CONTINUOUS-SPIN ISING FERROMAGNETS

by

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ABSTRACT

of

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We define and analyze the Gibbs measures of continuous-spin ferromagnetic Ising models. We obtain many inequalities interrelating the moments (spin expectations) of these measures. We investigate the dependence on temperature and magnetic field parameters, and find that at low temperature the first moment of the Gibbs measure (the magnetization) is discontinuous in the magnetic field parameter for all nontrivial models in two or more dimensions. Thus the appearance of a phase transition is generic: all nontrivial continuous-spin ferromagnets in at least two dimensions become spontaneously magnetized at sufficiently low temperature.

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Chapter I: Introduction

In this thesis we investigate continuous-spin ferromagnetic Ising models, with principal emphasis on the inequalities they obey and the remarkable low-temperature phenomena they exhibit. Mathematically, the study of these models amounts to the analysis of a physically-motivated class of probability measures, called Gibbs measures, carried on finite or infinite-dimensional product spaces $\prod_{i \in \mathcal{L}} \mathbb{R}$. The models we consider, which are rigorously defined at the close of the introduction, generalize the original notion of Ising and Lenz [21] in two ways: the spin variables σ_i may assume any real values with some a priori probability measure ν instead of merely assuming the values ± 1, and the energy of a configuration of spins may include many-body terms instead of only two-body terms. Physically, continuous-spin ferromagnets are of interest not so much because they resemble real crystals - with our degree of generality this resemblance is tenuous - but rather because they accurately approximate Euclidean scalar quantum fields [43], and so provide a simpler structure for developing conjectures and proving theorems that carry over in the limit to the more difficult models of quantum field theory. Mathematically, continuous-spin ferromagnets are of greatest interest for the striking dependence of the moments of the Gibbs measure on certain parameters representing physical variables such as temperature and magnetic field strength. One generally expects that the limit of a naturally-arising convergent sequence of continuous functions is continuous. By contrast, one of the main theorems in this thesis is a proof of precisely the opposite: certain moments of the Gibbs measure, which are defined as limits of sequences of continuous (in fact, real analytic) functions, are necessarily discontinuous.

We now give a synopsis of our results. Chapters II-IV deal with inequalities for finite Ising ferromagnets, whose Gibbs measures are defined on finite products $\prod_{i\in\Lambda} \mathbb{R}$. In Chapter II we introduce the convenient method of duplicate variables, and use it to give a simple, unified derivation for continuous-spin ferromagnets of inequalities proved by other methods in various special cases by Griffiths [17], Griffiths, Hurst, and Sherman [19], Ginibre [12], Lebowitz [28], Percus [39], and Ellis and Monroe [8]. With a different technique, we derive an inequality for change of single-spin measure which will be very useful in our subsequent analysis of low-temperature phenomena. While some inequalities of this chapter hold for all continuous-spin Ising ferromagnets, others are restricted in their domain of validity. Chapter III invokes combinatoric techniques to give a new simplified proof of a Gaussian-type inequality discovered in its present form by Newman [36]. In Chapter IV, we combine the method of duplicate variables with additional combinatoric techniques to investigate the signs of the Ursell functions u_n (generalized cumulants of the Gibbs measure) of spin-$\frac{1}{2}$ finite ferromagnetic Ising models. We represent these cumulants as moments of a measure on a larger space, and use this representation to prove complete results through order n=6. A reduction formula then gives partial results for higher orders. We present formulas for the Maclaurin coefficients of (functions closely related to) the Ursell functions when $n\leq 8$. Our methods yield additional inequalities, though we have no application for them at present. In a related appendix (Appendix B) we describe a computational algorithm for the evaluation of (functions closely related to) Ursell functions of all orders, and the results of a computer study using it. Chapters II-IV include, with one exception, proofs of all major inequalities for finite ferromagnetic Ising models of which the author is aware. This exception is the F.K.G. inequality [11], which we shall only use at one point in Chapter V. Although our interest lies in models with real-valued spins, in some cases our results extend to models with vector-valued spins, and where possible we try to point this out.

With the inequalities of Chapters II-IV serving as the primary investigative tools, we turn in Chapter V to the study of infinite continuous-spin Ising ferromagnets. After making some preliminary definitions, we construct the infinite-volume limit Gibbs measure for a very large class of models by using C*-algebraic techniques, and we give an easy proof that it has finite moments in many cases of interest. With these fundamental results established, we undertake an analysis of three closely-related low-temperature cooperative phenomena: long-range order, spontaneous magnetization, and phase separation. We begin with a discussion of the decay of spin correlations when the separation of two clusters of spins becomes large. For many models, we show that these correlations must decay to zero for almost all values of a parameter h representing the influence of an external magnetic field, and in some instances this set of potential exceptional points actually reduces further to the single point h=0. In fact, as we next prove, if h=0 and the parameter representing temperature is sufficiently low, then (nontrivial nearest-neighbor) models in two or more dimensions do have all their correlations bounded away from zero: they are long-range ordered. This is one of our main theorems. To coordinate our results on the decay of correlations we define the infinite-volume transfer matrix (for nearest-neighbor models), and characterize the cluster properties of an Ising ferromagnet in terms of spectral properties of its transfer matrix. We next consider the phenomenon of spontaneous magnetization (discontinuity of the moments of the Gibbs measure in the external field h), and show that it is a consequence of the long-range order previously established at low temperature. For certain models we combine inequalities of the previous chapter with an explicit computation by Onsager [37] to estimate the critical temperature; that is, the temperature for the onset of spontaneous magnetization. We establish the third cooperative phenomenon, phase separation, as a consequence (in three or more dimensions) of spontaneous magnetization. The final section of Chapter V treats some of the many applications to quantum field theory of the inequalities derived in Chapters II-IV.

In Chapter VI we present some unsolved problems, and make concluding remarks.

Let us now give some definitions of terms used in the remainder of this thesis, and some physical motivation for them. A finite continuous-spin ferromagnetic Ising model is a triple (Λ, H, ν), where:

(1) The set of sites Λ is a finite set. We associate with each site $i \in \Lambda$ a spin variable $\sigma_i \in \mathbb{R}$, and the product $\prod_{i\in\Lambda} \mathbb{R}$ is called the configuration space.

(2) The Hamiltonian H is a polynomial on the configuration space with negative coefficients. We write

$$H(\sigma) = -\sum_{K\in\mathfrak{F}_0(\Lambda)} J_K \sigma_K \quad , \quad J_K \geq 0 \, , \tag{1}$$

where the numbers J_K are called couplings (or bonds), $\mathfrak{F}_0(\Lambda)$ is the set of finite families ("sets" with repeated elements) in Λ , and

σ_K is by definition the product $\sigma_K = \prod_{i\in K} \sigma_i$.

(The distinction between sets and families is not important for our purposes, and we shall largely ignore it.)

(3) The single-spin measure ν is an even Borel probability measure on $\mathbb{R}$ which decays sufficiently rapidly that if d is the degree of the polynomial H, then

$$\int_{\mathbb{R}} \exp(a|\sigma|^d)\, d\nu(\sigma) < \infty \qquad \forall\, a \in \mathbb{R}. \tag{2}$$

The linear term $-\sum_{i\in\Lambda} J_i \sigma_i$ in H is usually thought of as describing the effect of an external magnetic field, while higher-order terms are considered to arise from the mutual interaction of the spins. We usually recognize this distinction by writing $-\sum_{i\in\Lambda} h_i\sigma_i$ in the Hamiltonian instead of $-\sum_{i\in\Lambda} J_i\sigma_i$. A pair interaction is a Hamiltonian of degree two. In connection with the decay condition (2) on the single-spin measure we define for $d \geq 0$

$$\Pi_d = \{\text{Even Borel probability measures } \nu : \int_{\mathbb{R}} \exp(a|\sigma|^d)\, d\nu(\sigma) < \infty \ \forall a \in \mathbb{R}\} \tag{3}$$

and set

$$\Pi_\infty = \bigcap_{d\in[0,\infty)} \Pi_d \quad . \tag{4}$$

A model (Λ, H, ν) is called connected if any pair of sites $i, j \in \Lambda$ is connected by a finite chain $K_1, K_2, \cdots, K_n \in \mathcal{B}_0(\Lambda)$ with $J_{K_1}, \cdots, J_{K_n} \neq 0$, $i \in K_1$, $j \in K_n$, and $\forall \ell$ $K_\ell \cap K_{\ell+1} \neq \phi$.

The Gibbs measure μ of (Λ, H, ν) is the measure on the configuration space $\mathbb{R}^\Lambda = \prod_\Lambda \mathbb{R}$ defined by

$$\mu(E) = \frac{\int_E \exp(-\beta H(\sigma)) \prod_{i\in\Lambda} d\nu(\sigma_i)}{\int_{\mathbb{R}^\Lambda} \exp(-\beta H(\sigma)) \prod_{i\in\Lambda} d\nu(\sigma_i)} , \qquad E \subset \mathbb{R}^\Lambda \text{ measurable}; \tag{5}$$

here $\beta \in [0,\infty)$ is a parameter representing inverse temperature. Note that this measure favors lower values of H. The normalization factor in (5) is called the partition function and traditionally denoted by Z:

$$Z = \int_{\mathbb{R}^\Lambda} \exp(-\beta H(\sigma)) \prod_{i\in\Lambda} d\nu(\sigma_i) . \tag{6}$$

We indicate (thermal) expectations with respect to the Gibbs measure at inverse temperature β by angular brackets $\langle\ ; H, \nu, \beta\rangle$, omitting the descriptive arguments H, ν, β when they are clear from context:

$$\langle f; H, \nu, \beta\rangle = \langle f\rangle = \int_{\mathbb{R}^\Lambda} f\, d\mu = Z^{-1} \int_{\mathbb{R}^\Lambda} f e^{-\beta H} d\nu . \tag{7}$$

Physically, the sites Λ may be thought of as atoms in a crystal, and the spin variable σ_i at each site $i\in\Lambda$ as a classical version of the quantum-mechanical spin each atom possesses. The single-spin measure describes the spin probability distribution of a completely isolated atom. A point σ in the configuration space $\mathbb{R}^\Lambda$ corresponds to a state of the system, and $H(\sigma)$ is the energy of that state. Note that the ferromagnetic condition $J_K \geq 0$ causes configurations in which all spins σ_i have the same sign to have generally lower (more negative) energies. If we allow the crystal to exchange energy (but not mass) with a large heat bath at temperature β^{-1} , it will reach eventual equilibrium. According to the principles of statistical mechanics, the probability of finding the equilibrium system in some subset $E \subset \mathbb{R}^\Lambda$ of the configuration space is given by the Gibbs measure $\mu(E)$.

We conclude the introduction by describing our notational conventions. Chapters are given Roman numerals I, II, etc., while sections within a chapter have Arabic numerals 1,2, etc. We use the standard decimal notation to show in which chapter a section appears. Thus, Section II.3 is the third section of the second chapter. Important formulas are enumerated sequentially within a section, the numbering beginning again when a new section starts. As before, we use the standard decimal convention, so that formula (IV.2.12) is the twelfth enumerated formula of the second section in the fourth chapter. Lemmas, propositions, theorems, and corollaries are similarly numbered within a section. If descriptive arguments of a section, formula, lemma, proposition, etc. are omitted in some reference, by convention they are taken to be the values in effect at the point of the reference. Thus, if in Section IV.4 we see a reference to Theorem 3.1, this means Theorem IV.3.1. References to the numbered bibliography are indicated by square brackets [ ].

Chapter II: Inequalities

Section 1: Introduction

In this chapter, taken largely from [46] , we exploit the method of duplicate variables to give a simple unified derivation of continuous-spin Ising ferromagnet inequalities established in various special cases by Griffiths [17], Griffiths, Hurst, and Sherman [19], Ginibre [12], Lebowitz [28], Percus [39], and Ellis and Monroe [8], obtaining them for a large class of single-spin measures. The single-spin measure and the Hamiltonian for which the inequalities may be proved become more restricted as the inequality becomes more complex. However, all inequalities hold for a model with ferromagnetic pair interactions, positive (nonuniform) external field, and single-spin measure either $\frac{1}{\ell+1}\sum_{j=0}^{\ell}\delta(-\ell+2j+\sigma)$ (spin $\frac{\ell}{2}$) or $\exp(-P(\sigma))d\sigma$, where P is an even polynomial all of whose coefficients must be positive except the quadratic, which is arbitrary. (Recent work by Ellis and Newman [9] elegantly relaxes this condition on P: it need only be an even continuously differentiable function whose derivative is convex on $[0,\infty)$.) The Percus inequality is akin to the Fortuin-Kasteleyn-Ginibre inequality [11] in that it holds for arbitrary external field, though the Hamiltonian is restricted to pair interactions. We exhibit interrelationships among these inequalities, deriving the Lebowitz correlation inequality from the Ellis-Monroe inequality in the same way the second Griffiths inequality may be derived from the Ginibre inequality. The G.H.S. inequality for concavity of magnetization is a corollary of the Lebowitz correlation inequality, as is an inequality which at zero external field shows the fourth Ursell function u_4 is negative. These basic results are all proved in Section 2. In Section 3 we comment on the restrictions in the hypotheses of the theorems proved in Section 2 and mention various generalizations. The final section is devoted to an inequality for change of single-spin measure, which will be useful in our later analysis of low-temperature cooperative phenomena. Combining this inequality with a result of Griffiths [18], we compare the spin expectations of a continuous-spin ferromagnet whose single-spin measure is absolutely continuous near zero with those of a related model having the same Hamiltonian, whose single-spin measure is concentrated at just two points.

Applications of the inequalities proved in this chapter are given in Chapter V.

Section II.2: Inequalities by Duplicate Variables

We now state and prove the inequalities for ferromagnetic Ising models mentioned in Section 1. The proofs employ the method of duplicate variables. Consider a finite ferromagnetic Ising model (Λ, H, ν). (See the final part of Chapter 1 for notation and definitions.) It is convenient to take

$$\Lambda = \{1, 2, \cdots, N\} ,$$

so that the spin variables are $\sigma_1, \sigma_2, \cdots, \sigma_N$. Construct the doubled system $(\Lambda \vee \Lambda, H \oplus H, \nu)$, where $\Lambda \vee \Lambda$ is the disjoint union of two copies of Λ, the 2N spin variables are $\sigma_1, \sigma_2, \cdots, \sigma_N, \tau_1, \tau_2, \cdots, \tau_N$, and the Hamiltonian $H \oplus H$ is $H(\sigma_1, \sigma_2, \cdots, \sigma_N) + H(\tau_1, \tau_2, \cdots, \tau_N)$. Thus, the doubled system consists of two copies of the original system that don't interact with each other. Define the transformed variables

$$t_i = \frac{1}{\sqrt{2}}(\sigma_i + \tau_i) \qquad q_i = \frac{1}{\sqrt{2}}(\sigma_i - \tau_i) , \qquad i \in \Lambda. \tag{1}$$

Construct also a redoubled system $(\Lambda \vee \Lambda \vee \Lambda \vee \Lambda, H \oplus H \oplus H \oplus H, \nu)$ consisting of four non-interacting copies of the original, with spins $\sigma_1, \cdots, \sigma_N$, $\tau_1, \cdots, \tau_N$, $\sigma_1', \cdots, \sigma_N'$, $\tau_1', \cdots, \tau_N'$, and Hamiltonian $H(\sigma_1, \cdots, \sigma_N) + H(\tau_1, \cdots, \tau_N) + H(\sigma_1', \cdots, \sigma_N') + H(\tau_1', \cdots, \tau_N')$. As before, define

$$\begin{aligned} t_i &= \frac{1}{\sqrt{2}}(\sigma_i + \tau_i) & q_i &= \frac{1}{\sqrt{2}}(\sigma_i - \tau_i) \\ t_i' &= \frac{1}{\sqrt{2}}(\sigma_i' + \tau_i') & q_i' &= \frac{1}{\sqrt{2}}(\sigma_i' - \tau_i') \qquad , \ i \in \Lambda . \end{aligned} \tag{2}$$

Now set

$$\begin{aligned} \alpha_i &= \frac{1}{\sqrt{2}}(t_i + t_i') & \beta_i &= \frac{1}{\sqrt{2}}(t_i - t_i') \\ \gamma_i &= \frac{1}{\sqrt{2}}(q_i' + q_i) & \delta_i &= \frac{1}{\sqrt{2}}(q_i' - q_i) \qquad , \ i \in \Lambda . \end{aligned} \tag{3}$$

Note the reversal of primes between α, β and γ, δ.

