

ANALYSIS OF DUALITY CONSTRUCTIONS FOR
VARIABLE DIMENSION FIXED POINT ALGORITHMS

by

NEDA F. EMAMI

A.B., Mathematics
University of California, Berkeley
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Submitted in Partial Fulfillment of
the Requirements of the Degree of
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September 1987

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Signature of Author _____
Interdepartmental Program in Operations Research
September 1987

Certified by _____
Robert M. Freund
Associate Professor of Operations Research and Management
Thesis Supervisor

Accepted by _____
Thomas L. Magnanti
Chairman, Interdepartmental Committee on Operations Research

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Abstract

Variable dimension algorithms are a class of algorithms for computation of fixed points. They normally start at a single point and generate a path of simplices of varying dimension until a simplex that contains an approximation of a fixed point is found.

This thesis analyzes, compares, and contrasts five duality models for variable dimension fixed point algorithms, namely, primal-dual subdivided manifolds, primal-dual pseudomanifolds, V-complexes and H-complexes, the framework \bar{K} , and antiprisms. Each framework is defined, examples are given, and the relation between them is discussed.

Thesis Supervisor: Dr. Robert M. Freund

Title: Associate Professor of Operations Research and Management

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Introduction

In 1910 the Dutch mathematician L.E.J. Brouwer proved that every continuous map of a closed unit ball to itself has a fixed point. This started the systematic development of fixed point theory. Today, the study of fixed points of maps is important not just in topology, but in other areas of mathematics. For instance, fixed point theorems are commonly used in the proofs of existence theorems in optimization and complementarity, as well as in the theory of differential equations.

It is only recently, however, that constructive procedures for computation of fixed points have been developed. Variable dimension algorithms are a class of such algorithms. They normally start at a single point and generate a path of simplices of varying dimension until a simplex that contains an approximation of a fixed point is found.

In this thesis we will analyze, compare, and contrast four duality models for variable dimension fixed point algorithms. In Chapter I we will define and give example of a primal-dual subdivided manifold. In Chapter II we will discuss the construction of a primal-dual pseudomanifold. In Chapter III we compare these two frameworks with each other. In Chapter IV we look at V-complexes and H-complexes. In Chapter V we discuss the relation between a V-complex and an H-complex with a primal-dual subdivided manifold and a primal-dual pseudomanifold. In Chapter VI we consider the framework \bar{K} , and show its relation with a primal-dual pseudomanifold. In Chapter VII we define the notion of an antiprism and show that it may be considered as a geometric version of a special case of a primal-dual pseudomanifold. In Chapter VIII we give a summary of what we covered in this thesis.

Chapter I

Primal-Dual Submanifolds

This chapter discusses the concept of a primal-dual pair of subdivided manifolds. This structure was developed by Kojima and Yamamoto [9] for studying variable dimension fixed point algorithms. As is shown in [9], several algorithms in complementarity theory, such as Lemke's algorithm [10] for linear complementarity problem, can be interpreted in terms of this model.

We will start by reviewing some definitions that provide a basis for the material discussed in this chapter. Next, we will study the structure of a primal-dual submanifold, and finally remark on why this model gives rise to variable dimension fixed point algorithms.

1.1. Definitions

A *cell* or a *polyhedral set* in some Euclidean space \mathbb{R}^k is defined to be the intersection of a finite number of closed half spaces. Therefore, a cell is necessarily a convex set. The *dimension of a cell* C , denoted $\dim C$, is the dimension of the affine subspace spanned by C . A cell of dimension m is often referred to as an m -cell.

Given a finite or a countable collection of m -cells M_o in \mathbb{R}^k , Let $M = |M_o|$ be the union of all the m -cells in M_o . Also, define $\bar{M} = \{B: B \text{ is a face of } C, C \in M\}$. The following definition of a subdivided-manifold is due to Eaves [3]:

M is called a *subdivided-manifold* if:

- (i) If A and B belong to M , then $A \cap B = \emptyset$ or $A \cap B$ is a common face of both A and B .
- (ii) Every $(m-1)$ -cell of \bar{M} lies in at most two cells of M .
- (iii) M is locally finite, i.e., each point in M_0 has a neighborhood that intersects only a finite number of cells in M .

If M is a subdivided-manifold, then we call it a *subdivision for M_0* .

If M is a subdivided-manifold, then the *boundary of M* , denoted ∂M , is defined to be the collection of all $(m-1)$ -cells of \bar{M} which lie in exactly one m -cell of M .
 $\partial M_0 = |\partial M|$ is the union of all $(m-1)$ -cells in ∂M .

1.2 Basic Framework: Primal-Dual Subdivided Manifolds

In this section we will examine the structure of a primal-dual pair of subdivided manifolds.

Let P and D be a pair of subdivided-manifolds. For any positive integer m , $(P, D; d)$ is called a primal-dual pair of subdivided manifolds with degree m and duality operator $d: \bar{P} \cup \bar{D} \rightarrow \bar{P} \cup \bar{D}$, denoted by PDM.

1. $X \in \bar{P}$, implies that $X^d = \emptyset$ or $X^d \in \bar{D}$
- 1'. $Y \in \bar{D}$, implies that $Y^d = \emptyset$ or $Y^d \in \bar{P}$
2. If $Z \in \bar{P} \cup \bar{D}$ and $Z^d \neq \emptyset$, then $(Z^d)^d = Z$
3. If X_1 and X_2 belong to \bar{P} , X_1 is a face of X_2 , $X_1^d \neq \emptyset$ and $X_2^d \neq \emptyset$, then X_2^d is a face of X_1^d .

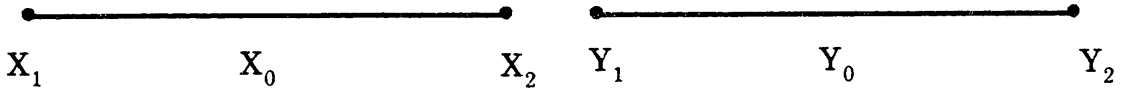
3. If Y_1 and Y_2 belong to \bar{D} , Y_1 is a face of Y_2 , $Y_1^d \neq \emptyset$ and $Y_2^d \neq \emptyset$, then Y_2^d is a face of Y_1^d .

4. If $Z \in \bar{P} \cup \bar{D}$ and $Z^d \neq \emptyset$, then $\dim Z + \dim Z^d = m$.

If $(P, D; d)$ is a PDM, then P and D are called the primal and dual subdivided manifolds respectively.

The following are examples of primal-dual submanifolds:

1.



$$P = \{X_0\}$$

$$\bar{P} = \{X_0, X_1, X_2\}$$

$$D = \{Y_0\}$$

$$\bar{D} = \{Y_0, Y_1, Y_2\}$$

$$X_1^d = Y_0$$

$$X_0^d = X_2^d = \emptyset$$

$$Y_0^d = X_1$$

$$Y_1^d = Y_2^d = \emptyset$$

$(P, D; d)$ is a PDM of degree 1.

1' . P, \bar{P}, D, \bar{D} are the same as in example 1.

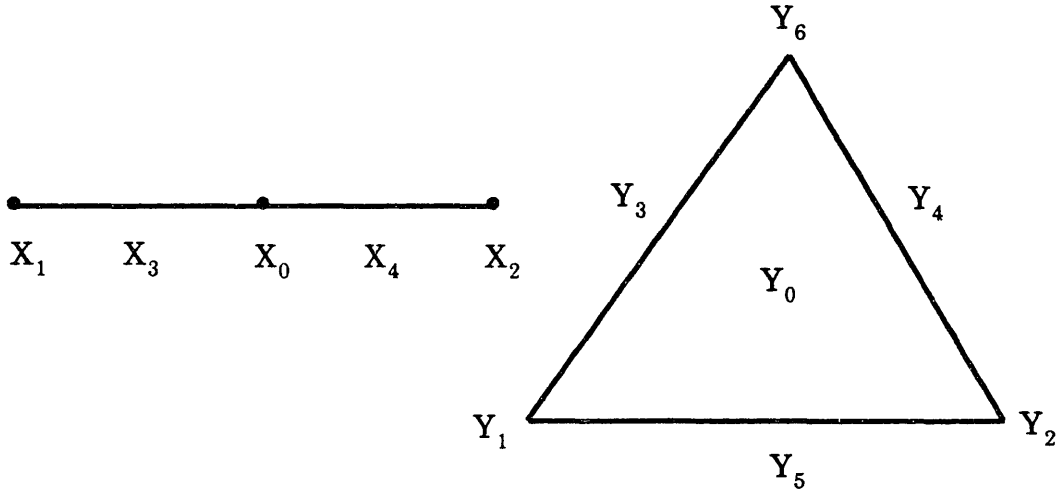
$$X_1^d = Y_0 \qquad Y_0^d = X_1$$

$$X_0^d = Y_1 \qquad Y_1^d = X_0$$

$$X_2^d = \emptyset \qquad Y_2^d = \emptyset$$

$(P, D; d)$ is a PDM of degree 1.

2.



$$P = \{X_3, X_4\}$$

$$\bar{P} = \{X_0, X_1, X_2, X_3, X_4\}$$

$$D = \{Y_0\}$$

$$\bar{D} = \{Y_0, Y_1, Y_2, Y_3, Y_4, Y_5, Y_6\}$$

$$X_0^d = Y_0$$

$$X_3^d = Y_3$$

$$X_4^d = Y_4$$

$$X_1^d = X_2^d = \emptyset$$

$$Y_0^d = X_0$$

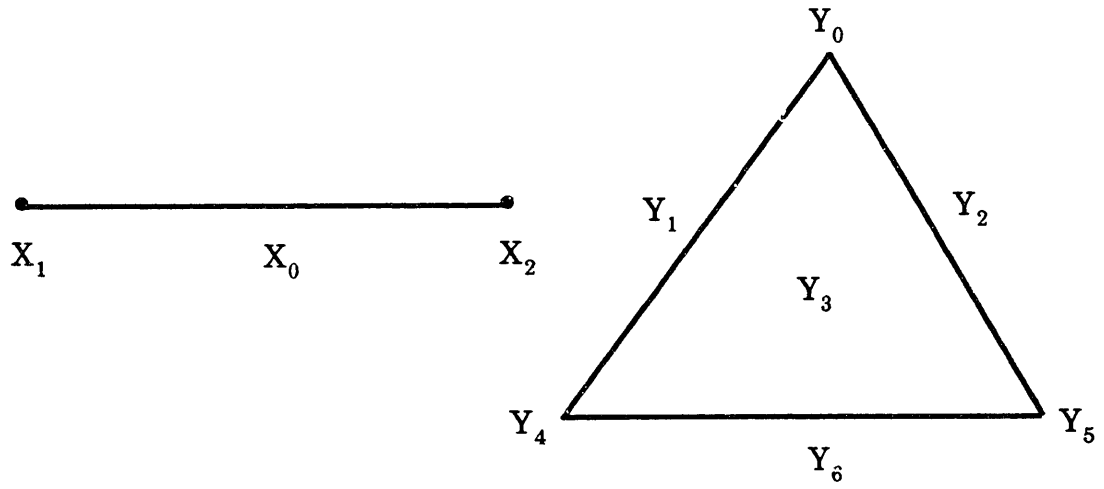
$$Y_3^d = X_3$$

$$Y_4^d = X_4$$

$$Y_1^d = Y_2^d = Y_5^d = Y_6^d = \emptyset$$

$(P, D; d)$ is a PDM of degree 2.

3.



$$P = \{X_0\}$$

$$\bar{P} = \{X_0, X_1, X_2\}$$

$$D = \{Y_3\}$$

$$\bar{D} = \{Y_i : i=0, \dots, 6\}$$

$$X_0^d = Y_0$$

$$X_1^d = Y_1$$

$$X_2^d = Y_2$$

$$Y_0^d = X_0$$

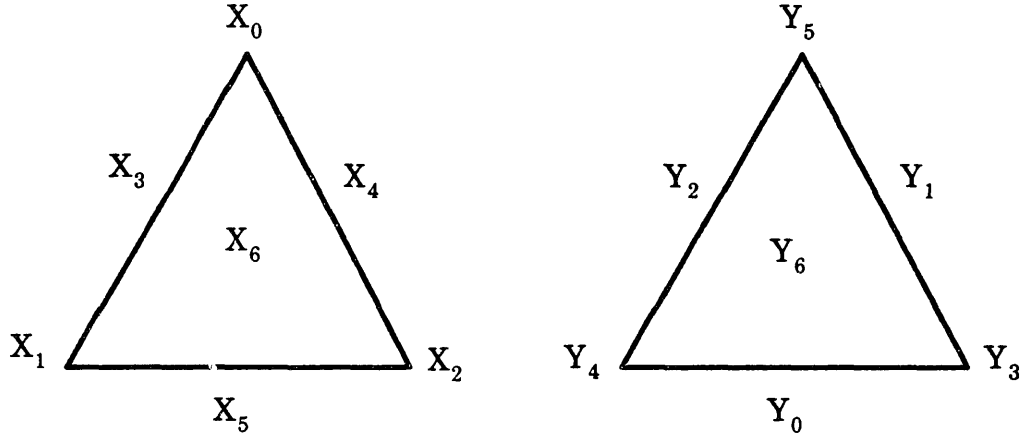
$$Y_1^d = X_1$$

$$Y_2^d = X_2$$

$$Y_3^d = Y_4^d = Y_5^d = Y_6^d = \emptyset$$

$(P, D; d)$ is a PDM of degree 1.

