. Our goal was to avoid making unnecessary approximations and unreasonable assumptions. Moreover, we attempted to find solution algorithms that, under certain assumptions on the data, converge to the desired solution either in finite time, or in infinite time but with good performance in practice. To the best of our knowledge, this is the first research work associating simultaneously optimal pricing and production strategies, fluid models, uncertainty via robust optimization, and competition.

The thesis is structured as follows. Chapter two provides a detailed description of the model of dynamic pricing and inventory control problem we propose, the assumptions of the model, and the notations used.

Chapter three describes a solution method for the deterministic problem in a monopoly setting.

Chapter four proposes a robust optimization model to take uncertainty into account. Furthermore, it illustrates the robust formulation is of the same order of "complexity" as the nominal formulation. In particular, we show how to adapt the method from Chapter three to get a robust solution.

Chapter five studies equilibria in continuous time in a duopoly market where demand is deterministic. Subsequently, it extends these results to incorporate uncertain demand. Moreover, this chapter studies an algorithm that converges to an equilibrium in discrete time, when demand is uncertain.

Chapter 2

Formulation

Throughout the thesis, we consider a finite time horizon $[0, T]$ and a dynamic setting. We assume that the suppliers produce multiple non-perishable differentiated products $i = 1, \ldots, N$. We assume that these products share a common resource which yields a production capacity constraint coupling the products together. Our goal is to determine at time zero the optimal pricing and production strategy for all products on the entire time horizon. The solution for prices and production rates is a set of functions of time on the time horizon. Furthermore, the dimension of the problem is infinite. Assuming that sellers are profit-maximizers, an optimal strategy is the strategy providing the highest net profits, given by the sum over time and across products of the demand multiplied by the price, after subtracting the production and inventory costs. We model the production cost either as a strictly convex increasing function, or more particularly as a quadratic function of the production rate. We model the inventory cost either as a linear or a quadratic function of the inventory level. The resulting problem is therefore nonlinear.

Note that all parameters of the problem are time dependent (including the bounds defining feasible ranges when there is uncertainty), thus allowing to model the dynamic aspect of such problems.

This chapter provides a detailed description of the general setting, assumptions on the data, and the motivation behind modeling assumptions.

2.1 Modeling assumptions

2.1.1 No backorders constraint on inventory levels

Inventory problems may allow or deny backorders, i.e. the possibility of having a negative inventory level. A key aspect of the problem under consideration is the no backorders constraint which ensures that inventory levels remain non negative at all times:

$$I_i(t) \geq 0, \quad i = 1, \ldots, N, \quad \forall t \in [0, T],$$

where T is the time horizon and $I_i(t)$ the inventory level of product i at time t. In a manufacturing system which does not allow backorders and where the demand rate is not external, but determined by a relationship with price, the price can be adjusted so that no demand is actually lost. That is, the price is set so that there is enough available inventory to satisfy the demand, in such a way that the selling rate equals the demand rate. This setting can be justified by the presence of a contract between a supplier and a retailer, or by very high fixed backlog cost. For example, Pekelman [105] studies a problem of optimal pricing and production for a single product with no backorders. Axsäter and Juntti [8] study echelon stock reorder policies with no backorders.

In the fluid model, the inventory level is given as the solution of a first order differential equation involving the production rate and the price in a linear relation. More specifically, the inventory level at a given time t is the difference between the cumulative production and the cumulative demand up to that time t, where the cumulative production depends on all production rate decisions from time zero up to time t, and the cumulative demand depends on all prices applied for that product from time zero up to time t. It is therefore *indirectly* related to the decision variables. Reversely, constraints such as production capacity and bounds on the prices involve directly and *instantaneously* the decision variables, and have as a result a simpler impact on the set of feasible strategies. The inventory level at a given time is a

state variable, i.e. it characterizes the state of the system *at that time* based on all choices of decisions variables *up to that time*. The no backorders constraint imposes a non negativity constraint on the inventory levels at each time. Therefore, as a state variable constraint, it is difficult to deal with and makes the problem significantly more complex.

2.1.2 Capacity constraint

We assume that multiple products share a single common production capacity by introducing the constraint

$$\sum_{i=1}^{N} u_i(t) \leq K(t), \quad \forall t \in [0, T],$$

where $u_i(t)$ is the production rate of product i at time t and $K(t)$ is the total production capacity rate at time t. This assumption is a standard one in the literature that considers multiclass systems. For example, Bertsimas and Paschalidis [28] consider a multiclass make-to-stock system and assume that a single facility produces several products, with the production process over time taken as an arbitrary stationary stochastic process. Also in a make-to-stock manufacturing setting with multiple products, Kachani and Perakis [82] suppose that the total production capacity rate across all products is bounded. Gilbert [68] addresses the problem of jointly determining prices and production schedules for a set of items that are produced on the same production equipment and with a limited capacity. Maglaras and Meissner [95] consider a monopolist firm that owns a fixed capacity of a resource that is consumed in the production of multiple products. Finally, Biller et al. [33] extend a single product model of dynamic pricing to cover supply chains with multiple products, each of which is assembled from a set of parts and shares common production capacity. In order to keep the model simple in this thesis, we make a similar assumption of a single production capacity constraint, and we leave as a direction of future research the case of multiple capacity constraints which could be applicable to certain production settings.

2.1.3 Cost structure

Monopoly setting

Producing and holding stocks of inventory incurs costs. A variety of costs structure assumptions can be found in the literature. In the monopoly setting (both with and without demand uncertainty), we assume that the production cost for a given product is a strictly convex increasing function of the production rate:

$$f_i(u_i(t)),$$

where $u_i(t)$ is the production rate of product i at time t.

We make the following technical assumption on the production cost functions.

Assumption 1. *For all products i, function $f_i(.)$ is assumed to be twice continuously differentiable, strictly convex, non-negative and increasing. Moreover, we assume $\lim_{u \to +\infty} f_i'(u) = +\infty$.*

Note that Assumption 1 holds for example if $f_i(.)$ is quadratic.

We model the inventory holding cost as a linear function of the inventory level:

$$h_i(t)I_i(t),$$

where $h_i(t)$ is the positive holding cost coefficient of product i at time t, and $I_i(t)$ is the inventory level of product i at time t. Pekelman [105] studies a problem of optimal pricing and production for a single product with strictly convex production costs and linear holding costs. Clark and Scarf [42] introduce the Multi-echelon Inventory Problem which includes linear holding costs. This model was used extensively in the literature. We provide below references of studies involving quadratic production cost (which are strictly convex).

Duopoly setting

In the duopoly setting, we assume that production and inventory costs are quadratic, and formulated respectively as

$$\gamma_i(t)(u_i(t))^2$$

and

$$h_i(t)(I_i(t))^2,$$

where $u_i(t)$ is the production rate of product i at time t, $I_i(t)$ is the inventory level of product i at time t, $h_i(t)$ is the positive holding cost coefficient of product i at time t, and $\gamma_i(t)$ is the positive production cost coefficient of product i at time t.

This type of cost model has been used often in the literature on inventory control. Goh [70] assumes that the holding cost is a nonlinear function of the amount of the on-hand inventory. He motivates the model by discussing its application to products whose inventory value is very high and many precautionary steps are to be taken to ensure its safety and quality. He cites in particular luxury items like expensive jewelry and designer watches, for which as the on-hand stock inventory grows, some firms employ higher dimensions of security such as hidden cameras and infrared sensors. Similarly, Giri and Chaudhuri [69] consider a model with nonlinear holding cost depending on the stock level with the form hI^n, $n > 1$, where I is the inventory level. They justify this assumption by taking the example of electronic components, radioactive substances, or volatile liquids which are costly and require more sophisticated arrangements for their security and safety.

Holt et al. [77] introduce a linear-quadratic inventory model in which the production and the holding cost are respectively the sum of a linear and a quadratic term in the production rate or the inventory rate. Our model is a particular case where the coefficient of the linear term is zero. They justify this approximation for production costs from a connection with workforce costs. They observe that the cost of hiring and training people rises with the number hired, and the cost of laying off workers, including terminal pay, reorganization, etc., rises with the number laid off. Moreover,

for fixed workforce, increasing production may incur overtime costs.

Pindyck [107] models production costs for commodities such as copper, lumber and heating oil as quadratic costs.Finally, Sethi et al. [114] assume general convex production and inventory costs. Our model is also a particular case of this model.

2.2 Demand model in a deterministic monopoly setting

In the monopolistic and deterministic setting, the demand for product i will be modeled as a linear decreasing function of the price of that product :

$$d_i(t) = \alpha_i(t) - \beta_i(t)p_i(t),$$

where $d_i(t)$ and $p_i(t)$ are respectively the demand and the price at time t for product i, and $\alpha_i(t)$ and $\beta_i(t)$ are known positive real valued functions of time. We will sometimes refer to $\alpha_i(t)$ as the fixed term of the demand, and to $\beta_i(t)$ as the price sensitivity or elasticity. Notice that since the demand must be non negative, prices may not exceed $\frac{\alpha_i(t)}{\beta_i(t)}$.

We make the following assumption on the production cost function and demand parameters.

Assumption 2. $f_i'(0) < \frac{\alpha_i(t)}{\beta_i(t)}, \quad i = 1, \ldots, N \quad \forall t \in [0, T].$

Assumption 2 means that the intercept of the marginal production cost function is smaller than the maximum price that may be charged at any fixed time. (Clearly if this is not the case for some time t, no production will take place at that time. Therefore this assumption simply states that producing may be relevant at all times.)

We also make some technical assumptions on the inputs of the model, that are satisfied in all our numerical examples.

Assumption 3. *For all products i, $\alpha_i(.), \beta_i(.), h_i(.)$ as well as $K(.)$ are assumed to be positive, continuous functions of the time. Moreover, $\alpha_i(.), \beta_i(.)$ and $K(.)$ are assumed to be continuously differentiable.*

Econometricians have considered the problem of estimating price elasticities over time. Senhadji and Montenegro [113] analyze time series to estimate short-run and long-run price elasticities via regression techniques. Slaughter [116] also considers elasticities that vary over time.

In the Operations Research literature, demand learning problems have motivated many researchers. In the case of models of demand linear with the price, the methods they propose can be applied to estimating the parameters $\alpha(.), \beta(.)$ in our model. For example, Kachani, Perakis and Simon [84] design an approach that enable to achieve dynamic pricing while learning the price-demand linear relationship in an oligopoly.

We have assumed that the demand for a product depends only on the price for this product and not on the prices of other products. This assumption is standard in multi-product pricing problems when the products are considered distinct so that they target distinct classes of customers. The automotive industry is one example of industry where such an assumption is valid (see [33]). Bertsimas and de Boer [26] study a joint pricing and resource allocation problem in which a finite supply of resource can be used to produce multiple products and the demand for each product depends on its price. They apply this problem to airline revenue management. Paschalidis and Liu [104] consider a communication network with fixed routing that can accommodate multiple service classes and in which the arrival rate of a given class (or demand for that class) depends on the price per call of that class only. In their multi-product case, Biller et al. [33] assume that there are no diversions among products, i.e. that a change in the price for one product does not affect the demand for another product. They motivate this assumption by focusing on items that appeal to various consumer market segments, such as luxury cars, SUV, small pickup, etc. for example of the automotive industry. We position this thesis in the same line of

research and make the similar assumption of a demand independent of prices of other products. A more general model would allow the demand to depend on all prices with various price elasticities. However, such a model would significantly increase the complexity of the problem. This problem would go beyond the scope of this thesis but could be the focus of follow-up research.

2.3 Uncertainty model in a monopoly setting

Chapter 3 of the thesis takes a *deterministic* approach in a monopoly setting. In Chapter 4, we introduce uncertainty on some input parameters. Specifically, we will assume that the term $\alpha_i(t)$ of the demand is subject to uncertainty. The motivation for considering that parameters $\alpha_i(t)$ are particularly subject to uncertainty comes from the difficulty in practice of forecasting the demand. As a result, in a model linking the demand with price, the parameters of this relationship may be quite difficult to estimate accurately. Other parameters characterizing the system, such as costs parameters or capacity rates, are typically easier to estimate. The results obtained under this model can be generalized to an uncertainty model where the slope of the demand with respect to the price is uncertain as well, as shown in Chapter 4.

2.3.1 An additive model

We first introduce an additive model of demand uncertainty as follows:

$$d_i(t) = \alpha_i(t) - \beta_i(t)p_i(t) + \epsilon_i(t),$$

where $\epsilon_i(t)$ is uncertain within a given interval, with unknown probability distribution on that interval. This model essentially assumes an additive demand model, i.e. that there is uncertainty on the demand parameters $\alpha_i(.)$, $i = 1, \ldots, N$. We denote $\alpha_i(.)$ the nominal function, $\tilde{\alpha}_i(.)$ the realization, and we suppose that the realization belongs to an interval centered around the nominal function with half-length $\hat{\alpha}_i(.)$. This model is illustrated in Figure 2-1.

$$\tilde{\alpha}_i(t) = \alpha_i(t) + z_i(t)\hat{\alpha}_i(t), \quad -1 \leq z_i(t) \leq 1$$

$$\alpha_i(t) - \hat{\alpha}_i(t) \qquad \alpha_i(t) \qquad \alpha_i(t) + \hat{\alpha}_i(t)$$

Allowed range for realization $\tilde{\alpha}_i(t)$
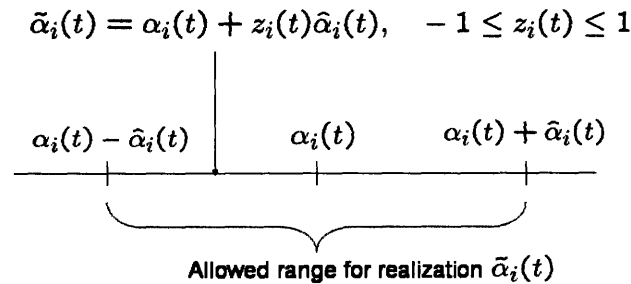
Figure 2-1: Uncertainty model: illustration for $\tilde{\alpha}_i(t)$

We assume that

$$0 < \hat{\alpha}_i(.) < \frac{1}{2}\alpha_i(.) \quad \forall i$$

and write the constraints as follows:

$$|\tilde{\alpha}_i(t) - \alpha_i(t)| \leq \hat{\alpha}_i(t) \quad \forall t, i$$

2.3.2 Budget of uncertainty

To avoid making assumptions that are difficult to satisfy in practice, we do not introduce a particular probability distribution on the range described above for the uncertain parameters. However, we do not want to take an overly conservative approach that would allow parameters to be at the value corresponding to the worst-case scenario (typically, one extreme of the allowed interval) at all times, since such a scenario is highly unlikely. We favor a more reasonable and realistic approach that would bound from above the cumulative deviation of the realization away from the nominal

value. Motivated by this observation, we introduce "budget of uncertainty" functions

$$\Gamma_i(.),$$

taking values on $[0, T]$ that are increasing functions of time and bound the cumulative dispersion of the realized values $\tilde{\alpha}_i(.)$ around the nominal values $\alpha_i(.)$ over time. The general notion has been used by several authors (see for example, Bertsimas and Sim [30], Bertsimas and Thiele [32]).

The motivation for introducing this notion is twofold. On the one hand, it does not require to assume a particular probability distribution on the uncertain data. This is an important feature since in practice it is very difficult to determine the probability distributions.

On the other hand, it provides an alternative to worst case reasoning, that can be seen as unnecessarily overly conservative. Indeed, assuming that the data takes, at all times, the least favorable values enables to be fully protected against data perturbation. However, this high protection level comes at the cost of performance: the solution obtained may be very suboptimal and yield poor objective values. Budgets of uncertainty are motivated by the observation that it is not necessary to protect against the highly unlikely event that data realizations take the worst values at all times. In other words, it allows to take advantage of risk pooling. The level of protection should be the choice of the decision-maker, based on her risk aversion.

The budget of uncertainty is an efficient way to measure the trade-off between conservativeness and performance. It represents a bound on the allowed spread of the realized data around the nominal value over time. The budget of uncertainty is input in the model. The modeler can decide whether she wants to obtain a more conservative solution (by choosing a large budget of uncertainty) while sacrificing optimality, if she is very risk averse. She may prefer a solution that performs well and is less immune to data uncertainty (by choosing a smaller budget of uncertainty), if

she is less risk averse. In other words, a risk averse decision maker will consider large uncertainty sets for the data and wants to be protected for any realization within this set. A risk taking decision maker will choose smaller uncertainty sets. The risk aversion may depend for instance on true preferences, confidence in data estimates, stakes of the decisions, etc.

We assume that

$$\dot{\Gamma}_i(t) \geq 0 \quad \forall i, \ t,$$

since the aggregate dispersion over time can only increase, and that

$$\dot{\Gamma}_i(t) \leq 1 \quad \forall i, \ t,$$

in order to ensure that the budgets of uncertainty do not grow faster than new variables are added.

The feasible realizations of the parameters must then satisfy the following budget of uncertainty constraints:

$$\int_0^t \frac{|\tilde{\alpha}_i(s) - \alpha_i(s)|}{\hat{\alpha}_i(s)} ds \leq \Gamma_i(t) \quad \forall i, \ t.$$

Denoting

$$z_i(t) \equiv \frac{\tilde{\alpha}_i(t) - \alpha_i(t)}{\hat{\alpha}_i(t)}$$

the scaled variations, it follows that the constraints can be rewritten

$$-1 \leq z_i(t) \leq 1 \quad \forall i, \ t$$
$$\int_0^t |z_i(s)| ds \leq \Gamma_i(t) \quad \forall i, \ t.$$

Uncertainty set $\mathcal{F}$ is defined as the set that contains all realizations $\tilde{\alpha}(.)$ satisfying the inequalities above.

We observe that any realization within the range of variation satisfies

$$\int_0^t |z_i(s)|ds \le t \quad \forall i, \ t.$$

This implies in particular, that if for some time t, the budget of uncertainty exceeds value t, the inequality $\int_0^t |z_i(s)|ds \le \Gamma_i(t)$ follows directly from the bounds imposed on the scaled variation. As a result, at the optimal solution to the robust optimization problem, $|z_i(.)|$ will be equal to 1 on $[0, t]$ as this corresponds to the worst case scenario on that interval and it is allowed by the budget of uncertainty constraint. In particular, the exact value of the budget of uncertainty (given that it is greater than or equal to t) will not matter in that case. We conclude from this remark that the *effective* budgets of uncertainty are

$$\min\{t, \Gamma_i(t)\}.$$

In order to measure the global uncertainty of the problem, we will introduce a quantity used as a metric representative of the budgets of uncertainty. As a result, instead of simply using the integral of the budget of uncertainty over the time horizon, we introduce the *cumulative effective budgets of uncertainty* defined for all products i by

$$\int_0^T \min\{t, \Gamma_i(t)\}dt.$$

2.4 Model in a duopoly setting

2.4.1 Formulation

The competitive model of duopoly places the problem in a game-theoretic framework. The dynamic aspect leads to consider a non-zero sum, two-person, *differential* game of pre-specified duration $[0, T]$. We assume that the game is non cooperative, with no leader / follower: the competitors make their choices simultaneously.

We focus on open-loop pure Nash equilibria: the competitors decide at time $t = 0$ their strategy for the entire time horizon. In an open-loop equilibrium, a firm makes an irreversible commitment to a future course of action. This situation may arise in practice if a contract with the buyer or with a labor union forces the firm to commit to prices or workforce at the beginning of the time horizon. In such an equilibrium, the policy depends on time and the *initial* state vector only, the players do not use any other information, on the state variable in particular. In contrast, a feedback/ Markovian Nash equilibrium induces strategies that are based on time and on the *current* state vector. Then the competitors can observe at all times their current inventory level to choose an optimal policy over the rest of the time horizon. In a closed-loop strategy, the firms may also review their course of action as time evolves based on the observation of their inventory level since time zero. A closed-loop Nash equilibrium yields optimal policies that depend on time and *all state vectors from time zero up to the current time.*

See [65] for more details on the difference between closed-loop, feedback, and open-loop equilibria. Note that a feedback solution may be approximated by deriving an open-loop solution and using rolling-horizon techniques.

In the problem we consider, the competitor's strategy plays a role in the demand, and as a result it appears not only in the objective function, but also in the set of feasible strategies via the upper bound on prices and the inventory level constraint of not allowing backorders. Therefore, the game under consideration is couple constrained. This features adds to the complexity of the problem because the feasible set is not fixed.

2.4.2 Demand model

To ease the exposition, we start by presenting the demand model in a deterministic setting, and we then incorporate uncertainty on the demand parameters.

Deterministic coefficients

In the duopoly case, the demand for a given product is modeled as a decreasing linear function of the supplier's price for that product, and an increasing linear function of the *competitor*'s price for that same product: the nominal demands are as follows:

$$d_i^A(t) = \alpha_i^A(t) - \beta_i^{A,A}(t)p_i^A(t) + \beta_i^{A,B}(t)p_i^B(t), \quad \forall i,\ t$$

$$d_i^B(t) = \alpha_i^B(t) - \beta_i^{B,B}(t)p_i^B(t) + \beta_i^{B,A}(t)p_i^A(t), \quad \forall i,\ t$$

where the superscripts A and B denote the two competitors, and $\beta_i^{k,-k}(.) > 0$ denotes the demand sensitivity of supplier k, for product i, with respect to the price of product i applied by the competitor $-k$.

The demand viewed as a function of prices must be constrained to be non negative at all times, which yields an upper bound on prices applied by the supplier.

Notice that this model implies in particular that, in general, the total demand for a product is *not* fixed: instead, it depends on the prices applied by the two suppliers. Customers would then buy the product if they believe the price is reasonable, but they would not make the purchase if the price is too high.

Assumption 4. *We assume that the following inequalities hold* $\forall i, t$:

$$0 \leq \beta_i^{k,-k}(t) \quad < \quad \beta_i^{k,k}(t), \quad k = A, B \tag{2.1}$$

$$0 \leq \beta_i^{k,-k}(t) \quad < \quad \beta_i^{-k,-k}(t), \quad k = A, B. \tag{2.2}$$

The first condition (2.1) states that the demand observed by a given supplier is more sensitive to that supplier's price rather than to her competitor's price of the same product. It is a fairly standard assumption in economics for this type of demand model. This assumption can be rephrased in the following way. Supplier A's demand decreases when her own price increases, or when supplier B's price decreases. If we consider a price increase of the same amount for both suppliers, supplier A's demand increase due to supplier B's price increase does not fully compensate supplier A's

demand decrease due to her own price increase. Therefore supplier A's demand actually decreases. The same is true for supplier B, as a result the total demand decreases.

Condition (2.2) has the following interpretation: if the price applied by supplier B increases, then the demand observed by supplier B decreases more than the demand observed by supplier A increases (so the total demand decreases). In other words, the price chosen by a given supplier affects her own demand more than it affects her competitor's demand. This is also a standard assumption in economics (see Vives [125]). It may be explained by a certain reluctance of customers to change supplier, and thus a price increase would make them give up the purchase rather than go to the competitor.

Remark:

Instead of assumping conditions (2.1) and (2.2), it would be sufficient to assume condition (2.1) and condition

$$(\beta_i^{A,B}(t) + \beta_i^{B,A}(t))^2 < 16\beta_i^{A,A}(t)\beta_i^{B,B}(t) \tag{2.3}$$

(which clearly holds under Assumption 4).

Condition (2.3) avoids too large an asymmetry on the suppliers' demand sensitivity to prices. We notice than in the case of a symmetric demand sensitivities, where suppliers are subject to similar market conditions, in the sense that they each have the same demand sensitivity with respect to their own price and to their competitor's price:

$$\beta_i^{A,B}(t) = \beta_i^{B,A}(t) = \beta_i'(t) \text{ and } \beta_i^{A,A}(t) = \beta_i^{B,B}(t) = \beta_i(t) \ \forall i, \ t,$$

condition (2.3) follows from condition (2.1). We can show that this implication is true even in the asymmetric case provided that

$$7 - \sqrt{48} < \frac{\beta_i^{A,A}(t)}{\beta_i^{B,B}(t)} < 7 + \sqrt{48},$$

47

in other words, that $\beta_i^{A,A}(t)$ and $\beta_i^{B,B}(t)$ are not disproportionately different.

Uncertain coefficients

We then consider a general additive and multiplicative demand model, where not only coefficients $\alpha_i^k(.)$ uncertain (like in the monopoly setting) but also the price elasticities $\beta_i^{k,k}(.)$, $\beta_i^{k,-k}(.)$. The realized demand of supplier k for product i is given by

$$\tilde{d}_i^k(t) = \tilde{\alpha}_i^k(t) - \tilde{\beta}_i^{k,k}(t)p_i^k(t) + \tilde{\beta}_i^{k,-k}(t)p_i^{-k}(t)$$

where $-k$ designates the competitor, and the realized values $\tilde{\alpha}_i^k(.)$, $\tilde{\beta}_i^{k,k}(.)$, $\tilde{\beta}_i^{k,-k}(.)$ are uncertain. As explained for the monopoly case, we do not assume a particular probability distribution, but rather assume parameters may vary within a range of values that is symmetric around some known nominal value, denoted respectively $\alpha_i^k(.)$, $\beta_i^{k,k}(.)$, $\beta_i^{k,-k}(.)$, with respective half-length $\hat{\alpha}_i^k(.)$, $\hat{\beta}_i^{k,k}(.)$, $\hat{\beta}_i^{k,-k}(.)$. This model is illustrated for $\tilde{\alpha}_i^k(t)$ in Figure 2-2.
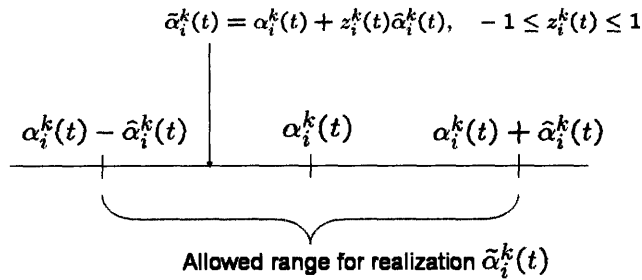
Figure 2-2: Uncertainty model: illustration for $\tilde{\alpha}_i^k(t)$

Without loss of generality, we assume that

$$0 < \hat{\alpha}_i^k(.) < \alpha_i^k(.) \quad \forall i, \ k$$

$$0 < \hat{\beta}_i^{k,k}(.) < \beta_i^{k,k}(.) \quad \forall i, \ k$$

$$0 < \hat{\beta}_i^{k,-k}(.) < \beta_i^{k,-k}(.) \quad \forall i, \ k$$

to ensure that the realized values remain non negative.

The feasible realization must therefore satisfy:

$$|\tilde{\alpha}_i^k(t) - \alpha_i^k(t)| \ \leq \ \hat{\alpha}_i^k(t) \ \forall t, \ i, \ k$$

$$|\tilde{\beta}_i^{k,k}(t) - \beta_i^{k,k}(t)| \ \leq \ \hat{\beta}_i^{k,k}(t) \ \forall t, \ i, \ k$$

$$|\tilde{\beta}_i^{k,-k}(t) - \beta_i^{k,-k}(t)| \ \leq \ \hat{\beta}_i^{k,-k}(t) \ \forall t, \ i, \ k.$$

2.4.3 Budget of uncertainty

Similarly to the monopoly setting, we introduce for each uncertain parameter budget of uncertainty functions

$$\Gamma_i^k(.), \ \Theta_i^{k,k}(.), \ \Theta_i^{k,-k}(.),$$

taking values on $[0, T]$ that are increasing functions of time and bound the cumulative dispersion over time of the realized values $\tilde{\alpha}_i^k(.)$, $\tilde{\beta}_i^{k,k}(.)$, $\tilde{\beta}_i^{k,-k}(.)$ respectively, around their nominal values. As explained before, these function are chosen such that

$$0 \ \leq \ \dot{\Gamma}_i^k(t), \ \dot{\Theta}_i^{k,k}(t), \ \dot{\Theta}_i^{k,-k}(t) \ \leq \ 1 \ \forall t, \ i, \ k$$

and the set $\mathcal{F}^k$ of feasible realizations $(\tilde{\alpha}^k(.), \ \tilde{\beta}^{k,k}(.), \ \tilde{\beta}^{k,-k}(.))$ must satisfy

$$z_i^k(t), \ y_i^{k,k}(t), \ y_i^{k,-k}(t) \ \in \ [-1,1] \ \ \forall t, \ i, \ 'k$$

$$\int_0^t |z_i^k(s)|ds \ \leq \ \Gamma_i^k(t) \ \ \forall t, \ i, \ k$$

$$\int_0^t |y_i^{k,k}(s)|ds \ \leq \ \Theta_i^{k,k}(t) \ \ \forall t, \ i, \ k$$

$$\int_0^t |y_i^{k,-k}(s)|ds \ \leq \ \Theta_i^{k,-k}(t) \ \ \forall t, \ i, \ k,$$

where

$$z_i^k(t) \equiv \frac{\tilde{\alpha}_i^k(t) - \alpha_i^k(t)}{\hat{\alpha}_i^k(t)}$$

$$y_i^{k,k}(t) \equiv \frac{\tilde{\beta}_i^{k,k}(t) - \beta_i^{k,k}(t)}{\hat{\beta}_i^{k,k}(t)}$$

$$y_i^{k,-k}(t) \equiv \frac{\tilde{\beta}_i^{k,-k}(t) - \beta_i^{k,-k}(t)}{\hat{\beta}_i^{k,-k}(t)}.$$

As explained in the monopoly setting, we define the effective budgets of uncertainty

$$\min\{t, \Gamma_i^k(t)\}, \quad \min\{t, \Theta_i^{k,k}(t)\}, \quad \min\{t, \Theta_i^{k,-k}(t)\}$$

and the cumulative effective budgets of uncertainty

$$\int_0^T \min\{t, \Gamma_i^k(t)\}dt, \quad \int_0^T \min\{t, \Theta_i^{k,k}(t)\}dt, \quad \int_0^T \min\{t, \Theta_i^{k,-k}(t)\}dt.$$

2.5 Objective function under uncertainty

When dealing with uncertainty on input parameters, a natural question is to decide what the objective value should be. A common approach when assuming that demand follows a particular probability distribution, is to maximize the *expected value* of the objective. One may also introduce constraints bounding the variance of the objective. Alternatively, one may minimize the variance subject to bounds on the profits. Depending on the context and on risk preferences, it is also possible to op-

timize the best case or worst case objective, or to define a utility function based on expected value and standard deviation of the objective.

In robust optimization, since no probability distribution is assumed, one traditional option is to aim at optimizing the *realized* objective, which is reformulated as maximizing the *worst case* objective within the uncertainty set of the parameters.

In this thesis, both in the monopoly and the duopoly setting with data uncertainty, we consider a robust approach that maximizes the *nominal* objective function. In other words, like in traditional stochastic optimization approaches, we maximize the "expected" objective function value - not in the probabilistic sense since we do not know the distribution, but in terms of considering the values at the center of the allowed range of variation for the realized values. However, we still consider demand uncertainty in the feasibility constraints. Another motivation for this approach comes from the following qualitative observation. The worst case objective in our model corresponds to high inventory costs and low revenues, i.e. *low* demand realization. However, the worst case realization for the no backorders constraint corresponds to low inventory levels, i.e. *high* demand realization. Therefore the worst case cannot occur simultaneously for both the objective and the constraints, and it would be overly conservative to protect against both occurrences simultaneously. As a result, we choose to focus on guaranteeing the *feasibility* of the problem, and to solve for the worst case of the constraints, but to maximize the nominal objective. The goal of the robust formulation is then to find the solution that maximizes the *nominal* objective value and that is feasible *for any feasible realization of the parameters*.

2.6 Notations

This section summarizes the main notations used throughout the thesis. Additional notations may be defined when they are introduced if they are used in limited parts. For a given set S, S^c denotes the complementary set.

For a given differentiable function of time $\varphi(.)$, the first order time derivative is denoted $\dot{\varphi}(.)$. If $\varphi(.)$ takes an argument that is not time, the first order derivative is denoted $\varphi'(.)$. For function of multiple variables $\varphi(x_1, .)$, the partial derivative is denoted by $\frac{\partial \varphi}{\partial x_1}(x_1, .)$.

For given functions $v_i(.)$, $i = 1, \ldots, N$ defined on $[0, T]$, $v(.)$ denotes the vector with components $(v_1(.), \ldots, v_N(.))$.

For given real numbers v_i, $i = 1, \ldots, N$, v denotes the vector with components $(v_1, \ldots, v_N)$.

For two given vectors v and w, $v'w$ denotes the inner product $\sum_i v_i w_i$ and $v \times w$ denotes the vector with components $v_i w_i$.

In the monopoly setting, if x is a decision or state variable, the value at the optimal solution is denoted x^*. In the competitive setting, x^* represents the value of the variable at a Nash equilibrium.

In the notations below, the subscript i refers to some product i, and time varying parameters or variables are defined at some given time t.

2.6.1 Monopolistic and deterministic setting

Inputs

$[0, T]$: time horizon;

N: number of products;

$K(t)$: shared production capacity rate;

I_i^0: initial non negative inventory level;

$h_i(t)$: holding cost of one unit;

$f_i(.)$: production cost function with respect to the production rate;

$\alpha_i(t), \beta_i(t)$: coefficients of the linear relationship between price and demand;

$g_i(t, z) = z - f_i'\left(\frac{\alpha_i(t) - \beta_i(t)z}{2}\right)$ defined for $t \in [0, T]$ and,
(for a given value of t) for $z \in \left(-\infty, \frac{\alpha_i(t)}{\beta_i(t)}\right]$;

$l_{i,t} : z \mapsto g_i(t, z)$;

$\phi_i(t, \eta)$: solution of the equation with unknown z: $g_i(t, z) = \eta$;

$\psi_i(t) = \phi_i(t, 0)$.

Outputs

$p_i(t)$:   price of one unit of product (control variable);

$u_i(t)$:   production flow rate (control variable);

$I_i(t)$:   inventory level (number of units) (state variable);

$d_i(t)$ $= \alpha_i(t) - \beta_i(t)p_i(t)$ demand rate;

$q_i(t)$:   adjoint variable used to write the Hamiltonian function;

q_i^0 $= q_i(0)$;

$\rho_i(t)$:   Lagrange multiplier dualizing the no backorder constraint in the Lagrangian function;

$\eta(t)$:   Lagrange multiplier dualizing the production capacity constraint in the Lagrangian function;

$\mathcal{I}(t)$:   set of indices of products with a positive production rate at time t;

j_0:   smallest index in $\mathcal{I}(t)$ if non-empty;

$S(t)$:   set of indices of products with inventory level at zero at time t;

$\mathcal{J}(t)$:   set of indices of products with a positive production rate and zero inventory level at time t $(= \mathcal{I}(t) \cap S(t))$;

i_1:   smallest index of $\mathcal{J}(t)$ if non-empty;

i_0:   index preceding i_1 in $S(t)$ (if $i_1 \neq \min S(t)$);

$\mathcal{J}'(t)$:   set of indices of products with a positive production rate and positive inventory level at time t (complementary set of $\mathcal{J}(t)$ in set $\mathcal{J}(t)$);

i_1':   smallest index of $\mathcal{J}'$ if non-empty;

i_0':   index preceding i_1' in $S(t)^c$ (when $i_1' \neq \min S(t)^c$);

t_i^0:   when the inventory level is positive, last time it was at zero;

t_i^1:   when the inventory level is positive, next time the inventory level reaches zero;

t_i^2:   when the inventory level is positive, next time after t_i^1 the inventory level becomes positive.

2.6.2 Additional notations for the monopolistic setting with uncertainty

Deterministic inputs

$\alpha_i(t)$: nominal value of the fixed term in the demand (center of the range of variation);

$\hat{\alpha}_i(t)$: half-length of the range of variation for the realization, centered around the nominal value $\alpha_i(t)$;

$\Gamma_i(t)$: budget of uncertainty;

$\mathcal{F}$: uncertainty set that contains all feasible realizations $\tilde{\alpha}(.)$

$J_i(t)$: minimum inventory security level;

$D_i(t) \quad = \dot{J}_i(t)$.

Uncertain parameters

$\tilde{\alpha}_i(t)$: realization of the parameter;

$z_i(t) \quad = \frac{\tilde{\alpha}_i(t) - \alpha_i(t)}{\hat{\alpha}_i(t)}$ scaled variation around the nominal value.

2.6.3 Notations for the duopolistic setting with uncertainty

The notations defined above will be used in the duopoly setting by adding the superscript k to denote supplier k, for price, production rate, inventory level, production capacity rate, nominal fixed term of the demand and its realized value, half range of variation, and budget of uncertainty. The superscript $-k$ refers to supplier k's competitor.

Some new notations for supplier k are defined below.

Deterministic inputs

$h_i^k(t)$ coefficient of quadratic holding cost;

$\gamma_i^k(t)$ coefficient of quadratic production cost;

$\beta_i^{k,k}(t)$, $\beta_i^{k,-k}(t)$ nominal values of price sensitivities of the demand with respect to respectively $p_i^k(t)$ and $p_i^{-k}(t)$;

$\hat{\beta}_i^{k,k}(t)$, $\hat{\beta}_i^{k,-k}(t)$: half length of the allowed range of variation;

$\Theta_i^{k,k}(t)$: budgets of uncertainty for $\tilde{\beta}_i^{k,k}(t)$;

$\Theta_i^{k,-k}(y)$: budgets of uncertainty for $\tilde{\beta}_i^{k,-k}(t)$;

$\Omega_i^k(t, p_i^k(.), p_i^{-k}(.))$: minimum inventory security level as a function of the prices;

$\mathcal{F}^k$: uncertainty set that contains all feasible realizations
$$(\tilde{\alpha}^k(.), \tilde{\beta}^{k,k}(.), \tilde{\beta}^{k,-k}(.).$$

Uncertain parameters

$\tilde{\beta}_i^{k,k}(t)$, $\tilde{\beta}_i^{k,-k}(t)$: realization of the parameter;

$y_i^{k,k}(t) = \frac{\tilde{\beta}_i^{k,k}(t) - \beta_i^{k,k}(t)}{\hat{\beta}_i^{k,k}(t)}$ scaled variation around the nominal value.

$y_i^{k,-k}(t) = \frac{\tilde{\beta}_i^{k,-k}(t) - \beta_i^{k,-k}(t)}{\hat{\beta}_i^{k,-k}(t)}$ scaled variation around the nominal value.

Ouputs

$d_i^k(t) = \alpha_i^k(t) - \beta_i^{k,k}(t) p_i^k(t) + \beta_i^{k,-k}(t) p_i^{-k}(t)$ demand rate;

$x^k = (p^k, u^k, I^k)$ strategy and state vector of supplier k;

(x^A, x^B): collective strategy and state vector

Additional notations

E_1: vector space containing vectors with $3N$ components that are real bounded functions defined over $[0, T]$;

$E = E_1 \times E_1$;

X^k: set of strategy and state vectors for player k satisfying the constraints that are independent of the competitor's strategy;

$X = X^A \times X^B$;

$Q^k(\bar{x}^{-k})$: set of feasible strategy and state vectors for player k, given some strategy and state vector $\bar{x}^{-k}$ of her competitor;

$Y = \{x \in X : x^k \in Q^k(x^{-k}), \ k = A, B\}$ set of collectively feasible strategy and state vectors for both players;

Q: mapping such that $Q(x) = Q^A(x^B) \times Q^B(x^A)$ (set of feasible strategies for each player, assuming that the competitor's strategy is unchanged;

$\Pi^k(x^k, x^{-k})$: objective value of supplier k corresponding to strategy x^k, when supplier $-k$ has a strategy x^{-k}

Chapter 3

Deterministic and monopolistic setting

In this section, all data are deterministic, and the market is a monopoly. Our goal is two-fold: to study the structure of the optimal solution on the finite time horizon, and to propose under some additional assumptions a method for computing it. We derive a continuous time optimal control solution that applies to the entire time horizon determining simultaneously the prices and the production rates of all products. Our approach does not introduce a time discretization. It provides an analytical way to compute optimal policies throughout the time horizon. Furthermore, our approach does not require that we observe the state of the system. It illustrates the effect of capacity in the problem as well as the effect of the dynamic nature of the problem.

3.1 Definitions, formulation, and description of the solution approach

We define:

Constrained interval: Interval of time where the inventory level equals zero (also called boundary interval).

Constrained product: Product that is on a constrained interval.

Unconstrained interval: Interval of time where the inventory level is positive.

Unconstrained product: Product that is on an unconstrained interval.

Active product: Product with a positive production rate.

Inactive product: Product with a production rate equal to zero.

We notice that for any pricing and production policy we consider (and in particular an optimal policy), the inventory level will be structured via a sequence of intervals, where the inventory level is successively positive and equal to zero, as illustrated in Figure 3-1. A constrained interval starts at an *entry time* and finishes at an *exit time*, i.e. the time the inventory level becomes again positive.

Assumption 5. *We will assume throughout this section that for each product, there is a finite number of entry and exit times.*
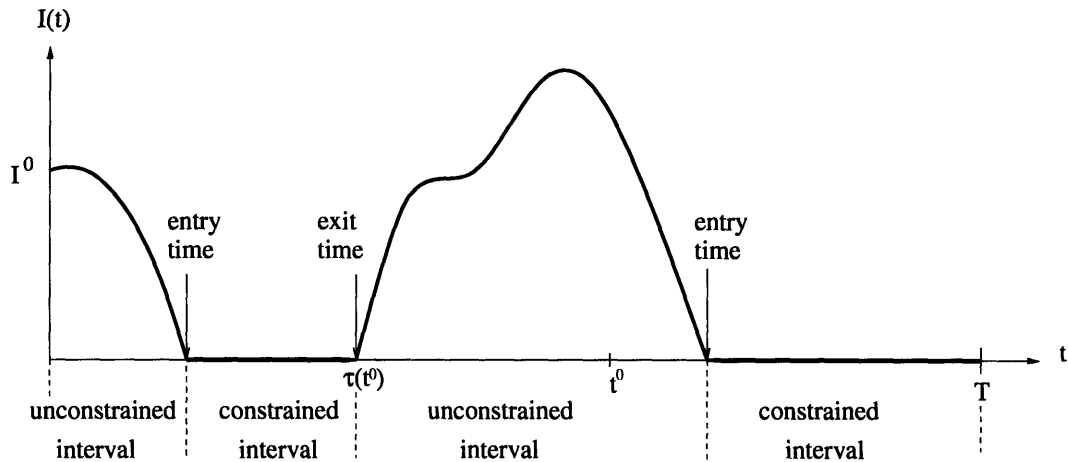
Figure 3-1: Example of evolution of the inventory level

3.1.1 Description of the model

This problem was solved by Pekelman [105] for the single product case with a holding cost coefficient constant in time. Nevertheless, the presence of multiple products

sharing production capacity makes the problem rather complex.

The problem seeks to maximize the revenues minus the inventory and production costs. As a result, it can be written as follows:

$$\max \quad \int_0^T \left[\sum_{i=1}^N \Big(p_i(t)d_i(t) - f_i(u_i(t)) - h_i(t)I_i(t) \Big) \right] dt \qquad (3.1)$$

$$\text{s.t.} \quad \dot{I}_i(t) = u_i(t) - d_i(t), \quad \forall t \in [0,T] \quad i = 1,\ldots,N,$$

$$d_i(t) = \alpha_i(t) - \beta_i(t)p_i(t), \quad \forall t \in [0,T] \quad i = 1,\ldots,N,$$

$$\sum_{i=1}^N u_i(t) \le K(t), \quad \forall t \in [0,T],$$

$$u_i(t), \quad p_i(t), \quad d_i(t), \quad I_i(t) \ge 0, \quad \forall t \in [0,T] \quad i = 1,\ldots,N,$$

$$I_i(0) = I_i^0, \quad i = 1,\ldots,N.$$

Equivalently:

$$\max \quad \int_0^T \left[\sum_{i=1}^N \Big(p_i(t)(\alpha_i(t) - \beta_i(t)p_i(t)) - f_i(u_i(t)) - h_i(t)I_i(t) \Big) \right] dt \quad (3.2)$$

$$\text{s.t.} \quad \dot{I}_i(t) = u_i(t) - \alpha_i(t) + \beta_i(t)p_i(t), \quad \forall t \in [0,T] \quad i = 1,\ldots,N, \qquad (3.3)$$

$$\sum_{i=1}^N u_i(t) \le K(t), \quad \forall t \in [0,T], \qquad (3.4)$$

$$I_i(t) \ge 0, \quad \forall t \in [0,T] \quad i = 1,\ldots,N, \qquad (3.5)$$

$$u_i(t) \ge 0, \quad \forall t \in [0,T] \quad i = 1,\ldots,N \qquad (3.6)$$

$$0 \le p_i(t) \le \frac{\alpha_i(t)}{\beta_i(t)}, \quad \forall t \in [0,T] \quad i = 1,\ldots,N, \qquad (3.7)$$

$$I_i(0) = I_i^0, \quad i = 1,\ldots,N.$$

We observe that in this continuous time optimal control model, constraint (3.3) is the dynamic equation that describes the evolution of the level of inventory, modeled as a continuous and differentiable function of time.

Constraint (3.4) corresponds to the common production capacity that is shared among all the products. This is the only constraint that is coupling the products and prevents us from simply solving N times a single-product problem.

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Constraint (3.5) represents the no backorders constraint. Notice that these are constraints on the state variables. This makes their treatment different from constraints on control variables but also harder. We will apply the Maximum Principle in the case of inequality constraints on the state variables (see [7], [85], [115]).

We introduce constraints (3.6) and (3.7) to ensure that prices and production rates are non-negative. Furthermore, the upper bounds on the prices reflect the fact that the demand should remain non-negative. These are constraints on the control variables, which are taken into account by simply restricting the feasible domain of admissible controls.

In what follows, we establish existence of an optimal policy. Furthermore, we will examine conditions of optimality. Specifically: we will simultaneously compute Lagrange multipliers and adjoint variables that satisfy these optimality conditions, and will allow us to compute an optimal policy.

3.1.2   Existence of an optimal solution

Theorem 1. *Under Assumptions 1, 2, 3 and 5, there exists an optimal solution* $u^*(.), p^*(.)$ *to Problem (3.1).*

This existence result follows similarly to [73]. To prove it, we consider the following control problem

$$\max \quad \int_0^T F(I(t), w(t), t)dt \tag{3.8}$$

$$\text{subject to} \quad \dot{I}(t) = \vartheta(I(t), w(t), t) \tag{3.9}$$

$$I(0) = I^0 \tag{3.10}$$

$$\xi(I(t), w(t), t) \geq 0 \tag{3.11}$$

$$\varpi(I(t), t) \geq 0 \tag{3.12}$$

where:

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$$T \quad \text{is the time horizon,}$$

$I(t) \in E^n$ is the vector of state variables at time t,

I^0 is the vector of initial conditions,

$w(t) \in E^m$ is the vector of control variables (prices and production rates) at time t,

$F : E^n \times E^m \times E \to E$ is a function assumed to be continuously differentiable,

$\vartheta : E^n \times E^m \times E \to E^n$ is a function assumed to be continuously differentiable,

$\xi : E^n \times E^m \times E \to E^a$ is a function assumed to be continuously differentiable in all its arguments and depends explicitly on $w(t)$,

$\varpi : E^n \times E \to E^b$ is a function assumed to be continuously differentiable.

We notice that constraint (3.11) involves control variables and possibly state variables as well (we refer to this as a mixed inequality constraint) while constraint (3.12) involves the state variable only (we refer to this as a pure state variable inequality constraint).

We define a control $w(.)$ to be admissible if it is piecewise continuous and if, together with the state trajectory $I(.)$ it generates through (3.9) and (3.10), it satisfies (3.11) and (3.12).

Inequality (3.12) represents by definition a set of constraints $\varpi_i(I(t), t) \geq 0$, $i = 1, \ldots, b$. The constraint $\varpi_i(I(t), t) \geq 0$ is called a constraint of rth order if the rth time derivative of $\varpi_i(I(t), t)$ is the first in which a term in control $w(.)$ appears. To make this dependency in the control variable clear, we will add $w(t)$ as an argument of the rth time derivative of $\varpi_i(I(t), t)$, even though $w(t)$ is not an argument of $\varpi_i(I(t), t)$. For the sake of simplicity and because it is satisfied in the application we are interested in, we will assume that each constraint $\varpi_i(I(t), t) \geq 0$ is of the first order.

Define the (state-dependent) control region

$$\Omega(I, t) = \{w \in E^m | \xi(I, w, t) \geq 0\} \subset E^m$$

and the set

$$Q(I,t) = \{(F(I,w,t) + c, \ \vartheta(I,w,t)) | c \leq 0, w \in \Omega(I,t)\} \subset E^{n+1}.$$

Theorem 2 (Filippov-Cesari Theorem). *Consider problem (3.8). Assume that F, ϑ, ξ and ϖ are continuous in all their arguments at all points $(I,w) \in E^n \times E^m$. Suppose that there exists an admissible pair and that the following Roxin's condition holds:*

set $Q(I,T)$ is convex, for all $I \in E^n$.

Suppose further that

there exists $\delta > 0$ such that $\|I(t)\| < \delta$, for all admissible $\{I(t), w(t)\}$ and t,

and that

there exists $\delta_1 > 0$ such that $\|w\| < \delta_1$, for all $w \in \Omega(I,t)$ with $\|I\| < \delta$.

Then there exists an optimal couple $\{I^*, w^*\}$ with $w^*(.)$ measurable.

Theorem 1. Under Assumptions refassump2, 2, 3 and 5, an optimal control policy exists for Problem (3.1).

Proof. We will show that conditions of Theorem 2 hold for Problem (3.1).

For Problem (3.1), the control region is independent of the state and may be expressed as:

$$
\begin{aligned}
\Omega(t) &= \{(u,p) \in \mathbb{R}^{2N} | \xi(u,p,t) \geq 0\} \\
&= \{(u,p) \in \mathbb{R}^{2N} | u \geq 0, \ p \geq 0, \ p_i \leq \frac{\alpha_i(t)}{\beta_i(t)}, \ i = 1,\ldots,N, \ \sum_{i=1}^{N} u_i \leq K(t).\}
\end{aligned}
$$

The set $Q(I, t)$ is given by

$$Q(I, t) = \left\{ \left(\sum_{i=1}^{N} \left(p_i(\alpha_i(t) - \beta_i(t)p_i) - f_i(u_i) - h_i(t)I_i \right) + c, \ u - \alpha(t) + \beta(t) \times p \right) \Big| \right.$$

$$\left. c \le 0, (u, p) \in \Omega(t) \right\}.$$

- Continuity:

  It is clear that

  (i) the integrand F of the objective function

  $$(I, u, p, t) \mapsto \sum_{i=1}^{N} \left(p_i(\alpha_i(t) - \beta_i(t)p_i) - f_i(u_i) - h_i(t)I_i \right),$$

  (ii) the function ϑ describing the dynamic evolution $(u, p, t) \mapsto u - \alpha(t) + \beta(t) \times p$,

  (iii) the function ξ giving rise to control inequality constraints

  $$(u, p) \mapsto \left(p, \ \frac{\alpha(t)}{\beta(t)} - p, \ u, \ K(t) - \sum_{i=1}^{N} u_i \right),$$

  and

  (iv) the function ϖ giving rise to the pure state variables inequality constraints $I \mapsto I$ are continuous functions in all their arguments.

- There exists an admissible pair:

  Consider the policy

  $$p_i(t) = \frac{\alpha_i(t)}{\beta_i(t)}, \ u_i(t) = 0, \ i = 1, \ldots, N, \ \forall t \in [0, T].$$

  Under this policy, $\dot{I}_i(t) = u_i(t) - \alpha_i(t) + \beta_i(t)p_i(t) = 0$, so the generated state trajectory satisfies $I_i(t) = I_i^0 \ge 0 \ \forall t \in [0, T]$. As a result, this policy satisfies all constraints so it is an admissible pair.

- Roxin's condition holds:

Let x_1, $x_2 \in \mathbb{R}$, y_1, $y_2 \in \mathbb{R}^N$ such that (x_1, y_1), $(x_2, y_2) \in Q(I, T)$ with

$$x_1 = \sum_{i=1}^{N} \left(p_i^1(\alpha_i(T) - \beta_i(T)p_i^1) - f_i(u_i^1) - h_i(T)I_i \right) + c^1$$

$$x_2 = \sum_{i=1}^{N} \left(p_i^2(\alpha_i(T) - \beta_i(T)p_i^2) - f_i(u_i^2) - h_i(T)I_i \right) + c^2$$

$$y_1 = u^1 - \alpha(T) + \beta(T) \times p^1$$

$$y_2 = u^2 - \alpha(T) + \beta(T) \times p^2$$

$$c^1 \leq 0$$

$$c^2 \leq 0$$

$$(u^1, p^1) \in \Omega(T)$$

$$(u^2, p^2) \in \Omega(T).$$

Let $\lambda \in [0, 1]$. We want to show that $(\bar{x}, \bar{y}) \equiv \lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) \in Q(I, T)$.

Let $(\bar{u}, \bar{p}) = \lambda(u^1, p^1) + (1 - \lambda)(u^2, p^2)$.

It is easy to verify that $(\bar{u}, \bar{p}) \in \Omega(T)$.

It is also clear that $\bar{y} = \bar{u} - \alpha(T) + \beta(T) \times \bar{p}$.

Since the function $(u, p) \mapsto \sum_{i=1}^{N} \left(p_i(\alpha_i(t) - \beta_i(t)p_i) - f_i(u_i) \right)$ is concave in (u, p), it follows that

$$\sum_{i=1}^{N} \left(\bar{p}_i(\alpha_i(T) - \beta_i(T)\bar{p}_i) - f_i(\bar{u}_i) - h_i(T)I_i \right)$$

$$\geq \lambda \sum_{i=1}^{N} \left(p_i^1(\alpha_i(T) - \beta_i(T)p_i^1) - f_i(u_i^1) - h_i(T)I_i \right)$$

$$+ (1 - \lambda) \sum_{i=1}^{N} \left(p_i^2(\alpha_i(T) - \beta_i(T)p_i^2) - f_i(u_i^2) - h_i(T)I_i \right).$$

By observing that the right hand side of this inequality may be rewritten as $\lambda(x_1 - c^1) + (1 - \lambda)(x_2 - c^2)$, we obtain that there exists $c^3 \leq 0$ such that

$$\sum_{i=1}^{N} \left(\bar{p}_i(\alpha_i(T) - \beta_i(T)\bar{p}_i) - f_i(\bar{u}_i) - h_i(T)I_i \right) + c^3 = \bar{x} - \lambda c^1 - (1 - \lambda)c^2$$

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Letting $\bar{c} \equiv \lambda c^1 + (1 - \lambda)c^2 + c^3 \leq 0$ implies that

$$\bar{x} = \sum_{i=1}^{N} \left(\bar{p}_i(\alpha_i(T) - \beta_i(T)\bar{p}_i) - f_i(\bar{u}_i) - h_i(T)I_i \right) + \bar{c}.$$

Therefore, $(\bar{x}, \bar{y}) \in Q(I, T)$ and $Q(I, T)$ is a convex set.

- The admissible controls are bounded:

  The constraints defining the admissible controls provide bounds to the prices $0 \leq p_i(t) \leq \frac{\alpha_i(t)}{\beta_i(t)}$ and to the production rates $0 \leq u_i(t) \leq K(t)$, where $\alpha_i(.), \beta_i(.)$ and $K(.)$ are positive- and finite-valued functions of time. Since the time horizon is finite, there exists bounds on the control variables at each time.

- The state variable is bounded:

  The inventory level is bounded below by zero. Moreover, the control variables $p_i(t)$ and $u_i(t)$ are bounded (as we discussed above). As a consequence, $\dot{I}_i(t) = u_i(t) - \alpha_i(t) + \beta_i(t)p_i(t)$ is bounded too. Since the time horizon is finite, it follows that there exists also an upper bound on the state variable $I_i(t)$ for all times t.

This proves that all assumptions in Theorem 2 hold for Problem (3.1). □

3.1.3 Solution approach

In order to solve problem (3.1), we will address its various stages of difficulty gradually. We will treat those difficulties by employing ideas from control theory and nonlinear optimization. First, the problem has a capacity constraint which couples the products. Second, the dynamics of the system are described through the dynamic equations which illustrate how the state variables (namely, the levels of inventory) evolve. Third, there are no backorders constraints on the state variables (i.e., the inventory level for each product).

Since we are dealing with a continuous-time control problem, we will first define the Hamiltonian function using adjoint variables corresponding to the dynamic equations. We will also introduce a Lagrangian function by dualizing the difficult constraints, i.e. the capacity constraint and the no backorders constraints. Dualizing the capacity constraint will enable us to decouple the problem, and reduce it to several single-product problems. Subsequently, we will use the Maximum Principle under constraints on the state variables and the indirect adjoining method to the Lagrangian function (see Hartl, Sethi and Vickson [73] or Sethi and Thompson [115]).

- We will assign adjoint variables $q_i(t)$ in order to dualize the dynamic constraint i at time t;

- We will write the Hamiltonian function;

- We will assign multipliers $\rho_i(t)$ to dualize the constraint on the non negativity of $I_i(t)$;

- We will assign a multiplier $\eta(t)$ to dualize the capacity constraint;

- Through these multipliers, we will construct the Lagrangian function (3.13).

Due to its complexity, we will present this solution approach in three stages of increasing complexity. In particular,

(i) We will first focus on the capacity constraint while "ignoring" the dynamic aspect of the problem and the no backorders constraints. To achieve this, at the first stage we will assume that $q_i(t) + \rho_i(t)$ are known. This will allow us to construct the optimal policy as a function of these multipliers. We will construct a procedure to compute the multiplier $\eta(t)$ (corresponding to the capacity constraint) and derive an optimal policy at time t, which clearly will depend on vector $q(t) + \rho(t)$.

(ii) At the second stage, we will augment the approach by also computing the vector $q(t) + \rho(t)$ instead of assuming it is given. We will solve the problem for any time t under the assumption of *observability* of the system, that is, that we know which

products have a non-zero inventory level at that time t (we will define this notion more accurately later). This allows us to "ignore" the state variable constraints but allows us to take into account both the dynamic equations as well as the capacity constraint. We will express vector $q(t) + \rho(t)$ as a function of multiplier $\eta(t)$, then solve for $\eta(t)$ - and thus obtain $q(t) + \rho(t)$. Finally, incorporating the solution approach in (i), will allow us to present a solution method for computing the optimal policy under the assumption of observability of the system.

(iii) Finally, we consider the problem solution over the whole time horizon and show how to compute the optimal pricing and production policy without assuming observability of the system at each time, but by imposing a no backorders constraint in the optimization problem.

In order to solve the problem under consideration, we first present some preliminary results.

3.2 Maximum Principle

3.2.1 Theoretical results

In this section, we state the maximum principle for optimal control problems with mixed inequality constraints and pure state variable inequality constraints. These results are described in more detail in Sethi and Thompson [115], Hartl, Sethi and Vickson [73], Arrow and Kurz [7].

Consider the problem (3.8) defined in Section 3.1.2.

We define

$$\varpi^1(I(t), w(t), t) = \frac{d\varpi}{dt}(I(t), t) = \frac{\partial \varpi}{\partial I}\vartheta(I(t), w(t)t) + \frac{\partial \varpi}{\partial t}(I(t), t).$$

With respect to the ith constraint $\varpi_i(I(t), t) \geq 0$, an interval $(\theta_i^1, \theta_i^2) \subset [0, T]$ is called an *interior* or *unconstrained* interval if $\varpi_i(x(t), t) > 0$, $\forall t \in (\theta_i^1, \theta_i^2)$. If the optimal trajectory "hits the boundary," i.e., satisfies $\varpi_i(x(t), t) = 0$, $\forall t \in (\tau_i^1, \tau_i^2)$, for some

i and some interval $(\tau_i^1, \tau_i^2) \subset [0, T]$, then $[\tau_i^1, \tau_i^2]$ is called a *boundary* or *constrained interval*. An instant τ_i^1 is called an *entry time* if there is an interior interval ending at time τ_i^1 and a boundary interval starting at time τ_i^1. Correspondingly, τ_i^2 is called an *exit time* if a boundary ends and an interior interval starts at time τ_i^2. If the trajectory touches the boundary at time τ_i, i.e., $\varpi_i(I(\tau_i), \tau_i) = 0$ for some i and if the trajectory is in the interior just before and just after τ_i, then τ_i is called a *contact time*. Taken together, entry, exit and contact times are called *junction times*.

We assume that the following constraint qualification holds:

$$\text{rank} \left[\frac{\partial \xi}{\partial w}, \text{diag}(\xi) \right] = a$$

as well as the full-rank condition on any boundary interval $[\tau_j^1, \tau_j^2]$:

$$\text{rank} \begin{bmatrix} \partial \varpi_1^1 / \partial w \\ \vdots \\ \partial \varpi_{\hat{b}}^1 / \partial w \end{bmatrix} = \hat{b},$$

where for $t \in [\tau_j^1, \tau_j^2]$, $\varpi_i(I^*(t), t) = 0 \quad i = 1, \ldots, \hat{b} \leq b$ and $\varpi_i(I^*(t), t) > 0 \quad i = \hat{b} + 1, \ldots, b$.

We define the Hamiltonian function $H : E^n \times E^m \times E^n \times E \to E$ as

$$H(I, w, q, t) \equiv F(I, w, t) + q\vartheta(I, w, t),$$

where $q \in E^n$ (a row vector). We also define the Lagrangian function $L : E^n \times E^m \times E^n \times E^q \times E \to E$ as[1]

$$L(I, w, q, \eta, \rho, t) = H(I, w, q, t) + \eta\xi(I, w, t) + \rho\varpi^1(I, w, t),$$

---

[1]We form the Lagrangian function by adjoining indirectly (i.e. via their first time derivative) the constraints on the state variable. This method is called indirect adjoining method. In the direct adjoining method, the Lagrangian function is formed by adjoining directly the constraints as follows: $L^d(I, w, q, \eta^d, \rho^d, t) = H(I, w, q, t) + \eta^d\xi(I, w, t) + \rho^d\varpi(I, t)$, with $H = F(I, w, t) + q^d\vartheta(I, w, t)$. It is shown in [73] that $\eta^d(t) = \eta(t)$, $q^d(t) = q(t) + \rho(t)\frac{\partial \varpi}{\partial I}(I^*(t), t)$.

where $\eta \in E^a$ and $\rho \in E^b$ are row vectors, whose components are called Lagrange multipliers. These Lagrange multipliers satisfy the complementary slackness conditions

$$\eta(t) \geq 0, \quad \eta(t)\xi(I(t), w(t), t) = 0,$$

$$\rho(t) \geq 0, \quad \dot{\rho}(t) \leq 0, \quad \rho(t)\varpi(I(t), t) = 0.$$

We now state the maximum principle for the problem under consideration.