With this notation we have the following theorems:

<u>Theorem 1</u>: (First Griffiths Inequality) Let $A \in \mathcal{J}_0(\Lambda)$ be a family of sites in a finite ferromagnetic Ising model (Λ, H, ν) with Hamiltonian

$$H = -\sum_{K \in \mathcal{J}_0(\Lambda)} J_K \sigma_K \qquad , \quad J_K \geq 0,$$

and arbitrary (symmetric) single-spin measure $\nu \in \mathcal{N}_d$. Then

$$\langle \sigma_A \rangle \geq 0. \tag{4}$$

<u>Theorem 2</u>: (Ginibre Inequality) Let $A, B \in \mathcal{J}_0(\Lambda)$ be families of sites in a finite ferromagnetic Ising model with Hamiltonian

$$H = -\sum_{K \in \mathcal{J}_0(\Lambda)} J_K \sigma_K \quad , \qquad J_K \geq 0,$$

and arbitrary (symmetric) single-spin measure $\nu \in \mathcal{N}_d$. Then

$$\langle q_A t_B \rangle \geq 0. \tag{5}$$

<u>Corollary 3</u>: (Second Griffiths Inequality) Let $A, B \in \mathcal{J}_0(\Lambda)$ be families of sites in the model of Theorem 2. Then

$$\frac{\partial}{\partial J_B} \langle \sigma_A \rangle \equiv \langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle \geq 0. \tag{6}$$

<u>Theorem 4</u>: (Percus Inequality) Let $A \in \mathcal{J}_0(\Lambda)$ be a family of sites in a finite ferromagnetic Ising model (Λ, H, ν) with pair Hamiltonian

$$H = -\sum_{i \leq j} J_{ij} \sigma_i \sigma_j - \sum_i h_i \sigma_i, \quad J_{ij} \geq 0 \qquad \text{and } h_i \text{ \underline{arbitrary}},$$

and arbitrary (symmetric) single-spin measure $\nu \in \mathcal{N}_2$. Then

$$\langle q_A \rangle \geq 0 \tag{7}$$

Corollary 5: Let i,j be sites in the model of Theorem 4. Then

$$\frac{\partial}{\partial h_j}\langle\sigma_i\rangle \equiv \langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle \geq 0. \tag{8}$$

Theorem 6: (Ellis-Monroe Inequality) Let $A,B,C,D \in \mathcal{Y}_0(\Lambda)$ be families of sites in a finite ferromagnetic Ising model (Λ, H, ν) with pair Hamiltonian

$$H = -\sum_{i\leq j} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i \,, \qquad\qquad J_{ij}\geq 0,\; h_i \geq 0$$

and single-spin measure either discrete and of the form

$$d\nu(\sigma) = \frac{1}{\ell+1}\sum_{j=0}^{\ell}\delta(-\ell+2j+\sigma)$$

(spin $\frac{\ell}{2}$), or continuous and of the form

$$d\nu(\sigma) = \exp(-P(\sigma))d\sigma \Big/ \int_{\mathbb{R}} \exp(-P(s))ds \quad , \tag{9}$$

where P is an even polynomial whose leading coefficient is positive, whose quadratic and constant coefficients are arbitrary, and whose remaining coefficients are nonnegative. (Situations where coefficients of P other than the quadratic may be negative are discussed in Appendix A.) Then

$$\langle\alpha_A\,\beta_B\,\gamma_C\,\delta_D\rangle \geq 0. \tag{10}$$

Corollary 7: (Lebowitz Correlation Inequality) Let $A,B \in \mathcal{Y}_0(\Lambda)$ be families of sites in the model of Theorem 6. Then

$$\langle t_A t_B\rangle - \langle t_A\rangle\langle t_B\rangle \geq 0 \tag{11a}$$

$$\langle q_A q_B\rangle - \langle q_A\rangle\langle q_B\rangle \geq 0 \tag{11b}$$

$$\langle t_A q_B\rangle - \langle t_A\rangle\langle q_B\rangle \leq 0. \tag{11c}$$

Corollary 8: (Griffiths-Hurst-Sherman Inequality) Let i,j,k be sites in the model of Theorem 6. Then

$$\frac{\partial^2}{\partial h_j \partial h_k}\langle\sigma_i\rangle \equiv \langle\sigma_i\sigma_j\sigma_k\rangle - \langle\sigma_i\rangle\langle\sigma_j\sigma_k\rangle - \langle\sigma_j\rangle\langle\sigma_i\sigma_k\rangle - \langle\sigma_k\rangle\langle\sigma_i\sigma_j\rangle + 2\langle\sigma_i\rangle\langle\sigma_j\rangle\langle\sigma_k\rangle \leq 0\,. \tag{12}$$

Corollary 9: Let i,j,k,l be sites in the model of Theorem 6. Then

$$\langle\sigma_i\sigma_j\sigma_k\sigma_\ell\rangle - \langle\sigma_i\sigma_j\rangle\langle\sigma_k\sigma_\ell\rangle - \langle\sigma_i\sigma_k\rangle\langle\sigma_j\sigma_\ell\rangle - \langle\sigma_i\sigma_\ell\rangle\langle\sigma_j\sigma_k\rangle + 2\langle\sigma_i\rangle\langle\sigma_j\rangle\langle\sigma_k\rangle\langle\sigma_\ell\rangle \leq 0. \tag{13}$$

The proofs of Theorems 1,2, and 6 all proceed similarly, by reduction to the case of a model with a single site and zero external field. The inverse temperature β is inessential and we set it to one. We must show that a thermal expectation

$$\langle f\rangle = \int f e^{-H} \prod d\nu \Big/ \int e^{-H} \prod d\nu$$

is nonnegative. The normalization factor (partition function) in the denominator is positive, so we ignore it. We first verify that in the transformed variables the Hamiltonian is a polynomial with nonpositive coefficients. Expanding exp(-H) in its Taylor series, we obtain a sum with nonnegative coefficients of integrals of products of the transformed variables against the product of the single-spin measures. Since each integral factors over the sites, it suffices to show that for a single site the integral of any product of the transformed variables is nonnegative; that is, that the theorem holds for one-site models with zero external field. This is what we do. In the proof of Theorem 4 the reduction cannot proceed quite as far, but essentially the same method prevails. This reduction procedure makes it clear that in all the results of this section we could allow a different single-spin measure at each site, though such models are not commonly studied. Corollary 7 follows from Theorem 6 just as Corollary 3 follows from Theorem 2. Corollary 5 and Corollaries 8,9 are important special cases of Theorem 4 and Corollary 7.

Proofs:

Theorem 1 (Prf):

We want to show

$$\int_{\mathbb{R}^N} \sigma_A \exp\Big(\sum_{K\in\mathcal{G}_0(\Lambda)} J_K \sigma_K\Big) d\nu(\sigma_1)\cdots d\nu(\sigma_N) \geq 0 . \tag{14}$$

By expanding the exponential in its Taylor series and factoring the integrals over the sites as described in the previous paragraph, we reduce the problem to showing

$$\int_{\mathbb{R}} \sigma^n \, d\nu(\sigma) \geq 0 \qquad \forall\, n. \tag{15}$$

By the symmetry of ν this vanishes when n is odd, and when n is even the integrand is nonnegative.

QED

Theorem 2 (Prf):

In terms of the transformed variables q and t the Hamiltonian $H(\sigma) + H(\tau)$ is

$$-\sum_{K\in\mathcal{G}_0(\Lambda)} J_K \Big[\Big(\frac{t+q}{\sqrt{2}}\Big)_K + \Big(\frac{t-q}{\sqrt{2}}\Big)_K \Big] . \tag{16}$$

This is a polynomial in the t's and q's with nonpositive coefficients, because when we expand the product $\prod_{k\in K}(t_k - q_k)$ any negative term which appears is cancelled by the corresponding term from the expansion of $\prod_{k\in K}(t_k + q_k)$. Now by expanding the exponential and factoring the integrals over the sites we reduce the problem to showing

$$\int_{\mathbb{R}^2} t^m q^n \, d\nu(\sigma)\, d\nu(\tau) \geq 0 \qquad \forall\ m,n. \tag{17}$$

This vanishes by symmetry unless m and n are both even, in which case the integrand is nonnegative.

QED

Theorem 4 (Prf):

The transformation (1) is orthogonal, so in terms of the transformed variables q and t the Hamiltonian $H(\sigma) + H(\tau)$ is

$$-\sum_{i \leq j} J_{ij}(q_i q_j + t_i t_j) - \sqrt{2} \sum_i h_i t_i \quad . \tag{18}$$

We want to show

$$\int_{\mathbb{R}^N} q_A \exp\Big(\sum_{i\leq j} J_{ij} q_i q_j\Big) \exp\Big(\sum_{i \leq j} J_{ij} t_i t_j + \sqrt{2}\sum_i h_i t_i\Big)\, d\nu(\sigma_1) d\nu(\tau_1) \cdots d\nu(\sigma_N) d\nu(\tau_N) \geq 0. \tag{19}$$

By expanding the first exponential $\exp(\sum_{i\leq j} J_{ij} q_i q_j)$ we see it suffices to show

$$\int_{\mathbb{R}^N} \Big(\prod_{k=1}^{N} [q_k]^{n_k}\Big) \exp\Big(\sum_{i\leq j} J_{ij} t_i t_j + \sqrt{2}\sum_i h_i t_i\Big)\, d\nu(\sigma_1) \cdots d\nu(\tau_N) \geq 0, \tag{20}$$

for all possible exponents n_k. But this integral vanishes by symmetry unless all the n_k are even, in which case the integrand is positive.

QED

Theorem 6 (Prf):

The transformation (3) is orthogonal, so in terms of the transformed variables $\alpha, \beta, \gamma, \delta$ the Hamiltonian $H(\sigma) + H(\tau) + H(\sigma') + H(\tau')$ is

$$-\sum_{i\leq j} J_{ij}(\alpha_i \alpha_j + \beta_i \beta_j + \gamma_i \gamma_j + \delta_i \delta_j) - 2\sum_i h_i \alpha_i \quad . \tag{21}$$

Since this is a polynomial with nonpositive coefficients, by expanding the exponential and factoring the integrals over the sites we reduce the problem to showing

$$\int_{\mathbb{R}^4} \alpha^k \beta^\ell \gamma^m \delta^n \, d\nu(\sigma) d\nu(\tau) d\nu(\sigma') d\nu(\tau') \geq 0 \qquad \forall\, k, \ell, m, n. \tag{22}$$

By symmetry this vanishes unless k, ℓ ,m,n all have the same parity. When this parity is even the integrand is nonnegative, so we restrict our further attention to the case of odd parity. At this point we distinguish between discrete and continuous spins.

In the discrete case it suffices to consider spin ½ spins,

$$d\nu(\sigma) = \tfrac{1}{2}\left(\delta(\sigma+1) + \delta(\sigma-1)\right) d\sigma\,, \tag{23}$$

for since our transformation of variables is linear the Griffiths "analog system" method [18] may be applied to generate the higher-spin results from the spin ½ case. (The analog system method represents a higher spin by a sum of spin ½ spins in a suitably enlarged model.) Because the exponents k, ℓ ,m,n are all odd we may factor out $\alpha\beta\gamma\delta$:

$$\alpha^k \beta^\ell \gamma^m \delta^n = \left[\alpha^{k-1} \beta^{\ell-1} \gamma^{m-1} \delta^{n-1}\right] \alpha\beta\gamma\delta\;. \tag{24}$$

The first factor is nonnegative since it has even exponents. The second factor is also nonnegative; since $\sigma^2 = \tau^2 = \sigma'^2 = \tau'^2$ for spin ½ spins we find

$$\alpha\beta\gamma\delta = \frac{1}{4}(\sigma\tau - \sigma'\tau')^2 \geq 0. \tag{25}$$

In the continuous case our problem is to show

$$\int_{\mathbb{R}^4} \alpha^k \beta^\ell \gamma^m \delta^n \exp(-P(\sigma) - P(\tau) - P(\sigma') - P(\tau'))\, d\sigma\, d\tau\, d\sigma'\, d\tau' \geq 0 \tag{26}$$

for odd k, ℓ, m, n. We claim that when $P(\sigma) + \cdots + P(\tau')$ is expressed in terms of $\alpha, \beta, \gamma, \delta$ it has the special form

$$P(\sigma) + \cdots + P(\tau') = Q(\alpha^2, \beta^2, \gamma^2, \delta^2) - \alpha\beta\gamma\delta \cdot R(\alpha^2, \beta^2, \gamma^2, \delta^2), \tag{27}$$

where Q and R are polynomials with nonnegative coefficients, except possibly for the coefficients of $\alpha^2, \beta^2, \gamma^2, \delta^2$ in Q. Temporarily accepting this claim, and recalling that transformation (3) is orthogonal, the integral (26) becomes

$$\int_{\mathbb{R}^4} \alpha^k \cdots \delta^n \exp[\alpha\beta\gamma\delta \cdot R(\alpha^2, \cdots, \delta^2) - Q(\alpha^2, \cdots, \delta^2)]\, d\alpha\, d\beta\, d\gamma\, d\delta. \tag{28}$$

Replacing α by $-\alpha$ and averaging gives

$$\int_{\mathbb{R}^4} [\alpha^{k-1}\beta^{\ell-1}\gamma^{m-1}\delta^{n-1}][\alpha\beta\gamma\delta \sinh(\alpha\beta\gamma\delta \cdot R(\alpha^2, \cdots, \delta^2))][\exp(-Q(\alpha^2, \cdots, \delta^2))]\, d\alpha \cdots d\delta. \tag{29}$$

The first factor in (29) is nonnegative since it has even exponents; the second is nonnegative because $R(\alpha^2, \cdots, \delta^2) \geq 0$; the third is obviously nonnegative.

It remains to verify claim (27). We need only consider the case of a monomial $P(X) = X^{2p}$. Expanding with the multinomial theorem gives

$$\begin{aligned} \sigma^{2p} + \tau^{2p} + \sigma'^{2p} + \tau'^{2p} &= \left(\frac{\alpha+\beta+\gamma-\delta}{2}\right)^{2p} + \left(\frac{\alpha+\beta-\gamma+\delta}{2}\right)^{2p} + \left(\frac{\alpha-\beta+\gamma+\delta}{2}\right)^{2p} + \left(\frac{\alpha-\beta-\gamma-\delta}{2}\right)^{2p} \\ &= 2^{-2p} \sum_{a+b+c+d=2p} \frac{(2p)!}{a!b!c!d!} \left[(-1)^d + (-1)^c + (-1)^b + (-1)^{d+c+b}\right] \alpha^a \beta^b \gamma^c \delta^d. \end{aligned} \tag{30}$$

The coefficient of $\alpha^a\beta^b\gamma^c\delta^d$ vanishes unless a,b,c,d all have the same parity; it is positive when this parity is even; and, it is negative when the parity is odd. This observation immediately yields claim (27).

QED

Corollary 3 (Prf):

We want to show

$$\langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle \geq 0.$$

Using the doubled system we have

$$\begin{aligned} \langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle &= \langle \sigma_A \sigma_B - \sigma_A \tau_B \rangle \\ &= \left\langle \left(\tfrac{t+q}{\sqrt{2}}\right)_A \left[\left(\tfrac{t+q}{\sqrt{2}}\right)_B - \left(\tfrac{t-q}{\sqrt{2}}\right)_B \right] \right\rangle . \end{aligned} \tag{31}$$

This is the expectation of a polynomial in the q's and t's which may be shown to have nonnegative coefficients just as (16) was shown to have nonpositive coefficients. By Theorem 2 this expectation is nonnegative.