4.



$$P = \{X_6\}$$

$$\bar{P} = \{X_i; i=0, \dots, 6\}$$

$$X_i^d = Y_i \quad i=0, \dots, 5$$

$$X_6^d = \emptyset$$

$$D = \{Y_6\}$$

$$\bar{D} = \{Y_i; i=0, \dots, 6\}$$

$$Y_i^d = X_i \quad i=0, \dots, 5$$

$$Y_6^d = \emptyset$$

$(P, D; d)$ is a PDM of degree 1.

Consider a PDM, $(P, D; d)$, of degree m and let

$$L = \langle P, D; d \rangle = \{Y^d \times Y; Y \in \bar{D} \text{ and } Y^d \neq \emptyset\}.$$

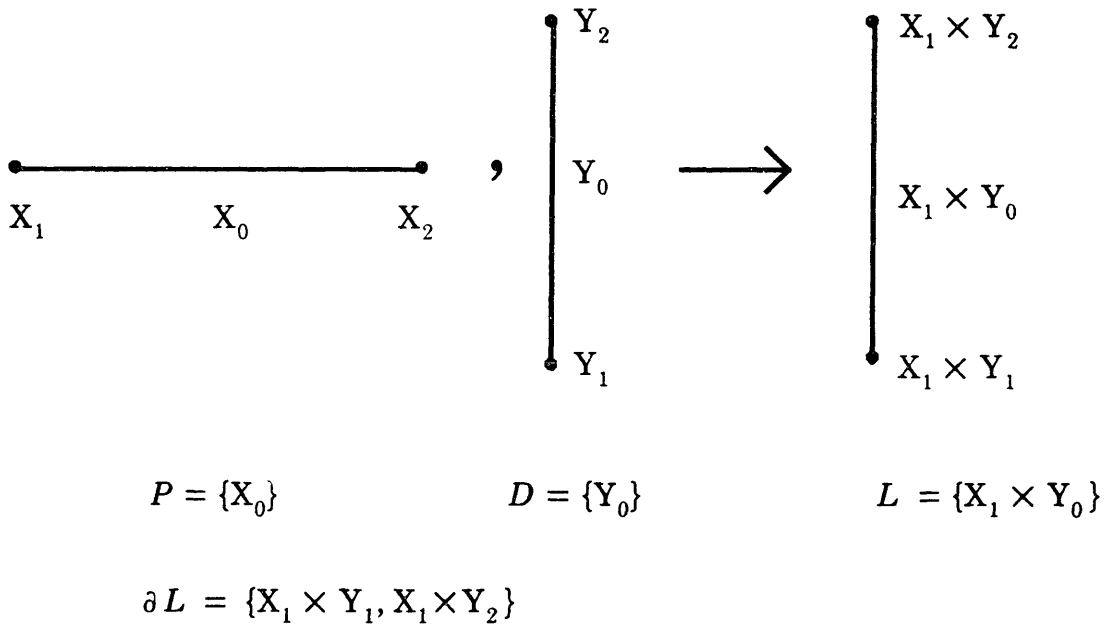
Condition (2) from the definition of a PDM states that if $Y \in \bar{D}$, the product $Y^d \times Y$ can also be written as $X \times X^d$ where $X \in \bar{P}$ and $X^d \in \bar{D}$. Hence, L may equivalently be defined as, $L = \{X \times X^d; X \in \bar{P} \text{ and } X^d \neq \emptyset\}$.

From condition (4) we know that if $z \in \bar{P} \cup \bar{D}$ and $z^d \neq \emptyset$, then $\dim z + \dim z^d = m$. Furthermore, if $B = X \times Y$ is an $(m-1)$ cell of \bar{L} , then any m -cell of L that has B as its face must be of the form $X \times X^d$ or $Y^d \times Y$. Therefore, it is clear that L is a subdivided manifold of dimension m .

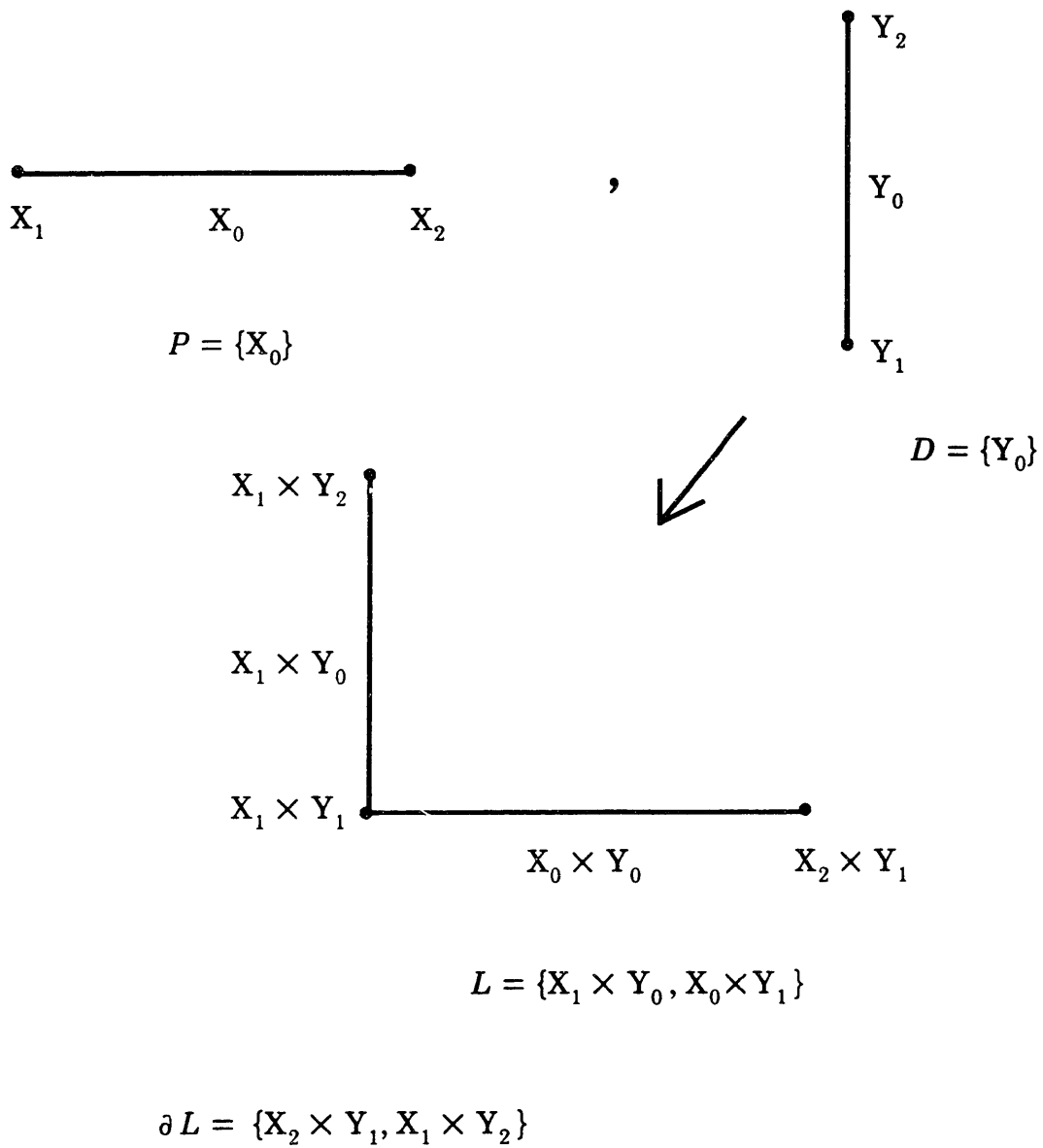
Recall that ∂L is defined to be the collection of all $(m-1)$ -cells of \bar{L} that lie in exactly one m -cell of L . So, from the above observations it follows that the $(m-1)$ -cell $B = X \times Y$, is in the boundary of L if and only if either $X^d = \emptyset$ or $Y^d = \emptyset$.

We now illustrate L and ∂L in each of the examples of the primal-dual pair of submanifolds given above.

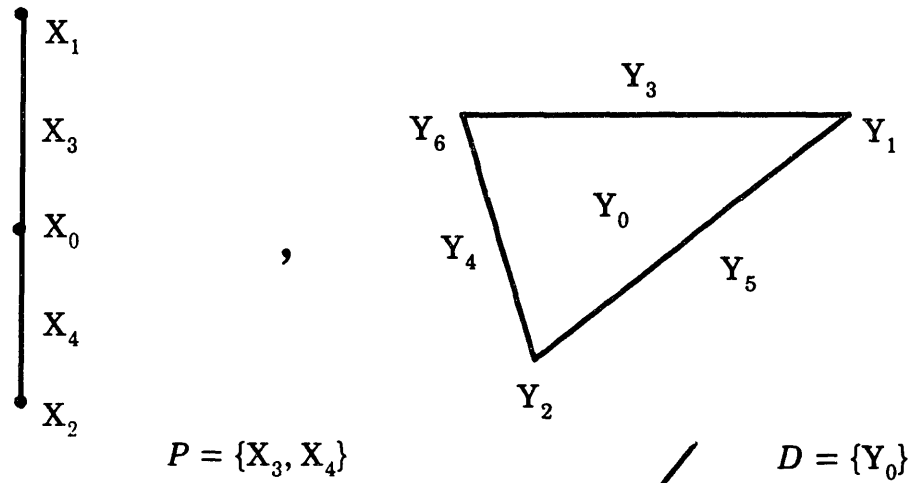
1.



1.

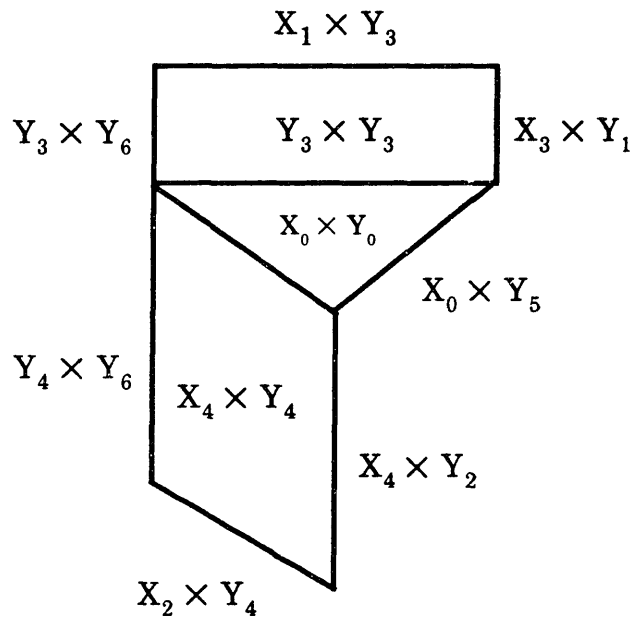


2.



$$P = \{X_3, X_4\}$$

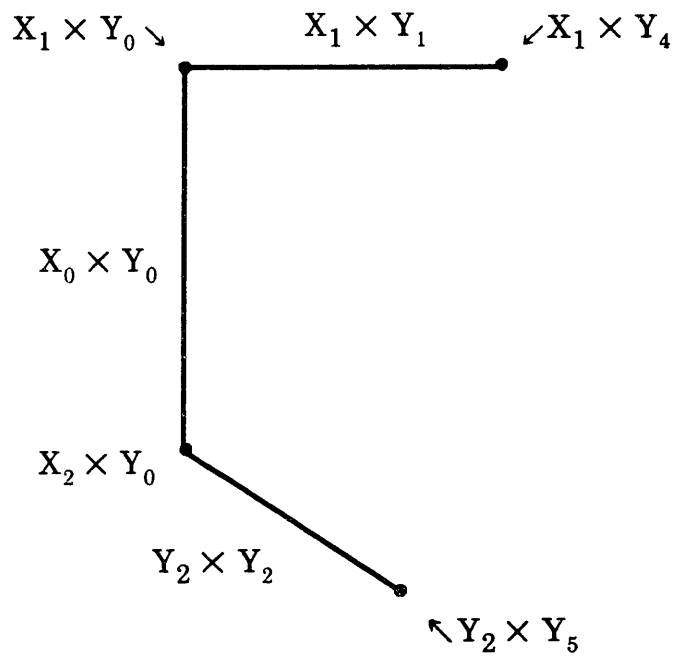
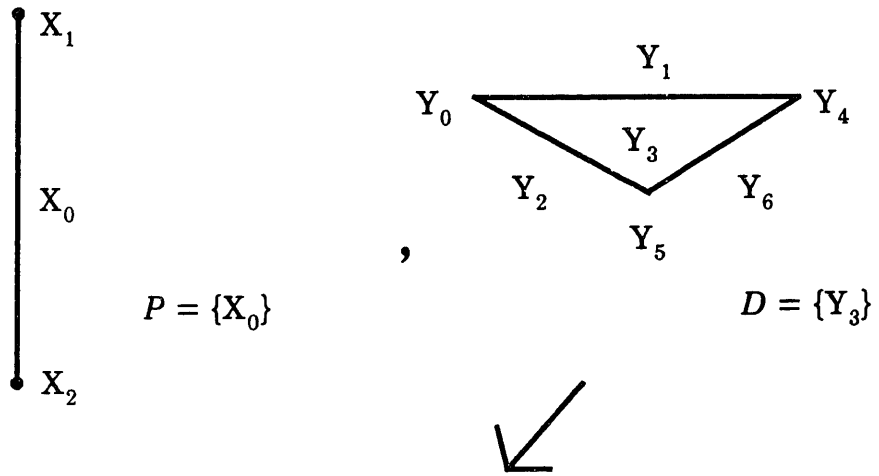
$$D = \{Y_0\}$$



$$L = \{X_0 \times Y_0, Y_3 \times Y_3, X_4 \times Y_4\}$$

$$\partial L = \{X_1 \times Y_3, X_3 \times Y_6, X_3 \times Y_1, X_4 \times Y_6, X_4 \times Y_2, X_2 \times Y_4, X_0 \times Y_5\}$$