Theorem 3. *(Maximum Principle) We suppose that $I^*(.)$ has only finitely many junction times, that each pure state constraint $\varpi_i(I(t), t)) \geq 0$ is of the first order, that constraint qualification holds, and that the full rank condition holds. The necessary conditions for w^* (with state trajectory I^*) to be an optimal control policy for the problem we defined above are the following:*

There exist piecewise continuous[2] and piecewise continuously differentiable adjoint variable $q(.)$, piecewise continuous multipliers $\eta(.)$, $\rho(.)$, parameter ν, and jump parameter $\zeta(.)$, such that the following conditions hold almost everywhere:

- $\dot{I}^*(t) = \vartheta(I^*(t), w^*(t), t), \quad I^*(0) = I_0,$ *satisfying constraints*

$$\xi(I^*(t), w^*(t), t) \geq 0, \quad \varpi(I^*, t) \geq 0;$$

- $\dot{q}(t) = -\frac{\partial L}{\partial I}(I^*(t), w^*(t), q(t), \eta(t), \rho(t), t)$ *except at entry/contact times, with transversality conditions[3]*

$$q(T^-) = \nu \frac{\partial \varpi}{\partial I}(I^*(T), T), \quad \nu \geq 0, \quad \nu \varpi(I^*(T), T) = 0;$$

---

[2] In the direct adjoining method, $q^d(.)$ is continuous.

[3] In the direct adjoining method, the transversality conditions are:

$$q^d(T) = \nu^d \frac{\partial \varpi}{\partial I}(I^*(T), T), \quad \nu^d \geq 0, \quad \nu^d \varpi(I^*(T), T) = 0.$$

- *the Hamiltonian maximizing condition*

$$H(I^*(t), w^*(t), q(t), t) \geq H(I^*(t), w(t), q(t), t),$$

at each $t \in [0, T]$, for all w satisfying

$$\xi(I^*(t), w, t) \geq 0, \ \text{and} \ \varpi_i^1(I^*(t), w, t) \geq 0,$$

$$\text{whenever} \ \varpi_i(I^*(t), t) = 0, \ i = 1, \ldots, b;$$

- *at any entry/contact time[4] τ, the adjoint variable q may have a discontinuity of the form*

$$q(\tau^-) = q(\tau^+) + \zeta(\tau)\frac{\partial \varpi}{\partial I}(I^*(\tau), \tau) \ \text{and}$$

$$H(I^*(\tau), w^*(\tau^-), q(\tau^-), \tau) = H(I^*(\tau), w^*(\tau^+), q(\tau^+), \tau) - \zeta(\tau)\frac{\partial \varpi}{\partial t}(I^*(\tau), \tau);$$

- *the Lagrange multipliers $\eta(t)$ are such that*

$$\frac{\partial L}{\partial w}(I^*(t), w^*(t), q(t), \eta(t), \rho(t), t) = 0$$

and the complementary slackness conditions

$$\eta(t) \geq 0, \ \eta\xi(I^*(t), w^*(t), t) = 0,$$

$$\rho \geq 0, \ \dot{\rho} \leq 0 \ \text{on boundary intervals of} \ \varpi, \ \rho\varpi(I^*(t), t) = 0, \ \text{and}$$

$$\zeta(\tau) \geq 0, \ \zeta(\tau)\varpi(I^*(\tau), \tau) = 0.$$

Theorem 4. *Let $(I^*(.), w^*(.), q(.), \eta(.), \rho(.), \nu, \zeta(.))$ satisfy the necessary conditions above. Suppose that constraint qualification and full-rank condition hold. Let $q^d(t) = q(t) + \rho(t)\frac{\partial \varpi}{\partial I}(I^*(t), t)$. If $H(I, w, q^d, t)$ is a concave function in (I, w), at each*

---

[4]We are using the convention specifying that the adjoint variable is continuous at exit times.

$t \in [0,T]$, $\xi(I,W,t)$ *is a quasi-concave function in* (I,w), $\varpi(I,t)$ *is a quasi-concave function in* I, *then policy* $(I^*(.), w^*(.))$ *is optimal.*

3.2.2 Necessary conditions for optimality

We express the Hamiltonian function as follows:

$$H(I,p,u,q,t) = \sum_{i=1}^{N} \Big(p_i\big(\alpha_i(t) - \beta_i(t)p_i\big) - f_i(u_i) - h_i(t)I_i + q_i\big(u_i - \alpha_i(t) + \beta_i(t)p_i\big) \Big)$$

where the arguments I, p, u, q are vectors with N components and t is the time argument.

The Lagrangian function relaxing the no backorders constraints and the capacity constraint can then be written as:

$$L(I,p,u,q,\rho,\eta,t) = H(I,p,u,q,t) + \sum_{i=1}^{N} \rho_i\Big(u_i - \alpha_i(t) + \beta_i(t)p_i\Big)$$

$$+ \eta\big(K(t) - \sum_{i=1}^{N} u_i\big) \qquad (3.13)$$

where the arguments I, p, u, q, ρ are vectors with N components, argument η is a non negative real number, and t is the time argument.

Notice that we dualized only the difficult constraints, i.e. the capacity constraint and the no backorders constraint, and not those that bound the admissible controls.

In what follows, we illustrate why the assumptions in Section 3.2 apply to the pricing problem (3.1) we are studying under Assumptions 1, 2, 3 and 5. In particular, we show that the assumption of constraints of the first order, constraint qualification, full-rank condition, and sufficiency conditions defined in Section 3.2 hold.

In Problem (3.1),

- the control variables are $(u_i(.)$, $p_i(.)$, $i = 1, \ldots, N)$ which are functions defined on $[0,T]$,

- the state variables are $(I_i(.),\ i = 1,\ldots,N)$ which are functions defined on $[0,T]$,

- the dynamic evolution of the system is given by

$$\dot{I}(t) = \vartheta(I(t),u(t),p(t),t) = u(t) - \alpha(t) + \beta(t) \times p(t),$$

- the mixed inequality constraints are $\xi(u(t),p(t),t) \geq 0$ where

$$\xi(u(t),p(t),t) = \left(p_1(t),\ldots,p_N(t),\ \frac{\alpha_1(t)}{\beta_1(t)} - p_1(t),\ldots,\frac{\alpha_N(t)}{\beta_N(t)} - p_N(t), \right.$$
$$\left. u_1(t),\ldots,u_N(t),\ K(t) - \sum_{i=1}^{N} u_i(t) \right)$$

(note that $\xi(u,p,t) \in \mathbb{R}^{3N+1}$),

- the pure state variable inequality constraint is given by
$$\varpi(I(t)) = (I_1(t),\ldots,I_N(t)) \geq 0.$$

We have $q_i^d(t) = q_i(t) + \rho_i(t)$ the adjoint variables within the framework of the direct adjoining method.

Lemma 1. *The pure state variable inequality constraints are of order 1.*

Proof. The ith pure state variable inequality constraint is $\varpi_i(I(t)) = I_i(t) \geq 0$. By taking the derivative with respect to time once, we obtain

$$\frac{d\varpi_i}{dt}(I(t)) = \frac{d\varpi_i}{dI_i}(I(t))\frac{dI_i}{dt}(t) = \frac{dI_i}{dt}(t) = u_i(t) - \alpha_i(t) + \beta_i(t)p_i(t).$$

We thus observe that the first time derivative depends explicitly on the controls, therefore the constraint is of the first order. $\qquad\square$

We define

$$\varpi^1(I(t),u(t),p(t),t) \equiv \frac{d\varpi}{dt}(I(t)) = \frac{d\varpi}{dI}(I(t))\dot{I}(t) = \dot{I}(t) = u(t) - \alpha(t) + \beta(t) \times p(t).$$

Lemma 2. *Constraint qualification holds.*

Proof. This condition guarantees that the gradients with respect to (u, p) of active constraints on controls are linearly independent.

We need to show that $\operatorname{rank}\left[\frac{\partial \xi}{\partial u}, \frac{\partial \xi}{\partial p}, \operatorname{diag}(\xi)\right] = 3N + 1$.

Let $M \equiv \left[\frac{\partial \xi}{\partial u}, \frac{\partial \xi}{\partial p}, \operatorname{diag}(\xi)\right]$. Then we can write matrix $M \in \mathbb{R}^{3N+1} \times \mathbb{R}^{5N+1}$ as follows (omitting the time argument for a lack of space):

the first $2N$ columns are:

$$
\begin{bmatrix}
 & & & 1 & & & p_1 & & \\
 & & & & \ddots & & & \ddots & \\
 & & & & & 1 & & & p_N \\
 & & & -1 & & & & & \\
 & & & & \ddots & & & & \\
 & & & & & -1 & & & \\
 1 & & & & & & & & \\
 & \ddots & & & & & & & \\
 & & 1 & & & & & & \\
 -1 & \cdots & -1 & & & & & &
\end{bmatrix}
$$

and the next $3N + 1$ columns are:

$$
\begin{bmatrix}
 \frac{\alpha_1(t)}{\beta_1(t)} - p_1 & & & & & & \\
 & \ddots & & & & & \\
 & & \frac{\alpha_N(t)}{\beta_N(t)} - p_N & & & & \\
 & & & u_1 & & & \\
 & & & & \ddots & & \\
 & & & & & u_N & \\
 & & & & & & K(t) - \sum_{i=1}^{N} u_i
\end{bmatrix}
$$

To show that $\operatorname{rank}(M) = 3N + 1$, we observe that:

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(i) there are $3N + 1$ rows in M, so the rank is at most $3N + 1$;

(ii) if there is no binding constraint on the control variables, the last $3N + 1$ columns are non zero and linearly independent, implying that the rank is equal to $3N + 1$;

(iii) for each i such that the price $p_i = 0$, column $2N + i$ is the zero vector; however we can replace it with column $N + i$ to obtain a set of linearly independent columns. Similarly, for each i such that $p_i = \alpha_i(t)/\beta_i(t)$, we replace column $3N + i$ with column $N + i$. We observe that the constraints $p_i = 0$ and $p_i = \alpha_i(t)/\beta_i(t)$ cannot be binding simultaneously.

Using the same reasoning, for each i such that $u_i = 0$, we replace column $4N + i$ with column i.

Finally, if $\sum_{i=1}^{N} u_i = K(t)$, we replace the last column with any column i, $1 \le i \le N$ such that $u_i > 0$. We observe that this is possible because when the capacity constraint is tight, at least one of the production rates must be positive (i.e. the last $N + 1$ columns cannot be simultaneously equal to the zero vector).

This implies that there are $3N + 1$ linearly independent columns. $\qquad\square$

Lemma 3. *Let $[\tau_1, \tau_2]$ be a boundary interval. The full-rank condition holds on $[\tau_1, \tau_2]$.*

Proof. Suppose that $I_i(t) = 0$, $i = 1, \ldots, n_0$, and $I_i(t) > 0$, $i = n_0 + 1, \ldots, N$, on $[\tau_1, \tau_2]$. Let

$$
Q \equiv \begin{bmatrix}
\frac{\partial \varpi_1^1}{\partial u} & \frac{\partial \varpi_1^1}{\partial p} \\
\frac{\partial \varpi_2^1}{\partial u} & \frac{\partial \varpi_2^1}{\partial p} \\
\vdots & \vdots \\
\frac{\partial \varpi_{n_0}^1}{\partial u} & \frac{\partial \varpi_{n_0}^1}{\partial p}
\end{bmatrix}
$$

We need to show that $\operatorname{rank}(Q) = n_0$. We can express matrix $Q \in \mathbb{R}^{n_0} \times \mathbb{R}^{2n_0}$ as follows:

$$
Q = \begin{bmatrix}
1 & & & & \beta_1(t) & & & \\
& 1 & & & & \beta_2(t) & & \\
& & \ddots & & & & \ddots & \\
& & & 1 & & & & \beta_{n_0}(t)
\end{bmatrix}
$$

which clearly has rank n_0. $\qquad\square$

Remark. In the direct adjoining method, $q_i^d(t) = q_i(t) + \rho_i(t)$ and the transversality conditions are given by

$$q^d(T) = \nu^d \frac{\partial \varpi}{\partial I}(I^*(T), T) = \nu^d, \quad \nu^d \geq 0, \quad \nu^d \varpi(I^*(T), T) = \nu^d I^*(T) = 0,$$

which can be rewritten

$$q_i(T) + \rho_i(T) \geq 0, \quad (q_i(T) + \rho_i(T))I^*(T) = 0.$$

Proposition 1. *Under Assumptions 1, 2, 3 and 5, the assumptions in Theorem 4 (see Section 3.2) hold for Problem (3.1).*

Proof. Notice that

- $H(I, u, p, q^d, t) = \sum_{i=1}^{N} \left(p_i(\alpha_i(t) - \beta_i(t)p_i) - f_i(u_i) - h_i(t)I_i + q_i^d(u_i - \alpha_i(t) + \beta_i(t)p_i) \right)$ is a concave function in (I, u, p);

- ξ is a linear function in (p, u), and thus quasi-concave in (I, p, u);

- ϖ is a linear (thus quasi-concave) function of I.

$\square$

Using Theorem 3 in Section 3.2, at the optimal solution,

- The state trajectory satisfies:

$$\begin{aligned}
I_i^*(0) &= I_i^0, \quad i = 1, \ldots, N \\
I_i^*(t) &\geq 0, \quad \forall t \in [0, T], \quad i = 1, \ldots, N \\
\dot{I}_i^*(t) &= u_i^*(t) - \alpha_i(t) + \beta_i(t)p_i^*(t), \quad \forall t \in [0, T], \quad i = 1, \ldots, N.
\end{aligned}$$

- The optimal control on $[0, T]$ is then given as a function of the adjoint variable and Lagrange multipliers by:

$$(p^*(t), u^*(t)) = \arg \max_{(p,u) \in W(t)} L(I^*(t), p, u, q(t), \rho(t), \eta(t)), \quad (3.14)$$

where $W(t)$ is the set of admissible controls (p, u) such that:

$$u_i \geq 0, \quad i = 1, \ldots, N,$$

$$0 \leq p_i \leq \frac{\alpha_i(t)}{\beta_i(t)}, \quad i = 1, \ldots, N.$$

- Additional feasibility constraints on $[0, T]$ include constraint (3.4), i.e.:

$$\sum_{i=1}^{N} u_i^*(t) \leq K(t),$$

as well as

$$\dot{I}_i^*(t) = 0 \quad \forall i, \ t \text{ such that } I_i^*(t) = 0.$$

- Complementary slackness conditions on $[0, T]$ give rise to:

$$
\begin{aligned}
\eta(t)\Big(K(t) - \sum_{i=1}^{N} u_i^*(t)\Big) &= 0 \\
\rho_i(t) I_i^*(t) &= 0, \quad i = 1, \ldots, N \\
\dot{\rho}_i(t) &\leq 0 \text{ on boundary interval of } I_i^*(.), \quad i = 1, \ldots, N \\
\rho_i(t) &\geq 0, \quad i = 1, \ldots, N \\
\eta(t) &\geq 0, \quad i = 1, \ldots, N.
\end{aligned}
$$

- The vector of adjoint variables $q(.)$ satisfies the adjoint equation almost everywhere (i.e. except at the entry times to the boundary condition $I_i^*(t) = 0$):

$$
\begin{aligned}
\dot{q}_i(t) &= -\nabla_{I_i} L(I^*(t), p^*(t), u^*(t), q(t), \rho(t), \eta(t), t) \\
&= h_i(t) \quad i = 1, \ldots, N,
\end{aligned}
$$

as well as transversality conditions[5]

$$(q_i + \rho_i)(T) \geq 0, \quad i = 1, \ldots, N$$

---

[5]Notice that the transversality conditions are written using the direct adjoining method.

$$I_i^*(T)(q_i + \rho_i)(T) = 0, \quad i = 1, \ldots, N.$$

- Finally, for $i = 1, \ldots N$,

  o $I_i^*(.)$ is a continuous function of time;

  o $q_i(.)$ is a continuous function of time except at the entry times to the boundary condition[6] $I_i^*(t) = 0$;

  o $(q_i + \rho_i)(.)$ is a continuous function of time everywhere[7].

3.3 Derivation of the solution method

3.3.1 The optimal policy as a function of the multipliers and adjoint variables

Proposition 2. *Under Assumptions 1, 2, 3 and 5, given $q(.), p(.)$ and $\eta(.)$, there exist at each time $t \in [0, T]$, unique optimal controls given by:*

$$
\begin{aligned}
p_i^*(t) &= \arg \max_{0 \le p_i \le \frac{\alpha_i(t)}{\beta_i(t)}} \left(\alpha_i(t) - \beta_i(t)p_i + (q_i(t) + \rho_i(t))\beta_i(t) \right) p_i \\
&= \begin{cases}
0 & \text{if } q_i(t) + \rho_i(t) \le -\frac{\alpha_i(t)}{\beta_i(t)}, \\
\frac{1}{2}\left(\frac{\alpha_i(t)}{\beta_i(t)} + q_i(t) + \rho_i(t) \right) & \text{if } -\frac{\alpha_i(t)}{\beta_i(t)} < q_i(t) + \rho_i(t) \le \frac{\alpha_i(t)}{\beta_i(t)}, \quad (3.15) \\
\frac{\alpha_i(t)}{\beta_i(t)} & \text{otherwise,}
\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
u^*(t) &= \arg \max_{u \ge 0} \left(q_i(t) + \rho_i(t) - \eta(t) \right) u_i - f_i(u_i) \\
&= \begin{cases}
0 & \text{if } q_i(t) + \rho_i(t) - \eta(t) < f_i'(0), \\
f_i'^{-1}(q_i(t) + \rho_i(t) - \eta(t)) & \text{otherwise.}
\end{cases} \quad (3.16)
\end{aligned}
$$

Proof. We solve optimization problem (3.14) in order to determine the optimal policy as a function of the multipliers and adjoint variables. We notice that the Lagrangian

---

[6]The adjoint variable may be discontinuous at the entry or exit times to constrained intervals. However, by convention, we impose continuity at the exit times. This allows to constrain the multiplier ρ to be non negative. See [73] for more details.

[7]This is a consequence from the theory of the direct adjoining method. See [97] or [105] for a justification.

function is separable across products and in p_i and u_i (we have dualized the coupling constraints). Furthermore, it is a strictly concave continuously differentiable function in p_i and u_i (since for every product i, f_i is a strictly convex function, which implies that the function $u_i \mapsto (q_i(t) + \rho_i(t) - \eta(t))u_i - f_i(u_i)$ is strictly concave on $\mathbb{R}^+$). Moreover, the remaining constraints (those not dualized, which are constraints on the control variables only) are linear, i.e. they constrain the control variables within a convex set.

Therefore, there are unique optimal controls $(u^*(t), p^*(t))$, which are the maximizers of the Lagrangian function over the feasible control variables. To compute them, we consider the partial derivatives of the Lagrangian function:

$$\frac{\partial L}{\partial u_i}(I, p, u, q, \rho, \eta, t) = -f_i'(u_i) + q_i + \rho_i - \eta$$

$$\frac{\partial L}{\partial p_i}(I, p, u, q, \rho, \eta, t) = \alpha_i(t) - 2\beta_i(t)p_i + \beta_i(t)q_i + \beta_i(t)\rho_i.$$

To obtain the optimal solution, we proceed as follows. We first solve the equations setting these partial derivatives to zero. If the solution obtained lies within the set of feasible controls (defined by the linear, not dualized constraints), then it is the optimal control. Otherwise, the optimal control lies on a boundary of the set of feasible controls, i.e. zero for production rates, and either zero or $\frac{\alpha_i(t)}{\beta_i(t)}$ for the price, depending on which value corresponds to the higher value of the Lagrangian.

Before ending the proof, we recall that we assumed f_i defined on $\mathbb{R}^+$ to be positive, strictly convex and increasing. Moreover, f_i' is also defined on $\mathbb{R}^+$ and is strictly increasing with range $[f_i'(0), +\infty)$. Therefore, on the one hand it is invertible and on the other hand $f_i'(u) \geq f_i'(0) \geq 0 \ \forall u \geq 0$. Moreover, it implies that $f_i'^{-1}$ is positive valued and $f_i'^{-1}(u)$ is defined for $u \geq f_i'(0)$. The result then follows.

$\square$

We can also derive the expression for $\dot{I}_i^*(t) = u_i^*(t) - \alpha_i(t) + \beta_i(t)p_i^*(t)$. Clearly if product i is on a constrained interval at time t, then $\dot{I}_i^*(t) = 0$. If product i is on an

unconstrained interval at time t, then $\rho_i(t) = 0$ and

$$
\dot{I}_i^*(t) = \begin{cases}
-\alpha_i(t) & \text{if } q(t) < -\frac{\alpha_i(t)}{\beta_i(t)} \\[2mm]
\frac{1}{2}\Big(-\alpha_i(t) + \beta_i(t)q_i(t)\Big) & \text{if } -\frac{\alpha_i(t)}{\beta_i(t)} \le q_i(t) \le \\[2mm]
& \qquad \min\{f_i'(0) + \eta(t); \frac{\alpha_i(t)}{\beta_i(t)}\} \\[2mm]
f_i'^{-1}\Big(q_i(t) - \eta(t)\Big) - \frac{\alpha_i(t)}{2} + \frac{\beta_i(t)}{2}q_i(t) & \text{if } f_i'(0) + \eta(t) \le q_i(t) \le \frac{\alpha_i(t)}{\beta_i(t)} \\[2mm]
0 & \text{if } \frac{\alpha_i(t)}{\beta_i(t)} \le q_i(t) \le f_i'(0) + \eta(t) \\[2mm]
f_i'^{-1}\Big(q_i(t) - \eta(t)\Big) & \text{if } q_i(t) > \max\{\frac{\alpha_i(t)}{\beta_i(t)}; f_i'(0) + \eta(t)\}.
\end{cases}
$$

$$(3.17)$$

3.3.2 First step: $(q_i + \rho_i)(t)$, $i = 1, \ldots, N$ are known

In this Section, we relax the difficulties associated with the dynamic aspect of the problem as well as the no backorders constraint by fixing the value of t and assuming that $q_i(t) + \rho_i(t)$, $i = 1, \ldots, N$ are known. The following algorithm $\mathcal{A}_1(t)$ takes vector $q(t) + \rho(t)$ as input and gives the optimal policy at time t: $u^*(t)$, $p^*(t)$ as output. The intuition behind the algorithm is to first check whether the components of the given vector $q(t) + \rho(t)$ imply that the capacity constraint is tight or not at time t. If it is not, then we set $\eta(t) = 0$ and compute the solution. Otherwise, we determine the value of $\eta(t) > 0$ by using the associated complementary slackness condition. To do so, we need to find the set of active products at time t which will be denoted by $\mathcal{I}(t)$. Once the value of $\eta(t)$ is computed, we obtain the optimal policy by using the results from the previous section. See Figure 3-2 for an illustration of algorithm $\mathcal{A}_1(t)$. More formally,

Initialization

1. Renumber the indices by increasing value of $q_i(t) + \rho_i(t) - f_i'(0)$.

   Initialize a set of indices $\mathcal{I}(t) = \{i : q_i(t) + \rho_i(t) - f_i'(0) \ge 0\}$.

   If $\mathcal{I}(t) = \emptyset$, set $\eta(t) = 0$ and go to step 5.

   Otherwise, initialize $j_0 = \min \mathcal{I}(t)$ the smallest index in $\mathcal{I}(t)$[8]; go to step 2.

---

[8]Index j_0 actually depends on time t but we omit the time argument in order to improve the

2. Compute

$$B(t) = \sum_{i \in \mathcal{I}(t)} f_i'^{-1}(q_i(t) + \rho_i(t)).$$

If $B(t) < K(t)$, set $\eta(t) = 0$, go to step 5.

Otherwise, go to step 3.

Determination of $\eta(t)$

3. If the following equation for η:

$$K(t) = \sum_{i \in \mathcal{I}(t)} f_i'^{-1}(q_i(t) + \rho_i(t) - \eta)$$

has a solution $\eta \in \left[0, \; q_{j_0}(t) + \rho_{j_0}(t) - f_{j_0}'(0)\right]$, set $\eta(t) = \eta$ and go to step 5.

Otherwise, do $\mathcal{I}(t) \leftarrow \mathcal{I}(t) \setminus \{j_0\}, \;\; j_0 \leftarrow j_0 + 1$ and go to step 4.

4. If the following equation for η:

$$K(t) = \sum_{i \in \mathcal{I}(t)} f_i'^{-1}(q_i(t) + \rho_i(t) - \eta)$$

has a solution $\eta \in \left(q_{j_0-1}(t) + \rho_{j_0-1}(t) - f_{j_0-1}'(0), \; q_{j_0}(t) + \rho_{j_0}(t) - f_{j_0}'(0)\right]$, set $\eta(t) = \eta$ and go to step 5.

Otherwise, do $\mathcal{I}(t) \leftarrow \mathcal{I}(t) \setminus \{j_0\}, \;\; j_0 \leftarrow j_0 + 1$ and go to step 4.

Computation of the optimal policy

5.
$$u_i^*(t) = \begin{cases} 0 & \text{if } \; i \notin \mathcal{I}(t) \\ f_i'^{-1}(q_i(t) + \rho_i(t) - \eta(t)) & \text{if } \; i \in \mathcal{I}(t) \end{cases}$$

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exposition.

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$$p_i^*(t) = \begin{cases} 0 & \text{if } q_i(t) + \rho_i(t) \le -\frac{\alpha_i(t)}{\beta_i(t)}, \\ \frac{1}{2}\left(\frac{\alpha_i(t)}{\beta_i(t)} + q_i(t) + \rho_i(t)\right) & \text{if } -\frac{\alpha_i(t)}{\beta_i(t)} < q_i(t) + \rho_i(t) \le \frac{\alpha_i(t)}{\beta_i(t)}, \\ \frac{\alpha_i(t)}{\beta_i(t)} & \text{if } q_i(t) + \rho_i(t) > \frac{\alpha_i(t)}{\beta_i(t)}, \end{cases}$$
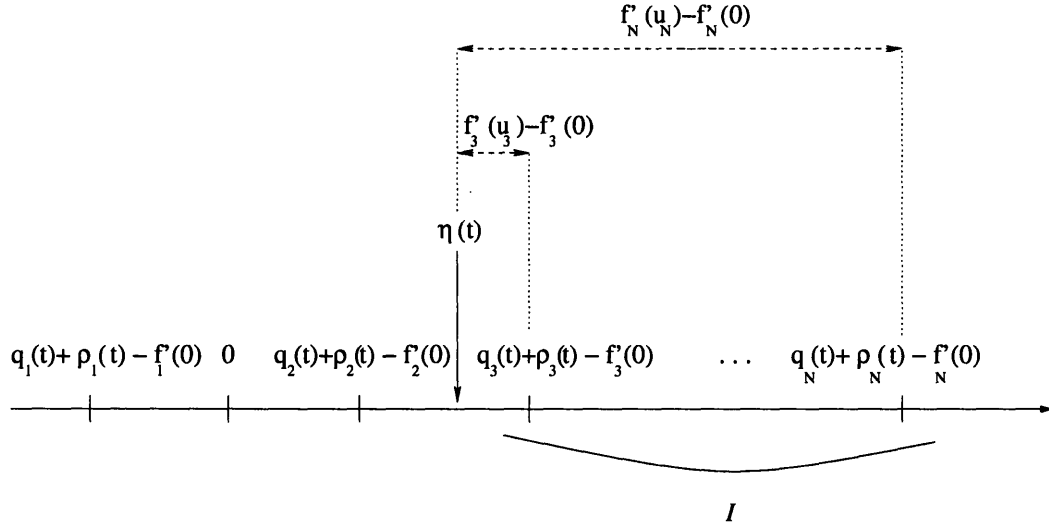
Figure 3-2: Algorithm $\mathcal{A}_1(t)$: Example with $\eta(t) > 0$, when products $3, \ldots, N$ are active.

Next we prove that, given $q(t) + \rho(t)$, this algorithm computes an optimal policy at time t.

Proposition 3. *Under Assumptions 1, 2, 3 and 5, given $q(t) + \rho(t)$, ($q_i(t)$ is the adjoint variable for dynamic constraint i at time t and $\rho_i(t)$ is the Lagrangian multiplier dualizing the non negativity constraint for the inventory level of product i at time t), then there exists a unique optimal policy $(u^*(t), p^*(t))$ at a fixed time $t \in [0, T]$, for problem (3.1) obtained via algorithm $\mathcal{A}_1(t)$.*

Proof. We will use the necessary conditions for optimality to find the optimal policy for the production rate as a function of the multipliers $q_i(t) + \rho_i(t)$.

$\eta(t) \ge 0$ is the Lagrange multiplier dualizing the capacity constraint. We have two possibilities:

- $\eta(t) = 0$, which implies that

$$u^*(t) = \begin{cases} 0 & \text{if } q_i(t) + \rho_i(t) < f_i'(0), \\ f'^{-1}(q_i(t) + \rho_i(t)) & \text{otherwise} \end{cases}$$

and for feasibility,

$$\sum_{i=1}^{N} u_i^*(t) = \sum_{i \text{ s.t. } q_i(t)+\rho_i(t) \geq f_i'(0)} f_i'^{-1}(q_i(t) + \rho_i(t)) \leq K(t).$$

- $\eta(t) > 0$ in which case, by complementary slackness,

$$\sum_{i=1}^{N} u_i^*(t) = K(t).$$

Furthermore, since $f_i'^{-1}$ is strictly increasing,

$$f_i'^{-1}(q_i(t) + \rho_i(t) - \eta(t)) < f_i'^{-1}(q_i(t) + \rho_i(t)).$$

This implies that

$$\begin{aligned} K(t) &= \sum_{i=1}^{N} u_i^*(t) = \sum_{i \text{ s.t. } q_i(t)+\rho_i(t)-\eta(t) \geq f_i'(0)} f_i'^{-1}(q_i(t) + \rho_i(t) - \eta(t)) \\ &< \sum_{i \text{ s.t. } q_i(t)+\rho_i(t)-\eta(t) \geq f_i'(0)} f_i'^{-1}(q_i(t) + \rho_i(t)). \end{aligned} \tag{3.18}$$

The fact that $q_i(t) + \rho_i(t) - \eta(t) \geq f_i'(0) \Rightarrow q_i(t) + \rho_i(t) \geq f_i'(0)$ and that $f_i'^{-1}$ has positive values, implies that

$$\begin{aligned} K(t) &< \sum_{i \text{ s.t. } q_i(t)+\rho_i(t)-\eta(t) \geq f_i'(0)} f_i'^{-1}(q_i(t) + \rho_i(t)) \\ &\leq \sum_{i \text{ s.t. } q_i(t)+\rho_i(t) \geq f_i'(0)} f_i'^{-1}(q_i(t) + \rho_i(t)). \end{aligned} \tag{3.19}$$

As a result, we have established that

$$\sum_{i \ s.t. \ q_i(t)+\rho_i(t) \geq f_i'(0)} f_i'^{-1}(q_i(t) + \rho_i(t)) \leq K(t) \iff \eta(t) = 0.$$

Therefore if $q_i(t) + \rho_i(t), i = 1, \ldots, N$ are given, we can compute the optimal policy by using the following procedure:

- Compute $B(t) = \sum_{i \ s.t. \ q_i(t)+\rho_i(t) \geq f_i'(0)} f_i'^{-1}(q_i(t) + \rho_i(t))$.

- If $B(t) < K(t)$, then $\eta(t) = 0$ and use the result from Proposition 1 to obtain $u^*(t)$.

- Otherwise, using (3.18), $\eta(t) > 0$ is the solution of

$$K(t) = \sum_{i \ s.t. \ q_i(t)+\rho_i(t)-\eta(t) \geq f_i'(0)} f_i'^{-1}(q_i(t) + \rho_i(t) - \eta(t)).$$

Once $\eta(t)$ is determined, use the result from Proposition 1 to obtain $u^*(t)$.

We also notice that we first checked whether or not the capacity constraint could be non tight, and if it cannot be non tight, we then repeatedly try to determine the value of $\eta(t) > 0$ by letting the set of active products decrease at each step. Since at that point we know that the capacity constraint must be tight, the set $\mathcal{I}(t)$ of active products cannot become empty (at least one product must be active) so the algorithm stops before we have removed all indices from the set, which guarantees that the algorithm terminates.

It can then be seen that algorithm $\mathcal{A}_1(t)$ provides $u^*(t)$ and $p^*(t)$ the optimal production and pricing policies at time t. $\qquad \square$

We observe that this algorithm (which applies for a fixed value of time t) terminates in at most $N + 1$ iterations.

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In what follows we execute the algorithm in an example at time $t = 1$.

Example

Let us consider the setting of two products, i.e. $N = 2$, a time horizon $[0, 10]$, i.e. $T = 10$, and the following data:

$$f_1(u) = 2u^2 + u + 0.2 \qquad f_2(u) = u^2 + 2u + 1$$
$$\alpha_1(t) = 1 + 0.5t \qquad \alpha_2(t) = 2 + 0.1t$$
$$\beta_1(t) = 0.4 + 0.3t \qquad \beta_2(t) = 0.1 + 0.1t$$
$$K(1) = 13.$$

We assume that $q_1(1) + \rho_1(1) = 27, \quad q_2(1) + \rho_2(1) = 18$ are given. Then we can compute

$$f_1'(u) = 4u + 1 \qquad f_2'(u) = 2u + 2$$
$$f_1'^{-1}(u) = \frac{u-1}{4} \qquad f_2'^{-1}(u) = u/2 - 1$$
$$f_1'(0) = 1 \qquad f_2'(0) = 2.$$

We compute the optimal pricing policy at time $t = 1$. Notice that

$$\alpha_1(1)/\beta_1(1) \approx 2.14 < q_1(1) + \rho_1(1), \quad \alpha_2(1)/\beta_2(1) = 10.5 < q_2(1) + \rho_2(1).$$

Therefore

$$p_i^*(1) = \frac{1}{2}\left(\alpha_i(1)/\beta_i(1) + q_i(1) + \rho_i(1)\right), \quad i = 1, 2,$$

which implies that $p_1^*(1) \approx 14.57, \quad p_2^*(1) = 14.25$.

Let us compute the optimal production policy at time $t = 1$ using algorithm $\mathcal{A}_1(1)$.

<u>Step 1:</u>

$$q_1(1) + \rho_1(1) - f_1'(0) = 27 - 1 = 26 > q_2(1) + \rho_2(1) - f_2'(0) = 18 - 2 = 16.$$

Therefore we reorder the products by using the indices: $k_1 = 2, \ k_2 = 1$.

Since $q_{k_1}(1) + \rho_{k_1}(1) - f_{k_1}'(0) \geq 0, \ q_{k_2}(1) + \rho_{k_2}(1) - f_{k_2}'(0) \geq 0$, we let $\mathcal{I}(1) = \{k_1, k_2\}, \ j_0 = k_1$.

Step 2: We compute

$$S(1) = f_{k_1}^{'-1}(q_{k_1}(1) + \rho_{k_1}(1)) + f_{k_2}^{'-1}(q_{k_2}(1) + \rho_{k_2}(1))$$

$$= 18/2 - 1 + \frac{27 - 1}{4} = 8 + 6.5 = 14.5 > K(1) = 13.$$

This implies that $\eta(1) > 0$.

Step 3: We consider the equation

$$K(1) = f_{k_1}^{'-1}(q_{k_1}(1) + \rho_{k_1}(1) - \eta) + f_{k_2}^{'-1}(q_{k_2}(1) + \rho_{k_2}(1) - \eta),$$

$$\Leftrightarrow 13 = \frac{18 - \eta}{2} - 1 + \frac{27 - \eta - 1}{4} \Leftrightarrow \eta = 2 \in [0, q_{k_1}(1) + \rho_{k_1}(1) - f_{k_1}'(0)].$$

Step 4: Therefore, $u_i^*(1) = f_i^{'-1}(q_i(1) + \rho_1(1) - \eta)$, $i = 1, 2$, and

$$u_1^*(1) = 6, \quad u_2^*(1) = 7.$$

3.3.3 Second step: the system is observable at each time t

So far in this chapter, we have illustrated how to compute the optimal production and pricing policies $p^*(t)$ and $u^*(t)$ at any fixed time t, as functions of vector $q(t) + \rho(t)$. In this subsection, we will introduce one additional level of difficulty, that is, we will not assume that $q(t) + \rho(t)$ is known. This allows us to incorporate the dynamic aspect of the problem. In particular, we extend the approach in the previous subsection by illustrating how to compute the quantities $q_i(t) + \rho_i(t)$ as well as $\eta(t)$.

However, in order to provide some intuition on the problem solution for the general case (i.e., when no information on the system is available, see Subsection 5.3), in this subsection we assume that at time t we know whether the current level of inventory $I_i(t)$ for each product i, is positive or equal to zero. Moreover, we assume that at any fixed time t, for each product i with positive inventory level we kept track of the last exit time t_i^1 and of the value $q_i(t_i^1)$ of the adjoint variable q_i at that

time[9]. If product i has a positive initial inventory level and at time t lies on its first unconstrained interval, then we define t_i^1 to be zero. These assumptions define the notion of *observability* we have referred to before (see Subsection 3.1.3).

Let us fix the value of time t. Perhaps after renumbering the products (see below for further details), we will use the following notations[10]:

$S(t)$ set of constrained products at time t;

$\mathcal{J}(t)$ set of active constrained products at time t $(\mathcal{J}(t) \subseteq S(t))$;

i_1 smallest index of $\mathcal{J}(t)$ (when $\mathcal{J}(t) \neq \emptyset$);

i_0 index preceding i_1 in $S(t)$ (when $i_1 \neq \min S(t)$);

$\mathcal{J}'(t)$ set of active unconstrained products at time t $(\mathcal{J}'(t) \subseteq S(t)^c)$;

i_1' smallest index of $\mathcal{J}'$ (when $\mathcal{J}'(t) \neq \emptyset$);

i_0' index preceding i_1' in $S(t)^c$ (when $i_1' \neq \min S(t)^c$).

	active	inactive
constrained (set $S(t)$)	$\mathcal{J}(t)$, i_1	i_0
unconstrained (set $S(t)^c$)	$\mathcal{J}'(t)$, i_1'	i_0'

The procedure that follows starts by checking if the observation of the system implies that the capacity constraint is not tight (i.e. if $\eta(t) = 0$). If this is not valid, we increase the value of $\eta(t)$ gradually, until all the necessary conditions (see Subsection 3.2.2) are satisfied at time t. Figure 3-3 provides an illustration of algorithm $\mathcal{A}_2(t)$. Before describing algorithm $\mathcal{A}_2(t)$, we show some properties that will be useful.

Let

$$g_i(t, z) = z - f_i'\left(\frac{\alpha_i(t) - \beta_i(t)z}{2}\right)$$

defined for $t \in [0, T]$ and, (for a given value of t) for $z \in \left(-\infty, \frac{\alpha_i(t)}{\beta_i(t)}\right]$. We define $l_{i,t} : z \mapsto g_i(t, z)$.

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[9]Time t_i^1 and thus multiplier $q_i(t_i^1)$ actually depend on time t but we omit the time argument in order to improve the exposition.

[10]Indices i_1, i_0, i_1', i_0' actually depend on time t but we omit the time argument in order to improve the exposition.

Proposition 4. *Under Assumptions 1, 2, 3 and 5, function $l_{i,t}(.)$ is an invertible mapping on $\left(-\infty, \frac{\alpha_i(t)}{\beta_i(t)} \right]$ and its range for a fixed t is $\left(-\infty, \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0) \right]$.*

Proof. It is clear that $l_{i,t}(.)$ is continuous and differentiable.

It easily follows from the definition that $l_{i,t}\left(\frac{\alpha_i(t)}{\beta_i(t)}\right) = \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0)$. Moreover, since $f_i'(.)$ takes only non negative arguments, the definition of $l_{i,t}(z)$ requires $\frac{\alpha_i(t) - \beta_i(t)z}{2} \geq 0$, or equivalently, that $z \leq \frac{\alpha_i(t)}{\beta_i(t)}$.

We recall that f_i' is a strictly increasing function, so it is clear that $\lim_{z \to -\infty} l_{i,t}(z) = -\infty$.

We compute

$$l_{i,t}'(z) = 1 + \frac{\beta_i(t)}{2} f_i''\left(\frac{\alpha_i(t) - \beta_i(t)z}{2}\right) > 0$$

since f_i is strictly convex, so $l_{i,t}(.)$ is strictly increasing (for a fixed t), and hence invertible. Also, since it is increasing, it follows that its range for a fixed t is $\left(-\infty, \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0) \right]$. $\qquad\square$

Corollary 1. *Under Assumptions 5, 3, 1 and 2, given $0 \leq \eta \leq \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0)$, equation $g_i(t,z) = \eta$ for argument z (and fixed t) has a unique solution $z_0 \equiv \phi_i(t,\eta)$ satisfying $f_i'(0) < \phi_i(t,\eta) \leq \frac{\alpha_i(t)}{\beta_i(t)}$. Moreover, $\phi_i(.,.)$ is continuous in both arguments.*

Proof. Since $\eta \leq \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0)$ is in the range of $l_{i,t}(.)$ (for a fixed t), using the previous proposition $l_{i,t}^{-1}(\eta)$ is well defined. As a result, the solution of the equation is uniquely defined by $z_0 = l_{i,t}^{-1}(\eta)$ and we have $\eta = l_{i,t}(z_0) = g_i(t,z_0)$.

Let $\phi_i(t,\eta) = l_{i,t}^{-1}(\eta)$. In particular, since $l_{i,t}(.)$ is continuous, $\phi_i(.,.)$ is continuous in its second argument. The continuity of $\phi_i(.,.)$ with respect to its first argument follows from the fact that $g_i(.,.)$ is continuous in both arguments and from the relation $g_i(t, \phi_i(t,\eta)) - \eta = 0$.

Using the previous proposition, it is clear that $l_{i,t}^{-1}(\eta) \leq \frac{\alpha_i(t)}{\beta_i(t)}$. Therefore, $\phi_i(t,\eta) \leq \frac{\alpha_i(t)}{\beta_i(t)}$.

Suppose $\phi_i(t, \eta) \leq f_i'(0)$. Then, using Assumption 2, $\phi_i(t, \eta) < \frac{\alpha_i(t)}{\beta_i(t)}$. Moreover, since $\eta \geq 0$, we have $\phi_i(t, \eta) - \eta \leq f_i'(0)$. It can be seen using the definition of $\phi_i(t, \eta)$ that

$$\phi_i(t, \eta) - \eta = f_i'\left(\frac{\alpha_i(t) - \beta_i(t)\phi_i(t, \eta)}{2}\right),$$

and therefore

$$f_i'\left(\frac{\alpha_i(t) - \beta_i(t)\phi_i(t, \eta)}{2}\right) \leq f_i'(0).$$

This implies $\alpha_i(t) - \beta_i(t)\phi_i(t, \eta) \leq 0$ since f_i' is increasing, or equivalently $\phi_i(t, \eta) \geq \frac{\alpha_i(t)}{\beta_i(t)}$, which is a contradiction. It follows that $\phi_i(t, \eta) > f_i'(0)$. $\qquad\square$

Corollary 2. *(i) Function $\phi_i(., .)$ is continuously differentiable in its first argument and*

$$\frac{\partial \phi_i}{\partial t}(t, \eta) = \frac{\frac{1}{2}(\alpha_i'(t) - \beta_i'(t)\phi_i(t, \eta))f_i''\left(\frac{\alpha_i(t) - \beta_i(t)\phi(t, \eta)}{2}\right)}{1 + \frac{\beta_i(t)}{2}f_i''\left(\frac{\alpha_i(t) - \beta_i(t)\phi(t, \eta)}{2}\right)}.$$

(ii) Function $\phi_i(., .)$ is continuously differentiable in its second argument and

$$\frac{\partial \phi_i}{\partial \eta}(t, \eta) = \frac{1}{1 + \frac{\beta_i(t)}{2}f_i''\left(\frac{\alpha_i(t) - \beta_i(t)\phi(t, \eta)}{2}\right)}.$$

Proof. (i) Differentiability follows from the differentiability of $g(., .)$ with respect to both arguments and from the relation $g_i(t, \phi_i(t, \eta)) - \eta = 0$. The expression is obtained by observing that the relation $g_i(t, \phi_i(t, \eta)) - \eta = 0$ implies by differentiating with respect to t:

$$\frac{\partial g_i}{\partial t}(t, \phi_i(t, \eta)) + \frac{\partial \phi_i}{\partial t}(t, \eta)\frac{\partial g_i}{\partial z}(t, \phi_i(t, \eta)) = 0$$

and since $\frac{\partial g_i}{\partial z}(t, \phi_i(t, \eta)) = l_{i,t}'(\phi_i(t, \eta))$,

$$\frac{\partial \phi_i}{\partial t}(t, \eta) = -\frac{\frac{\partial g_i}{\partial t}(t, \phi_i(t, \eta))}{l_{i,t}'(\phi_i(t, \eta))}.$$

(ii) The result follows immediately from $\phi_i(t, \eta) = l_{i,t}^{-1}(\eta)$ and the expression of the derivative of $l_{i,t}(.)$. $\qquad\square$

Let $\psi_i(t) = \phi_i(t, 0)$. In particular, function $\psi(.)$ is continuously differentiable.

In what follows, we will show that $\phi_i(t, \eta(t))$ represents $q_i(t) + \rho_i(t)$ expressed as a function of $\eta(t)$ for constrained products.

Algorithm $\mathcal{A}_2(t)$ has input:

- the data of the problem,

- whether each product i is constrained or unconstrained at time t,

- for each unconstrained product i at time t, the last exit time t_i^1 (defined by time 0 if t belongs to the first unconstrained interval for that product),

- for each unconstrained product i at time t, the value $q_i(t_i^1)$ of q_i at time t_i^1,

and output $(u^*(t), p^*(t))$ as described below.

Recall that we have fixed time t. We use the convention that $\min \emptyset = \infty$.

Initialization

1. Let $S(t)$ be the set of constrained products.
   For each unconstrained product i, compute $\bar{q}_i(t) = q_i(t_i^1) + \int_{t_i^1}^t h_i(s)ds$.
   Reorder the indices by increasing values of $\{\frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0), i \in S(t)\} \bigcup \{\bar{q}_i(t) - f_i'(0), i \notin S(t)\}$.
   Initialize $\mathcal{J}(t) = S(t)$, $\mathcal{J}'(t) = \{i \in S(t)^c : \bar{q}_i(t) - f_i'(0) \geq 0\}$.
   Let $j = \min \mathcal{J}(t)$, $j' = \min \mathcal{J}'(t)$ (by convention $\frac{\alpha_\infty(t)}{\beta_\infty(t)} - f_\infty'(0) = \bar{q}_\infty - f_\infty'(0) = \infty$).

2. **Test of the hypothesis $\eta(t) = 0$.**
   If
   $$\sum_{i \in \mathcal{J}'(t)} f_i'^{-1}(\bar{q}_i(t)) + \frac{1}{2} \sum_{i \in \mathcal{J}(t)} (\alpha_i(t) - \beta_i(t)\phi_i(t, 0)) \leq K(t),$$
   set $\eta(t) = 0$ and go to step 8.
   Otherwise, go to step 3.

Determination of $\eta(t)$.

3. Let $i_1 = j$, $i'_1 = j'$ and $k = \min \{i_1, i'_1\}$[11].

   If the following equation for η

   $$\sum_{i \in \mathcal{J}'(t)} f_i'^{-1}(\bar{q}_i(t) - \eta) + \frac{1}{2} \sum_{i \in \mathcal{J}(t)} (\alpha_i(t) - \beta_i(t)\phi_i(t, \eta)) = K(t)$$

   has a solution $\eta \in \left[0, \frac{\alpha_{i_1}(t)}{\beta_{i_1}(t)} - f'_{i_1}(0)\right] \bigcap \left[0, \bar{q}_{i'_1}(t) - f'_{i'_1}(0)\right]$, set $\eta(t) = \eta$ and go to step 8.

   Otherwise, go to step 4.

4. If $k = i_1$, do $\mathcal{J}(t) \leftarrow \mathcal{J}(t) \setminus \{i_1\}$, update $i_1 = \min \mathcal{J}(t)$ and $k = \min \{i_1, i'_1\}$.
   Otherwise ($k = i'_1$), do $\mathcal{J}'(t) \leftarrow \mathcal{J}'(t) \setminus \{i'_1\}$, update $i'_1 = \min \mathcal{J}'(t)$ and $k = \min \{i_1, i'_1\}$.

   If $j' < k \leq j$, go to step 5.

   If $j < k \leq j'$, go to step 6.

   Otherwise ($k > j$, j'[12]), go to step 7.

5. Let $i'_0 = \max S(t)^c \setminus \mathcal{J}'(t)$[13].

   If the following equation for η

   $$\sum_{i \in \mathcal{J}'(t)} f_i'^{-1}(\bar{q}_i(t) - \eta) + \frac{1}{2} \sum_{i \in \mathcal{J}(t)} (\alpha_i(t) - \beta_i(t)\phi_i(t, \eta)) = K(t)$$

   has a solution $\eta \in \left[0, \frac{\alpha_{i_1}(t)}{\beta_{i_1}(t)} - f'_{i_1}(0)\right] \bigcap \left(\bar{q}_{i'_0}(t) - f'_{i'_0}(0), \ \bar{q}_{i'_1}(t) - f'_{i'_1}(0)\right]$ set $\eta(t) = \eta$ and go to step 8.

   Otherwise, go to step 4.

---

[11]Index k is always finite because at this stage, the capacity constraint must be tight, therefore the set of active constrained products and the set of active unconstrained products cannot both be empty.

[12]At this stage of the algorithm, one index has been removed from either $\mathcal{J}(t)$ or $\mathcal{J}'(t)$, therefore we cannot have $k \leq j$, j'.

[13]Index i'_0 is well defined because at this stage of the algorithm, the index that has just been removed from set $\mathcal{J}'(t)$ remains in set $S(t)^c$.

6. Let $i_0 = \max S(t) \setminus \mathcal{J}(t)^{14}$.

   If the following equation for η

   $$\sum_{i \in \mathcal{J}'(t)} f_i'^{-1}(\bar{q}_i(t) - \eta) + \frac{1}{2} \sum_{i \in \mathcal{J}(t)} (\alpha_i(t) - \beta_i(t)\phi_i(t, \eta)) = K(t)$$

   has a solution $\eta \in \left(\frac{\alpha_{i_0}(t)}{\beta_{i_0}(t)} - f_{i_0}'(0) \ , \ \frac{\alpha_{i_1}(t)}{\beta_{i_1}(t)} - f_{i_1}'(0)\right] \bigcap [0, \ \bar{q}_{i_1'}(t) - f_{i_1'}'(0)]$, set $\eta(t) = \eta$ and go to step 8.

   Otherwise, go to step 4.

7. Let $i_0' = \max S(t)^c \setminus \mathcal{J}'(t)$, $i_0 = \max S(t) \setminus \mathcal{J}(t)$.

   If the following equation for η

   $$\sum_{i \in \mathcal{J}'(t)} f_i'^{-1}(\bar{q}_i(t) - \eta) + \frac{1}{2} \sum_{i \in \mathcal{J}(t)} (\alpha_i(t) - \beta_i(t)\phi_i(t, \eta)) = K(t)$$

   has a solution $\eta \in \left(\frac{\alpha_{i_0}(t)}{\beta_{i_0}(t)} - f_{i_0}'(0) \ , \ \frac{\alpha_{i_1}(t)}{\beta_{i_1}(t)} - f_{i_1}'(0)\right] \bigcap \left(\bar{q}_{i_0'}(t) - f_{i_0'}'(0) \ , \ \bar{q}_{i_1'}(t) - f_{i_1'}'(0)\right]$, set $\eta(t) = \eta$ and go to step 8.

   Otherwise, go to step 4.

Computation of the optimal policy.

8. Do
   $$u_i^*(t) = \begin{cases} 0, & i \notin \mathcal{J}(t), i \notin \mathcal{J}'(t), \\ f_i'^{-1}(\bar{q}_i(t) - \eta(t)), & i \in \mathcal{J}'(t), \\ \frac{1}{2}\Big(\alpha_i(t) - \beta_i(t)\phi_i(t, \eta(t))\Big), & i \in \mathcal{J}(t), \end{cases}$$

   $$p_i^*(t) = \begin{cases} \frac{\alpha_i(t)}{\beta_i(t)}, & \left(i \in S(t)^c \text{ s.t. } \bar{q}_i(t) \geq \frac{\alpha_i(t)}{\beta_i(t)}\right) \text{ or } i \in S(t) \setminus \mathcal{J}(t), \\ \frac{1}{2}\left(\frac{\alpha_i(t)}{\beta_i(t)} + \bar{q}_i(t)\right), & i \in S(t)^c \text{ s.t. } -\frac{\alpha_i(t)}{\beta_i(t)} \leq \bar{q}_i(t) \leq \frac{\alpha_i(t)}{\beta_i(t)}, \\ \frac{1}{2}\left(\frac{\alpha_i(t)}{\beta_i(t)} + \phi_i(t, \eta(t))\right), & i \in \mathcal{J}(t), \\ 0, & i \in S(t)^c \text{ s.t. } \bar{q}_i(t) \leq -\frac{\alpha_i(t)}{\beta_i(t)}, \end{cases}$$

   and stop.

---

^{14}Index i_0 is well defined because at this stage of the algorithm, the index that has just been removed from set $\mathcal{J}(t)$ remains in set $S(t)$.

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$$f_N^c(u_N) - f_N^c(0) - \phi_N(\eta, t) + \frac{\alpha_N(t)}{\beta_N(t)}$$

$$f_3^c(u_3) - f_3^c(0)$$

$$\eta(t)$$

$$\bar{q}_1(t) - f_1^c(0) \qquad 0 \qquad \frac{\alpha_2(t)}{\beta_2(t)} - f_2^c(0) \qquad \bar{q}_3(t) - f_3^c(0) \qquad \cdots \qquad \frac{\alpha_N(t)}{\beta_N(t)} - f_N^c(0)$$

$$I$$
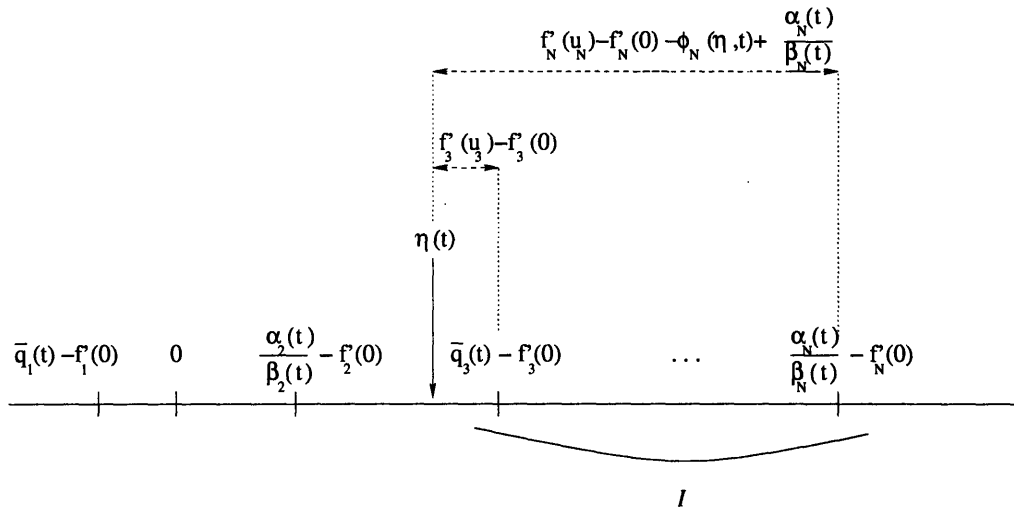
Figure 3-3: Algorithm $\mathcal{A}_2(t)$: Example with $\eta > 0$, when products $3, \ldots N$ are active at time t. In particular, products 1 and 3 are unconstrained and products 2 and N are constrained.

In what follows we prove that algorithm $\mathcal{A}_2(t)$ finds an optimal policy at time t for problem (3.1). We will show that algorithm $\mathcal{A}_2(t)$ has a unique solution.

Theorem 5. *Assume the following is known at a fixed time t:*

- *the information of whether each product i is constrained or unconstrained,*

- *t_i^1 the last exit time for each unconstrained product i (defined as zero if product i is on its first unconstrained interval),*

- *the value $q_i(t_i^1)$ of the adjoint variable q_i at time t_i^1 for each unconstrained product i.*

Then under Assumptions 1, 2, 3 and 5, there exists a unique optimal policy for problem (3.1) at time t computed through algorithm $\mathcal{A}_2(t)$.

In order to prove this result, we first need to show some preliminary results that hold under Assumptions 1, 2, 3 and 5, and that connect the multipliers with the notions of unconstrained and active products.

Lemma 4. *For each unconstrained product i at time t with last exit time t_i^1 (possibly 0), the following equality holds:*

$$q_i(t) + \rho_i(t) = q_i(t) = q_i(t_i^1) + \int_{t_i^1}^t h_i(s)ds.$$

Proof. On the current unconstrained interval (including t, starting at t_i^1), the inventory level for product i is positive, therefore by complementary slackness, $\rho_i(.)$ takes value 0 on that interval. Moreover, the adjoint equation (which holds everywhere except at entry times[15]) gives $\dot{q}_i(s) = h_i(s)$. Since $q_i(.)$ is continuous at exit time t_i^1 implies in particular that this differential equation is valid on $[t_i^1, t]$, which gives rise to the result. $\square$

Corollary 3. *For each unconstrained product i at time t with last exit time t_i^1 (possibly zero),*

$$u_i^*(t) = \begin{cases} 0 & \text{if } q_i(t) - \eta(t) < f_i'(0), \\ f_i'^{-1}(q_i(t) - \eta(t)) & \text{otherwise,} \end{cases}$$

$$p_i^*(t) = \begin{cases} 0 & \text{if } q_i(t) \leq -\frac{\alpha_i(t)}{\beta_i(t)}, \\ \frac{\alpha_i(t)}{2\beta_i(t)} + \frac{1}{2}q_i(t) & \text{if } -\frac{\alpha_i(t)}{\beta_i(t)} \leq q_i(t) \leq \frac{\alpha_i(t)}{\beta_i(t)}, \\ \frac{\alpha_i(t)}{\beta_i(t)} & \text{if } q_i(t) \geq \frac{\alpha_i(t)}{\beta_i(t)}, \end{cases}$$

where $q_i(t) \equiv q_i(t_i^1) + \int_{t_i^1}^t h_i(s)ds$.

Lemma 5. *If $\eta(t) = 0$, then for each constrained product i at time t,*

$$q_i(t) + \rho_i(t) = \psi_i(t) = \phi_i(t, 0) > f_i'(0).$$

Proof. We consider the products i that have a zero inventory level at time t. The condition $\dot{I}_i(s) = 0$ must hold on the interior of the current constrained interval (which includes time t) since the inventory level remains at the value 0. Therefore, on that interval,

$$u_i^*(s) - \alpha_i(s) + \beta_i(s)p_i^*(s) = 0, \tag{3.20}$$

---

[15]When writing the necessary conditions for optimality, by convention, the adjoint variables may be discontinuous only at entry times of constrained intervals.

equality which holds in particular at time t. We suppose $\eta(t) = 0$. If i is an inactive product at time t, then $u_i^*(t) = 0$. This is equivalent to $q_i(t) + \rho_i(t) \leq f_i'(0)$, using (3.16). In that case, (3.20) gives rise to $p_i^*(t) = \frac{\alpha_i(t)}{\beta_i(t)}$, which implies, using (3.15), that $q_i(t) + \rho_i(t) \geq \frac{\alpha_i(t)}{\beta_i(t)}$. This contradicts Assumption 2.

Therefore, all constrained products must be active, implying further that

$$q_i(t) + \rho_i(t) > f_i'(0) \qquad (3.21)$$

since from (3.16), $u_i^*(t) = {f_i'}^{-1}(q_i(t) + \rho_i(t)) > 0$. Relation (3.20) then implies that $-\alpha_i(t) + \beta_i(t)p_i^*(t) < 0$. Consequently, relations (3.15) and (3.21) give rise to

$$p_i^*(t) = \frac{1}{2}\left(\frac{\alpha_i(t)}{\beta_i(t)} + q_i(t) + \rho_i(t)\right).$$

As a result, (3.20) can be rewritten as:

$${f_i'}^{-1}(q_i(t) + \rho_i(t)) + \frac{1}{2}\Big(-\alpha_i(t) + \beta_i(t)(q_i(t) + \rho_i(t))\Big) = 0. \qquad (3.22)$$

Recalling the definition of $\phi_i(t, \eta(t)) > f_i'(0)$ as the unique solution of the equation

$${f_i'}^{-1}\big(q_i(t) + \rho_i(t) - \eta(t)\big) + \frac{1}{2}\Big(-\alpha_i(t) + \beta_i(t)\big(q_i(t) + \rho_i(t)\big)\Big) = 0$$

gives rise to the result. $\qquad \square$

Corollary 4. *If $\eta(t) = 0$, then each constrained product i at time t is active and*

$$u_i^*(t) = {f_i'}^{-1}(\psi_i(t)) = \frac{1}{2}\Big(\alpha_i(t) - \beta_i(t)\psi_i(t)\Big),$$

$$p_i^*(t) = \frac{1}{2}\Big(\frac{\alpha_i(t)}{\beta_i(t)} + \psi_i(t)\Big).$$

Lemma 6. *If $\eta(t) > 0$, then each constrained product i at time t is active if and only if*

$$\frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0) > \eta(t).$$

Proof. For any constrained product i at time t, as we saw in the proof of Lemma 2, equality (3.20), i.e. $u_i^*(t) - \alpha_i(t) + \beta_i(t)p_i^*(t) = 0$, holds.

First, we assume that product i is constrained and inactive at time t.

Then using (3.16), $q_i(t) + \rho_i(t) - \eta(t) \le f_i'(0)$. Since an inactive product is defined by $u_i^*(t) = 0$, (3.20) implies that $p_i^*(t) = \frac{\alpha_i(t)}{\beta_i(t)}$. Relation (3.15) leads to $q_i(t) + \rho_i(t) \ge \frac{\alpha_i(t)}{\beta_i(t)}$. Combining these inequalities, we get

$$\frac{\alpha_i(t)}{\beta_i(t)} - \eta(t) \le q_i(t) + \rho_i(t) - \eta(t) \le f_i'(0)$$

and therefore $\frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0) \le \eta(t)$.

For the converse, assume product i is constrained and active at time t.

Then (3.16) implies that $q_i(t) + \rho_i(t) - \eta(t) > f_i'(0)$ and $u_i^*(t) = f_i'^{-1}(q_i(t) + \rho_i(t) - \eta(t)) > 0$. To satisfy (3.20), we need $-\alpha_i(t) + \beta_i(t)p_i^*(t) < 0$. This implies that $q_i(t) + \rho_i(t) < \frac{\alpha_i(t)}{\beta_i(t)}$ (see (3.15)).