QED

Corollary 5 (Prf):

Corollary 5 is a special case of Theorem 4:

$$0 \leq \langle q_i q_j \rangle = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \; . \tag{32}$$

QED

Corollary 7 (Prf):

We want to show

$$\begin{aligned} &\langle t_A t_B \rangle - \langle t_A \rangle \langle t_B \rangle \geq 0 \\ &\langle q_A q_B \rangle - \langle q_A \rangle \langle q_B \rangle \geq 0 \\ &\langle t_A \rangle \langle q_B \rangle - \langle t_A q_B \rangle \geq 0 . \end{aligned}$$

Using the redoubled system we have

$$\langle t_A t_B \rangle - \langle t_A \rangle \langle t_B \rangle = \langle t_A t_B - t_A t'_B \rangle = \left\langle \left(\tfrac{\alpha+\beta}{\sqrt{2}}\right)_A \left[\left(\tfrac{\alpha+\beta}{\sqrt{2}}\right)_B - \left(\tfrac{\alpha-\beta}{\sqrt{2}}\right)_B \right] \right\rangle \tag{33a}$$

$$\langle q_A q_B\rangle - \langle q_A\rangle\langle q_B\rangle = \langle q_A q_B' - q_A' q_B\rangle = \left\langle \left(\frac{\gamma+\delta}{\sqrt{2}}\right)_A \left[\left(\frac{\gamma+\delta}{\sqrt{2}}\right)_B - \left(\frac{\gamma-\delta}{\sqrt{2}}\right)_B\right]\right\rangle \tag{33b}$$

$$\langle t_A\rangle\langle q_B\rangle - \langle t_A q_B\rangle = \langle t_A q_B' - t_A q_B\rangle = \left\langle \left(\frac{\alpha+\beta}{\sqrt{2}}\right)_A \left[\left(\frac{\gamma+\delta}{\sqrt{2}}\right)_B - \left(\frac{\gamma-\delta}{\sqrt{2}}\right)_B\right]\right\rangle . \tag{33c}$$

In each case the right-hand side is the expectation of a polynomial in $\alpha, \beta, \gamma, \delta$ with nonnegative coefficients. By Theorem 6, these expectations are nonnegative.

QED

Corollary 8 (Prf):

As noted by Lebowitz, Corollary 8 is a special case of Corollary 7 [28]:

$$0 \geq \langle q_i q_j t_k\rangle - \langle q_i q_j\rangle\langle t_k\rangle = \frac{1}{\sqrt{2}} \frac{\partial^2}{\partial h_j \partial h_k} \langle \sigma_i\rangle . \tag{34}$$

QED

Corollary 9 (Prf):

Corollary 9 is obtained by symmetrizing the special case

$$\langle t_i t_j q_k q_\ell\rangle - \langle t_i t_j\rangle\langle q_k q_\ell\rangle \leq 0 \tag{35}$$

of Corollary 7.

QED

Section II.3: Discussion

In this section we discuss the range of validity of the theorems of Section 2. We indicate generalizations where we can, and illustrate by example the role played by various restrictive hypotheses.

Theorem 2.1 states that for any family of sites A in a suitable model,

$$\langle \sigma_A \rangle \geq 0 . \tag{1}$$

The same proof shows that the spins in the product σ_A may be replaced by more general functions. Let $\{F_i : \mathbb{R} \to \mathbb{R},\ i \in \Lambda\}$ be a set of (measurable) functions such that $F_i([0,\infty)) \subset [0,\infty)$ and F_i has definite parity (is either even or odd). Define

$$F_A = \prod_{a \in A} F_a \quad .$$

Then

$$\langle F_A \rangle \geq 0 \quad . \tag{2}$$

Also, note that Theorem 2.1 generalizes easily to ferromagnetic models with vector spins taking values in $\mathbb{R}^n$, provided that the single-spin measure ν is invariant under the n coordinate reflections.

Theorem 2.2 states that for any families of sites A,B in a suitable model,

$$\langle q_A t_B \rangle \geq 0 . \tag{3}$$

As remarked by Nelson [35], the spins in the product $q_A t_B$ may be replaced by more general functions. Let $\{F_i : \mathbb{R} \to \mathbb{R},\ i \in \Lambda\}$ be a set of functions satisfying the restrictions of the preceding paragraph (invariance of $[0,\infty)$; definite parity) and the additional restriction of monotone increase on $[0,\infty)$. Define

$$T_i^F = \frac{1}{\sqrt{2}}[F_i(\sigma_i)+F_i(\tau_i)] \quad , \quad Q_i^F = \frac{1}{\sqrt{2}}[F_i(\sigma_i)-F_i(\tau_i)], \quad i\in\Lambda, \tag{4}$$

and if $K\in\mathfrak{Y}_0(\Lambda)$ set

$$T_K^F = \prod_{k\in K} T_k^F \quad , \quad Q_K^F = \prod_{k\in K} Q_k^F \quad , \quad i\in\Lambda .$$

Then

$$\langle Q_A^F T_B^F\rangle \geq 0 \ , \tag{5}$$

which has the immediate corollary

$$\langle F_A F_B\rangle - \langle F_A\rangle\langle F_B\rangle \geq 0. \tag{6}$$

We state this as a proposition:

Proposition 1 (Nelson): Let (Λ, H, ν) be an Ising ferromagnet with Hamiltonian

$$H = -\sum_{K\in\mathfrak{Y}_0(\Lambda)} J_K \sigma_K \quad , \quad J_K \geq 0 \ ,$$

and arbitrary (symmetric) single-spin measure ν . Let $\{F_i:\mathbb{R}\to\mathbb{R}, i\in\Lambda\}$, $\{G_i:\mathbb{R}\to\mathbb{R}, i\in\Lambda\}$ be (exponentially bounded measurable) functions such that each F_i, G_i has definite parity, leaves the interval $[0,\infty)$ invariant, and is monotone increasing there. Then

$$\langle \prod_\Lambda F_i(\sigma_i) \cdot \prod_\Lambda G_i(\sigma_i)\rangle - \langle \prod_\Lambda F_i(\sigma_i)\rangle\langle \prod_\Lambda G_i(\sigma_i)\rangle \geq 0. \tag{7}$$

This extension of the second Griffiths inequality will be useful in the construction in Chapter V of the infinite-volume limit by virtue of its monotonicity corollary,

Corollary 2: Let (Λ, H, ν) be an Ising ferromagnet with Hamiltonian

$$H = -\sum_{K\in\mathcal{Y}_0(\Lambda)} J_K \sigma_K \qquad , \quad J_K \geq 0, \tag{8}$$

let $\Lambda' \subset \Lambda$, and let $(\Lambda', H_{\Lambda'}, \nu)$ be the Ising ferromagnet with Hamiltonian

$$H_{\Lambda'} = -\sum_{K'\in\mathcal{Y}_0(\Lambda')} J_{K'} \sigma_{K'} \qquad , \quad J_{K'} \geq 0 \tag{9}$$

(same J_K as in (8); the sum is just restricted to families in $\mathcal{Y}_0(\Lambda')$). If $\{F_i : \mathbb{R} \to \mathbb{R}, i \in \Lambda'\}$ is a set of functions obeying the hypothesis of Proposition 1, then

$$\langle \prod_{\Lambda'} F_i(\sigma_i); H_{\Lambda'} \rangle \leq \langle \prod_{\Lambda'} F_i(\sigma_i); H \rangle \; . \tag{10}$$

In particular,

$$\langle \sigma_A ; H_{\Lambda'} \rangle \leq \langle \sigma_A ; H \rangle \qquad \forall \; A \in \mathcal{Y}_0(\Lambda'). \tag{11}$$

Proof:

By Proposition 1,

$$\frac{\partial}{\partial J_B} \langle \prod_{\Lambda'} F_i(\sigma_i) \rangle = \langle \sigma_B \cdot \prod_{\Lambda'} F_i \rangle - \langle \sigma_B \rangle \langle \prod_{\Lambda'} F_i \rangle \geq 0 \qquad \forall \; B \in \mathcal{Y}_0(\Lambda).$$

Thus, if we increase from zero to their final values all coupling constants J_K appearing in (8) but not (9), $\langle \prod_{\Lambda'} F_i(\sigma_i); H_{\Lambda'} \rangle$ must increase to $\langle \prod_{\Lambda'} F_i(\sigma_i); H \rangle$.

QED

Theorem 2.2 and Corollary 2.3 only have been generalized to vector spin models having spins in two (plane rotor [12]) and three (Heisenberg ferromagnet

[26]) dimensions.

Theorem 2.4 and Corollary 2.5 generalize to products of functions of the type for which Theorem 2.2 and Corollary 2.3 are valid. The hypotheses of Theorem 2.4 and Corollary 2.5 are somewhat unusual in that the single-spin measure is arbitrary while the Hamiltonian is restricted to pair interactions. To see that this restriction is valid, note that Corollary 2.5 fails for a spin ½ model with three sites $\{1,2,3\}$ and Hamiltonian

$$H=-\sigma_1\sigma_2\sigma_3+h\sigma_3\,, \qquad h>0. \tag{12}$$

(We find

$$\langle\sigma_1\rangle=\langle\sigma_2\rangle=0 \tag{13}$$

but

$$\langle\sigma_1\sigma_2\rangle=-\tanh(1)\cdot\tanh(h)<0 \quad .) \tag{14}$$

Theorem 2.6 states that if A,B,C,D are families of sites in a suitable model, then

$$\langle\alpha_A\beta_B\gamma_C\delta_D\rangle\geq 0\,. \tag{15}$$

In contrast to the previous results, the same method of proof does not seem to admit a more general class of functions in the product. (For example, it is easy to see that if $F:\mathbb{R}\to\mathbb{R}$ is any C^2 function such that

$$[F(x_1)-F(x_2)-F(x_3)+F(x_4)]\cdot[x_1-x_2-x_3+x_4]\geq 0 \qquad \forall\ (x_1,\ldots,x_4)\in\mathbb{R}^4,$$

which is a key inequality in the proof of Theorem 2.6, then

$$F(x)=ax\,,\quad a\geq 0\,;$$

that is, F must be of the form already considered.)

The hypothesis of Theorem 2.6 contains restrictions on both the Hamiltonian and the single-spin measure. Example 7.3 of [23] shows that the restriction of the Hamiltonian to pair interactions is needed. However, the constraint on the single-spin measure is more severe than necessary. A certain polynomial $R(\alpha^2, \dots, \delta^2)$ arises naturally from the single-spin polynomial P, and for the method of proof to work $R(\alpha^2, \cdots, \delta^2)$ must be nonnegative. The hypothesis we made ensured this by causing R to have positive coefficients. Clearly, negative coefficients in P, and hence R, are permitted provided the positive coefficients are large enough to ensure $R(\alpha^2, \cdots, \delta^2) \geq 0$. Restrictions on the coefficients of P were studied from this viewpoint in the appendix of [46], reproduced here for convenience as Appendix A. After this work was done, an elegant criterion was obtained by Ellis and Newman [9]. They show that Theorem 2.6 and its corollaries hold provided P is an even continuously differentiable function whose first derivative is convex on $[0,\infty)$. Theorem 2.6 is also valid for single-spin measures obtained by limiting procedures from those explicitly permitted. For example, Lebesgue measure on the interval $[-b,b]$ may be obtained as the limit

$$\frac{1}{2b}\chi_{[-b,b]}(\sigma)\,d\sigma = \lim_{n\to\infty}\left(\exp[-(\sigma/b)^{2n}]\,d\sigma \Big/ \int_{\mathbb{R}} \exp[-(s/b)^{2n}]\,ds\right). \qquad (16)$$

(Here of course $\chi_{[-b,b]}$ is the characteristic function of the interval $[-b,b]$.) However, some constraint on the single-spin measure is necessary. For example, Corollary 2.9 fails for a one-site model with zero external field having single-spin measure

$$a\,\delta(\sigma+1) + (1-2a)\,\delta(\sigma) + a\,\delta(\sigma-1)\ , \qquad 0<a<\tfrac{1}{6}\ , \tag{17}$$

since

$$\langle\sigma^4\rangle - 3\langle\sigma^2\rangle^2 = 2a\,(1-6a)\ . \tag{18}$$

It also fails for a one-site model having single-spin measure $\exp(-P(\sigma))d\sigma$, where

$$P(\sigma) = q\cdot\sigma^2(\sigma-1)^2(\sigma+1)^2 + \sigma^2\log\left(\frac{1}{2a}-1\right) + \log\left(\frac{1}{1-2a}\right),\ 0<a<\tfrac{1}{6} \tag{19}$$

and q is sufficiently large, because as $q\to\infty$ this distribution converges to the preceding one.

Finally, we remark that Theorem 2.6 may be reinterpreted as a theorem about plane rotors. Specifically, we find

Proposition 3: Let $A,B,C,D\in\mathcal{G}_0(\Lambda)$ be families of sites in a ferromagnetic plane rotor (Λ, H, ν) with Hamiltonian

$$H = -\sum_{K\in\mathcal{G}_0(\Lambda)}\left[J_K^x\sigma_K^x + J_K^y\sigma_K^y\right]\ , \qquad J_K^x, J_K^y \geq 0 \tag{20}$$

of degree d and single-spin measure ν on $\mathbb{R}^2$ which is invariant under the two coordinate reflections and is either

(i) concentrated on the unit circle, or

(ii) of the form

$$d\nu(\sigma^x,\sigma^y) = \frac{\exp[-P([\sigma^x]^2,[\sigma^y]^2)]\,d\nu_x(\sigma_x)\,d\nu_y(\sigma_y)}{\int_{\mathbb{R}^2}\exp[-P([s^x]^2,[s^y]^2)]\,d\nu_x(s^x)\,d\nu_y(s^y)}\ , \qquad \nu_x, \nu_y \in \mathcal{M}_d\ , \tag{21}$$

where P is a polynomial all of whose coefficients are nonnegative, except for those of $(\sigma^x)^2$, $(\sigma^y)^2$, which are arbitrary. Construct a duplicate system using primed variables, and define

$$\begin{aligned} \alpha_i &= \frac{1}{\sqrt{2}}(\sigma_i^x + \sigma_i^{x'}) & \beta_i &= \frac{1}{\sqrt{2}}(\sigma_i^x - \sigma_i^{x'}) \\ \gamma_i &= \frac{1}{\sqrt{2}}(\sigma_i^{y'} + \sigma_i^y) & \delta_i &= \frac{1}{\sqrt{2}}(\sigma_i^{y'} - \sigma_i^y)\ , \quad i \in \Lambda . \end{aligned} \tag{22}$$

Then

$$\langle \alpha_A \beta_B \gamma_C \delta_D \rangle \geq 0 . \tag{23}$$

<u>Corollary 4</u>: Let A,B,C,D be families of sites in the plane rotor of Theorem 3. Then

$$\langle \sigma_A^x \sigma_B^x \rangle - \langle \sigma_A^x \rangle \langle \sigma_B^x \rangle \geq 0 \tag{24a}$$

$$\langle \sigma_A^y \sigma_B^y \rangle - \langle \sigma_A^y \rangle \langle \sigma_B^y \rangle \geq 0 \tag{24b}$$

$$\langle \sigma_A^x \sigma_B^y \rangle - \langle \sigma_A^x \rangle \langle \sigma_B^y \rangle \leq 0 . \tag{24c}$$

Related inequalities for vector spin models are given in [12],[26].

Section II.4: Change of Single-Spin Measure

This section, taken mainly from [2], is devoted to an inequality for change of single-spin measure. We may view this inequality as a mathematical rendering of the physical notion that the moments of the Gibbs measure $\langle\sigma_A\rangle$ decrease when the single-spin measure ν becomes more concentrated near the origin. By combining the inequality with a result of Griffiths [18], we compare spin expectations of a continuous-spin ferromagnet whose single-spin measure is absolutely continuous near zero with those of a related model whose single-spin measure is concentrated at just two points. Chapter V contains an application of the inequality to the study of phase transitions.

Theorem 1: Let (Λ, H, ν) be a finite ferromagnetic Ising model, let f be a nonnegative even function monotonically decreasing on $[0,\infty)$ which is identically 1 on $[-c,c]$, $c\geq 0$, and let $\Delta\nu$ be an even measure supported in $[-c,c]$ which is normalized such that

$$\nu_c = \Delta\nu + f\cdot\nu \tag{1}$$

is a probability measure:

$$(\Delta\nu + f\cdot\nu)(\mathbb{R}) = \nu(\mathbb{R}) = 1\,. \tag{2}$$

Then the moments of the Gibbs measure decrease when ν is replaced by ν_c :

$$\langle\sigma_A; H, \nu_c, \beta\rangle \leq \langle\sigma_A; H, \nu, \beta\rangle \qquad \forall\ A\in\mathcal{A}. \tag{3}$$

Proof:

We show that in an Ising model generalized so that the single-spin measures are permitted to be different at different sites, the replacement of ν by ν_c at a single site causes the spin expectations $\langle\sigma_A\rangle$ to decrease. The theorem then follows by successively applying this result to each site in the model.