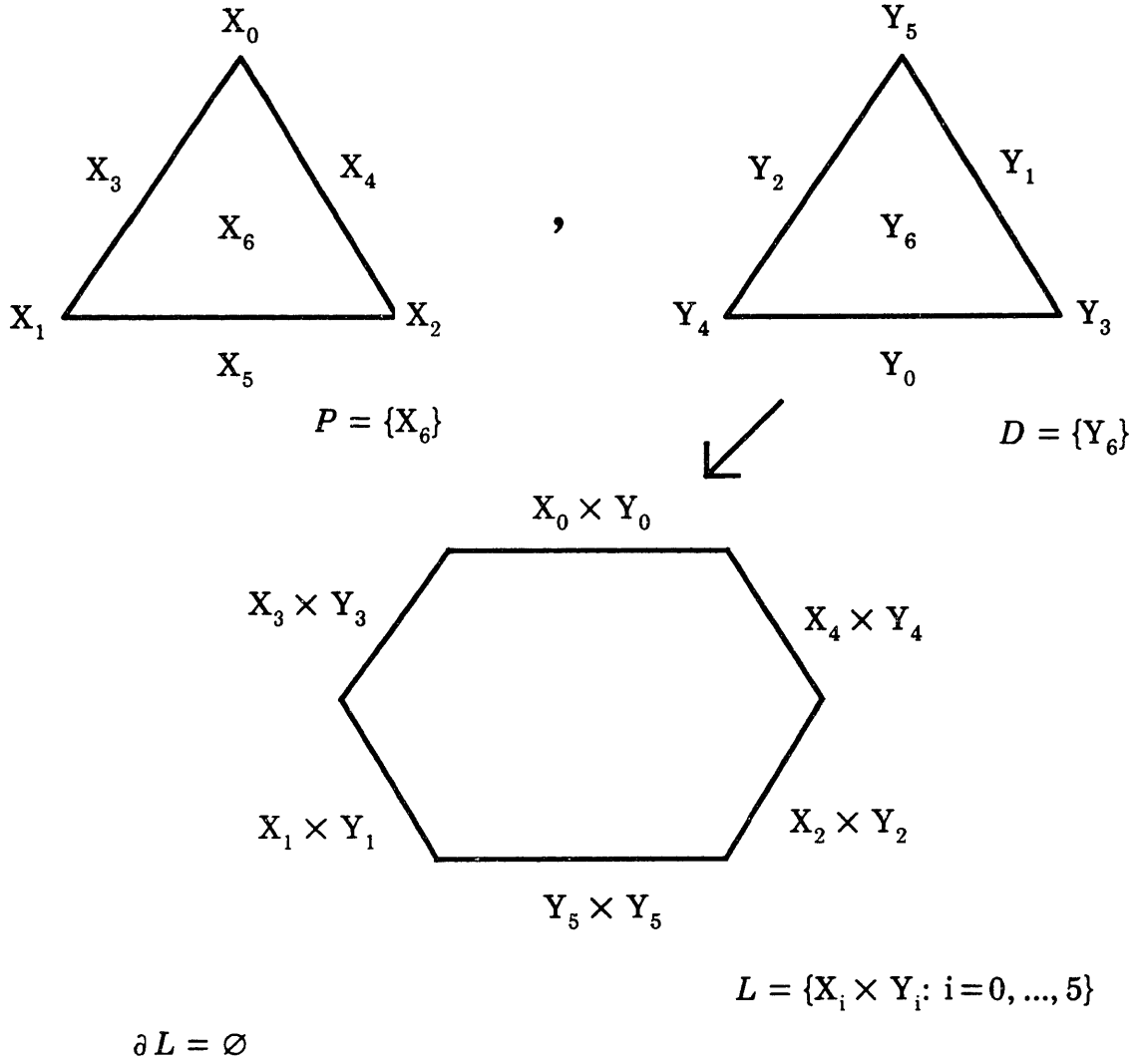
3.



$$L = \{X_0 \times Y_0, X_1 \times Y_1, X_2 \times Y_2\}$$

$$\partial L = \{X_1 \times Y_4, X_2 \times Y_5\}$$

4.



We will end this section by showing that for any PDM $(P, D; d)$, we can construct a refinement of L from any refinement of P . But first we define the refinement of a subdivided manifold.

Definition 1.1 Let M and M' be subdivisions of an m -dimensional manifold in \mathbb{R}^k . M' is said to be a *refinement* of M if each m -cell in M' lies in some cell in M .

Given a refinement P^* of P , note that for any k -dimensional cell $X \in \bar{P}$, the collection $\bar{P}^*|X = \{\sigma \in \bar{P}^*: \sigma \in X \text{ and } \dim \sigma = k\}$ subdivides X . Therefore if we let $K = \{\sigma \times Y: Y \in \bar{D}, Y^d \neq \emptyset, \sigma \in \bar{P}^*|Y^d\}$, K will clearly be a refinement of L .

1.3 PDM: A Model for Variable Dimension Algorithms

In [9], Kojima and Yamamoto show that a PDM provides a model for variable dimension algorithms. We therefore, conclude our discussion of PDM's by explaining this variable nature of PDM's.

Given a PDM $(P, D; d)$ of degree m , Let C^1 and C^2 be two cells in L with the $(m-1)$ cell, $B = X \times Y$ ($X \in \bar{P}, Y \in \bar{D}$), as their common face. Hence, $C^1 = X \times X^d$ and $C^2 = Y^d \times Y$ where X and Y^d are in \bar{P} and X^d and Y are in \bar{D} .

From the definition of a PDM we note the following facts:

$$C^1 \text{ is an } m\text{-cell, therefore } \dim X + \dim X^d = m \quad (1)$$

$$B \text{ is an } (m-1)\text{-cell, therefore } \dim X + \dim Y = m-1 \quad (2)$$

$$C^2 \text{ is an } m\text{-cell, therefore } \dim Y^d + \dim Y = m \quad (3)$$

Conditions (1) and (2) imply that $\dim Y = \dim X^d - 1$. Similarly, conditions (2) and (3) imply that $\dim Y^d = \dim X + 1$. Each time we move from C^1 to C^2 , we go

from X to Y^d in \bar{P} and from X^d to Y in \bar{D} . So, the dimension of the cells in \bar{P} increase by one, while the dimension of the cells in \bar{D} decrease by one.

Notice, however, that if K is a refinement of L resulting from P^* , a refinement of P , then every m -cell of K in C^1 is of the form $\sigma \times X^d$ where $\sigma \in \bar{P}^*|X$, and every m -cell in C^2 has the form $\tau \times Y$ where $\tau \in \bar{P}^*|Y^d$. Hence as we cross m -cells in C^1 , the dimension of the σ 's remain the same as $\dim X$. But as we move into a new m -cell in C^2 , the dimension of τ 's will equal $\dim Y^d$. Since $\dim Y^d = \dim X + 1$, every time we go from an m -cell of L into a new m -cell in L , the dimension of the cells we cross in \bar{P}^* varies by plus one. This explains the variable nature of PDM's.

Chapter II

Primal-Dual Pseudomanifolds

In this chapter we will examine the structure of a primal-dual pseudomanifold. This model was introduced by Yamamoto [11], and consists of a pair of pseudomanifolds and an operator that relates them. In contrast to the primal-dual subdivided manifold of Kojima and Yamamoto [9], this structure is combinatorial and not geometric, and therefore, suitable for studying combinatorial theorems in topology. In fact, Freund [7] and Yamamoto [11] generalize Sperner lemma on a general convex polytope by using primal-dual pseudomanifolds.

2.1 Definitions

This section is concerned with defining concepts that are central in the development of the material covered in this chapter.

Let $S = \{v_1, \dots, v_m\}$ be a set of m affinely independent points in \mathbb{R}^n . The convex hull of S is called an *m -dimensional simplex*, or more simply an *m -simplex*. If R is any subset of S consisting of k ($0 \leq k \leq m$) points, then the convex hull of R is said to be a *k -dimensional or a k -face of S* .

Given an m -dimensional convex set C in \mathbb{R}^n , set K of m -simplices σ together with all their faces is said to be a triangulation of C if:

$$(i) \quad C = \bigcup_{\sigma \in K} \sigma,$$

$$(ii) \quad \sigma, \tau \in K \text{ imply } \sigma \cap \tau \text{ is a face of both } \sigma \text{ and } \tau$$

- (iii) If σ is an $(m-1)$ -simplex of K , σ is a face of at most two m -simplices of K .

An m -dimensional abstract simplex σ consists of a set of $(m+1)$ points, i.e., $\sigma = \{v_1, \dots, v_{m+1}\}$. Any subset of σ is called a *face* of σ . A set L of abstract m -simplices together with all their faces is called an m -pseudomanifold if and only if each $(m-1)$ -simplex of L is contained in at most two m -simplices in L . An m -pseudomanifold L is *finite* if it consists of a finite number of m -simplices; it is *locally finite* if each vertex in L is contained in a finite number of m -simplices of L . The *boundary* of L denoted ∂L , is the set of all simplices in L that are contained in an $(m-1)$ -simplex τ in L , and τ is a face of exactly one m -simplex in L .

Given an m -dimensional convex set C in \mathbb{R}^n together with a triangulation K of C , the set $K' = \bigcup_{\sigma \in K} \{v: v \text{ is the vertex of } \sigma\}$ is an m -pseudomanifold.

The primal-dual structure that we will discuss in the next section is based on the pseudomanifold corresponding to a triangulation of a convex polyhedral set.

2.2 Basic Model

In this section we will define a primal-dual pseudomanifold as in Yamamoto [11]. We will give examples of primal-dual pseudomanifolds and characterize their boundaries.

As the concept of a partition of a pseudomanifold is used in the construction of a primal-dual pseudomanifold, we will start by giving a precise definition of such a partition.

Definition 2.1. A partition of an m -pseudomanifold K , is a decomposition of K into m -pseudomanifolds such that every m -simplex of K is in exactly one of the m -pseudomanifolds.

Let K and L be pseudomanifolds with dimensions p and d respectively. Let \hat{P}_p be a finite partition of K into p -pseudomanifolds. For each $k \in \{0, \dots, p-1\}$, let \hat{P}_k be a finite partition of $\bigcup_{P \in \hat{P}_{k+1}} \partial P$ into k -dimensional pseudomanifolds such

that if P_1 and P_2 are in \hat{P}_{k+1} , then $\partial P_1 \cap \partial P_2$ is also partitioned by \hat{P}_k . Similarly, we define $\hat{D}_0, \dots, \hat{D}_d$ for L . Let

$$\hat{P} = \bigcup_{k=0}^p \hat{P}_k$$

and

$$\hat{D} = \bigcup_{\ell=0}^d \hat{D}_\ell.$$

Given a set $C \subseteq \hat{P} \cup \hat{D} \setminus \{\emptyset\}$ and a positive integer n , $(\hat{P}, \hat{D}, C, *, n)$ is called an n -*primal-dual pseudomanifold* (abbreviated by n -pdpm) with operator $(*)$ if:

1. $P \in \hat{P}_k \cap C$ ($D \in \hat{D}_k \cap C$) where $0 \leq k \leq n$, implies that $P^* \in \hat{D}_{n-k-1}$
($D^* \in \hat{P}_{n-k-1}$)

So, each k -pseudomanifold in C corresponds to an $(n-k-1)$ -pseudomanifold.

2. If $P \in C$ and $P^* \neq \emptyset$, then $(P^*)^* = P$.
3. If $P, Q \in C$ and $P \subset \partial Q$, then $Q^* \subset \partial P^*$.

This condition states that (*) is inclusion reversing.

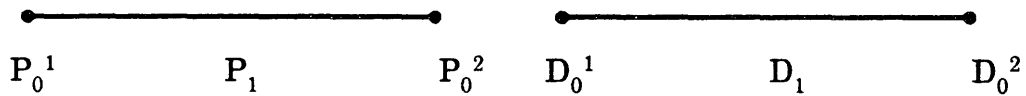
4. $P \in \hat{P}_{n-1} \cap C (D \in \hat{D}_{n-1} \cap C)$ implies that there is at most one $Q \in \hat{P}_n \cap C$ ($R \in \hat{D}_n \cap C$) such that $P \subset \partial Q (D \subset \partial R)$.

This condition implies that any (n-1)-dimensional partition in C is contained in the boundary of at most one n-dimensional partition in C.

Note that for fixed \hat{P} and \hat{D} we can obtain different pseudomanifolds by varying C.