Moreover, since $q_i(t) + \rho_i(t) - \eta(t) > f_i'(0)$ we obtain

$$\frac{\alpha_i(t)}{\beta_i(t)} - \eta(t) > q_i(t) + \rho_i(t) - \eta(t) > f_i'(0)$$

which implies $\frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0) > \eta(t)$. $\qquad\square$

Corollary 5. *If* $\eta(t) \ge \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0)$ (> 0) *and product i is constrained at time t, then*

$$u_i^*(t) = 0; \quad p_i^*(t) = \frac{\alpha_i(t)}{\beta_i(t)}.$$

Lemma 7. *If* $0 < \eta(t) < \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0)$ *and product i is constrained at time t, then*

$$u_i^*(t) = f_i'^{-1}(\phi_i(t, \eta(t)) - \eta(t)) = \frac{1}{2}\big(\alpha_i(t) - \beta_i(t)\phi_i(t, \eta(t))\big);$$

$$p_i^*(t) = \frac{1}{2}\Big(\frac{\alpha_i(t)}{\beta_i(t)} + \phi_i(t, \eta(t))\Big).$$

Proof. We saw in the proof of Lemma 3 that $q_i(t) + \rho_i(t) - \eta(t) > f_i'(0) \ge 0$, which implies $q_i(t) + \rho_i(t) > 0$. Also, $q_i(t) + \rho_i(t) < \frac{\alpha_i(t)}{\beta_i(t)}$ holds.

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As a result, (3.15) implies that $p_i^*(t) = \frac{1}{2}\left(\frac{\alpha_i(t)}{\beta_i(t)} + q_i(t) + \rho_i(t)\right)$. Relation (3.20) can then be rewritten as

$$f_i'^{-1}(q_i(t) + \rho_i(t) - \eta(t)) + \frac{1}{2}\left(-\alpha_i(t) + \beta_i(t)(q_i(t) + \rho_i(t))\right) = 0.$$

From the definition of $\phi_i(.,.)$, we obtain $q_i(t) + \rho_i(t) = \phi_i(t, \eta(t))$. $\qquad\qquad \square$

Proposition 5. *If a product i enters (resp. exits) a constrained interval, it does so as an active product.*

Proof. Suppose product i is unconstrained on some interval $[\tau - \delta, \tau)$, and is constrained inactive on $[\tau, \tau + \delta']$, where δ, $\delta' > 0$. Since τ is the time when product i becomes constrained, we assume without loss of generality that δ is small enough so that we have $\dot{I}_i(t) < 0$ on $[\tau - \delta, \tau)$. Since we supposed that i is inactive constrained on $[\tau, \tau + \delta']$, we have $p_i^*(t) = \frac{\alpha_i(t)}{\beta_i(t)}$ on that interval (using (3.20)) and thus $q_i(t) + \rho_i(t) > \frac{\alpha_i(t)}{\beta_i(t)}$ on $[\tau, \tau + \delta']$. Continuity of $q_i + \rho_i$ implies that, for δ small enough, $q_i(t) + \rho_i(t) = q_i(t) \geq \frac{\alpha_i(t)}{\beta_i(t)}$ on $[\tau - \delta, \tau)$. As a result, we have on $[\tau - \delta, \tau)$:

- $\frac{\alpha_i(t)}{\beta_i(t)} \leq q_i(t) \leq f_i'(0) + \eta(t)$ if i is inactive, in which case expression (3.17) implies $\dot{I}_i(t) = 0$ and leads to a contradiction, and

- $q_i(t) \geq \max\{\frac{\alpha_i(t)}{\beta_i(t)}; f_i'(0) + \eta(t)\}$ if i is active, in which case expression (3.17) implies
  $$\dot{I}_i(t) = f_i'^{-1}(q_i(t) - \eta(t)) \geq 0 \text{ and leads to a contradiction as well.}$$

Therefore i is active on $[\tau, \tau + \delta']$.

The proof for exiting a constrained interval is similar. $\qquad\qquad \square$

Corollary 6. *A necessary condition for an unconstrained product i to enter (resp. exit) a constrained interval at time τ is*

$$\lim_{t \to \tau^-} q_i(t) = \phi_i(\tau, \eta(\tau))$$

$$\left(\text{resp. } \lim_{t \to \tau^+} q_i(t) = \phi_i(\tau, \eta(\tau)).\right)$$

Remark.

This result implies that a constrained interval begins and ends as active. While active, $q_i(t) + \rho_i(t) = \phi_i(t, \eta(t))$. It is possible that during the course of the constrained interval, the product becomes inactive (if $\eta(t) > \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0)$), in which case the optimal policy is known - but $q_i(t) + \rho_i(t)$ is undetermined. However, we claim that supposing $q_i(t) + \rho_i(t) = \phi_i(t, \eta(t))$ throughout the entire constrained interval, *including the inactive part of it,* leads to the same optimal policy, provided that we can extend the interval on which f_i' is defined and differentiable to negative real numbers, while keeping the assumption that f_i' is strictly increasing on $\mathbb{R}$. (We do not make assumption on whether f_i' is lower bounded.)

This will allow to define $l_{i,t}(z)$ on $\mathbb{R}$ instead of $(-\infty, \frac{\alpha_i(t)}{\beta_i(t)}]$ and therefore $l_{i,t}(.)$ has range $\mathbb{R}$ instead of being upper bounded by $\frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0)$. As a result, $l_{i,t}$ is strictly increasing and thus invertible on $\mathbb{R}$ and we can define $\phi_i(t, \eta)$ for $\eta > \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0)$. Therefore, under that assumption, if $\eta(t) > \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0)$ $(= l_{i,t}(\frac{\alpha_i(t)}{\beta_i(t)}))$, then $\phi_i(t, \eta(t)) = l_{i,t}^{-1}(\eta(t)) > \frac{\alpha_i(t)}{\beta_i(t)}$ and therefore $p_i^*(t) = \frac{\alpha_i(t)}{\beta_i(t)}$. Since the product is constrained, $u_i^*(t) = 0$ in order to ensure $\dot{I}_i(t) = 0$.

Therefore, in order to simplify the analysis, we will consider in the remaining of the chapter that $q_i(t) + \rho_i(t) = \phi_i(t, \eta(t))$ on any entire constrained interval, whether the product is active or not.

Assumption 6. *The domain of f_i can be extended to $\mathbb{R}^-$ such that f_i' is strictly increasing and differentiable on $\mathbb{R}$.*

Notice that there is an automatic extension for the case of f_i quadratic since its derivative is linear and can be directly extended to $\mathbb{R}^-$.

We are now ready to proceed with the proof of Theorem 5.

Proof. Set $S(t)$ denotes the set of constrained products at time t.

(a) We first suppose $\eta(t) = 0$.

Using the optimal policy from Corollaries 2 and 3 we will check whether this assumption satisfies the capacity constraint, i.e. if the following inequality holds:

each product i. We need to validate the hypothesis on $\eta(t)$ by checking whether the solution is feasible, in particular if the capacity constraint holds.

$$\sum_{i \notin S(t) \text{ s.t. } q_i(t) \geq f_i'(0)} f_i'^{-1}(q_i(t)) + \frac{1}{2} \sum_{i \in S(t)} (\alpha_i(t) - \beta_i(t)\psi_i(t)) \leq K(t),$$

where $q_i(t) \equiv q_i(t_i^1) + \int_{t_i^1}^t h_i(s)ds$ for $i \notin S(t)$.

If this holds, then since all the conditions are satisfied, $\eta(t)$ is indeed equal to 0 and we found the optimal policy. Otherwise, we must determine the value of $\eta(t) > 0$.

(b) $\eta(t) > 0$.

Now by complementary slackness the capacity constraint must be tight. Using Corollaries 3 and 5 and Lemma 4, the fact that the capacity constraint is binding implies that:

$$\sum_{i \notin S(t) \text{ s.t. } q_i(t) - \eta(t) \geq f_i'(0)} f_i'^{-1}(q_i(t) - \eta(t))$$
$$+ \frac{1}{2} \sum_{i \in S(t) \text{ s.t. } \eta(t) < \frac{\alpha_i(t)}{\beta_i(t)} - f_i'(0)} \left(\alpha_i(t) - \beta_i(t)\phi_i(t, \eta(t)) \right) = K(t),$$

where $q_i(t) \equiv q_i(t_i^1) + \int_{t_i^1}^t h_i(s)ds$ for $i \notin S$. This equation allows us to compute the value of $\eta(t)$.

It is then easy to see that algorithm $\mathcal{A}_2(t)$ gives rise to the optimal solution at time t.

We also notice that we first checked whether or not the capacity constraint could be non tight, and if it cannot be non tight, we then repeatedly try to determine the value of $\eta(t) > 0$ by letting the set of active products decrease at each step, via either the set of active constrained products or the set of active unconstrained products. Since at that point we know that the capacity constraint must be tight, the set $\mathcal{J}(t) \cup \mathcal{J}'(t)$ of active products cannot become empty (at least one product must be active) so the algorithm stops before we have removed all indices from this

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set, which guarantees that the algorithm terminates. □

Notice that this algorithm (that applies for a fixed value of t) terminates in at most $N + 1$ iterations.

Remark.

It follows from this proof that $\eta(.)$ is piecewise differentiable. On an interval of time where the set of active products and the set of constrained products do not change, and the capacity constraint remains tight, its derivative is obtained by differentiating with respect to t the equation providing the value $\eta(t)$:

$$\sum_{i \text{ active, unconstrained}} \frac{h_i(t) - \eta'(t)}{f_i''(f_i'^{-1}(q_i(t) - \eta(t)))}$$
$$+ \frac{1}{2} \sum_{i \text{ active, constrained}} \left(\alpha_i'(t) - \beta_i'(t)\phi_i(t, \eta(t)) - \beta_i(t)\left[\frac{\partial \phi_i}{\partial t}(t, \eta(t)) + \eta'(t)\frac{\partial \phi_i}{\partial \eta}(t, \eta(t))\right] \right)$$
$$= K'(t)$$

where the partial derivatives of $\phi_i(.,.)$ are given in Corollary 2 (clearly, in the interior of an interval where the capacity constraint is not tight, $\eta(t) = \eta'(t) = 0$). By denoting $\mathcal{J}'(t)$ (resp. $\mathcal{J}(t)$) the set of active unconstrained (resp. constrained) products on the considered interval, this leads to $\eta'(t) =$

$$\frac{\sum_{i \in \mathcal{J}'(t)} \frac{h_i(t)}{f_i''(f_i'^{-1}(q_i(t) - \eta(t)))} + \frac{1}{2}\sum_{i \in \mathcal{J}(t)}\left(\alpha_i'(t) - \beta_i'(t)\phi_i(t, \eta(t)) - \beta_i(t)\frac{\partial \phi_i}{\partial t}(t, \eta(t)) \right) - K'(t)}{\sum_{i \in \mathcal{J}'(t)} \frac{1}{f_i''(f_i'^{-1}(q_i(t) - \eta(t)))} + \frac{1}{2}\sum_{i \in \mathcal{J}(t)} \beta_i(t)\frac{\partial \phi_i}{\partial \eta}(t, \eta(t))}.$$

In what follows we execute the algorithm in an example at time $t = 1$.

Example

Let us consider the setting of three products, i.e. $N = 3$, a time horizon $[0, 10]$, i.e.

$T = 10$, and the input data

$$f_1(u) = 2u^2 + u + 0.2 \qquad f_2(u) = u^2 + 2u + 1 \qquad f_3(u) = u^2 + u - 2$$

$$\alpha_1(t) = 1 + 0.5t \qquad \alpha_2(t) = 2 + 0.1t \qquad \alpha_3(t) = 1 + 0.8t$$

$$\beta_1(t) = 0.4 + 0.3t \qquad \beta_2(t) = 0.1 + 0.1t \qquad \beta_3(t) = 0.3 + 0.5t$$

$$h_1(t) = 2t \qquad h_2(t) = 2.5t \qquad h_3(t) = 3t$$

$$K(1) = 13.$$

We assume

$$I_1(1) > 0, \quad t_1^1 = 0.6, \quad q_1(t_1^1) = 2,$$

$$I_2(1) = 0$$

$$I_3(1) = 0.$$

We first compute

$$f_1'(u) = 4u + 1 \qquad f_2'(u) = 2u + 2 \qquad f_3'(u) = 2u + 1$$

$$f_1'^{-1}(u) = \frac{u-1}{4} \qquad f_2'^{-1}(u) = u/2 - 1 \qquad f_3'^{-1}(u) = \frac{u-1}{2}.$$

It is easy to verify that condition $f'(0) < \frac{\alpha_i(t)}{\beta_i(t)}$, $i = 1, 2, 3$, $0 \leq t \leq 10$ holds. In what follows we compute the optimal policy using algorithm $\mathcal{A}_2(1)$.

Step 1: Notice that $S(1) = \{2, 3\}$. We compute

$$\bar{q}_1(1) = 2 + \int_{0.6}^1 2s \, ds = 2 + 1^2 - 0.6^2 = 2.64$$

$$\bar{q}_1(1) - f_1'(0) = 2.64 - 1 = 1.64$$

$$\frac{\alpha_2(1)}{\beta_2(1)} - f_2'(0) = 10.5 - 2 = 8.5$$

$$\frac{\alpha_3(1)}{\beta_3(1)} - f_3'(0)) = 2.25 - 1 = 1.25.$$

We reorder the products as follows: $k_1 = 3$, $k_2 = 1$, $k_3 = 2$. We thus have $S(1) = \{k_1, k_3\}$ and $i_1 = k_1 = 3$, $i_1' = k_2 = 1$.

Step 2:

$$f_2'^{-1}(\phi_2(1,0)) + \tfrac{1}{2}\big(-\alpha_2(1) + \beta_2(1)\phi_2(1,0)\big) = 0$$

$$\Leftrightarrow 0.5\phi_2(1,0) - 1 + 0.5(-2.1 + 0.2\phi_2(1,0)) = 0 \Leftrightarrow \phi_2(1,0) = 41/12 \approx 3.417$$

$$f_3'^{-1}(\phi_3(1,0)) + \tfrac{1}{2}\big(-\alpha_3(1) + \beta_3(1)\phi_3(1,0)\big) = 0$$

$$\Leftrightarrow 0.5\phi_3(1,0) - 0.5 + 0.5(-1.8 + 0.8\phi_3(1,0)) = 0 \Leftrightarrow \phi_3(1,0) = 14/9 \approx 1.556.$$

We now compute

$$\begin{aligned}
&f_1'^{-1}(\bar{q}_1(1)) + 0.5(\alpha_2(1) - \beta_2(1)\phi_2(1,0) + \alpha_3(1) - \beta_3(1)\phi_3(1,0)) \\
&= \frac{2.64 - 1}{4} + 0.5(2.1 - 0.2 \times 3.417 + 1.8 - 0.8 \times 1.556) = 1.396 < K(1).
\end{aligned}$$

Since $\bar{q}_1(1) > \frac{\alpha_1(1)}{\beta_1(1)} \approx 2.143$

$$u_1^*(1) = f_1'^{-1}(\bar{q}_1(1)) = 0.41$$

$$u_2^*(1) = 0.5(\alpha_2(1) - \beta_2(1)\phi_2(1,0)) \approx 0.708$$

$$u_3^*(1) = 0.5(\alpha_3(1) - \beta_3(1)\phi_3(1,0)) \approx 0.278$$

$$p_1^*(1) = \frac{\alpha_1(1)}{\beta_1(1)} \approx 2.143$$

$$p_2^*(1) = 0.5(\tfrac{\alpha_2(1)}{\beta_2(1)} + \phi_2(1,0)) \approx 6.959$$

$$p_3^*(1) = 0.5(\tfrac{\alpha_3(1)}{\beta_3(1)} + \phi_3(1,0)) \approx 1.903.$$

3.3.4 Third step: No external information is given

Finally, in this subsection we also relax the assumption that the system is observable at each point of time. In what follows we will not take an instantaneous approach as we did before. To make our analysis more accessible, we first show the following results, under Assumptions 1, 2, 3 and 5.

Proposition 6. *If a product i has a positive level of inventory at time T, then the inventory level is positive throughout the entire time horizon.*

Proof. Consider a product i that has a positive inventory level at time T. This means by complementary slackness that $\rho_i(T) = 0$. Moreover, using the transversality

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conditions, it follows that $q_i(T) + \rho_i(T) = q_i(T) = 0$. Suppose that the inventory level of that product has reached zero at some point within the time horizon. Let $\tau < T$ be the last exit time from a constrained interval. We have $I_i(\tau) = 0$ and $I_i(t) > 0 \quad \forall t \in (\tau, T]$, therefore $\rho_i(t) = 0 \quad \forall t \in (\tau, T]$. In particular, $\rho_i(\tau^+) = 0$. Then, since $q_i + \rho_i$ is continuous everywhere and since product i is constrained at time τ^-,

$$0 \leq f_i'(0) < \phi_i(\tau^-, \eta(\tau^-)) = q_i(\tau^-) + \rho_i(\tau^-) = q_i(\tau^+) + \rho_i(\tau^+) = q_i(\tau^+)$$

and therefore, using the adjoint equation valid on $[\tau, T]$,

$$q_i(T) = q_i(\tau^+) + \int_\tau^T h_i(s)ds > q_i(\tau^+) > 0.$$

This is a contradiction. $\qquad \square$

We notice that this result makes sense at an intuitive level. There is no reward at the end of the time horizon for any remaining inventory. Moreover, incurring inventory that is not sold incurs cost but not revenue. Therefore, if the retailer follows an optimal pricing and production policy, she will not incur any inventory that will not be sold by time T. As a result, if there is some remaining inventory at time T, it means that this inventory was not incurred by some additional production, but was incurred from the initial inventory. In other words, no production took place throughout the entire time horizon and therefore, since there is some inventory at time T, the inventory level was positive all along.

Corollary 7. *There exists $\bar{I}_i > 0$ defined as $\bar{I}_i \equiv -\int_0^T v_i(t)dt > 0$ such that*

$$I_i^0 > \bar{I}_i \iff \text{product } i \text{ is unconstrained on the entire time horizon,}$$

where

$$0 \geq v_i(t) = \begin{cases} -\alpha_i(t) & \text{if } G_i(t) < -\frac{\alpha_i(t)}{\beta_i(t)} \\ \frac{1}{2}\left(-\alpha_i(t) + \beta_i(t)G_i(t)\right) & \text{if } G_i(t) \geq -\frac{\alpha_i(t)}{\beta_i(t)} \end{cases}$$

$$G_i(t) = - \int_t^T h_i(s) \, ds \leq 0.$$

Proof. If a product i is such that $I_i^*(t) > 0$, $\forall t \in [0, T]$, then $\rho_i(t) = 0$, $\forall t \in [0, T]$ and for this product i there is a unique unconstrained interval, on which the adjoint equation is valid.

Since $q_i(t) = q_i(0) + \int_0^t h_i(s)ds$ and $q_i(T) = 0$, it follows that $q_i(0) = -\int_0^T h_i(s)ds$ and therefore

$$q_i(t) + \rho_i(t) = q_i(t) = - \int_t^T h_i(s)ds \equiv G_i(t) \leq 0 \leq f_i'(0), \quad \forall t \in [0, T].$$

Therefore, $u_i^*(t) = 0$, $\forall t \in [0, T]$ and

$$p_i^*(t) = \begin{cases} 0 & \text{if } G_i(t) < -\frac{\alpha_i(t)}{\beta_i(t)} \\ \frac{1}{2}\left(G_i(t) + \frac{\alpha_i(t)}{\beta_i(t)}\right) & \text{if } G_i(t) \geq -\frac{\alpha_i(t)}{\beta_i(t)}. \end{cases} \tag{3.23}$$

We will denote $\mathcal{P}$ this pricing and production policy on $[0, T]$.

Therefore, since $\dot{I}^*_i(t) = u_i^*(t) - \alpha_i(t) + \beta_i(t)p_i^*(t)$, it follows that

$$\dot{I}_i^*(t) = \begin{cases} -\alpha_i(t) & \text{if } G_i(t) < -\frac{\alpha_i(t)}{\beta_i(t)} \\ \frac{1}{2}\left(-\alpha_i(t) + \beta_i(t)G_i(t)\right) & \text{if } G_i(t) \geq -\frac{\alpha_i(t)}{\beta_i(t)} \end{cases}$$

i.e. $\dot{I}_i^*(t) = v_i(t)$. Moreover,

$$0 < I_i^*(T) = I_i^0 + \int_0^T \dot{I}_i^*(t)dt \equiv I_i^0 - \bar{I}_i.$$

For the converse, suppose $I_i^0 > \bar{I}_i$ and that the inventory level reaches zero within the time horizon. Let $\tau < T$ the first time the inventory level becomes equal to zero. On the first unconstrained interval $[0, \tau)$, since $I_i^*(t) > 0$, by complementary slackness $\rho_i(t) = 0$, $\forall t \in [0, \tau)$. The adjoint equation then implies

$$q_i(t) = q_i(\tau^-) - \int_t^T h_i(s)ds \quad \forall t \in [0, \tau).$$

The adjoint variable $q_i(.)$ may be discontinuous at entry time τ, but $(q_i + \rho_i)(.)$ is

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continuous and in particular at the entry to the constrained interval, $(q_i + \rho_i)(\tau) = \phi_i(\tau, \eta(\tau)) > 0$.

Continuity of $(q_i + \rho_i)(.)$ along with the fact that $\rho_i(\tau^-) = 0$ then imply

$$q_i(\tau^-) = (q_i + \rho_i)(\tau^-) = \phi_i(\tau, \eta(\tau)) > 0.$$

Therefore,

$$q_i(t) = q_i(\tau^-) - \int_t^\tau h_i(s)ds > -\int_t^\tau h_i(s)ds > -\int_t^T h_i(s)ds = G_i(t) \quad \forall t \in [0, \tau).$$

As a result, since we notice in expression (3.17) that the derivative of the inventory level on an unconstrained interval is non decreasing with $q_i(t)$, this implies that $\dot{I}_i^*(t) > v_i(t) \quad \forall t \in [0, \tau)$. In other words, in this case the inventory does not decrease as fast on $[0, \tau]$ as with policy $\mathcal{P}$. However, $\bar{I}_i$ represents the total inventory consumed on $[0, T]$ with policy $\mathcal{P}$. Our assumptions imply that $I_i^0 > \bar{I}_i$ was consumed during $[0, \tau]$, which is a contradiction.

Therefore, $I_i^0 > \bar{I}_i$ implies $I_i^*(t) > 0 \quad \forall t \in [0, T]$.

$\square$

Remark.

If $I_i^0 = \bar{I}_i$, then the same policy holds and we obtain $I_i(T) = 0$. (The inventory level reaches zero for the first time at time T, and the optimal strategy is given by policy $\mathcal{P}$.)

This result suggests that there exists for each product a critical value of the initial inventory level above which it is optimal to never produce on the entire time horizon. This critical value depends only on the demand parameters and the holding cost of that product.

This result will also be used in its negative form, i.e. if $I_i^0 < \bar{I}_i$ then the inventory level of product i reaches zero on $[0, T]$, and is at zero level at the end of the time horizon T. We will distinguish two possible cases then:

case a: the inventory level of product i reaches zero for the first time before the end of the time horizon T, i.e. enters a constrained interval of non zero length within the time horizon, and as we proved earlier it is on a constrained interval at the end of the time horizon T

case b: the inventory level of product i reaches zero for the first time at the end of the time horizon T (without entering a constrained interval). Then the product is unconstrained on $[0, T)$, and the initial inventory level is totally consumed by the end of the time horizon T.

We will refer to these two cases in the remaining of the chapter and the description of the algorithm. Note that if $I_i^0 = \bar{I}_i$, the inventory level also reaches zero for the first time at time T (like in case **b**) but the optimal strategy is to idle while in case **b** the optimal strategy will not be to idle in general.

Let $\tilde{\phi}_i(t) \equiv \phi_i(t, \eta(t))$. Since $\eta(.)$ is piecewise differentiable, and $\phi_i(,.,)$ is differentiable with respect to both arguments, then $\tilde{\phi}_i(.)$ is piecewise differentiable. We have

$$\frac{d\tilde{\phi}_i}{dt}(t) = \frac{\partial \phi_i}{\partial t}(t, \eta(t)) + \eta'(t)\frac{\partial \phi_i}{\partial \eta}(t, \eta(t))$$

and we gave the expression of those derivatives earlier in the chapter.

We will call *transitive time* for product i a time such that either $\tilde{\phi}_i(.)$ is differentiable with $\frac{d\tilde{\phi}_i}{dt}(t) \leq h_i(t)$, or, if t is a time where $\eta(.)$ and thus $\tilde{\phi}_i(.)$ is not differentiable, a time such that $\lim_{s \to t^-} \frac{d\tilde{\phi}_i}{dt}(s) \leq h_i(t)$.

Proposition 7. *Product i may enter a constrained interval only at a transitive time.*

This is simply saying that the *transition* from an unconstrained interval to a constrained interval may occur only at a transitive time. Intuitively, this is due to the fact that it may not be possible to optimally maintain the inventory level of a product at zero under any circumstances.

Proof. On a constrained interval, $q_i(t) + \rho_i(t) = \phi_i(t, \eta(t)) = \tilde{\phi}_i(t)$. By taking derivative with respect to t and using the adjoint equation as well as the fact that $\dot{\rho}_i(t) \le 0$, the result follows. $\qquad\square$

Description of the method for determining the optimal policy

In what follows, we describe how to derive the optimal solution. The reader should refer to Section 3.4 for more details.

First we eliminate the products whose initial inventory level is high enough (i.e. higher that $\bar{I}_i$) so that it is optimal to never produce them. The pricing policy is as shown in the proof of Corollary 7. Therefore, without loss of generality, we assume that all products have an initial inventory level low enough so that their inventory level reaches zero by time T.

As we discussed earlier in the chapter, once for all i the sum of the adjoint variables q_i, $i = 1, \ldots, N$,, multipliers ρ_i, $i = 1, \ldots, N$, and η are known, the optimal pricing and production policies p_i^* and u_i^* are easy to compute.

We first ignore the capacity constraint; compute the solution for all products, and subsequently, check whether the capacity constraint is violated within the time horizon if we apply that solution. If it is not, we can stop. Otherwise, we will have to take into account the capacity constraint.

To do so, we solve N single product problems. We proceed as follows for each product i:

Method for a single product i

Step 1: (first unconstrained interval)

If there is a non zero initial inventory level I_i^0, we start on an unconstrained interval. (If there is no initial inventory level, we start on a constrained interval: set $t_i^0 = 0$ and go to Step 2.)

On that unconstrained interval, $\rho_i(t) = 0$ and $q_i(t) + \rho_i(t) = q_i(t)$. Using the adjoint

equation, the value of $q_i(t) + \rho_i(t)$ on that interval can be determined as a function of time t and the initial value of the adjoint variable $q_i^0 \equiv q_i(0)$. Precisely, we have

$$q_i(t) = q_i^0 + \int_0^t h_i(s)ds.$$

Supposing we are in case **a**, this interval ends at the first entry time t_i^0, the time when the product becomes constrained. By continuity of $(q_i + \rho_i)(.)$, we have

$$q_i(t_i^{0-}) = (q_i + \rho_i)(t_i^{0-}) = (q_i + \rho_i)(t_i^{0+}) = \phi_i(t_i^0, 0) = \psi_i(t_i^0).$$

To determine simultaneously q_i^0 and t_i^0, we solve the nonlinear system of equations and an inequality that ensures that the change of inventory on $[0, t_i^0]$ equals $-I_i^0$ and that the adjoint variable intersects for the first time at a transitive time the function $\psi_i(.)$ at time t_i^0.

More specifically, we attempt to solve the following system of two equations for t_i^0 and q_i^0 such that t_i^0 is the smallest positive number satisfying the equations and the inequality:

$$\int_0^{t_i^0} \dot{I}_i(t)dt = -I_i^0$$
$$q_i^0 + \int_0^{t_i^0} h_i(s)ds = \psi_i(t_i^0)$$
$$\psi_i'(t_i^0) \leq h_i(t_i^0)$$

where $\dot{I}_i(t)$ is given by expression (3.17) in which we use $\eta(t) = 0$ and $q_i(t) = q_i^0 + \int_0^t h_i(s)ds$.

If this system has a solution, once we have solved this system, we know the bound of the first unconstrained interval and the expression of $q_i(t) + \rho_i(t) = q_i(t)$ on that interval, and we can calculate the optimal policy.

If this system has no solution, we are in case **b** and we must only determine q_i^0 such

that

$$\int_0^T \dot{I}_i(t)dt = -I_i^0.$$

In particular, $q_i(t)$ does not reach $\psi_i(t)$ on $[0,T]$ in that case. Then $[0,T)$ is unconstrained and we have determined $q_i(t)$ on that interval, so we can calculate the optimal strategy on the entire time horizon.

Step 2: (constrained interval and following unconstrained interval)

On a constrained interval, the trajectory of $q_i + \rho_i$ follows that of ψ_i. In order to determine whether this constrained interval is followed by another unconstrained interval, we will attempt to compute the exit time t_i^1 ($> t_i^0$) of this constrained interval, and the next entry time t_i^2 ($> t_i^1$) (if there is another unconstrained interval, it must be followed by a constrained interval since all products are constrained at time T). If we find no solution we will conclude that product i remains constrained until the end of the time horizon.

We first suppose that there is an unconstrained interval (t_i^1, t_i^2). We have $\rho_i(t) = 0$ and $q_i(t) + \rho_i(t) = q_i(t)$ $\forall t \in (t_i^1, t_i^2)$. Using the adjoint equation, the value of $q_i(t) + \rho_i(t)$ on that interval can be determined as a function of time t and the initial value of the adjoint variable $q_i(t_i^1)$. Using the necessary conditions $\rho_i(t) = 0$ $\forall t \in (t_i^1, t_i^2)$, $(q_i + \rho_i)(t) = \psi_i(t)$ on the constrained interval (t_i^0, t_i^1), and the continuity of $(q_i + \rho_i)(.)$, we obtain

$$q_i(t_i^{1+}) = (q_i + \rho_i)(t_i^{1+}) = (q_i + \rho_i)(t_i^{1-}) = \phi_i(t_i^1, 0) = \psi_i(t_i^1),$$

$$q_i(t_i^{2-}) = (q_i + \rho_i)(t_i^{2-}) = (q_i + \rho_i)(t_i^{2+}) = \phi_i(t_i^2, 0) = \psi_i(t_i^2).$$

We then attempt to solve the nonlinear system of equations that ensures that the change of inventory on $[t_i^1, t_i^2]$ equals zero and that the adjoint variable intersects for the first time on a transitive interval the function ψ_i at time t_i^2.

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More specifically, we want to solve the following system of two equations for t_i^1 and t_i^2 such that $t_i^1 \in [t_i^0, T]$ and t_i^2 is the smallest number in $[t_i^1, T]$ satisfying the equations and the inequality:

$$\int_{t_i^1}^{t_i^2} \dot{I}_i(t)dt = 0$$

$$\psi_i(t_i^1) + \int_{t_i^1}^{t_i^2} h_i(s)ds = \psi_i(t_i^2)$$

$$\psi_i'(t_i^2) \leq h_i(t_i^2)$$

where $\dot{I}_i(t)$ is given by expression (3.17) in which we use $\eta(t) = 0$ and $q_i(t) = \psi_i(t_i^1) + \int_{t_i^1}^{t} h_i(s)ds$.

If we can solve this system, we know the bounds on the constrained interval and the following unconstrained interval. Moreover, we have

$$q_i(t) + \rho_i(t) = \psi_i(t) \quad \forall t \in (t_i^0, t_i^1)$$

$$q_i(t) + \rho_i(t) = \psi_i(t_i^1) + \int_{t_i^1}^{t} h_i(s)ds \quad \forall t \in (t_i^1, t_i^2)$$

which we can calculate and as a result we can obtain the optimal policy on these intervals. We set $t_i^0 \leftarrow t_i^2$ and we repeat Step 2.

If we cannot solve this system we conclude that product i is constrained on $(t_i^0, T]$. We set $t_i^1 = T$ and stop.

We assumed that the number of junction times is finite (see Assumption 5), so that this process iterates only a finite number of times (i.e. there is a finite number of constrained and unconstrained intervals).

Method for multiple products

Once this algorithm has been executed for each product, we compute the aggregated production rate over time and compare it with the capacity rate. If the capacity rate

exceeds the aggregated production rate at all times, the solution obtained for each of the single product cases is optimal in the capacitated setting as well. Otherwise, we proceed as follows.

Re-calculation of the parameters

Let's first assume that we know for each product, the *phase* within which the product lies when the capacity constraint becomes tight at time τ in the capacitated setting, i.e. first unconstrained interval in case **a**, first unconstrained interval in case **b**, subsequent unconstrained interval, or constrained interval (see Assumption 7 for a more formal description). We will discuss below how we can make such an assumption. If a product i is on the first unconstrained interval at time τ, the parameters q_i^0 and t_i^0 (or only q_i^0 if we are in case **b**) must be recomputed taking into account the fact that the capacity constraint becomes active during interval $[0, t_i^0]$. If product i is on a constrained or on a subsequent unconstrained interval at time τ, the corresponding parameters t_i^1 and t_i^2 must be recomputed taking into account the fact that the capacity constraint becomes active during that interval. Furthermore, since time τ depends on the values of these parameters, it must be recomputed as well as soon as not all products are constrained at time τ. (It need not be recomputed if all products are constrained at time τ because of the following reason: time τ is determined the first time the aggregate production rates reach $K(.)$. We showed that on a constrained interval, while the capacity constraint is not tight, a product is active and its production rate depends only on the data of the problem (see Corollary 4). Therefore the time when the capacity constraint becomes tight is independent of all parameters t_i^1, t_i^2 that must be recalculated.)

Those parameters along with time τ are all simultaneously recomputed by solving the nonlinear system of equalities and inequalities that, similarly to the single-product algorithm, includes constraints on the change of inventory, and the condition to enter a constrained interval when $t \mapsto (q_i + \rho_i)(t)$ intersects $t \mapsto \phi_i(t, \eta(t))$ at a transitive time, for each product i. Moreover, we have the additional constraint that the capac-

ity becomes tight at time τ (and possibly, as we explain below, non tight later on, etc.) This system involves multiplier $\eta(.)$ when the capacity constraint is tight, and its derivative in order to determine transitive times.

Derivation of multiplier $\eta(.)$ and its derivative

The derivation of multiplier $\eta(.)$ and its derivative when the capacity constraint is tight is based on the algorithm described in Subsection 5.1. This algorithm takes as inputs multipliers $(q_i + \rho_i)(.)$, $i = 1\ldots, N$, which can be expressed depending on the parameters to be recalculated, the activity of products, and the phase they are on (constrained or unconstrained). The idea is to start by making the assumption that the capacity constraint remains tight after time τ until the end of the time horizon, and that the activity of products is the same as what it is when the capacity constraint becomes tight (i.e. either active or inactive on the rest of the time horizon). However, we notice that the presence of a capacity constraint may lead some products to become inactive at time τ while they were active in the uncapacitated setting, so we will have to first use the procedure described in algorithm $\mathcal{A}_2(\tau)$ to know which products are active at time τ.

Under those assumptions, we can derive $\eta(.)$ and its derivative as a function of all the parameters to recalculate. Therefore we can solve the system that recomputes time τ and the parameters t_i^1, t_i^2 (or q_i^0, t_i^0) for all products. We first do not impose the times to be necessarily smaller than time T (the end of the time horizon).

Checking the assumption on the capacity constraint and activity status of products

Then we check if the assumptions on tightness of capacity and activity of products were violated by time T. In order to check whether the capacity constraint should indeed have remained tight until the end of the time horizon, we check whether the expression giving $\eta(t)$ under all those assumptions takes only non negative values on $[\tau, T]$ (using the recalculated time τ). In order to check whether the assumption that the activity of each product does not change, we check whether $q_i(t)+\rho_i(t)-\eta(t)-f_i'(0)$ changes sign.

If there is a violation, we consider only the first one, we make the corresponding correction in our assumptions, and solve the system again under these corrected assumptions. For example, if the first violation is a negative value of the expression giving $\eta(t)$, then we add to the system of equalities and inequalities the assumption that the capacity constraint becomes non tight at a time $\tau' > \tau$ that will have to be determined when solving the corrected system - and we assume capacity remains non tight until time T. This new assumption will change the way $\eta(.)$ and its derivative are calculated, and time τ' is determined by an equality ensuring that $\eta(.)$ reaches value 0 for the first time at time τ'.

If the first violation is $q_i(t) + \rho_i(t) - \eta(t) - f_i'(0)$ becoming negative within an interval where the product was supposed to be active, we add the constraint that this product goes from being active to being inactive on the interval, and the time when this occurs is a variable to be redetermined by the time when $q_i(t) + \rho_i(t) - \eta(t) - f_i'(0)$ becomes negative. Changes in activity also impact the derivation of $\eta(.)$ and its derivative. We treat similarly the case where the first violation is $q_i(t) + \rho_i(t) - \eta(t) - f_i'(0)$ becoming positive for some product assumed to be inactive.

It can be seen at this point that in the most general setting of input parameters as functions of time, the solution algorithm may be extremely complex. In order to simplify the problem, we will make some assumptions that will allow us to reduce the complexity of the computations and simplify the description of the algorithm. These assumptions are not necessary for the general solution approach to be adapted, but they allow significant simplification, while being consistent with most real-life scenarios.

Phase of a product when the capacity constraint becomes tight

We recall that this process relies on the knowledge of which products are constrained when the capacity constraint becomes tight. We recognize that in the capacitated case, the system may tend to produce in advance in order to compensate for a foreseen shortage of production capacity later on. In that case, a product that lies on its second unconstrained interval at the time when the capacity constraint becomes tight

in the uncapacitated case may be constrained or on its first unconstrained interval in the capacitated case. Therefore, we cannot assume in general that the phase of a product is the same as the phase in the uncapacitated case at time τ.

In order to simplify the problem, we assume that one of the following holds:

Assumption 7.
- *The capacity rate is high enough so that at every time τ when the capacity constraint becomes tight in the capacitated setting, the phase of a product (constrained interval, m^{th} unconstrained interval, $m \geq 1$) is the same as the phase it is on in the uncapacitated setting.*

- *The capacity rate is low enough so that the capacity rate is tight from time 0 or becomes tight during the first phase of each product (first unconstrained interval if $I_i^0 > 0$, first constrained interval if $I_i^0 > 0$) and then either remains tight until time T, or until some time τ' such that the capacity is non tight on $[\tau', T]$.*

This assumption is saying that if the capacity is high, there may be multiple changes in the tightness of the capacity constraint, but the capacitated case will not incur critical changes in the phases of products compared with the uncapacitated case. Therefore, we can use the phases observed in the uncapacitated case. Moreover, if the capacity is very low, then the capacity constraint is a hard constraint and it will be tight most of the time, except possibly at the beginning while the system can use the existing initial inventory, and at the end when the approaching end of the time horizon makes it unnecessary to keep producing for more than for keeping the inventory level at zero. In most applications, the second case of this assumption is satisfied, since resources are expensive and it does not make sense to have available capacity that is not necessary. We chose the parameters of the numerical implementation to illustrate that case. In order to determine whether a product that was in case **b** in the uncapacitated setting, is in case **a** in the capacitated setting, we first assume the product is still in case **b** and check whether the system is solvable under that assumption. Otherwise, the product is in case **a**.

Iteration of the process

This process of checking assumptions and correction if necessary is repeated until no more violation of assumption is observed as explained above. Clearly, the number of times we must iterate this process depends on the variability of input parameters such as holding cost, capacity rate, and demand parameters. Therefore, in principle there could be an infinite number of iterations even though we already assumed that the number of junction times is finite (see Assumption 5). We make the assumption that the parameters are *reasonably variable*, in the sense that they are somewhat smooth/steady, so that the capacity constraint does not change status infinitely many times, as well as product activity, and that products do not enter a constrained interval infinitely many times.

Assumption 8. *We assume that the activity status of each product does not change infinitely many times, and that the tightness of the capacity constraint does not change infinitely many times.*

We do not formalize these conditions in order to avoid making too restrictive and unnecessary assumptions. However, it can be seen in the numerical computations section of this chapter that for reasonable inputs that make practical sense, the number of iterations is small and the calculations can be carried out quickly by using a software solving nonlinear systems of equalities and inequalities.

To simplify the problem, we make a seasonality effect assumption as follows:

Assumption 9. *There exist seasonality effects such that on any constrained interval, there exists a time at which all products are constrained.*

This assumption means that the demand is somewhat cyclical with cycles not influenced by the holding cost in ways that differ for the various products. Therefore there are periods at which it is optimal to not hold any inventory for any products. In other words, the mth constrained interval of all products intersect, for any integer $m \geq 1$.

This assumption simplifies the problem because it allows us to solve multiple smaller

systems of equalities and inequalities (one for each *season*) instead of one much bigger system that would involve the entire time horizon. Under this assumption, we need to recompute simultaneously 2 parameters at a time for each product (initial value of q_i and entry time, or exit time and entry time), then we reach a time when all products are constrained, and we iterate to determine the next exit and entry time (if they exist) for all products, that correspond to the next cycle.

Section 3.4 provides a formal description of this algorithm.

Finally, notice that this algorithm computes the unique optimal control policy. This follows since earlier in the chapter we had illustrated both the existence and uniqueness of the optimal policy through the necessary conditions for optimality. We therefore have the following theorem.

Theorem 6. *Under Assumptions 1-9, the algorithm described in detail in Section 3.4 computes the unique optimal control policy for problem (3.1).*

3.4 Algorithm

We recall and introduce some notations:

$\bar{I}_i$ critical value of the initial inventory above which it is optimal to idle for product i and to price according to policy $\mathcal{P}$ on the entire time horizon;

τ first time the capacity constraint becomes tight;

$First_i(t)$ binary variable equal to 1 when product i has an initial inventory level $0 < I_i^0 < \bar{I}_i^0$ and is on the first unconstrained interval at time t, equal to 0 otherwise;

$Const_i(t)$ binary variable equal to 1 when product i is on a constrained interval at time t, equal to 0 otherwise;

t_i^0 entry time on a constrained interval for product i;

t_i^2 subsequent entry time on a constrained interval for product i;

t_i^1 exit time from a constrained interval for product i;

$\mathcal{J}(t)$ set of active constrained products at time t;

$\mathcal{J}'(t)$ set of active unconstrained products at time t;

S set of constrained products when the capacity constraint becomes tight;

$S_1 \subset S^c$ set of products on their first unconstrained interval when the capacity constraint becomes tight;

$S_2 \subset S^c$ set of products on a subsequent unconstrained interval when the capacity constraint becomes tight;

Algorithm for a single product i

Initialization

1. Let

$$G_i(t) = - \int_t^T h_i(s) \, ds, \quad t \in [0, T].$$

Let

$$v_i(t) = \begin{cases} -\alpha_i(t) & \text{if } G_i(t) < -\frac{\alpha_i(t)}{\beta_i(t)} \\ \frac{1}{2}\Big(-\alpha_i(t) + \beta_i(t)G_i(t) \Big) & \text{if } G_i(t) \geq -\frac{\alpha_i(t)}{\beta_i(t)} \end{cases}, \quad t \in [0, T].$$

Let

$$\bar{I}_i = - \int_0^T v_i(s) \, ds.$$

- If $I_i^0 \geq \bar{I}_i$, go to 2.

- If $0 < I_i^0 < \bar{I}_i$, go to 3.

- If $I_i^0 = 0$, go to 8.

2. **Large initial inventory level**

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Let

$$q_i(t) = G_i(t) \quad \forall t \in [0, T];$$

$$\rho_i(t) = 0 \quad \forall t \in [0, T];$$

$$Const_i(t) = 0 \quad \forall t \in [0, T];$$

$$First_i(t) = 1 \quad \forall t \in [0, T].$$

Go to 9.

3. **Small initial inventory level**
   (First unconstrained interval, case a)
   Define

$$q_i(t) = q_i^0 + \int_0^t h_i(s)\, ds$$

$$y_i(t) = \begin{cases} -\alpha_i(t) & \text{if } q_i(t) < -\frac{\alpha_i(t)}{\beta_i(t)} \\ \frac{1}{2}\big(-\alpha_i(t) + \beta_i(t)q_i(t)\big) & \text{if } -\frac{\alpha_i(t)}{\beta_i(t)} \le q_i(t) \le f_i'(0) \\ f_i'^{-1}\big(q_i(t)\big) - \frac{\alpha_i(t)}{2} + \frac{\beta_i(t)}{2}q_i(t) & \text{if } f_i'(0) \le q_i(t) \le \frac{\alpha_i(t)}{\beta_i(t)} \\ f_i'^{-1}\big(q_i(t)\big) & \text{if } q_i(t) > \frac{\alpha_i(t)}{\beta_i(t)} \end{cases}$$

Solve for q_i^0 and t_i^0 (smallest feasible solution) the following nonlinear system:

$$\begin{cases} \psi_i(t_i^0) = q_i(t_i^0) \\ \psi'(t_i^0) \le h_i(t_i^0)) \\ \int_0^{t_i^0} y_i(t)\, dt = -I_i^0 \\ t_i^0 \in (0, T) \end{cases}$$

If there are multiple solutions for t_i^0, choose the smallest solution. Go to 5.

If there is no solution, go to 4.

4. **Small initial inventory level**
   (First unconstrained interval, case b)

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Define

$$q_i(t) \;=\; q_i^0 + \int_0^t h_i(s)\,ds$$

$$y_i(t) \;=\; \begin{cases} -\alpha_i(t) & \text{if } q_i(t) < -\frac{\alpha_i(t)}{\beta_i(t)} \\[2mm] \frac{1}{2}\big(-\alpha_i(t) + \beta_i(t)q_i(t)\big) & \text{if } -\frac{\alpha_i(t)}{\beta_i(t)} \le q_i(t) \le f_i'(0) \\[2mm] {f_i'}^{-1}\big(q_i(t)\big) - \frac{\alpha_i(t)}{2} + \frac{\beta_i(t)}{2}q_i(t) & \text{if } f_i'(0) \le q_i(t) \le \frac{\alpha_i(t)}{\beta_i(t)} \end{cases}$$

(Note that in this case, $q_i(t)$ never reaches $\psi_i(t)$ which lies in $[{f_i'}^{-1}(0), \frac{\alpha_i(t)}{\beta_i(t)}]$, so $q_i(t)$ can never be greater than $\frac{\alpha_i(t)}{\beta_i(t)}$.)

Solve for q_i^0 the following equation:

$$\int_0^T y_i(t)\,dt = -I_i^0.$$

Let

$$q_i(t) \;=\; q_i^0 + \int_0^t h_i(s)\,ds \quad \forall t \in [0,T];$$

$$\rho_i(t) \;=\; 0 \quad \forall t \in [0,T];$$

$$Const_i(t) \;=\; 0 \quad \forall t \in [0,T];$$

$$First_i(t) \;=\; 1 \quad \forall t \in [0,T].$$

Go to 9.

5. Let $q_i(t)$ as given above, $First_i(t) = 1$, $Const_i(t) = 0$ and $\rho_i(t) = 0$ on $[0, t_i^0]$.
   Go to 6.

6. **(Next intervals)**
   Define

$$q_i(t) \;=\; \psi_i(t_i^1) + \int_{t_i^1}^t h_i(s)\,ds$$

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$$\bar{y}_i(t) = \begin{cases} -\alpha_i(t) & \text{if } q_i(t) < -\frac{\alpha_i(t)}{\beta_i(t)} \\ \frac{1}{2}\Big(-\alpha_i(t) + \beta_i(t)q_i(t)\Big) & \text{if } -\frac{\alpha_i(t)}{\beta_i(t)} \le q_i(t) \le f_i'(0) \\ f_i'^{-1}\Big(q_i(t)\Big) - \frac{\alpha_i(t)}{2} + \frac{\beta_i(t)}{2}q_i(t) & \text{if } f_i'(0) \le q_i(t) \le \frac{\alpha_i(t)}{\beta_i(t)} \\ f_i'^{-1}\Big(q_i(t)\Big) & \text{if } q_i(t) > \frac{\alpha_i(t)}{\beta_i(t)} \end{cases}$$

Solve for t_i^1 and t_i^2 (smallest feasible solutions) the following nonlinear system:

$$\begin{cases} \psi_i(t_i^2) = q_i(t_i^2) \\ \psi_i'(t_i^2) \le h_i(t_i^2) \\ \int_{t_i^1}^{t_i^2} \bar{y}_i(t) \, dt = 0 \\ t_i^0 \le t_i^1 < t_i^2 \end{cases}$$

If there are multiple solutions, choose the smallest one (giving priority to t_i^2).
If there is no solution such that $t_i^2 < T$, let $t_i^1 = t_i^2 = T$ and go to 9. If there is
a solution such that $t_i^2 < T$, go to 7.

7. Let

$$\begin{aligned} (q_i + \rho_i)(t) &= \psi_i(t) \ \ \forall t \in [t_i^0, t_i^1] \\ Const_i(t) &= 0 \ \ \forall t \in [t_i^0, t_i^1] \\ First_i(t) &= 1 \ \ \forall t \in [t_i^0, t_i^1] \\ \rho_i(t) &= 0 \ \ \forall t \in [t_i^1, t_i^2] \\ q_i(t) & \quad \text{as described in step 6} \ \ \forall t \in [t_i^1, t_i^2] \\ Const_i(t) &= 0 \ \ \forall t \in [t_i^1, t_i^2] \\ First_i(t) &= 0 \ \ \forall t \in [t_i^1, t_i^2] \end{aligned}$$

Do $t_i^0 \leftarrow t_i^2$; go to 6.

8. **Zero initial inventory level**

Let $t_i^0 = 0$; go to 6.

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9. **Final step** Let

$$
p_i(t) = \begin{cases} 0 & \text{if } q_i(t) + \rho_i(t) < -\frac{\alpha_i(t)}{\beta_i(t)} \\ \frac{1}{2}\left(q_i(t) + \rho_i(t) + \frac{\alpha_i(t)}{\beta_i(t)}\right) & \text{if } -\frac{\alpha_i(t)}{\beta_i(t)} \leq q_i(t) + \rho_i(t) \leq \frac{\alpha_i(t)}{\beta_i(t)} \quad \forall t \in [0, T] \\ \frac{\alpha_i(t)}{\beta_i(t)} & \text{if } q_i(t) + \rho_i(t) > \frac{\alpha_i(t)}{\beta_i(t)} \end{cases}
$$

$$
u_i(t) = \begin{cases} 0 & \text{if } q_i(t) + \rho_i(t) \leq f_i'(0) \\ f_i'^{-1}\big(q_i(t) + \rho_i(t)\big) & \text{if } q_i(t) + \rho_i(t) > f_i'(0). \end{cases} \quad \forall t \in [0, T]
$$

Algorithm for multiple products:

Initialization

1. Do the algorithm for a single product above, for each of the N products. Output $Const_i(.)$, $First_i(.)$, $u_i(.)$, and the successive values of the entry times t_i^0. Remove indices such that $I_i^0 > \bar{I}$, update the value of N as the number of remaining products, and possibly renumber the indices from 1 to the new value of N.

2. Let

$$
\tau = \min\left\{ \min\left\{ t : \sum_{i=1}^{N} u_i(t) = K(t) \right\}; T \right\}
$$

3. If $\tau = T$, stop.

   Otherwise, based on Assumption 6, either the phase of products when the capacity becomes tight is the same in the capacitated case as it was in the uncapacitated case, or all products are in their first phase. In either case, we are able to determine the phase in the capacitated case. (Also, we first assume that products in case **b** remain in that case. If there is no solution under this assumption, we will assume they are in case **a**. To ease the exposition in the following we do not distinguish cases **a** and **b** and provide the description of the algorithm for products in case **a** only. The difference can be easily extended

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from the description in the single product case.)

Let

$$S_1 \equiv \{i: \; First_i(\tau) = 1, \;\; Const_i(\tau) = 0\}$$

$$S_2 \equiv \{i: \; First_i(\tau) = 0, \;\; Const_i(\tau) = 0\}$$

$$S \equiv \{i: \; Const_i(\tau) = 1)\}$$

if the first case of the assumption is satisfied, and let

$$S_1 \equiv \{i: \; I_i^0 > 0\}, \;\; S \equiv \{i: I_i^0 = 0\}, \;\; S_2 = \emptyset$$

if the second case of the assumption is satisfied.

Parameters τ, q_i^0, t_i^0, $i \in S_1$ and t_i^1, t_i^2, $i \in S_2 \bigcup S$ need to be updated simultaneously with the computation of $\eta(t)$, $t \geq \tau$. Note that for $i \in S_2 \bigcup S$, we know the value of the last entry time t_i^0 which is not to be recalculated.

4. We determine $\eta(.)$ along with τ, $q_i^0, t_i^0, i \in S_1$ and t_i^1, t_i^2, $i \in S_2 \bigcup S$ where t_i^0 $(i \in S_1)$, t_i^2 $(i \in S_2 \bigcup S)$ are the smallest solutions such that all of the following holds:

- $\forall i \in S_1$, we have

  - $\tau \in [0, t_i^0]$

  - $q_i(t) = q_i^0 + \int_0^t h_i(s) \, ds$, $\quad \rho_i(t) = 0$, $\quad t \in [0, t_i^0]$

  - $q_i(t_i^0) = \phi_i(t_i^0, \eta(t_i^0))$

  - $\psi_i'(t_i^0) \leq h_i(t_i^0)$

  - $\int_0^{t_i^0} y_i(t) \, dt = -I_i^0$ where

$$
y_i(t) = \begin{cases}
-\alpha_i(t) & \text{if } q_i(t) < -\frac{\alpha_i(t)}{\beta_i(t)} \\[2mm]
\frac{1}{2}\Big(-\alpha_i(t) + \beta_i(t)q_i(t)\Big) & \text{if } -\frac{\alpha_i(t)}{\beta_i(t)} \leq q_i(t) \leq \\[1mm]
& \qquad \min\{\eta(t) + f_i'(0), \frac{\alpha_i(t)}{\beta_i(t)}\} \\[2mm]
f_i'^{-1}\Big(q_i(t) - \eta(t)\Big) - \frac{\alpha_i(t)}{2} + \frac{\beta_i(t)}{2}q_i(t) & \\[1mm]
& \text{if } f_i'(0) + \eta(t) \leq q_i(t) \leq \frac{\alpha_i(t)}{\beta_i(t)} \\[2mm]
0 & \text{if } \frac{\alpha_i(t)}{\beta_i(t)} \leq q_i(t) \leq \eta(t) + f_i'(0) \\[2mm]
f_i'^{-1}\Big(q_i(t) - \eta(t)\Big) & \text{if } q_i(t) > \max\{\frac{\alpha_i(t)}{\beta_i(t)}, \eta(t) + f_i'(0)\}
\end{cases}
$$

- $\forall i \in S_2$, we have

  - $\tau \in [t_i^1, t_i^2]$

  - $q_i(t) + \rho_i(t) = \psi_i(t), \quad t \in [t_i^0, t_i^1]$ (t_i^0 is given, not a variable)

  - $q_i(t) = \psi(t_i^1) + \int_{t_i^1}^{t} h_i(s)ds, \quad \rho_i(t) = 0, \quad t \in [t_i^1, t_i^2]$

  - $q_i(t_i^2) = \phi_i(t_i^2, \eta(t_i^2))$

  - $\lim_{t \to t_i^{2-}} \frac{d\tilde{\phi}_i}{dt}(t) \leq h_i(t_i^2)$

  - $\int_{t_i^1}^{t_i^2} y_i(t)\, dt = 0$ with $y_i(.)$ as above

- $\forall i \in S$, we have

  - $\tau \in [t_i^0, t_i^1]$ (t_i^0 is given, not a variable)

  - $q_i(t) + \rho_i(t) = \psi_i(t), \quad t \in [t_i^0, \tau]$

  - $q_i(t) + \rho_i(t) = \phi_i(t, \eta(t)), \quad t \in [\tau, t_i^1]$

  - $q_i(t) = \phi_i(t_i^1, \eta(t_i^1)) + \int_{t_i^1}^{t} h_i(s)ds, \quad \rho_i(t) = 0, \quad t \in [t_i^1, t_i^2]$

  - $q_i(t_i^2) = \phi_i(t_i^2, \eta(t_i^2))$

  - $\lim_{t \to t_i^{2-}} \frac{d\tilde{\phi}_i}{dt}(t) \leq h_i(t_i^2)$ and

  - $\int_{t_i^1}^{t_i^2} y_i(t)\, dt = 0$

- $\eta(t) = 0$ on $[0, \tau]$,

- $\sum_{i=1}^{N} u_i^*(t)$ reaches $K(t)$ for the first time at time τ, where

$$u_i^*(t) = \begin{cases} 0 & \text{if } q_i(t) + \rho_i(t) < f_i'(0), \\ f_i'^{-1}(q_i(t) + \rho_i(t)) & \text{otherwise.} \end{cases}$$

- Let $\eta(t) = \eta'(t) = 0, \quad t \in [0, \tau)$.

  The set of active products at time τ is determined by the procedure described in algorithm $\mathcal{A}_2(\tau)$ in Section 5.2.

  Then the sets $\mathcal{J}(t)$ (resp $\mathcal{J}'(t)$) of active constrained (resp. unconstrained) products are determined over time by supposing that the activity status remains the same as the activity at time τ and by using the entry and exit times t_i^0 ($i \in S_1$), t_i^2 ($i \in S_2 \bigcup S$) that define whether products are constrained or not.

  We then compute $\eta(t), \quad t \geq \tau$ by solving

$$\sum_{i \in \mathcal{J}'(t)} f_i'^{-1}(q_i(t) - \eta(t)) + \frac{1}{2} \sum_{i \in \mathcal{J}(t)} (\alpha_i(t) - \beta_i(t)\phi_i(t, \eta(t))) = K(t)$$

  and its derivative is

$$\begin{aligned} \eta'(t) \ = \ & \frac{1}{\sum_{i \in \mathcal{J}'(t)} \frac{1}{f_i''(f_i'^{-1}(q_i(t)-\eta(t)))} + \frac{1}{2}\sum_{i \in \mathcal{J}(t)} \beta_i(t)\frac{\partial \phi_i}{\partial \eta}(t, \eta(t))} \\ & \cdot \Bigg(\sum_{i \in \mathcal{J}'(t)} \frac{h_i(t)}{f_i''(f_i'^{-1}(q_i(t) - \eta(t)))} \\ & + \frac{1}{2} \sum_{i \in \mathcal{J}(t)} \Big(\alpha_i'(t) - \beta_i'(t)\phi_i(t, \eta(t)) - \beta_i(t)\frac{\partial \phi_i}{\partial t}(t, \eta(t)) \Big) - K'(t) \Bigg). \end{aligned}$$

- If at a time $t > \tau$, we observe that either $\eta(t)$ takes negative value, or for some product i, $q_i(t) + \rho_i(t) - \eta(t) - f_i'(0)$ changes sign, we consider the first time τ_1 such an event occurs.

  In the former case, we update the system to be solved in step 4 by including time τ_1 as a variable and letting $\eta(t) = 0, \quad t > \tau_1$. Once we have solved

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the system, we check that the aggregated production rates do not exceed the capacity rate at a time greater than τ_1 if they do, we introduce in the system a time τ_2 when the capacity becomes tight again (assuming that the phase of all products is unchanged) and we calculate $\eta(.)$ for $t > \tau_2$; we iterate the process.

In the latter case occurring for some product i, and we update sets $\mathcal{J}(t)$ and $\mathcal{J}'(t)$ for $t > \tau_1$ by either removing index i if $q_i(t) + \rho_i(t) - \eta(t) - f'_i(0)$ became negative, or if it became positive, by adding i to $\mathcal{J}(t)$ when i is constrained, and to $\mathcal{J}'(t)$ when i is unconstrained. We update the system to be solved in step 4 by including time τ_1 as a variable.

Every time we re-run step 4 after modification, we do this step again to check that all assumptions are satisfied.

- We iterate the process to determine the times of further exit and entry times on the next cycle if there is one.

5. **Final step**

Let

$$
p_i(t) = \begin{cases} 0 & \text{if } q_i(t) + \rho_i(t) < -\frac{\alpha_i(t)}{\beta_i(t)} \\ \frac{1}{2}\left(q_i(t) + \rho_i(t) + \frac{\alpha_i(t)}{\beta_i(t)}\right) & \text{if } -\frac{\alpha_i(t)}{\beta_i(t)} \leq q_i(t) + \rho_i(t) \leq \frac{\alpha_i(t)}{\beta_i(t)} \quad \forall t \in [0,T] \\ \frac{\alpha(t)}{\beta_i(t)} & \text{if } q_i(t) + \rho_i(t) > \frac{\alpha_i(t)}{\beta_i(t)} \end{cases}
$$

$$
u_i(t) = \begin{cases} 0 & \text{if } q_i(t) + \rho_i(t) - \eta(t) \leq f'_i(0) \\ f_i'^{-1}\big(q_i(t) + \rho_i(t) - \eta(t)\big) & \text{if } q_i(t) + \rho_i(t) - \eta(t) > f'_i(0) \end{cases} \quad \forall t \in [0,T].
$$

	β	h	γ	I^0
product 1	1	1	10	10
product 2	1	2	20	10

Table 3.1: Choice of input parameters in the deterministic, monopoly setting

3.5 Computational results and insights

3.5.1 Example 1: Impact of a demand peak and of the capacity constraint

Input parameters

In order to illustrate our results, we consider an example with 2 products and 3 different maximum demand scenarios (coefficient $\alpha_i(.)$), on a time horizon $[0, 10]$. In each scenario, we let the capacity take 3 different values chosen as we illustrate below. For each demand scenario we keep the capacity constant throughout the time horizon. For simplicity, and in a similar fashion as in numerical results from the literature (see [40], [64], [68], [94], [95], [120]), we consider coefficients $\beta_i(t)$ (describing the elasticity of the demand with respect to the price) and holding cost coefficients that are constant. We also assume that the production cost is quadratic, that is,

$$f_i(u_i) = \frac{\gamma_i}{2} u_i^2,$$

with coefficients γ_i, $i = 1$, 2, constant.

The inputs chosen are summarized in the following table:

Product 1 has smaller holding and production costs, but both products start with the same initial inventory and their demands have the same sensitivity to price. This is to ease the comparison of results.

In a similar fashion as in the literature, we model the maximal demand (coefficient α) increasing on the first half of the time horizon and decreasing on the second half

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	Scenario 1	Scenario 2	Scenario 3
$\alpha_1(t)$	$30 + 10t - t^2$	$13.33 + 20t - 2t^2$	$30 + 10t - t^2$
$\alpha_2(t)$	$30 + 10t - t^2$	$30 + 10t - t^2$	$13.33 + 20t - 2t^2$

Table 3.2: Scenarios of evolution of parameter $\alpha(t)$

to study the effect of a demand peak in the middle of the time horizon. We will consider 3 scenarios. In all scenarios, the average demand for both products is the same (equal to 46.67). However, the amplitude differs: in scenario 1, $\alpha_1(t)$ and $\alpha_2(t)$ both have an amplitude of 25; in scenario 2, we double the amplitude of $\alpha_2(t)$ only, while in scenario 3, we double the amplitude of $\alpha_1(t)$ only, as shown in the following table:

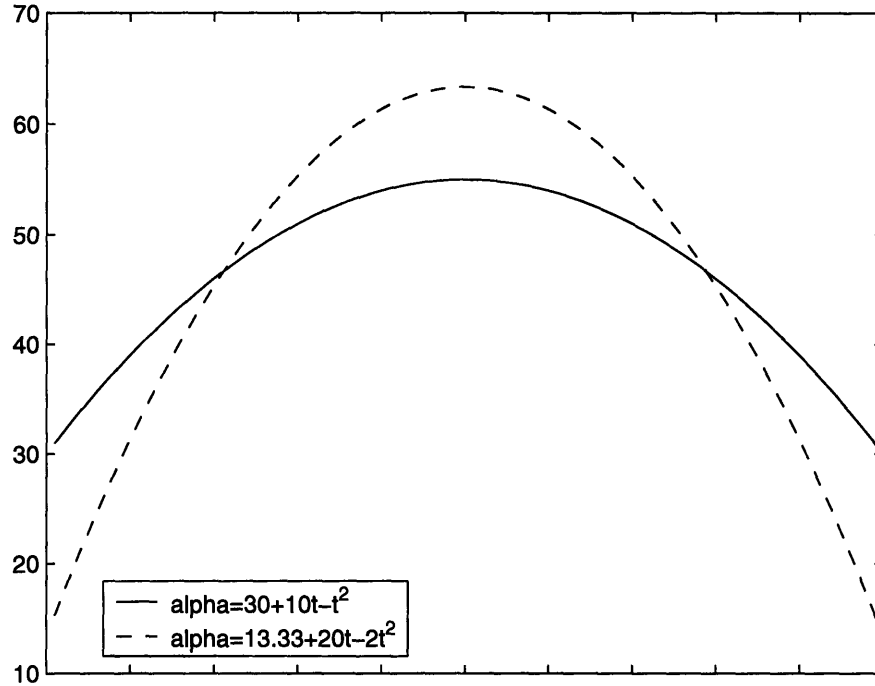The corresponding plots are shown in Figure 3-4.