Consider a ferromagnet on Λ with Hamiltonian H and single-spin measure ν_i at each site $i\in\Lambda$. Select a distinguished site $1\in\Lambda$, at which we assume the single-spin measure is ν. We want to show

$$Z_c^{-1}\int_{\mathbb{R}^\Lambda}\sigma_A e^{-\beta H(\sigma)}d\nu_c(\sigma_1)\prod_{i\neq 1}d\nu_i(\sigma_i)\leq Z^{-1}\int_{\mathbb{R}^\Lambda}\sigma_A e^{-\beta H(\sigma)}d\nu(\sigma_1)\prod_{i\neq 1}d\nu_i(\sigma_i), \quad (4)$$

where of course Z_c and Z are the partition functions

$$Z_c=\int_{\mathbb{R}^\Lambda}e^{-\beta H(\sigma)}d\nu_c(\sigma_1)\prod_{i\neq 1}d\nu_i(\sigma_i)\ \ ;\ \ Z=\int_{\mathbb{R}^\Lambda}e^{-\beta H(\sigma)}d\nu(\sigma_1)\prod_{i\neq 1}d\nu_i(\sigma_i). \quad (5)$$

We rewrite the expectations in (4) to display the dependence on ν, ν_c :

$$\langle\sigma_A;\nu_c\rangle=\int_{[0,\infty)}\langle\sigma_A\rangle_s\, d\rho_c(s) \quad (6)$$

$$\langle\sigma_A;\nu\rangle=\int_{[0,\infty)}\langle\sigma_A\rangle_s\, d\rho(s)\ , \quad (7)$$

where $\langle\sigma_A\rangle_s$, ρ, and ρ_c are defined by

$$\langle\sigma_A\rangle_s=Z(s)^{-1}\int_{\mathbb{R}^{\Lambda-1}}\tfrac{1}{2}\sum_{\sigma_1=\pm s}\sigma_A e^{-\beta H(\sigma)}\prod_{i\neq 1}d\nu_i(\sigma_i) \quad (8)$$

$$Z(s)=\int_{\mathbb{R}^{\Lambda-1}}\tfrac{1}{2}\sum_{\sigma_1=\pm s}e^{-\beta H(\sigma)}\prod_{i\neq 1}d\nu_i(\sigma_i) \quad (9)$$

$$\begin{aligned} d\rho&=Z^{-1}Z(s)d\hat\nu & d\hat\nu(s)&=d\nu(s)-\tfrac{1}{2}\nu(\{0\})\delta(s)\\ d\rho_c&=Z_c^{-1}Z(s)d\hat\nu_c & d\hat\nu_c(s)&=d\nu_c(s)-\tfrac{1}{2}\nu_c(\{0\})\delta(s)\,. \end{aligned} \quad (10)$$

The functions Z(s), $\langle\sigma_A\rangle_s$ and the measures ρ , ρ_c have simple interpretations: Z(s) and $\langle\sigma_A\rangle_s$ are the partition function and expectation of σ_A in the model where the measure ν_1 at site 1 is $\frac{1}{2}[\delta(\sigma+s)+\delta(\sigma-s)]$, and ρ , ρ_c are the density measures of the random variable $|\sigma_1|$ in the models where the single-spin measures at site 1 are ν , ν_c respectively. Note that by the Griffiths inequalities (Theorem 2.1 and Corollary 2.3), in the region of integration $[0,\infty)$ we consider in (6) and (7), both Z(s) and $\langle\sigma_A\rangle_s$ are nonnegative increasing functions of s.

Let μ_1 , μ_2 be finite measures on $[0,\infty)$ of equal total mass, and let $I\subset[0,\infty)$ be a finite interval containing 0 (either open or closed at the right endpoint). Suppose the inequalities

$$\mu_1(E)\leq\mu_2(E) \qquad \forall \text{ measurable } E\subset I \tag{11}$$

$$\mu_2(E)\leq\mu_1(E) \qquad \forall \text{ measurable } E\subset\tilde{I}=[0,\infty)-I \tag{12}$$

hold. Then if $F:[0,\infty)\longrightarrow[0,\infty)$ is a nonnegative monotone increasing function,

$$\int_{[0,\infty)} F(s)\,d\mu_1(s) \geq \int_{[0,\infty)} F(s)\,d\mu_2(s), \tag{13}$$

because

$$\begin{aligned}\int_{[0,\infty)} F\,d(\mu_1-\mu_2) &= \int_I F\,d(\mu_1-\mu_2) + \int_{\tilde{I}} F\,d(\mu_1-\mu_2)\\ &\geq \left[\sup_I F(s)\right]\cdot\left[(\mu_1-\mu_2)(I)\right] + \left[\sup_I F(s)\right]\cdot\left[(\mu_1-\mu_2)(\tilde{I})\right]\\ &= 0\ .\end{aligned} \tag{14}$$

From (13) we conclude immediately that

$$Z=\int_{[0,\infty)} Z(s)\,d\hat{\nu}(s) \geq \int_{[0,\infty)} Z(s)\,d\hat{\nu}_c(s) = Z_c \quad , \tag{15}$$

since by (1) and (10), $\hat{\nu}\leq\hat{\nu}_c$ on $[0,c]$ and $\hat{\nu}\geq\hat{\nu}_c$ on (c,∞) .

Let I be the interval

$$I=\{s\in[0,\infty):f(s)\geq Z_c/Z\} \quad , \tag{16}$$

which contains $[0,c]$ since by (15) $Z_c/Z \leq 1$. We claim that $\rho_c \geq \rho$ on I and $\rho \geq \rho_c$ on $\tilde{I}$. This is easily verified: if $E \subset I$ then

$$\begin{aligned} \rho_c(E) &= \int_E Z(s) Z_c^{-1} \left[f(s) d\hat{\nu}(s) + d\hat{\Delta\nu}(s) \right] \\ &\geq \int_E Z(s) Z^{-1} d\hat{\nu}(s) = \rho(E) \end{aligned} \tag{17}$$

by (16), while if $E \subset \tilde{I}$ then

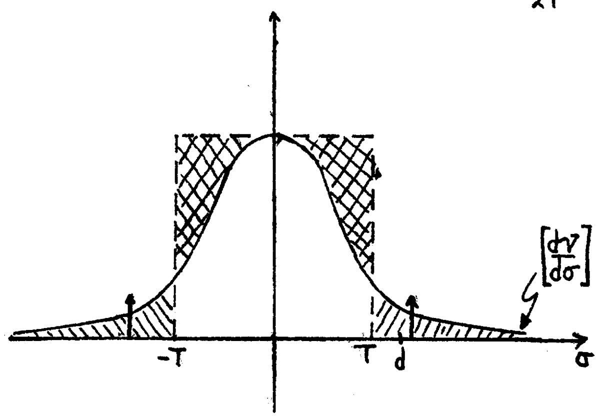
$$\begin{aligned} \rho_c(E) &= \int_E Z(s) Z_c^{-1} f(s) d\hat{\nu}(s) \\ &\leq \int_E Z(s) Z^{-1} d\hat{\nu}(s) = \rho(E) \quad , \end{aligned} \tag{18}$$

again by (16). If we now apply (13) to the integral $\int_{[0,\infty)} \langle \sigma_A \rangle_s d\rho(s)$, we find

$$\langle \sigma_A ; \nu \rangle = \int_{[0,\infty)} \langle \sigma_A \rangle_s d\rho(s) \geq \int_{[0,\infty)} \langle \sigma_A \rangle_s d\rho_c(s) = \langle \sigma_A ; \nu_c \rangle \ . \tag{19}$$

QED

Loosely speaking, Theorem 1 says that if we cut off the single-spin measure by multiplying it by an even nonnegative function which is one on some interval $[-C,C]$ and monotone decreasing on the right half-line, then redistributing the probability mass eliminated by the cutoff in any (symmetric) way in $[-C,C]$ causes the expectations $\langle \sigma_A \rangle$ to decrease. As a special case, suppose the single-spin measure ν of (Λ, H, ν) is absolutely continuous with respect to Lebesgue measure in some interval $[-d,d]$, $d>0$, and that its Radon-Nikodym derivative has finite essential supremum there. Then, as we see in Figure 1, by cutting off ν completely outside some sufficiently small interval $[-T,T] \subset [-d,d]$, and properly redistributing the eliminated probability mass inside $[-T,T]$, we may reshape ν into Lebesgue measure $\frac{d\sigma}{2T}\upharpoonright[-T,T]$ restricted to $[-T,T]$.



Hatched area = Crosshatched area

Figure 1

The largest possible T is given by

$$T = \sup\{t\in[-d,d] : 2t\cdot \operatorname*{ess\,sup}_{\sigma\in[-t,t]}\left[\tfrac{d\nu}{d\sigma}\right]\leq 1\}. \tag{20}$$

For $t\in\mathbb{R}$ let b_t be the two-point measure

$$b_t = \tfrac{1}{2}\left[\delta(\sigma+t)+\delta(\sigma-t)\right]. \tag{21}$$

A result of Griffiths [18] shows that if $(\Lambda, \mathrm{H}, \frac{d\sigma}{2T}\upharpoonright[-T,T])$ is a ferromagnet with arbitrary polynomial Hamiltonian and Lebesgue single-spin measure, then

$$\langle\sigma_A; H, b_{T/2}\rangle \leq \langle\sigma_A; H, \tfrac{d\sigma}{2T}\upharpoonright[-T,T]\rangle \leq \langle\sigma_A; H, b_T\rangle \qquad , A\in\mathcal{G}_o(\Lambda). \tag{22}$$

Thus, with our choice (20) of T,

$$\langle\sigma_A; H, b_{T/2}\rangle \leq \langle\sigma_A; H, \nu\rangle \quad , \qquad A\in\mathcal{G}_o(\Lambda). \tag{23}$$

We state this inequality as a proposition:

Proposition 2: Let (Λ, H, ν) be a finite Ising ferromagnet such that the single-spin measure ν is absolutely continuous with respect to

Lebesgue measure on some interval $[-d, d]$, $d>0$, and has essentially bounded Radon-Nikodym derivative $\left[\frac{d\nu}{d\sigma}\right]$ there. Let $T = \sup\{t\in[-d,d] : 2t \cdot \operatorname{ess\,sup}_{[-t,t]} \left[\frac{d\nu}{d\sigma}\right] \leq 1\}$ and let b_t be the two-point measure defined by (21). Then for all families $A \in \mathfrak{F}_0(\Lambda)$,

$$\langle \sigma_A ; H, b_{T/2}, \beta\rangle \leq \langle \sigma_A ; H, \nu, \beta\rangle. \tag{24}$$

Finally, we remark that Theorem 1 also holds in the case where the spins in the product σ_A are replaced by more general functions of the type considered in Proposition 3.1. In addition, the proof of Theorem 1 goes through with minor modifications to give an analogous result for plane rotors.

Chapter III: Gaussian Inequalities

Section 1: Introduction

In this short chapter, taken largely from [47], we use combinatoric methods to prove an inequality bounding expectations of products of many spins by sums of products of simpler expectations. As a special case of a more general result, we show that the higher moments of the Gibbs measure μ of a finite Ising ferromagnet (Λ,H,b) with spin ½ spins $(b=\frac{1}{2}[\delta(\sigma+1)+\delta(\sigma-1)])$, a pair Hamiltonian, and zero external field are bounded in terms of the covariance of μ :

$$\langle\sigma_A\rangle \leqslant \sum_{\mathcal{P}\in\mathcal{G}} \prod_{\{k,k'\}\in\mathcal{P}} \langle\sigma_k\sigma_{k'}\rangle \ , \qquad A\in\mathcal{J}_0(\Lambda). \quad (1)$$

Here $\mathcal{G}$ is the set of all partitions $\mathcal{P}$ of A into pairs $\{k,k'\}$. Inequality (1) is called a Gaussian inequality because the right-hand side $\sum_{\mathcal{P}\in\mathcal{G}} \prod_{\{k,k'\}\in\mathcal{P}} \langle\sigma_k\sigma_{k'}\rangle$ is the expectation of σ_A with respect to a Gaussian measure on $\mathbb{R}^\Lambda$ having mean zero and the same covariance $\langle\sigma_k\sigma_{k'};H,b,\beta\rangle$ as the Gibbs measure of (Λ,H,b). It is closely related to Corollary II.2.7, and may indeed follow from Theorem II.2.6, though this is not presently known. The Griffiths "analog system" method [18] (described in Section II.2) shows that in addition to spin ½ models, (1) holds for ferromagnets (Λ,H,ν) whose single-spin measure ν may be approximated by spin ½ models, including

$$\nu(\sigma)=\frac{1}{\ell+1}\sum_{j=0}^{\ell}\delta(-\ell+2j+\sigma) \qquad ([18], \text{spin } \tfrac{\ell}{2}) \quad (2a)$$

$$\nu(\sigma)=\frac{d\sigma}{2T}\restriction[-T,T] \qquad ([18], \text{Lebesgue measure on } [-T,T]) \quad (2b)$$

$$\nu(\sigma)=\exp(-a\sigma^4+b\sigma^2)d\sigma\Big/\int_{\mathbb{R}}\exp(-as^4+bs^2)ds \ , \quad a\geq 0 \quad ([43]). \quad (2c)$$

Inequality (1) was discovered in its present form by Newman [36], though a special case was established much earlier by Khintchine [24]. The proof given here is similar in spirit to that of Newman, but conceptually and technically simpler.

In Section 2 we prove the Gaussian (or Khintchine) inequality, comment on the roles played by various hypotheses in it, and mention possible improvements.

Section 2: Proof of Gaussian Inequality

We derive the Gaussian inequality from a more general result. Let us first define admissibility. Fix a finite family A of even cardinality, and use $\sim$ to denote complementation in A. A collection $\mathcal{B}$ of even subfamilies of A is called admissible if and only if every partition of A into pairs is a refinement of some two-element partition $\{B,\tilde{B}\}$ with $B\in\mathcal{B}$. For example, an admissible partition of $A = \{1,2,3,4\}$ is $\mathcal{B} = \{\{1,2\},\{1,3\},\{1,4\}\}$.

Theorem 1: Let A be an even family of sites in a finite ferromagnetic Ising model (Λ, H, ν) with pair Hamiltonian

$$H = -\sum J_{ij}\sigma_i\sigma_j - \sum h_i\sigma_i \,, \qquad J_{ij}, h_i \geq 0 \,,$$

and single-spin measure ν of the form

$$\nu(\sigma) = \frac{1}{\ell+1}\sum_{j=0}^{\ell} \delta(-\ell+2j+\sigma) \tag{1a}$$

$$\nu(\sigma) = \frac{d\sigma}{2T} \upharpoonright [-T,T] \tag{1b}$$

$$\nu(\sigma) = \exp(-a\sigma^4 + b\sigma^2) \Big/ \int_{\mathbb{R}} \exp(-as^4+bs^2)\,ds \,, \quad a>0. \tag{1c}$$

If a collection $\mathcal{B}$ of subfamilies of A is admissible, then

$$\langle\sigma_A\rangle \leq \sum_{B\in\mathcal{B}} \langle\sigma_B\rangle\langle\sigma_{\tilde{B}}\rangle \,. \tag{2}$$

Proof:

By the "analog system" method [18] it suffices to prove Theorem 1 for the simplest measure of the form (1), namely

$$\nu = b = \tfrac{1}{2}[\delta(\sigma+1) + \delta(\sigma-1)] \,.$$

Furthermore, the "ghost spin" method of Griffiths [17], which creates the effect of an external field by coupling to an extra "ghost" spin, permits us to assume the magnetic field h_i is zero. As a final simplification, we reduce to the case when the family A is a set (all members distinct). If $k_1=k_2$ are members of A, let

$$\mathcal{B}_1 = \{B \in \mathcal{B} : \{k_1, k_2\} \subset B \text{ or } \{k_1, k_2\} \subset \tilde{B}\}, \tag{3}$$

in abusive notation. We may assume without loss of generality that $\{k_1, k_2\}$ always lies in B, not $\tilde{B}$. With this assumption, define

$$\hat{\mathcal{B}} = \{B_1 - \{k_1, k_2\} : B_1 \in \mathcal{B}_1\}. \tag{4}$$

Then $\hat{\mathcal{B}}$ is admissible with respect to $\hat{A}=A-\{k_1, k_2\}$. Since $\langle\sigma_A\rangle = \langle\sigma_{\hat{A}}\rangle$ and

$$\sum_{\hat{B}\in\hat{\mathcal{B}}} \langle\sigma_{\hat{B}}\rangle\langle\sigma_{\tilde{\hat{B}}}\rangle = \sum_{B_1\in\mathcal{B}_1} \langle\sigma_{B_1}\rangle\langle\sigma_{\tilde{B}_1}\rangle \leqslant \sum_{B\in\mathcal{B}} \langle\sigma_B\rangle\langle\sigma_{\tilde{B}}\rangle, \tag{5}$$

this reduction procedure allows us to suppose that all members of A are distinct. With these simplifications in hand, we turn to the body of the proof.