Examples:

1.



$$\hat{P} = \{P_0^1, P_0^2, P_1\}$$

$$\hat{D} = \{D_0^1, D_0^2, D_1\}$$

$$C = \{P_0^1, D_1\}$$

$$(P_0^1)^* = D_1$$

$$(D_1)^* = P_0^1$$

$(\hat{P}, \hat{D}, C, *)$ is a 2-pdpm.

1.

\hat{P} and \hat{D} are the same as in example 1.

$$C = \{P_0^1, P_1, D_0^1, D_1\}$$

$$(P_0^1)^* = D_1$$

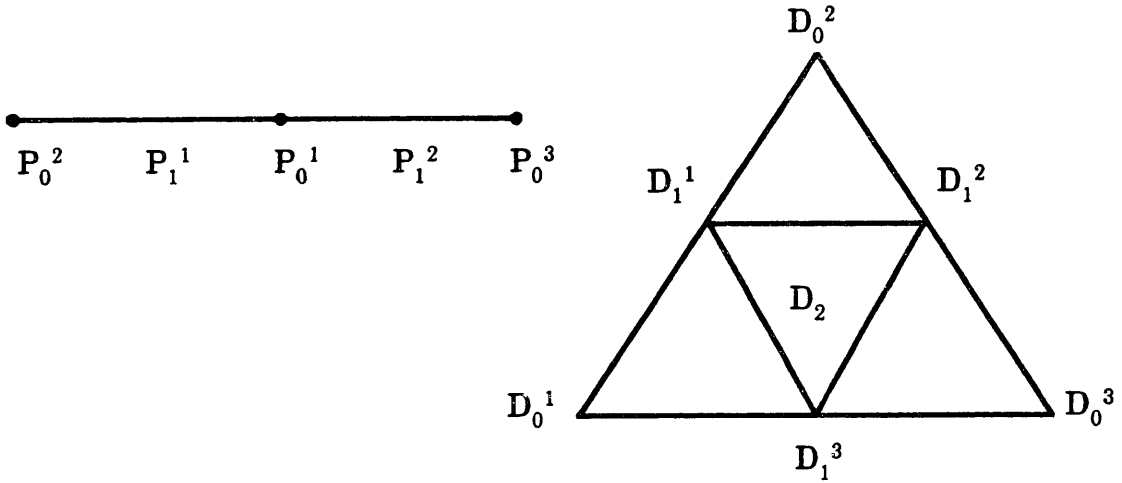
$$(P_1)^* = D_0^1$$

$$(D_1)^* = P_0^1$$

$$(D_0^1)^* = P_1$$

$(\hat{P}, \hat{D}, C, *)$ is a 2-pdpm.

2.



$$\hat{P} = \{P_0^1, P_0^2, P_0^3, P_1^1, P_1^2\}$$

$$\hat{D} = \{D_0^1, D_0^2, D_0^3, D_1^1, D_1^2, D_1^3, D_2\}$$

$$C = \{P_0^1, P_1^1, P_1^2, D_1^1, D_1^2, D_2\}$$

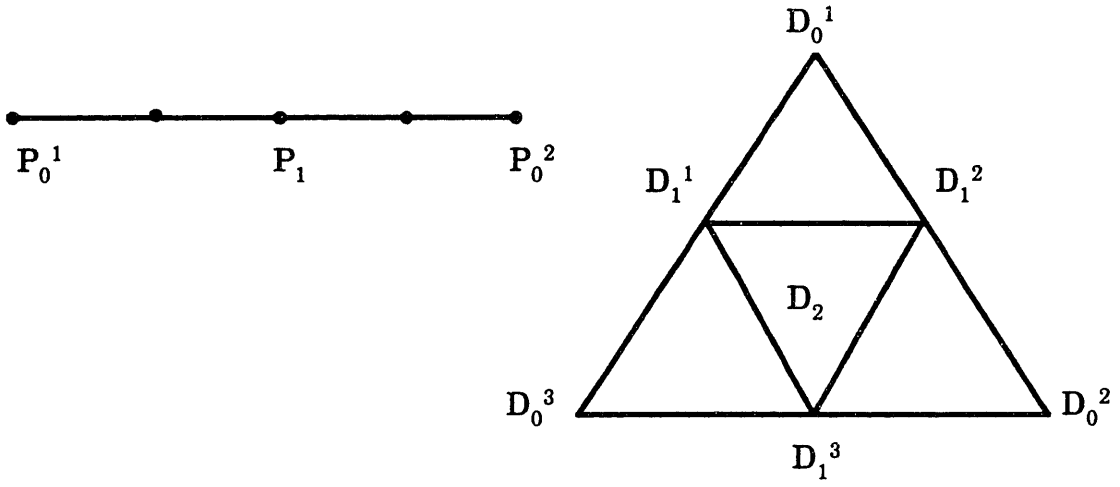
$$(P_0^1)^* = D_2$$

$$(P_1^1)^* = D_1^1$$

$$(P_1^2)^* = D_1^2$$

$(\hat{P}, \hat{D}, C, *)$ is a 3-pdpm.

3.



$$\hat{P} = \{P_0^1, P_0^2, P_1\}$$

$$\hat{D} = \{D_0^1, D_0^2, D_0^3, D_1^1, D_1^2, D_1^3, D_2\}$$

$$C = \{P_0^1, P_0^2, P_1, D_0^1, D_1^1, D_1^2\}$$

$$(P_0^1)^* = D_1^1$$

$$(D_1^1)^* = P_0^1$$

$$(P_1)^* = D_0^1$$

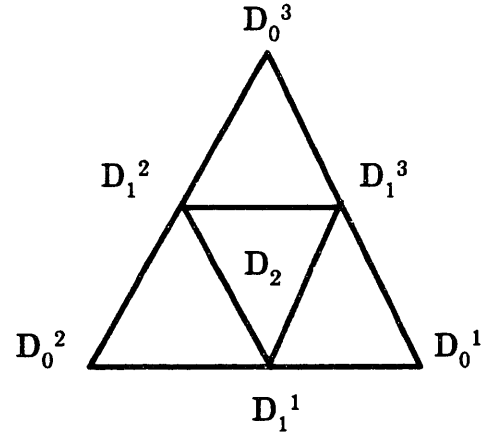
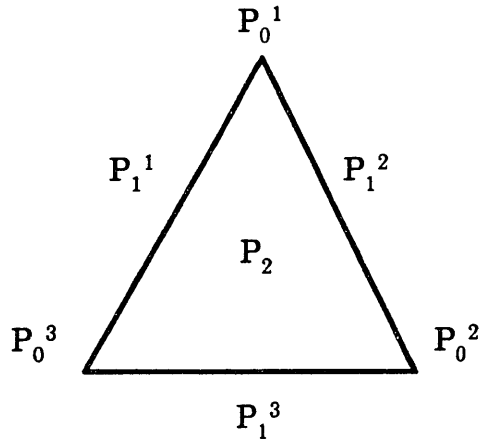
$$(D_0^1)^* = P_1$$

$$(P_0^2)^* = D_1^2$$

$$(D_1^2)^* = P_0^2$$

$(\hat{P}, \hat{D}, C, *)$ is a 2-pdpm.

4.



$$\hat{P} = \{P_0^1, P_0^2, P_0^3, P_1^1, P_1^2, P_1^3, P_2\}$$

$$\hat{D} = \{D_0^1, D_0^2, D_0^3, D_1^1, D_1^2, D_1^3, D_2\}$$

$$C = \{P_0^i, P_1^i, D_0^i, D_1^i, P_2, D_2 : i=1, \dots, 3\}$$

$$(P_i^j)^* = D_{1-i}^j$$

$$(D_i^j)^* = P_{1-i}^j$$

$$(P_2)^* = \emptyset$$

$$(D_2)^* = \emptyset$$

$(\hat{P}, \hat{D}, C, *)$ is a 2-pdpm.

For any $P \in \hat{P} \cap C$ we define

$$P \circ P^* = \{\sigma \cup N \mid \sigma \in P, N \in P^*\}$$

Note that $P \circ P^* = P^{**} \circ P$ if $P^* \neq \emptyset$. With this definition we have:

- (i) $P \circ P^*$ is an n-pseudomanifold.

$$(ii) \quad \partial(P \circ P^*) = (\partial P \circ P^*) \cup (P \circ \partial P^*)$$

(iii) If $P, Q \in \hat{P}$ and $P \neq Q$, then $P \circ P^*$ and $Q \circ Q^*$ have no n -simplices in common.

Condition (iii) implies that if $P \in \hat{P}_k \cap C$ and $Q \in \hat{P}_\ell \cap C$ ($\ell \leq k$), are two distinct partitions in \hat{P} , then $P \circ P^*$ and $Q \circ Q^*$ share a common $(n-1)$ -simplex if and only if $k = \ell + 1$ and $Q \subset \partial P$.

Now if we define

$$M = \left(\bigcup_{\substack{P \in \hat{P} \cap C \\ P^* \neq \emptyset}} P \circ P^* \right) \cup \left(\bigcup_{\substack{P \in \hat{P} \cap C \\ P^* = \emptyset}} P \right) \cup \left(\bigcup_{\substack{D \in \hat{D} \cap C \\ D^* = \emptyset}} D \right)$$

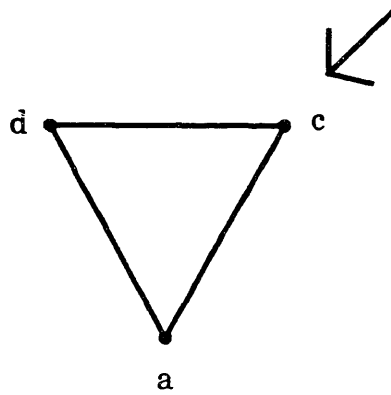
The above conditions imply that M is an n -pseudomanifold.

The boundary of M , ∂M , can be characterized as follows: An $(n-1)$ simplex $\tau \cup \gamma \in \partial M$ if and only if $\tau \cup \gamma$ is an $(n-1)$ -simplex in $P \circ P^*$ for some $P \in \hat{P}_k$, $P^* \in \hat{D}_\ell$ ($\ell = n-k-1$) and one of the following conditions is satisfied:

1. $k > 0$, $\tau \in \partial P$ and if $\tau \in Q \subset \partial P$, then $Q \notin C$.
2. $\ell > 0$, $\gamma \in \partial P^*$ and if $\gamma \in D \subset \partial P^*$, then $D \notin C$.
3. $k = 0$, $\tau = \emptyset$ and if $P^* \subset \partial D$, then $D \notin C$.
4. $\ell = 0$, $\gamma = \emptyset$ and if $P \subset \partial Q$, then $Q \notin C$.

We will conclude this chapter by identifying M and ∂M for the examples of primal-dual pseudomanifolds given above:

1.



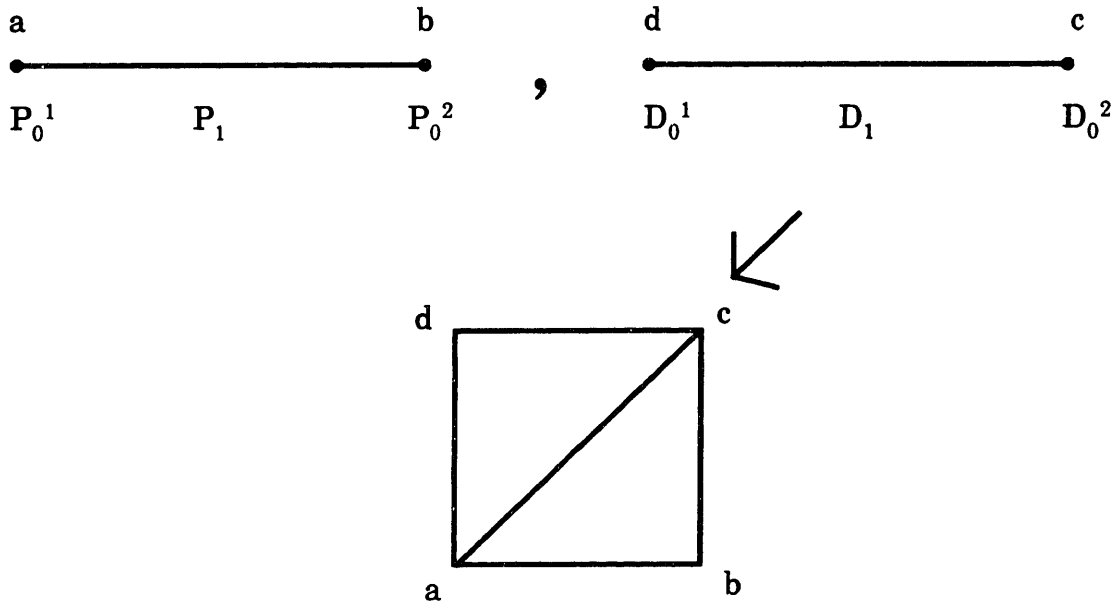
$$M = P_0^1 \circ D_1$$

$$\partial M = \{ \{a\}, \{d\}, \{c\}, \{d,a\}, \{d,c\}, \{c,a\} \}$$

$\{a,d\}$ and $\{a,d\}$ are of type (2)

$\{c,d\}$ if of type (3).