Figure 3-4: Choices of parameters α

In each scenario, we first compute the optimal pricing and production policy for

both products separately based on the assumption that there is no capacity constraint. We then determine the maximum value of the total production $\max_{t \in [0,T]} u_1(t) + u_2(t) \equiv u^*$. Clearly, if the capacity remains greater than or equal to u^*, the policies obtained are optimal.

Then in each demand scenario, we compute the solution for lower values of the capacity, that is, for a capacity equal to $0.75u^*$ and to $0.5u^*$. In these two cases, the capacity will be binding at least at some point within the time horizon.

Interpretation of the results

The results can be seen in Figures 3-5, 3-6, and 3-7. These figures show the evolution of inventory levels, production rates, and prices for both products in the optimal solution for each scenario, and in each scenario for three values of the capacity as explained above.

We also report the objective value (profit) under each scenario and for each value of the capacity available, as well as the proportion of the total profit generated by product 1.

We observe that in all cases, the system builds up some inventory at the beginning of the time horizon because of the upcoming demand peak, and then maintains a level of inventory at zero for the remaining time. Of course, the lower the capacity, the least the system has the ability to build up inventory.

We observe that the prices increase and production rates decrease when capacity decreases.

We also observe that in Scenario 1, the capacity is tight from the beginning of the time horizon both for capacity levels of $0.75u^*$ and $0.5u^*$, and only in the latter case it is tight over the whole time horizon.

In scenario 2 and 3, in both cases the capacity is tight from the beginning, but is not tight near the end of the time horizon.

Moreover, by comparing the scenarios, we notice that the amplitude of variation for prices increases when the amplitude of the coefficient $\alpha(t)$ increases.

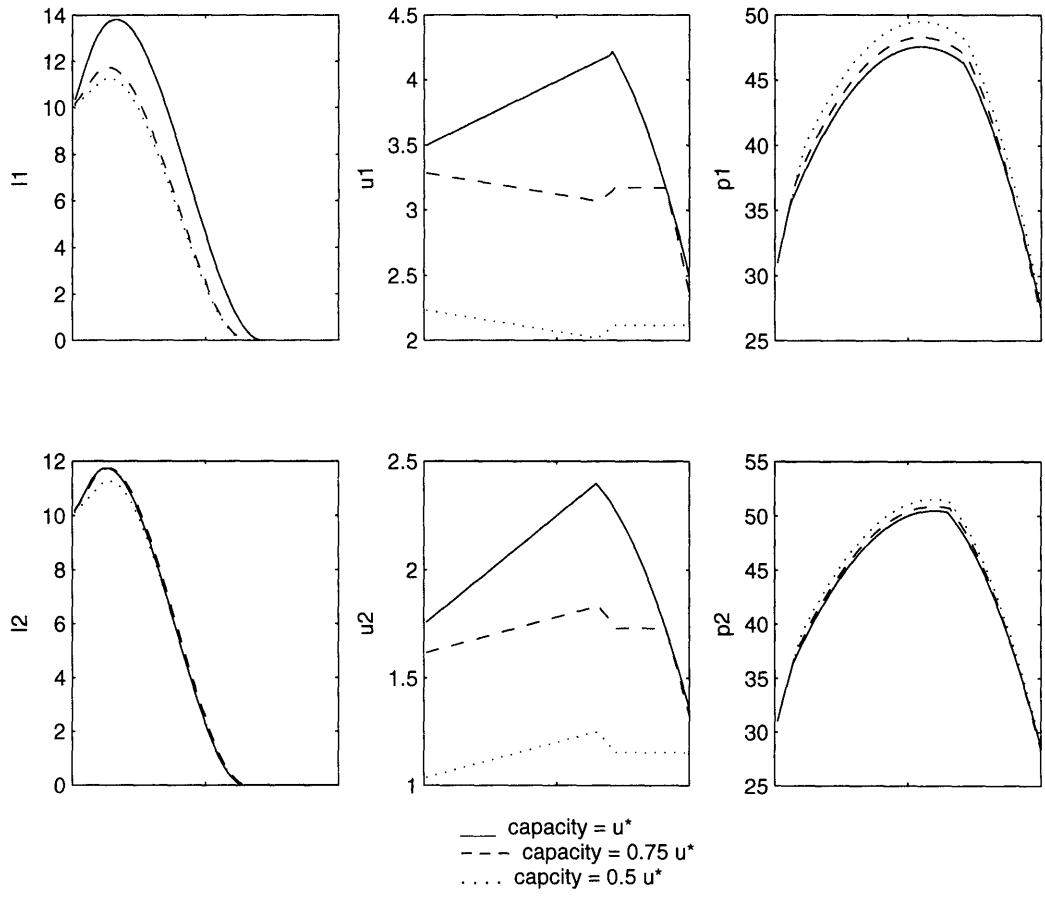Finally, it is worth noticing that in all scenarios, under no capacity constraint, the
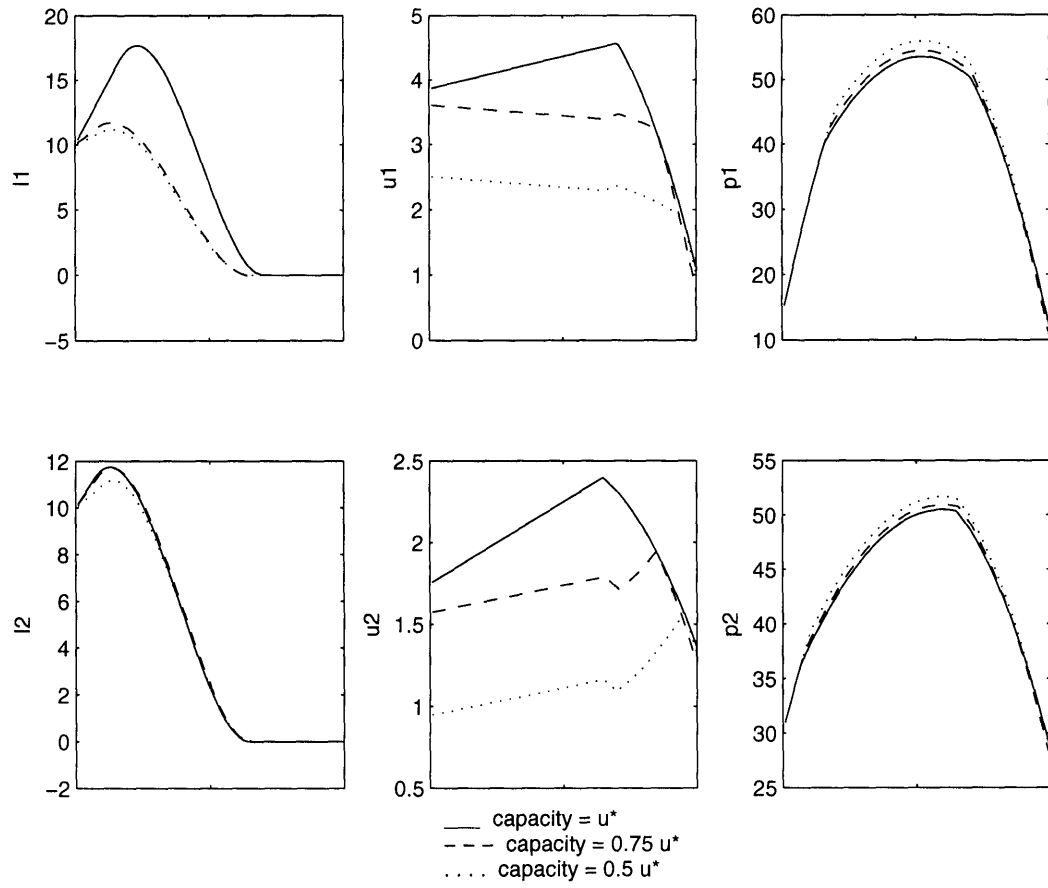
Figure 3-5: Solution for demand scenario 1
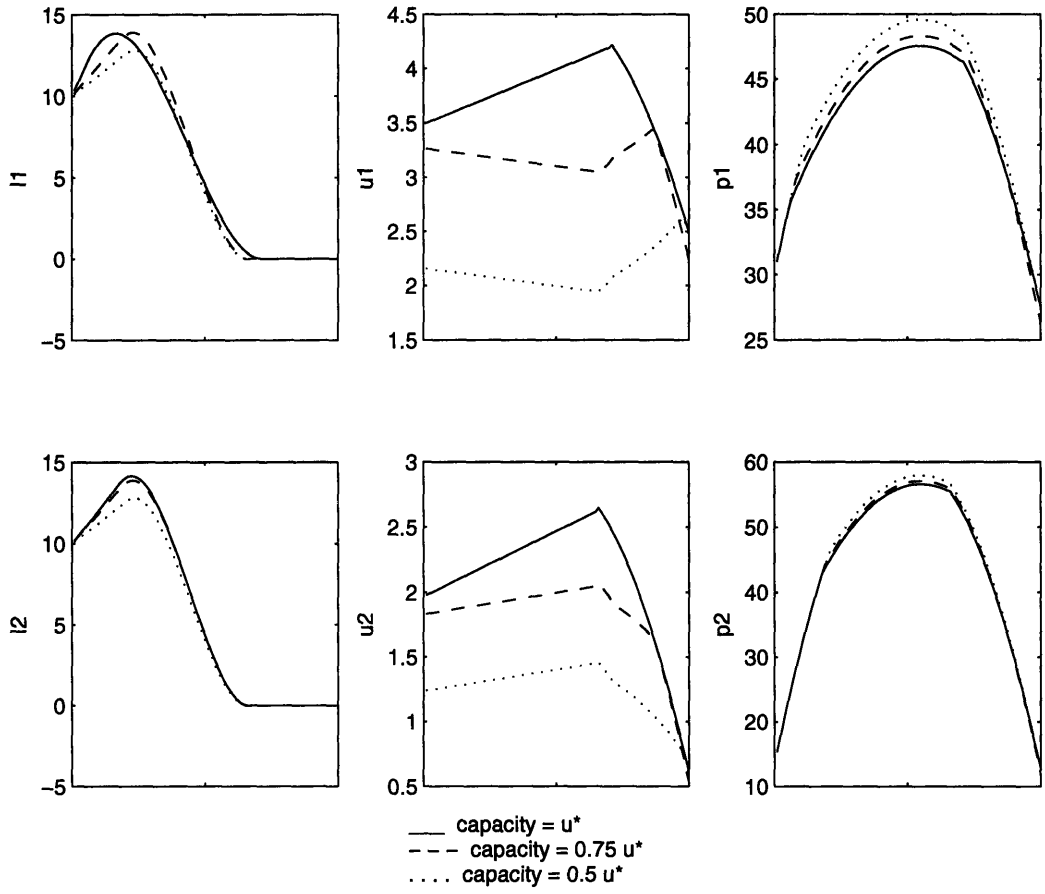
Figure 3-6: Solution for demand scenario 2

Figure 3-7: Solution for demand scenario 3

		Scenario 1	Scenario 2	Scenario 3
		$u^* = 6.5352$	$u^* = 6.9099$	$u^* = 6.7935$
$K = u^*$	π_1	1327.5 (54.6%)	1523.4 (58.01%)	1327.5 (51.24%)
	π_2	1102.8	1102.8	1263.1
	Total profit	2430.3	2626.2	2590.5
$K = 0.75u^*$	π_1	1311.6 (59.79%)	1520.4 (62.78%)	1328.5 (56.46%)
	π_2	881.9	901.5	1024.4
	Total profit	2193.6	2421.9	2352.9
$K = 0.5u^*$	π_1	1160.2 (59.19%)	1435.0 (67.11%)	1156.3 (55.30%)
	π_2	799.9	793.2	934.9
	Total profit	1960.1	2138.3	2091.1

Table 3.3: Numerical results: Profits under different scenarios. π_i denotes the profits due to product i.

production rate for both products increases in the first part of the time horizon (while the inventory level is non zero) since the system attempts to build up some inventory due to the upcoming demand peak. However, under lower capacity, the production rate for product 1 decreases in that phase, while production rate for product 2 keeps increasing (but is lower in all cases). The fact that the system tends to produce more of the less expensive product is quite natural. Therefore, introducing a capacity constraint has more effect on the production for that product. It can also be seen that the level of inventory changes much more for product 1 than for product 2 in the presence of a capacity constraint, with maybe the exception of scenario 3 where a noticeable peak of demand for the expensive product justifies to stock some inventory in the beginning of the time horizon, despite the higher holding and production costs.

We observe that profits decrease as the capacity decreases (when the capacity drops by 25% and 50% respectively, there is a 9.74% and a 19.35% decrease in scenario 1, 7.78% and 18.58% in scenario 2 and 9.17% and 19.28% in scenario 3.

We also notice that the capacity constraint increases the proportion of total profit due to product 1, which is the least expensive product (to hold in inventory and to produce).

When a product has a demand that is more time varying (but with the same aver-

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age), the proportion of profit that product generates is also greater. Also the total profit increases if the demand for one of the products is more varying, compared with demands that are both less varying.

Finally, the maximum demand satisfied under no capacity constraint increases when the demand for one of the products is more varying. This effect is more marked when the demand for product 1 (the cheapest product) is more varying.

To conclude this section, the major insights from the numerical tests we performed are the following:

1. The optimal solution tends to build up some inventory prior to the demand peak (and more so for the cheapest product), and subsequently lets the inventory level remain at zero.

2. As the capacity decreases (i.e. the capacity constraints more the system), inventory levels and production rates tend to decrease, prices tend to increase, and profits decrease.

3. As the capacity decreases, the proportion of profits due to the cheapest product increases.

4. As the capacity decreases, the production rate of the most expensive product decreases less than the other product, while remaining smaller.

5. The shape of the evolution of prices over time is similar to the shape of the evolution of coefficient $\alpha(t)$.

6. As the amplitude of the coefficient α for a product increases, the amplitude of prices increases as well and the proportion of profits this product generates increases. Moreover, the maximal demand satisfied over the time horizon increases.

3.5.2 Example 2: Impact of constant price sensitivities (coefficients $\beta_i(.)$) with a demand peak

We consider the same inputs as above in scenario 1 of coefficients $\alpha_i(.)$ and a capacity level constant and equal to 1, but with 3 different cases of coefficients $\beta_i(.)$, defined by

- $\beta_1(t) = \beta_2(t) = 1, \quad t \in [0, T]$

- $\beta_1(t) = \beta_2(t) = 2, \quad t \in [0, T]$

- $\beta_1(t) = \beta_2(t) = 3, \quad t \in [0, T]$.
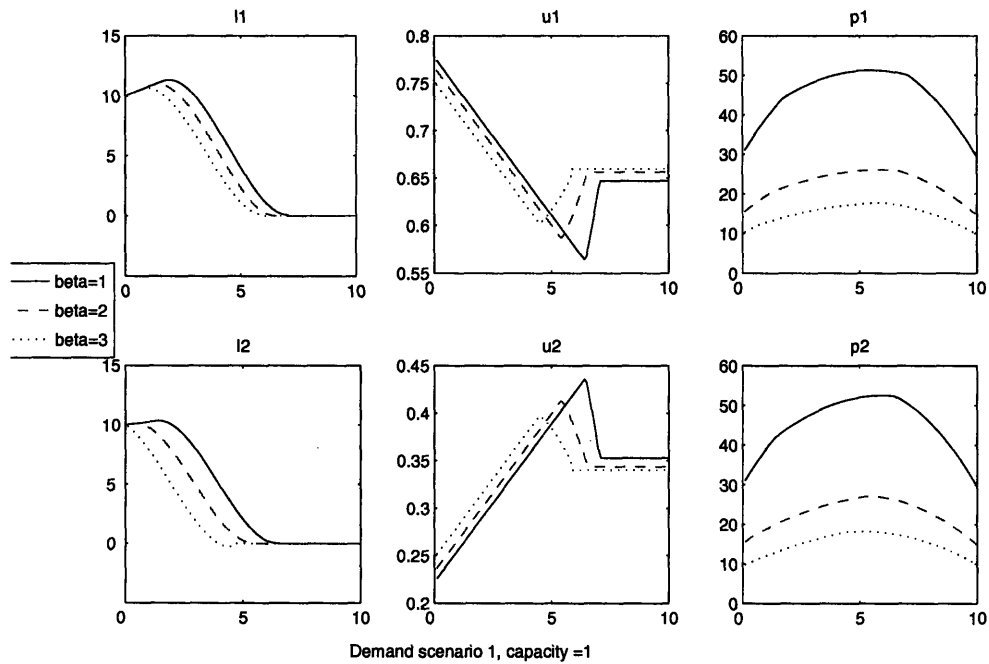
The results are shown in Figure 3-8.

Figure 3-8: Solution for price sensitivities equal to 1

In all three cases the capacity was tight all along the time horizon. We observe that, as suggested by intuition, the prices decrease when the price sensitivities increase, and the inventory levels reach zero earlier. Moreover, the amplitude of prices decrease as well.

3.5.3 Example 3: Impact of time-varying price sensitivities (coefficients $\beta_i(.)$) with a constant maximum demand

We now consider that coefficients $\alpha_i(.)$ are fixed at 15, $t \in [0,T]$ and that the capacity level is constant and equal to 1. We want to study the effect of time varying price sensitivities, both increasing and decreasing. We will run the solution method for

- $\beta_1(t) = \beta_2(t) = 0.5 + 0.1t$

- $\beta_1(t) = \beta_2(t) = 1.5 - 0.1t$.

Price sensitivities that increase with time correspond to products that become less attractive to the customer towards the end of the time horizon, for example products subject to a seasonality effect, or such that there have appeared on the market newer products that can serve as a substitute. Price sensitivities that increase with time correspond to products that become more attractive to the customer towards the end of the time horizon, for example because of a marketing campaign or an appearing trend.

The results are shown in Figure 3-9. The capacity level was tight all along the time horizon (except in one case at the very beginning). Similarly as above, the trend of prices is intuitive: the prices evolve with time in an way opposite to the way the price sensitivities evolve with time. Notice that for decreasing price sensitivities, the products are in case **b**, i.e. the inventory level is positive on $[0,T)$ and reach zero at time T. Indeed, it is optimal to save inventory to be sold at the end of the time horizon when the price sensitivity is lower and the products can be sold at a higher price. This effect is stronger for product 1 which has a lower holding cost. When price sensitivities increase with time, the inventory levels reach zero faster that when
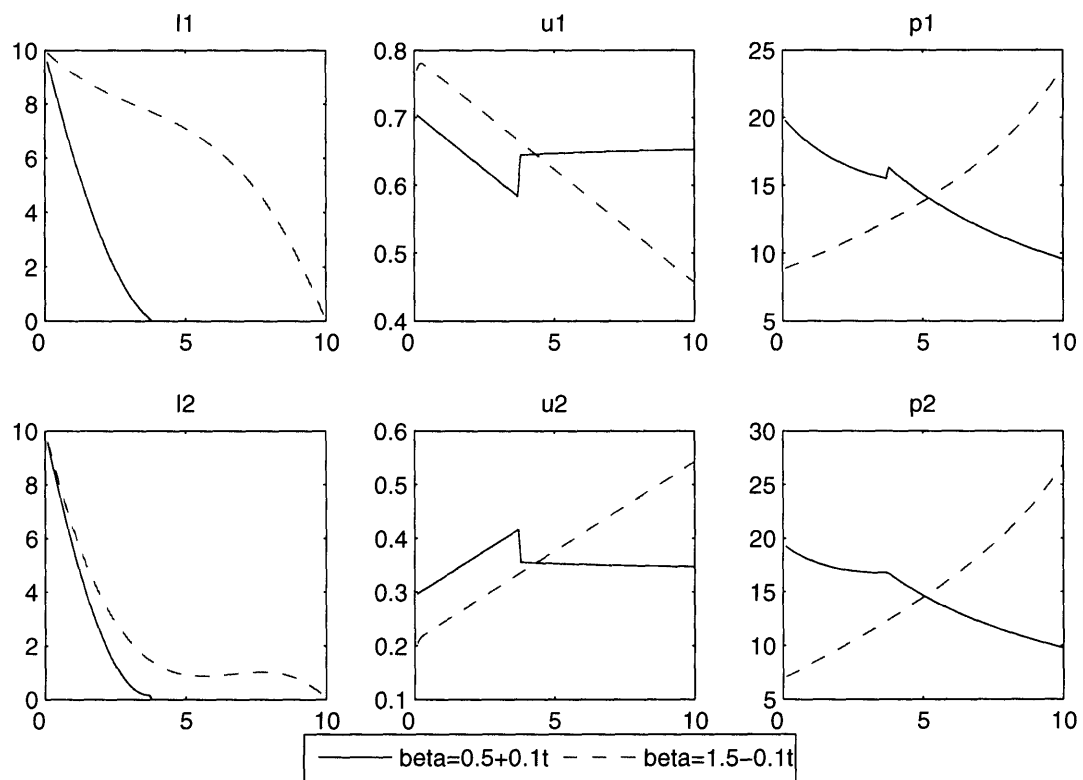
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Figure 3-9: Solution for price sensitivities equal to 1

they were constant since it is optimal to sell all inventory before price sensitivities become too high and the prices are low.

Chapter 4

Uncertain data in a monopoly setting: a robust optimization approach

In this chapter, we consider formulation (3.1) where the demand parameter $\alpha(.)$ is uncertain. The model of uncertainty is detailed in Chapter 2. In particular, we recall that the realization may be written

$$\tilde{\alpha}_i(t) = \alpha_i(t) + z_i(t)\hat{\alpha}_i(t)$$

where the scaled variation $z_i(t)$ is constrained by

$$-1 \leq z_i(t) \leq 1, \quad \int_0^t |z_i(s)|ds \leq \Gamma_i(t) \quad \forall t, i.$$

The set $\mathcal{F}$ is the set of realizations $\tilde{\alpha}(.)$ that satisfy the conditions above.

The results we obtain under this additive model of uncertainty can be generalized to a more general additive and multiplicative model of demand uncertainty. Details are provided in Chapter 5.

In this chapter, we show how to reformulate this problem deterministically. The equivalent deterministic problem is called *robust counterpart*. For reasons detailed in

Chapter 2, we do not maximize the worst-case objective, but the nominal objective, such that the constraints are satisfied for any realization of the parameter within the specified uncertainty set $\mathcal{F}$. However, we provide in Section 4.2 a formulation of the robust counterpart problem under the traditional approach that seeks to maximize the worst-case objective function.

We then show that the solution method detailed in Chapter 3 can be adapted to solve the robust counterpart without increasing the complexity.

4.1 Derivation of the robust counterpart problem

4.1.1 Equivalent deterministic formulation

The problem with uncertainty is the following.

$$\max \quad \int_0^T \sum_{i=1}^N \Big(p_i(t)(\alpha_i(t) - \beta_i(t)p_i(t)) - f_i(u_i(t)) - h_i(t)I_i(t) \Big) dt \qquad (4.1)$$

$$\text{s.t.} \quad \dot{\tilde{I}}_i(t) = u_i(t) - \tilde{\alpha}_i(t) + \beta_i(t)p_i(t), \quad \forall t \in [0,T] \ \ i = 1,\ldots,N \qquad (4.2)$$

$$\dot{I}_i(t) = u_i(t) - \alpha_i(t) + \beta_i(t)p_i(t), \quad \forall t \in [0,T] \ \ i = 1,\ldots,N \qquad (4.3)$$

$$\tilde{I}_i(0) = I_i(0) = I_i^0, \quad i = 1,\ldots,N. \qquad (4.4)$$

$$\tilde{I}_i(t) \geq 0, \quad \forall t \in [0,T] \ \ i = 1,\ldots,N \ \ \forall \tilde{\alpha} \in \mathcal{F} \qquad (4.5)$$

$$p_i(t) \leq \frac{\tilde{\alpha}_i(t)}{\beta_i(t)}, \quad \forall t \in [0,T] \ \ i = 1,\ldots,N \ \ \forall \tilde{\alpha} \in \mathcal{F} \qquad (4.6)$$

$$u_i(t), \ p_i(t) \geq 0, \quad \forall t \in [0,T] \ \ i = 1,\ldots,N$$

$$\sum_{i=1}^N u_i(t) \leq K(t), \quad \forall t \in [0,T]$$

We observe that constraints (4.5) and (4.6) are the ones where uncertainty has an impact, (equation (4.2), along with the initial conditions (4.4), simply defines the realized state variable $\tilde{I}$.)

Observe that the objective involves the *nominal* inventory level, defined by equations (4.3) and (4.4), while the no backorders constraint (4.5) states that *realized* inventory levels must be non negative.

Notice also that the inventory level depends on all previous control decisions. As a result, constraint (4.5) links the time instants together by involving the inventory level. In contrast, constraint (4.6) is separable across time. This will have an impact in the way we derive the robust counterpart in the following sense: a constraint that does not link together time instants needs to be satisfied at each time for the worst realization, while for a constraint that links time, the worst realization may not occur at each time because of the budget constraint involving $\Gamma_i(.)$.

In order to reformulate this problem deterministically, we need, for each constraint where uncertainty is involved, to determine the realization of $\tilde{\alpha}$, that is the "worst-case scenario", i.e. that makes it hardest to satisfy (for example for constraint (4.6), the realization that minimizes the right hand side). Then we will be guaranteed that the constraint is satisfied for any realization.

We start by considering constraint (4.6) for a given product i and at a given time t. Clearly, the worst case is obtained when the numerator is the smallest, i.e. $z_i(t) = -1$. It may be seen that for any given time t and index i, it is possible to find a vector of functions z such that $z_i(t) = -1$, and $\tilde{\alpha} \in \mathcal{F}$. As a result, in the robust counterpart, constraint (4.6) is written as

$$p_i(t) \leq \frac{\alpha_i(t) - \hat{\alpha}_i(t)}{\beta_i(t)} \quad \forall i, \ t.$$

In constraint (4.5), at fixed time t, we seek for the deviation z_i on $[0, t]$ that minimizes $\tilde{I}_i(t)$. We observe that we may write the realized inventory level at time t as follows:

$$
\begin{aligned}
\tilde{I}_i(t) &= I_i^0 + \int_0^t (u_i(s) - \alpha_i(s) - z_i(s)\hat{\alpha}_i(s) + \beta_i(s)p_i(s))ds \\
&= I_i(t) - \int_0^t z_i(s)\hat{\alpha}_i(s)ds,
\end{aligned}
$$

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where

$$I_i(t) = I_i^0 + \int_0^t (u_i(s) - \alpha_i(s) + \beta_i(s)p_i(s))ds$$

is the nominal inventory level.

Therefore, we must seek for the feasible deviation z_i that minimizes

$$-\int_0^t z_i(s)\hat{\alpha}_i(s)ds.$$

Clearly in the optimal solution, $z_i \geq 0$, so we can rewrite this subproblem as follows, for each product i and at any given time t:

$$-J_i(t) = \min_{z_i(.)} \quad -\int_0^t z_i(s)\hat{\alpha}_i(s)ds$$

$$\text{s.t.} \quad \int_0^t z_i(s)ds \leq \Gamma_i(t)$$

$$0 \leq z_i(s) \leq 1 \quad \forall s \in [0,t],$$

where the decision variable is the function $z_i(.)$ over $[0,t]$. Equivalently,

$$-J_i(t) = -\max_{z_i(.)} \quad \int_0^t z_i(s)\hat{\alpha}_i(s)ds$$

$$\text{s.t.} \quad \int_0^t z_i(s)ds \leq \Gamma_i(t)$$

$$0 \leq z_i(s) \leq 1 \quad \forall s \in [0,t].$$

This is a particular instance of a continuous linear program. This class of problems was introduced by Bellman [12], [13] to model some economic processes. A dual formulation for this class of problems was studied by Tyndall [122]. Some results by Tyndall also establish strong duality under some regularity assumptions on the data of the problem. Using these results, we have strong duality, with a dual problem

given by:

$$-J_i(t) = - \min_{\omega_i(t), r_i(.,t)} \quad \omega_i(t)\Gamma_i(t) + \int_0^t r_i(s,t)ds$$

$$\text{s.t.} \quad \omega_i(t) + r_i(s,t) \geq \hat{\alpha}_i(s) \quad \forall s \in [0,t]$$

$$\omega_i(t) \geq 0$$

$$r_i(s,t) \geq 0 \quad \forall s \in [0,t]$$

or equivalently

$$-J_i(t) = \max_{\omega_i(t), r_i(.,t)} \quad -\omega_i(t)\Gamma_i(t) - \int_0^t r_i(s,t)ds \qquad (4.7)$$

$$\text{s.t.} \quad \omega_i(t) + r_i(s,t) \geq \hat{\alpha}_i(s) \quad \forall s \in [0,t]$$

$$\omega_i(t) \geq 0$$

$$r_i(s,t) \geq 0 \quad \forall s \in [0,t].$$

We notice that in this case the primal (and thus the dual) subproblem takes as inputs only the known parameters $\Gamma_i(.)$ and $\hat{\alpha}_i(.)$.

In the robust formulation, strong duality allows us to replace the minimization problem objective (primal subproblems) in the constraint

$$I_i(t) - J_i(t) \geq 0,$$

by its dual maximization subproblem objective:

$$I_i(t) - \omega_i(t)\Gamma_i(t) - \int_0^t r_i(s,t)ds \geq 0.$$

Indeed, at the optimum, the maximum will be realized as it makes the constraint easier to satisfy, therefore, we can simply replace the maximization subproblem by its objective function and integrate the constraints on the dual variables into the constraints of the robust counterpart.

Therefore we obtain the following.

Theorem 7. *The robust counterpart problem for problem (3.1) is:*

$$\max \quad \int_0^T \sum_i \Big(p_i(t)(\alpha_i(t) - \beta_i(t)p_i(t)) - f_i(u_i(t)) - h_i(t)I_i(t) \Big) dt$$

$$s.t. \quad \dot{I}_i(t) = u_i(t) - \alpha_i(t) + \beta_i(t)p_i(t) \quad \forall i \ \forall t \in [0,T]$$

$$I_i(0) = I_i^0 \quad \forall i$$

$$I_i(t) \geq \omega_i(t)\Gamma_i(t) + \int_0^t r_i(s,t)ds \quad \forall i \ \forall t \in [0,T]$$

$$\omega_i(t) + r_i(s,t) \geq \hat{\alpha}_i(s) \quad \forall i \ \forall s \in [0,t] \ \forall t \in [0,T]$$

$$p_i(t) \leq \frac{\alpha_i(t) - \hat{\alpha}_i(t)}{\beta_i(t)} \quad \forall i \ \forall t \in [0,T]$$

$$\sum_{i=1}^N u_i(t) \leq K(t) \quad \forall t \in [0,T]$$

$$p_i(t), \ u_i(t) \geq 0 \quad \forall i \ \forall t \in [0,T]$$

$$\omega_i(t) \geq 0 \quad \forall i \ \forall t \in [0,T]$$

$$r_i(s,t) \geq 0 \quad \forall i \ \forall s \in [0,t] \ \forall t \in [0,T]$$

4.1.2 Solution of the dual subproblem

We denote $(\omega_i^*(t), r_i^*(.,t))$ the optimal solution of problem (4.7).

Case 1 :

If $\omega_i^*(t) = 0$, then $r_i^*(s,t) = \hat{\alpha}_i(s)$, $\forall s \in [0,t]$ and

$$J_i(t) = J_i^1(t) \equiv \int_0^t \hat{\alpha}_i(s)ds.$$

Case 2 :

If $\omega_i^*(t) \geq \sup_{s \in [0,t]} \hat{\alpha}(s)$, then $r_i^*(s,t) = 0$, $\forall s \in [0,t]$ and $J_i(t) = J_i^2(t) \equiv \Gamma(t)\omega_i^*(t)$. Therefore if this case is optimal, $\omega_i^*(t) = \sup_{s \in [0,t]} \hat{\alpha}_i(s)$ and

$$J_i^2(t) = \Gamma(t). \sup_{s \in [0,t]} \hat{\alpha}_i(s).$$

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Note: a necessary condition for Case 2 to be better than Case 1 is $\Gamma(t) < t$.

Case 3 :

If $0 < \omega_i^*(t) < \sup_{s \in [0,t]} \hat{\alpha}_i(s)$, then

$$r_i^*(s,t) = \begin{cases} 0 & \text{if } \omega_i^*(t) \geq \hat{\alpha}_i(s) \\ \hat{\alpha}_i(s) - \omega_i^*(t) & \text{if } \omega_i^*(t) < \hat{\alpha}_i(s) \end{cases} = (\hat{\alpha}_i(s) - \omega_i^*(t))^+$$

and

$$J_i(t) = J_i^3(t) \equiv \Gamma(t)\omega_i^*(t) + \int_0^t (\hat{\alpha}_i(s) - \omega_i^*(t))^+ ds.$$

In other words, by denoting $D_{\omega_i^*(t)}$ the domain

$$D_{\omega_i^*(t)} = \{s \in [0,t] : \hat{\alpha}_i(s) > \omega_i^*(t)\}$$

and $l_{\omega_i^*(t)}$ its measure, then

$$J_i^3(t) = (\Gamma(t) - l_{\omega_i^*(t)})\omega_i^*(t) + \int_{s \in D_{\omega_i^*(t)}} \hat{\alpha}_i(s) ds.$$

As a result, $\omega_i^*(t)$ takes the value in $(0, \sup_{s \in [0,t]} \hat{\alpha}_i(s))$ that minimizes the expression above for $J_i^3(t)$.

Notice that if $\Gamma(t) \geq t$, then

$$\begin{aligned} J_i^1(t) - J_i^3(t)) &= -(\Gamma(t) - l_{\omega_i^*(t)})\omega_i^*(t) + \int_{s \in D_{\omega_i^*(t)}^c} \hat{\alpha}_i(s) ds \\ &< -(\Gamma(t) - l_{\omega_i^*(t)})\omega_i^*(t) + (t - l_{\omega_i^*(t)})\omega_i^*(t) = (t - \Gamma(t))\omega_i^*(t) < 0 \end{aligned}$$

so Case 1 is optimal.

- **Case 3a:** if $\omega_i^*(t) \leq \inf_{s \in [0,t]} \hat{\alpha}(s)$, then

$$J_i^{3a}(t) = (\Gamma(t) - t)\omega_i^*(t) + \int_0^t \hat{\alpha}_i(s) ds.$$

If $\Gamma(t) \geq t$, the best value is $\omega_i^*(t) = 0$ and this leads to Case 1.

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If $\Gamma(t) < t$, the best value is $\omega_i^*(t) = \inf_{s\in[0,t]} \hat{\alpha}_i(s)$ and

$$J_i^{3a}(t) = -(t - \Gamma(t)) \cdot \inf_{s\in[0,t]} \hat{\alpha}_i(s) + \int_0^t \hat{\alpha}_i(s)ds.$$

Clearly then the objective value is lower than in Case 1. Note that depending on actual data, if $\Gamma(t) < t$, either Case 2 or Case 3a may be optimal. Indeed,

$$J_i^2(t) - J_i^{3a}(t) = -\int_0^t \hat{\alpha}_i(s)ds + t\inf_{s\in[0,t]} \hat{\alpha}_i(s) + \Gamma(t)\Big(\sup_{s\in[0,t]} \hat{\alpha}_i(s) - \inf_{s\in[0,t]} \hat{\alpha}_i(s)\Big)$$

which tends to a non positive value as $\Gamma(t) \to 0^+$ and to a non negative value as $\Gamma(t) \to t^-$.

- **Case 3b**: if $\inf_{s\in[0,t]} \hat{\alpha}_i(s) < \omega_i^*(t) < \sup_{s\in[0,t]} \hat{\alpha}_i(s)$: notice that at the extreme points of this range, we are respectively in Case 3a and Case 2. However, it is possible that for some value of $\omega_i^*(t)$ in this range, $J_i^3(t)$ takes an even lower value than in those two other cases.

  In particular, if $\hat{\alpha}_i(.)$ is strictly increasing and differentiable, then $D_{\omega_i^*(t)} = (\hat{\alpha}_i^{-1}(\omega_i^*(t)), t]$, $l_{\omega_i^*(t)} = t - \hat{\alpha}_i^{-1}(\omega_i^*(t))$ and by seeing $J_i^{3b}(t)$ as some function $g(\omega_i^*(t))$, we have

  $$g(\omega_i^*(t)) = (\Gamma(t) - t + \hat{\alpha}_i^{-1}(\omega_i^*(t)))\omega_i^*(t) + \int_{\hat{\alpha}_i^{-1}(\omega_i^*(t))}^t \hat{\alpha}_i(s)ds.$$

We have

$$\begin{aligned}
\frac{\partial g}{\partial \omega_i^*(t)}(\omega_i^*(t)) &= \Gamma(t) - t + \hat{\alpha}_i^{-1}(\omega_i^*(t)) + \frac{\omega_i^*(t)}{\hat{\alpha}_i'(\hat{\alpha}_i^{-1}(\omega_i^*(t)))} - \frac{\hat{\alpha}_i(\hat{\alpha}_i^{-1}(\omega_i^*(t)))}{\hat{\alpha}_i'(\hat{\alpha}_i^{-1}(\omega_i^*(t)))} \\
&= \Gamma(t) - t + \hat{\alpha}_i^{-1}(\omega_i^*(t))
\end{aligned}$$

so $g(\omega_i^*(t))$ reaches a minimum on the considered range if $\Gamma(t) < t$, and then

$$J_i^{3b}(t) = \int_{t-\Gamma(t)}^t \hat{\alpha}_i(s)ds$$

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(note: we verify that this case yields indeed a smaller objective value than Case 3a and Case 2)

Finally, we have to compare which of Cases 1, 2, 3 provides the smallest value of $J_i(t)$.

Note in particular that $J_i(t) \leq J_i^1(t) = \int_0^t \hat{\alpha}_i(s)ds$.

We observe that if $\Gamma(t) \geq t$, then Case 1 is optimal. If $\Gamma(t) < t$, in general, either of the cases may be optimal depending on the data.

In the particular case where $\hat{\alpha}_i(.)$ is strictly increasing and differentiable,

$$
J_i(t) = \begin{cases} \int_{t-\Gamma(t)}^t \hat{\alpha}_i(s)ds & \text{if } \Gamma(t) < t \\ \int_0^t \hat{\alpha}_i(s)ds & \text{else.} \end{cases}
$$

Conclusions

To summarize, in the general case, $J_i(t)$, depending on t, $\Gamma(t)$ and $\hat{\alpha}(.)$, is as follows:

- if $\Gamma(t) \geq t$, then $J_i(t) = \int_0^t \hat{\alpha}_i(s)ds$.

- else, $J_i(t) = \min\{J_i^2(t), J_i^{3a}(t), J_i^{3b}(t)\}$ where

$$
J_i^2(t) = \Gamma(t). \sup_{s \in [0,t]} \hat{\alpha}_i(s)
$$

$$
J_i^{3a}(t) = -(t - \Gamma(t)). \inf_{s \in [0,t]} \hat{\alpha}_i(s) + \int_0^t \hat{\alpha}_i(s)ds
$$

and

$$
J_i^{3b}(t) = \min_{\omega_i^*(t) \in (\inf_{s \in [0,t]} \hat{\alpha}_i(s), \sup_{s \in [0,t]} \hat{\alpha}_i(s))} \left[\Gamma(t)\omega_i^*(t) + \int_0^t (\hat{\alpha}_i(s) - \omega_i^*(t))^+ ds \right].
$$

Notice that this can be rewritten as

$$
J_i(t) = \min_{\omega_i^*(t) \in [\inf_{s \in [0,t]} \hat{\alpha}_i(s), \sup_{s \in [0,t]} \hat{\alpha}_i(s)]} \left[\Gamma(t)\omega_i^*(t) + \int_0^t (\hat{\alpha}_i(s) - \omega_i^*(t))^+ ds \right]
$$

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(the minimum is taken over the closed interval).

4.1.3 Properties

Proposition 8. *The function $J_i(.)$ is non decreasing on $[0, T]$ for all i.*

Proof. We consider the dual problems (P_t) and (P_{t+dt}) respectively at times t and $t + dt$:

$$-J_i(t) = \max_{\omega_i(t), r_i(., t)} \quad -\omega_i(t)\Gamma_i(t) - \int_0^t r_i(s, t)ds$$

$$\text{s.t.} \quad \omega_i(t) + r_i(s, t) \geq \hat{\alpha}_i(s) \quad \forall s \in [0, t]$$

$$\omega_i(t) \geq 0$$

$$r_i(s, t) \geq 0 \quad \forall s \in [0, t],$$

$$-J_i(t + dt) = \max_{\omega_i(t+dt), r_i(., t+dt)} \quad -\omega_i(t + dt)\Gamma_i(t + dt) - \int_0^{t+dt} r_i(s, t + dt)ds$$

$$\text{s.t.} \quad \omega_i(t + dt) + r_i(s, t + dt) \geq \hat{\alpha}_i(s) \quad \forall s \in [0, t + dt]$$

$$\omega_i(t + dt) \geq 0$$

$$r_i(s, t + dt) \geq 0 \quad \forall s \in [0, t + dt].$$

We denote by $(\omega_i^*(t), r_i^*(., t))$, $(\omega_i^*(t+dt), r_i^*(., t+dt))$ the respective optimal solutions. It is clear that $(\omega_i^*(t + dt), r_i^*(., t + dt))$ is feasible for (P_t), therefore, we have

$$-\omega_i^*(t)\Gamma_i(t) - \int_0^t r_i^*(s, t)ds \geq -\omega_i^*(t + dt)\Gamma_i(t) - \int_0^t r_i^*(s, t + dt)ds$$

or equivalently

$$\omega_i^*(t + dt)\Gamma_i(t) + \int_0^t r_i^*(s, t + dt)ds \geq \omega_i^*(t)\Gamma_i(t) + \int_0^t r_i^*(s, t)ds.$$

As a result, we observe that

$$
\begin{aligned}
J_i(t+dt) &= \omega_i^*(t+dt)\Gamma_i(t+dt) + \int_0^{t+dt} r_i^*(s,t+dt)ds \\
&= \omega_i^*(t+dt)\Gamma_i(t) + \omega_i^*(t+dt)\dot{\Gamma}_i(t)dt + \int_0^t r_i^*(s,t+dt)ds \\
&\qquad\qquad\qquad\qquad + \int_t^{t+dt} r_i^*(s,t+dt)ds \\
&\geq \omega_i^*(t)\Gamma_i(t) + \int_0^t r_i^*(s,t)ds + \omega_i^*(t+dt)\dot{\Gamma}_i(t)dt + \int_t^{t+dt} r_i^*(s,t+dt)ds \\
&= J_i(t) + \omega_i^*(t+dt)\dot{\Gamma}_i(t)dt + \int_t^{t+dt} r_i^*(s,t+dt)ds.
\end{aligned}
$$

Since $\dot{\Gamma}_i(t) \geq 0$ by assumption, and for feasibility of (P_{t+dt}), we have $\omega_i^*(t+dt) \geq 0$ and $r_i^*(s,t+dt) \geq 0 \ \forall s \in [t, t+dt]$, we obtain that

$$
J_i(t+dt) \geq J_i(t).
$$

$\square$

Assumption 10. *We assume that $(\omega_i^*(t), r_i^*(.,t)$ are differentiable with respect to variable t.*

Proposition 9. *Under Assumption 10, function $J_i(.)$ is differentiable on $[0,T]$ for all i and has derivative*

$$
D_i(t) = \dot{\omega}_i^*(t)\Gamma_i(t) + \omega_i^*(t)\dot{\Gamma}_i(t) + r_i^*(t,t) + \int_0^t \frac{\partial r_i^*}{\partial t}(s,t)ds.
$$

Proof. The result follows directly from taking the derivative with respect to t of $-J_i(t) = -\omega_i^*(t)\Gamma_i(t) - \int_0^t r_i^*(s,t)ds$. $\square$

4.1.4 Robust counterpart problem

As a result, we obtain the following robust optimization formulation:

$$\max \quad \sum_i \left[\int_0^T \Big(p_i(t)(\alpha_i(t) - \beta_i(t)p_i(t)) - f_i(u_i(t)) - h_i(t)I_i(t) \Big) dt \right] \quad (4.8)$$

$$\text{s.t.} \quad \dot{I}_i(t) = u_i(t) - \alpha_i(t) + \beta_i(t)p_i(t) \quad \forall i \ \forall t \in [0,T] \quad (4.9)$$

$$I_i(0) = I_i^0 \quad \forall i$$

$$I_i(t) \geq J_i(t) \quad \forall i \ \forall t \in [0,T] \quad (4.10)$$

$$p_i(t) \leq \frac{\alpha_i(t) - \hat{\alpha}_i(t)}{\beta_i(t)} \quad \forall i \ \forall t \in [0,T] \quad (4.11)$$

$$\sum_{i=1}^N u_i(t) \leq K(t) \quad \forall t \in [0,T]$$

$$p_i(t), \ u_i(t) \geq 0 \quad \forall i \ \forall t \in [0,T]$$

In this formulation, the uncertainty of demand has an effect only on the no backo-rders constraint and the upper limit on prices. The uncertainty of demand translates into protection levels for the prices and the inventory levels (see (4.10), (4.11)) that are stronger than in the nominal case. That is, protection levels ensure that the inventory remains above level $J_i \geq 0$, and prices below the limit $\frac{\alpha_i(t) - \hat{\alpha}_i(t)}{\beta_i(t)} < \frac{\alpha_i(t)}{\beta_i(t)}$. As a result, even with some variation in the demand - within the introduced uncertainty constraints - the inventory level will remain positive, and prices will remain below their upper bound. These protection levels depend on the budget of uncertainty Γ and on the half length $\hat{\alpha}$ of the interval of variation for the demand parameter α. They are determined through the solution of the dual subproblem (4.7).

The following theorem follows after a change of variable $I_i \leftarrow I_i - J_i$.

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Theorem 8. *Under Assumption (10), problem (4.1) is equivalent to:*

$$\max \quad \sum_i \left[\int_0^T \Big(p_i(t)(\alpha_i(t) - \beta_i(t)p_i(t)) - f_i(u_i(t)) - h_i(t)I_i(t) \Big) dt \right]$$

$$s.t. \quad \dot{I}_i(t) = u_i(t) - \alpha_i(t) + \beta_i(t)p_i(t) - D_i(t) \quad \forall i \quad \forall t \in [0, T]$$

$$I_i(t) \geq 0 \quad \forall i \quad \forall t \in [0, T]$$

$$I_i(0) = \bar{I}_i^0 \quad \forall i$$

$$p_i(t) \leq \frac{\alpha_i(t) - \hat{\alpha}_i(t)}{\beta_i(t)} \quad \forall i \quad \forall t \in [0, T]$$

$$\sum_{i=1}^N u_i(t) \leq K(t) \quad \forall t \in [0, T]$$

$$p_i(t), \; u_i(t) \geq 0 \quad \forall i \quad \forall t \in [0, T]$$

where $\bar{I}_i^0 = I_i^0 - J_i(0) = I_i^0$.

This problem is very similar to the original problem, in terms of type and number of constraints and variables. However, it differs in the fact that we cannot introduce a parameter $\bar{\alpha}_i(t)$ such that we have:

- $\dot{I}_i(t) = u_i(t) - \bar{\alpha}_i(t) + \beta_i(t)p_i(t)$ be the evolution of inventory levels

- $\frac{\bar{\alpha}_i(t)}{\beta_i(t)}$ be the upper limit of the price $p_i(t)$

- the revenue term of the objective function be of the form $p_i(t)(\bar{\alpha}_i(t) - \beta_i(t)p_i(t))$.

Therefore, a straightforward application of the algorithm we propose in Chapter 3 is not possible. In what follows we explain how to modify algorithm solving the nominal problem in order to solve this new robust optimization reformulation. The discussion that follows also allows us to illustrate that solving the robust optimization model is of the same difficulty as the nominal one.

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4.2 Traditional worst-case objective robust counterpart

In general, a robust optimization formulation seeks a solution that is feasible for any realization of the data within the uncertainty set and maximizes the realized objective function. Under this approach, the robust counterpart problem of problem (3.1) can be written as

$$
\begin{aligned}
\max \quad & \mu \\
\text{s.t.} \quad & \mu \leq \int_0^T \sum_{i=1}^N \Big(p_i(t)(\tilde{\alpha}_i(t) - \beta_i(t)p_i(t)) - f_i(u_i(t)) - h_i(t)\tilde{I}_i(t) \Big) dt \quad \forall \tilde{\alpha} \in \mathcal{F}
\end{aligned}
$$

(4.12)

$$
\dot{\tilde{I}}_i(t) = u_i(t) - \tilde{\alpha}_i(t) + \beta_i(t)p_i(t), \quad \forall t \in [0,T] \quad i = 1, \ldots, N
$$

$$
\tilde{I}_i(0) = I_i^0, \quad i = 1, \ldots, N.
$$

$$
\tilde{I}_i(t) \geq 0, \quad \forall t \in [0,T] \quad i = 1, \ldots, N, \quad \forall \tilde{\alpha} \in \mathcal{F}
$$

$$
p_i(t) \leq \frac{\tilde{\alpha}_i(t)}{\beta_i(t)}, \quad \forall t \in [0,T] \quad i = 1, \ldots, N, \quad \forall \tilde{\alpha} \in \mathcal{F}
$$

$$
u_i(t), \; p_i(t) \geq 0, \quad \forall t \in [0,T] \quad i = 1, \ldots, N
$$

$$
\sum_{i=1}^N u_i(t) \leq K, \quad \forall t \in [0,T]
$$

Recall that the realized inventory level at time t can be written:

$$
\tilde{I}_i(t) = I_i(t) - \int_0^t z_i(s)\hat{\alpha}_i(s)ds,
$$

where

$$
I_i(t) = I_i^0 + \int_0^t (u_i(s) - \alpha_i(s) + \beta_i(s)p_i(s))ds
$$

is the nominal inventory level.

Moreover, the constraints must be satisfied for all scaled variations $z(.)$ such that

$$
-1 \leq z_i(t) \leq 1 \quad \forall i, \; t
$$

and

$$\int_0^t |z_i(s)| ds \leq \Gamma_i(t) \quad \forall i, \ t,$$

where $\tilde{\alpha}_i(t) = \alpha_i(t) + z_i(t)\hat{\alpha}_i(t)$.

In order to reformulate this problem, we proceed in a way similar to Section 4.1.1. Specifically, for each constraint where uncertainty in involved, we seek the feasible realization of $\tilde{\alpha}(.)$ that makes the constraint hardest to satisfy.

We may rewrite constraint (4.12) as follows:

$$\mu \ \leq \ \int_0^T \sum_{i=1}^N \Big(p_i((t)\alpha_i(t) - \beta_i(t)p_i(t)) - f_i(u_i(t)) - h_i(t)I_i(t) + p_i(t)\hat{\alpha}_i(t)z_i(t)$$

$$+ h_i(t)\int_0^t z_i(s)\hat{\alpha}_i(s)ds \Big) dt \ \ \forall \tilde{\alpha} \in \mathcal{F}$$

$$\Leftrightarrow \mu \ \leq \ C + \int_0^T \sum_{i=1}^N \Big(p_i(t)\hat{\alpha}_i(t)z_i(t) + h_i(t)\int_0^t z_i(s)\hat{\alpha}_i(s)ds \Big) dt \ \ \forall \tilde{\alpha} \in \mathcal{F}, \quad (4.13)$$

where

$$C = \int_0^T \sum_{i=1}^N \Big(p_i(t)(\alpha_i(t) - \beta_i(t)p_i(t)) - f_i(u_i(t)) - h_i(t)I_i(t) \Big) dt$$

is the nominal objective function, and is independent of $z(.)$.

We seek the feasible realization of $z(.)$ that minimizes the right-hand side in inequality (4.13). We notice that we may consider each product separately; in other words we minimize each term of the sum across products. We thus need to find, for each i, the feasible realization of $z_i(.)$ that solves

$$\min_{z_i(.)} \int_0^T \Big(p_i(t)\hat{\alpha}_i(t)z_i(t) + h_i(t)\int_0^t z_i(s)\hat{\alpha}_i(s)ds \Big) dt.$$

Since $p_i(.)$, $\hat{\alpha}_i(.)$ and $h_i(.)$ are positive valued, it is easy to see that in the optimal solution, $z_i(.) \leq 0$. Therefore, after transformation of variables, we may rewrite the

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constraints on the new variable $z_i(.)$ as follows

$$0 \leq z_i(t) \leq 1 \quad \forall t$$

$$\int_0^t z_i(s)ds \leq \Gamma_i(t) \quad \forall t$$

(note that we abuse notations to avoid confusion) and rewrite the minimization problem as

$$\min_{z_i(.)} \quad \int_0^T \left(-p_i(t)\hat{\alpha}_i(t)z_i(t) + h_i(t)\int_0^t (-z_i(s)\hat{\alpha}_i(s))ds \right)dt$$

$$\Leftrightarrow -\max_{z_i(.)} \quad \int_0^T p_i(t)\hat{\alpha}_i(t)z_i(t)dt + \int_0^T \left(h_i(t)\int_0^t z_i(s)\hat{\alpha}_i(s)ds \right)dt$$

$$\Leftrightarrow -\max_{z_i(.)} \quad \int_0^T p_i(t)\hat{\alpha}_i(t)z_i(t)dt + \int_{t=0}^T \left(\int_{s=0}^t h_i(t)z_i(s)\hat{\alpha}_i(s)ds \right)dt$$

$$\Leftrightarrow -\max_{z_i(.)} \quad \int_0^T p_i(t)\hat{\alpha}_i(t)z_i(t)dt + \int_{s=0}^T \left(\int_{t=s}^T h_i(t)z_i(s)\hat{\alpha}_i(s)dt \right)ds$$

$$\Leftrightarrow -\max_{z_i(.)} \quad \int_0^T p_i(t)\hat{\alpha}_i(t)z_i(t)dt + \int_{s=0}^T H_i(s)z_i(s)\hat{\alpha}_i(s)ds$$

$$\Leftrightarrow -\max_{z_i(.)} \quad \int_0^T (p_i(t) + H_i(t))z_i(t)\hat{\alpha}_i(t)dt,$$

where $H_i(t) \equiv \int_t^T h_i(s)ds$.

Therefore, we obtain the equivalent subproblem

$$-\max_{z_i(.)} \quad \int_0^T (p_i(t) + H_i(t))z_i(t)\hat{\alpha}_i(t)dt$$

$$\text{s.t.} \quad 0 \leq z_i(t) \leq 1 \quad \forall t$$

$$\int_0^t z_i(s)ds \leq \Gamma_i(t) \quad \forall t.$$

We notice at this point that this subproblem depends on the control variable $p_i(.)$. This dependency comes from the fact that the revenue term is non linear in both the control variable $p_i(.)$ and the uncertain parameter $\alpha_i(.)$.

This is an instance of a *separated continuous linear program* (as introduced by An-

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derson [3], [4]). It was shown that if the functions $(p_i(.) - H_i(.))\hat{\alpha}_i(.)$, and $\Gamma_i(.)$ are piecewise analytic, and we search for the optimal $z_i(.)$ in the space of of measurable bounded functions, there is strong duality and the dual problem is given by:

$$- \min_{\Pi_i(.), w_i(.)} \quad \int_0^T \Gamma_i(t) d\Pi_i(t) + \int_0^T w_i(t) dt$$

$$\text{s.t.} \quad -\Pi_i(t) + w_i(t) \geq (p_i(t) - H_i(t))\hat{\alpha}_i(t) \quad \forall t$$

$$w_i(t) \geq 0 \quad \forall t$$

$$\Pi_i(T) = 0$$

$$\Pi_i \text{ non-decreasing.}$$

To ease the exposition, let's assume that the optimal solution $\Pi_i(.)$ is differentiable and let $v_i(.)$ its derivative. The dual problem above is then equivalent to

$$\max_{\Pi_i(.), v_i(.), w_i(.)} \quad -\int_0^T \Gamma_i(t) v_i(t) - w_i(t) dt$$

$$\text{s.t.} \quad -\Pi_i(t) + w_i(t) \geq (p_i(t) - H_i(t))\hat{\alpha}_i(t) \quad \forall t$$

$$\dot{\Pi}_i(t) = v_i(t) \quad \forall t$$

$$\Pi_i(T) = 0$$

$$w_i(t) \geq 0 \quad \forall t$$

$$v_i(t) \geq 0 \quad \forall t$$

The deterministic robust version of the no backorders constraint and the upper bound on prices is obtained identically to Section 4.1.1.

Therefore, under the traditional approach where the worst case objective is maximized, the robust counterpart problem takes as decision variables $p(.), u(.), v(.), w(.)$

and as state variables $I(.), \Pi(.)$ as follows:

$$\max \quad \int_0^T \sum_i \Big(p_i(t)(\alpha_i(t) - \beta_i(t)p_i(t)) - f_i(u_i(t)) - h_i(t)I_i(t) - w_i(t) - \Gamma_i(t)v_i(t) \Big) dt$$

s.t.
$$\dot{I}_i(t) = u_i(t) - \alpha_i(t) + \beta_i(t)p_i(t), \quad I_i(0) = I_i^0 \quad \forall i \;\; \forall t \in [0,T]$$

$$\dot{\Pi}_i(t) = v_i(t), \quad \Pi_i(T) = 0 \quad \forall i \;\; \forall t \in [0,T]$$

$$I_i(t) \geq J_i(t) \quad \forall i \;\; \forall t \in [0,T]$$

$$-\Pi_i(t) + w_i(t) - (p_i(t) + H_i(t))\hat{\alpha}_i(t) \geq 0 \quad \forall i \; \forall t \in [0,T] \tag{4.14}$$

$$\sum_{i=1}^N u_i(t) \leq K(t) \quad \forall t \in [0,T]$$

$$p_i(t) \leq \frac{\alpha_i(t) - \hat{\alpha}_i(t)}{\beta_i(t)} \quad \forall i \;\; \forall t \in [0,T]$$

$$w_i(t), \; v_i(t) \geq 0 \quad \forall i \; \forall t \in [0,T]$$

$$p_i(t), \; u_i(t) \geq 0 \quad \forall i \;\; \forall t \in [0,T]$$

This new robust optimization problem is still a convex fluid model with linear constraints, and with twice as many variables as the nominal problem (3.1). However, the solution method developed in Chapter 3 for solving (3.1) is not easily adaptable. The main reason is the inequality constraint (4.14) that couples the new state variable with a decision variable. As a result, the order of complexity is higher than the nominal problem.

We notice that if pricing was not a decision, or if demand was external (i.e., not depending on pricing), the general model of uncertainty (i.e. with realized revenues) would yield a tractable formulation (see for example [32] for a discretized version of the problem). This discussion, along with Section 2.5 on modeling the objective function under uncertainty, motivates the model we introduce and study in Section 4.1.1 and in the remainder of this chapter.

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4.3 Solving the robust counterpart problem

To avoid repetition, we do not derive in detail the solution method to solve this problem, as it is very similar to the one described in Chapter 2. We however mention that an additional assumption is necessary for the robust counterpart problem to be feasible. Indeed, while the nominal problem is always feasible, the robust counterpart may not be feasible, for example if the production capacity $K(t)$ is close to zero, the production rate must remain close to zero, and even by setting prices at their maximum in an attempt to slow the decrease of inventory level, it may not be possible to always satisfy the minimum inventory security level guarantee.

Assumption 11. *The following inequality holds at all times* t

$$\sum_{i \ s.t. \ 0 > \frac{\alpha_i(t) - 2\hat{\alpha}_i(t)}{\beta_i(t)} - f_i'(\hat{\alpha}_i(t) + D_i(t))} \left(\hat{\alpha}_i(t) + D_i(t)\right) \leq K(t).$$

This assumption ensures that the production capacity level is sufficiently large to guarantee that the minimum inventory level constraints can be satisfied, i.e. that there exists a feasible solution to this problem.

One difference with the method detailed in Chapter 2 is that constrained products never idle in the robust problem, while in the nominal problem they either (a) idle or (b) are produced in order to satisfy the demand. We do recognize two possible states however, one (a') in which they are produced in order to keep the inventory level at zero and allow for demand uncertainty (the rate depends on D_i and $\hat{\alpha}_i$ only), and a second one (b') where they are produced in order to satisfy the demand with uncertainty. This discussion leads to conclude that the modified method for solving the robust optimization formulation is of the same order of complexity as the algorithm for solving the nominal problem.

4.4 Numerical results

4.4.1 Choice of parameters

In this section, we consider a numerical example for two products on a time horizon $[0,10]$ that is similar to the example we considered in Chapter 2. In this chapter we also introduce demand uncertainty. Our goal is to understand the relationship between the optimal objective value and the budget of uncertainty $\Gamma_i(.)$. As a result, we will consider only one demand scenario and demand uncertainty model, and a capacity level that is constant at 75% of the maximum of the cumulative production rate achieved in the nominal case under not capacity constraint: $K(t) = 4.9014$. This guarantees that the capacity constraint is tight for most of the time horizon.

We consider the same production cost structure where the cost $f_i(.)$ is a quadratic function of the production rate:

$$f_i(u_i) = \frac{\gamma_i}{2} u_i^2.$$

$\hat{\alpha}_i(.)$, which represents the half-length of the allowed range for parameter $\alpha_i(.)$, must satisfy $0 < \hat{\alpha}_i(.) < \frac{1}{2}\alpha_i(.)$. For ease of computations in this example, we consider input parameters $\hat{\alpha}_i(.)$ that are linear functions of the time (nevertheless, the linearity assumption is not necessary):

$$\hat{\alpha}_i(t) \;\; = \;\; a_i t + b_i,$$

where a_i, $b_i \geq 0$. Indeed, it is reasonable to consider that in practice, the accuracy of a forecast for the demand is non increasing on the time horizon, i.e. that the length of the interval of feasible outcomes is non decreasing, hence $a_i \geq 0$.