We claim that all derivatives with respect to coupling constants J_{ij} of $Z^2(\langle\sigma_A\rangle - \sum_{\mathcal{B}}\langle\sigma_B\rangle\langle\sigma_{\tilde{B}}\rangle)$ are nonpositive when evaluated at zero coupling, and hence throughout the ferromagnetic region $J_{ij} \geqslant 0$. It is convenient to represent a differential operator $D = \frac{\partial^m}{\partial J_{i_1 j_1} \cdots \partial J_{i_m j_m}}$ in the coupling constants by a graph Γ. The vertices of Γ are sites in the model, and for each derivative $\frac{\partial}{\partial J_{ij}}$ appearing in D we place an edge between vertices (sites) i and j. Sites with no incident edges are then suppressed. For example, the differential operator $\frac{\partial^4}{\partial J_{12}\partial J_{12}\partial J_{13}\partial J_{23}}$ would be represented by the graph

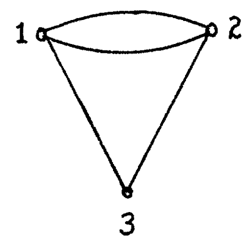


Figure 1.

To simplify the notation, given a family of sites K we write [K] for $\int \sigma_K e^{-\beta H} db(\sigma)$ and $\Gamma \cdot [K]$ for the action of the derivative associated with the graph Γ on $\int \sigma_K e^{-\beta H} db(\sigma)$. Finally, define the ($\mathbb{Z}_2$ reduced) boundary $\partial\Gamma$ of a graph Γ to be the set of all vertices of Γ having an odd number of incident edges.

With this notation our claim becomes

$$\Gamma \cdot \left([\emptyset][A] - \sum_{\mathcal{B}} [B][\tilde{B}]\right)\Big|_{J=0} \leqslant 0 \qquad (6)$$

for all derivative graphs Γ , and is a consequence of the following three statements.

(1) $\Gamma \cdot ([\emptyset][A])\big|_{J=0}$ and $\Gamma \cdot ([B][\tilde{B}])\big|_{J=0}$ both vanish unless $\partial\Gamma = A$.

(2) If $\partial\Gamma = A$ then there exists a subgraph G of Γ and a set $B \in \mathcal{B}$ with $\partial G = B$.

(3) $G \cdot ([\emptyset][A] - [B][\tilde{B}]) \equiv 0$ so $\Gamma \cdot ([\emptyset][A] - [B][\tilde{B}]) \equiv 0$.

Since the remaining terms on the left of inequality (6) are manifestly negative, this cancellation verifies the claim.

Statement (1) is obvious, since we are dealing with spin ½ spins.

Statement (2) is a straightforward induction. Since $\partial\Gamma = A$, given a site $k \in A$ there exists a site $k' \in A$ connected with k by some path γ in Γ .

Upon removing k and k' from A and γ from Γ, we see that by repeating the argument we may produce a partition of A into pairs $\{k,k'\}$ connected in Γ by edge-disjoint paths γ. Since $\mathcal{B}$ is admissible there exists $B \in \mathcal{B}$ which is a union of some of these pairs; for G we just take the paths γ connecting them.

Statement (3) is a simple calculation. Using Δ for symmetric difference we find

$$\begin{aligned} G\cdot([\emptyset][A]) &= \sum_{G_1 \oplus G_2 = G} (G_1\cdot[\emptyset])(G_2\cdot[A]) \\ &= \sum [\partial G_1][(\partial G_2)\Delta A] \\ &= \sum [(\partial G_2)\Delta B][(\partial G_1)\Delta \tilde{B}] \\ &= G\cdot([B][\tilde{B}]) \end{aligned} \tag{7}$$

since $\partial G_1 = (\partial G_2)\Delta B$ and $(\partial G_2)\Delta A = (\partial G_1)\Delta \tilde{B}$.

QED

Corollary 2: Let A be a family of sites with even cardinality in the model of Theorem 1, and let $\mathcal{C}$ be the set of all partitions of A into pairs. Then

$$\langle \sigma_A \rangle \leq \sum_{\mathcal{P}\in\mathcal{C}} \prod_{\{k,k'\}\in\mathcal{P}} \langle \sigma_k \sigma_{k'} \rangle , \qquad A \in \mathcal{Y}_0(\Lambda). \tag{8}$$

Proof:

This corollary is immediate from successive applications of Theorem 1.

QED

Note that for a family of jointly Gaussian random variables with mean zero, Corollary 2 is an equality. In this sense, it is a best-possible result. However, Corollary II.2.9, which states that

$$\langle \sigma_i\sigma_j\sigma_k\sigma_\ell \rangle \leq \langle \sigma_i\sigma_j\rangle\langle\sigma_k\sigma_\ell\rangle + \langle\sigma_i\sigma_k\rangle\langle\sigma_j\sigma_\ell\rangle + \langle\sigma_i\sigma_\ell\rangle\langle\sigma_j\sigma_k\rangle - 2\langle\sigma_i\rangle\langle\sigma_j\rangle\langle\sigma_k\rangle\langle\sigma_\ell\rangle,$$

makes it clear that Corollary 2 may be improved for nonzero external field. Unfortunately, the proofs of Corollary II.2.9 and Corollary 2 are dissimilar. The former uses duplicate variables, while the latter is combinatoric in nature. A combinatoric proof of Corollary II.2.9 might be valuable, and could lead to a new family of correlation inequalities.

Finally, we remark that some restriction on the Hamiltonian in Theorem 1 is necessary, because Corollary 2 fails for the four-site model (Λ, H, b), where

$$\begin{aligned} &\Lambda = \{1,2,3,4\} \\ &H = -J\sigma_1\sigma_2\sigma_3 - h\sigma_4 \qquad J, h \geq 0 \\ &b = \tfrac{1}{2}[\delta(\sigma+1) + \delta(\sigma-1)] \,. \end{aligned} \tag{9}$$

This is because the corollary demands that

$$\langle\sigma_1\sigma_2\sigma_3\sigma_4\rangle \leq \langle\sigma_1\sigma_2\rangle\langle\sigma_3\sigma_4\rangle + \langle\sigma_1\sigma_3\rangle\langle\sigma_2\sigma_4\rangle + \langle\sigma_1\sigma_4\rangle\langle\sigma_2\sigma_3\rangle, \tag{10}$$

but computing explicitly we find

$$\langle\sigma_1\sigma_2\rangle\langle\sigma_3\sigma_4\rangle + \langle\sigma_1\sigma_3\rangle\langle\sigma_2\sigma_4\rangle + \langle\sigma_1\sigma_4\rangle\langle\sigma_2\sigma_3\rangle = 0 \tag{11}$$

and

$$\langle\sigma_1\sigma_2\sigma_3\sigma_4\rangle = \tanh(J)\tanh(h), \tag{12}$$

in contradiction to (10).

Applications of Theorem 1 and Corollary 2 are given in Chapter V.

Chapter IV: Ursell Functions

Section 1: Introduction

In this chapter, taken largely from [47], we use the method of duplicate variables already exploited in Chapter II to study the Ursell functions of finite ferromagnetic Ising models with spin ½ spins and pair interactions. Let us recall the definition of Ursell functions. The Ursell function $u_n(\sigma_1, \cdots, \sigma_n)$ of a family $\{\sigma_i\}$ of n random variables may be defined by means of a generating function as

$$u_n(\sigma_1, \cdots, \sigma_n) = \frac{\partial^n}{\partial\lambda_1 \cdots \partial\lambda_n} \log \mathcal{E}\left(\exp\left[\sum_{i=1}^{n} \lambda_i \sigma_i\right]\right) . \tag{1}$$

Here $\mathcal{E}$ is the expectation integral; we assume the necessary expectations are finite. The Ursell function may be defined recursively by

$$\mathcal{E}(\sigma_1 \sigma_2 \cdots \sigma_n) = \sum_{\mathcal{P} \in \coprod(\{1,\cdots,n\})} \prod_{P \in \mathcal{P}} u_{|P|}(\sigma_{p_a}, \sigma_{p_b}, \cdots) . \tag{2}$$

Here $\coprod(\{1,\cdots,n\})$ is the set of partitions of $\{1,\cdots,n\}$. A set P in a partition $\mathcal{P} \in \coprod(\{1,\cdots,n\})$ has elements p_a, p_b, etc., and $|P|$ denotes the cardinality of P. Finally, $u_n(\sigma_1, \cdots, \sigma_n)$ may be defined explicitly by

$$u_n(\sigma_1, \cdots, \sigma_n) = \sum_{\mathcal{P} \in \coprod(\{1,\cdots,n\})} (-1)^{|\mathcal{P}|-1} (|\mathcal{P}|-1)! \prod_{P \in \mathcal{P}} \mathcal{E}\left(\prod_{p \in P} \sigma_p\right) . \tag{3}$$

Combinatorially, the Ursell functions are related to expectations in the same way that cumulants are related to moments and connected Green's functions (truncated vacuum expectation values) are related to Green's functions (vacuum expectation values). As examples, we have

$$u_1(\sigma_1) = \mathcal{E}(\sigma_1) \tag{4a}$$

$$u_2(\sigma_1, \sigma_2) = \mathcal{E}(\sigma_1 \sigma_2) - \mathcal{E}(\sigma_1)\mathcal{E}(\sigma_2) \tag{4b}$$

$$u_3(\sigma_1, \sigma_2, \sigma_3) = \mathcal{E}(\sigma_1\sigma_2\sigma_3) - \mathcal{E}(\sigma_1)\mathcal{E}(\sigma_2\sigma_3) - \mathcal{E}(\sigma_2)\mathcal{E}(\sigma_1\sigma_3) - \mathcal{E}(\sigma_3)\mathcal{E}(\sigma_1\sigma_2) + 2\mathcal{E}(\sigma_1)\mathcal{E}(\sigma_2)\mathcal{E}(\sigma_3) . \tag{4c}$$

Also, if the family $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ has even symmetry (that is, the expectation of the product of an odd number of σ's is zero), we have

$$u_4(\sigma_1, \cdots, \sigma_4) = \mathcal{E}(\sigma_1\sigma_2\sigma_3\sigma_4) - \mathcal{E}(\sigma_1\sigma_2)\mathcal{E}(\sigma_3\sigma_4) - \mathcal{E}(\sigma_1\sigma_3)\mathcal{E}(\sigma_2\sigma_4) - \mathcal{E}(\sigma_1\sigma_4)\mathcal{E}(\sigma_2\sigma_3). \quad (4d)$$

In Section 2 we describe and investigate representations involving duplicate variables for the Ursell function of a general family of random variables $\{\sigma_i\}$. Let $\{\sigma_i^\alpha\}$, $\alpha \in \{0, 1, \cdots, n-1\}$, be a collection of n independent but identically distributed copies of the family $\{\sigma_i\}$, let ω be a primitive n^{th} root of unity, and define

$$s_i = \sum_{\alpha=0}^{n-1} \omega^\alpha \sigma_i^\alpha \quad . \quad (5)$$

We shall find that

$$u_n(\sigma_1, \cdots, \sigma_n) = \frac{1}{n} \mathcal{E}(s_1 s_2 \cdots s_n) \quad , \quad (6)$$

a result previously obtained in another way by Cartier [4]. Thus we represent an Ursell function as an expectation. In the event that the family $\{\sigma_i\}$ has even symmetry and n is even we can cut the number of copies in half. (Of course, if $\{\sigma_i\}$ has even symmetry and n is odd, $u_n(\sigma_1, \cdots, \sigma_n)$ vanishes.) Defining

$$t_i = \sum_{\alpha=0}^{\frac{n}{2}-1} \omega^\alpha \sigma_i^\alpha , \quad (7)$$

we find the simplified representation

$$u_n(\sigma_1, \cdots, \sigma_n) = \frac{2}{n} \mathcal{E}(t_1 t_2 \cdots t_n) \quad . \quad (8)$$

(The variable t introduced here has no relation to the variable t introduced in transformation (II.2.1)) We conclude the section by demonstrating a method to produce additional representations.

In Section 3 we use the general results of Section 2 to study the even Ursell functions of finite ferromagnetic Ising models with spin ½ spins, pair interactions, and zero external field. It has been conjectured that these Ursell functions obey the inequality

$$(-1)^{\frac{n}{2}+1} u_n(\sigma_{k_1},\cdots,\sigma_{k_n}) \geq 0 \, . \tag{9}$$

We have seen that this conjecture is correct for n=2 and n=4 (Theorem II.2.1; Corollaries II.2.9, III.2.2). In a few very simple models it is known for all n [40,42], essentially by explicit calculation. Using the representation (8) we prove here that in addition to n=2,4 inequality (9) holds for n=6. (We actually establish the stronger result that all the coefficients $\frac{\partial^m}{\partial J_{i_1j_1}\cdots\partial J_{i_mj_m}}(-1)^{\frac{n}{2}+1} Z^{\frac{n}{2}} u_n(\sigma_{k_1},\cdots,\sigma_{k_n})\Big|_{J=0}$ of the Maclaurin expansion of $Z^{n/2}u_n$ in the couplings J_{ij} are nonnegative for n=2,4,and 6.) Other independent proofs that (9) is valid for n=6 recently have been given by Percus [39] and Cartier (unpublished). We use combinatoric methods to derive a reduction formula for Ursell functions with repeated arguments. This allows us to conclude that conjecture (9) holds for arbitrary n provided the spin arguments of the Ursell function are selected from at most seven distinct sites. We finish Section 3 by noting some additional inequalities which follow from the methods we have developed. Although our results are derived explicitly for models with spin ½ spins, by the "analog system" method of [18] they extend immediately to the more general single-spin measures (III.1.2) of the preceding chapter.

Section 4 investigates in more detail the results of Section 3. We establish a graphical notation for the derivatives of $Z^{n/2}u_n$ with respect to couplings, and give formulas for the evaluation of these graphs when n=4,6,and 8. The formulas make clear why our method of proof works for n=2,4,6 but is inadequate for higher n. We present partial results showing that derivatives of $Z^{n/2}u_n$ which are sufficiently simple in a graphical sense have the anticipated sign. We conclude with the asymptotic result that if all couplings J_{ij} are nonzero and the inverse temperature β is sufficiently small or sufficiently large, then the conjectured inequalities hold. This result, however, is not uniform in the order n or the system size.

In Appendix B we describe algorithms for calculating the derivatives of $Z^{n/2}u_n$. We tabulate the results of a computer study using these algorithms on derivatives not controlled by the methods of Sections 3 and 4; they all have the expected sign. The study, however, is indicative but not exhaustive. This is because the long running time for the evaluation of even a moderately complex derivative - on the order of an hour - made a thorough study impractical.

Section IV.2: Representations of Ursell Functions

We describe and analyze representations for the Ursell function of a family of random variables $\{\sigma_i\}_{i\in\{1,\cdots,n\}}$. These representations employ independent but identically distributed copies of the original family. Let $\{\sigma_i^\alpha\}$, $\alpha\in\{0,1,\cdots,c\}$, be (c+1) such independent copies of the family $\{\sigma_i\}$, each copy having the same joint distributions as $\{\sigma_i\}$. Given a set of coefficients $S_{i\alpha}\in\mathbb{C}$ we may define a new family of random variables $\{s_i\}_{i\in\{1,\cdots,n\}}$ by $s_i = \sum_{\alpha=0}^{c} S_{i\alpha}\sigma_i^\alpha$. We shall see that up to a simple factor the family $\{s_i\}$ has the same Ursell function as the original family $\{\sigma_i\}$. By judicious choice of the transformation coefficients $S_{i\alpha}$ we may cause all but the leading term in the Ursell function of the family $\{s_i\}$ to vanish, thereby transforming an Ursell function into an expectation. In the event that the family $\{\sigma_i\}$ has an even symmetry the representation simplifies, the number of copies employed being halved.

To exhibit the proportionality between the Ursell functions of $\{\sigma_i\}$ and $\{s_i\}$ we recall that if a family of random variables may be split into two mutually independent subfamilies, its Ursell function vanishes. (This is immediate from definition (1.1) because the expectation factors.) Thus,

$$
\begin{aligned}
u_n(s_1,\cdots,s_n) &= \sum_{\alpha_1,\cdots,\alpha_n} S_{1\alpha_1}\cdots S_{n\alpha_n}\, u_n(\sigma_1^{\alpha_1},\cdots,\sigma_n^{\alpha_n}) \\
&= \left\{\sum_{\alpha=0}^{c} S_{1\alpha}\cdots S_{n\alpha}\right\}\cdot u_n(\sigma_1,\cdots,\sigma_n),
\end{aligned} \tag{1}
$$

since only those terms for which $\alpha_1=\alpha_2=\cdots=\alpha_n$ survive.