1' .



$$M = (P_0^1 \circ D_1) \cup (P_1 \circ D_0^1)$$

$$\partial M = \{ \{a\}, \{b\}, \{c\}, \{d\}, \{a,d\}, \{d,c\}, \{c,b\}, \{a,b\} \}$$

$\{a,d\}$ is of type (2)

$\{d,c\}$ is of type (3)

$\{c,b\}$ is of type (1)

$\{a,b\}$ is of type (4).

2.

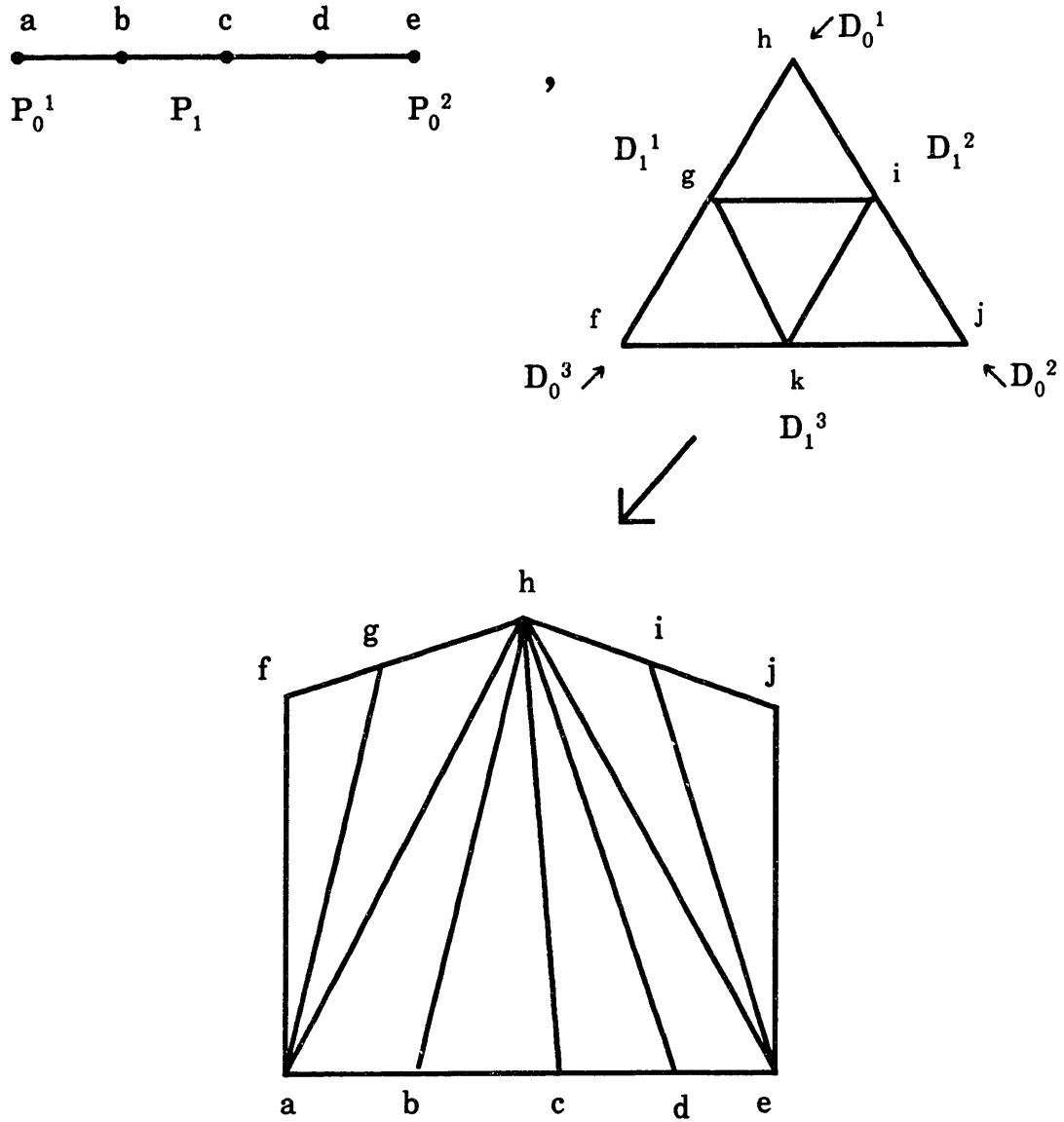
$$M = (P_0^1 \circ D_2) \cup (P_1^1 \circ D_1^1) \cup (P_1^2 \circ D_1^2)$$

$$\partial M = (P_0^2 \circ D_1^1) \cup (P_1^1 \circ D_0^2) \cup (P_1^1 \circ D_0^3) \cup (P_1^2 \circ D_0^3) \cup (P_0^3 \circ D_1^2) \cup$$

$$(P_1^2 \circ D_0^1) \cup (P_0^1 \circ D_1^3) \cup D_2$$

The (n-1)-simplices in $P_0^2 \circ D_1^1$ and $P_0^3 \circ D_1^2$ are of type (1). Those in $P_1^1 \circ D_0^2$, $P_1^1 \circ D_0^3$, $P_1^2 \circ D_0^3$, $P_1^2 \circ D_0^1$ and $P_0^1 \circ D_1^3$ are of type (2) and the one in D_2 are of type (3).

3.



$$M = (P_0^1 \circ D_1^1) \cup (P_1 \circ D_0^1) \cup (P_0^2 \circ D_1^2)$$

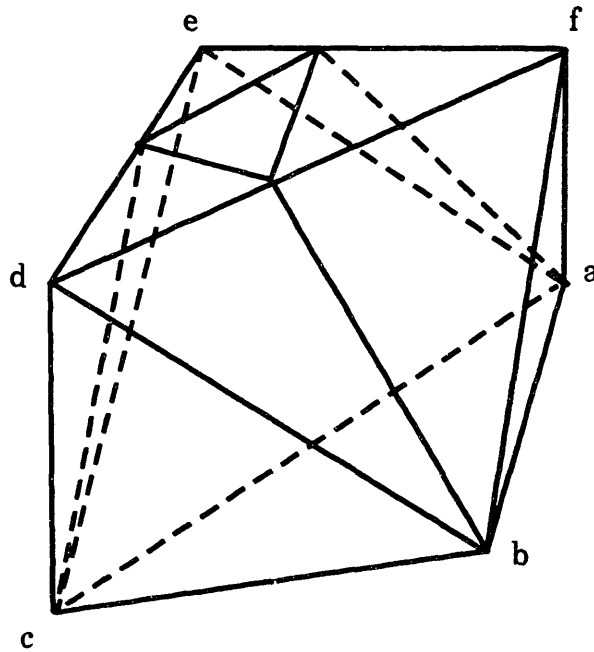
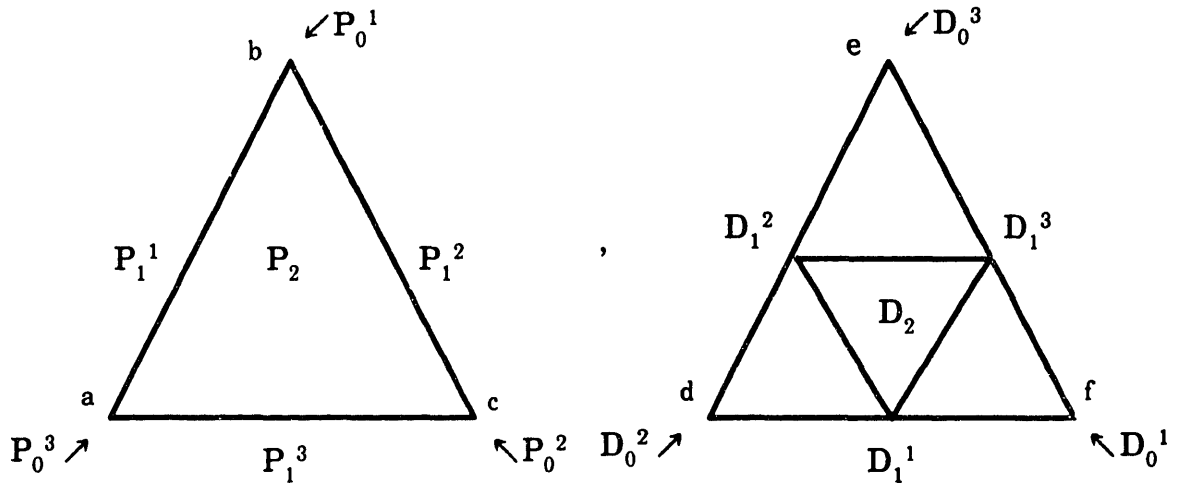
$$\partial M = \{ \{a,f\}, \{f,g\}, \{g,h\}, \{h,i\}, \{i,j\}, \{j,e\}, \{a,b\}, \{b,c\}, \{c,d\}, \{d,e\}, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}, \{j\} \}$$

$\{a,f\}$ and $\{e,j\}$ are of type (1)

$\{f,g\}, \{g,h\}, \{h,i\}, \{i,j\}$ are of type (3)

$\{a,b\}, \{b,c\}, \{c,d\}, \{d,e\}$ are of type (4).

4.



$$M = \bigcup_{j=1}^3 \left[\bigcup_{i=0}^1 P_0^j D_{1-i}^j \right] \cup P_2 \cup D_2$$

$$\partial M = \emptyset.$$

Chapter III

A Comparison of Primal-Dual Subdivided Manifolds and Primal-Dual Pseudomanifolds

In the last two chapters we discussed the construction of primal-dual subdivided manifolds of Kojima and Yamamoto [9] and the primal-dual pseudomanifolds of Yamamoto [11]. In this chapter, we analyze, compare, and contrast these two models.

3.1 Review of Major Results

This section presents a review of the major results concerning primal-dual pseudomanifolds and primal-dual subdivided manifolds.

Recall that a primal-dual pair of subdivided manifolds (PDM) of degree m and operator d is denoted by $(P, D; d)$ where P and D are subdivided manifolds in a Euclidean space \mathbf{R}^n , and d relates their faces as stated in Section 1.2. Given a PDM $(P, D; d)$ of degree m , we define $\bar{P} = \{B: B \text{ is a face of } C, C \in P\}$ and let $L = \langle P, D; d \rangle = \{Y^d \times Y: Y \in \bar{D} \text{ and } Y^d \neq \emptyset\} = \{X \times X^d: X \in \bar{P} \text{ and } X^d \neq \emptyset\}$. L is an m -dimensional subdivided manifold. Given a refinement P^* of P , $K = \{\sigma \times T: Y \in \bar{D} \text{ and } Y^d \neq \emptyset, \sigma \text{ belongs to the restriction of } \bar{P}^* \text{ to } Y^d\}$ is a refinement of L .

The construction of a primal-dual pseudomanifold (pdpm) is based on two pseudomanifold K and L of dimensions p and d respectively. We partition $K(L)$ into pseudomanifold $\hat{P}_\ell(\hat{D}_\ell)$ of dimension $k(\ell)$, where $0 \leq k \leq p$ ($0 \leq \ell \leq d$), as specified in Section 2.2. We denote

$$\hat{P} = \bigcup_{k=0}^p \hat{P}_k$$

and

$$\hat{D} = \bigcup_{\ell=0}^d \hat{D}_\ell.$$

Given a set $C \subset \hat{P} \cup \hat{D}$ and an operator $(*)$ relating the members of \hat{P} and \hat{D} , we call $(P, D, C, *)$ an m -pdpm if it satisfies the conditions given in Section 2.2. Given an m -pdpm $(P, D, C, *)$, the set

$$M = \left(\bigcup_{\substack{P \in \hat{P} \cap C \\ P^* \neq \emptyset}} P \circ P^* \right) \cup \left(\bigcup_{\substack{P \in \hat{P} \cap C \\ P^* = \emptyset}} P \right) \cup \left(\bigcup_{\substack{D \in \hat{D} \cap C \\ D^* = \emptyset}} D \right) \text{ is an } m\text{-pseudomanifold.}$$

3.2 A Comparison Between PDM's and PDPM's

In this section we start by analyzing the primal-dual subdivided manifold framework in Kojima and Yamamoto [9]. Then we will look at the primal-dual pseudomanifold of Yamamoto [11] and compare its characteristics with the first model.

Given a PDM $(P, D; d)$ of degree m , we observe the following:

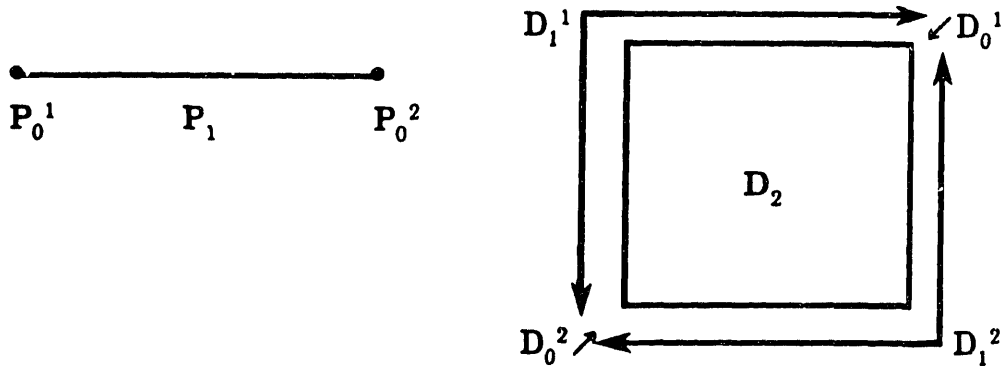
1. The structure is based on a primal and a dual subdivided manifold each imbedded in a Euclidean space \mathbb{R}^n .
2. The underlying polytopes for both P and D are general polytopes and not restricted to be simplices.
3. The partitions in both subdivided manifolds P and D are restricted to be the faces of the cells in the subdivisions.