We choose a_i, b_i, $i = 1, 2$ such that, the uncertainty on α_i is rather small initially, when the forecast should be rather accurate, and is about 4% of its nominal value at the end of the time horizon, as illustrated in Figure 4-1.

	β	h	γ	I^0	$\alpha(t)$	$\hat{\alpha}(t)$
product 1	1	1	10	10	$30 + 10t - t^2$	0.1t+0.2
product 2	1	2	20	10	$30 + 10t - t^2$	0.1t+0.2

Table 4.1: Data chosen as input in the numerical implementation for the robust formulation
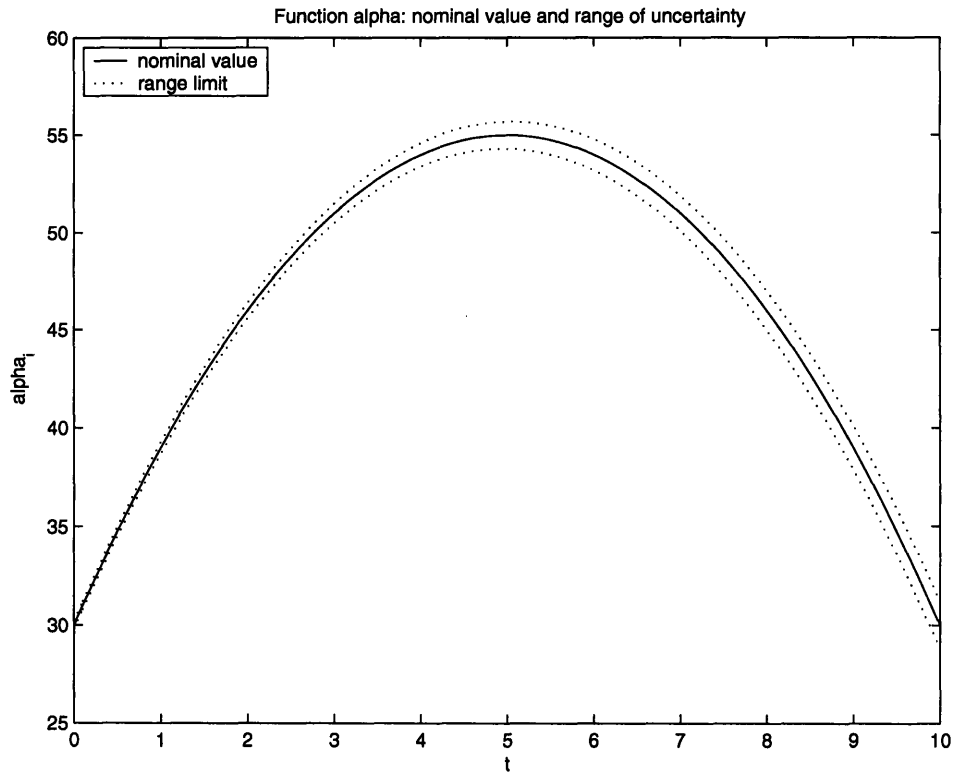
The input data are summarized in Table 4.1.

Figure 4-1: Demand uncertainty

The choice of $\Gamma_i(.)$ satisfies $0 \leq \dot{\Gamma}_i(t) \leq 1$. In this example, we first consider these input parameters $\Gamma_i(.)$ to be linear functions of the time:

$$\Gamma_i(t) = g_i t + c_i,$$

where g_i, $c_i \geq 0$, $g_i < 1$.

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Indeed, it can be seen that as soon as the graph of $\Gamma_i(.)$ is above the $45°$ line with the horizontal axis, its actual value does not matter and $|z_i(t)|$ takes the value 1, meaning that the realized $\tilde{\alpha}_i$ lies at the extreme of its allowed range (worst case scenario). To avoid this from happening on much of the time horizon, we choose $g_i < 1$.
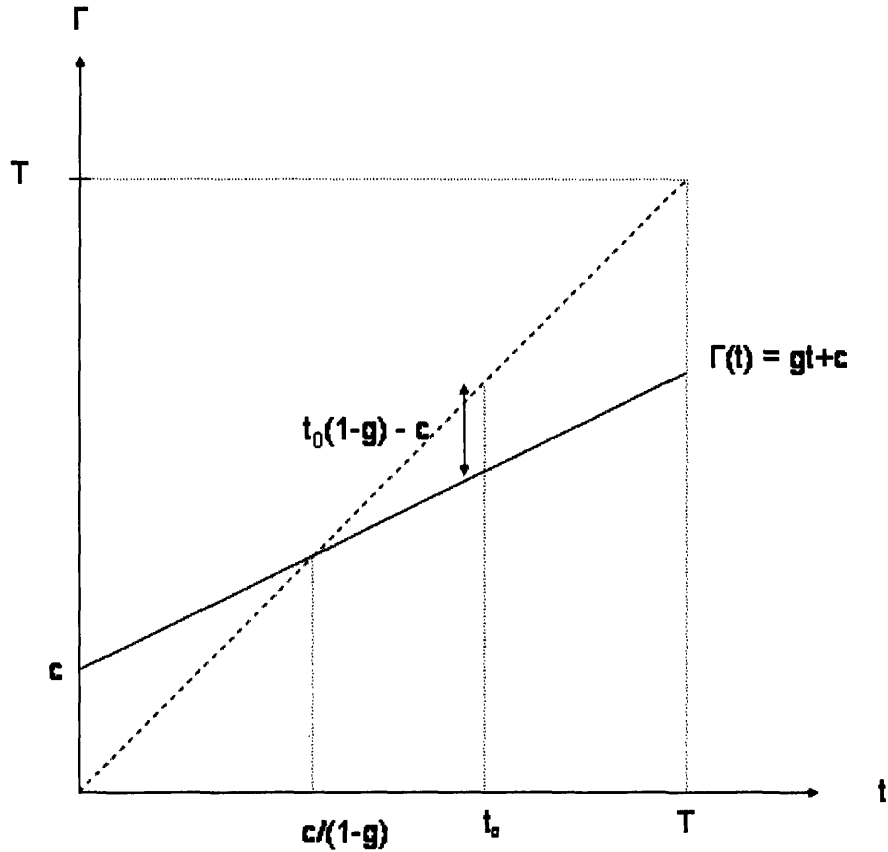

Figure 4-2: Choice of budget uncertainty function $\Gamma(.)$.

In order to study the effect of the budget uncertainty on the optimal objective value (i.e. performance), we will consider multiple scenario in which only the parameter $\Gamma_i(.)$ varies, and in which it varies in the following way: on the one hand we let the value at time 0 change but keep a constant slope, and on the other hand we keep a constant slope, and change the value at time 0. In all scenarios we assign the same

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	$\Gamma^k(t)$	$\int_0^T \min\{t, \Gamma^k(t)\}dt$
scenario 1	1+0.8t	47.5
scenario 2	1+0.5t	34
scenario 3	1+0.2t	19.38
scenario 4	0.5+0.8t	44.38
scenario 5	0.5+0.5t	29.75
scenario 6	0.5+0.2t	14.84

Table 4.2: Multiple scenarios of budget of uncertainty

budget of uncertainty to both products. The scenarios are shown in Table 4.2 We will compute the cumulative effective budget of uncertainty $\int_0^T \min\{t, \Gamma_i(t)\}dt$ as a measure of the global uncertainty in each scenario.

4.4.2 Closed-form solution for the dual subproblem (4.7)

In order to implement our algorithm, we need to compute the derivative of the inventory safety level $D_i(.)$. After calculations, we have

$$D_i(t) = \begin{cases} at + b & \text{if } 0 \le t < \frac{c}{1-g} \\ a(1-g)(gt+c) + g(at+b) & \text{if } \frac{c}{1-g} < t \le T. \end{cases}$$

It is also easy to verify that in all scenarios we have at all times

$$\sum_{i=1,\,2} (\hat{\alpha}_i(t) + D_i(t)) < K(t)$$

which implies that Assumption 11 holds.

4.4.3 Results

Using these inputs, we run the algorithm and obtain the production rates, prices, and inventory levels under the optimal policy for the robust formulation we described in (4.8). Recall that in this formulation, the inventory levels are constrained to remain above a safety level $J_i(t) \ge 0$.
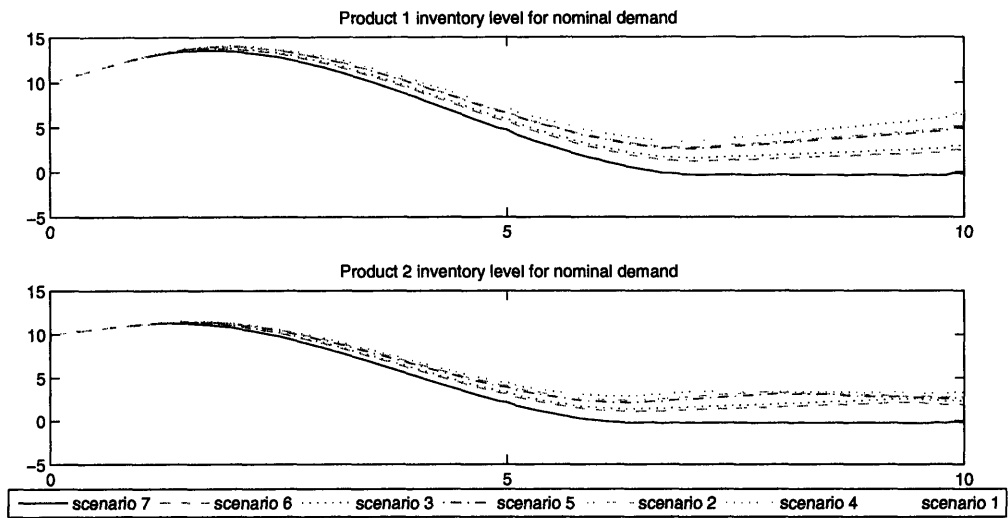
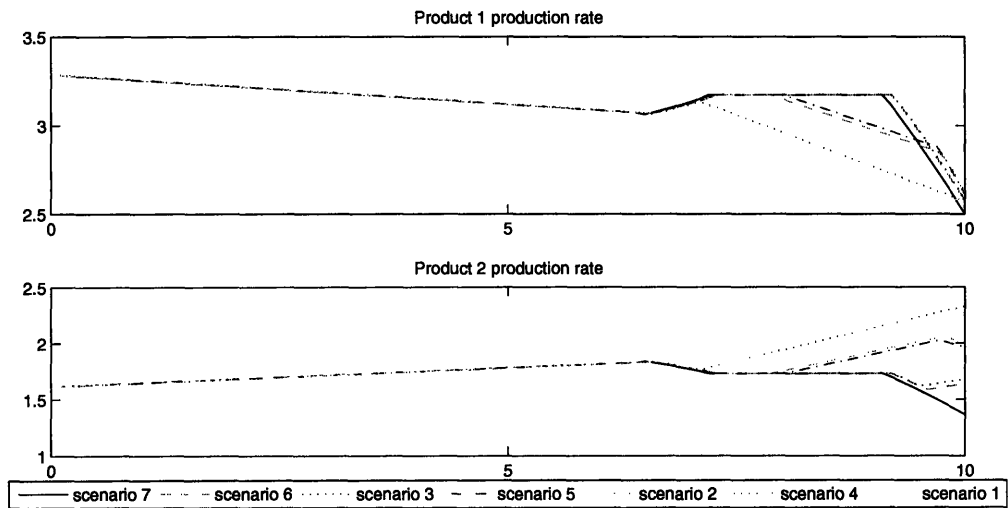Figure 4-3: Optimal inventory levels over time results

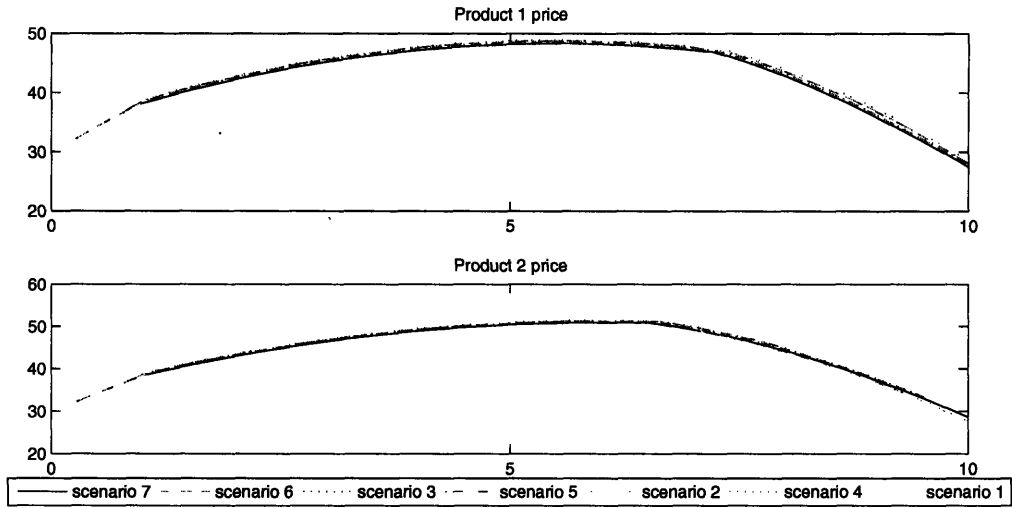Figure 4-4: Optimal production rates over time results

Figure 4-5: Optimal prices over time results

Similarly to the nominal problem, the system starts by building up inventory in anticipation of the demand peak that occurs in the middle of the time horizon. Then the inventory levels are kept at the minimum level - zero in the deterministic model, but $J_i(t)$ in the robust formulation - for the remaining time. Notice that prices have the same shape as the demand curves.

We notice that in all scenarios, the capacity constraint is tight except at the very end of the time horizon. Since product 2 is cheaper to produce and to hold, its production rate is maintained at a lower value than for product 1 as the products are unconstrained. In that stage, the budget of uncertainty has no significant influence on the production rates. This makes sense because we saw in formulating the problem that uncertainty played a role in the no backorder constraint and the upper bound on prices. As a result, while the inventory level is positive, the uncertainty does not matter.

In the constrained phase, the safety level $J_i(t)$ for the inventory increases as the budget of uncertainty increases, which is consistent with our interpretation that the more demand uncertainty we impose to the system, the larger should be the min-

imum value of the inventory level guaranteeing no backorder for any realization of the demand. We observe that in that phase, as the cumulative budget of uncertainty increases, the production rate for product 2 increases more and more because maintaining the level of inventory for that product at $J_2(t)$ requires to produce more and more, given that production rate for product 2 starts from a lower value that production rate for product 1.

Finally, we notice that prices slightly increase as the cumulative budget of uncertainty increases, reflecting the increasing difficulty of satisfying all constraints as the uncertainty increases.

These numerical results suggest that the cumulative effective budget of uncertainty might be a relevant metric for measuring the global uncertainty. This is further confirmed by the following computation on the objective value.

We compute in each scenario the optimal objective value and the cumulative effective budget of uncertainty (see Table 4.3).

	Budget of uncertainty $\Gamma_i(t)$, $i = 1, 2$	Cumulative effective budget of uncertainty $\int_0^T \min\{t, \Gamma_i(t)\}dt$	Objective value
scenario 1	1+0.8t	47.5	1693.5
scenario 2	1+0.5t	34	1791.5
scenario 3	1+0.2t	19.4	1928.7
scenario 4	0.5+0.8t	44.4	1707.7
scenario 5	0.5+0.5t	29.8	1812.7
scenario 6	0.5+0.2t	14.8	1972.3
scenario 7	0	0	2198.4

Table 4.3: Objective value results

Table 4.3 and Figure 4-6 seem to suggest that the cumulative effective budget of uncertainty may be a reasonable way of measuring the uncertainty in the problem. Notice that as the uncertainty increases, the optimal objective value decreases. This illustrates the trade-off between optimality (high optimal objective value) and robustness (high level of uncertainty).
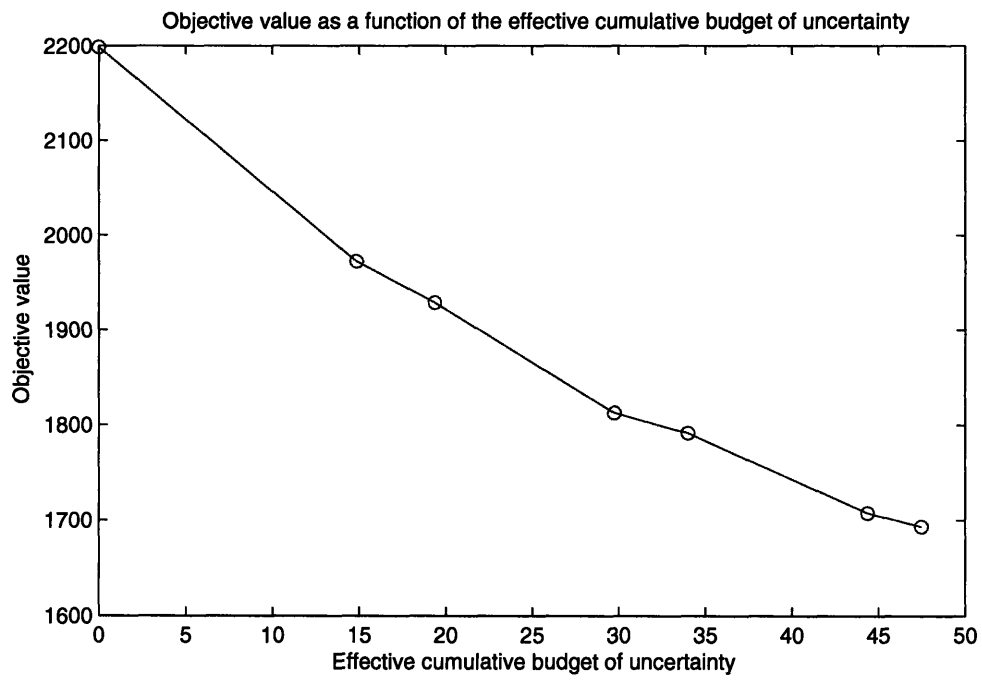
Figure 4-6: Trade-off between robustness and performance

Chapter 5

Uncertain data in a duopoly setting

Our goal is now to extend the results of the previous chapter to a duopoly setting, and address competition together with the presence of demand uncertainty. We assume that the demand observed by a given supplier is a linear function not only of the price applied by this supplier, but also of the price applied by her competitor. Moreover, we model in this section the inventory and production costs are quadratic. Finally, we consider an uncertainty model that is more general than in the previous chapter, by also introducing a *multiplicative* uncertainty factor in addition to the additive one. This form of uncertainty is more general but also more difficult to address than the additive uncertainty.

We first reformulate the robust problem faced by each supplier as a fluid model of a form similar to the deterministic fluid model in a monopoly setting. We show existence of a Nash equilibrium in continuous time by using an equivalent variational inequalities formulation. We then discuss issues of uniqueness and address how to compute a particular Nash equilibrium, i.e. the *normalized* Nash Equilibrium (this term will be defined in this chapter).

To achieve these goals, we gradually increase complexity and start by showing existence of an equilibrium in a duopoly setting *without* data uncertainty. Secondly, we

generalize the results to the problem with uncertainty on all the demand parameters. Thirdly, we study the model in discretized time and study uniqueness of equilibria. Finally, we describe an algorithm and show its convergence to a particular equilibrium.

We recall that Chapter 2 details the notations for the problem in this setting, and that generally speaking, the superscript $k = A$ or B is used to designate a supplier, and $-k$ for her competitor.

5.1   Duopoly setting with deterministic data

In this section, we assume that the demand parameters are deterministic and equal to their nominal values.

5.1.1   Formulation

Since the demand depends on the competitor's prices, a given supplier's optimization problem also depends on these prices. In this setting, the term *best-response* corresponds to the optimal strategy of the supplier *assuming that the prices of her competitor are fixed and known.*

Supplier k needs to determine the optimal decision variables $p^k(.)$, $u^k(.)$ and resulting state variables $I^k(.)$. The best response problem she faces, given price $p_i^{-k}(.)$

for product i applied by the competitor, may be formulated as follows

$$\max \quad \int_0^T \sum_{i=1}^N \Big(p_i^k(t)(\alpha_i^k(t) - \beta_i^{k,k}(t)p_i^k(t) + \beta_i^{k,-k}(t)p_i^{-k}(t)) - \gamma_i^k(t)(u_i^k(t))^2$$

$$-h_i^k(t)(I_i^k(t))^2 \Big) dt \quad (5.1)$$

$$\text{s.t.} \quad \dot{I}_i^k(t) = u_i^k(t) - \alpha_i^k(t) + \beta_i^{k,k}(t)p_i^k(t) - \beta_i^{k,-k}(t)p_i^{-k}(t),$$

$$I_i^k(0) = I_i^{k^0}, \quad \forall t \in [0,T], \quad i = 1, \ldots, N \quad (5.2)$$

$$\sum_{i=1}^N u_i^k(t) \le K^k(t), \quad \forall t \in [0,T] \quad (5.3)$$

$$p_i^k(t) \le \frac{\alpha_i^k(t) + \beta_i^{k,-k}(t)p_i^{-k}(t)}{\beta^{k,k}i(t)}, \quad \forall t \in [0,T] \quad i = 1, \ldots, N \quad (5.4)$$

$$I_i^k(t) \ge 0, \quad \forall t \in [0,T] \quad i = 1, \ldots, N \quad (5.5)$$

$$u_i^k(t), \ p_i^k(t) \ge 0, \quad \forall t \in [0,T] \quad i = 1, \ldots, N$$

In this formulation, objective (5.1) describes the profit by adding over all products and over the time horizon the price multiplied by the demand, and subtracting all costs, that is the quadratic production and inventory costs. Fluid equation (5.2) along with the initial condition, determines the inventory level at all times by defining the change of inventory level as the difference between the production rate and the demand rate. Constraint (5.5) ensures that there are no backorders. The upper bound on the price (5.4) comes from the fact that the demand rate should remain non negative. Finally, constraint (5.3) is the capacity constraint on the total production.

We notice that the revenues, the inventory level, and the upper bound on prices involve the demand. Furthermore, the demand involves the competitor's prices. As a result, not only the objective function, but also the set of feasible strategies for a given player, depend on the pricing strategy of the other player, via the no backorders constraint and the upper bound on the price. We assume that the competitors make their choices simultaneously. As a result, we have to study a *coupled constraint* game. As we will see in the remaining of this chapter, existence, uniqueness, and determination of Nash equilibria for games of this type are more difficult to study than for

games where the feasible strategy set is independent of the competitor's strategy.

Remark.

Notice that combining the feasibility conditions in the best-response of both suppliers

$$p_i^A(t) \leq \frac{\alpha_i^A(t)}{\beta_i^{A,A}(t)} + \frac{\beta_i^{A,B}(t)}{\beta_i^{A,A}(t)} p_i^B(t)$$

and

$$p_i^B(t) \leq \frac{\alpha_i^B(t)}{\beta_i^{B,B}(t)} + \frac{\beta_i^{B,A}(t)}{\beta_i^{B,B}(t)} p_i^A(t)$$

gives rise to the following bounds on prices

$$p_i^B(t) \leq \frac{\alpha_i^B(t)\beta_i^{A,A}(t) + \alpha_i^A(t)\beta_i^{B,A}(t)}{\beta_i^{B,B}(t)\beta_i^{A,A}(t) - \beta_i^{B,A}(t)\beta_i^{A,B}(t)}$$

and

$$p_i^A(t) \leq \frac{\alpha_i^A(t)\beta_i^{B,B}(t) + \alpha_i^B(t)\beta_i^{A,B}(t)}{\beta_i^{B,B}(t)\beta_i^{A,A}(t) - \beta_i^{B,A}(t)\beta_i^{A,B}(t)}.$$

In particular, this shows that the feasible prices are bounded with an upper bound independent of the competitor's strategy. Also, the production rates are bounded by the capacity rate. Moreover, since the time horizon and initial inventory level are finite, and the inventory levels are differentiable and as such continuous functions of time, they are bounded as well. Therefore, the strategy and state space are bounded.

5.1.2   Definitions and properties

Vector space and associated norm

Let E_1 be the vector space such that any element of E_1 has $3N$ components (price, production and inventory vectors) that are real bounded functions defined over $[0, T]$. The integral of the square of their absolute value is well-defined. Let $E = E_1 \times E_1$ be

the Hilbert space (we use the L^2 norm on E so we have a reflexive Banach space):

$$\|(x^1, x^2)\| = \sqrt{\int_0^T \sum_{i=1}^{3N} \sum_{k=1,2} (x_i^k(t))^2 dt} \qquad .$$

with x^1, $x^2 \in E_1$, associated with the inner product

$$< (x^1, x^2), \ (\bar{x}^1, \bar{x}^2) > = \int_0^T \sum_{i=1}^{3N} \sum_{k=1,2} x_i^k(t) \bar{x}_i^k(t) dt.$$

Note that this space has an infinite dimension.

We will denote by $x^k \in E_1$ the vector representing a pricing and production strategy along with the state variables of player k in the following way:

$$x^k = (p^k, \ u^k, \ I^k),$$

where

$$p^k = (p_1^k(.), \ldots, p_N^k(.)), \quad u^k = (u_1^k(.), \ldots, u_N^k(.)), \quad I^k = (I_1^k(.), \ldots, I_N^k(.)).$$

The vector $(x^A, x^B) \in E$ represents the *collective* strategy and state vector. As a result the norm for a collective strategy and state vector is given by:

$$\|x\| = \sqrt{\int_0^T \sum_{i=1}^{N} \sum_{k=A,B} \Big[(p_i^k(t))^2 + (u_i^k(t))^2 + (I_i^k(t))^2 \Big] dt}, \quad x \in E$$

associated with the inner product

$$< x, \ \bar{x} > = \int_0^T \sum_{i=1}^{N} \sum_{k=A,B} \Big(p_i^k(t) \bar{p}_i^k(t) + u_i^k(t) \bar{u}_i^k(t) + I_i^k(t) \bar{I}_i^k(t) \Big) dt, \quad x, \bar{x} \in E.$$

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Feasible set

Let's denote $X^k \subset E_1$ the set of strategy and state vectors for player k satisfying the constraints that are independent of the competitor's strategy:

$$X^k = \{ \quad x = (p, u, I) \in E_1 :$$

$$u_i(t),\ p_i(t),\ I_i(t) \geq 0\ \forall i,\ t$$

$$\sum_{i=1}^{N} u_i(t) \leq K^k(t)\ \forall t$$

$$I_i(0) = I_i^{k^0}\ \forall i\ \}.$$

Let $X \subset E$ such that $X = X^A \times X^B$.

The following lemma follows directly from the definition of X.

Lemma 8. X *is a non empty, convex, closed subset of* E.

For a fixed strategy and state vector of the competitor, let's denote $Q^k(\bar{x}^{-k}) \subset X^k$ the subset of all feasible strategy and state vectors for player k, given the strategy and state vector $\bar{x}^{-k}$ of her competitor:

$$Q^k(\bar{x}^{-k}) = \{ \quad x = (p_1(.), \ldots, p_N(.),\ u_1(.), \ldots, u_N(.),\ I_1(.), \ldots, I_N(.)) \in X^k :$$

$$p_i(t) \leq \frac{\alpha_i^k(t) + \beta_i^{k,-k}(t)\bar{p}_i^{-k}(t)}{\beta_i^{k,k}(t)}\ \forall i,\ t$$

$$\dot{I}_i(t) = u_i(t) - \alpha_i^k(t) + \beta_i^{k,k}(t)p_i(t) - \beta_i^{k,-k}(t)\bar{p}_i^{-k}(t)\ \forall i,\ t\ \}.$$

Lemma 9. *For all* $\bar{x}^{-k} \in X^{-k}$, $Q^k(\bar{x}^{-k})$ *is a non empty, closed, convex subset of* X^k.

Proof. Convexity follows from the fact that the constraints defining the set are linear. To see that the set is closed, note that if we take a convergent sequence of vectors of X^k (even not uniformly convergent), since the inventory levels are bounded and the time horizon is finite, we can interchange the limit and the integral, and as a result the limit belongs to the set as well.

It is easy to verify that the vector $x = (p, u, I)$ such that

$$p_i(t) = \frac{\alpha_i^k(t) + \beta_i^{k,-k}(t)\bar{p}_i^{-k}(t)}{\beta_i^{k,k}(t)}, \quad u_i(t) = \frac{K^k(t)}{N}, \quad I_i(t) = I_i^{k^0} + \int_0^t \frac{K^k(s)}{N} \, ds \quad \forall i, \ t$$

is an element of $Q^k(\bar{x}^{-k})$. $\qquad\qquad\qquad\qquad\qquad\square$

We denote $Y \subset X$ the set of *collectively feasible* strategy and state vectors for both players:

$$Y = \{x \in X : x^k \in Q^k(x^{-k}), \ k = A, B\}.$$

Lemma 10. *Y is a convex, closed, non empty subset of X.*

Proof. Convexity and closedness follow from the fact that sets X and $Q^k(\bar{x}^{-k})$ are convex and closed.

It is easy to verify that vector $x = (p, u, I)$ such that

$$p_i^k(t) = \frac{\alpha_i^k(t)\beta_i^{-k,-k}(t) + \alpha_i^{-k}(t)\beta_i^{k,-k}(t)}{\beta_i^{B,B}(t)\beta_i^{A,A}(t) - \beta_i^{B,A}(t)\beta_i^{A,B}(t)}, \quad u_i^k(t) = \frac{K^k(t)}{N}$$

$$I_i^k(t) = I_i^{k^0} + \int_0^t \frac{K^k(s)}{N} \, ds \quad \forall i, \ t$$

is an element of Y. $\qquad\qquad\qquad\qquad\qquad\square$

We denote $Q : X \mapsto 2^X$ the mapping such that $Q(x) = Q^A(x^B) \times Q^B(x^A)$. $Q(x)$ represents the set of *unilaterally feasible* strategy and state vectors for both players when the competitor keeps her strategy fixed at x^{-k}.

The following lemma follows from Lemma 9.

Lemma 11. *$Q(x)$ is a non empty convex closed subset of X.*

The following proposition is immediate.

Proposition 10. *The following equivalence holds:*

$$x \in Q(x) \Leftrightarrow x \in Y.$$

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Objective function

Let Π^k the payoff function of player k. Note that, since the objective function depends on the competitor's strategy, Π^k depends not only on x^k but also on x^{-k}. Therefore Π^k is defined on set X and is real-valued. We recall that for $x = (x^A, x^B) \in X$, we have for $k = A,\ B$,

$$\Pi^k(x) \ = \ \int_0^T \sum_{i=1}^N \Big(p_i^k(t)(\alpha_i^k(t) - \beta_i^{k,k}(t)p_i^k(t) + \beta_i^{k,-k}(t)p_i^{-k}(t)) - \gamma_i^k(t)(u_i^k(t))^2$$
$$-h_i^k(t)(I_i^k(t))^2 \Big) dt.$$

Definition 1. *A Nash equilibrium is a vector* $x = (x^A, x^B) \in Y$ *such that*

$$\Pi^k(x) \geq \Pi^k(\bar{x}^k, x^{-k}) \quad \forall \bar{x}^k \in Q^k(x^{-k}), \quad k = A,\ B. \tag{5.6}$$

In other words, at a Nash equilibrium, no supplier can increase her profits by unilaterally deviating from the equilibrium solution, when the competitor keeps her strategy fixed.

Proposition 11. *Vector* $x = (x^A, x^B)$ *is a Nash equilibrium if and only if* $x \in Y$ *and*

$$\forall \bar{x} = (\bar{x}^A, \bar{x}^B) \in Q(x), \quad \Pi^A(x^A, x^B) + \Pi^B(x^A, x^B) \geq \Pi^A(\bar{x}^A, x^B) + \Pi^B(x^A, \bar{x}^B).$$

Proof. It is clear by adding (5.6) for $k = A$ and $k = B$ that the inequality above is a necessary condition for x to be a Nash equilibrium.

For the reverse, suppose $x \in Y$ satisfies the inequality above for all $\bar{x} \in Q(x)$ but it is not a Nash equilibrium, i.e.

$$\exists k, \ \bar{x}^k \in Q^k(x^{-k}) \text{ such that } \Pi^k(x^k, x^{-k}) < \Pi^k(\bar{x}^k, x^{-k}).$$

Let y such that $y^k = \bar{x}^k$, $y^{-k} = x^{-k}$. Then clearly $y \in Q(x)$. Moreover,

$$
\begin{aligned}
\Pi^A(y^A, x^B) + \Pi^B(x^A, y^B) &= \Pi^k(\bar{x}^k, x^{-k}) + \Pi^{-k}(x^k, x^{-k}) \\
&> \Pi^k(x^k, x^{-k}) + \Pi^{-k}(x^k, x^{-k}) = \Pi^A(x) + \Pi^B(x)
\end{aligned}
$$

which is a contradiction. $\qquad\square$

We observe that the payoff function of player k may be formulated as follows:

$$
\Pi^k(x) = -a^k(x^k, x^k) - 2b^k(x^{-k}, x^k) + 2L^k(x^k)
$$

where

- $a^k : E_1 \times E_1 \mapsto \mathbb{R}$ is the continuous bilinear form, symmetric and non-negative along the diagonal such that

$$
a^k(x, \bar{x}) = \int_0^T \sum_{i=1}^N \big(\beta_i^{k,k}(t) p_i(t) \bar{p}_i(t) + \gamma_i^k(t) u_i(t) \bar{u}_i(t) + h_i^k(t) I_i(t) \bar{I}_i(t) \big) dt
$$

- $b^k : E_1 \times E_1 \mapsto \mathbb{R}$ is the continuous bilinear form such that

$$
b^k(x, \bar{x}) = -\frac{1}{2} \int_0^T \sum_{i=1}^N \beta_i^{k,-k}(t) \bar{p}_i(t) p_i(t) dt
$$

- $L^k : E_1 \mapsto \mathbb{R}$ is the continuous linear functional such that

$$
L^k(x) = \frac{1}{2} \int_0^T \sum_{i=1}^N \big(\alpha_i^k(t) p_i(t) \big) dt.
$$

Let $a : E \times E \mapsto \mathbb{R}$ defined by

$$
\begin{aligned}
a(x, \bar{x}) &= a^A(x^A, \bar{x}^A) + a^B(x^B, \bar{x}^B) + b^B(x^A, \bar{x}^B) + b^A(\bar{x}^A, x^B) \qquad (5.7) \\
&= \int_0^T \sum_{i=1}^N \sum_{k=A,B} \Big(\beta_i^{k,k}(t) p_i^k(t) \bar{p}_i^k(t) + \gamma_i^k(t) u_i^k(t) \bar{u}_i^k(t) \\
&\qquad\qquad + h_i^k(t) I_i^k(t) \bar{I}_i^k(t) - \frac{1}{2} \beta_i^{k,-k}(t) \bar{p}_i^k(t) p_i^{-k}(t) \Big) dt.
\end{aligned}
$$

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Let $L : E \mapsto \mathbb{R}$ the linear functional defined by

$$L(x) = L^A(x^A) + L^B(x^B). \tag{5.8}$$

5.1.3 Quasi-variational inequality formulation

The following theorem reformulates the Nash equilibrium problem as a quasi-variational inequality problem.

Proposition 12. *There exists $f \in E$ and a linear operator A on E such that*

$$a(x, \bar{x}) = \; < Ax, \bar{x} > \quad \forall x, \bar{x} \in E$$

$$L(x) = \; < f, x > \quad \forall x \in E$$

with

$$
\begin{aligned}
f \;=\; & \frac{1}{2}\Big(\alpha_1^A(.), \ldots, \alpha_N^A(.), 0, \ldots, 0, 0, \ldots, 0, \alpha_1^B(.), \ldots, \alpha_N^B(.), 0, \ldots, 0, 0, \ldots, 0\Big) \\
Ax \;=\; & \Big(\beta_1^{A,A}(.)p_1^A(.) - \frac{1}{2}\beta_1^{A,B}(.)p_1^B(.), \ldots, \beta_N^{A,A}(.)p_N^A(.) - \frac{1}{2}\beta_N^{A,B}(.)p_N^B(.), \\
& \gamma_1^A(.)u_1^A(.), \ldots, \gamma_N^A(.)u_N^A(.), \\
& h_1^A(.)I_1^A(.), \ldots, h_N^A(.)I_N^A(.), \\
& \beta_1^{B,B}(.)p_1^B(.) - \frac{1}{2}\beta_1^{B,A}(.)p_1^A(.), \ldots, \beta_N^{B,B}(.)p_N^B(.) - \frac{1}{2}\beta_N^{B,A}(.)p_N^A(.), \\
& \gamma_1^B(.)u_1^B(.), \ldots, \gamma_N^B(.)u_N^B(.), \\
& h_1^B(.)I_1^B(.), \ldots, h_N^B(.)I_N^B(.) \; \Big).
\end{aligned}
$$

Proof. Follows from (5.7) and (5.8). $\qquad\square$

Theorem 9. *[9] $x \in Y$ is a solution of (5.6) if and only if*

$$a(x, x - \bar{x}) \leq L(x - \bar{x}) \quad \forall \bar{x} \in Q(x). \tag{5.9}$$

Corollary 8. $x \in Y$ *is a solution of (5.9) if and only if*

$$< Ax - f, x - \bar{x} > \quad \leq \quad 0 \quad \forall \bar{x} \in Q(x). \tag{5.10}$$

We observe that the problem is thus reformulated as a *quasi*-variational inequality (QVI), since the set $Q(x)$ in which the inequality must be satisfied depends on the QVI solution x. (If that set was independent of x, we would simply have a variational inequality.) The fact that the inequality is quasi-variational results from the fact that the feasible strategy set depends on the competitor's strategy, i.e. that the game has coupled constraints.

Properties

Definition 2. *Operator* v *defined on* V *is coercive if and only if there exists real* $\lambda > 0$ *such that*

$$v(x) \geq \lambda ||x||^2 \quad \forall x \in V.$$

Lemma 12. *Under Assumption 4, operator* a *(and thus* A*) is coercive.*

Proof. Let $\lambda > 0$ a constant.

$$\begin{aligned}
a(x,x) - \lambda||x||^2 &= \int_0^T \sum_{i=1}^N \Big[(\beta_i^{A,A}(t) - \lambda)(p_i^A(t))^2 + (\beta_i^{B,B}(t) - \lambda)(p_i^B(t))^2 \\
&\quad - \frac{1}{2}(\beta_i^{A,B}(t) + \beta_i^{B,A}(t))p_i^A(t)p_i^B(t) \\
&\quad + (\gamma_i^A(t) - \lambda)(u_i^A(t))^2 + (\gamma_i^B(t) - \lambda)(u_i^B(t))^2 \\
&\quad + (h_i^A(t) - \lambda)(I_i^A(t))^2 + (h_i^B(t) - \lambda)(I_i^B(t))^2 \Big] dt.
\end{aligned}$$

Let

$$\lambda_1 = \min_k \min_i \inf_{t \in [0,T]} h_i^k(t), \quad \lambda_2 = \min_k \min_i \inf_{t \in [0,T]} \gamma_i^k(t).$$

A sufficient condition for the expression above to be positive for all $x \in E$ is that

$\lambda < \lambda_1, \quad \lambda < \lambda_2$ and the symmetric matrix (defined at fixed i, t)

$$M = \begin{bmatrix} \beta^{A,A} - \lambda & -\frac{(\beta^{A,B} + \beta^{B,A})}{4} \\ -\frac{(\beta^{A,B} + \beta^{B,A})}{4} & \beta^{B,B} - \lambda \end{bmatrix}$$

is positive semi-definite for all i, t (we omit the product index and time variable for the sake of clarity).

We notice that

$$M \succeq 0 \quad \Leftrightarrow \quad \left(\text{Tr}(M) \geq 0 \text{ and } \text{Det}(M) \geq 0 \right)$$

$$\Leftrightarrow \quad \begin{cases} \beta^{A,A} + \beta^{B,B} - 2\lambda \geq 0 \\ (\beta^{A,A} - \lambda)(\beta^{B,B} - \lambda) - \frac{1}{16}(\beta^{A,B} + \beta^{B,A})^2 \geq 0 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \lambda \leq \frac{\beta^{A,A} + \beta^{B,B}}{2} \\ \lambda^2 - \lambda(\beta^{A,A} + \beta^{B,B}) + \beta^{A,A}\beta^{B,B} - \frac{1}{16}(\beta^{A,B} + \beta^{B,A})^2 \geq 0. \end{cases}$$

The discriminant of the polynomial above is

$$\begin{aligned} \Delta &= (\beta^{A,A} + \beta^{B,B})^2 - 4\left(\beta^{A,A}\beta^{B,B} - \frac{1}{16}(\beta^{A,B} + \beta^{B,A})^2 \right) \\ &= (\beta^{A,A} - \beta^{B,B})^2 + \frac{1}{4}(\beta^{A,B} + \beta^{B,A})^2 > 0 \end{aligned}$$

so the polynomial has two real roots $\frac{\beta^{A,A} + \beta^{B,B} \pm \sqrt{\Delta}}{2}$ and only one, which we denote κ, satisfies $\kappa \leq \frac{\beta^{A,A} + \beta^{B,B}}{2}$. The polynomial takes positive values below the smaller root and above the larger root. Since we are interested in positive parameters λ, we obtain that

$$(\lambda > 0 \text{ and } M \succeq 0) \quad \Leftrightarrow \quad \begin{cases} \kappa = \frac{\beta^{A,A} + \beta^{B,B} - \sqrt{\Delta}}{2} > 0 \\ 0 < \lambda < \kappa = \frac{\beta^{A,A} + \beta^{B,B} - \sqrt{\Delta}}{2} \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} (\beta^{A,A} - \beta^{B,B})^2 + \frac{1}{4}(\beta^{A,B} + \beta^{B,A})^2 < (\beta^{A,A} + \beta^{B,B})^2 \\ 0 < \lambda < \frac{\beta^{A,A} + \beta^{B,B}}{2} - \frac{\sqrt{4\Delta}}{4} \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \frac{1}{4}(\beta^{A,B} + \beta^{B,A})^2 < 4\beta^{A,A}\beta^{B,B} \\ 0 < \lambda < \frac{\beta^{A,A} + \beta^{B,B}}{2} - \frac{\sqrt{4\Delta}}{4} \end{cases}$$

which is satisfied under Assumption 4 and provided that $0 < \lambda < \lambda_3$ where

$$\lambda_3 = \min_i \inf_{t \in [0,T]} \frac{\beta_i^{A,A}(t) + \beta_i^{B,B}(t)}{2} - \frac{1}{4}\sqrt{4(\beta_i^{A,A}(t) - \beta_i^{B,B}(t))^2 + (\beta_i^{A,B}(t) + \beta_i^{B,A}(t))^2}.$$

As a result, by taking $0 < \lambda < \min\{\lambda_1, \lambda_2, \lambda_3\}$, we obtain that

$$a(x, x) - \lambda ||x||^2 > 0 \quad \forall x \in E.$$

$\square$

5.1.4 Existence of a Nash equilibrium

Definition 3. $Q : X \mapsto 2^X$ is lower semi continuous on D_0 if and only if

for a generalized sequence x_n converging to x in D_0, for every $\bar{x} \in Q(x)$, there exists an integer n_0 and a sequence $\bar{x}_n \in X$ converging to $\bar{x}$, such that $\bar{x}_n \in Q(x_n)$, for all

$$n \geq n_0.$$

Definition 4. $Q : X \mapsto 2^X$ is upper semi continuous on D_0 if and only if

for every generalized sequence $(x_n, \bar{x}_n)$ converging to $(x, \bar{x})$ in $D_0 \times D_0$ and satisfying $\bar{x}_n \in Q(x_n)$, then in the limit $\bar{x} \in Q(x)$.

Definition 5. $Q : X \mapsto 2^X$ is continuous on $D_0 \subset X$ if and only if it is lower semi continuous and upper semi continuous on D_0.

Let S the selection map corresponding to the quasi-variational inequality (5.10): $S : X \mapsto E$ associates with any fixed vector $u \in X$ the unique solution $v \in E$ of the following variational inequality:

$$v \in Q(u), \quad < Av - f, \ v - w > \leq 0 \ \forall w \in Q(u).$$

Definition 6. [99] A set D_0 is stable under selection map S if set $S(u)$ is contained in set D_0 whenever u belongs to D_0.

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Theorem 10. *[99] If*

- *$a(.,.)$ is a coercive continuous bilinear form on the Hilbert space E*

- *f is a continuous linear functional on E*

- *Q is a map that associates with each vector u of the convex closed subset X of E a non empty convex closed subset $Q(u)$ of E*

- *There exists a Hilbert space E_0, which has a continuous injection $\hookrightarrow$ into E, and a non empty convex closed subset D_0 of E_0, with $D_0 \hookrightarrow X$, such that*

    - *D_0 is stable under S*

    - *Q is continuous on D_0*

    - *$S(D_0)$ is bounded in E_0*

then (5.10) admits a solution.

We are going to show that the assumptions from this theorem hold with $E_0 = E$, $D_0 = X$ and the injection $\hookrightarrow$ being the identity function.

Since the space E consists of bounded functions, it is immediate that $S(X)$ is bounded in E.

Proposition 13. *X is stable under S.*

Proof. Let $x \in X$ and let $\bar{x} \equiv S(x)$. Then $\bar{x} \in Q(x) \subset X$. As a result, $S(X) \subset X$. $\square$

Proposition 14. *The mapping Q is upper semi continuous on X.*

Proof. Consider a sequence $(x_n, \bar{x}_n)$ converging to $(x, \bar{x}) = (p, u, I, \bar{p}, \bar{u}, \bar{I})$ in $X \times X$ such that $\bar{x}_n \in Q(x_n)$, i.e. $\bar{x}_n \in X$ and $\forall\, n, i, t$

$$\bar{p}_{n,i}^k(t) \leq \frac{\alpha_i^k(t) + \beta_i^{-k}(t)p_{n,i}^{-k}(t)}{\beta_i^k(t)} \quad \text{and} \quad \dot{\bar{I}}_{n,i}^k(t) = \bar{u}_{n,i}^k(t) - \alpha_i^k(t) + \beta_i^{k,k}(t)\bar{p}_{n,i}^k(t) - \beta_i^{k,-k}(t)p_{n,i}^{-k}(t)$$

where

$$x_n = (x_n^A, x_n^B) = (p_n^A, u_n^A, I_n^A, p_n^B, u_n^B, I_n^B)$$

$$\bar{x}_n = (\bar{x}_n^A, \bar{x}_n^B) = (\bar{p}_n^A, \bar{u}_n^A, \bar{I}_n^A, \bar{p}_n^B, \bar{u}_n^B, \bar{I}_n^B)$$

$$x = (x^A, x^B) = (p^A, u^A, I^A, p^B, u^B, I^B)$$

$$\bar{x} = (\bar{x}^A, \bar{x}^B) = (\bar{p}^A, \bar{u}^A, \bar{I}^A, \bar{p}^B, \bar{u}^B, \bar{I}^B)$$

Since $\lim p_{n,i}^{-k}(t) = p_i^{-k}(t)$, $\lim \bar{p}_{n,i}^k(t) = \bar{p}_i^k(t)$, $\lim \bar{u}_{n,i}^k(t) = \bar{u}_i^k(t)$, and $\lim \bar{I}_{n,i}^k(t) = \bar{I}_i^k(t)$, we obtain that $\bar{I}$ is differentiable and

$$\dot{\bar{I}}_i^k(t) = \bar{u}_i^k(t) - \alpha_i^k(t) + \beta_i^{k,k}(t)\bar{p}_i^k(t) - \beta_i^{k,-k}(t)p_i^{-k}(t) \ \forall i, \ t$$

$$\text{and } \bar{p}_i^k(t) \leq \frac{\alpha_i^k(t) + \beta_i^{k,-k}(t)p_i^{-k}(t)}{\beta_i^{k,k}(t)}.$$

As a result, $\bar{x} \in Q(x)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Proposition 15. *The mapping Q is lower semi continuous on X.*

Proof. Consider a sequence $x_n = (x_n^A, x_n^B) = (p_n^A, u_n^A, I_n^A, p_n^B, u_n^B, I_n^B) \in X$ converging to $x = (x^A, x^B) = (p^A, u^A, I^A, p^B, u^B, I^B)$. Let $\bar{x} = (\bar{x}^A, \bar{x}^B) = (\bar{p}^A, \bar{u}^A, \bar{I}^A, \bar{p}^B, \bar{u}^B, \bar{I}^B) \in Q(x)$. Since X is closed, $x \in X$. Let $\bar{x}_n = (\bar{x}_n^A, \bar{x}_n^B) = (\bar{p}_n^A, \bar{u}_n^A, \bar{I}_n^A, \bar{p}_n^B, \bar{u}_n^B, \bar{I}_n^B)$ such that $\forall \ n, i, t$,

$$\bar{u}_{n,i}^k(t) = \bar{u}_i^k(t)$$

$$\bar{p}_{n,i}^k(t) = \begin{cases} \bar{p}_i^k(t) & \text{if } \bar{p}_i^k(t) = 0 \text{ and } p_{n,i}^{-k}(t) - p_i^{-k}(t) < 0 \\ \bar{p}_i^k(t) + \frac{\beta_i^{k,-k}(t)}{\beta_i^{k,k}(t)}(p_{n,i}^{-k}(t) - p_i^{-k}(t)) & \text{if } \bar{p}_i^k(t) > 0 \text{ or } p_{n,i}^{-k}(t) - p_i^{-k}(t) > 0 \end{cases}$$

$$\bar{I}_{n,i}^k(t) = I_i^{k^0} + \int_0^T (\bar{u}_{n,i}^k(s) - \alpha_i^k(s) + \beta_i^{k,k}(s)\bar{p}_{n,i}^k(s) - \beta_i^{k,-k}(s)p_{n,i}^{-k}(s))ds.$$

We want to show that $\bar{x}_n$ constructed above satisfies $\bar{x}_n \to \bar{x}$ and $\bar{x}_n \in Q(x_n)$. We clearly have $\bar{I}_n(0) = I^0$, $\bar{u}_n(.) \geq 0$ and $\sum_i \bar{u}_{n,i}^k(t) \leq K^k(t)$.

We notice that

$$
\begin{aligned}
\bar{I}_{n,i}^k(t) - \bar{I}_i^k(t) &= \int_0^t (\beta_i^{k,k}(s)(\bar{p}_{n,i}^k(s) - \bar{p}_i^k(s)) - \beta_i^{k,-k}(s)(p_{n,i}^{-k}(s) - p_i^{-k}(s)))ds \\
&= -\int_{D_i^k \cap [0,t]} \beta_i^{k,-k}(s)(p_{n,i}^{-k}(s) - p_i^{-k}(s)))ds \geq 0
\end{aligned}
$$

where $D_i^k = \{t \in [0,T] : \bar{p}_i^k(t) = 0 \text{ and } p_{n,i}^{-k}(t) - p_i^{-k}(t) < 0\}$.

Therefore, $\bar{I}_n \geq \bar{I} \geq 0$.

Also, when $\bar{p}_{n,i}^k(t)$ is equal to the expression $\bar{p}_i^k(t) + \frac{\beta_i^{k,-k}(t)}{\beta_i^{k,k}(t)}(p_{n,i}^{-k}(t) - p_i^{-k}(t))$, since $p_{n,i}^{-k}(t) - p_i^{-k}(t) \to 0$, we notice that for n sufficiently large $\bar{p}_{n,i}^k(t) \geq 0$ (either it is equal to a positive term to which we add a term that tends to zero, or it is zero plus a positive term that tends to zero). Clearly, when $\bar{p}_{n,i}^k(t)$ is given by the first expression, this still holds.

As a result, $\bar{x}_n \in X$.

Moreover, $\bar{u}_n \to \bar{u}$ and since $p_n^{-k} - p^{-k} \to 0$ we also have $\bar{I}_n \to \bar{I}$, $\bar{p}_n \to \bar{p}$, so that $\bar{x}_n \to \bar{x}$.

Finally, we notice that when $\bar{p}_{n,i}^k(t)$ is equal to the expression $\bar{p}_i^k(t) + \frac{\beta_i^{k,-k}(t)}{\beta_i^{k,k}(t)}(p_{n,i}^{-k}(t) - p_i^{-k}(t))$, then

$$
\bar{p}_{n,i}^k(t) - \frac{\alpha_i^k(t) + \beta_i^{k,-k}(t)p_{n,i}^{-k}(t)}{\beta_i^{k,k}(t)} = \bar{p}_i^k(t) - \frac{\alpha_i^k(t) + \beta_i^{k,-k}(t)p_i^{-k}(t)}{\beta_i^{k,k}(t)} \leq 0.
$$

Clearly, when $\bar{p}_{n,i}^k(t)$ is equal to $\bar{p}_i^k(t)$, then $\bar{p}_{n,i}^k(t) = \bar{p}_i^k(t) = 0$ so the inequality

$$
\bar{p}_{n,i}^k(t) \leq \frac{\alpha_i^k(t) + \beta_i^{k,-k}(t)p_{n,i}^{-k}(t)}{\beta_i^{k,k}(t)}
$$

it is also satisfied. As a result $\bar{x}_n \in Q(x_n)$. $\square$

Corollary 9. *Q is continuous on X.*

The following result then follows from Theorem 10.

Theorem 11. *Under Assumption 4, there exists a Nash equilibrium to the deterministic problem under competition (5.6).*

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5.2 Duopoly setting with uncertain data

In this section, we assume that all demand parameters are uncertain within the model detailed in Chapter 2. We recall that for each supplier k, the realized demand for product i is given by

$$\tilde{d}_i^k(t) = \tilde{\alpha}_i^k(t) - \tilde{\beta}_i^{k,k}(t)p_i^k(t) + \tilde{\beta}_i^{k,-k}(t)p_i^{-k}(t)$$

Denoting

$$z_i^k(t) \equiv \frac{\tilde{\alpha}_i^k(t) - \alpha_i^k(t)}{\hat{\alpha}_i^k(t)}$$

$$y_i^{k,k}(t) \equiv \frac{\tilde{\beta}_i^{k,k}(t) - \beta_i^{k,k}(t)}{\hat{\beta}_i^{k,k}(t)}$$

$$y_i^{k,-k}(t) \equiv \frac{\tilde{\beta}_i^{k,-k}(t) - \beta_i^{k,-k}(t)}{\hat{\beta}_i^{k,-k}(t)}$$

as the scaled variations, it follows that the constraints can be rewritten

$$z_i^k(t), \ y_i^{k,k}(t), \ y_i^{k,-k}(t) \ \in \ [-1,1] \ \ \forall t,i,k$$

$$\int_0^t |z_i^k(s)|ds \ \leq \ \Gamma_i^k(t) \ \ \forall t,i,k$$

$$\int_0^t |y_i^{k,k}(s)|ds \ \leq \ \Theta_i^{k,k}(t) \ \ \forall t,i,k$$

$$\int_0^t |y_i^{k,-k}(s)|ds \ \leq \ \Theta_i^{k,-k}(t) \ \ \forall t,i,k.$$

Uncertainty set $\mathcal{F}^k$ contains all realizations $(\tilde{\alpha}^k, \ \tilde{\beta}^{k,k}, \ \tilde{\beta}^{k,-k})$ satisfying the constraints above.

We aim at maximizing the nominal objective such that the constraints are satisfied for any feasible realization of the data.

5.2.1 Robust counterpart problem

The best response problem faced by supplier k under uncertainty can be written as follows (we omit the time argument for the sake of clarity in each of the terms below,

that are all time dependent - except the initial inventory level):

$$\max_{p^k(.),u^k(.)} \quad \int_0^T \sum_{i=1}^N \left(p_i^k(\alpha_i^k - \beta_i^{k,k}p_i^k + \beta_i^{k,-k}p_i^{-k}) - \gamma_i^k(u_i^k)^2 - h_i^k(I_i^k)^2 \right)dt \tag{5.11}$$

$$\text{s.t.} \quad \dot{I}_i^k = u_i^k - \alpha_i^k + \beta_i^{k,k}p_i^k - \beta_i^{k,-k}p_i^{-k}, \quad \forall t \in [0,T] \ \ \forall i \tag{5.12}$$

$$\sum_{i=1}^N u_i^k \leq K^k, \quad \forall t \in [0,T]$$

$$u_i^k, \ p_i^k \geq 0, \quad \forall t \in [0,T] \ \ \forall i$$

$$\forall \ (\tilde{\alpha}^k, \ \tilde{\beta}^{k,k}, \ \tilde{\beta}^{k,-k}) \ \in \ \mathcal{F}^k,$$

$$\dot{\tilde{I}}_i^k = u_i^k - \tilde{\alpha}_i^k + \tilde{\beta}_i^{k,k}p_i^k - \tilde{\beta}_i^{k,-k}p_i^{-k}, \quad \forall t \in [0,T] \ \ \forall i \tag{5.13}$$

$$p_i^k \leq \frac{\tilde{\alpha}_i^k + \tilde{\beta}_i^{k,-k}p_i^{-k}}{\tilde{\beta}_i^{k,k}}, \quad \forall t \in [0,T] \ \ \forall i \tag{5.14}$$

$$\tilde{I}_i^k \geq 0, \quad \forall t \in [0,T] \ \ \forall i \tag{5.15}$$

$$\tilde{I}_i^k(0) = I_i^k(0) = I_i^0, \quad \forall i$$

Notice that the fluid equation (5.12) describes the evolution of the nominal inventory level, while the fluid equation (5.13) describes the evolution of the realized inventory level. The no backorder constraint (5.15) is a constraint on the realized inventory level. The non negativity of the realized demand leads to an upper bound on prices (5.14). Finally, as we discussed before, the objective function involves the nominal inventory level.

Theorem 12. *The robust counterpart of Problem (5.11) is the following:*

$$\max \quad \int_0^T \sum_{i=1}^N \Big(p_i^k(t)(\alpha_i^k(t) - \beta_i^{k,k}(t)p_i^k(t) + \beta_i^{k,-k}(t)p_i^{-k}(t)) - \gamma_i^k(u_i^k(t))^2 $$
$$ -h_i^k(t)(I_i^k(t))^2 \Big) dt$$

$$s.t. \quad \dot{I}_i^k(t) = u_i^k(t) - \alpha_i^k(t) + \beta_i^{k,k}(t)p_i^k(t) - \beta_i^{k,-k}(t)p_i^{-k}(t) \quad \forall i \quad \forall t \in [0,T]$$

$$\sum_{i=1}^N u_i^k(t) \le K^k(t) \quad \forall t \in [0,T]$$

$$p_i^k(t) \le \frac{\alpha_i^k(t) - \hat{\alpha}_i^k(t) + (\beta_i^{k,-k}(t) - \hat{\beta}_i^{k,-k}(t))p_i^{-k}(t)}{\beta_i^{k,k}(t) + \hat{\beta}_i^{k,k}(t)} \quad \forall i \quad \forall t \in [0,T]$$

$$I_i^k(t) \ge \Omega_i^k(t, p_i^k(.), p_i^{-k}(.)) \quad \forall i \quad \forall t \in [0,T]$$

$$p_i^k(t), \ u_i^k(t) \ge 0 \quad \forall i \quad \forall t \in [0,T]$$

$$I_i^k(0) = I_i^{k^0} \quad \forall i$$

where $\Omega_i^k(t, p_i^k(.), p_i^{-k}(.))$ *is a minimum inventory security level given by (5.17), (5.18), (5.19), and (5.20).*

The problem written in this form is intuitive in the sense that the uncertainty on the demand parameters translates into an upper bound on the prices and on a minimum inventory level that are tighter than in the deterministic model (5.1). As a result, even in the presence of data perturbation (as defined by the budget of uncertainty and ranges of variation), these stronger constraints guarantee that the bounds will be satisfied.

Proof. Similarly to the reasoning detailed in Chapter 4, we obtain that in the robust counterpart, the price constraint (5.14) is written as

$$p_i^k(t) \le \frac{\alpha_i^k(t) - \hat{\alpha}_i^k(t) + (\beta_i^{k,-k}(t) - \hat{\beta}_i^{k,-k}(t))p_i^{-k}(t)}{\beta_i^{k,k}(t) + \hat{\beta}_i^{k,k}(t)} \quad \forall i, \ t.$$

We observe that we may write the realized inventory level at time t as follows:

$$\tilde{I}_i^k(t) = I_i^k(t) - \int_0^t (z_i^k(s)\hat{\alpha}_i^k(s) - y_i^{k,k}(s)\hat{\beta}_i^{k,k}(s)p_i^k(s) + y_i^{k,-k}(s)\hat{\beta}_i^{k,-k}(s)p_i^{-k}(s))ds$$

where

$$I_i^k(t) = I_i^{k^0} + \int_0^t (u_i^k(s) - \alpha_i^k(s) + \beta_i^{k,k}(s)p_i^k(s) - \beta_i^{k,-k}(s)p_i^{-k}(s))ds$$

is the nominal inventory level.

The no backorders constraint at time t indirectly involves the control decisions on prices and production rates from time 0 up to time t, and as such, the budget of uncertainty has an impact on it. We obtain the equivalent deterministic constraint by seeking the feasible realization that makes the no backorders constraint most difficult to satisfy (i.e. that yields the smallest inventory level). As a result, constraint (5.15) is equivalent to

$$I_i^k(t) \geq \Omega_i^k(t) \quad \forall i, t \tag{5.16}$$

where $\Omega_i^k(t)$ can be viewed as a minimum inventory security level that can be computed via the following deterministic continuous linear program

$$\Omega_i^k(t) = \max_{z_i^k(.),y_i^{k,k}(.),y_i^{k,-k}(.)} \int_0^t (z_i^k(s)\hat{\alpha}_i^k(s) - y_i^{k,k}(s)\hat{\beta}_i^{k,k}(s)p_i^k(s)$$
$$+ y_i^{k,-k}(s)\hat{\beta}_i^{k,-k}(s)p_i^{-k}(s))ds$$

$$\text{s.t.} \quad z_i^k(s), \; y_i^{k,k}(s), \; y_i^{k,-k}(s) \in [-1,1] \quad \forall s \in [0,t]$$

$$\int_0^t |z_i^k(s)|ds \leq \Gamma_i^k(t)$$

$$\int_0^t |y_i^{k,k}(s)|ds \leq \Theta_i^{k,k}(t)$$

$$\int_0^t |y_i^{k,-k}(s)|ds \leq \Theta_i^{k,-k}(t).$$

Notice that $\Omega_i^k(t)$ depends on the pricing strategies of both suppliers on $[0, t]$ via the objective function. To make this dependence clear, we will denote it by $\Omega_i^k(t, p^k(.), p^{-k}(.))$.