Next we give a specific choice for the transformation coefficients $S_{i\alpha}$ such that $u_n(s_1,\cdots,s_n) = \mathcal{E}(s_1 s_2\cdots s_n)$. Take n copies of the original family $\{\sigma_i\}$, and for $S_{i\alpha}$ choose ω^α, ω being a primitive n^{th} root of unity. Thus we have

$$s_i = \sum_{\alpha=0}^{n-1} \omega^\alpha \sigma_i^\alpha \quad . \tag{2}$$

We claim that $\mathcal{E}(s_1 \cdots s_k) = 0$ unless $k \equiv 0 \bmod(n)$. In establishing this it is convenient to regard the superscripts α as running through the elements of $\mathbb{Z}_n$. Notice that $\mathcal{E}(\sigma_1^{\alpha_1} \cdots \sigma_k^{\alpha_k})$ is unaltered if we subtract (in $\mathbb{Z}_n$) the same constant $\beta \in \mathbb{Z}_n$ from each α_i. Thus,

$$\begin{aligned}
\mathcal{E}(s_1 \cdots s_k) &= \sum_{\alpha_1,\cdots,\alpha_k \in \mathbb{Z}_n} \omega^{\alpha_1+\cdots+\alpha_k} \mathcal{E}(\sigma_1^{\alpha_1} \cdots \sigma_k^{\alpha_k}) \\
&= \sum_{\alpha_1,\cdots,\alpha_k \in \mathbb{Z}_n} \omega^{\alpha_1+\cdots+\alpha_k} \mathcal{E}(\sigma_1^{\alpha_1-\beta} \cdots \sigma_k^{\alpha_k-\beta}) \\
&= \sum_{\alpha_1,\cdots,\alpha_k \in \mathbb{Z}_n} \omega^{\alpha_1+\cdots+\alpha_k+k\beta} \mathcal{E}(\sigma_1^{\alpha_1} \cdots \sigma_k^{\alpha_k}) \\
&= \frac{1}{n} \sum_{\alpha_1,\cdots,\alpha_k,\beta \in \mathbb{Z}_n} \omega^{\alpha_1+\cdots+\alpha_k+k\beta} \mathcal{E}(\sigma_1^{\alpha_1} \cdots \sigma_k^{\alpha_k}) \\
&= 0
\end{aligned} \tag{3}$$

unless $k \equiv 0 \bmod(n)$, since $\sum_{\beta=0}^{n-1} \omega^{k\beta} = 0$ unless $k \equiv 0 \bmod(n)$. With this choice of variables we have

$$u_n(\sigma_1, \cdots, \sigma_n) = \frac{1}{n} \mathcal{E}(s_1 s_2 \cdots s_n) \quad . \tag{4}$$

It may happen that the family $\{\sigma_i\}$ has even symmetry; that is, the expectation of any product of an odd number of σ's is zero. In this case a simpler representation involving only $\frac{n}{2}$ copies of the family $\{\sigma_i\}$ is possible. (We take n even since for n odd by symmetry $u_n(\sigma_1, \cdots, \sigma_n) = 0$.) Let

$$t_i = \sum_{\alpha=0}^{\frac{n}{2}-1} \omega^\alpha \sigma_i^\alpha , \tag{5}$$

where again ω is a primitive n^{th} root of unity. To apply the preceding argument to show $\mathcal{E}(t_1 t_2 \cdots t_k)$ vanishes unless $k \equiv 0 \bmod(n)$ we note that the superscripts α essentially may be regarded as elements of $\mathbb{Z}_{n/2}$ because the ambiguity in the definition of $\omega^{\alpha_1+\cdots+\alpha_k}$ is obviated by the even symmetry of the family $\{\sigma_i\}$. Thus with even symmetry we find

$$u_n(\sigma_1,\cdots,\sigma_n) = \frac{2}{n}\,\mathcal{E}(t_1 t_2 \cdots t_n)\,. \tag{6}$$

Finally, we remark that if one chooses $S_{i\alpha} = \omega^{f_i \alpha}$, $f_i \in \mathbb{Z}_n$, only those terms $\prod_{P\in\mathcal{P}} \mathcal{E}(\prod_{i\in P} s_i)$ in the definition (1.3) of $u_n(s_1,\cdots,s_n)$ survive which satisfy the condition $\sum_{i\in P} f_i \equiv 0 \bmod(n)\ \forall\, P \in \mathcal{P}$. By varying the f_i, different representations for $u_n(\sigma_1,\cdots,\sigma_n)$ may be obtained. For example, the representations above have $f_i = 1\ \forall\, i$, and only the leading term survives. On the other hand, with even symmetry by choosing $f_1 = f_2 = 0$ and $f_3 = f_4 = 2$ two terms survive, and we recover the transformation (II.2.1) and the representation (II.2.35) of Chapter II.

Section IV.3: Signs of Ursell Functions for Ising Ferromagnets

We employ the representation (2.6) to analyze the Ursell functions u_n of a finite ferromagnetic Ising model (Λ, H, b) having spin ½ spins

$$b = \tfrac{1}{2}[\delta(\sigma+1) + \delta(\sigma-1)]$$

and pair Hamiltonian

$$H = -\sum_{i,j} J_{ij}\sigma_i\sigma_j \qquad J_{ij} \geq 0$$

with zero external field. Construct for each even n the enlarged model $(\vee_{\alpha=1}^{n/2}\Lambda, \oplus_{\alpha=1}^{n/2} H, b)$ consisting of $\frac{n}{2}$ non-interacting copies of the original model (Λ, H, b): the set of sites $\vee_{\alpha=1}^{n/2}\Lambda$ is just the disjoint union of $n/2$ copies of Λ, and if we denote the spin at site i in the $\alpha^{\underline{th}}$ copy by σ_i^α the Hamiltonian $\oplus_{\alpha=1}^{n/2} H$ is

$$\bigoplus_{\alpha=1}^{n/2} H = H(\sigma_1^1, \cdots, \sigma_N^1) + H(\sigma_1^2, \cdots, \sigma_N^2) + \cdots + H(\sigma_1^{n/2}, \cdots, \sigma_N^{n/2}).$$

Extend the definition (2.5) of the variables t_i by setting

$$t_i^\alpha = \sum_{\beta=0}^{\frac{n}{2}-1} \omega^{\alpha\beta}\sigma_i^\beta \quad , \quad \alpha \in \{1, 3, 5, \cdots, n-1\}. \tag{1}$$

Thus what we called t_i in (2.5) is t_i^1 here. Note that $(t_i^\alpha)^* = t_i^{n-\alpha}$.
For $\alpha \in \{1, 3, 5, \cdots, n-1\}$ and $\beta \in \{0, 1, \cdots, \frac{n}{2}-1\}$ the matrix $\sqrt{\frac{2}{n}}\,\omega^{\alpha\beta}$ is unitary.
Thus,

$$\bigoplus_1^{n/2} H = \sum_{i,j,\beta} J_{ij}\sigma_i^\beta\sigma_j^\beta = \frac{2}{n}\sum_{i,j,\alpha} J_{ij} t_i^\alpha t_j^{n-\alpha}, \tag{2}$$

and in the t-variables the representation (2.6) becomes

$$Z^{\frac{n}{2}} u_n(\sigma_{k_1}, \cdots, \sigma_{k_n}) = \frac{2}{n} \mathrm{Tr}\left(t_{k_1}^1 \cdots t_{k_n}^1 \exp\left[\frac{2}{n}\sum_{i,j,\alpha} J_{ij} t_i^\alpha t_j^{n-\alpha}\right]\right), \tag{3}$$

where we follow customary usage and write $\mathrm{Tr}(\cdot)$ for $\int(\cdot)db$. The derivative of (3) with respect to coupling constants $J_{i_1j_1},\dots,J_{i_mj_m}$ is

$$\frac{\partial^m}{\partial J_{i_1j_1}\cdots\partial J_{i_mj_m}} Z^{\frac{n}{2}} u_n(\sigma_{k_1},\dots,\sigma_{k_n}) =$$

$$= \sum_{\alpha_1,\dots,\alpha_m} \left(\frac{2}{n}\right)^{m+1} \mathrm{Tr}\left(t^1_{k_1}\cdots t^1_{k_n}\cdot t^{\alpha_1}_{i_1} t^{n-\alpha_1}_{j_1}\cdots t^{\alpha_m}_{i_m} t^{n-\alpha_m}_{j_m}\cdot e^{\sum_{i,j,\alpha}\frac{2}{n}J_{ij}t^{\alpha}_i t^{n-\alpha}_j}\right). \quad (4)$$

In order to show that all these derivatives have a certain sign when evaluated at arbitrary $J_{ij} \geq 0$ it suffices to show they all have this sign when the couplings J_{ij} are set to zero, and this is what we do for n=2,4, and 6.

<u>Theorem 1</u>: Let u_n be the Ursell function of a finite Ising ferromagnet (Λ, H, b) with

$$b = \tfrac{1}{2}[\delta(\sigma+1) + \delta(\sigma-1)]$$

$$H = \sum_{i,j} J_{ij}\sigma_i\sigma_j \qquad\qquad J_{ij} \geq 0.$$

Let Z denote the partition function $\int e^{-\beta H}db$ of (Λ, H, b). Then for n=2,4, and 6

$$\frac{\partial^m}{\partial J_{i_1j_1}\cdots\partial J_{i_mj_m}} (-1)^{\frac{n}{2}+1} Z^{\frac{n}{2}} u_n(\sigma_{k_1},\dots,\sigma_{k_n}) \geq 0 \qquad \forall i_1,j_1,\dots,i_m,j_m,k_1,\dots,k_n \quad (5)$$

Moreover, if (Λ, H, b) is connected, the inequality (5) is strict.

<u>Remark</u>: These inequalities, which as they stand involve factors of Z, may be converted to inequalities involving the spins alone by dividing by $Z^{n/2}$.

<u>Proof</u>:

We give the proof only for the case n=6. The case n=4 may be done in a similar way, and the case n=2 is trivial.

We want to show that the sum $\sum_{\alpha_1,\cdots,\alpha_m} \mathrm{Tr}(t^1_{k_1}\cdots t^1_{k_6}\, t^{\alpha_1}_{i_1}\, t^{6-\alpha_1}_{j_1}\cdots t^{\alpha_m}_{i_m}\, t^{6-\alpha_m}_{j_m})$ arising from the evaluation of (4) at J=0 is nonnegative. It is actually true that an individual term is nonnegative: $\mathrm{Tr}(t^1_{k_1}\cdot\cdots\cdot t^{6-\alpha_m}_{j_m}) \geqslant 0$. Since this trace factors over sites, we break it up into a product of traces of the form $\mathrm{Tr}(t^{\gamma_1}\cdot\cdots\cdot t^{\gamma_a})$, with the common site subscript suppressed. By an argument given in Section 2 in connection with the representations (2.2) and (2.6), this trace vanishes unless $\gamma_1+\cdots+\gamma_a \equiv 0 \bmod (6)$. Assume this condition is satisfied at all sites. We claim that the function $t^{\gamma_1}\cdots t^{\gamma_a}$ obeys the inequality

$$(-1)^{|\{i:\gamma_i=3\}|}\cdot t^{\gamma_1}\cdots t^{\gamma_a} \geqslant 0. \tag{6}$$

To see this is true, we note that since $(t^1)^* = t^5$ and $(t^3)^* = t^3$, pairing t^1's with t^5's and t^3's with one another reduces the problem to showing that $(t^1)^6 \geqslant 0$ and $(t^1)^3\, t^3 \leqslant 0$. This may be done by explicit verification of cases. It now follows immediately that the product over the sites of the terms $t^{\gamma_1}\cdots t^{\gamma_a}$ is nonnegative and so has nonnegative trace, because the total number of γ's appearing with value 3 is even.

The strict positivity may be seen in several ways. One simple one is to resurrect $\beta = 1/kT$, which we have set to one to this point. Note that if a finite ferromagnetic Ising model with spin ½ spins is connected (see Chapter I for definition), then for any function of the spins $F(\sigma_1,\cdots,\sigma_N)$

$$\lim_{\beta\to\infty} \langle F\rangle = \tfrac{1}{2}[F(-1,\cdots,-1)+F(1,\cdots,1)]. \tag{7}$$

Thus in such a model, $\lim_{\beta\to\infty} Z^{-3} \frac{\partial^m}{\partial J_{i_1 j_1} \cdots \partial J_{i_m j_m}} Z^3 u_6 = 3^m \cdot 16$. But since all the coefficients in the Maclaurin expansion of $Z^3 u_6$ in the couplings are nonnegative, if the above derivative were zero for $\beta = 1$ it would remain so for all β and, when normalized by Z^3, could not converge to $3^m \cdot 16$ as $\beta\to\infty$.

QED

We remark that by using the "ghost spin" method of Griffiths [17] described in Section III.2, we may extend Theorem 1 to the case of positive (nonuniform) external field, provided that the Ursell functions for nonzero field are modified by dropping all terms involving the expectation of an odd number of spins. (Such terms of course vanish by symmetry when there is no field.) Also, as we noted in Section 1, the "analog system" method permits the extension of Theorem 1 to models with single-spin measure ν of the form

$$\nu(\sigma) = \sum_{j=0}^{\ell} \frac{1}{\ell+1} \delta(-\ell+2j+\sigma) \qquad (\text{spin } \tfrac{\ell}{2}) \tag{8a}$$

$$\nu(\sigma) = \frac{d\sigma}{2T} \upharpoonright [-T,T] \qquad (\text{Lebesgue measure restricted to } [-T,T]) \tag{8b}$$

$$\nu(\sigma) = \exp(-a\sigma^4 + b\sigma^2)d\sigma \Big/ \int_{\mathbb{R}} \exp(-as^4+bs^2)ds , \quad a>0. \tag{8c}$$

Next we state a corollary of this theorem. The corollary extends the theorem to Ursell functions of arbitrary order, provided that at most seven distinct spin sites appear among the arguments, by means of a reduction formula. The reduction formula provides the necessary combinatorics for expressing Ursell functions with repeated arguments in terms of simpler Ursell functions. To state it we need some notation. Let $\{\sigma_i\}_{i\in\{1,\cdots,n\}}$

be a family of n random variables, and let $\mathcal{P}, \mathcal{Q}$ be partitions of $\{1,\cdots,n\}$. Define

$$u_{\mathcal{Q}}(\sigma_1,\cdots,\sigma_n) = \prod_{Q\in\mathcal{Q}} u_{|Q|}(\sigma_{q_a},\sigma_{q_b},\cdots) \quad , \tag{9}$$

where q_a, q_b, etc. are the elements of Q. Define the family $\{\sigma_P\}_{P\in\mathcal{P}}$ of random variables by

$$\sigma_P = \prod_{i\in P} \sigma_i \ . \tag{10}$$

Let $\mathcal{P}\vee\mathcal{Q}$ denote the finest partition coarser than both $\mathcal{P}$ and $\mathcal{Q}$, and let $\mathbb{1}$ be the one-element partition $\{\{1,\cdots,n\}\}$. A simple combinatoric calculation with Möbius functions gives the following lemma.

Lemma 2: Let $\{\sigma_i\}$ be a family of n random variables. Then, with the above notation,

$$u_{|\mathcal{P}|}(\{\sigma_P\}) = \sum_{\mathcal{Q}:\, \mathcal{Q}\vee\mathcal{P}=\mathbb{1}} u_{\mathcal{Q}}(\sigma_1,\cdots,\sigma_n) \, . \tag{11}$$

To avoid interrupting the main flow of argument, we defer the proof of this lemma to the technical appendix following this chapter.