4. If $X \in \bar{P} (Y \in \bar{D})$ has dimension $0 \leq k \leq m$, and if $X^d \neq \emptyset (Y^d \neq \emptyset)$, then $X^d (Y^d)$ must have dimension $m-k$.
5. Each member of L is the cross-product of an element in \bar{P} with an element in \bar{D} , therefore the structure is geometric and the resulting subdivided manifold depend on the facet structure of P and D . Furthermore, even if P and D are simplicial, in general L will not be simplicial. Thus there is no way of representing L combinatorially.
6. The structure is based on subdivided manifolds and the notion of a triangulated manifold is not used.

Now let $(\hat{P}, \hat{D}, C, *)$ be an m -pdpm. In comparing this model with a subdivided manifold we see:

1. The construction of a pdpm is based on two pseudomanifolds, and therefore free of imbedding in any Euclidean space \mathbb{R}^n .
2. As in a PDM, the underlying polytopes for both the primal and dual pseudomanifolds (K and L) in a pdpm are not restricted to be simplices.
3. In a pdpm, the members of both P and D are not restricted to be the faces of the subdivisions. The following example illustrates this point:

Example:



$$\hat{P} = \{P_0^1, P_1, P_0^2\}$$

$$\hat{D} = \{D_0^1, D_0^2, D_1^1, D_1^2, D_2\}$$

D_1^1 and D_1^2 are valid partitions.

4. As we are considering pseudomanifolds in Yamamoto [11], the structure is combinatorial and the facet structure of the underlying subdivided polytopes does not necessarily affect the structure of the resulting pdpm.
5. In a pdpm, if P belongs to a k -dimensional partition with $P^* \neq \emptyset$, then P^* has dimension $m-k-1$, i.e., 1 less than the dimension of the matched element in a PDM.

In view of the above observations, we conclude therefore, that the pdpm framework in Yamamoto [11] is a more general model than the subdivided manifold framework of Kojima and Yamamoto [9].

Chapter IV

V-Complexes and H-Complexes

The notion of a V-complex was introduced in Freund [4] [5], where it was used to develop constructive proofs for combinatorial analogs of certain fixed point theorems such as Sperner lemma. Later in [6], the same framework was used to extend similar results on the simplotope which is the coss-product of simplices.

4.1 Definitions

This section reviews the basic definitions that are necessary for the construction of a V-complex.

Given a set of vertices K° , an *abstract complex* is a set of finite non-empty subsets of K° , denoted by K , such that the following conditions are satisfied:

- (i) If $v \in K^\circ$, then $\{v\} \in K$,
- (ii) $\emptyset \neq x \subset y \in K$ implies that $x \in K$.

If in addition the set K° is finite, then K is called a *finite abstract complex*. Any element of K is referred to as a n (*abstract*) *simplex*. Letting $|\cdot|$ denote cardinality, $x \in K$ is defined to be an *n-simplex* if $|x| = n + 1$. An abstract complex is *locally finite* if each element of K° is contained in a finite number of simplices of K .

A complex K , is called an *n-dimensional pseudomanifold* or an *n-pseudomanifold* ($n \geq 1$) if each simplex of K is contained in an n -simplex of K , and each $(n-1)$ -simplex is contained in at most two n -simplices. For an n -

pseudomanifold ($n \geq 1$), K , the *boundary of K* , denoted ∂K , is defined to be the set of simplices that are contained in an $(n-1)$ -simplex which in turn is contained in exactly one n -simplex of K .

In the case where $n = 0$, the following definition for a 0-pseudomanifold is given in Freund []:

K is a 0-pseudomanifold if one of the following two conditions holds:

$$(i) \ K^\circ = \{v\}, K = \{\emptyset, \{v\}\}.$$

$$(ii) \ K^\circ = \{u, v\}, K = \{\emptyset, \{u\}, \{v\}\}.$$

In the first case $\partial K = \{\emptyset\}$, and in the latter case $\partial K = \emptyset$.

4.2 Construction of A V-Complex

In this section we define and give examples of a V-complex.

Let N be a finite set called the *label set*, and choose τ to be a collection of subsets of N such that if S and $T \in \tau$, then $S \cap T \in \tau$. Starting with a locally finite complex K , define $A(\cdot) : \tau \rightarrow 2^K \setminus \{\emptyset\}$ to be a mapping from τ into the set of all non-empty subsets of K . We call K a V-complex with admissible sets τ and operator $A(\cdot)$ if:

1. $x \in K$ implies that $x \in A(T)$ for some $T \in \tau$.

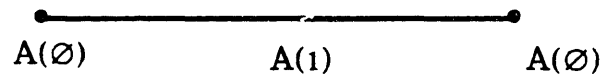
So, for each element x in K , there must be some $T \in \tau$ such that x belongs to $A(T)$.

2. $S, T \in \tau$ implies that $A(S \cap T) = A(S) \cap A(T)$.

3. If $T \in \tau$, then $A(T)$ is a pseudomanifold of dimension $|T|$.
4. $T \in \tau, T \cup \{j\} \in \tau (j \notin T)$ implies that $A(T)$ is contained in the boundary of $A(T \cup \{j\})$.

Examples:

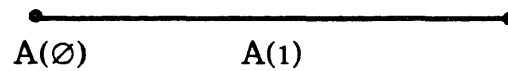
1.



$$N = \{1\}$$

$$\tau = \{\emptyset, \{1\}\}$$

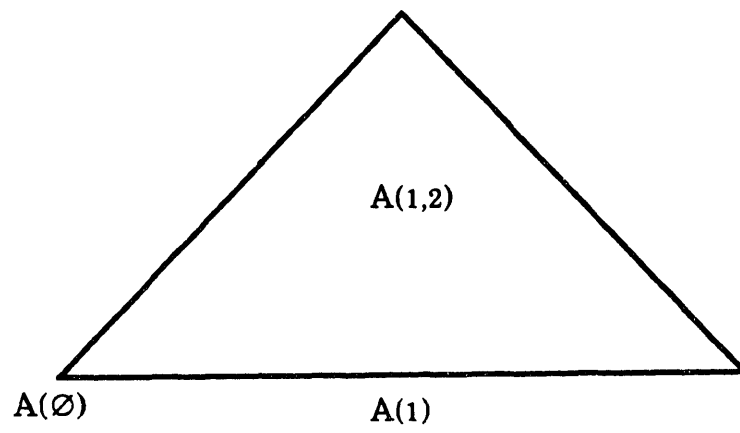
2.



$$N = \{1\}$$

$$\tau = \{\emptyset, \{1\}\}$$

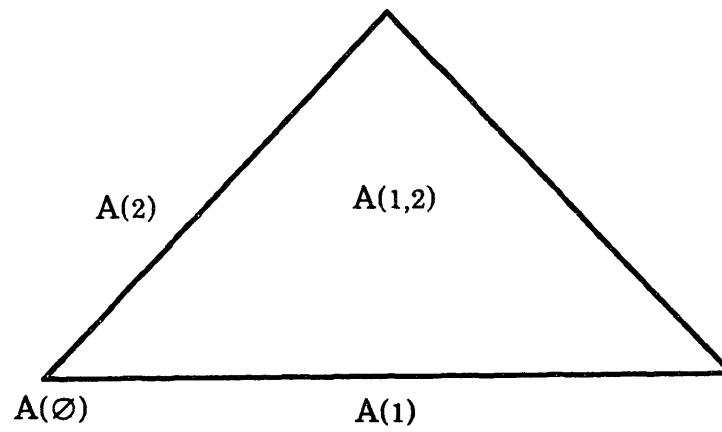
3.



$$N = \{1,2\}$$

$$\tau = \{\emptyset, \{1\}, \{1,2\}\}$$

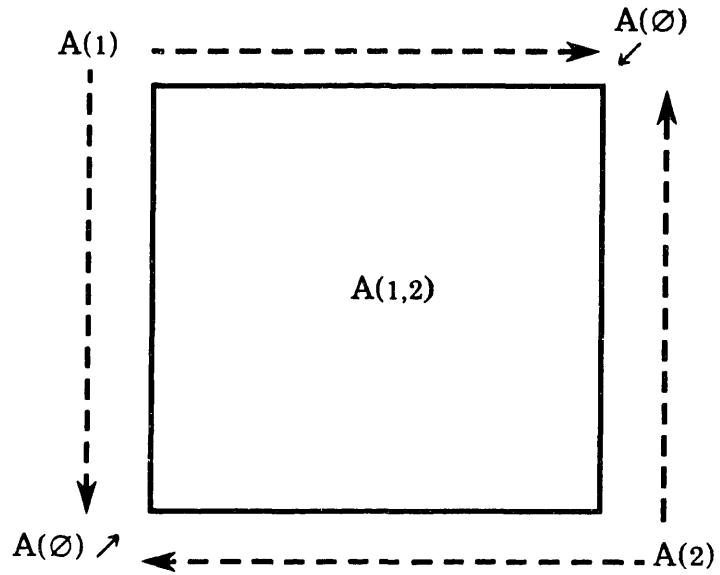
4.



$$N = \{1,2,3\}$$

$$\tau = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$$

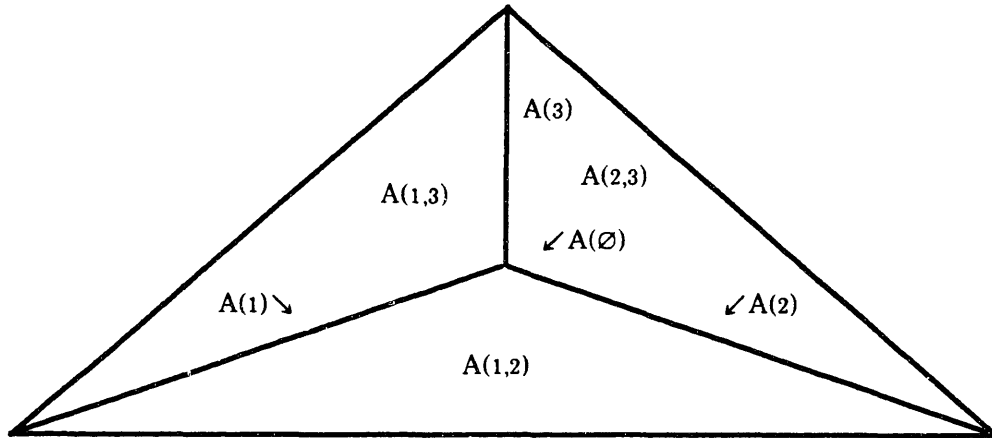
5.



$$N = \{1,2\}$$

$$\tau = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$$

6.



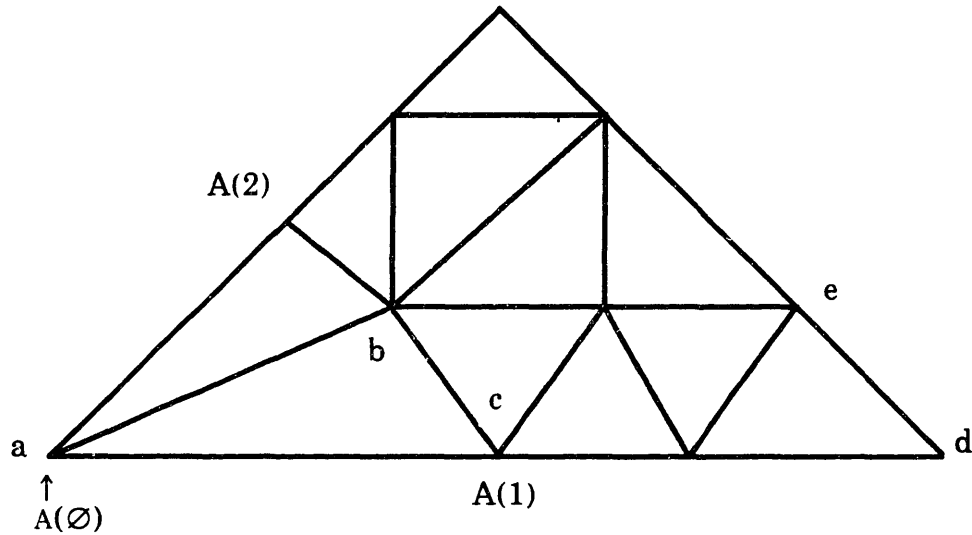
$$N = \{1,2,3\}$$

$$\tau = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}\}$$

Definition 4.1 Given a V-complex K, for each $x \in K$, we define T_x to be the smallest set T where $x \in A(T)$; mathematically:

$$T_x = \bigcap_{\substack{T \in \tau \\ x \in A(T)}} T$$

Example:



$$\begin{aligned} \tau &= \{\emptyset, \{1\}, \{2\}, \{1,2\}\} \\ T_{\{a\}} &= \emptyset \\ T_{\{a,b,c\}} &= T_{\{b,c\}} = T_{\{e,d\}} = \{1,2\} \\ T_{\{a,c\}} &= T_{\{d\}} = \{1\} \end{aligned}$$

Definition 4.2 For a V-complex K, $x \in K$ is called *full* is $|x| = |T_x| + 1$.