Equivalently, after a change of variables, $\Omega_i^k(t, p^k(.), p^{-k}(.)) =$

$$\max_{z_i^k(.), y_i^{k,k}(.), y_i^{k,-k}(.)} \int_0^t (z_i^k(s)\hat{\alpha}_i^k(s) + y_i^{k,k}(s)\hat{\beta}_i^{k,k}(s)p_i^k(s) + y_i^{k,-k}(s)\hat{\beta}_i^{k,-k}(s)p_i^{-k}(s))ds$$

$$\text{s.t.} \quad z_i^k(t), \ y_i^{k,k}(t), \ y_i^{k,-k}(t) \ \in [0,1] \ \ \forall s \in [0,t]$$

$$\int_0^t z_i^k(s)ds \le \Gamma_i^k(t)$$

$$\int_0^t y_i^{k,k}(s)ds \le \Theta_i^{k,k}(t)$$

$$\int_0^t y_i^{k,-k}(s)ds \le \Theta_i^{k,-k}(t).$$

This problem separates into three subproblems, that are continuous linear programs, as follows:

$$\Omega_i^k(t, p^k(.), p^{-k}(.)) = \Omega_i^{k^1}(t) + \Omega_i^{k^2}(t, p^k(.)) + \Omega_i^{k^3}(t, p^{-k}(.)) \qquad (5.17)$$

with

$$\Omega_i^{k^1}(t) = \max_{z_i^k(.)} \int_0^t z_i^k(s)\hat{\alpha}_i^k(s)ds$$

$$\text{s.t.} \quad 0 \le z_i^k(t) \le 1 \ \ \forall s \in [0,t]$$

$$\int_0^t z_i^k(s)ds \le \Gamma_i^k(t)$$

$$\Omega_i^{k^2}(t, p^k(.)) = \max_{y_i^{k,k}(.)} \int_0^t y_i^{k,k}(s)\hat{\beta}_i^{k,k}(s)p_i^k(s)ds$$

$$\text{s.t.} \quad 0 \le y_i^{k,k}(t) \le 1 \ \ \forall s \in [0,t]$$

$$\int_0^t y_i^{k,k}(s)ds \le \Theta_i^{k,k}(t)$$

$$\Omega_i^{k^3}(t, p^{-k}(.)) = \max_{y_i^{k,-k}(.)} \int_0^t y_i^{k,-k}(s)\hat{\beta}_i^{k,-k}(s)p_i^{-k}(s)ds$$

$$\text{s.t.} \quad 0 \le y_i^{k,-k}(t) \le 1 \ \ \forall s \in [0,t]$$

$$\int_0^t y_i^{k,-k}(s)ds \le \Theta_i^{k,-k}(t).$$

Notice that $\Omega_i^{k^2}(t, p^k(.))$ depends on $p_i^k(s)$, $0 \leq s \leq t$ and $\Omega_i^{k^3}(t, p^{-k}(.))$ depends on $p_i^{-k}(s)$, $0 \leq s \leq t$.

Under regularity assumptions (see Chapter 4) we have strong duality and the respective dual subproblems are given by the continuous linear programs:

$$\Omega_i^{k^1}(t) = \min_{\omega_i^k(t), r_i^k(.,t)} \quad \omega_i^k(t)\Gamma_i^k(t) + \int_0^t r_i^k(s,t)ds \qquad (5.18)$$

$$\text{s.t.} \quad \omega_i^k(t) + r_i^k(s,t) \geq \hat{\alpha}_i^k(s) \quad \forall s \in [0,t]$$

$$\omega_i^k(t) \geq 0$$

$$r_i^k(s,t) \geq 0 \quad \forall s \in [0,t]$$

$$\Omega_i^{k^2}(t, p^k(.)) = \min_{\theta_i^{k,k}(t), q_i^{k,k}(.,t)} \quad \theta_i^{k,k}(t)\Theta_i^{k,k}(t) + \int_0^t q_i^{k,k}(s,t)ds \qquad (5.19)$$

$$\text{s.t.} \quad \theta_i^{k,k}(t) + q_i^{k,k}(s,t) \geq \hat{\beta}_i^{k,k}(s)p_i^k(s) \quad \forall s \in [0,t]$$

$$\theta_i^{k,k}(t) \geq 0$$

$$q_i^{k,k}(s,t) \geq 0 \quad \forall s \in [0,t]$$

$$\Omega_i^{k^3}(t, p^{-k}(.)) = \min_{\theta_i^{k,-k}(t), q_i^{k,-k}(.,t)} \quad \theta_i^{k,-k}(t)\Theta_i^{k,-k}(t) + \int_0^t q_i^{k,-k}(s,t)ds \qquad (5.20)$$

$$\text{s.t.} \quad \theta_i^{k,-k}(t) + q_i^{k,-k}(s,t) \geq \hat{\beta}_i^{k,-k}(s)p_i^{-k}(s) \quad \forall s \in [0,t]$$

$$\theta_i^{k,-k}(t) \geq 0$$

$$q_i^{k,-k}(s,t) \geq 0 \quad \forall s \in [0,t].$$

$\square$

The uncertainty of demand thus translates into protection levels for the prices (via an upper bound) and for the inventory levels (via a lower bound) that are stronger than in the nominal case so that, even with some variation in the demand parameters - within the introduced uncertainty constraints - the realized inventory levels will remain positive, and realized demand will remain non negative.

5.2.2 Properties of the minimum inventory security level

In Chapter 4, we show how each of the three dual subproblems can be solved. Even though there is no closed-form solution in general, we show how to calculate the objective value of the dual subproblems as a function of the data - and the prices for $\Omega_i^{k^2}(t, p^k(.))$ and $\Omega_i^{k^3}(t, p^{-k}(.))$.

Referring to that chapter, the reader may notice that the notion of effective budget of uncertainty we introduced is validated. Indeed the robust counterpart involves the actual value of the budget of uncertainty for each parameter only if the budget of uncertainty at time t is smaller than t. Otherwise, the robust formulation does not depend on it, and the security level requirement corresponds to the worst case scenario on $[0, t]$. In the case where the actual budget of uncertainty does matter (i.e. has a value smaller than t), then the security level is lower, thus easier to satisfy, which makes the solution less conservative and yields a higher objective.

In particular, we note from Chapter 4 that

$$\Omega_i^k(t, p_i^k(.), p_i^{-k}(.)) \leq \int_0^t \left(\hat{\alpha}_i^k(s) + \hat{\beta}_i^{k,k}(s) p_i^k(s) + \hat{\beta}_i^{k,-k}(s) p_i^{-k}(s) \right) ds. \tag{5.21}$$

Proposition 16. *The minimum inventory security level $\Omega_i^k(t, p_i^k(.), p_i^{-k}(.))$ is non decreasing with time.*

Proof. We prove this result by showing that $\Omega_i^{k^1}, \Omega_i^{k^2}$, and $\Omega_i^{k^3}$ are each non decreasing with time. The proof for $\Omega_i^{k^1}$ is similar to the one given in detail in Chapter 4 for the minimum inventory security level in a monopoly setting. We show similarly that $\Omega_i^{k^2}$ and $\Omega_i^{k^3}$ are non decreasing with time. $\qquad\square$

Proposition 17. *The minimum inventory security level $\Omega_i^k(t, p_i^k(.), p_i^{-k}(.))$ is non decreasing with each budget of uncertainty $\Gamma_i^k(t)$, $\Theta_i^{k,k}(t)$, $\Theta_i^{k,-k}(t)$.*

Proof. We prove this result by showing that $\Omega_i^{k^1}, \Omega_i^{k^2}$, and $\Omega_i^{k^3}$ are respectively non decreasing with $\Gamma_i^k(t)$, $\Theta_i^{k,k}(t)$ and $\Theta_i^{k,-k}(t)$. We give below the detailed proof for $\Omega_i^{k^1}$,

and it is similar for $\Omega_i^{k^2}$ and $\Omega_i^{k^3}$.

Let $\Gamma_i'^k(t)$ such that $\Gamma_i^k(t) < \Gamma_i'^k(t)$, and let $(\omega(t), r(., t))$ and $(\omega'(t), r'(., t))$ the respective optimal solutions of the dual subproblems, which we denote (D) and (D'), and $\Omega_i^{k^1}(t)$ and $\Omega_i'^{k^1}(t)$ the respective optimal objective values. Notice that $(\omega'(t), r'(., t))$ is feasible for (D), since (D) and (D') have the same feasible sets. Therefore

$$
\begin{aligned}
\Omega_i^{k^1}(t) &= \omega(t)\Gamma_i^k(t) + \int_0^t r(s,t)ds \\
&\leq \omega'(t)\Gamma_i^k(t) + \int_0^t r'(s,t)ds, \quad \text{since } (\omega'(t), r'(., t)) \text{ is feasible suboptimal} \\
&\leq \omega'(t)\Gamma_i'^k(t) + \int_0^t r'(s,t)ds, \quad \text{since } \omega'(t) \geq 0, \quad \Gamma_i^k(t) < \Gamma_i'^k(t) \\
&= \Omega_i'^{k^1}(t).
\end{aligned}
$$

$\square$

These properties make sense at an intuitive level since the more uncertainty in the problem, the higher the protection level should be. Moreover, we observe that the minimum inventory level is also non decreasing in the half-lengths of allowed ranges of variation $\hat{\alpha}_i^k(.)$, $\hat{\beta}_i^{k,k}(.)$, $\hat{\beta}_i^{k,k}(.)$, which makes sense for the same reason. Furthermore, the cumulative uncertainty over time can only increase, therefore the protection level should also increase over time.

Proposition 18. *The minimum inventory security level* $\Omega_i^k(t, p_i^k(.), p_i^{-k}(.))$ *is convex in* $p_i^k(.)$ *(and thus, by symmetry, in* $p_i^{-k}(.))$.

Proof. Let us, for example, show the convexity of $\Omega_i^k(t, p_i^k(.), p_i^{-k}(.))$ in $p_i^k(.)$.

$\Omega_i^k(t, p_i^k(.), p_i^{-k}(.))$ is the sum of the objective values of the three corresponding dual subproblems. Two of these subproblems are independent of $p_i^k(.)$ (the ones addressing uncertainty on respectively $\hat{\alpha}_i^k$ and $\hat{\beta}_i^{k,-k}$). Therefore, it is necessary and sufficient to show that $\Omega_i^{k^2}(t, p_i^k(.))$ is convex in $p_i^k(.)$.

The subproblem involving prices $p_i^k(.)$ is

$$\Omega_i^{k^2}(t, p_i^k(.)) = \min_{\theta_i^{k,k}(t), q_i^{k,k}(.,t)} \quad \theta_i^{k,k}(t)\Theta_i^{k,k}(t) + \int_0^t q_i^{k,k}(s,t)ds$$

$$\text{s.t.} \quad \theta_i^{k,k}(t) + q_i^{k,k}(s,t) \geq \hat{\beta}_i^{k,k}(s)p_i^k(s) \quad \forall s \in [0,t]$$

$$\theta_i^{k,k}(t) \geq 0$$

$$q_i^{k,k}(s,t) \geq 0 \quad \forall s \in [0,t].$$

Let $p_i^{k^1}(.)$, $p_i^{k^2}(.)$, $\lambda \in (0,1)$ and $p_i^{k^3}(.) = \lambda p_i^{k^1}(.) + (1-\lambda)p_i^{k^2}(.)$. We will use superscripts 1, 2, 3 similarly to denote the optimal solutions of the corresponding subproblems with input $p_i^{k^1}(.)$, $p_i^{k^2}(.)$, $p_i^{k^3}(.)$. Clearly,

$$\lambda\theta_i^{k,k^1}(t) + \lambda q_i^{k,k^1}(s,t) \geq \hat{\beta}_i^{k,k}(s)\lambda p_i^{k^1}(s) \quad \forall s \in [0,t]$$

$$(1-\lambda)\theta_i^{k,k^2}(t) + (1-\lambda)q_i^{k,k^2}(s,t) \geq \hat{\beta}_i^{k,k}(s)(1-\lambda)p_i^{k^2}(s) \quad \forall s \in [0,t].$$

Adding these inequalities shows that $\lambda\theta_i^{k,k^1}(t) + (1-\lambda)\theta_i^{k,k^2}(t)$ along with $\lambda q_i^{k,k^1}(.,t) + (1-\lambda)q_i^{k,k^2}(.,t)$, is feasible for the subproblem with input $p_i^{k^3}(.)$ (non negativity is clearly satisfied). Since it may not yield the optimal objective value, we have

$$\Omega_i^{k^2}(t, p_i^{k^3}(.)) \leq (\lambda\theta_i^{k,k^1}(t) + (1-\lambda)\theta_i^{k,k^2}(t))\Theta_i^{k,k}(t)$$
$$+ \int_0^t (\lambda q_i^{k,k^1}(s,t) + (1-\lambda)q_i^{k,k^2}(s,t))ds$$
$$= \lambda\Omega_i^{k^1}(t, p_i^{k^1}(.)) + (1-\lambda)\Omega_i^{k^1}(t, p_i^{k^2}(.)).$$

$\square$

Corollary 10. *For a fixed $p^{-k}(.)$, the feasible set of the robust counterpart above is convex.*

Proof. All constraints are linear except the minimum inventory security level constraint. This constraint can be written as $\Omega_i^k(t, p_i^k(.), p_i^{-k}(.)) - I_i^k(t) \leq 0$. Since $\Omega_i^k(t, p_i^k(.), p_i^{-k}(.))$ is convex in terms of the prices, the left hand side is convex in the variables, which yields the result. $\square$

In this formulation of the robust counterpart, we have a deterministic fluid model with no more variables than the initial problem. Nevertheless, the constraints are no longer linear. However, we still have a convex optimization problem since the objective to maximize is concave (quadratic) over a convex set.

5.2.3 Existence of a Nash equilibrium

We can show the following using the upper bound on feasible prices described in the previous section.

Proposition 19. *Prices $p_i^k(t)$ are bounded from above by $p_{i_{max}}^k(t) =$*

$$\frac{(\alpha_i^k(t) - \hat{\alpha}_i^k(t))(\beta_i^{-k,-k}(t) + \hat{\beta}_i^{-k,-k}(t)) + (\alpha_i^{-k}(t) - \hat{\alpha}_i^{-k}(t))(\beta_i^{k,-k}(t) - \hat{\beta}_i^{k,-k}(t))}{(\beta_i^{k,k}(t) + \hat{\beta}_i^{k,k}(t))(\beta_i^{-k,-k}(t) + \hat{\beta}_i^{-k,-k}(t)) - (\beta_i^{k,-k}(t) - \hat{\beta}_i^{k,-k}(t))(\beta_i^{-k,k}(t) - \hat{\beta}_i^{-k,k}(t))}.$$

In particular, all control and state variables are bounded. (The production rates are bounded above by the production capacity rate, and the inventory level evolve continuously on a finite time horizon from a finite initial value.)

Assumption 12. *We assume that*

$$2\sum_{i=1}^{N}\left(\hat{\alpha}_i^k(t) + \hat{\beta}_i^{k,k}(t)p_{i_{max}}^k(t) + \hat{\beta}_i^{k,-k}(t)p_{i_{max}}^{-k}(t)\right) \le K^k(t) \quad \forall t, \ k = A, B.$$

This assumption guarantees that the feasible set is not empty (see proof of Theorem 13). More specifically, it ensures that the capacity level is sufficient to guarantee the feasibility of the problem. Intuitively, the larger the range of allowed variation for the parameters, the higher the security level for the inventory, and the lower the upper bound on prices. To stop the inventory from decreasing and going below its minimum security level, we can either increase production or increase prices. Since the prices are bounded from above, the more uncertainty, the higher the production rates will be required to be in order to satisfy such a guarantee. This is the reason why we have to impose a minimum production capacity available. In other words, if the production capacity level is too low, we are not able to immune the solution

against too much uncertainty on data parameters.

Theorem 13. *Under Assumptions 4 and 12, there exists a Nash equilibrium to the general robust formulation.*

Proof. We start by reformulating the robust counterpart problem.

The following lemma follows from the derivation of the robust counterpart by integrating the dual subproblems into the main optimization problem.

Lemma 13. *The robust counterpart for the best response problem faced by supplier k (at $p_i^{-k}(.)$ fixed) can be written:*

$$\max \quad \int_0^T \sum_{i=1}^N \left(p_i^k(t)(\alpha_i^k(t) - \beta_i^{k,k}(t)p_i^k(t) + \beta_i^{k,-k}(t)p_i^{-k}(t)) - \gamma_i^k(u_i^k(t))^2 \right.$$
$$\left. -h_i^k(t)(I_i^k(t))^2 \right) dt \qquad (5.22)$$

$$s.t. \quad \dot{I}_i^k(t) = u_i^k(t) - \alpha_i^k(t) + \beta_i^{k,k}(t)p_i^k(t) - \beta_i^{k,-k}(t)p_i^{-k}(t) \quad \forall i \ \ \forall t \in [0,T]$$

$$I_i^k(0) = I_i^{k0} \quad \forall i$$

$$\sum_{i=1}^N u_i^k(t) \le K^k(t) \quad \forall t \in [0,T]$$

$$p_i^k(t) \le \frac{\alpha_i^k(t) - \hat{\alpha}_i^k(t) + (\beta_i^{k,-k}(t) - \hat{\beta}_i^{k,-k}(t))p_i^{-k}(t)}{\beta_i^{k,k}(t) + \hat{\beta}_i^{k,k}(t)} \quad \forall i \ \ \forall t \in [0,T]$$

$$p_i^k(t), \ u_i^k(t) \ge 0 \quad \forall i \ \ \forall t \in [0,T]$$

$$I_i^k(t) \ge \omega_i^k(t)\Gamma_i^k(t) + \theta_i^{k,k}(t)\Theta_i^{k,k}(t) + \theta_i^{k,-k}(t)\Theta_i^{k,-k}(t)$$
$$+ \int_0^t (r_i^k(s,t) + q_i^{k,k}(s,t) + q_i^{k,-k}(s,t))ds \quad \forall i \ \ \forall t \in [0,T]$$

$$\omega_i^k(t) + r_i^k(s,t) \ge \hat{\alpha}_i^k(s) \quad \forall i \ \ \forall s \in [0,t] \ \ \forall t \in [0,T]$$

$$\theta_i^{k,k}(t) + q_i^{k,k}(s,t) \ge \hat{\beta}_i^{k,k}(s)p_i^k(s) \quad \forall i \ \ \forall s \in [0,t] \ \ \forall t \in [0,T]$$

$$\theta_i^{k,-k}(t) + q_i^{k,-k}(s,t) \ge \hat{\beta}_i^{k,-k}(s)p_i^{-k}(s) \quad \forall i \ \ \forall s \in [0,t] \ \ \forall t \in [0,T]$$

$$\omega_i^k(t), \ \theta_i^{k,k}(t), \ \theta_i^{k,k}(t) \ge 0 \quad \forall i \ \ \forall t \in [0,T]$$

$$r_i^k(s,t), \ q_i^{k,k}(s,t), \ q_i^{k,-k}(s,t) \ge 0 \quad \forall i \ \ \forall s \in [0,t] \ \ \forall t \in [0,T].$$

Adding the constraint that prices must be lower than their maximum price leaves the problem unchanged. After introducing new variables in order to reformulate the

constraint involving an integral expression, we obtain the following:

Lemma 14. *The robust counterpart problem can be reformulated equivalently as a deterministic fluid model with linear constraints:*

$$\max \quad \int_0^T \sum_{i=1}^N \left(p_i^k(t)(\alpha_i^k(t) - \beta_i^{k,k}(t)p_i^k(t) + \beta_i^{k,-k}(t)p_i^{-k}(t)) - \gamma_i^k(u_i^k(t))^2 \right.$$

$$\left. - h_i^k(t)(I_i^k(t))^2 \right) dt \qquad (5.23)$$

$$s.t. \quad \dot{I}_i^k(t) = u_i^k(t) - \alpha_i^k(t) + \beta_i^{k,k}(t)p_i^k(t) - \beta_i^{k,-k}(t)p_i^{-k}(t) \quad \forall i \ \ \forall t \in [0,T]$$

$$I_i^k(0) = I_i^{k^o} \quad \forall i$$

$$\sum_{i=1}^N u_i^k(t) \le K^k(t) \quad \forall t \in [0,T]$$

$$p_i^k(t) \le \frac{\alpha_i^k(t) - \hat{\alpha}_i^k(t) + (\beta_i^{k,-k}(t) - \hat{\beta}_i^{k,-k}(t))p_i^{-k}(t)}{\beta_i^{k,k}(t) + \hat{\beta}_i^{k,k}(t)} \quad \forall i \ \ \forall t \in [0,T]$$

$$p_i^k(t) \le p_{i_{max}}^k(t) \quad \forall i \ \ \forall t \in [0,T]$$

$$p_i^k(t), \ u_i^k(t) \ge 0 \quad \forall i \ \ \forall t \in [0,T]$$

$$I_i^k(t) \ge \omega_i^k(t)\Gamma_i^k(t) + \theta_i^{k,k}(t)\Theta_i^{k,k}(t) + \theta_i^{k,-k}(t)\Theta_i^{k,-k}(t) + R_i^k(t,t)$$

$$+ S_i^{k,k}(t,t) + S_i^{k,-k}(t,t) \quad \forall i \ \ \forall t \in [0,T]$$

$$\omega_i^k(t) + r_i^k(s,t) \ge \hat{\alpha}_i^k(s) \quad \forall i \ \ \forall s \in [0,t] \ \ \forall t \in [0,T]$$

$$\theta_i^{k,k}(t) + q_i^{k,k}(s,t) \ge \hat{\beta}_i^{k,k}(s)p_i^k(s) \quad \forall i \ \ \forall s \in [0,t] \ \ \forall t \in [0,T]$$

$$\theta_i^{k,-k}(t) + q_i^{k,-k}(s,t) \ge \hat{\beta}_i^{k,-k}(s)p_i^{-k}(s) \quad \forall i \ \ \forall s \in [0,t] \ \ \forall t \in [0,T]$$

$$\omega_i^k(t), \ \theta_i^{k,k}(t), \ \theta_i^{k,k}(t) \ge 0 \quad \forall i \ \ \forall t \in [0,T]$$

$$R_i^k(0,t) = S_i^{k,k}(0,t) = S_i^{k,-k}(0,t) = 0 \quad \forall i \ \ \forall t \in [0,T]$$

$$\frac{\partial R_i^k}{\partial s}(s,t) = r_i^k(s,t) \quad \forall i \ \ \forall s \in [0,t], \ \ \forall t \in [0,T]$$

$$\frac{\partial S_i^{k,k}}{\partial s}(s,t) = q_i^{k,k}(s,t) \quad \forall i \ \ \forall s \in [0,t], \ \ \forall t \in [0,T]$$

$$\frac{\partial S_i^{k,-k}}{\partial s}(s,t) = q_i^{k,-k}(s,t) \quad \forall i \ \ \forall s \in [0,t], \ \ \forall t \in [0,T]$$

$$r_i^k(s,t), \ q_i^{k,k}(s,t), \ q_i^{k,-k}(s,t) \ge 0 \quad \forall i \ \ \forall s \in [0,t], \ \ \forall t \in [0,T].$$

Note that the fluid equations as well as the constraints remain linear, even though

there are more variables than in the nominal problem: only the size has increased, but the complexity is of the same order.

We now prove Theorem 13.

Proof. We will show that the existence theorem (Theorem 10) used in Section 5.1.4 holds for this problem as well.

The variables space is now the space of vectors $x = (x^A, x^B)$ with

$$x^k = (p_i^k(.),\ u_i^k(.),\ I_i^k(.),\ \omega_i^k(.),\ \theta_i^{k,k}(.),\ \theta_i^{k,-k}(.),\ r_i^k(.,.),\ q_i^{k,k}(.,.),\ q_i^{k,-k}(.,.),$$

$$R_i^k(.,.),\ S_i^{k,k}(.,.),\ S_i^{k,-k}(.,.),\ i = 1,\dots,N).$$

To ease the exposition, we will denote $y^k = (p_i^k(.),\ u_i^k(.),\ I_i^k(.),\ i = 1,\dots,N)$ and $\lambda^k = (\lambda_i^k,\ i = 1,\dots,N)$, where

$$\lambda_i^k = (\omega_i^k(.),\ \theta_i^{k,k}(.),\ \theta_i^{k,-k}(.),\ r_i^k(.,.),\ q_i^{k,k}(.,.),\ q_i^{k,-k}(.,.), R_i^k(.,.),\ S_i^{k,k}(.,.),\ S_i^{k,-k}(.,.))$$

so that $x^k = (y^k, \lambda^k)$.

By defining X^k as the set of variables x^k that satisfy the constraints that are independent from p^{-k}, and X such that $X = X^A \times X^B$, it is clear that X is convex and closed. We notice that it is non empty by taking all variables equal to 0 except $r_i^k(s,t) = \hat{\alpha}_i^k(s)$, $R_i^k(\tau,t) = \int_0^\tau \hat{\alpha}_i^k(s)ds$, and $I_i^k(t) = I_i^{k0} + R_i^k(t,t)\ \forall i, t,\ k = A, B$.

As previously, we denote $Q^k(\bar{x}^{-k}) \subset X^k$ the subset of all feasible strategy and state vectors x^k for player k including all constraints, given the strategy and state vector $\bar{x}^{-k}$ of her competitor. Again, it is clear that for all $\bar{x}^{-k} \in X^{-k}$, $Q^k(\bar{x}^{-k})$ is a closed and convex subset of X^k. We will prove that it is non empty by showing that the solution (feasible under Assumption 12) such that $\forall i,\ t$:

$$\omega_i^k(t) = \theta_i^{k,k}(t) = \theta_i^{k,-k}(t) = 0$$

$$p_i^k(t) = \frac{\alpha_i^k(t) - \hat{\alpha}_i^k(t) + (\beta_i^{k,-k}(t) - \hat{\beta}_i^{k,-k}(t))\bar{p}_i^{-k}(t)}{\beta_i^{k,k}(t) + \hat{\beta}_i^{k,k}(t)}$$

$$r_i^k(s,t) = \hat{\alpha}_i^k(s), \quad q_i^{k,k}(s,t) = \hat{\beta}_i^{k,k}(s)p_i^k(s), \quad q_i^{k,-k}(s,t) = \hat{\beta}_i^{k,-k}(s)\bar{p}_i^{-k}(s) \quad \forall s \in [0,t]$$

$$R_i^k(\tau,t) = \int_0^\tau r_i^k(s,t)ds$$

$$S_i^{k,k}(\tau,t) = \int_0^\tau q_i^{k,k}(s,t)ds, \quad S_i^{k,-k}(\tau,t) = \int_0^\tau q_i^{k,-k}(s,t)ds \quad \forall \tau \in [0,t]$$

$$u_i^k(t) = 2\hat{\alpha}_i^k(t) + 2\hat{\beta}_i^{k,k}(t)p_i^k(t) + 2\hat{\beta}_i^{k,-k}(t)\bar{p}_i^{-k}(t)$$

$$I_i^k(t) = I_i^{k^0} + \int_0^t (u_i^k(s) - \alpha_i^k(s) + \beta_i^{k,k}(s)p_i^k(s) - \beta_i^{k,-k}(s)\bar{p}_i^{-k}(s))ds$$

belongs to the set $Q^k(\bar{x}^{-k})$ for $\bar{x}^{-k} \in X^{-k}$.

Since $\bar{x}^{-k} \in X^{-k}$, $\bar{p}_i^{-k}(t) \leq p_{i_{max}}^{-k}(t)$ and therefore $p_i^k(t) \leq p_{i_{max}}^k(t)$.

Since both prices are below their maximum threshold, it is clear that under Assumption 12, $\sum_i u_i^k(t) \leq K^k(t)$.

Finally, it is easy to derive that $\dot{I}_i^k(t) = \hat{\alpha}_i^k(t) + \hat{\beta}_i^{k,k}(t)p_i^k(t) + \hat{\beta}_i^{k,-k}(t)\bar{p}_i^{-k}(t)$, and since $I_i^{k^0} \geq 0 = R_i^k(0,0) + S_i^{k,k}(0,0) + S_i^{k,-k}(0,0)$, using inequality (5.21), the security level for $I_i^k(t)$ is satisfied.

We denote $Y \subset X$ the set of feasible collective strategy and state vectors:

$$Y = \{x \in X : x^k \in Q^k(x^{-k}), \; k = A, B\}.$$

Then clearly Y is a convex closed subset of X. To show that it is non empty, we take the same solution as above except that both prices are set to their maximum threshold. Using the same reasoning, this point is an element of set Y.

The objective function is unchanged, so all the properties we proved regarding it earlier still hold for this problem.

The proof of upper semi continuity of Q can be adapted from the proof of Proposition 14 in a straightforward way.

Now let's prove that Q is lower semi continuous. Consider $x_n \in X$ such that

$x_n \to x (\in X)$ and $\bar{x} \in Q(x)$. We want to construct $\bar{x}_n$ such that $\bar{x}_n \to \bar{x}$ and for n large enough, $\bar{x}_n \in Q(x_n)$. We observe that the difficulty comes from the inventory security level guarantee; it is straightforward to satisfy the constraints that are involving directly the control variables.

Let's denote

$$m_i^k(t, \lambda) \equiv \omega_i^k(t)\Gamma_i^k(t) + \theta_i^{k,k}(t)\Theta_i^{k,k}(t) + \theta_i^{k,-k}(t)\Theta_i^{k,-k}(t) + R_i^k(t,t) + S_i^{k,k}(t,t) + S_i^{k,-k}(t,t)$$

the minimum security level for the inventory of product i at time t for supplier k. The constraint guaranteeing no backorders is written $I_i^k(t) \geq m_i^k(t, \lambda)$. Note that $m_i^k(t, .)$ is a continuous function.

First, we notice that if $\bar{I}_i^k(t) > m_i^k(t, \bar{\lambda})$ $\forall i, t$, then for any $\bar{x}_n$ such that $\bar{x}_n \to \bar{x}$, we will have $\bar{I}_{n,i}^k(t) > m_i^k(t, \bar{\lambda}_n)$ for n large enough since $\bar{I}_n(t) \to \bar{I}(t)$ and $m_i^k(t, \bar{\lambda}_n) \to m_i^k(t, \bar{\lambda})$. It is therefore easy to construct $\bar{x}_n$ such that $\bar{x}_n \to \bar{x}$ and $\bar{x}_n \in Q(x_n)$ in that case, so let's assume we have a time t and a product i such that $\bar{I}_i^k(t) = m_i^k(t, \bar{\lambda})$ for supplier k.

To prove the result, it would be sufficient to construct feasible $\bar{x}_n$ such that in particular

$$\bar{I}_{n,i}^k(t) - \bar{I}_i^k(t) \geq m_i^k(t, \bar{\lambda}_n) - m_i^k(t, \bar{\lambda}),$$

(in addition to other feasibility constraints) i.e.

$$\int_0^t (\bar{u}_{n,i}^k(t) - \bar{u}_i^k(t)) + (\beta_i^{k,k}(s)(\bar{p}_{n,i}^k(s) - \bar{p}_i^k(s)) - \beta_i^{k,-k}(s)(p_{n,i}^{-k}(s) - p_i^{-k}(s)))ds \geq$$
$$(\bar{\omega}_{n,i}^k(t) - \bar{\omega}_i^k(t))\Gamma_i^k(t) + (\bar{\theta}_{n,i}^{k,k}(t) - \bar{\theta}_i^{k,k}(t))\Theta_i^{k,k}(t) + (\bar{\theta}_{n,i}^{k,-k}(t) - \bar{\theta}_i^{k,-k}(t))\Theta_i^{k,-k}(t)$$
$$+ \int_0^t (\bar{r}_{n,i}^k(s,t) - \bar{r}_i^k(s,t) + \bar{q}_n^{k,k}(s,t) - \bar{q}^{k,k}(s,t) + \bar{q}_n^{k,-k}(s,t) - \bar{q}^{k,-k}(s,t))ds. \quad (5.24)$$

In order to satisfy this inequality, we should attempt to choose $\bar{x}_n$ such that $\bar{u}_{n,i}^k - \bar{u}_i^k, \bar{p}_{n,i}^k - \bar{p}_i^k$ are as large as possible (while converging to zero) and $m_i^k(t, \bar{\lambda}_n) - m_i^k(t, \bar{\lambda})$ as small as possible (while converging to zero). Our goal is thus to construct $\bar{\lambda}_n$ by modifying $\bar{\lambda}$ (this modification converging to zero) while decreasing its value is possible, and satisfying all feasibility constraints.

Note that for a given x and $\bar{y}$ (in particular their price components), the vector $\bar{\lambda}$ that minimizes $m_i^k(t, \bar{\lambda})$ under the constraint $\bar{x} \in Q(x)$ is obtained if $\bar{\lambda}$ is formed by the variables that solve the dual subproblems presented in section 5.2. Let's denote $\Omega_i^{k*}(t, \bar{p}^k(.), p^{-k}(.))$ the minimum value of the security level obtained with the solution above. Let $\bar{\lambda}_i^{k*}(t, \bar{p}^k(.), p^{-k}(.))$ the corresponding components (we write explicitly the arguments for the same reason as just explained).

Let $\epsilon > 0$. We claim that given $x_n \to x$ and $\bar{y}_n \to \bar{y}$, if $\bar{\lambda}_i^k \neq \bar{\lambda}_i^{k*}(t, \bar{p}^k(.), p^{-k}(.))$ (and thus $m_i^k(t, \bar{\lambda}) > \Omega_i^{k*}(t, \bar{p}^k(.), p^{-k}(.))$), there exists $\bar{\bar{\lambda}}_{n,i}^k \to \bar{\lambda}_i^k$ such that for n large enough, $m_i^k(t, \bar{\lambda}_n) = m_i^k(t, \bar{\lambda}) - \epsilon m_{n,i}^k$ for some positive $m_{n,i}^k$ that converges toward zero, and such that $\bar{\bar{\lambda}}_{n,i}^k$ satisfies the feasibility constraints depending on $\bar{p}_{n,i}^k$ and $p_{n,i}^{-k}$ for n sufficiently large. To see this, notice that $\bar{\lambda}$ is not the optimal solution of the continuous LPs shown above; therefore the linearity of the problem implies that it is possible to perturb its components (in a way that converges to zero at $n \to \infty$) while decreasing the objective value. Furthermore, the linearity of the constraints satisfied by $\bar{\lambda}$ that involve prices $\bar{p}_i^k$ and p_i^{-k} implies that it is again possible to perturb the components of $\bar{\lambda}$ (in a way that converges to zero at $n \to \infty$) to make the perturbed solution feasible with $\bar{p}_{n,i}^k$ and $p_{n,i}^{-k}$, since $x_n \to x$ and $\bar{y}_n \to \bar{y}$.

Moreover, the only situation in which we cannot choose $\bar{u}_{n,i}^k$ strictly greater than $\bar{u}_i^k$ is when the capacity constraint is tight under $\bar{u}^k$ and the inventory security level guarantee is satisfied with equality for all products. Indeed, otherwise we can increase infinitesimally the production rate by shifting production from a product that has a non tight inventory level constraint (shifting production from that product will slightly decrease its inventory level, but as long as the security level constraint is not tight we can do it infinitesimally and remain feasible).

Similarly, the only situation in which we cannot perturb $\bar{p}_i^k$ by increasing it while remaining feasible is when the price is already at its maximum (for fixed p^{-k}).

Let's define on $[0, t]$ (omitting the time argument for the sake of clarity) for some

$\epsilon' > 0$

$$\bar{u}_{n,i}^k = \begin{cases} \bar{u}_i^k & \text{if } \sum_i \bar{u}_i^k = K^k \text{and } \bar{I}_j^k(t) = m_j(t,\bar{\lambda}) \ \forall j \\ \bar{u}_i^k + m_{n,i}^k \epsilon' & \text{else} \end{cases}$$

$$\bar{p}_{n,i}^k = \begin{cases} \frac{\alpha_i^k(t)-\hat{\alpha}_i^k(t)+(\beta_i^{k,-k}(t)-\hat{\beta}_i^{k,-k}(t))p_{n,i}^{-k}(t)}{\beta_i^{k,k}(t)+\hat{\beta}_i^{k,k}(t)} & \text{if } \bar{p}_i^k = \frac{\alpha_i^k(t)-\hat{\alpha}_i^k(t)+(\beta_i^{k,-k}(t)-\hat{\beta}_i^{k,-k}(t))p_i^{-k}(t)}{\beta_i^{k,k}(t)+\hat{\beta}_i^{k,k}(t)} \\ \bar{p}_i^k + m_{n,i}^k \epsilon' & \text{else} \end{cases}$$

$$\bar{\lambda}_{n,i}^k = = \begin{cases} \bar{\lambda}_i^{k*}(t,\bar{p}_n^k(.),p_n^{-k}) & \text{if } \bar{\lambda}_i^k = \bar{\lambda}_i^{k*}(t,\bar{p}^k(.),p^{-k}) \\ \bar{\bar{\lambda}}_i^k & \text{else.} \end{cases}$$

We observe that if for some product i, either $\bar{u}_{n,i}^k$ or $\bar{p}_{n,i}^k$ is given by its second expression on a domain with positive measure, or if $\bar{\lambda}_{n,i}^k$ is given by its second expression, then the inequality (5.24) that we want to prove can be rewritten $\epsilon'' m_{n,i}^k + A_{n,i}^k \geq 0$, with ϵ'' depends on ϵ, ϵ' and the measure of that domain. If $A_n \geq 0$, taking $m_{n,i}^k = 0$ will satisfy the inequality. Otherwise, we take $m_{n,i}^k = -A_{n,i}^k/\epsilon''$.

So now let's suppose that for all products i, $\bar{u}_{n,i}^k$ and $\bar{p}_{n,i}^k$ are given at all times by their first expression and that so is $\bar{\lambda}_{n,i}^k$. Therefore we are supposing that at time t, the production capacity is tight, that for all products the inventory security level is tight, and for some product i prices are equal to their upper bounds, and the variables introduced by the dual subproblems are at their optimum. We will show that this situation is impossible by displaying a contradiction.

In this case, the fact that the inventory security level is tight can be rewritten as

$$I_i^{k0} + \int_0^t \left(\bar{u}_i^k(s) - \alpha_i^k(s) + \beta_i^{k,k}(s)\frac{\alpha_i^k(s) - \hat{\alpha}_i^k(s) + (\beta_i^{k,-k}(s) - \hat{\beta}_i^{k,-k}(s))p_i^{-k}(s)}{\beta_i^{k,k}(s) + \hat{\beta}_i^{k,k}(s)} \right.$$
$$\left. -\beta_i^{k,-k}(s)p^{-k}(s) \right) ds$$

$$= \Omega_i^k\left(t, \frac{\alpha_i^k(s) - \hat{\alpha}_i^k(s) + (\beta_i^{k,-k}(s) - \hat{\beta}_i^{k,-k}(s))p_i^{-k}(s)}{\beta_i^{k,k}(s) + \hat{\beta}_i^{k,k}(s)}, \ s \in [0,t], \ p_i^{-k}(.) \right)$$

$$\leq \int_0^t \left(\hat{\alpha}_i^k(s) + \hat{\beta}_i^{k,k}(s) \frac{\alpha_i^k(s) - \hat{\alpha}_i^k(s) + (\beta_i^{k,-k}(s) - \hat{\beta}_i^{k,-k}(s))p_i^{-k}(s)}{\beta_i^{k,k}(s) + \hat{\beta}_i^{k,k}(s)} \right.$$

$$\left. + \hat{\beta}_i^{k,-k}(s)p_i^{-k}(s) \right) ds$$

which (after calculations) implies that

$$I_i^{k^0} + \int_0^t \left(\bar{u}_i^k(s) - 2\hat{\alpha}_i^k(s) - 2\hat{\beta}_i^{k,-k}(s)p_i^{-k}(s) \right.$$

$$\left. - 2\hat{\beta}_i^{k,k}(s) \frac{\alpha_i^k(s) - \hat{\alpha}_i^k(s) + (\beta_i^{k,-k}(s) - \hat{\beta}_i^{k,-k}(s))p_i^{-k}(s)}{\beta_i^{k,k}(s) + \hat{\beta}_i^{k,k}(s)} \right) ds \ \leq \ 0.$$

Note that this expression is lower bounded by

$$I_i^{k^0} + \int_0^t \left(\bar{u}_i^k(s) - 2\hat{\alpha}_i^k(s) - 2\hat{\beta}_i^{k,-k}(s)p_{i_{max}}^{-k}(s) - 2\hat{\beta}_i^{k,k}(s)p_{i_{max}}^k(s) \right) ds$$

which, after adding over all products, and under Assumption 12, since the capacity is tight, is lower bounded by $\sum_{i=1}^N I_i^{k^0} > 0$. This is a contradiction since the right hand side in the last equality is negative. $\qquad \square$

$\square$

5.3 Formulation in discrete time with uncertain data

5.3.1 Formulation and properties

We rewrite the best response problem for player k by dicretizing time. To improve the exposition, we consider without loss of generality a time step length of 1 and assume that T is an integer, such that the discrete time instants are $t = 0, 1, \ldots, T$. The first decisions are made at time $t = 1$. For each player, we now have $2NT$ control variables (prices and production rates for all products at all times) and NT state variables (inventory levels).

We obtained in the previous section (see Theorem 12) the robust counterpart prob-

lem, and we extend to the following formulation in discrete time (where the constraints must be satisfied for $t = 1, \ldots, T$ and, except the capacity constraint, for $i = 1, \ldots, N$):

$$\max_{u_i^k(t), p_i^k(t), \forall i, t} \sum_{t=1}^{T} \sum_{i=1}^{N} \Big(p_i^k(t)(\alpha_i^k(t) - \beta_i^{k,k}(t)p_i^k(t) + \beta_i^{k,-k}(t)p_i^{-k}(t)) - \gamma_i^k(u_i^k(t))^2$$

$$-h_i^k(t)(I_i^k(t))^2 \Big) \qquad (5.25)$$

$$\text{s.t.} \quad I_i^k(t) - I_i^k(t-1) = u_i^k(t) - \alpha_i^k(t) + \beta_i^{k,k}(t)p_i^k(t) - \beta_i^{k,-k}(t)p_i^{-k}(t)$$

$$I_i^k(0) = I_i^{k0}$$

$$\sum_{i=1}^{N} u_i^k(t) \leq K^k(t)$$

$$p_i^k(t) \leq \frac{\alpha_i^k(t) - \hat{\alpha}_i^k(t) + (\beta_i^{k,-k}(t) - \hat{\beta}_i^{k,-k}(t))p_i^{-k}(t)}{\beta_i^{k,k}(t) + \hat{\beta}_i^{k,k}(t)}$$

$$p_i^k(t), \ u_i^k(t) \geq 0$$

$$I_i^k(t) \geq \omega_i^k(t)\Gamma_i^k(t) + \theta_i^{k,k}(t)\Theta_i^{k,k}(t) + \theta_i^{k,-k}(t)\Theta_i^{k,-k}(t)$$

$$+ \sum_{s=1}^{t} (r_i^k(s,t) + q_i^{k,k}(s,t) + q_i^{k,-k}(s,t))$$

$$\omega_i^k(t) + r_i^k(s,t) \geq \hat{\alpha}_i^k(s), \quad s = 1, \ldots, t$$

$$\theta_i^{k,k}(t) + q_i^{k,k}(s,t) \geq \hat{\beta}_i^{k,k}(s)p_i^k(s), \quad s = 1, \ldots, t$$

$$\theta_i^{k,-k}(t) + q_i^{k,-k}(s,t) \geq \hat{\beta}_i^{k,-k}(s)p_i^{-k}(s), \quad s = 1, \ldots, t$$

$$\omega_i^k(t), \ \theta_i^{k,k}(t), \ \theta_i^{k,k}(t) \geq 0$$

$$r_i^k(s,t), \ q_i^{k,k}(s,t), \ q_i^{k,-k}(s,t) \geq 0, \quad s = 1, \ldots, t.$$

This problem was shown to have a simpler equivalent formulation (where the constraints must be satisfied for $t = 1, \ldots, T$ and, except the capacity constraint, for $i = 1, \ldots, N$):

$$\max_{u_i^k(t), p_i^k(t), \forall i, t} \quad \sum_{t=1}^{T} \sum_{i=1}^{N} \Big(p_i^k(t)(\alpha_i^k(t) - \beta_i^{k,k}(t)p_i^k(t) + \beta_i^{k,-k}(t)p_i^{-k}(t)) - \gamma_i^k(t)(u_i^k(t))^2$$

$$-h_i^k(t)(I_i^k(t))^2 \Big) \qquad (5.26)$$

s.t. $\quad I_i^k(t) - I_i^k(t-1) = u_i^k(t) - \alpha_i^k(t) + \beta_i^{k,k}(t)p_i^k(t) - \beta_i^{k,-k}(t)p_i^{-k}(t)$

$$I_i^k(0) = I_i^{k^0}$$

$$\sum_{i=1}^{N} u_i^k(t) \le K^k(t)$$

$$p_i^k(t) \le \frac{\alpha_i^k(t) - \hat{\alpha}_i^k(t) + (\beta_i^{k,-k}(t) - \hat{\beta}_i^{k,-k}(t))p_i^{-k}(t)}{\beta_i^{k,k}(t) + \hat{\beta}_i^{k,k}(t)}$$

$$I_i^k(t) \ge \Omega_i^k(t, p^k(.), p^{-k}(.))$$

$$u_i^k(t), \ p_i^k(t) \ge 0$$

where $\Omega_i^k(t, p^k(.), p^{-k}(.))$ is extended in discrete time from the expression given in Section 5.2, and was shown to be convex in $p^k(.)$ and in $p^{-k}(.)$, non decreasing with time and with each budget of uncertainty as well as each half length of range of variation.

We will first briefly recall some properties of the problem that were shown in Section 5.2 in a continuous-time setting. We extend them to a discretized time setting.

Let's denote x^k the vector of control and state variables for supplier k with $3NT$ components

$$x^k = (p_i^k(t), u_i^k(t), I_i^k(t), \ t = 1, \ldots, T, \ i = 1, \ldots, N).$$

We denote $x = (x^A, x^B) \in \mathbb{R}^{6NT}$. We will use the usual norm on this vector space. We recall that the formal definition of the set Y of jointly feasible strategies and the product $Q(x)$ of feasible sets of strategies for a player when her competitor keeps strategy x^{-k}:

$$X^k = \left\{ x = (p, u, I) \in \mathbb{R}^{3NT} : \quad u_i(t), \ p_i(t) \geq 0 \ \forall i, \ t \right.$$

$$\sum_{i=1}^{N} u_i(t) \leq K^k(t) \ \forall t$$

$$\left. I_i(0) = I_i^{k^0} \ \forall i \ \right\}$$

$$X = X^A \times X^B \subset \mathbb{R}^{6NT}$$

$$Q^k(\bar{x}^{-k}) = \left\{ x = (p, u, I) \in X^k : \quad p_i(t) \leq \frac{\alpha_i^k(t) + \beta_i^{k,-k}(t)\bar{p}_i^{-k}(t)}{\beta_i^{k,k}(t)} \ \forall i, \ t \right.$$

$$I_i(t) - I_i(t-1) = u_i(t) - \alpha_i^k(t) + \beta_i^{k,k}(t)p_i(t) - \beta_i^{k,-k}(t)\bar{p}_i^{-k}(t) \ \forall i, \ t$$

$$\left. I_i(t) \geq \Omega_i^k(t, p^k(.), p^{-k}(.)) \ \forall i, \ t \ \right\}$$

$$Y = \{ x = (x^A, x^B) \in X : x^k \in Q^k(x^{-k}), \ k = A, B \}$$

$$Q(x) = Q(x^A, x^B) = Q^A(x^B) \times Q^B(x^A).$$

Notice that we have

$$Q(x) = Q(x^A, x^B) = \{ z = (z^A, z^B) : (z^k, x^{-k}) \in Y, \ k = A, B \}.$$

Clearly, $x \in Q(x) \Leftrightarrow x \in Y$.

Lemma 15. *The set Y is convex.*

Proof. This is clear under formulation (5.25) since all constraints are linear. In formulation (5.26), the result follows from the convexity of $\Omega_i^k(t, p^k(.), p^{-k}(.))$ with respect to $p^k(.)$ and $p^{-k}(.)$. $\qquad\square$

We also showed in Section 5.2 (and this remains correct in discrete time) that jointly feasible prices are bounded from above by $p_{i_{max}}^k(t) =$

$$\frac{(\alpha_i^k(t) - \hat{\alpha}_i^k(t))(\beta_i^{-k,-k}(t) + \hat{\beta}_i^{-k,-k}(t)) + (\alpha_i^{-k}(t) - \hat{\alpha}_i^{-k}(t))(\beta_i^{k,-k}(t) - \hat{\beta}_i^{k,-k}(t))}{(\beta_i^{k,k}(t) + \hat{\beta}_i^{k,k}(t))(\beta_i^{-k,-k}(t) + \hat{\beta}_i^{-k,-k}(t)) - (\beta_i^{k,-k}(t) - \hat{\beta}_i^{k,-k}(t))(\beta_i^{-k,k}(t) - \hat{\beta}_i^{-k,k}(t))}.$$

Production rates are bounded from below by zero and bounded from above by the capacity rate. Prices are non negative, and as we just showed, they are also bounded from above. Also, inventory level start from a finite value and can be expressed as a linear combination of prices and production rates. Therefore, since the horizon is finite, they are bounded from above as well. As a result, set Y is bounded. It is also clearly a closed set. Therefore the following corollary holds.

Corollary 11. *Y is a compact set.*

We prove in Section 5.2 that under Assumptions 4 and 12, Y is a non empty set.

We denote the objective function for supplier k, when supplier $-k$ has a strategy x^{-k}, as

$$\Pi^k(x^k, x^{-k}) = \sum_{t=1}^{T} \sum_{i=1}^{N} \Big(p_i^k(t)(\alpha_i^k(t) - \beta_i^{k,k}(t)p_i^k(t) + \beta_i^{k,-k}(t)p_i^{-k}(t)) - \gamma_i^k(t)(u_i^k(t))^2$$
$$- h_i^k(t)(I_i^k(t))^2 \Big).$$

We observe that $\Pi^k(x)$ is continuous with x, and is concave with x^k for a fixed x^{-k}.

Example 1:
We illustrate the sets Y and $Q(x)$ on a simple example. Let's simplify our problem and consider the case of one product and one time period in the space of prices only. We ignore the production rates and inventory levels. The goal is then to maximize revenues so that price and demand are non negative. The best response problem for supplier k is then written as follows:

$$\max_{p^k} \quad p^k(\alpha^k - \beta^{k,k}p^k + \beta^{k,-k}p^{-k})$$
$$\text{s.t.} \quad 0 \le p^k \le \frac{\alpha^k + \beta^{k,-k}p^{-k}}{\beta^{k,k}}.$$

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We then have

$$Q(p^{*A}, p^{*B}) = \left\{ (p^A, p^B) \geq 0 : \ p^A \leq \frac{\alpha^A + \beta^{A,B} p^{*B}}{\beta^{A,A}}, \ p^B \leq \frac{\alpha^B + \beta^{B,A} p^{*A}}{\beta^{B,B}} \right\}$$

and

$$Y = \left\{ (p^A, p^B) \geq 0 : \ \alpha^A - \beta^{A,A} p^A + \beta^{A,B} p^B \geq 0, \ \alpha^B - \beta^{B,B} p^B + \beta^{B,A} p^A \geq 0 \right\}.$$

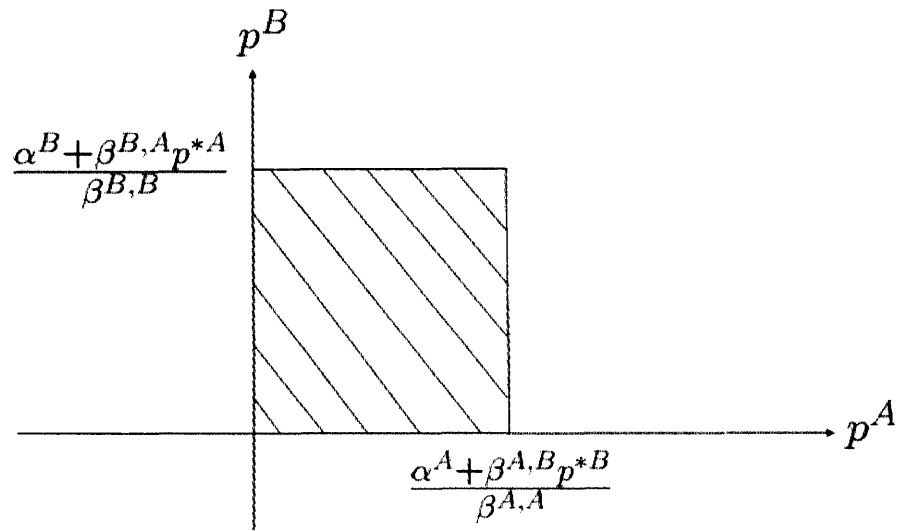See Figures 5-1 and 5-2 for an illustration.

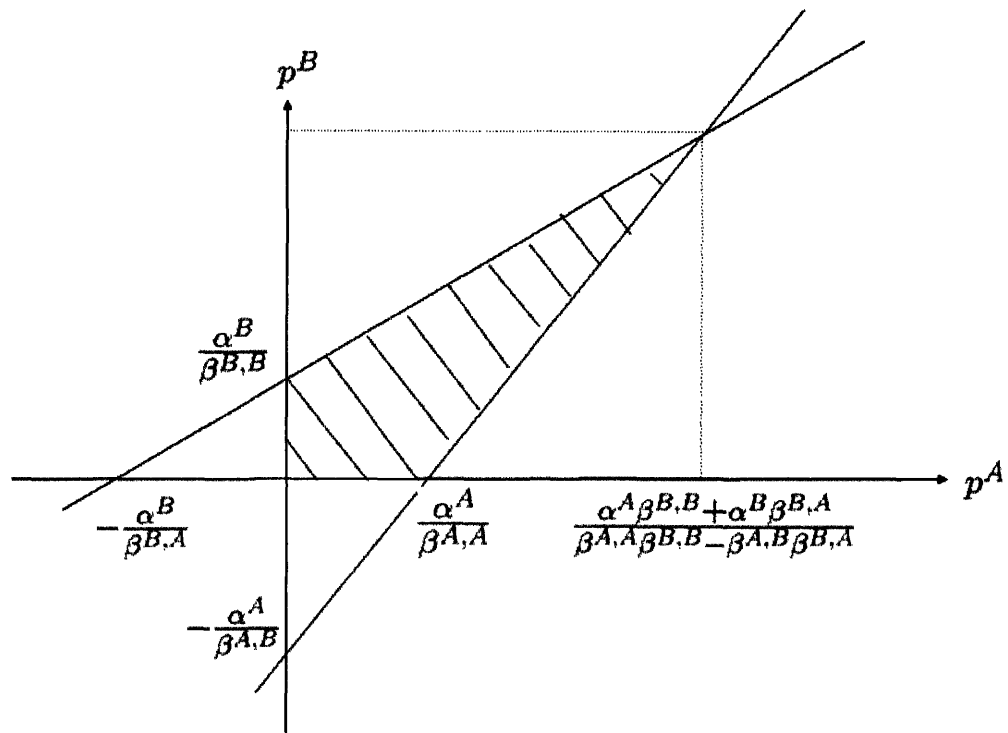Figure 5-1: Example 1 in the space of prices: set $Q(p^{*A}, p^{*B})$

Figure 5-2: Example 1 in the space of prices: set Y

5.3.2 Nash equilibria

We recall the definition of a Nash equilibrium.

Definition 7. *A Nash equilibrium is a vector* $x = (x^A, x^B) \in Y$ *such that*

$$\Pi^k(x^k, x^{-k}) \geq \Pi^k(\bar{x}^k, x^{-k}) \quad \forall \bar{x}^k \in Q^k(x^{-k}), \quad k = A, B. \tag{5.27}$$

In other words, at a Nash equilibrium, no supplier can increase her profits by unilaterally deviating from the solution.

The following existence result was proven by Rosen [111] based on Kakutani's fixed point theorem, using the facts that the joint strategy set is closed, convex, bounded, the payoff function of a player is continuous in all arguments and concave with the player's strategy vector. In particular, it holds for a 2-person game.

Theorem 14. *[111] An equilibrium point exists for every concave n-person game.*

We introduce the Nikaido-Isoda function

$$\psi(x, z) = \sum_{k=A,B} \left(\Pi^k(z^k, x^{-k}) - \Pi^k(x^k, x^{-k}) \right), \quad x, \ z \in \mathbb{R}^{6NT}.$$

Notice that if $x \in Y$, $z \in Q(x)$, this expression represents the sum of the changes in the players' payoff function when they change unilaterally their strategy and state from x^k to z^k while the other player keeps strategy and state x^{-k}.

In the expression below we omit the time argument in order to ease the reading.

$$
\begin{aligned}
\psi(x, \bar{x}) &= \sum_{t=1}^{T} \sum_{i=1}^{N} \sum_{k=A,B} \left[\bar{p}_i^k(\alpha_i^k - \beta_i^{k,k} \bar{p}_i^k + \beta_i^{k,-k} p_i^{-k}) - \gamma_i^k \bar{u}_i^{k^2} - h_i^k \bar{I}_i^{k^2} \right. \\
&\qquad\qquad\qquad \left. - p_i^k(\alpha_i^k - \beta_i^{k,k} p_i^k + \beta_i^{k,-k} p_i^{-k}) + \gamma_i^k u_i^{k^2} + h_i^k I_i^{k^2} \right] \\
&= \sum_{t=1}^{T} \sum_{i=1}^{N} \sum_{k=A,B} \left[(\alpha_i^k + \beta_i^{k,-k} p_i^{-k})(\bar{p}_i^k - p_i^k) - \beta_i^{k,k}(\bar{p}_i^{k^2} - p_i^{k^2}) \right. \\
&\qquad\qquad\qquad \left. - \gamma_i^k(\bar{u}_i^{k^2} - u_i^{k^2}) - h_i^k(\bar{I}_i^{k^2} - I_i^{k^2}) \right].
\end{aligned}
$$

We observe that ψ is a continuous function in each one of its arguments.

It is clear from the definition that $\psi(x, x) = 0 \ \ \forall x \in \mathbb{R}^{6NT}$.

Consider a fixed $x \in Y$. Since $x \in Q(x)$, we have

$$\max_{z \in Y} \psi(x, z) \geq 0, \quad \max_{z \in Q(x)} \psi(x, z) \geq 0.$$

We now introduce another characterization of a Nash equilibrium.

Proposition 20. $x^* \in Y$ *is a Nash equilibrium if and only if*

$$\max_{z \in Q(x^*)} \psi(x^*, z) = 0.$$

Proof. Let x^* a Nash equilibrium and $z \in Q(x^*)$. By definition of a Nash equilibrium, since $(z^k, x^{*-k}) \in Y$, $k = A, B$, we have $\Pi^k(z^k, x^{*-k}) - \Pi^k(x^{*k}, x^{*-k}) \leq 0$, $k = A, B$. Summing these inequalities over $k = A, B$ leads to $\psi(x^*, z) \leq 0 \ \ \forall z \in Q(x^*)$. Since $x^* \in Q(x^*)$ and $\psi(x^*, x^*) = 0$, we have $\max_{z \in Q(x^*)} \psi(x^*, z) = 0$.

To show the reverse, let's assume that $x^* \in Y$ is given and $\max_{z \in Q(x^*)} \psi(x^*, z) = 0$, and suppose x^* is not a Nash equilibrium. Then there exists k_0 and z^{k_0} such that $(z^{k_0}, x^{-k_0}) \in Y$ and

$$\Pi^{k_0}(z^{k_0}, x^{*-k_0}) - \Pi^{k_0}(x^{*k_0}, x^{*-k_0}) > 0.$$

Let z^* such that $z^{*k_0} = z^{k_0}$ and $z^{*-k_0} = x^{-k_0}$. Then $z^* \in Q(x^*)$ and

$$
\begin{aligned}
\psi(x^*, z^*) &= \sum_k \Pi^k(z^{*k}, x^{*-k}) - \Pi^k(x^{*k}, x^{*-k}) \\
&= \Pi^{k_0}(z^{*k_0}, x^{*-k_0}) - \Pi^{k_0}(x^{*k_0}, x^{*-k_0}) + \sum_{k \neq k_0} \Pi^k(z^{*k}, x^{*-k}) - \Pi^k(x^{*k}, x^{*-k}) \\
&= \Pi^{k_0}(z^{k_0}, x^{*-k_0}) - \Pi^{k_0}(x^{*k_0}, x^{*-k_0}) + \sum_{k \neq k_0} \Pi^k(x^{*k}, x^{*-k}) - \Pi^k(x^{*k}, x^{*-k}) \\
&= \Pi^{k_0}(z^{k_0}, x^{*-k_0}) - \Pi^{k_0}(x^{*k_0}, x^{*-k_0}) \\
&> 0
\end{aligned}
$$

which contradicts $\max_{z \in Q(x^*)} \psi(x^*, z) = 0.$ $\qquad \square$

In other words, we showed that x^* a Nash equilibrium if and only if

$$x^* = \text{Arg} \max_{z \in Q(x^*)} \Pi^A(z^A, x^{B*}) + \Pi^B(x^{A*}, z^B).$$

This property presents the advantage of combining into a single condition the usual definition of a Nash equilibrium that involves conditions for each player. Note that the set we are maximizing over depends on x^*.

Example

As an illustration, consider *Example 1* described in the previous section. In order to determine the Nash equilibria using the traditional definition, we solve for $k = A, B$, the following optimization problem (assuming p^{-k} fixed):

$$\max_{p^k} \quad p^k(\alpha^k - \beta^{k,k} p^k + \beta^{k,-k} p^{-k})$$

$$\text{s.t.} \quad 0 \le p^k \le \frac{\alpha^k + \beta^{k,-k} p^{-k}}{\beta^{k,k}}.$$

which yields

$$p^{*k} = \frac{\alpha^k + \beta^{k,-k} p^{-k}}{2\beta^{k,k}}.$$

Therefore a Nash equilibrium is solution to the following system of two equations with two unknowns p^{*A}, p^{*B}:

$$p^{*A} = \frac{\alpha^A + \beta^{A,B} p^{*B}}{2\beta^{A,A}}, \quad p^{*B} = \frac{\alpha^B + \beta^{B,A} p^{*A}}{2\beta^{B,B}}$$

This system has a unique solution:

$$p^{*A} = \frac{2\beta^{B,B}\alpha^A + \beta^{A,B}\alpha^B}{4\beta^{A,A}\beta^{B,B} - \beta^{A,B}\beta^{B,A}}, \quad p^{*B} = \frac{2\beta^{A,A}\alpha^B + \beta^{B,A}\alpha^A}{4\beta^{A,A}\beta^{B,B} - \beta^{A,B}\beta^{B,A}}.$$

We conclude that there exists a unique Nash equilibrium $p^* = (p^{*A}, p^{*B})$ as given above, that is located in the interior of Y and in the center of the rectangle repre-

senting $Q(p^*)$.

We could obtain the result similarly using Proposition 20, imposing (p^{*A}, p^{*B}) as the solution of:

$$\max_{p^A, p^B} \quad p^A(\alpha^A - \beta^{A,A}p^A + \beta^{A,B}p^{*B}) + p^B(\alpha^B - \beta^{B,B}p^B + \beta^{B,A}p^{*A})$$

$$\text{s.t.} \quad 0 \leq p^A \leq \frac{\alpha^A + \beta^{A,B}p^{*B}}{\beta^{A,A}}$$

$$0 \leq p^B \leq \frac{\alpha^B + \beta^{B,A}p^{*A}}{\beta^{B,B}}$$

Indeed, the problem is equivalent to the two subproblems that were solved using the traditional definition.