As a special case of Lemma 2 we have

$$u_{n-1}(\sigma_1\cdot\sigma_2,\sigma_3,\cdots,\sigma_n) = u_n(\sigma_1,\cdots,\sigma_n) + \sum_{P<\{1,\cdots,n\}\,\big|\,\substack{1\in P\\ 2\notin P}} u_k(\sigma_{p_1},\cdots,\sigma_{p_k})\, u_\ell(\sigma_{q_1},\cdots,\sigma_{q_\ell}) \, , \tag{12}$$

where $P = \{p_1,\cdots,p_k\}$ and the complement $\tilde{P} = \{q_1,\cdots,q_\ell\}$. If $\sigma_1\cdot\sigma_2$ is independent of the remaining random variables, as is the case when σ_1 and σ_2 are spins from the same site, the left-hand side of (12) is zero and we obtain the reduction

$$u_n(\sigma_1,\cdots,\sigma_n) = -\sum_{P<\{1,\cdots,n\}\,\big|\,\substack{1\in P\\ 2\notin P}} u_k(\sigma_{p_1},\cdots,\sigma_{p_k})\, u_\ell(\sigma_{q_1},\cdots,\sigma_{q_\ell}) \, . \tag{13}$$

We use this reduction to prove

Corollary 3: Let $u_n(\sigma_{k_1},\cdots,\sigma_{k_n})$ be an Ursell function of the model of Theorem 1. If the n spins used as arguments are selected from at most seven different sites, then

$$\frac{\partial^m}{\partial J_{i_1j_1}\cdots\partial J_{i_mj_m}}(-1)^{\frac{n}{2}+1}Z^{\frac{n}{2}}u_n(\sigma_{k_1},\cdots,\sigma_{k_n})\geq 0 \qquad \forall\, i_1,j_1,\cdots,i_m,j_m,k_1,\cdots,k_n. \tag{14}$$

Moreover, if the model is connected the inequality is strict.

Proof:

We use induction on n. By the theorem, (14) is obviously true if $n\leq 6$. If $n>6$, two spins must be selected from the same site, say $k_1=k_2$. By reduction (13)

$$u_n(\sigma_{k_1},\cdots,\sigma_{k_n})=-\sum_{P}u_k(\sigma_{p_1},\cdots,\sigma_{p_k})\,u_\ell(\sigma_{q_1},\cdots,\sigma_{q_\ell})$$

and so

$$(-1)^{\frac{n}{2}+1}Z^{\frac{n}{2}}u_n=\sum_{P}\left[(-1)^{\frac{k}{2}+1}Z^{\frac{k}{2}}u_k\right]\left[(-1)^{\frac{\ell}{2}+1}Z^{\frac{\ell}{2}}u_\ell\right], \tag{15}$$

with notation as above. From (15) the corollary is immediate.

QED

As with Theorem 1, the "ghost spin" method allows immediate extension of Corollary 3 to the case of positive external field provided the Ursell functions are modified by dropping all terms involving the expectation of an odd number of spins.

To conclude this section, we state a general inequality which follows from the methods we have developed here. It includes Theorem 1 as a special case.

Theorem 4: Let $k_1,\cdots,k_m \in \Lambda$ be sites in a finite Ising ferromagnet (Λ, H, ν) with Hamiltonian

$$H = -\sum_{i,j} J_{ij}\sigma_i\sigma_j \qquad\qquad J_{ij} \geq 0$$

and single-spin measure ν of the form

$$\nu(\sigma) = \frac{1}{\ell+1}\sum_{j=0}^{\ell} \delta(-\ell+2j+\sigma)$$

$$\nu(\sigma) = \frac{d\sigma}{2T} \restriction [-T,T]$$

$$\nu'(\sigma) = exp(-a\sigma^4+b\sigma^2)/\int_{\mathbb{R}} exp(-as^4+bs^2)\,ds, \qquad a>0.$$

Define the transformed variables t_k^{α} by (1); then for n=2,4, and 6

$$\left\langle \prod_{i=1}^{m} \left(t_{k_i}^{\alpha_i} \cdot \sum_{\beta=0}^{\frac{n}{2}-1} \omega^{\beta\alpha_i} \right) \right\rangle \geq 0 \quad . \tag{16}$$

As a corollary, we restate this inequality in terms of the original spin variables σ when all the superscripts α_i are one. First we make some preliminary definitions. If A_4 is a set whose cardinality is a multiple of four, let $\coprod_2^e(A_4)$ be the set of all partitions of A_4 into at most two subsets, each of which must have even cardinality. Define $F: \coprod_2^e(A_4) \rightarrow \mathbb{R}$ by

$$F(\mathcal{P}) = (-1)^{|P|/2} \qquad , \mathcal{P} \in \coprod_2^e(A_4), \tag{17}$$

where P is any element of $\mathcal{P}$. If A_6 is a set whose cardinality is a multiple of six, let $\coprod_3^e(A_6)$ be the set of all partitions of A_6 into at most three subsets, each of which must have even cardinality. Define $S: \coprod_3^e(A_6) \rightarrow \mathbb{R}$ by

$$\mathcal{S}(\mathcal{P}) = \begin{cases} 1, & |\mathcal{P}| = 1 \\ 2, & |\mathcal{P}| \geq 2 \;\&\; |P_1| \equiv |P_2| \bmod(6) \\ -1, & |\mathcal{P}| \geq 2 \;\&\; |P_1| \not\equiv |P_2| \bmod(6) \end{cases} \quad , \; \mathcal{P} \in \amalg_3^e(A_6), \tag{18}$$

where P_1, P_2 are any two distinct elements of $\mathcal{P}$. With this notation, we have

<u>Corollary 5</u>: Let A_4, A_6 be families of sites in the model of Theorem 4 with $|A_4| \equiv 0 \bmod(4)$ and $|A_6| \equiv 0 \bmod(6)$. Then, defining F and S by (17) and (18),

$$(-1)^{|A_4|/4} \sum_{\mathcal{P} \in \amalg_2^e(A_4)} F(\mathcal{P}) \prod_{P \in \mathcal{P}} \langle \sigma_P \rangle \geq 0 \tag{19a}$$

$$\sum_{\mathcal{P} \in \amalg_3^e(A_6)} S(\mathcal{P}) \prod_{P \in \mathcal{P}} \langle \sigma_P \rangle \geq 0 . \tag{19b}$$

As usual, the "ghost spin" method may be used to extend these inequalities to the case of positive external field.

Section IV.4: Miscellaneous Results

In this section we describe a graphical notation for the derivatives

$$\frac{\partial^m}{\partial J_{i_1 j_1} \cdots \partial J_{i_m j_m}} Z^{\frac{n}{2}} u_n(\sigma_{k_1}, \cdots, \sigma_{k_n}) \Big|_{J=0} \tag{1}$$

of Ursell functions in Ising models (Λ,**H**,b) with spin ½ spins, pair interactions, and zero external field. We give formulas for the evaluation of these graphs when n=4,6,8. Turning from explicit calculations, we inductively combine Theorem 3.1 with reduction (3.13) to show that derivatives (1) whose graphs are sufficiently simple topologically have the conjectured sign $(-1)^{\frac{n}{2}+1}$. As a consequence, we obtain the asymptotic result that $(-1)^{\frac{n}{2}+1} u_n \geq 0$ if the inverse temperature β is sufficiently small or sufficiently large.

The graphical notation we use for derivatives (1) is a refinement of that introduced in Chapter III. We regard the sites of our Ising model as vertices of a linear graph, and for each $\frac{\partial}{\partial J_{ij}}$ appearing in the derivative we put an edge between sites i and j. This specifies the differential operator. To specify the arguments σ_{k_a} of u_n, introduce n dummy vertices - one for each k_a - and put an edge between each site k_a and its associated dummy vertex. Finally, suppress all vertices not touched by an edge. The resulting graph G is called the graph of the derivative, and the derivative the value [G] of the graph. (This use of square brackets $[\cdot]$ is not related with the notation of Chapter III employing the same brackets.) As an example, the graph of

$$\frac{\partial^2}{(\partial J_{12})^2} Z^2 u_4(\sigma_1, \sigma_1, \sigma_2, \sigma_2) \Big|_{J=0} \tag{2}$$

is

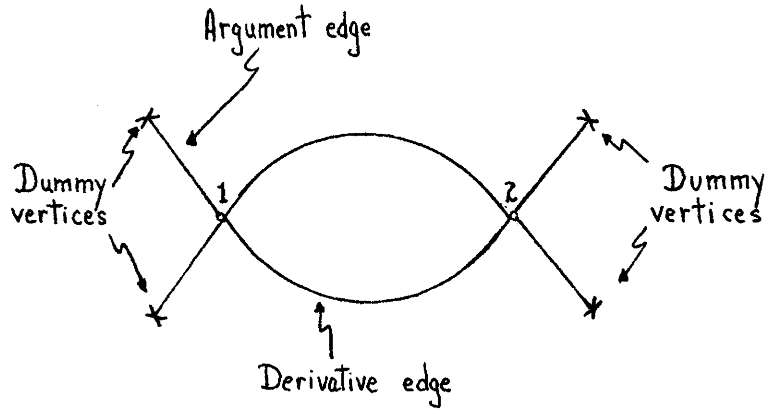


Figure 1

and has value -4.

Recall that (3.4) represents each derivative as a sum:

$$\frac{\partial^m}{\partial J_{i_1j_1}\cdots\partial J_{i_mj_m}} Z^{\frac{n}{2}} u_n(\sigma_{k_1},\cdots,\sigma_{k_n})\Big|_{J=0} = \left(\frac{2}{n}\right)^{m+1} \sum_{\alpha_1,\cdots,\alpha_m} \mathrm{Tr}\left(t^1_{k_1}\cdots t^1_{k_n} t^{\alpha_1}_{i_1} t^{n-\alpha_1}_{j_1}\cdots t^{\alpha_m}_{i_m} t^{n-\alpha_m}_{j_m}\right). \quad (3)$$

We may identify each term

$$\mathrm{Tr}\left(t^1_{k_1}\cdots t^1_{k_n} t^{\alpha_1}_{i_1} t^{n-\alpha_1}_{j_1}\cdots t^{\alpha_m}_{i_m} t^{n-\alpha_m}_{j_m}\right) \quad (4)$$

in the sum with a network of odd $\mathbb{Z}_n$-valued currents on the graph of the associated derivative. The current carried by an edge into a vertex is the superscript of the associated t-variable, and the dummy vertices are regarded as unit sources. For example, the term $\mathrm{Tr}(\, t^1_1 t^1_1 t^1_2 t^1_2 [t^1_1 t^3_2]^2\,)$ appearing in the derivative (2) is represented by the network

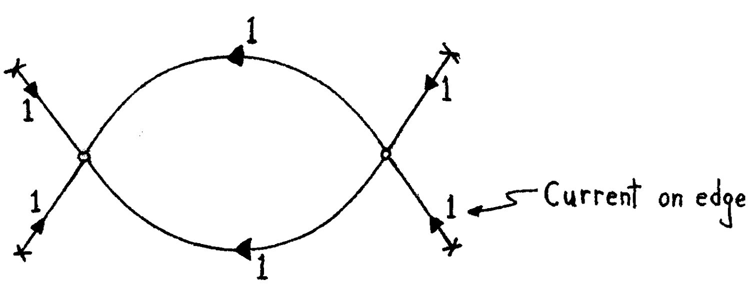


Figure 2

and has value -16. (Subsequently, as in this example, we shall always use the word "network" to mean a graph with currents.) We saw in the proof of Theorem 3.1 that for a term (4) to be nonzero the associated network must obey the Kirchoff current law in $\mathbb{Z}_n$: the sum of the currents at a vertex vanishes. Networks obeying this law will be called nontrivial. Any graph admitting a nontrivial network must have all argument edges in the same connected component and an even number of edges incident at every vertex (except the dummies, which have one each). Such graphs will be called nontrivial. Once a nontrivial graph has been selected, all nontrivial networks on it may be readily generated by means of the well-known method of loop currents. In this method, the currents on the edges of the complement in the graph of a spanning tree are assigned independently, and the remaining currents are calculated from them by applying the Kirchoff current law at each vertex. Thus, the value of a nontrivial graph with λ independent loops is the sum of its $(\frac{n}{2})^{\lambda}$ nontrivial networks, reduced by a factor of $(\frac{2}{n})^{m+1}$.

We turn now to explicit formulas for the evaluation of networks when n=4,6,8.(The case n=2 is trivial and we omit it.) The trace factors over the vertices of the network (sites of the model), so we need only consider a single vertex Tr($t^{\alpha_1} \cdots t^{\alpha_v}$), the common site subscript being suppressed. If each such vertex had the sign sgn($\prod_{i=1}^{v} [\sum_{\beta=0}^{\frac{n}{2}-1} \omega^{\beta\alpha_i}]$) - roughly, if we could somehow factor out $\sum_{\beta=0}^{n/2-1} \omega^{\beta\alpha_i}$ from $t^{\alpha_i} = \sum_{\beta=0}^{n/2-1} \omega^{\beta\alpha_i}\sigma^{\beta}$ - then the whole network would have the conjectured sign $(-1)^{\frac{n}{2}+1}$. This is because each derivative edge engenders a complex conjugate pair of factors in the product over the vertices, while the argument edges give rise to an overall factor with sign $\mathrm{sgn}\left[\left(\sum_{\beta=0}^{n/2-1} \omega^{\beta}\right)^n\right] = (-1)^{\frac{n}{2}+1}$. In the following formulas we shall tabulate Tr($\prod_{i=1}^{v} t^{\alpha_i}$)/sgn($\prod_{i=1}^{v} [\sum_{\beta=0}^{n/2-1} \omega^{\beta\alpha_i}]$); thus, negative values will be suspect.

For n=4, we find that

$$t^{(1)} = 2^{\frac{1}{2}} e^{i\frac{\pi}{4}} \cdot f$$
$$t^{(3)} = 2^{\frac{1}{2}} e^{-i\frac{\pi}{4}} \cdot (f)^3, \tag{5}$$

where f: $\overset{1}{\underset{\alpha=0}{\times}} \{-1,1\} \to \mathbb{C}$ takes for its values the four fourth roots of unity. (Here we have emphasized with parentheses the distinction between the superscripts appearing on the left of (5) and the power appearing on the right.) If $A+3B \equiv 0 \bmod(4)$ (to satisfy the Kirchoff current law) then it follows from (5) that

$$\frac{\mathrm{Tr}\left[(t^1)^A (t^3)^B\right]}{\mathrm{sgn}\left[(1+i)^A (1-i)^B\right]} = e^{-i\frac{\pi}{4}(A-B)}\, \mathrm{Tr}\left[(t^1)^A (t^3)^B\right] = 2^{\frac{1}{2}(A+B)} \geq 0. \tag{6}$$

This formula is simple enough so that we may perform the sum over all networks of any nontrivial fourth-order graph G to find

$$[G] = -2^{\lambda+1} \tag{7}$$

where λ is the cyclomatic number of G (number of independent loops).

If n=6 there are g,h: $\overset{2}{\underset{0}{\times}} \{-1,1\} \to \mathbb{C}$ such that

$$t^{(1)} = 2e^{i\frac{\pi}{3}} \cdot g$$
$$t^{(3)} = (g)^3 + 3 \cdot h$$
$$t^{(5)} = 2e^{-i\frac{\pi}{3}} \cdot (g)^5. \tag{8}$$

The function g runs through the six sixth roots of unity on six of the eight points of $\overset{2}{\underset{0}{\times}} \{-1,1\}$ and vanishes on the remaining two. The function h takes the values ± 1 on these two points and vanishes on the first six. If $A+3B+5C \equiv 0 \bmod(6)$ it follows from (8) that

$$\frac{\mathrm{Tr}\left[(t^1)^A (t^3)^B (t^5)^C\right]}{\mathrm{sgn}\left[(1+\omega+\omega^2)^A\, 1^B (1-\omega^2-\omega)^C\right]} = \left\{ \begin{array}{ll} \frac{3}{4}(1+3^{B-1}) & , A=C=0 \\ \frac{3}{4} \cdot 2^{A+C} & \text{otherwise} \end{array} \right\} \geq 0, \quad \omega = e^{i\frac{\pi}{3}}. \tag{9}$$

When n=8 we find functions $u, v : \overset{3}{\underset{0}{\times}}\{-1,1\} \to \mathbb{C}$ such that

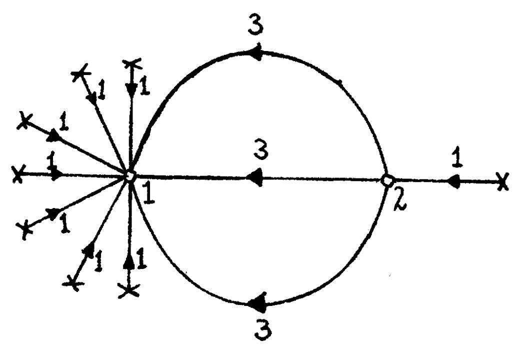
$$
\begin{aligned}
t^{(1)} &= 2^{\frac{3}{2}} e^{i\frac{3\pi}{8}} \left(u \cdot \cos\tfrac{\pi}{8} + v \cdot \sin\tfrac{\pi}{8} \right) \\
t^{(3)} &= 2^{\frac{3}{2}} e^{i\frac{\pi}{8}} \left((u)^3 \sin\tfrac{\pi}{8} - (v)^3 \cos\tfrac{\pi}{8} \right) \\
t^{(5)} &= 2^{\frac{3}{2}} e^{-i\frac{\pi}{8}} \left((u)^5 \sin\tfrac{\pi}{8} - (v)^5 \cos\tfrac{\pi}{8} \right) \\
t^{(7)} &= 2^{\frac{3}{2}} e^{-i\frac{3\pi}{8}} \left((u)^7 \cos\tfrac{\pi}{8} + (v)^7 \sin\tfrac{\pi}{8} \right) .
\end{aligned}
\tag{10}
$$

The functions u and v are supported on complementary halves of $\overset{3}{\underset{0}{\times}}\{-1,1\}$, and each runs through the eight eighth roots of unity on its support. If $A+3B+5C+7D \equiv 0 \bmod (8)$, then it follows from (10) that

$$
\begin{aligned}
&\frac{\mathrm{Tr}\left[(t^1)^A (t^3)^B (t^5)^C (t^7)^D\right]}{\mathrm{sgn}\left[\left(\sum_{\beta=0}^{3}\omega^{\beta}\right)^A \left(\sum_{\beta=0}^{3}\omega^{3\beta}\right)^B \left(\sum_{\beta=0}^{3}\omega^{5\beta}\right)^C \left(\sum_{\beta=0}^{3}\omega^{7\beta}\right)^D\right]} = \\
&\qquad = 2^{\frac{3}{2}(A+B+C+D)-1} \left(\cos\tfrac{\pi}{8}\right)^{A+D} \left(\sin\tfrac{\pi}{8}\right)^{B+C} \left[1+(-1)^{B+C}\left(\tan\tfrac{\pi}{8}\right)^{A+D-(B+C)}\right] .
\end{aligned}
\tag{11}
$$

When $B+C$ is odd and $B+C > A+D$, the right-hand side of (11) is negative. This contrasts with (6) and (9), which were always positive. The source of the trouble in (11) is the minus signs in (10). With formula (11) as a guide, we may easily devise positive eighth-order networks. An example is

$\mathrm{Tr}\left[(t_1^1)^7 (t_2^1)(t_1^3 t_2^5)^3\right] =$



$> 0 .$

Figure 3

Nevertheless, it is known by other reasoning that the derivative from which this network is derived is negative, as one conjectures it should be.