In the above example, $\{a,c\}$ is a full simplex.

Definition 4.3 Given a V-complex K with admissible sets τ and operator $A(\cdot)$, for each $T \in \tau$ we define:

$$\partial'A = \{x \in \partial A(T) : T_x = T\}$$

Again, in our example, $\{d\} = \partial'A(1)$ and $\{e\}, \{e,d\} \in \partial'A(1)$.

1.4.3 H-Complexes

As shown in Freund [4], a V-complex K , with label set N and admissible sets τ may be lifted into a pseudomanifold, K , of dimension $|N|$. The procedure is as follows:

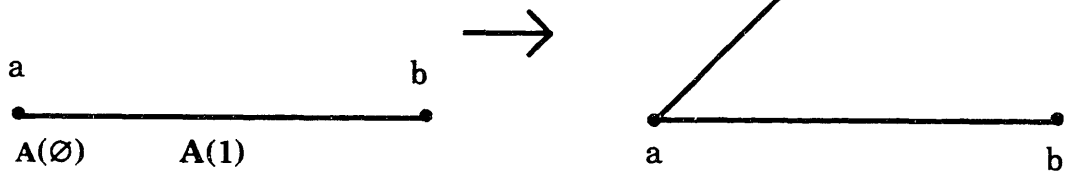
Let $Q = \{q_1, \dots, q_n\}$ be a set of "artificial vertices." For each $x \in K$, define $Q_x = \{q_i \in Q: i \in N \setminus T_x\}$. Then $K = \{x \cup Q: x \in K, Q \subset Q_x\}$ is the H-complex corresponding to the V-complex K .

Examples:

1.

$$N = \{1\}$$

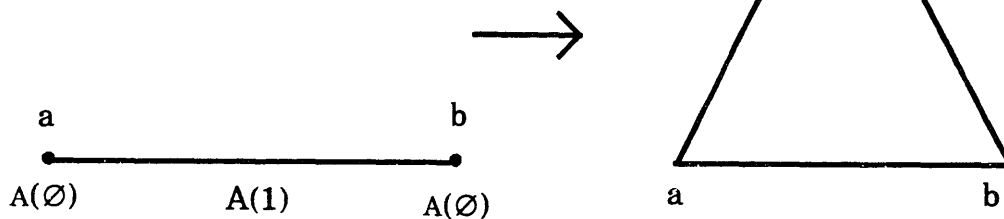
$$\tau = \{\emptyset, \{1\}\}$$



2.

$$N = \{1\}$$

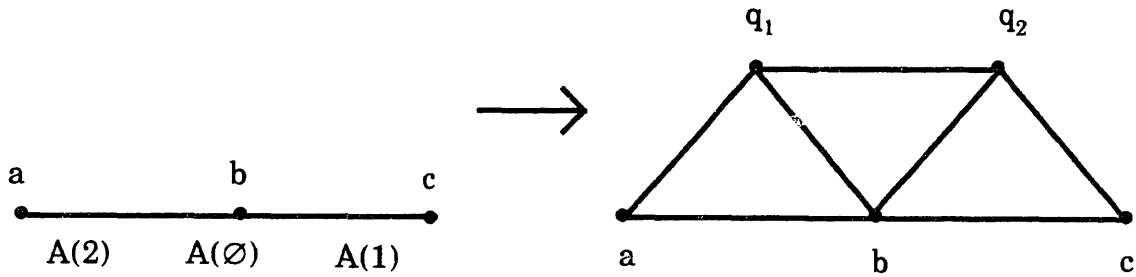
$$\tau = \{\emptyset, \{1\}\}$$



3.

$$N = \{1, 2\}$$

$$\tau = \{\emptyset, \{1\}, \{2\}\}$$



Definition 4.4 Given an H-complex, K , the boundary of K can be characterized as follows:

$$\text{Let } S_1 = \{y \cup Qy : y \in \partial'A(Ty)\}$$

$$S_2 = \{y \cup Qy \in K : N \setminus \{i : q_i \in Qy\} \notin \tau\}$$

$$\text{Then } \partial K = S_1 \cup S_2.$$

The boundaries of the H-complexes in the examples above are as follows:

$$1. \quad \partial'A(\emptyset) = \{\emptyset\}, \quad \partial'A(1) = \{b\}$$

$$S_1 = \{q_1, \{b\}\}, \quad S_2 = \emptyset$$

$$\partial K = \{\{q_1\}, \{b\}\}$$

$$2. \quad \partial'A(\emptyset) = \{\emptyset\}, \quad \partial'A(1) = \{\emptyset\}$$

$$S_1 = \{\emptyset\}, \quad S_2 = \emptyset$$

K has no boundary.

$$3. \quad \partial'A(\emptyset) = \{\emptyset\}, \quad \partial'A(1) = \{c\}, \quad \partial'A(2) = \{a\},$$

$$S_1 = \{\{a, q_1\}, \{q_1, q_2\}, \{c, q_2\}\},$$

$$S_2 = \{\{a, b\}, \{b, c\}\}$$

$$\partial K = \{\{a, q_1\}, \{q_1, q_2\}, \{c, q_2\}, \{a, b\}, \{b, c\}\}.$$

Chapter V

A Comparison of V-Complex and H-Complex With A PDM and PDPM

The main objective of this chapter is to study the relationship between a V-complex and a primal-dual subdivided manifold (PDM), and to give an interpretation of the H-complex in terms of the primal-dual pseudomanifold (pdpm).

5.1 A Comparison Between a V-Complex and a PDM

Recall from the last chapter that given a locally finite complex K , the construction of a V-complex is based on a label set N together with a collection of its subsets τ , and a map $A(\cdot): \tau \rightarrow 2^k \setminus \{\emptyset\}$ from τ to the set of all non-empty subsets of K . The sets in τ and the mapping $A(\cdot)$ must satisfy the conditions listed in Section 4.2.

We begin our discussion in this section by showing that a V-complex K , with admissible sets τ and operator $A(\cdot)$, cannot necessarily identify all facets of the geometric realization of a region $A(T)$, where $T \in \tau$.

Proposition 5.1 Let P be an n -dimensional polytope. then P has at least $(n + 1)$ facets.

Proof: see e.g., Brondsted [2].

In view of the above fact we can prove the following proposition:

Proposition 5.2 Suppose P is an n -dimensional polytope. Let $\{F_\alpha\}$ denote the set of all facets of P . Consider a triangulation T of P that triangulates each face of P , and let K be the pseudomanifold corresponding to T . We can cover at most (n) facets of P with a V -complex with admissible sets τ and operator $A(\cdot)$.

Proof: With no loss of generality let $N = \{1, \dots, n\}$ be the label set. Set $T_0 = N$ and define $A(T_0) = K$. From the definition of a V -complex, we now that for any $T \in \tau$, $A(T)$ must have dimension $|T|$. Furthermore, if $S, T \in \tau$, $A(S \cap T) = A(S) \cap A(T)$. Therefore for each facet F_α of P , there must exist a subset T_α of T_0 with cardinality $(n-1)$ such that $A(T_\alpha) = F_\alpha$. Since there are exactly n of these subsets, we can identify at most n facets of P .

Q.E.D.

Since in the study of certain combinatorial theorems in topology, such as a generalization of Sperner's lemma for a general polyhedral set, one usually must identify all the facets, the V -complex is not an appropriate model for studying such fixed point theorems.

With the PDM however, one can identify all facets of both the primal and the dual polytopes. Furthermore, neither of the primal and dual subdivided manifolds are required to be simplices, whereas in the construction of an H -complex one considers the subsets (simplices) of the artificial set of vertices Q . A short coming of the framework of Kojima and Yamamoto [9] is that its construction is based on two subdivided manifolds that are imbedded in a Euclidean space \mathbb{R}^n . The V -complex is free of this imbedding since its structure is combinatorial. Finally, we remark that in constructing an H -complex, each $x \in A(T_x)$ is matched with a subset of $Q_x \subseteq Q$. Note that as in the "pdpm" model,

Q_x has cardinality $|N|-|T_x|$, i.e., it is an $(|N|-|T_x|-1)$ -simplex. We saw before that this is one less than the dimension of the matched element in a PDM.

5.2 An Interpretation of an H-Complex In Terms of a PDPM

Let K be a V -complex with label set N and admissible sets τ . Note that for each $T \in \tau$, $A(T)$ is a pseudomanifold of dimension $|T|$. Let

$$\hat{P} = \left[\bigcup_{T \in \tau} A(T) \right] \cup \left[\bigcup_{T \in \tau} \partial' A(T) \right]$$

Let $Q = \{q_1, \dots, q_{|N|}\}$ be an artificial set of vertices and note that Q together with all its subsets is an $(|N|-1)$ dimensional pseudomanifold. For $i = 1, \dots, |N|$, let \hat{D}_{i-1} be the set of all subsets of Q with cardinality i . Define

$$\hat{D} = \bigcup_{i=1}^{|N|-1} \hat{D}_i$$

and let $C = \{A(T) : T \in \tau\} \cup \{\{q_i \in Q : i \in N \setminus T, T \in \tau\} \setminus \{\emptyset\}\}$. For each $P \in C \cap \hat{P}$ let $P^* = \{q_i \in Q : i \in N \setminus T\}$. Then $(\hat{P}, \hat{D}, C, *, |N|)$ is an $|N|$ -pdpm that corresponds to the H -complex obtained from the V -complex K .

Chapter VI

The Framework “ \bar{K} ”

In [7] Freund gives generalizations of several combinatorial theorems such as Scarf’s Dual Sperner lemma, Sperner’s lemma and a generalized Sperner lemma on a general bounded polyhedral set. Each one of these theorems implies the corresponding theorems on the simplex and the simplotope. In proving such theorems, one usually needs to identify all facets of a given polytope. As we noted in the last chapter, given an n -dimensional polytope, a V -complex can cover at most n of its facets. hence the V -complex is not an appropriate framework in these cases. To overcome this shortcoming of the V -complexes, a new framework is developed in [7].

In this chapter we will define this new model and show its relationship with the primal-dual pseudomanifold framework of Yamamoto [11].

6.1 Some Facts About Convex Polytopes

In this section we review some concepts from the theory of convex sets. This material will be used in the development of the frameworks discussed in this and the following chapter.

Let K be a nonempty set in \mathbb{R}^n . The *polar set* K^* of K is defined by $K^* = \{y \in \mathbb{R}^n : x^T y \leq 1 \text{ for all } x \in K\}$. Note that K^* is closed, convex, and contains the origin. If in addition K is closed, convex, and contains the origin, then $(K^*)^* = K$. Finally, if K is a compact convex set that contains the origin as an interior point, then K^* is also compact and the origin belongs to the interior of K^* .

Given a polytope P , any polytope R is called *dual* to P if there exists a one-to-one inclusion reversing correspondence between the set of faces of P and the set of faces of R . Note that this definition implies that R is dual to P if and only if their face lattices are anti-isomorphic. It can be shown that if P is an n -dimensional polytope in \mathbb{R}^n containing the origin in its interior, then the polar set P^* of P is an n -dimensional polytope dual to P (see e.g., Grünbaum [8]). In fact, for each face F of P , $F^* = \{y \in P^* : y^T F = 1\}$ is the associated face of P^* ; $\dim F^* = n-1-\dim F$.

An n -dimensional polytope P is *simplicial* if each facet of P is a simplex. P is called *simple* if each vertex of P is incident to precisely n facets. If P and R are dual polytopes, then P is *simple* if and only if R is *simplicial*.

6.2 The Construction of \bar{K}

Throughout this chapter, A and b are a given $(m \times n)$ matrix and an m -vector. We denote the i -th row of A and i -th component of b by A_i and b_i respectively.