In this example, the Nash equilibrium is unique. However, as we will illustrate, this is not necessarily true for a coupled constraint game.

5.3.3 Normalized Nash equilibrium

Definition and illustration

We now introduce normalized Nash equilibria as a particular case of the ones defined by Rosen [111].

Definition 8. *We will refer to x^* as a **normalized** Nash equilibrium if and only if*

$$\max_{z \in Y} \psi(x^*, z) = 0.$$

In other words, in this chapter, x^* is a normalized Nash equilibrium if and only if

$$x^* = \text{Arg}\max_{z \in Y} \Pi^A(z^A, x^{B*}) + \Pi^B(x^{A*}, z^B).$$

Note that in this definition the maximum is taken over Y, and not $Q(x^*)$ like for a Nash equilibrium. This means that we consider responses that are *jointly feasible*, and not responses that are *unilaterally feasible* given the current competitor's strategy.

Example

In *Example 1*, the normalized Nash equilibrium (p^{*A}, p^{*B}) is determined as the solution to :

$$\max_{p^A, p^B} \quad p^A(\alpha^A - \beta^{A,A}p^A + \beta^{A,B}p^{*B}) + p^B(\alpha^B - \beta^{B,B}p^B + \beta^{B,A}p^{*A})$$

$$\text{s.t.} \quad \alpha^A - \beta^{A,A}p^A + \beta^{A,B}p^B \geq 0$$

$$\alpha^B - \beta^{B,B}p^B + \beta^{B,A}p^A \geq 0$$

$$p^A, p^B \geq 0$$

This problem does not decouple into two subproblems here, because there are constraints that involve simultaneously the controls of both players (coupling constraints). We can solve and obtain the solution that is the unique Nash equilibrium presented in the previous section.

Lemma 16. *[111] A normalized Nash equilibrium is a Nash equilibrium.*

Proof. We give the proof for an arbitrary number of players.

Suppose x^* is a normalized Nash equilibrium but not a Nash equilibrium. Therefore,

$$\exists k_0, \quad z^{k_0} \in Q^{k_0}(x^{*-k_0}) \quad : \quad \Pi^{k_0}(z^{k_0}, x^{*-k_0}) > \Pi^{k_0}(x^{*k_0}, x^{*-k_0}).$$

Notice that $z^{k_0} \in Q^{k_0}(x^{*-k_0})$ implies $(z^{k_0}, x^{*-k_0}) \in Y$.

Let z^* such that $z^{*k_0} = z^{k_0}$, $z^{*-k_0} = x^{*-k_0}$. Then $z^* \in Y$ and

$$
\begin{aligned}
\psi(x^*, z^*) &= \sum_k \left(\Pi^k(z^{*k}, x^{*-k}) - \Pi^k(x^{*k}, x^{*-k}) \right) \\
&= \sum_{k \neq k_0} \left(\Pi^k(z^{*k}, x^{*-k}) - \Pi^k(x^{*k}, x^{*-k}) \right) + \Pi^{k_0}(z^{*k_0}, x^{*-k_0}) - \Pi^{k_0}(x^{*k_0}, x^{*-k_0}) \\
&= \sum_{k \neq k_0} \left(\Pi^k(x^{*k}, x^{*-k}) - \Pi^k(x^{*k}, x^{*-k}) \right) + \Pi^{k_0}(z^{k_0}, x^{*-k_0}) - \Pi^{k_0}(x^{*k_0}, x^{*-k_0}) \\
&= \Pi^{k_0}(z^{k_0}, x^{*-k_0}) - \Pi^{k_0}(x^{*k_0}, x^{*-k_0}) \\
&> 0
\end{aligned}
$$

which contradicts $\max_{z \in Y} \psi(x^*, z) = 0$. □

As a side remark, Rosen [111] showed that the converse is true when the feasible strategy set is not a coupled constraint set, i.e. when the feasible strategy of each player is independent of the competitors' strategies. For a coupled constraint game, a Nash equilibrium is not necessarily a normalized Nash equilibrium (when there are multiple Nash equilibria).

The normalized Nash equilibrium can be interpreted qualitatively as the Nash equilibrium where the coupling constraints have the same shadow price (or Lagrange multiplier) in the best response problem faced by each player. In other words, the marginal profit change per unit change of the right hand side of a coupling constraint is the same for the two players (when the constraint gets tighter, i.e. harder to satisfy, each supplier observes the same profit loss after re-adjusting her strategy).

In this sense, the normalized Nash equilibrium may seem "fair", since both players contribute equally to the global constraints, relative to the marginal profit loss. As a result, this Nash equilibrium may be the most desirable to attain. Moreover, in some settings, couplings constraints may be due to social requirements that a central authority wants to impose, but that players have no direct incentive to achieve. It is therefore of interest to design a way to give incentives to the players in order for them to reach the normalized Nash equilibrium, assuming that they will ignore the coupling constraints, and thus they participate in a decoupled game.

To this end, Haurie ([74], [75]) presents a tax scheme that modifies the payoff functions and leads the decoupled game to have as a Nash equilibrium the original normalized Nash equilibrium, and achieves the coupling constraints. Indeed, the common multiplier (or shadow price) to a coupling constraint can be economically interpreted as a Pigouvian tax that modifies the agents' profits if this coupling constraint is violated, and that induces them to satisfy it. After introduction of this tax, the game is non cooperative and decoupled. In this thesis, the normalized Nash equilibrium is calculated by assigning the same weight to the two players. Therefore, the authority that imposes the taxes treats the two players in the same way and as equally responsible

of the burden of achieving the coupling constraints.

The tax is designed such that, assuming the agents solve a best response problem *ignoring* coupled constraints, and considering only non coupled constraints, the Nash equilibrium corresponds to the normalized Nash equilibrium under the presence of the coupled constraints. This tax is proportional to the amplitude of violation of the coupled constraint, with a coefficient of proportionality that equals the common value of the shadow price for this constraint at the normalized Nash equilibrium.

In this thesis, the coupled constraints for a player are the non negativity of her inventory levels and of her demand rates (that imply an upper bound on her prices). Observe that the coupled constraints faced by a given player's best response problem do *not* appear in the competitor's best response problem. However, a Nash equilibrium must be jointly feasible, and therefore satisfies both sets of coupled constraints. As a result, adding to player k's best response problem the coupled constraints from the competitor's best response problem leave the set of Nash equilibria unchanged. The normalized Nash equilibrium is then the Nash equilibrium that shares equally the burden of satisfying each one of those coupled constraints, i.e. keeping inventory levels and demand rates of both players, all products, at all times, non negative.

Remark:

Rosen [111] introduced a more general definition of a normalized Nash equilibrium, where the competitors do *not* share equally the responsibility of achieving the coupling constraint, but do so according to some given (arbitrary) weights. Specifically, he assumes as given the weights r^k for player k. The corresponding normalized Nash equilibrium $x = (x^k, x^{-k})$ is then the solution of:

$$x = \text{Arg} \max_{z = (z^k, z^{-k}) \in Y} r^A \Pi^A(z^A, x^B) + r^B \Pi^B(x^A, z^B).$$

This normalized Nash equilibrium presents the following property: the Lagrange multiplier vectors λ^A and λ^B (with dimension m) of the m coupling constraint $h_j(x) \leq$

$0, j = 1, \ldots, m$ in each of the two best response problems faced respectively by A and B, satisfy

$$r^A \lambda^A = r^B \lambda^B \equiv \lambda^0,$$

where λ^0 is a vector of dimension m.

The tax scheme is then as follows: if the jth coupling constraint is violated ($h_j(x) > 0$), we impose to player k a tax $T_j^k h_j(x)$ proportional to the amplitude of violation, for $k = A, B$. The coefficient of proportionality is defined by $T_j^k = \lambda_j^k = \frac{\lambda_j^0}{r^k}$. Therefore the taxes are different for the different players in general. Observe that as the weight r^k for player k increases, player k's burden of achieving the common constraints decreases with respect to the other player.

Discussion and insights

To understand the notion of normalized equilibrium more intuitively, we put it into perspective by comparing it to other types of equilibria or optima. We recall

- x_{NE} is a Nash equilibrium if and only if

$$x_{NE} = \text{Arg} \max_{z \in Q(x_{NE})} \Pi^A(z^A, x_{NE}^B) + \Pi^B(x_{NE}^A, z^B).$$

  This means that at a Nash equilibrium, no player has an incentive to unilaterally deviate.

- x_{NNE} is a normalized Nash equilibrium if and only if

$$x_{NNE} = \text{Arg} \max_{z \in Y} \Pi^A(z^A, x_{NNE}^B) + \Pi^B(x_{NNE}^A, z^B).$$

Definition 9. *The system optimum is defined as*

$$x_{SO} = Arg \max_{z \in Y} \Pi^A(z^A, z^B) + \Pi^B(z^A, z^B).$$

The system optimum corresponds to the solution that the players would choose if they were to fully cooperate, and aimed at maximizing the overall profits instead

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of having each player maximize her own. Another way to interpret it is to imagine that a central authority has the power to assign decisions to the players, and aims at maximizing the global objective function, that is the sum of the players' payoff functions.

In *Example 1*, we can determine the system optimum by solving

$$\max_{p^A,p^B} \quad p^A(\alpha^A - \beta^{A,A}p^A + \beta^{A,B}p^B) + p^B(\alpha^B - \beta^{B,B}p^B + \beta^{B,A}p^A)$$

$$\text{s.t.} \quad \alpha^A - \beta^{A,A}p^A + \beta^{A,B}p^B \geq 0$$

$$\alpha^B - \beta^{B,B}p^B + \beta^{B,A}p^A \geq 0$$

$$p^A, p^B \geq 0$$

To simplify the calculations, let's focus on the symmetric case: $\beta^{A,A} = \beta^{B,B} = \beta$, $\beta^{A,B} = \beta^{B,A} = \beta'$, $\alpha^A = \alpha^B = \alpha$. In that case the unique Nash equilibrium (and normalized Nash equilibrium) is $p^A_{NE} = p^B_{NE} = \frac{\alpha}{2\beta-\beta'}$.

At the system optimum, by symmetry, $p^A_{SO} = p^B_{SO}$ is the solution to

$$\max_{p} \quad p(\alpha - (\beta - \beta')p)$$

$$\text{s.t.} \quad \alpha - (\beta - \beta')p \geq 0$$

$$p \geq 0$$

which leads to $p^A_{SO} = p^B_{SO} = \frac{\alpha}{\beta-\beta'} > p^k_{NE}$. In particular, the system optimum is not a Nash equilibrium and is not the normalized Nash equilibrium.

Definition 10. *A Pareto optimal point is a solution $x_{PO} \in Y$ such that*

$$\Pi^A(z) > \Pi^A(x_{PO}) \Rightarrow \Pi^B(z) < \Pi^B(x_{PO}).$$

In other words, at a Pareto optimal point, it is impossible to increase one player's payoff without decreasing another player's payoff. This notion is used to determine solutions that are *socially desirable*. Indeed, a point that is not Pareto optimal is

never a good outcome, since it is possible to select another solution that is strictly better for a player, and leaves the other player indifferent or better off as well.

Clearly, the system optimum is Pareto optimal. A Nash equilibrium is not necessarily Pareto optimal.

For example, consider the Prisoner's dilemma problem where players (A,B) receive penalties as given in Table 5.1: ($x^k = 1$ means that k admits the crime, $x^k = 0$ that she denies)

	$x^B = 1$	$x^B = 0$
$x^A = 1$	(3,3)	(8,0)
$x^A = 0$	(0,8)	(5,5)

Table 5.1: Prisoner's dilemma penalties: first case

In this example, $x^A = x^B = 0$ is the unique Nash equilibrium, with penalties (5,5). The system optimum is $x^A = x^B = 0$ with penalties (3,3).

All points are Pareto optimal except the Nash equilibrium, since the system optimum strictly dominates the Nash equilibrium.

Let's now consider a modified version where the penalties are given in Table 5.2:

	$x^B = 1$	$x^B = 0$
$x^A = 1$	(2,6)	(8,0)
$x^A = 0$	(0,8)	(5,5)

Table 5.2: Prisoner's dilemma penalties: second case

$x^A = x^B = 0$ is still the unique Nash equilibrium, with penalties (5,5). The three other solutions are system optima. All points, including the Nash equilibrium, are Pareto optimal.

Notice that if we change the penalty (2,6) above to (1,6), $x^A = x^B = 0$ is still the unique Nash equilibrium, $x^A = x^B = 1$ is the unique system optimum, but all points are still Pareto optimal.

Therefore, a Nash equilibrium may or may not be Pareto optimal, the system optimum

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is always Pareto optimal, and there may be Pareto optimal points that are neither a system optimum nor a Nash equilibrium.

Example

We illustrate these notions on another example inspired from [111]. Figure 5-3 summarizes the results for that example.

Example 2:

Consider two players A and B who must decide their respective strategies x and y in $\mathbb{R}$ subject to the non coupling constraints $x \geq 0, \quad y \geq 0$ and the coupling constraint $x + y \geq 1$, with payoff functions

$$\Pi^A(x, y) = -\frac{1}{2}x^2 + xy, \quad \Pi^B(x, y) = -y^2 - xy.$$

In this example,

$$Y = \{x, \ y \geq 0 : x + y \geq 1\}$$

while

$$Q(x^0, y^0) = [m(y^0), \infty) \times [m(x^0), \infty),$$

where $m(z) = \max\{0, 1 - z\}$.

Nash equilibrium :

We first determine the Nash equilibria, by deriving the best response of each players. For a fixed $(x^0, y^0) \in Y$, we solve for A:

$$\begin{aligned} \max_x \quad & \Pi^A(x, y^0) = -\frac{1}{2}x^2 + xy^0 \\ \text{s.t.} \quad & x \geq 1 - y^0 \\ & x \geq 0 \end{aligned}$$

We dualize the coupling constraint $x \geq 1 - y^0$ by introducing the Lagrangian
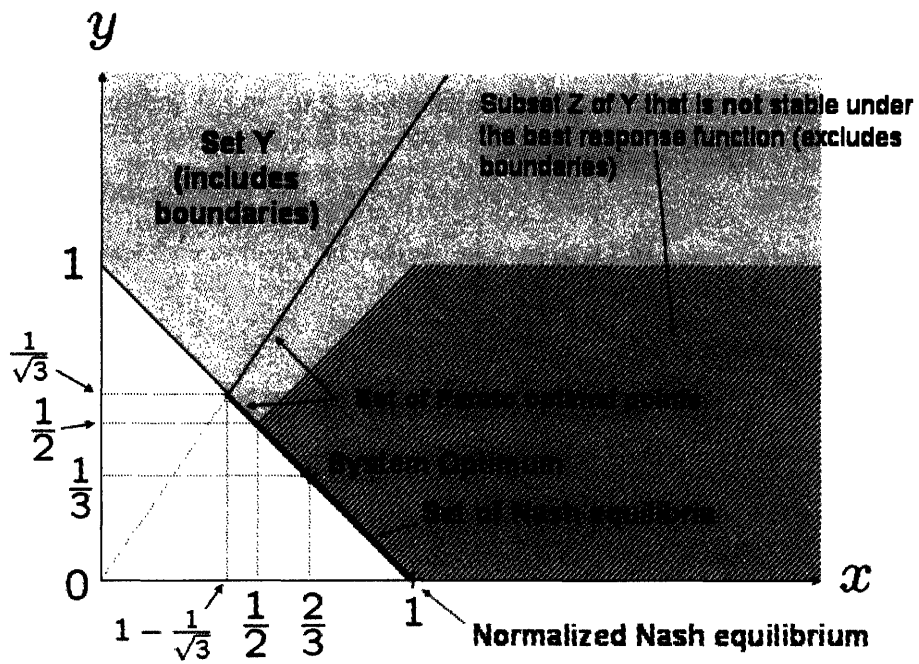
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Figure 5-3: Example 2: illustration of jointly feasible set, set unstable under the best response function, Nash equilibria, system optimum, normalized Nash equilibrium, and Pareto optimal points

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multiplier $\lambda^A \geq 0$. The Lagrangian function is

$$L^A(x, \lambda^A) = -\frac{1}{2}x^2 + xy^0 + \lambda^A(x + y^0 - 1)$$

and it is maximized over $\{x : x \geq 0\}$ for

$$x = \begin{cases} \lambda^A + y^0 & \text{if } \lambda^A \geq -y^0 \\ 0 & \text{if } \lambda^A < -y^0 \end{cases}$$

The complementary slackness condition leads to

- if $y^0 \leq \frac{1}{2}$, then $x = 1 - y^0$, $\lambda^A = 0$ and $\Pi^A = \frac{1}{2}(y^0)^2$
- if $y^0 \geq \frac{1}{2}$, then $x = y^0$, $\lambda^A = 1 - 2y^0$ and $\Pi^A = (1 - y^0)(\frac{3}{2}y^0 - \frac{1}{2})$.

Similarly, we solve for B:

$$\begin{aligned} \max_{y} \quad & \Pi^B(x^0, y) = -y^2 - x^0 y \\ \text{s.t.} \quad & y \geq 1 - x^0 \\ & y \geq 0 \end{aligned}$$

We dualize the coupling constraint $y \geq 1 - x^0$ by introducing the Lagrangian multiplier $\lambda^B \geq 0$. The Lagrangian function is

$$L^B(y, \lambda^B) = -y^2 - x^0 y + \lambda^B(x^0 + y - 1)$$

and it is maximized for

$$y = \begin{cases} \frac{\lambda^B - x^0}{2} & \text{if } \lambda^B \geq x^0 \\ 0 & \text{if } \lambda^B < x^0 \end{cases}$$

The complementary slackness condition leads to

- if $x^0 \leq 1$, then $y = 1 - x^0$, $\lambda^B = 0$ and $\Pi^B = 0$
- if $x^0 \geq 1$, then $y = 0$ $\lambda^B = 2 - x^0$ and $\Pi^B = x^0 - 1$.

As a result, $(x, 1-x)$ is a Nash equilibrium for all $x \in [\frac{1}{2}, 1]$.

We easily derive that at the Nash equilibrium $(x, 1-x)$ for some $x \in [\frac{1}{2}, 1]$,

$$\lambda^A = 2x - 1$$

$$\Pi^A(x, 1-x) = x - \frac{3}{2}x^2$$

$$\lambda^B = 2 - x$$

$$\Pi^B(x, 1-x) = x - 1$$

$$\Pi^A(x, 1-x) + \Pi^B(x, 1-x) = 2x - 1 - \frac{3}{2}x^2.$$

In particular, the Nash equilibrium that yields the maximum joint profits (equal to $\frac{1}{3}$) is $(\frac{2}{3}, \frac{1}{3})$.

We also obtain from this analysis that the best response to $(x^0, y^0) \in Y$ is

$$BR(x^0, y^0) = \begin{cases} (y^0, 0) & \text{if } x^0 \geq 1, \ y^0 \geq \frac{1}{2} \\ (y^0, 1 - x^0) & \text{if } x^0 \leq 1, \ y^0 \geq \frac{1}{2} \\ (1 - y^0, 0) & \text{if } x^0 \geq 1, \ y^0 \leq \frac{1}{2} \\ (1 - y^0, 1 - x^0) & \text{if } x^0 \leq 1, \ y^0 \leq \frac{1}{2} \end{cases}$$

In particular, .

$$BR(x^0, y^0) \notin Y \Leftrightarrow \left(0 < y^0 < 1, \ x^0 + y^0 > 1, \ y^0 < x^0\right).$$

We denote $Z = \{(x, y) : 0 < y < 1, \ x + y > 1, \ y < x\}$ the subset of points of Y that have a best response *outside* Y. We say that set Z is not stable under the best response function. Clearly, a Nash equilibrium is not in set Z since it is a fixed point of the best response function.

Normalized Nash equilibrium :

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Let's now find a normalized Nash equilibrium by solving for a fixed $(x^0, y^0) \in Y$

$$\max \quad \Pi^A(x, y^0) + \Pi^B(x^0, y) = -\frac{1}{2}x^2 + xy^0 - y^2 - x^0 y$$

$$\text{subject to} \quad x + y \geq 1$$

$$x, \ y \geq 0.$$

Dualizing the coupling constraint with multiplier $\lambda \geq 0$, we write the Lagrangian function

$$L(x, y) = -\frac{1}{2}x^2 + xy^0 - y^2 - x^0 y + \lambda(x + y - 1),$$

and

$$\frac{\partial L}{\partial x} = -x + y^0 + \lambda, \quad \frac{\partial L}{\partial y} = -2y - x^0 + \lambda$$

therefore the optimal solution is $x = y^0 + \lambda$, $\quad y = \begin{cases} \frac{\lambda - x^0}{2} & \text{if } \lambda \geq x^0 \\ 0 & \text{if } \lambda \leq x^0. \end{cases}$

Using the complementary slackness condition, we obtain

- if $y^0 \geq 1$, then $x = y^0$, $y = 0$

- if $y^0 < 1$, then $\lambda = 1 - y^0 \leq x^0$ and $x = y^0 + \lambda = 1$, $y = 0$.

(Note: the case $\lambda \geq x^0$ is impossible because it leads to $y^0 + \frac{3}{2}\lambda - \frac{x^0}{2} = 1$ by complementary slackness, thus $\lambda = \frac{x^0 - 2y^0}{3} < x^0$ which is a contradiction.)
Therefore $(x^0 = 1, \ y^0 = 0)$ is the only normalized Nash equilibrium. It has joint profit equals $-\frac{1}{2}$. Notice that it is also a Nash equilibrium.
Furthermore, the normalized Nash equilibrium is Pareto optimal here. Indeed, $\Pi^B(1, 0) = 0$ so only points such that $y = 0$ would not decrease player B's payoff. However, on the half-line $y = 0$ in Y, any point with $x > 1$ decreases Π^A.
Moreover, it is the only Nash equilibrium for which the Lagrange multiplier of the coupling constraint are equal. Indeed, the shadow price at the normalized

equilibrium $\left(\frac{2}{3}, \frac{1}{3}\right)$ is

$$\lambda^A = 1 - 2y^0 = \lambda^B = 2 - x^0 = 1.$$

Therefore to induce players to satisfy the coupling constraint if they do not include it in their program, we should impose to each a tax $T = 1$ per unit of violation of the coupled constraint. In other words, the players' best response problems is now

$$\text{for A: } \max \quad -\frac{1}{2}x^2 + xy^0 - T(1 - x - y^0)$$
$$\text{subject to} \quad x \geq 0$$

and

$$\text{for B: } \max \quad -y^2 - x^0 y - T(1 - y - x^0)$$
$$\text{subject to} \quad y \geq 0.$$

We easily derive the respective optimal respective best responses, for $(x^0, y^0) \in Y$:

$$x = 1 + y^0, \quad y = \begin{cases} \frac{1-x^0}{2} & \text{if } x^0 \leq 1 \\ 0 & \text{if } x^0 > 1 \end{cases}$$

It follows that there is a unique Nash equilibrium to this game: $x^0 = 1$, $y^0 = 0$. We verify that this Nash equilibrium is the normalized equilibrium of the coupled constraint game.

System optimum :

Consider

$$\max \quad \Pi^A(x, y) + \Pi^B(x, y) = -\frac{1}{2}x^2 - y^2$$
$$\text{subject to} \quad x + y \geq 1$$
$$x, \ y \geq 0.$$

There is a unique solution $x_{SO} = (\frac{2}{3}, \frac{1}{3})$, with joint profit of $-\frac{1}{3}$.

Notice that the system optimum is a Nash equilibrium here (but this is not necessarily true in general), because it is not in set Z.

Indeed, suppose it is not a Nash equilibrium. Then it is not a fixed point of the best response function: $BR(x_{SO}) \neq x_{SO}$. We have $x_{SO} \notin Z$, so $BR(x_{SO}) \in Y$, so by definition of the system optimum,

$$\Pi^A(x_{SO}) + \Pi^B(x_{SO}) \geq \Pi^A(BR(x_{SO})) + \Pi^B(BR(x_{SO})).$$

However, since the best response of x_{SO} is not the point x_{SO} itself, $BR(x_{SO})$ has a greater joint profit, which is a contradiction.

Therefore, when the system optimum has a best response that belongs to Y, it is a Nash equilibrium.

Clearly, the system optimum is Pareto optimal, since no point yields a strictly higher joint profit.

Pareto optimal points :

If the optimal solution to

$$\begin{aligned} \max \quad & \Pi^k(x, y) && (5.28) \\ \text{subject to} \quad & x + y \geq 1 \\ & x, \ y \geq 0. \end{aligned}$$

exists and is unique, then it is Pareto optimal, for $k = A, B$. Indeed, at this optimal solution, the only way to increase Π^{-k} is to move to a different point, that necessarily yields a lower value of Π^k.

For $k = A$, the problem is unbounded. For $k = B$, the optimal solution is $(1, 0)$.

We can find other Pareto optimal points by solving

$$\max \quad a\Pi^A(x,y) + \Pi^B(x,y) = -\frac{a}{2}x^2 + (a-1)xy - y^2$$

$$\text{subject to} \quad x + y \geq 1$$

$$x,\ y \geq 0.$$

for fixed parameter $a > 0$.

Indeed, suppose the optimal solution to the problem above was not Pareto optimal. Then there would be a feasible point that, compared with the optimal solution, strictly increases the payoff of one player while increasing or keeping constant the other player's payoff. This is a contradiction since this new point would yield a strictly higher value for $a\Pi^A(x,y) + \Pi^B(x,y)$.

This quadratic problem is convex if the matrix given by

$$\begin{bmatrix} \frac{a}{2} & -\frac{a-1}{2} \\ -\frac{a-1}{2} & 1 \end{bmatrix}$$

is positive semidefinite, i.e. iff

$$\frac{a}{2} - \frac{(a-1)^2}{4} \geq 0 \qquad\qquad (5.29)$$

$$\Leftrightarrow \quad 2a - (a-1)^2 \geq 0$$

$$\Leftrightarrow \quad -a^2 + 4a - 1 \geq 0$$

$$\Leftrightarrow \quad 2 - \sqrt{3} \leq a \leq 2 + \sqrt{3}$$

Notice that the problem has been solved for $a = 1$ when we determined the system optimum.

The Lagrangian function is

$$L(x,y) = -\frac{a}{2}x^2 + (a-1)xy - y^2 + \lambda(x + y - 1)$$

and we have

$$\frac{\partial L}{\partial x} = -ax + (a-1)y + \lambda, \quad \frac{\partial L}{\partial y} = -2y + (a-1)x + \lambda$$

so, for $2 - \sqrt{3} \le a \le 2 + \sqrt{3}$, the Lagrangian function is maximized for

$$x = \begin{cases} \frac{(a-1)y+\lambda}{a} & \text{if } \lambda \ge -(a-1)y \\ 0 & \text{if } \lambda < -(a-1)y, \end{cases} \qquad y = \begin{cases} \frac{(a-1)x+\lambda}{2} & \text{if } \lambda \ge -(a-1)x \\ 0 & \text{if } \lambda < -(a-1)x \end{cases}$$

- if $\lambda < -(a-1)y$ and $\lambda < -(a-1)x$, then $x = y = 0$ which is infeasible.

- if $\lambda \ge -(a-1)y$ and $\lambda \ge -(a-1)x$, then

$$x = \frac{(a-1)y + \lambda}{a}, \quad y = \frac{(a-1)x + \lambda}{2}$$

and therefore

$$
\begin{aligned}
x &= \frac{\lambda}{a} + \frac{\lambda(a-1)}{2a} + \frac{(a-1)^2}{2a}x \\
x(2a - (a-1)^2) &= 2\lambda + (a-1)\lambda \\
x &= \lambda\frac{a+1}{4a - a^2 - 1}, \quad a \ne 2 \pm \sqrt{3} \\
&\quad \text{or } (a = 2 \pm \sqrt{3}, \quad \lambda = 0 \text{ and } x \ge 0) \\
y &= \frac{\lambda}{2} + \lambda\frac{a^2 - 1}{2(4a - a^2 - 1)} = \lambda\frac{2a - 1}{4a - a^2 - 1}, \quad a \ne 2 \pm \sqrt{3} \\
&\quad \text{or } (a = 2 \pm \sqrt{3}, \quad \lambda = 0 \text{ and } y = \frac{(a-1)x}{2})
\end{aligned}
$$

and $y \ge 0$ implies: either $a = 2 + \sqrt{3}$ and $\lambda = 0$, or $\frac{1}{2} \le a < 2 + \sqrt{3}$.

By complementary slackness, if $a \ne 2 + \sqrt{3}$, then $\lambda > 0$ and

$$\lambda\frac{3a}{4a - a^2 - 1} = 1 \Rightarrow \lambda = \frac{4a - a^2 - 1}{3a}$$

($\lambda = 0$ would imply $x = y = 0$ which is infeasible).

If $a = 2 + \sqrt{3}$, then any point $(x, \frac{(1+\sqrt{3})x}{2})$ with $x \ge \frac{2}{3+\sqrt{3}}$ is solution. In

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the case $a \neq 2 + \sqrt{3}$, we must verify

$$\lambda > 0 \quad \Leftrightarrow \quad 4a - a^2 - 1 > 0 \quad \Leftrightarrow \quad 2 - \sqrt{3} < a < 2 + \sqrt{3}$$

- if $-(a-1)x \leq \lambda < -(a-1)y$, then $x = 0, y = \frac{\lambda}{2}$. For feasibility, $\lambda > 0$, and by complementary slackness, $\lambda = 2$. We must verify $2 < -(a-1)$, i.e. $a < -1$ which is impossible.

- if $-(a-1)y \leq \lambda < -(a-1)x$, then $y = 0, x = \frac{\lambda}{a}$. For feasibility, $\lambda > 0$, and by complementary slackness, $\lambda = a$. We must verify $a < -(a-1)$, i.e. $a < \frac{1}{2}$.

As a result, we obtain:

- if $a = 2 + \sqrt{3}$, the optimal solutions are $(x, \frac{(1+\sqrt{3})x}{2})$ with $x \geq \frac{2}{3+\sqrt{3}} = \frac{3-\sqrt{3}}{3}$,

- if $\frac{1}{2} \leq a < 2 + \sqrt{3}$, the optimal solution is $(\frac{a+1}{3a}, \frac{2a-1}{3a})$. In other words, $(x, 1-x)$ is optimal solution corresponding to $a = \frac{1}{3x-1}$ if $\frac{3+\sqrt{3}}{3(2+\sqrt{3})} = \frac{3-\sqrt{3}}{3} < x \leq 1$.

- if $0 \leq a < \frac{1}{2}$, the optimal solution is $(1, 0)$.

Therefore, any point in the set

$$\left\{ (x, 1-x) : 1 - \frac{1}{\sqrt{3}} < x \leq 1 \right\} \cup \left\{ (x, \frac{(1+\sqrt{3})x}{2}) : x \geq 1 - \frac{1}{\sqrt{3}} \right\}$$

is Pareto optimal. In particular, this proves that any Nash equilibrium is Pareto optimal in this example.

Notice that if $a > 2 + \sqrt{3}$, the problem is unbounded, so there is not Pareto optimal point. In particular, along the line $y = \frac{a-1}{2}x$, $x \geq \frac{2}{a+1}$, the problem is feasible and the profit increases as x increases since on this line,

$$a\Pi^A(x, y) + \Pi^B(x, y) = x^2 \left(\frac{(a-1)^2}{4} - \frac{a}{2} \right) = \frac{x^2}{4}(a^2 - 4a + 1)$$

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and $a^2 - 4a + 1 > 0$ for $a > 2 + \sqrt{3}$.

Now suppose that $z = (x, y)$ is Pareto optimal. Then the system of inequalities with unknown z'

$$\begin{aligned} \Pi^k(z') &> \Pi^k(z) \\ \Pi^{-k}(z') &> \Pi^{-k}(z) \end{aligned}$$

has no solution on Y, for $k = A, B$.

Since Y is convex and Π^A, Π^B are concave, by a fundamental property showed in [23],

$$\exists \lambda \in [0, 1] : \forall z' \in Y, \quad \lambda \Pi^A(z') + (1 - \lambda)\Pi^B(z') \le \lambda \Pi^A(z) + (1 - \lambda)\Pi^B(z).$$

The cases $\lambda = 0$ and $\lambda = 1$ correspond to the problem (5.28) for $k = A$ and $k = B$.

If $\lambda \in (0, 1)$, dividing each side by $(1 - \lambda)$ yields

$$\exists a > 0 : \forall z' \in Y, a\Pi^A(z') + \Pi^B(z') \le a\Pi^A(z) + \Pi^B(z).$$

Therefore, any Pareto optimal point is the optimal solution to the problem (5.28) for $k = A$ or $k = B$ or to problem (5.29). As a result, we have found all the Pareto optimal points.

To conclude, in this example, each payoff function to maximize is concave, and the jointly feasible set is convex. We obtain that there are infinitely many Nash equilibria, but there is a unique normalized Nash equilibrium. The normalized Nash equilibrium is included in the set of Nash equilibria.
The (unique) system optimum is a particular Nash equilibrium (thus the socially best one) in this example, but not the normalized Nash equilibrium. In addition, it is not the Nash equilibrium that maximizes the joint payoffs.

There are infinitely many Pareto optimal points. All Nash equilibria are Pareto optimal (in particular the system optimum and the normalized Nash equilibrium are Pareto optimal). In addition, there exist Pareto optimal points that are not a Nash equilibrium.

Generally however, the system optimum may not be a Nash equilibrium, and there may be Nash equilibria that are not Pareto optimal.

Conclusion

There are many notions that characterize equilibria or the concept of optimality in a game. In this thesis, we will focus on the normalized Nash equilibrium. The motivation is the following:

- as we show in the following section, it is unique in the problem under consideration;

- it is a Nash equilibrium;

- it is the Nash equilibrium such that the two competitors contribute *fairly* to achieving the coupling constraints, in the sense of how much they would benefit from violating it in terms of objective value change;

- assuming that it is impossible to directly force the competitors to satisfy these constraints, we can design a tax scheme which would give incentives to the players to reach a solution that achieves these constraints.

5.3.4 Uniqueness results

Theorem 15. *[111] There exists a normalized Nash equilibrium point to a concave n-person game.*

This theorem applies in particular to the problem we are considering in this chapter.

We observe that set Y can be written as a set of inequalities $\{x : H(x) \geq 0\}$ with H being concave. (This is clear for formulation (5.25) given the linearity of the constraints. For formulation (5.26), the result follows from the convexity of Ω_i^k with respect to both $p^k(.)$ and $p^{-k}(.)$.)

We argue that the following constraint qualification holds: $\exists x \in Y : H_j(x) > 0 \; \forall j$. This can be shown in a way similar to the how we proved that Y is non empty. We also notice that $H_j(x)$ possesses continuous first derivatives for $x \in Y$ in formulation (5.25) since all constraints are linear.

For a scalar function $J(x^A, x^B) = J(x)$ taking arguments in $\mathbb{R}^{3NT} \times \mathbb{R}^{3NT}$, we will denote by $\nabla_{x^k} J(x)$ the gradient of $J(x)$ with respect to x^k. Thus $\nabla_{x^k} J(x) \in \mathbb{R}^{3NT}$.

We denote $\sigma(x) = \Pi^A(x) + \Pi^B(x)$ and

$$g(x) = \begin{pmatrix} \nabla_{x^A} \Pi^A(x) \\ \nabla_{x^B} \Pi^B(x) \end{pmatrix}$$

$g(x)$ is called the pseudogradient of $\sigma(x)$.

Definition 11. *The function* $\sigma(x) = \Pi^A(x) + \Pi^B(x)$ *is called diagonally strictly concave for* $x \in Y$ *if for every* x^0, $x^1 \in Y$ *we have*

$$(x^1 - x^0)' g(x^0) + (x^0 - x^1)' g(x^1) > 0.$$

In this problem we have (we abuse notations by mentioning only the component

for product i at time t in order to ease the exposition):

$$g(x) = \begin{pmatrix} \nabla_{x^A} \Pi^A(x) \\ \nabla_{x^B} \Pi^B(x) \end{pmatrix} = \begin{pmatrix} \alpha_i^A(t) - 2\beta_i^{A,A}(t)p_i^A(t) + \beta_i^{A,B}(t)p_i^B(t) \\ -2\gamma_i^A(t)u_i^A(t) \\ -2h_i^A(t)I_i^A(t) \\ \alpha_i^B(t) - 2\beta_i^{B,B}(t)p_i^B(t) + \beta_i^{B,A}(t)p_i^A(t) \\ -2\gamma_i^B(t)u_i^B(t) \\ -2h_i^B(t)I_i^B(t) \end{pmatrix}$$

Using the Karush Kuhn Tucker conditions, Rosen shows the following:

Theorem 16. *[111] If $\sigma(x)$ is diagonally strictly concave, then the normalized Nash equilibrium point is unique.*

Note that the strict diagonal concavity of $\sigma(x)$, which here implies uniqueness of the normalized Nash equilibrium for coupled constraint games, would imply uniqueness of the Nash equilibrium if the constraints were not coupled. For coupled constraint games, the Nash equilibrium is in general not unique. We illustrate this in the following example.

Example 1 modified:

We now modify *Example 1* in order to show that there may be multiple Nash equilibria, but a unique normalized Nash equilibrium, when a Nash equilibrium lies on the boundary of the feasible set. We will assume here that all data are symmetric between A and B and we will simplify the notation as follows:

$$\beta^{A,A} = \beta^{B,B} = \beta, \quad \beta^{A,B} = \beta^{B,A} = \beta', \quad \alpha^A = \alpha^B = \alpha.$$

We add the constraint:

$$p^A + p^B \leq \frac{\alpha}{\beta}$$

in addition to the previous ones defined in *Example 1*. It is easy to verify that the solution for the Nash equilibrium obtained in *Example 1* does not satisfy this

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constraint.

The sets $Q(p^{*A}, p^{*B})$ and Y for the modified example are illustrated in Figures 5-4 and 5-5 below.
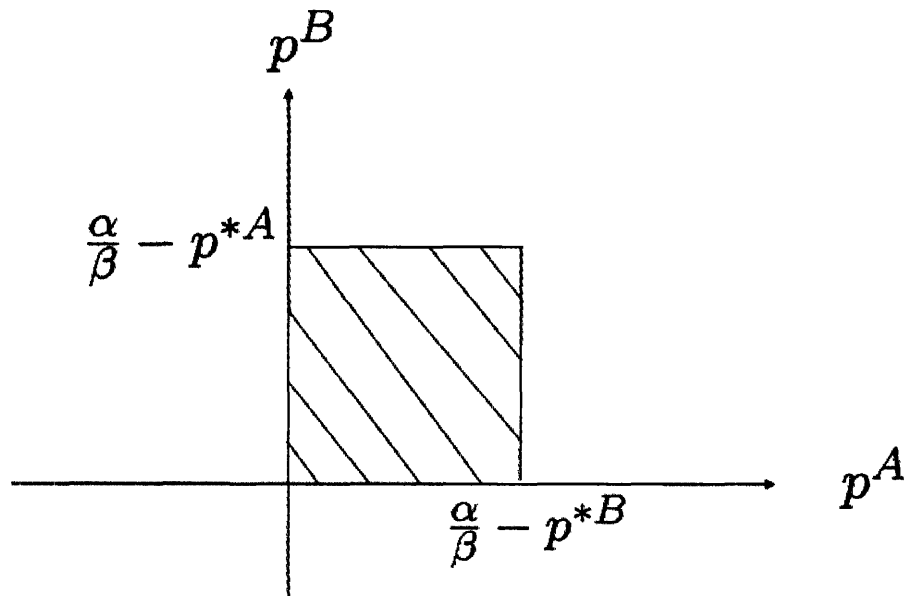

Figure 5-4: Modified example in space of prices: set $Q(p^{*A}, p^{*B})$

We can show that there is an infinite number of Nash equilibria as indicated on Figure 5-5, that correspond to prices such that

$$p^A + p^B = \frac{\alpha}{\beta} \quad \text{and} \quad \frac{\alpha}{\beta' + 2\beta} \leq p^k \leq \frac{\alpha}{\beta} - \frac{\alpha}{\beta' + 2\beta}, \quad k = A,\ B.$$
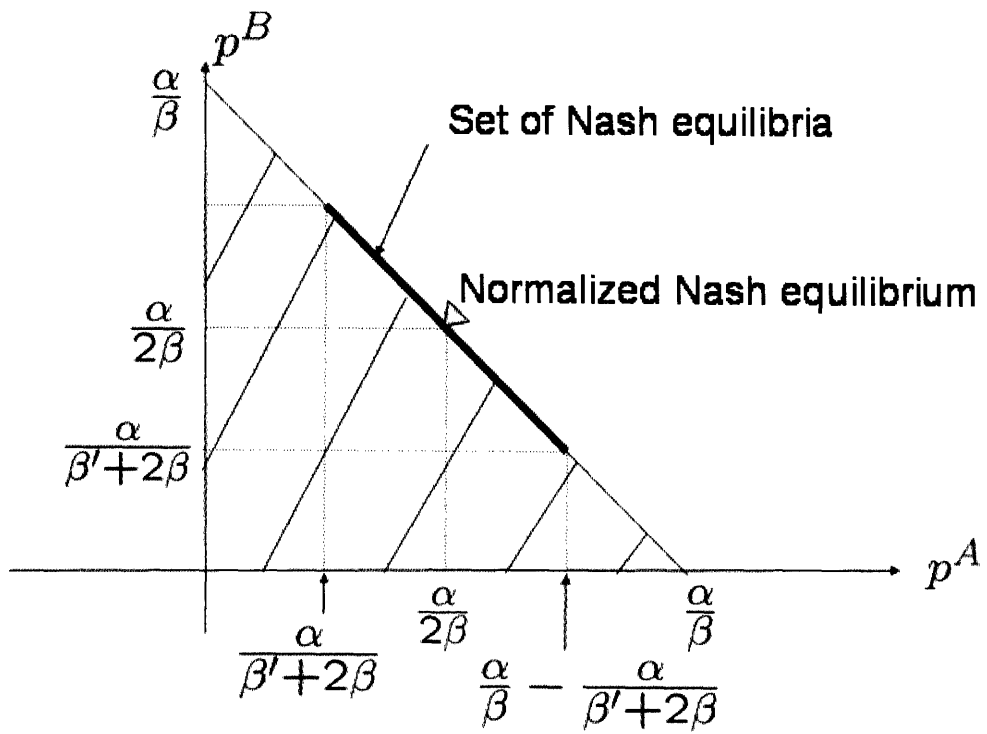
Figure 5-5: Modified example in space of prices: set Y

The unique normalized Nash equilibrium is

$$p^{*A} = p^{*B} = \frac{\alpha}{2\beta}.$$

Intuitively, what makes the modified problem have multiple Nash equilibria, including the normalized Nash equilibrium that is unique, is the fact that a coupling constraint is binding at the normalized equilibrium. Indeed, in that case, the optimization problem determining the normalized Nash equilibrium is not separable into the two subproblems that determine the Nash equilibrium.

Notice in particular that in *Example 1* the unique Nash equilibrium is in the interior of Y while in *Example 2* and *Example 1 modified* there are multiple Nash equilibria that lie on the boundary of Y. In all three examples however, there was, as expected, a unique normalized Nash equilibrium.

Theorem 17. *Under Assumptions 4 and 12, the normalized Nash equilibrium point is unique.*

To prove this theorem, we first mention a result from [111].

Let $G(x) \in \mathbb{R}^{6NT} \times \mathbb{R}^{6NT}$ the Jacobian of $g(x)$. We have

$$G(x) = \begin{pmatrix} -2\beta_i^{A,A}(t) & & & \beta_i^{A,B}(t) & & \\ & -2\gamma_i^A(t) & & & & \\ & & -2h_i^A(t) & & & \\ \beta_i^{B,A}(t) & & & -2\beta_i^{B,B}(t) & & \\ & & & & -2\gamma_i^B(t) & \\ & & & & & -2h_i^B(t) \end{pmatrix}$$

Theorem 18. *[111] A sufficient condition that $\sigma(x)$ be diagonally strictly concave for $x \in Y$ is that the symmetric matrix $G(x) + G^T(x)$ be negative definite for $x \in Y$.*

We now prove Theorem 17.

Proof. $G(x) + G^T(x)$ is a symmetric and strictly diagonally dominant matrix under

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Assumption 4 for $x \in Y$, with negative elements on the diagonal. Therefore, it is negative definite. The result then follows from Theorem 18. $\qquad\square$

To summarize, although the Nash equilibrium is in general not unique, the normalized Nash equilibrium is unique. Hence we will focus on how to compute the normalized Nash equilibrium, via the algorithm presented in the following section.

5.4 Solution algorithm

5.4.1 Description of the algorithm

We consider an iterative relaxation algorithm in which at each iteration, a current pricing policy is given for both suppliers, and the suppliers respond by determining simultaneously a new strategy such that

- the objective function involves the *current* given pricing policy of her competitor, but

- in the feasibility constraints, the responses must be jointly feasible.

The intuition behind this algorithm is to reflect actual practices in which suppliers adapt their policy in order to improve their performance based on their competitor's current strategy. Nevertheless since they adapt their policies simultaneously, they are constrained by the competitor's reaction as well.

As we will show, this algorithm converges to the unique normalized Nash equilibrium, which as we proved above is a particular Nash equilibrium.

We define the simultaneous best response function $BR(x) : Y \mapsto Y$ by

$$BR(x) = \operatorname*{Arg\,max}_{z \in Y} \psi(x, z)$$

(or equivalently: $BR(x) = \operatorname{Arg\,max}_{z \in Y} \Pi^A(z^A, x^B) + \Pi^B(x^A, z^B)$.)

Therefore the unique normalized Nash equilibrium is defined as the fixed point of

map BR.

$BR(x)$ represents the vector with components the respective optimal strategies each player should simultaneously adopt (and are thus jointly feasible, since the maximization is taken over set Y) when their payoffs are based on the fact that the other player keeps strategy x^{-k}. We observe that to compute the optimum response we only need to solve a constrained concave maximization quadratic program with a convex bounded non empty set, which is tractable. Therefore the solution $BR(x)$ exists and is uniquely defined.

We introduce an algorithm in order to find the fixed point of map BR which is the unique normalized Nash equilibrium.

Starting from an initial feasible collective strategy and state vector $x_0 \in Y$, we will use the following relaxation algorithm.

Algorithm $\mathcal{A}$:

1. Start at $x_0 \in Y$; let $s = 0$.

2. Let

$$x_{s+1} \;=\; (1 - \delta_s)x_s + \delta_s BR(x_s) \tag{5.30}$$

and do $s \leftarrow s + 1$.

3. if $\|x_s - BR(x_s)\| < \eta$, stop. Else, go to 2.

We will use this as a stopping criterion with $\eta = 10^{-6}$.

We choose step sizes $\delta_s \in (0,1)$, $s = 0, 1, \ldots$ such that

$$\delta_s > 0, \quad s = 0, 1, \ldots$$
$$\sum_{s=0}^{\infty} \delta_s = \infty \qquad (5.31)$$
$$\delta_s \to 0 \text{ as } s \to \infty.$$

We observe that since the starting point x_0 is chosen in the set Y, and since $BR(x) \in Y$ for $x \in Y$, the sequence of vectors produced by this algorithm remains in set Y.

We showed in Section 5.2 that the vector such that $\forall i$, t

$$p_i^k(t) = p_{i_{max}}^k(t)$$

$$u_i^k(t) = 2\hat{\alpha}_i^k(t) + 2\hat{\beta}_i^{k,k}(t)p_{i_{max}}^k(t) + 2\hat{\beta}_i^{k,-k}(t)p_{i_{max}}^{-k}(t)$$

$$I_i^k(t) = I_i^{t-1} + u_i^k(s) - \alpha_i^k(s) + \beta_i^{k,k}(s)p_{i_{max}}^k(s) - \beta_i^{k,-k}(s)p_{i_{max}}^{-k}(s)$$

is a feasible vector and we will use it as a possible starting point for Algorithm $\mathcal{A}$.

5.4.2 Theorem of convergence

Theorem 19. *[123] If*

1. *Y is a convex compact subset of $\mathbb{R}^{6NT}$;*

2. *the function ψ is a continuous weakly convex concave function and $\psi(x,x) = 0$, $x \in Y$;*

3. *the optimum response function $BR(.)$ is single valued and continuous;*

4. *the residual terms v and μ (defined below) satisfy the inequality*

$$v(x,y) - \mu(y,x) \geq \zeta(\|x - y\|), \quad x, y \in Y,$$

where ζ is a strictly monotonically increasing function such that $\zeta(0) = 0$;

5. *the step sizes satisfy condition (5.31)*

then Algorithm $\mathcal{A}$ described in Section 5.4.1 converges to the unique normalized Nash equilibrium.

In order to apply this theorem and prove convergence of algorithm $\mathcal{A}$, we first show some properties.

5.4.3 Continuity of the best response function

The following result is obtained in a way similar to Daniel [50].

Proposition 21. *BR is continuous on Y.*

Proof. Let's consider formulation (5.26). Let $x \in Y$; we can rewrite $BR(x)$ as the solution of the quadratic program

$$\min \quad z^T M z - c(x)^T z$$

$$\text{subject to} \quad z \in Y$$

where $M \in \mathbb{R}^{6NT}$ is positive definite, $c(x)$ is a vector such that $c(.)$ a continuous mapping. (M is a diagonal matrix with components $\beta_i^{k,k}(t)$, $\gamma_i^k(t)$, $h_i^k(t)$, and $c(x)$ has components $\alpha_i^k(t) + \beta_i^{k,-k}(t)p_i^{-k}(t)$, with the prices $p_i^{-k}(t)$ taken from vector x). Based on the fact that the feasible set is convex and independent of x, Daniel[1] showed using variational inequalities that if $x' \in Y$, $\epsilon \equiv ||c(x) - c(x')||$ and κ is the smallest eigenvalue of matrix M, then we have

$$||BR(x') - BR(x)|| \leq \epsilon(\kappa - \epsilon)^{-1}(1 + ||BR(x)||)$$

for $\epsilon < \kappa$.

Since $c(.)$ is a continuous mapping, it follows that BR is continuous. $\qquad\square$

---

[1]Daniel [50] proved the result under linear equality and inequality constraints. Nevertheless, it is easy to see that the proof remains the same under a more general convex set.

5.4.4 Weak convexity with respect to the first argument

Definition 12. *A function $f(.) : S \mapsto \mathbb{R}$ is said to be weakly convex on a convex set S if and only if $\forall x,\ y \in S,\quad 0 \le \theta \le 1$, the following inequality holds:*

$$\theta f(x) + (1 - \theta)f(y) \ge f(\theta x + (1 - \theta)y) + \theta(1 - \theta)r(x,y)$$

where the remainder $v : S \times S \mapsto \mathbb{R}$ satisfies

$$\frac{v(x,y)}{\|x - y\|} \to 0 \ as \ x \to w, \ y \to w$$

for all $w \in S$.

Proposition 22. *The Nikaido-Isoda function $\psi(x,y)$ is weakly convex on Y with respect to the first argument.*

Proof. Let $\theta_1 \in [0,1]$ and $\theta_2 = 1 - \theta_1$. To ease the exposition, we omit the time argument.

$$
\begin{aligned}
\Delta &= \theta_1 \psi(x, \bar{\bar{x}}) + \theta_2 \psi(\bar{x}, \bar{\bar{x}}) - \psi(\theta_1 x + \theta_2 \bar{x}, \bar{\bar{x}}) \\
&= \sum_{t=1}^{T} \sum_{i=1}^{N} \sum_{k=A,B} \Big[\theta_1 \Big((\alpha_i^k + \beta_i^{k,-k} p_i^{-k})(\bar{p}_i^k - p_i^k) - \beta_i^{k,k}(\bar{p}_i^{k^2} - p_i^{k^2}) - \gamma_i^k(\bar{u}_i^{k^2} - u_i^{k^2}) \\
&\hspace{8cm} - h_i^k(\bar{I}_i^{k^2} - I_i^{k^2}) \Big) \\
&\quad + \theta_2 \Big((\alpha_i^k + \beta_i^{k,-k}\bar{p}_i^{-k})(\bar{p}_i^k - \bar{p}_i^k) - \beta_i^{k,k}(\bar{p}_i^{k^2} - \bar{p}_i^{k^2}) - \gamma_i^k(\bar{u}_i^{k^2} - \bar{u}_i^{k^2}) - h_i^k(\bar{I}_i^{k^2} - \bar{I}_i^{k^2}) \Big) \\
&\quad - (\alpha_i^k + \beta_i^{k,-k}(\theta_1 p_i^{-k} + \theta_2 \bar{p}_i^{-k}))(\bar{p}_i^k - \theta_1 p_i^k - \theta_2 \bar{p}_i^k) + \beta_i^{k,k}(\bar{p}_i^{k^2} - (\theta_1 p_i^k + \theta_2 \bar{p}_i^k))^2) \\
&\quad - \gamma(\bar{u}_i^{k^2} - (\theta_1 u_i^k + \theta_2 \bar{u}_i^k)^2) - h_i^k(\bar{I}_i^{k^2} - (\theta_1 I_i^k + \theta_2 \bar{I}_i^k)^2) \Big].
\end{aligned}
$$

After calculations,

$$
\begin{aligned}
\Delta = \sum_{t=1}^{T} \sum_{i=1}^{N} \sum_{k=A,B} \Big[&-\theta_1 \theta_2 \beta_i^{k,-k}(p_i^{-k} - \bar{p}_i^{-k})(p_i^k - \bar{p}_i^k) + \theta_1 \theta_2 \beta_i^{k,k}(p_i^k - \bar{p}_i^k)^2 \\
&+ \gamma_i^k \theta_1 \theta_2 (u_i^k - \bar{u}_i^k)^2 + h_i^k \theta_1 \theta_2 (I_i^k - \bar{I}_i^k)^2 \Big] = \theta_1 \theta_2 v(x, \bar{x})
\end{aligned}
$$

where

$$v(x, \bar{x}) = \sum_{t=1}^{T} \sum_{i=1}^{N} \sum_{k=A,B} \left[- \beta_i^{k,-k}(p_i^{-k} - \bar{p}_i^{-k})(p_i^k - \bar{p}_i^k) + \beta_i^{k,k}(p_i^k - \bar{p}_i^k)^2 \right.$$
$$\left. + \gamma_i^k(u_i^k - \bar{u}_i^k)^2 + h_i^k(I_i^k - \bar{I}_i^k)^2 \right].$$

We have

$$v(x, \bar{x}) = - \sum_{t=1}^{T} \sum_{i=1}^{N} (\beta_i^{A,B} + \beta_i^{B,A})(p_i^B - \bar{p}_i^B)(p_i^A - \bar{p}_i^A) + O_1(||x - \bar{x}||^2),$$

and thus

$$\frac{v(x, \bar{x})}{||x - \bar{x}||} = - \frac{\sum_{t=1}^{T} \sum_{i=1}^{N} (\beta_i^{A,B} + \beta_i^{B,A})(p_i^B - \bar{p}_i^B)(p_i^A - \bar{p}_i^A)}{\sqrt{\sum_{t=1}^{T} \sum_{i=1}^{N} \sum_{k=A,B} \left[(p_i^k - \bar{p}_i^k)^2 + (u_i^k - \bar{u}_i^k)^2 + (I_i^k - \bar{I}_i^k)^2 \right]}}$$
$$+ O_1(||x - \bar{x}||)$$

which tends to 0 as $x, \bar{x} \to w \in Y$ since in the ratio above, the numerator is of the order of ϵ^2 and the denominator is of the order of ϵ, with $\epsilon \to 0$. $\qquad \square$

5.4.5 Concavity with respect to the second argument

Definition 13. *A function* $f(.) : S \mapsto \mathbb{R}$ *is said to be weakly concave on a convex set* S *if and only if* $\forall x, y \in S, \ 0 \leq \theta \leq 1$, *the following inequality holds:*

$$\theta f(x) + (1 - \theta)f(y) \leq f(\theta x + (1 - \theta)y) + \theta(1 - \theta)\mu(x, y)$$

where the remainder $\mu : S \times S \mapsto \mathbb{R}$ *satisfies*

$$\frac{\mu(x, y)}{||x - y||} \to 0 \ \text{as} \ x \to z, \ y \to w$$

for all $w \in S$.

The function is concave if $\mu(x, y) \leq 0 \ \forall x, y \in S$.

Proposition 23. *The Nikaido-Isoda function $\psi(x, y)$ is concave on Y with respect to the second argument.*

Proof. This is clear considering that in the expression of $\psi(x, \bar{x})$, the dependence in $\bar{x}$ appears as negative quadratic terms in $\bar{p}$, $\bar{u}$ and $\bar{I}$. More explicitly, let $\theta_1 \in [0, 1]$ and $\theta_2 = 1 - \theta_1$. To ease the exposition, we omit the time argument.

$$
\begin{aligned}
\Delta &= \theta_1 \psi(\bar{\bar{x}}, x) + \theta_2 \psi(\bar{\bar{x}}, \bar{x}) - \psi(\bar{\bar{x}}, \theta_1 x + \theta_2 \bar{x}) \\
&= \sum_{t=1}^{T} \sum_{i=1}^{N} \sum_{k=A,B} \Big[\theta_1 \Big((\alpha_i^k + \beta_i^{k,-k} \bar{\bar{p}}_i^{-k})(p_i^k - \bar{p}_i^k) - \beta_i^{k,k}(p_i^{k^2} - \bar{p}_i^{k^2}) - \gamma(u_i^{k^2} - \bar{\bar{u}}_i^{k^2}) \\
&\hspace{9cm} - h_i^k(I_i^{k^2} - \bar{\bar{I}}_i^{k^2}) \Big) \\
&\quad + \theta_2 \Big((\alpha_i^k + \beta_i^{k,-k} \bar{\bar{p}}_i^{-k})(\bar{p}_i^k - \bar{p}_i^k) - \beta_i^{k,k}(\bar{p}_i^{k^2} - \bar{p}_i^{k^2}) - \gamma_i^k(\bar{u}_i^{k^2} - \bar{\bar{u}}_i^{k^2}) - h_i^k(\bar{I}_i^{k^2} - \bar{\bar{I}}_i^{k^2}) \Big) \\
&\quad - (\alpha_i^k + \beta_i^{k,-k} \bar{\bar{p}}_i^{-k})(\theta_1 p_i^k + \theta_2 \bar{p}_i^k - \bar{p}_i^k) + \beta_i^{k,k}((\theta_1 p_i^k + \theta_2 \bar{p}_i^k)^2 - \bar{p}_i^{k^2}) \\
&\quad - \gamma_i^k((\theta_1 u_i^k + \theta_2 \bar{u}_i^k)^2 - \bar{\bar{u}}_i^{k^2}) - h_i^k((\theta_1 I_i^k + \theta_2 \bar{I}_i^k)^2 - \bar{\bar{I}}_i^{k^2}) \Big].
\end{aligned}
$$

After calculations,

$$
\begin{aligned}
\Delta &= \sum_{t=1}^{T} \sum_{i=1}^{N} \sum_{k=A,B} \Big[-\theta_1 \theta_2 \Big(\beta_i^{k,k}(p_i^k - \bar{p}_i^k)^2 + \gamma_i^k(u_i^k - \bar{u}_i^k)^2 + h_i^k(I_i^k - \bar{I}_i^k)^2 \Big) \Big] \\
&= \theta_1 \theta_2 \mu(x, \bar{x})
\end{aligned}
$$

where

$$
\mu(x, \bar{x}) = -\sum_{t=1}^{T} \sum_{i=1}^{N} \sum_{k=A,B} \Big[\beta_i^{k,k}(p_i^k - \bar{p}_i^k)^2 + \gamma_i^k(u_i^k - \bar{u}_i^k)^2 + h_i^k(I_i^k - \bar{I}_i^k)^2 \Big].
$$

We have

$$
\mu(x, \bar{x}) = O_1(\|x - \bar{x}\|^2),
$$

and thus

$$
\frac{\mu(x, \bar{x})}{\|x - \bar{x}\|} = O_1(\|x - \bar{x}\|)
$$

which tends to 0 as $x, \bar{x} \to w \in Y$.

In particular, ψ is weakly concave with respect to the second argument. $\quad\square$

Since ψ is weakly convex with respect to the first argument and weakly concave with respect to the second argument, ψ is said to be weakly convex concave.

5.4.6 An inequality

In what follows, we illustrate that the inequality in Theorem 19 indeed holds.

Proposition 24. *There exists a strictly monotonically increasing function* $\zeta : \mathbb{R} \mapsto \mathbb{R}$ *such that* $\zeta(0) = 0$ *and*

$$v(x,y) - \mu(y,x) \geq \zeta(||x - y||), \quad x, y \in Y.$$

Proof. Consider $\zeta(x) = \lambda x^2$ for some $\lambda > 0$. Then $v(x, \bar{x}) - \mu(\bar{x}, x) - \lambda ||x - \bar{x}||^2 =$

$$\sum_{t=1}^{T} \sum_{i=1}^{N} \left[(2\beta_i^{A,A}(t) - \lambda)(p_i^A(t) - \bar{p}_i^A(t))^2 + (2\beta_i^{B,B}(t) - \lambda)(p_i^B(t) - \bar{p}_i^B(t))^2 \right.$$
$$- (\beta_i^{A,B}(t) + \beta_i^{B,A}(t))(p_i^A(t) - \bar{p}_i^A(t))(p_i^B(t) - \bar{p}_i^B(t))$$
$$+ (2\gamma_i^A(t) - \lambda)(u_i^A(t) - \bar{u}_i^A(t))^2 + (2\gamma_i^B(t) - \lambda)(u_i^B(t) - \bar{u}_i^B(t))^2$$
$$\left. + (2h_i^A(t) - \lambda)(I_i^A(t) - \bar{I}_i^A(t))^2 + (2h_i^B(t) - \lambda)(I_i^B(t) - \bar{I}_i^B(t))^2 \right].$$

Let

$$\lambda_1 = 2 \min_k \min_i \min_t h_i^k(t), \quad \lambda_2 = 2 \min_k \min_i \min_t \gamma_i^k(t).$$

A sufficient condition for the expression above to be positive is that $\lambda < \lambda_1$, $\lambda < \lambda_2$ and the symmetric matrix (defined for fixed i, t)

$$N = \begin{bmatrix} 2\beta_i^{A,A}(t) - \lambda & -\frac{(\beta_i^{A,B}(t) + \beta_i^{B,A}(t))}{2} \\ -\frac{(\beta_i^{A,B}(t) + \beta_i^{B,A}(t))}{2} & 2\beta_i^{B,B}(t) - \lambda \end{bmatrix}$$

is positive semi-definite for all i, t.

We notice that

$$N \succeq 0 \quad \Leftrightarrow \quad \big(\mathrm{Tr}(N) \geq 0 \text{ and } \mathrm{Det}(N) \geq 0\big)$$

$$\Leftrightarrow \begin{cases} (\beta_i^{A,A}(t) + \beta_i^{B,B}(t) - \lambda \geq 0 \\ (2\beta_i^{A,A}(t) - \lambda)(2\beta_i^{B,B}(t) - \lambda) - \frac{1}{4}(\beta_i^{A,B}(t) + \beta_i^{B,A}(t))^2 \geq 0) \end{cases}$$

$$\Leftrightarrow \begin{cases} \lambda \leq \beta_i^{A,A}(t) + \beta_i^{B,B}(t) \\ \lambda^2 - 2\lambda(\beta_i^{A,A}(t) + \beta_i^{B,B}(t)) + 4\beta_i^{A,A}(t)\beta_i^{B,B}(t) - \frac{1}{4}(\beta_i^{A,B}(t) + \beta_i^{B,A}(t))^2 \\ \hspace{10cm} \geq 0. \end{cases}$$

The simplified discriminant of the polynomial above is (for fixed i, t)

$$\begin{aligned} \Delta &= (\beta_i^{A,A}(t) + \beta_i^{B,B}(t))^2 - 4\beta_i^{A,A}(t)\beta_i^{B,B}(t) + \frac{1}{4}(\beta_i^{A,B}(t) + \beta_i^{B,A}(t))^2 \\ &= (\beta_i^{A,A}(t) - \beta_i^{B,B}(t))^2 + \frac{1}{4}(\beta_i^{A,B}(t) + \beta_i^{B,A}(t))^2 > 0 \end{aligned}$$

so the polynomial has two real roots $\beta_i^{A,A}(t) + \beta_i^{B,B}(t) \pm \sqrt{\Delta}$ and only one satisfies $\lambda \leq \beta_i^{A,A}(t) + \beta_i^{B,B}(t)$. Since we are interested in positive parameters λ, and since the polynomial is non negative when evaluated at point below the smaller root and above the larger root, we obtain that

$$(\lambda > 0 \text{ and } N \succeq 0) \quad \Leftrightarrow \quad \begin{cases} \beta_i^{A,A}(t) + \beta_i^{B,B}(t) - \sqrt{\Delta} > 0 \\ 0 < \lambda \leq \beta_i^{A,A}(t) + \beta_i^{B,B}(t) - \sqrt{\Delta} \end{cases}$$

$$\Leftrightarrow \begin{cases} (\beta_i^{A,A}(t) - \beta_i^{B,B}(t))^2 + \frac{1}{4}(\beta_i^{A,B}(t) + \beta_i^{B,A}(t))^2 \\ \hspace{5cm} < (\beta_i^{A,A}(t) + \beta_i^{B,B}(t))^2 \\ 0 < \lambda \leq \beta_i^{A,A}(t) + \beta_i^{B,B}(t) - \sqrt{\Delta} \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{1}{4}(\beta_i^{A,B}(t) + \beta_i^{B,A}(t))^2 < 4\beta_i^{A,A}(t)\beta_i^{B,B}(t) \\ 0 < \lambda \leq \beta_i^{A,A}(t) + \beta_i^{B,B}(t) - \sqrt{\Delta}. \end{cases}$$

The first condition is satisfied under Assumption 4. Let

$$\lambda_3 = \min_i \inf_{t \in [0,T]} \beta_i^{A,A}(t) + \beta_i^{B,B}(t) - \sqrt{(\beta_i^{A,A}(t) - \beta_i^{B,B}(t))^2 + \frac{1}{4}(\beta_i^{A,B}(t) + \beta_i^{B,A}(t))^2} > 0.$$

As a result, by taking $0 < \lambda < \min\{\lambda_1, \lambda_2, \lambda_3\}$, we obtain that

$$v(x, \bar{x}) - \mu(\bar{x}, x) - \lambda\|x - \bar{x}\|^2 > 0 \quad \forall x, \ \bar{x} \in Y.$$

$\square$

We have shown that all the conditions in Theorem 19 hold for our problem.

Corollary 12. *Under Assumptions 4 and 12, Algorithm $\mathcal{A}$ converges to the unique normalized Nash equilibrium (which is a particular Nash equilibrium).*

5.5 Numerical results

5.5.1 Deterministic model

We implement the robust competitive reformulation discussed in Section 5.1 and Algorithm $\mathcal{A}$ we introduced in the previous section. We first consider an example with no data uncertainty, on a time horizon $T = 10$. Our goal in this chapter is to study the role of input parameters on the equilibrium solution. In order to isolate the effect of these parameters, we will implement the algorithm for $N = 1$ product. We will thus omit the subscript i in this section.

We want to study the effect of:

- the price sensitivities (coefficients $\beta(.)$)

- the capacity level $K(.)$

- the initial inventory level I^0.

We will consider symmetric suppliers on the one hand (i.e. subject to similar market conditions, so all inputs are the same for both), and asymmetric suppliers on the other hand, in the sense that they are subject to different price sensitivities.

We choose δ_s as the constant 0.99 for the first 50 iterations, and then equal to $\frac{1}{s-49}$. (In practice, the stopping criterion is reached before the 50th iteration in the numerical examples. Most solutions were obtained in less that 20 iterations of the algorithm.)

5.5.2   Effect of the price sensitivities

We will consider on the one hand price sensitivities that are symmetric for the suppliers (scenarios a through d), i.e. $\beta^{A,A}(.) = \beta^{B,B}(.)$ and $\beta^{A,B}(.) = \beta^{B,A}(.)$, and on the other hand asymmetric price sensitivities (scenarios e through j).

In both the symmetric and the asymmetric case, we will consider successively

1. demand sensitivities to prices that increase with time (i.e. when closer to the end time, an increase on prices affects the demand more): scenarios a, b, e through g

2. demand sensitivities to prices that decrease with time (i.e. products whose high price matters less at the end of the horizon): scenarios c, d, h through j.

Price sensitivities that increase with time correspond to products that become less attractive to the customer towards the end of the time horizon, for example products subject to a seasonality effect, or such that there have appeared on the market newer products that can serve as a substitute. Price sensitivities that increase with time correspond to products that become more attractive to the customer towards the end of the time horizon, for example because of a marketing campaign or an appearing trend.

Also in both cases, we will choose the data such that the ratio $\rho^k = \frac{\beta^{k,-k}}{\beta^{k,k}}$ of the sensitivity to the competitor's price over the sensitivity to the supplier's price is constant across time (and we will consider two possible values for this ratio). In the symmetric case, this ratio will clearly be the same for both suppliers. In the asymmetric case, we will consider inputs such that either this ratio is the same for the two suppliers (with two values), or it is different.

Scenario	$\beta^{A,A}(t)$	$\beta^{B,B}(t)$	$\beta^{A,B}(t)$	$\beta^{B,A}(t)$	ρ^A	ρ^B
a	$1+0.2t$	$1+0.2t$	$0.5+0.1t$	$0.5+0.1t$	0.5	0.5
b	$1+0.2t$	$1+0.2t$	$0.25+0.05t$	$0.25+0.05t$	0.25	0.25
c	$3-0.2t$	$3-0.2t$	$1.5-0.1t$	$1.5-0.1t$	0.5	0.5
d	$3-0.2t$	$3-0.2t$	$0.75-0.05t$	$0.75-0.05t$	0.25	0.25
e	$1+0.2t$	$0.6+0.2t$	$0.5+0.1t$	$0.3+0.1t$	0.5	0.5
f	$1+0.2t$	$0.6+0.2t$	$0.5+0.1t$	$0.15+0.05t$	0.5	0.25
g	$1+0.2t$	$0.6+0.2t$	$0.25+0.05t$	$0.15+0.05t$	0.25	0.25
h	$3-0.2t$	$2.2-0.2t$	$1.5-0.1t$	$1.1-0.1t$	0.5	0.5
i	$3-0.2t$	$2.2-0.2t$	$1.5-0.1t$	$0.55-0.05t$	0.5	0.25
j	$3-0.2t$	$2.2-0.2t$	$0.75-0.05t$	$0.55-0.05t$	0.25	0.25

Table 5.3: Scenarios of price sensitivities

We use the following parameter values for $k = A, B, \quad t = 1, \dots, 10$.

- $\alpha^k(t) = 15$

- $\gamma^k(t) = 0.01$

- $h^k(t) = 0.01$

- $I^{0k} = 10$

- capacity $K^k(t) = 10$

- demand sensitivities to prices as shown in Table 5.3. (We verified that Assumption 4 holds in all cases.)

However, one must be careful when interpreting the objective values across scenarios since the demand sensitivities have an effect on the total demand.

The results are shown in Table 5.4 and Figures 5-6 and 5-7.

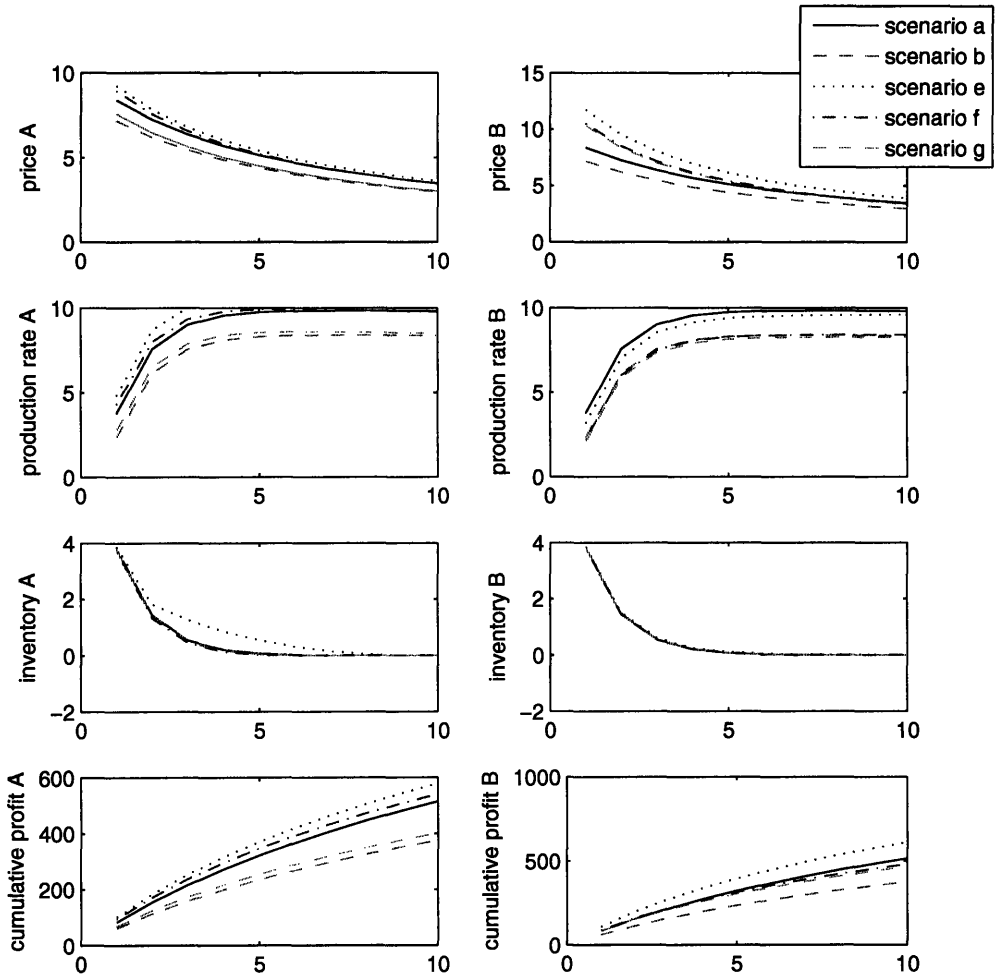The insights are summarized as follows.

Figure 5-6: Results: Effect of price sensitivities. Equilibrium in the case of price sensitivities increasing with time
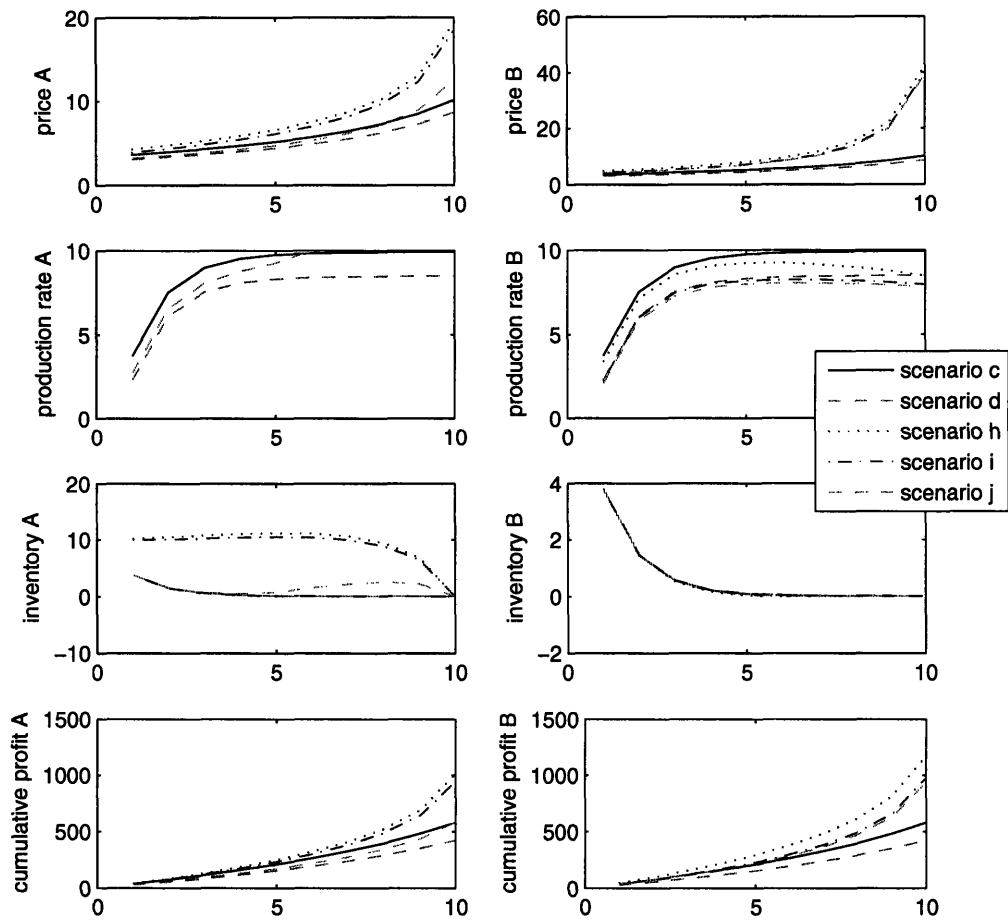
Figure 5-7: Results: Effect of price sensitivities. Equilibrium in the case of price sensitivities decreasing with time

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Scenario	Supplier A's obj.	Supplier B's obj.	Total obj.
a	514.83	514.83	1023.8
b	375.83	375.83	750.8
c	581.49	581.49	1157.1
d	484.80	484.80	848.7
e	578.86	611.73	1183.6
f	540.71	481.20	1018.2
g	401.45	465.57	866.1
h	1012.9	1171.2	2097.5
i	944.24	975.63	1866.5
j	586.77	934.24	1518.8

Table 5.4: Objective values: effect of price sensitivities

- Prices evolve with time with a trend opposite to the price sensitivities, i.e. prices are higher when the sensitivities are lower. In most cases (when the sensitivities are not too high), the inventory levels decrease from the initial value to zero, and then remain at that level. Production rates are adjusted in order to maintain the zero inventory level. When the price sensitivities are very high (A in scenario h, i), the inventory level remains high for most of the time horizon and is sold at the very end only. Selling earlier would not result in high enough profits because the high price sensitivities would force to price too low.

- We observe that when the cross price sensitivities are assigned in a scenario lower values than in another scenario, the equilibrium prices and production rates decrease, and profits decrease whether only one cross sensitivity is lowered or both. This remark holds for symmetric and asymmetric scenarios, and with sensitivities increasing and decreasing with time. If only one of the cross-sensitivities is decreased, the effect is stronger on the supplier subject to the decrease.

- Moreover, comparing the asymmetric case with the symmetric case (a vs. e, b vs. g, c vs. h, d vs. j), we notice that supplier B's share of the total objective is greater when she has lower price sensitivities than A. It seems that this is due to the decrease in the sensitivity to her own price though since comparing

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e and f, and h and j, we observe that her share of total revenues decreases when only her cross sensitivity decreases.

5.5.3 Effect of capacity

To study the effect of the capacity level, we choose the sensitivities as given by scenario f and h (asymmetric, respectively increasing and decreasing with time. Supplier A has higher price sensitivities.) and the coefficients $\alpha^k = 15$, $I^{0^k} = 10$ also fixed as in the previous section. We compute the equilibrium for a capacity limit taking values 6, 8, 10, 12, 14, 16 (identical for the two suppliers). The results are presented in Table 5.5 and Figures 5-8 and 5-9.

	K	6	8	10	12	14	16
	Supplier A's profit	540.86	540.22	540.71	540.71	540.71	540.71
Scenario f	Supplier B's profit	505.56	492.58	481.20	481.20	481.20	481.20
	Total profits	1046.4	1032.8	1021.9	1021.9	1021.9	1021.9
	Supplier A's profit	1031.8	1006.6	1012.9	1018.5	1018.5	1018.4
Scenario h	Supplier B's profit	1270.1	1239.2	1171.2	1141.8	1135.4	1134.8
	Total profits	2283.9	2245.8	2184.1	2160.3	2153.9	2153.1

Table 5.5: Results: Effect of production capacity. Profits for various symmetric capacity levels

First, we notice that under scenario f, when the capacity equals 10, the production rate for supplier A never reaches 10. As a result, the optimal policy is identical for higher capacity levels.

In scenario f (sensitivities increasing with time), supplier A gets higher profits than supplier B, but the reverse is true in scenario h (sensitivities increasing with time). Overall, we observe that when capacity increases the prices decrease at the equilibrium.

Interestingly, supplier B's profits tend to slightly decrease as the capacity increase. This may look surprising since a higher capacity gives more flexibility. This illustrates
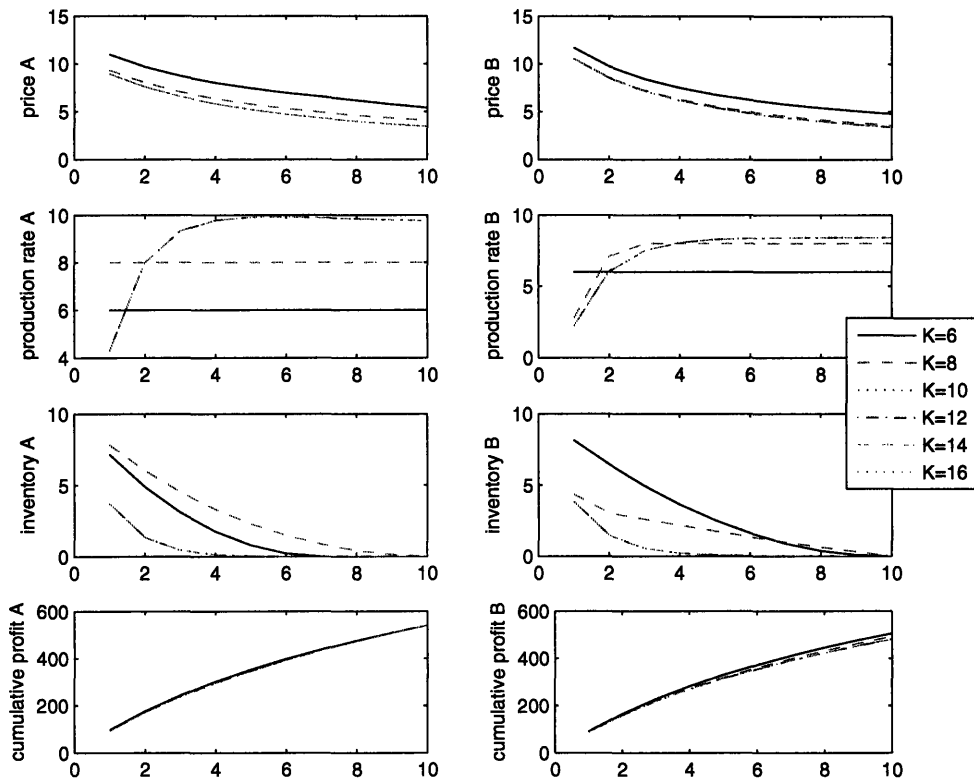
Figure 5-8: Results: Effect of production capacity. Equilibrium in the case of scenario f for various symmetric capacity levels
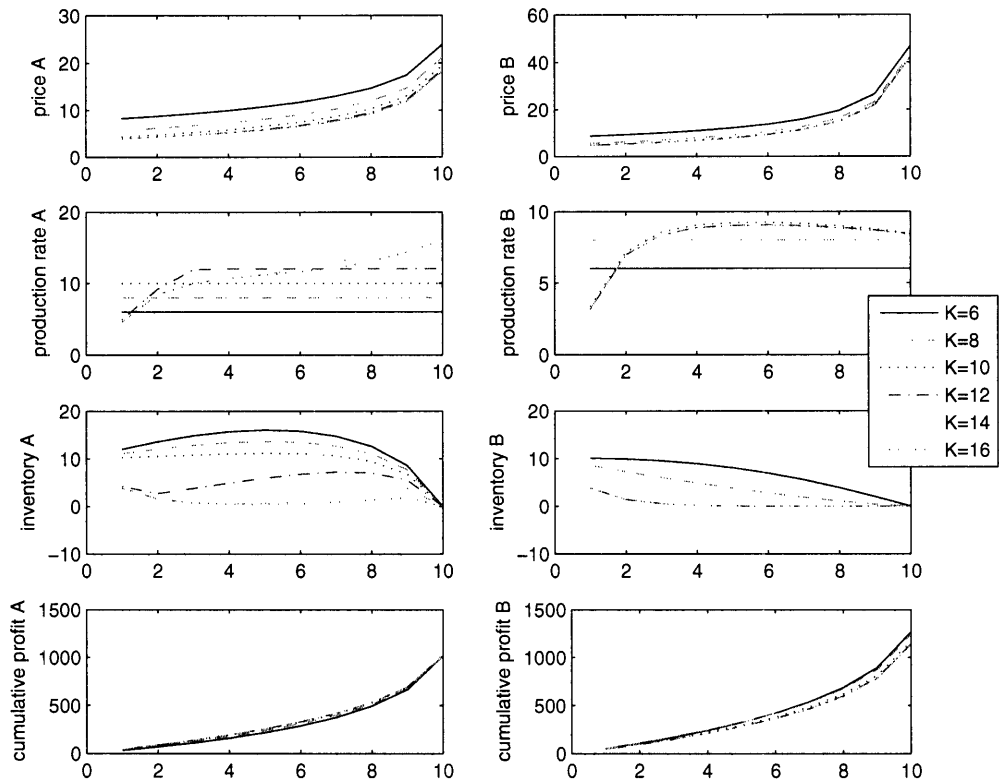
Figure 5-9: Results: Effect of production capacity. Equilibrium in the case of scenario h for various symmetric capacity levels

that the presence of competition may not yield an equilibrium that is unilaterally optimal for a given supplier.

Notice also that in scenario h, when the capacity is high enough in order to enable to meet the no backorders constraint towards the end of the time horizon (when sensitivities are lower and prices can increase) without using all available capacity, at the equilibrium the inventory levels decrease from the beginning of the time horizon in order to decrease holding costs. However when the capacity is low, inventories are kept around the initial value until the sensitivities become lower, so that selling yields more significant profits.

5.5.4 Effect of initial inventory level

To study the effect of the initial inventory level, we choose the sensitivities as given by scenario f and h (asymmetric, increasing and decreasing with time) and the coefficients $\alpha^k = 15$, $K^k = 10$ also fixed. We compute the optimal solution for an initial inventory level identical for both suppliers, and taking values 6, 8, 10, 12, 14, 16 (identical for the two suppliers). The results are presented in Table 5.6 and Figures 5-10 and 5-11.

	I^0	6	8	10	12	14	16
	Supplier A's profit	540.51	540.63	540.71	540.73	540.67	540.43
Scenario f	Supplier B's profit	481.03	481.14	481.20	481.21	481.17	481.00
	Total profits	1021.5	1021.8	1021.9	1021.9	1021.8	1021.4
	Supplier A's profit	1011.8	1012.7	1012.9	1012.5	1011.5	1011.5
Scenario h	Supplier B's profit	1181.6	1176.4	1171.2	1166.0	1160.9	1160.2
	Total profits	2193.4	2189.1	2184.1	2178.5	2172.4	2171.8

Table 5.6: Results: Effect of initial inventory level. Profits for various symmetric initial inventory levels

We observe that in scenario f where the capacity constraint is not tight, when the initial inventory level changes, the production rates changes in such a way that the effects cancel out (i.e. the production rate increases by as much as the initial inventory level decreased), and the prices and cumulative profits do not vary. Therefore, the
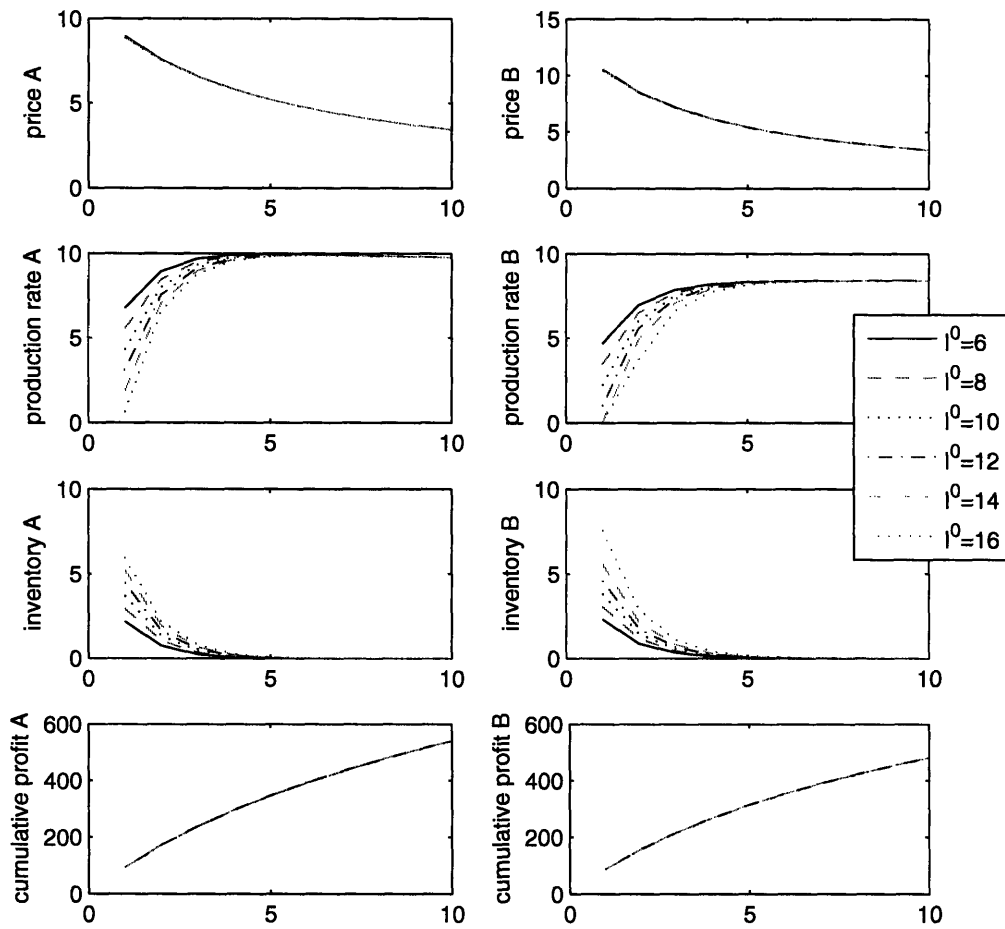
Figure 5-10: Results: Effect of initial inventory level. Equilibrium in the case of scenario f for various initial inventory levels
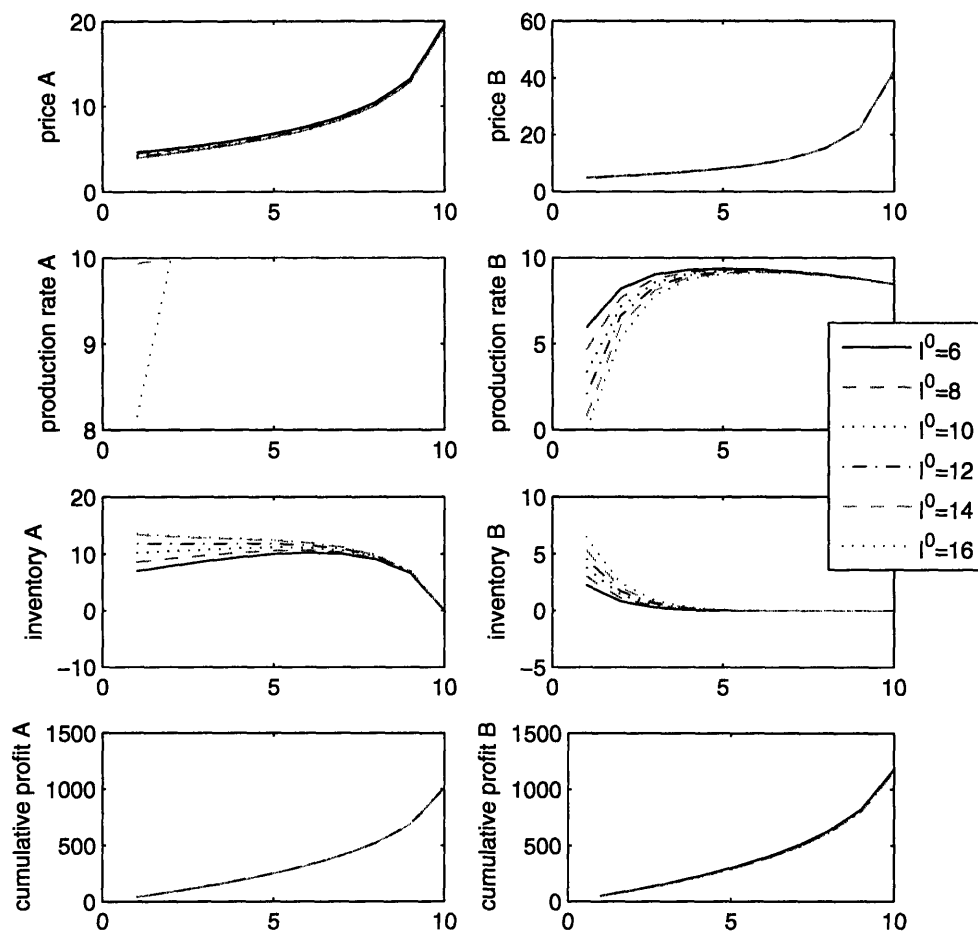
Figure 5-11: Results: Effect of initial inventory level. Equilibrium in the case of scenario h for various initial inventory levels

initial inventory levels seem to have a low impact.

The same is true for supplier B in scenario h for the same reason. However, for supplier A in case h, the capacity constraint is tight, the only way to compensate for the increase in initial inventory level is by decreasing prices. The profits slightly increase for supplier B, and slightly decrease for supplier A. As a result, suppliers with high price sensitivities are slightly advantaged by having low initial inventories, if the capacity is a binding constraint.

5.5.5 Robust formulation

We will implement the algorithm for $N = 1$ product (therefore we will omit the subscript i) on a time horizon $T = 10$ for symmetric and asymmetric suppliers in the case with uncertainty.

We want to study the effect of the budget of uncertainty so we will fix the parameters as given in the beginning of the previous section, and prices sensitivities as in scenarios f and h.

We will consider $\tilde{\alpha}(t)$ uncertain and we take input parameters $\hat{\alpha}(.)$ and $\Gamma(.)$ that are linear functions of the time (although the linearity assumption is not necessary in general). To be able to isolate the effect of uncertainty, we will suppose that the parameters $\beta^{k,-k}(.)$ are certain, i.e. $\hat{\beta}^{k,-k}(.) = 0$.

In these computations, we choose for both suppliers

$$\hat{\alpha}(t) = b + at = 0.1 + 0.2t.$$

Indeed, it is reasonable to suppose that in practice, the inaccuracy of a forecast for the demand increases on the time horizon, i.e. that the length of the interval of feasible outcomes increases with time. This choice of inputs represents an uncertainty on the nominal value of parameter α of $\pm j\%$ where j equals 2 at time $t = 1$ and 14 at time

$t = 10$.

We will consider the input parameters $\Gamma^k(.)$ to be linear functions of the time and identical for the two suppliers:

$$\Gamma^k(t) \;=\; gt + c$$

where g, $c \geq 0$, $g < 1$. We will compute the cumulative effective budget of uncertainty $\int_0^T \min\{t, \Gamma^k(t)\} dt$ as a measure of the global uncertainty in each scenario. See Table 5.7 for the values we consider.

We now have to compute the corresponding inventory security levels $\Omega^A(.)$, $\Omega^B(.)$. We proved in Chapter 4 that with $\hat{\alpha}^k(.)$ increasing with time and $\hat{\beta}^{k,k}(.) = \hat{\beta}^{k,-k}(.) = 0$, then

$$\Omega^k(t) = \begin{cases} \int_{t-\Gamma^k(t)}^t \hat{\alpha}^k(s)ds, & \Gamma^k(t) < t \\ \int_0^t \hat{\alpha}^k(s)ds, & \Gamma^k(t) \geq t. \end{cases}$$

Therefore

$$\Omega^k(t) = \begin{cases} \int_{t-gt-c}^t (as+b)ds = (gt+c)(at+b) - \frac{a}{2}(gt+c)^2, & \frac{c}{1-g} < t \leq T \\ \int_0^t (as+b)ds = \frac{a}{2}t^2 + bt, & 0 \leq t \leq \frac{c}{1-g} \end{cases}$$

after calculations. In particular, $\Omega^k(0) = 0 \leq I^{k^0}$. We also verify that with these inputs Assumption 12 is satisfied.

he results are presented in Figures 5-12 and 5-13.

Recall that in the deterministic case, in scenario f, at the equilibrium the inventory levels of supplier A and supplier B decrease from their initial value until they reached zero (before time $t = 5$) and then are kept at zero. In the case under uncertainty, the inventory levels must satisfy the minimum inventory security level, which is increasing with time. As a result, they do decrease from the initial value, until they reach the
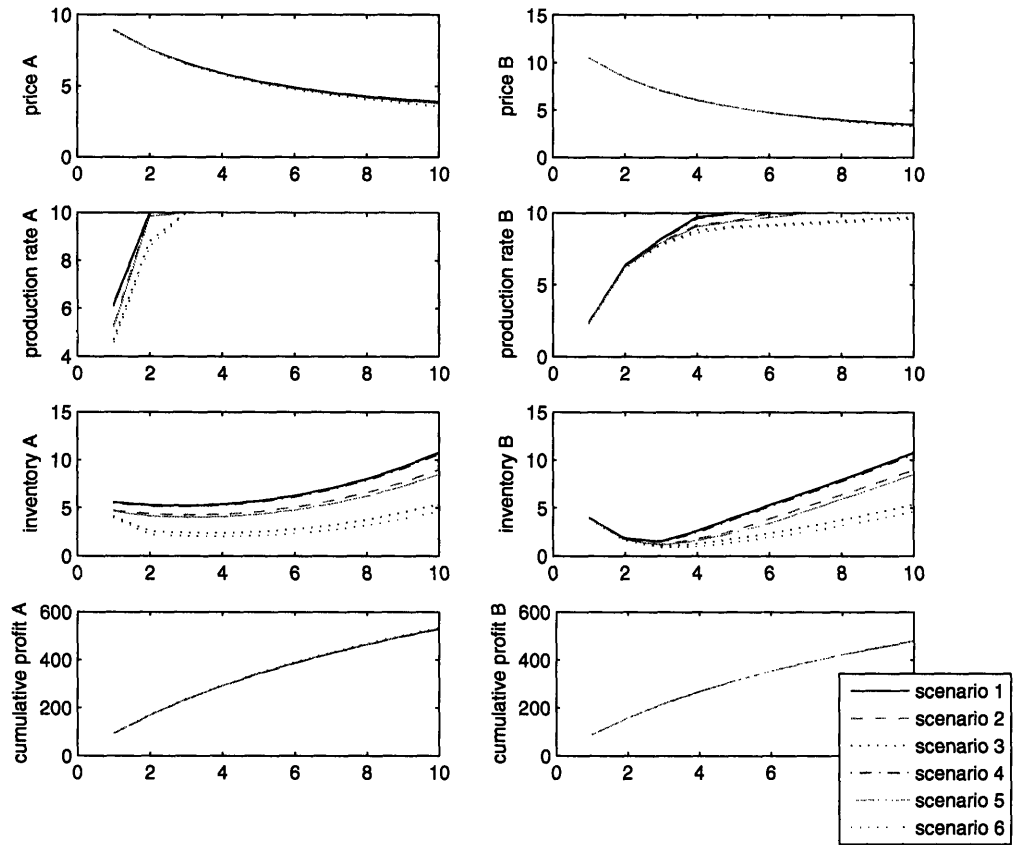
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Figure 5-12: Results: Robust formulation. Equilibrium in the case of price sensitivities scenario f for various scenarios of budget of uncertainty
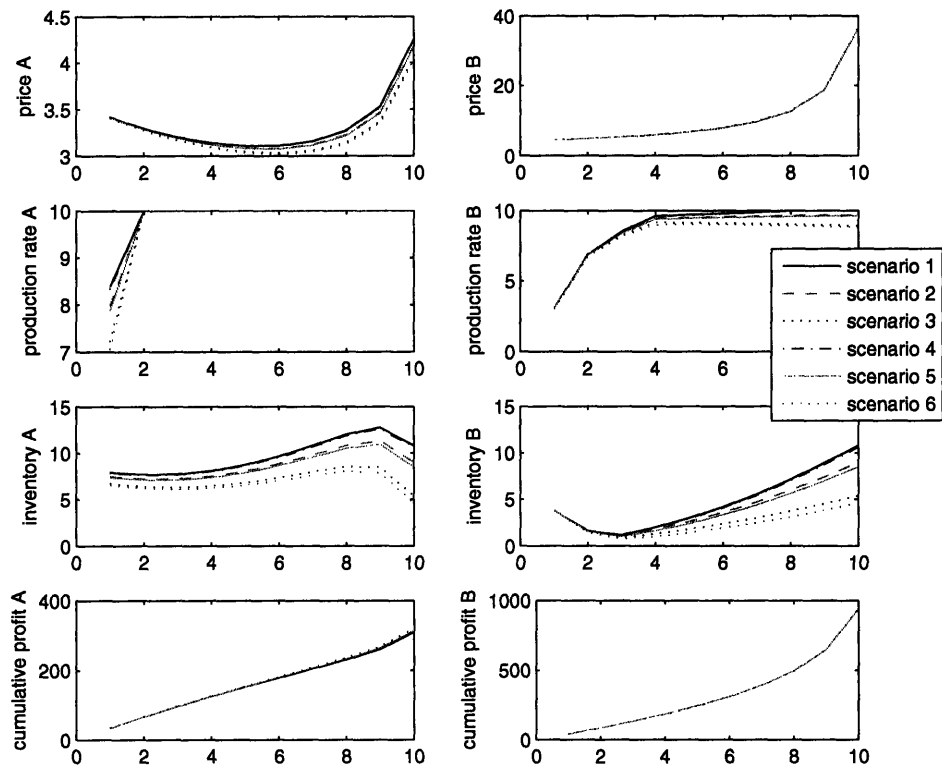
Figure 5-13: Results: Robust formulation. Equilibrium in the case of price sensitivities scenario h for various scenarios of budget of uncertainty

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minimum security level, and then they are kept at that security level, which explains why they increase with time in the remaining of the time horizon. This effect is stronger as the budget of uncertainty increases because the minimum inventory level increases with the budget of uncertainty. In order to produce this higher inventory, the production rates increase with time faster that they do in the deterministic case, and for both suppliers, they quickly use all the production capacity (which was not the case when data are deterministic). Once the capacity is tight, the only way to keep more inventory in stock is to decrease prices, and we observe that prices are lower in the robust formulation than in the deterministic one.

In scenario h, price sensitivities are decreasing with time, therefore in the deterministic case, both suppliers have prices that increase with time. Supplier A has sensitivities twice bigger than supplier B. Therefore, we observed that they are high enough for A to justify keeping most of her initial inventory until the end of the time horizon, when the sensitivities are low, and then only sharply increase prices and sell all the inventory. Supplier B however uses the strategy of selling the initial inventory and then keeping it at zero. In the robust formulation, supplier B's strategy is affected in the way explained before, i.e. guaranteeing the minimum inventory security levels forces her to have the inventory level that increases with time, thus to have lower prices and produce more than in the deterministic case.

Supplier A's strategy however was to keep a high inventory all along except at the very end when she would bring it to zero. The production capacity is too low to enable meeting the minimum inventory security level at the end of the time horizon if the inventory was kept at that level earlier. Therefore when data is uncertain, the minimum inventory security level affects her inventory level only at the last time period when instead of bringing it to zero she brings it to the final minimum level. Since less sales will be allowed at the end of the time horizon (to meet the minimum level), supplier A sells more before, therefore the inventory is kept at a level lower than in the deterministic case. To sell more, supplier A must produce more, and quickly reaches the production capacity, and as a result must significantly decrease

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	$\Gamma^k(t)$	$\int_0^T \min\{t, \Gamma^k(t)\}dt$	$\Pi^A + \Pi^B$ (sc. f)	$\Pi^A + \Pi^B$ (sc. h)
scenario 1	1+0.8t	47.5	1008.5	1253.5
scenario 2	1+0.5t	34	1010.3	1257.2
scenario 3	1+0.2t	19.38	1013.8	1264.9
scenario 4	0.5+0.8t	44.38	1008.8	1253.6
scenario 5	0.5+0.5t	29.75	1011.0	1258.4
scenario 6	0.5+0.2t	14.84	1014.8	1266.3

Table 5.7: Total objective value for various budgets of uncertainty

prices. Notice that supplier A realizes much lower profits than supplier B, as shown in Table 5.7.

We observe moreover that as there is more uncertainty on the data (i.e. as the cumulative effective budget of uncertainty increases), the inventory levels, production rates and prices overall increase, but the profits decrease. Figures 5-14 and 5-15 show that the sum of the suppliers' profits decrease as the cumulative effective budget of uncertainty increases, which illustrates the trade-off between performance (profit) and conservativeness (amount of uncertainty allowed), as well as confirms that the cumulative effective budget of uncertainty is a valid metrics to measure the global uncertainty.

In order to illustrate that the robust formulation is beneficial when data is uncertain, we want to show that the nominal solution may yield infeasibility when the realized value differs from the nominal one.

We focus on scenario f and h with the same inputs as in the previous numerical examples, and scenario 5 of budget of uncertainty.

We generate 1000 realizations of parameter $\tilde{\alpha}^k(.)$ according to both (i) a uniform distribution on $[\alpha^k(.) - \hat{\alpha}^k(.), \alpha^k(.) + \hat{\alpha}^k(.)]$ and (ii) a normal distribution with mean $\alpha^k(.)$ and standard deviation $0.5\hat{\alpha}^k(.)$. The realized values are generated in-
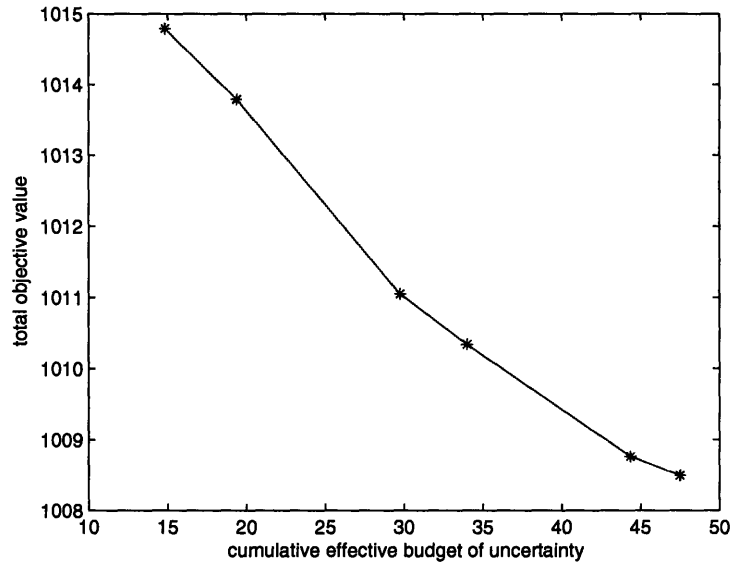
Figure 5-14: Results: Total objective value of the equilibrium under scenario f as a function of the cumulative effective budget of uncertainty
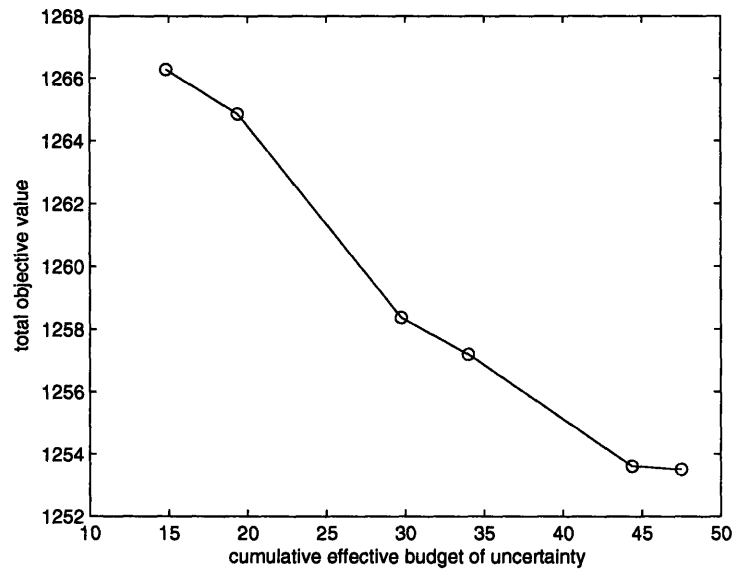

Figure 5-15: Results: Total objective value of the equilibrium under scenario h as a function of the cumulative effective budget of uncertainty

dependently over time and for the two suppliers. See Figure 5-16 for an example of realizations.
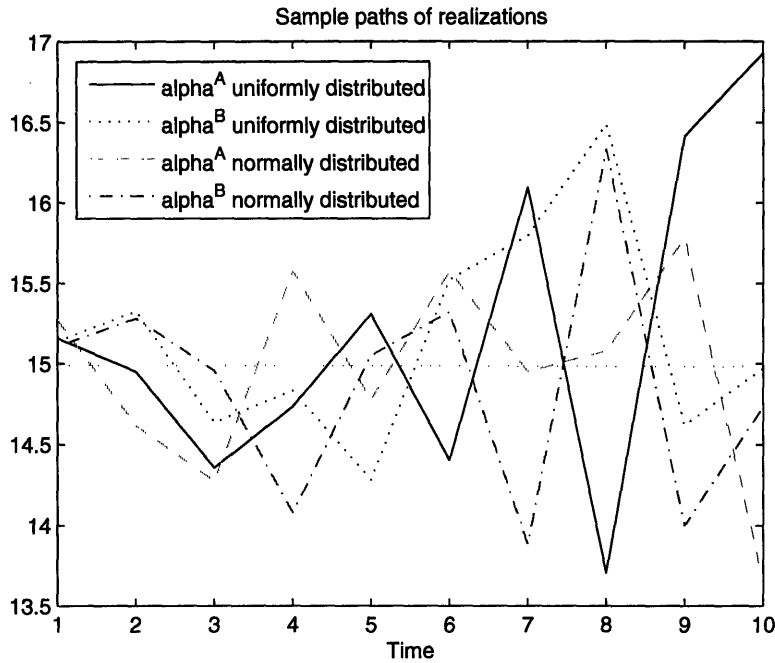

Figure 5-16: Example of realizations for alpha under uniform and normal distributions

We apply the nominal solution to these realizations and we record whether the no backorders constraint and upper bound on prices were violated. We calculate empirically the probability of violation of the constraints.

We obtain that, under either distribution, the upper bound on prices is never violated. However, the no backorders constraint is violated very frequently. See Table 5.8 for the numerical results. These high probabilities of a violation of the no backorders constraint were expected since, in the deterministic case, often the inventory level was on the boundary (zero level), so when the data are slightly perturbed, the actual inventory level may easily become negative. The question of interest is to determine by how much this constraint is violated.

In order to have an idea of the amplitude of violation of the no backorders con-

Distribution	Uniform	Normal
supplier A, sc. f	83.3%	82.4%
supplier B, sc. f	82.9%	81.4%
supplier A, sc. h	88.1%	91.2%
supplier B, sc. h	100%	100%

Table 5.8: Probability of violation of the no backorders constraint for the nominal solution

straint, we record the lowest value taken by the inventory level on the time horizon. The results are presented as histograms in Figures 5-17, 5-18, 5-19 and 5-20. We observe that the inventory levels may reach values significantly below the zero level. As a result, it seems relevant to use robust optimization to avoid a situation where backorders are likely to occurs at a significant level when there is uncertainty on the data.

As a comparison, we apply the robust solution (for each budget of uncertainty) to the generated simulations $\tilde{\alpha}(.)$ and calculate the probability of constraint violations. We obtain that the upper bound on the price is never violated, and the probability that the no backorder constraint is violated is given in Table 5.9. Note that the robust solution are designed so that there is no violation for any realization that satisfies the bounds and the budget of uncertainty constraints. Since the generated realizations may not satisfy these constraints, it is possible that the no backorders constraint or the upper bound on prices are violated by the robust solution. We expect to verify that the higher the budget of uncertainty, the least likely it is to have violation of the no backorders constraint, since the protection levels increase with the budget of uncertainty.

We also computed the average minimum inventory level attained over the time horizon for each robust solution and under both scenarios of price sensitivities. We obtain that the only case where the average is negative is supplier B, with scenario h
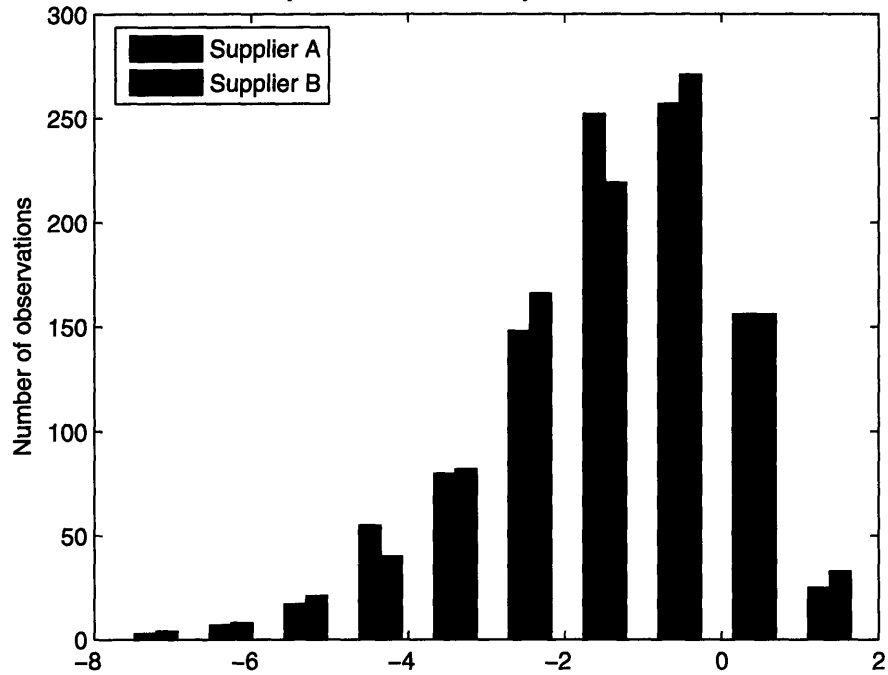
Figure 5-17: Histogram of minimum inventory level reached for uniformly distributed realization in scenario f

Budget of uncertainty	Distribution	supplier A scenario f	supplier B scenario f	supplier A scenario h	supplier B scenario h
scenario 1	Uniform	0%	0%	0%	0%
	Normal	0%	0%	0.1%	0%
scenario 2	Uniform	0%	0.8%	1.3%	0.2%
	Normal	0%	0.6%	0.5%	0%
scenario 3	Uniform	5.1%	11.8%	20.8%	28.2%
	Normal	1.8%	5.0%	15.2%	21.0%
scenario 4	Uniform	0%	0%	0%	0%
	Normal	0%	0%	0.2%	0%
scenario 5	Uniform	0%	1.5%	2.1%	0.3%
	Normal	0%	1.0%	0.6%	0%
scenario 6	Uniform	10.6%	21.6%	28.1%	48.9%
	Normal	5.5%	13.3%	25.4%	44.7%

Table 5.9: Probability of violation of the no backorders constraint for the robust solution
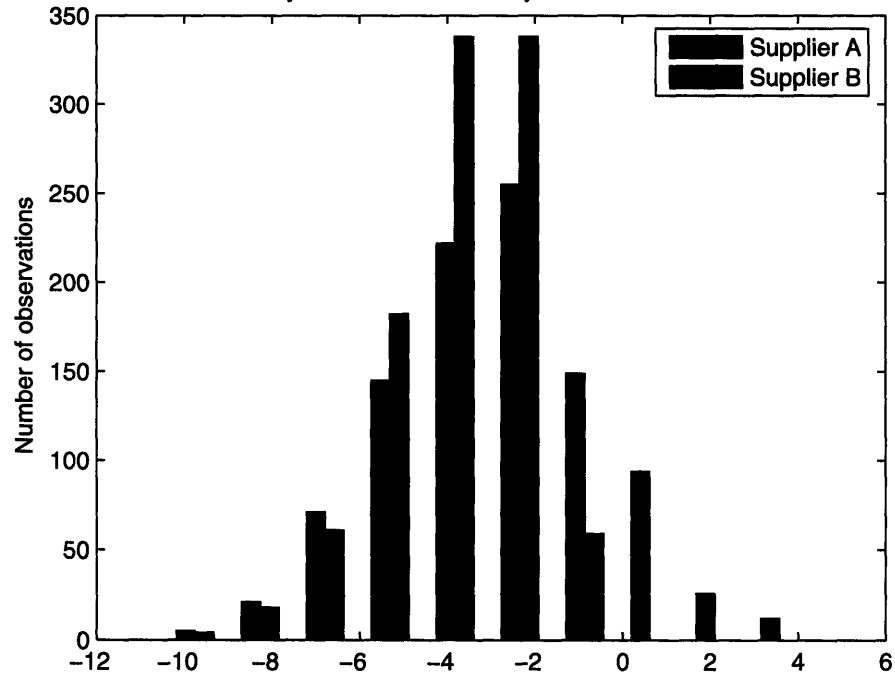
Figure 5-18: Histogram of minimum inventory level reached for uniformly distributed realization in scenario h
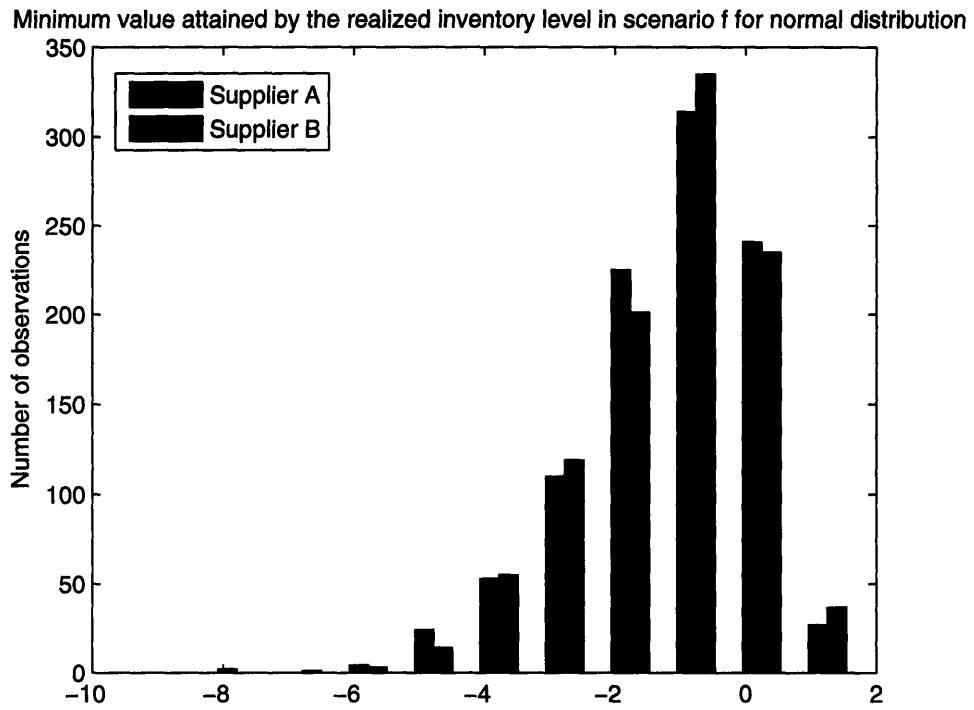
Figure 5-19: Histogram of minimum inventory level reached for normally distributed realization in scenario f
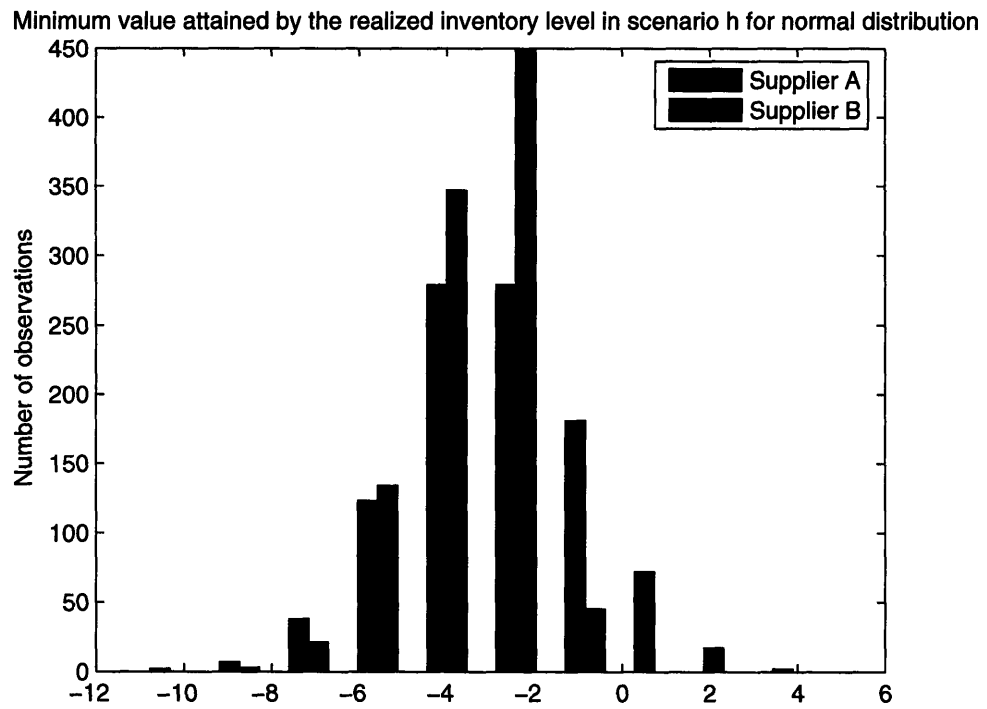
Figure 5-20: Histogram of minimum inventory level reached for normally distributed realization in scenario h

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of price sensitivities and scenario 6 of budget of uncertainty (which has the smallest cumulative effective value), when the realization is normally distributed. In that case, the average is -0.19, which shows that the violation tends not to have a very large amplitude. In all other cases, the average was positive.

We observe that the probability of violation of the no backorders constraint for the robust solutions is small in most cases, while it was very large for the nominal solution. Moreover, these probabilities are a decreasing function of the cumulative effective budget of uncertainty: the least overall uncertainty is allowed in the model, the least protected the system is against constraint violations, the higher the probabilities are.

Chapter 6

Conclusions

In this thesis, we studied a continuous time optimal control model for a dynamic pricing and inventory management problem with no backorders. In particular, we studied a demand based model in a make-to-stock system and a multi-product capacitated dynamic setting. We considered a particular cost structure, allowing time flexibility in the demand, production capacity, and cost parameters. A particular feature of the model we considered is that it does not allow backorders.

We introduced a continuous time solution approach utilizing the KKT conditions and an extension of Pontryagin's Maximum Principle for state-constrained problems. Through numerical examples, we illustrate the role of capacity and of the dynamic nature of demand in the model.

We have then proposed and studied a robust optimization approach for incorporating demand uncertainty into the fluid model. Using ideas from robust optimization, we reformulated the problem as a deterministic problem (*robust counterpart problem*) of a similar form as the original nominal formulation. Our approach does not make assumptions on the probabilistic distribution of the demand but rather assumes a general demand model whose uncertain parameters lies in a given interval, and a budget of uncertainty that allows to control the level of conservativeness sought. Furthermore, we were able to adapt to the robust formulation the algorithm for solving

the deterministic problem and show that the adapted algorithm is no more complex than the original one. We implemented this algorithm on a numerical example that illustrates the trade-off between robustness and performance.

Finally, we introduced competition by considering the problem with demand uncertainty in a duopoly setting as a non cooperative differential game. We first use ideas from robust optimization to reformulate the problem as a deterministic one, and we show the existence of a Nash equilibrium for the robust formulation in continuous time. We then study issues of uniqueness of a Nash equilibrium solution, and of a particular equilibrium: the normalized Nash equilibrium. Furthermore, we present a relaxation algorithm for solving the problem in discrete time and prove its convergence to the normalized Nash equilibrium. Finally we perform some computations and discuss some insights.

Throughout the thesis, our goal was to study a model that would be as realistic as possible, but remain tractable. In particular, practitioners often face issues such as multi-product systems, production capacity, price dependent demand, uncertainty on data and presence of competition. Our approach aimed at incorporating these constraints, without making assumptions that in practice are difficult to satisfy, such as probability distributions of uncertain data. The numerical experiments show that the methods that are presented allow, under reasonable assumptions, to compute the solution and gain insight.

We also believe that the interest of this thesis lies not only in the implications in a dynamic pricing and inventory control setting, but also in the optimization techniques presented. Indeed, these techniques may be adapted to other areas of application.

Future directions of research include studying further the role of the production capacity limit and the no backorders constraint, how they may affect the overall profits and policies.

In the thesis, suppliers have open-loop strategies: they commit to pricing and production decisions at the beginning of the time horizon. That is: they must make decisions at time zero for the entire time horizon. In closed-loop strategies, the suppliers may adapt the future strategy based on the current state of the system and on modified data forecasts. One could adapt an open-loop approach by applying it on a rolling horizon. An interesting direction of research would be to evaluate how good of an approximation this would be.

To address uncertainty without considering an open-loop setting, one could focus on *adjustable* robust formulation, where the decision maker may observe realization of past uncertain data before making decisions concerning the future. It would be interesting to compare this approach with results obtained with a Dynamic Programming model, in terms of average profits realized, tractability, and necessary assumptions. Similarly, the non adjustable robust formulation could be compared with the stochastic optimization approach.

In the competition setting, an interesting area of research would be to quantify the loss of efficiency incurred by the presence of competition, also called " price of anarchy" in the literature. This can be done by comparing the overall profits at an equilibrium with the overall profits if one could impose the suppliers a strategy in order to maximize the overall profits. Then it could be possible to find incentives for the suppliers to decrease the loss of efficiency.

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