Let P be a nonempty polytope in \mathbb{R}^n of the form $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, where none of the constraints are redundant. with no loss of generality assume that the origin is an interior point of P . Consider a triangulation T of P and let K be the pseudomanifold corresponding to T . Also, let K^0 be the set of all vertices of T , and let $M = \{1, \dots, m\}$ be the set of constraint row indices of A . Given a subset S of P , the carrier of S , denoted $C(S)$ is the set $\{i \in M : A_i x = b_i \text{ for all } x \in S\}$. \bar{K} is defined by

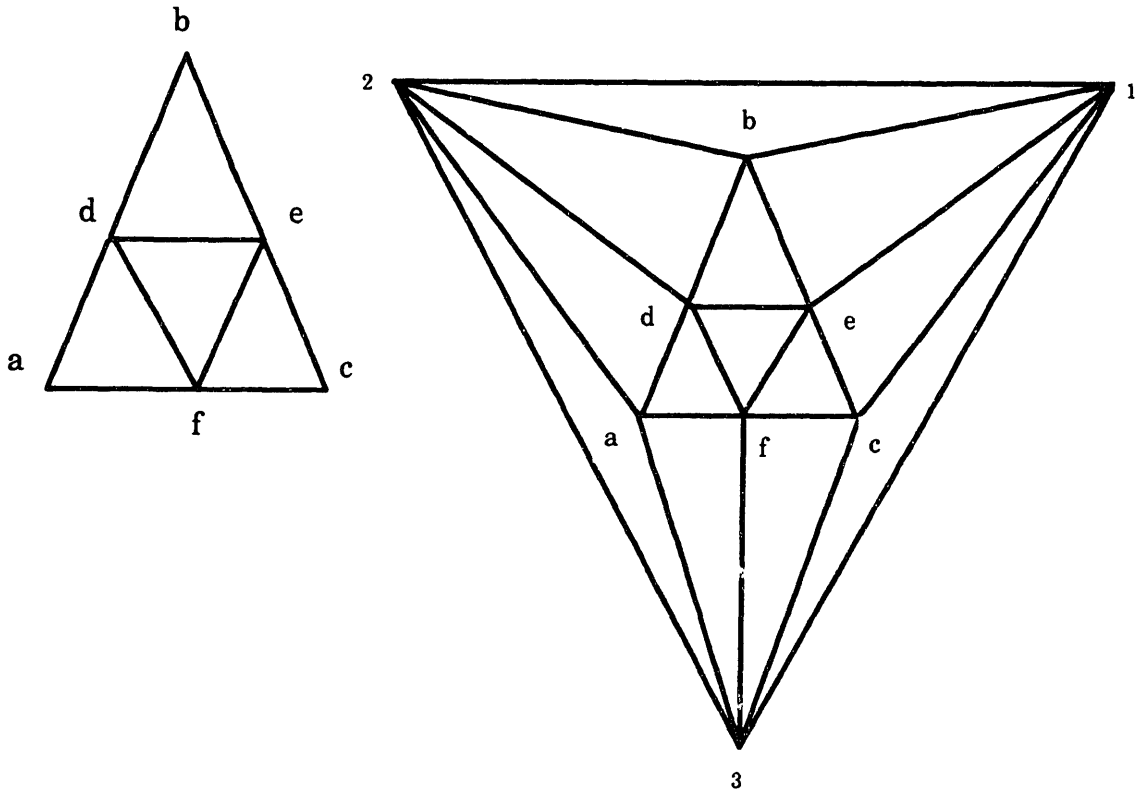
$$\bar{K} = \{\bar{\sigma} : \bar{\sigma} \neq \emptyset, \bar{\sigma} \subseteq (\sigma \cup C(\sigma)), \sigma \in K\}$$

and

$$\bar{K}^o = K^o \cup M.$$

The boundary of \bar{K} , denoted $\partial\bar{K}$, is the set $\{\beta: \beta = C(x) \text{ for some } x \in K\}$. It is shown in Freund [7] that when P is a simple polytope, \bar{K} is in fact an n -dimensional pseudomanifold.

The following example illustrates the construction of \bar{K} .



6.3 The Relationship Between \bar{K} and PDPM

Let P be a polytope defined as in Section 6.2, and let P^* be a dual to P . In the case where P is a simplex polytope, \bar{K} may be interpreted as a special case of the primal-dual pseudomanifold.

Consider a triangulation T of P , and let K be the corresponding pseudomanifold. Also, let T^* be a triangulation of P^* that does not introduce any new vertices. Note that since P is simple, P^* is a simplicial polytope, and therefore, its faces are already triangulated. Let K^* be the pseudomanifold corresponding to T^* . For each face $F(G)$ of $P(P^*)$, let $P_F(D_G)$ denote the pseudomanifold corresponding to the restriction of $T(T^*)$ to $F(G)$. For $k = 0, \dots, n$ define

$$\hat{P}_k = \{P_F: F \text{ is a } k\text{-face of } P\}$$

$$\hat{D}_k = \{D_G: G \text{ is a } k\text{-face of } P^*\}$$

$$\hat{P} = \bigcup_{k=0}^n \hat{P}_k$$

$$\hat{D} = \bigcup_{k=0}^n \hat{D}_k$$

Let ϕ be a one-to-one inclusion-reversing map from the set of faces of P to that of P^* such that for any face F of P , $\dim \phi(F) = n-1-\dim F$. For each $\sigma \in K$, let F_σ denote the lowest dimensional face of P containing σ . Note that the set of all subsets of $C(\sigma)$ corresponds to $D_{\phi(F_\sigma)}$. Now if we let $C = \hat{P} \cup \hat{D} \setminus (\{\emptyset\} \cup \hat{D}_n)$ and define $*$ by $(P_F)^* = D_{\phi(F)}$, $(\hat{P}, \hat{D}, C, *, n)$ becomes a pdpm corresponding to \bar{K} .

The above analysis was predicated on the fact that P is simple. If P is not simple there is no automatic construction of a pdpm. However, using perturbation methods such as “pulling” the vertices, it is possible to construct such a pdpm; see Freund [7].

Chapter VII

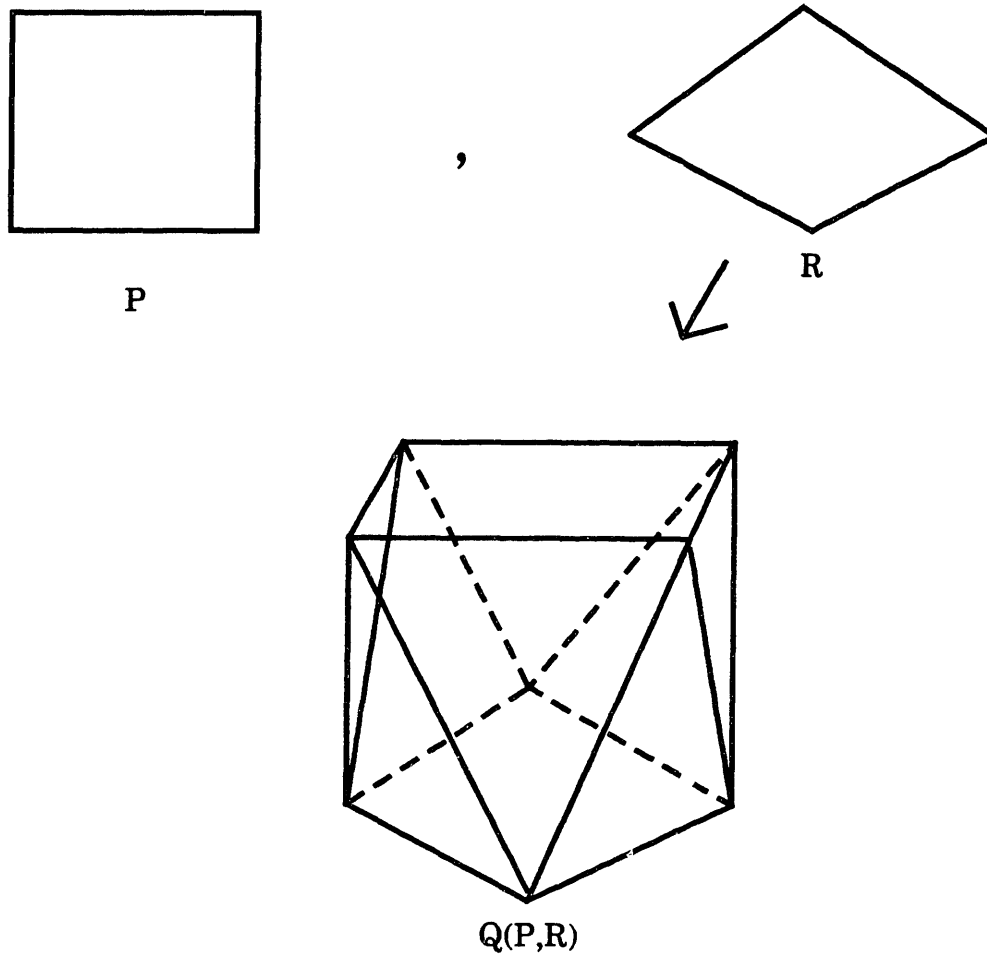
Antiprisms

In [1], Broadie suggested the structure of an antiprism for obtaining subdivisions for piecewise-linear homotopy algorithms. In this chapter, we will define the notion of an antiprism and show that it may be considered as a geometric version of a special case of a primal-dual pseudomanifold.

7.1 Definition of An Antiprism

Let P be an n -dimensional polytope and let R be a polytope dual to P . Let ϕ be a one-to-one inclusion reversing function relating the set of faces of P with that of R . For a set S , we let $\text{Conv}(S)$ denote the convex hull of S . Then $Q(P,R) = \text{Conv}\{P \times \{1\}, R \times \{0\}\}$ is called an antiprism if the facets of $Q(P,R)$ are precisely those of the form $\text{Conv}\{F \times \{1\}, \phi(F) \times \{0\}\}$, where F is a face of P . If $Q(P,R)$ is an antiprism, the sets of the form $\text{Conv}\{F \times \{1\}, \phi(F) \times \{0\}\}$: F is a face of P indeed form a subdivided manifold.

Example



Recall from the last chapter that if P contains the origin in its interior, then P^* , the polar set of P , is a polytope dual to P . Furthermore, for each face F of P , $F^* = \{y \in p: y^t F = 1\}$ is the corresponding dual face of P^* , and $\dim F^* = n-1-\dim F$. In [1], Broadie considers the polar set P^* , of such a polytope P , in the construction of an antiprism: For each face F of P let $Q(F, F^*) = \text{Conv}\{F \times \{1\}, F^* \times \{0\}\}$, then $Q(P, P^*)$ is an antiprism if the set of facets of $Q(P, P^*)$ is $\{Q(F, F^*): F \text{ is a face of } P\}$. Broadie also shows that $Q(P, P^*)$

is an antiprism if for each face F of P , the orthogonal projection of the origin onto the affine hull of F belongs to the relative interior of F .

7.2 Antiprism as a Geometric Version of a PDPM

Let P be a polytope containing the origin in its interior, and let P^* be the polar set of P . Consider triangulations T of P , and T^* of P^* , that do not introduce new vertices. Let K and K^* be pseudomanifolds corresponding to T and T^* respectively. For each face $F(G)$ of $P(P^*)$, let $P_F(D_G)$ denote the pseudomanifold corresponding to the restriction of $T(T^*)$ to $F(G)$. For $k=0, \dots, n$ define

$$\hat{P}_k = \{P_F: F \text{ is a } k\text{-face of } P\}$$

$$\hat{D}_k = \{D_G: G \text{ is a } k\text{-face of } P^*\}$$

$$\hat{P} = \bigcup_{k=0}^n \hat{P}_k$$

$$\hat{D} = \bigcup_{k=0}^n \hat{D}_k$$

Let $C = \hat{P} \cup \hat{D} \setminus \{\emptyset\}$ and define the operator $*$ by $(P_F)^* = D_{F^*}$. $(\hat{P}, \hat{D}, C, *, n)$ is a pdpm with no boundary.

Note that in this construction, for each face F of P , the simplices in P_F are joined with those in D_{F^*} . So each facet, $Q(F, F^*)$, of the antiprism $Q(P, P^*)$, corresponds to the underlying polytope in a geometric realization of $P_F \circ D_{F^*}$, and therefore, an antiprism may be considered as a geometric version of this pdpm.

Chapter VIII

Concluding Remarks

In this thesis we have looked at four duality models, namely, PDM's, pdpm's, V-complexes and H-complexes, and antiprisms. We showed that a pdpm is a more general model than a PDM. In comparing V-complexes with PDM's we demonstrated that a V-complex is neither a special case nor a more general case of a PDM. We proved, however, that an H-complex may be interpreted as a special case of a pdpm. We looked at the framework \bar{K} , and showed that if we start with a simple polytope, then this framework becomes a special case of the pdpm framework. Finally, we discussed the concept of an antiprism and noted that the collection of its facets may be considered as a geometric version of a pdpm with no boundary.

References

- [1] Broadie, M.N., "A Theorem About Antiprisms," Linear Algebra and Its Applications, Vol. 66, pp. 99-111, 1985.
- [2] Brøndsted, A., An Introduction to Convex Polytopes, Springer-Verlag, New York, 1983.
- [3] Eaves, B.C., "A Short Course in Solving Equations with PL Homotopies," SIAM-AMS Proceedings, Vol. 9, pp. 73-143, 1976.
- [4] Freund, R.M., "Variable Dimension Complexes, Part I: Basic Theory," Mathematics of Operations Research, Vol. 9, pp. 479-497, 1984.
- [5] Freund, R.M., "Variable Dimension Complexes, Part II: A Unified Approach to Some Combinatorial Lemmas in Topology," Mathematics of Operations Research, Vol. 9, pp. 448-509, 1984.
- [6] Freund, R.M., "Combinatorial Theorems on the Simplotope that Generalize Results on the Simplex and Cube," Mathematics of Operations Research, Vol. 11, pp. 169-179, 1986.
- [7] Freund, R.M., "Combinatorial Analogs of Brouwer's Fixed Point Theorem on a Bounded Polyhedron," Under Review at Journal of Combinatorial Theory, Series B.

- [8] Grünbaum, B., Convex Polytopes, Wiley, New York, 1967.
- [9] Kojima, M., and Yamamoto, Y., "Variable Dimension Algorithms: Basic Theory, Interpretations and Extensions of Some Existing Methods," Mathematical Programming, Vol. 24, pp. 177-215, 1982.
- [10] Lemke, C.E., and Howson, Jr., J.T., "Equilibrium Points of Bimatrix Games," SIAM Journal on Applied Mathematics, Vol. 12, pp. 413-423, 1964.
- [11] Yamamoto, Y., "Orientability of a Pseudomanifold and Generalization of Sperner's Lemma," Report No. 86411-OR, Institut für Ökonometrie und Operations Research, Bonn, West Germany, January 1986.