Algorithms for calculating graphs and networks of arbitrary order are presented in Appendix B, together with the results of a computer study making use of them.

We conclude with some partial results showing that derivatives whose graphs are sufficiently simple have the expected sign. We begin by interpreting the reduction (3.13) graphically. Differentiating this identity with respect to couplings, we find that if two argument edges e_1, e_2 in a graph G share a common vertex then

$$[G] = -\sum_{\substack{H_1 \cup H_2 = G \\ H_1 \cap H_2 = \emptyset \\ e_i \in H_i}} [H_1][H_2] \; . \tag{12}$$

By this notation we mean that H_1 and H_2 are the elements of a partition of G into two subgraphs, with edge e_i in subgraph H_i; the sum extends over all such partitions. Making use of this interpretation, we may now prove

<u>Proposition 1</u>: In a spin ½ Ising ferromagnet (Λ, H, b) with pair Hamiltonian and zero external field, if the graph of the derivative $\left.\frac{\partial^m}{\partial J_{i_1 j_1} \cdots \partial J_{i_m j_m}} Z^{\frac{n}{2}} u_n(\sigma_{k_1}, \cdots, \sigma_{k_n})\right|_{J=0}$ is nontrivial and has at most four independent loops in the component of the argument edges (cyclomatic number at most four), then

$$(-1)^{\frac{n}{2}+1} \frac{\partial^m}{\partial J_{i_1 j_1} \cdots \partial J_{i_m j_m}} Z^{\frac{n}{2}} u_n(\sigma_{k_1}, \cdots, \sigma_{k_n}) \Big|_{J=0} \geqslant 0 \quad .$$

<u>Proof</u>:

We use induction on the total number of edges. Since the trace factors over sites, connected components without argument edges merely contribute positive factors to the value, so it suffices to prove the theorem for connected graphs. By Theorem 3.1 we may assume at least 8 argument edges. If any two argument edges share a common vertex, we may use the reduction (12). Also, if any argument edge is incident on a vertex with only one other incident edge, we may simply erase the argument edge and call the other edge an argument edge without changing the value of the graph. There remains only the case in which each argument edge shares a vertex with at least three other edges, all of which must be derivative edges.

We claim that in this situation with at most four independent loops there can be at most six argument edges. We restrict our attention to the subgraph G' of G which contains only the derivative edges; let it have E' edges and V' vertices. The number of independent loops λ' is $\lambda' = E' - V' + 1$. Of course, this number is the same for G and G'. With the restrictions in the case at hand, we see easily that

$$E' \geqslant \frac{1}{2}\left[3n + 2(V'-n)\right] = \frac{n}{2} + V' \quad ; \tag{13}$$

consequently

$$\lambda' \geqslant \frac{n}{2} + 1 \quad , \tag{14}$$

which verifies the claim.

$\overline{\underline{\text{QED}}}$

Combining this proposition with Corollary 3.3, we may say that derivatives of $Z^{\frac{n}{2}}u_n$ have the conjectured sign provided either they are simple in not having argument edges at too many vertices in the associated graph, or in not having graphs which are too connected.

With a little more work, one may show that the inequality in Proposition 1 is actually strict. Thus we have the asymptotic result

Corollary 2: Let $u_n(\sigma_{k_1}, \cdots, \sigma_{k_n})$ be an Ursell function of a finite ferromagnetic Ising model (Λ, H, b) with spin ½ spins, a pair Hamiltonian with zero external field, and all couplings J_{ij} nonzero. Then, if the inverse temperature β is sufficiently small or sufficiently large,

$$(-1)^{\frac{n}{2}+1} u_n(\sigma_{k_1}, \cdots, \sigma_{k_n}) \geqslant 0 \;.$$

Proof:

For small β, expand $Z^{\frac{n}{2}} u_n(\sigma_{k_1}, \cdots, \sigma_{k_n})$ as a power series in βJ_{ij}. We may use the reduction (3.13) to assume the sites $k_1, \cdots, k_n$ are distinct. For distinct sites, the lowest order nonzero graphs are trees, which by Proposition 1 have the claimed sign.

For large β we use (3.7) to conclude that $u_n(\sigma_{k_1}, \cdots, \sigma_{k_n}) \to \frac{d^n}{d\lambda^n} \log \cosh \lambda \Big|_{\lambda=0}$ as $\beta \to \infty$. This derivative has the asserted sign.

QED

Technical Appendix: Proof of Lemma IV.3.2

In this appendix we use the properties of Möbius functions to prove Lemma 3.2. To set the notation and review the ideas involved, we begin with a brief summary of this method.

Let X be a finite partially ordered set, whose order relation $\leqslant$ is reflexive, antisymmetric, and transitive. Let $\mathbb{R}^X$ be the finite-dimensional vector space consisting of all real-valued functions on X. Define the indefinite sum linear transformation $\sum^1 : \mathbb{R}^X \to \mathbb{R}^X$ by

$$\left(\sum^1 f\right)(x) = \sum_{y \leqslant x} f(y) \quad . \tag{1}$$

The kernel (matrix) ζ of the linear map $\sum^1$ is given by

$$\zeta(x,y) = \begin{cases} 1, & y \leqslant x \\ 0 & \text{otherwise} \end{cases} \quad . \tag{2}$$

We claim that $\sum^1$ has determinant one. To see this, note that if we enumerate the elements of X as $x_1, x_2, \cdots$ in such a way that x_i is minimal in $\{x_j : j \geqslant i\}$ then $\zeta(x_i, x_j)$ is a lower triangular matrix with 1's along the diagonal. The inverse Δ of $\sum^1$ is a generalization of the difference operator. The Möbius function of X is the matrix $\mu(x,y)$ of Δ. Since the inverse of a lower triangular matrix with 1's along the diagonal is of the same form, we find

$$\begin{aligned} &\mu(x,x) = 1 \quad \forall\, x \in X \\ &\mu(x,y) = 0 \quad \text{unless} \quad y \leqslant x \quad . \end{aligned} \tag{3}$$

The remaining values of μ may be computed recursively by either of the formulas

$$\mu(x,y) = -\sum_{z \in (y,x]} \mu(x,z) \tag{4a}$$

$$\mu(x,y) = -\sum_{z \in [y,x)} \mu(z,y) \quad , \tag{4b}$$

which follow from the definition of μ as the matrix of Σ^{-1} and its lower triangularity. Note that for fixed $x,y \in X$, $\mu(x,y)$ is completely determined by the structure of the interval $[x,y]$.

We now concentrate on a particular partially ordered set. To enhance clarity, we give very explicit definitions. If F is a finite set, a partition $\mathcal{P}$ of F is a set of disjoint nonempty subsets of F whose union is F. The collection of all partitions of F is denoted by $\coprod(F)$. We partially order $\coprod(F)$ by refinement: $\mathcal{P} \leq \mathcal{Q} \Leftrightarrow \forall P \in \mathcal{P}\, \exists Q \in \mathcal{Q} : P \subset Q$. That is, $\mathcal{P} \leq \mathcal{Q}$ if and only if $\mathcal{P}$ refines $\mathcal{Q}$. With this ordering $\coprod(F)$ becomes a lattice: any pair $\mathcal{P},\mathcal{Q} \in \coprod(F)$ has a least upper bound $\mathcal{P} \vee \mathcal{Q}$ and a greatest lower bound $\mathcal{P} \wedge \mathcal{Q}$. We denote the least element $\{\{i\} : i \in F\}$ of $\coprod(F)$ by 0 and the greatest element $\{F\}$ by $\mathbb{1}$.

The Mobius function of $\coprod(F)$ has reduction and factorization properties which will be useful in the forthcoming proof. Given $\mathcal{P} \in \coprod(F)$ and $\mathcal{Q} \in [\mathcal{P}, \mathbb{1}]$, for each $Q \in \mathcal{Q}$ define the partition $\mathcal{P}_Q \in \coprod(Q)$ to be

$$\mathcal{P}_Q = \{P \in \mathcal{P} : P \subset Q\} \quad . \tag{5}$$

Thus $\mathcal{P}_Q$ is just the restriction of the refinement $\mathcal{P}$ of $\mathcal{Q}$ to the set $Q \in \mathcal{Q}$. Further, define the partition $\mathcal{P}_{\mathcal{Q}} \in \coprod(\mathcal{P})$ to be

$$\mathcal{P}_{\mathcal{Q}} = \{\mathcal{P}_Q : Q \in \mathcal{Q}\} \, . \tag{6}$$

Roughly, $\mathcal{P}_{\mathcal{Q}}$ is the partition of $\mathcal{P}$ obtained from $\mathcal{Q}$ by reducing the sets $P \in \mathcal{P}$ to points in the sets $Q \in \mathcal{Q}$ containing them. The interval $[\mathcal{P}, \mathbb{1}]$ is naturally isomorphic with $\coprod(\mathcal{P})$ under the correspondence $\mathcal{Q} \leftrightarrow \mathcal{P}_{\mathcal{Q}}$. Recalling that $\mu(x,y)$ is completely determined by the structure of the interval $[x,y]$, we find that for $\mathcal{Q}_1, \mathcal{Q}_2 \in [\mathcal{P}, \mathbb{1}]$,

$$\mu(\mathcal{Q}_1, \mathcal{Q}_2) = \mu(\mathcal{P}_{\mathcal{Q}_1}, \mathcal{P}_{\mathcal{Q}_2}) . \tag{7}$$

Here by a common abuse of notation we use the same letter μ for the Möbius functions of two different partially ordered sets (in this case $\amalg(F)$ and $\amalg(\mathcal{P})$), relying on the function arguments to make the set involved clear. This is the reduction mentioned above. Important special cases are

$$\mu(\mathcal{Q}, \mathcal{P}) = \mu(\mathcal{P}_{\mathcal{Q}}, 0) \tag{8}$$

$$\mu(\mathbb{1}, \mathcal{Q}) = \mu(\mathbb{1}, \mathcal{P}_{\mathcal{Q}}) . \tag{9}$$

To obtain the factorization, we note that by induction on formula (4) we may prove

$$\mu(\mathcal{Q}, \mathcal{P}) = \prod_{Q \in \mathcal{Q}} \mu(\mathbb{1}, \mathcal{P}_Q) . \tag{10}$$

Here as usual the $\mathbb{1}$ which appears in the factor $\mu(\mathbb{1}, \mathcal{P}_Q)$ is the greatest element of the lattice $\amalg(Q)$, in which $\mathcal{P}_Q$ lies. (We shall not actually need to compute $\mu(\mathcal{Q}, \mathcal{P})$, which by our reductions is now determined once $\mu(\mathbb{1}, 0)$ is known for sets F of arbitrary cardinality. As an aside, we remark that for the lattice $\amalg(F)$, $\mu(\mathbb{1}, 0) = (-1)^{|F|-1} (|F| - 1)!$.) This concludes our preparatory remarks on Möbius functions. More detail and further references may be found in [5]. We turn now to Lemma 3.2. With notation as in Section 3 we have

<u>Lemma IV.3.2</u>: Let $\{\sigma_i\}$ be a family of n random variables. Then

$$u_{|\mathcal{P}|}(\{\sigma_P\}) = \sum_{\mathcal{Q} : \mathcal{Q} \vee \mathcal{P} = \mathbb{1}} u_{\mathcal{Q}}(\{\sigma_i\}) . \tag{11}$$

Proof:

With the machinery established above, the proof is a straightforward calculation. Given a family of random variables $\{\tau_f\}_{f\in F}$ indexed by a finite set F, for any partition $\mathcal{R}\in\coprod(F)$ define

$$\mathcal{E}_{\mathcal{R}}(\{\tau_f\}) = \prod_{R\in\mathcal{R}} \mathcal{E}(\tau_R) \ , \tag{12}$$

where $\mathcal{E}$ is the expectation integral and as usual $\tau_R = \prod_{f\in R} \tau_f$. Using this notation, it follows from definition (1.2) that

$$u_{|\mathcal{P}|}(\{\sigma_P\}) = \sum_{\mathcal{R}\in\coprod(\mathcal{P})} \mu(\mathbb{1},\mathcal{R})\, \mathcal{E}_{\mathcal{R}}(\{\sigma_P\}) \ . \tag{13}$$

Recalling that $\coprod(\mathcal{P})$ is naturally isomorphic with $[\mathcal{P},\mathbb{1}]\subset\coprod(\{1,2,\cdots,n\})$, we rewrite this as

$$u_{|\mathcal{P}|}(\{\sigma_P\}) = \sum_{\mathcal{S}\in[\mathcal{P},\mathbb{1}]} \mu(\mathbb{1},\mathcal{P}_{\mathcal{S}})\, \mathcal{E}_{\mathcal{P}_{\mathcal{S}}}(\{\sigma_P\}) \ . \tag{14}$$

Tracing through the definitions we find $\mathcal{E}_{\mathcal{P}_{\mathcal{S}}}(\{\sigma_P\}) = \mathcal{E}_{\mathcal{S}}(\{\sigma_i\})$, and by (9) we have $\mu(\mathbb{1},\mathcal{P}_{\mathcal{S}}) = \mu(\mathbb{1},\mathcal{S})$. Thus (14) becomes

$$u_{|\mathcal{P}|}(\{\sigma_P\}) = \sum_{\mathcal{S}\in\coprod(\{1,\cdots,n\})} \mu(\mathbb{1},\mathcal{S})\, \zeta(\mathcal{S},\mathcal{P})\, \mathcal{E}_{\mathcal{S}}(\{\sigma_i\}) \ , \tag{15}$$

where we have inserted the factor $\zeta(\mathcal{S},\mathcal{P})$ and allowed $\mathcal{S}$ to range over all $\coprod(\{1,2,\cdots,n\})$. It follows from the factorization property (10) of μ that

$$\mathcal{E}_{\mathcal{S}}(\{\sigma_i\}) = \sum_{\mathcal{Q}\in\coprod(\{1,\cdots,n\})} \zeta(\mathcal{S},\mathcal{Q})\, u_{\mathcal{Q}}(\{\sigma_i\}) \ . \tag{16}$$

Using this in (15), we find

$$u_{|\mathcal{P}|}(\{\sigma_P\}) = \sum_{\mathcal{S},\mathcal{Q}} \mu(\mathbb{1},\mathcal{S})\, \zeta(\mathcal{S},\mathcal{P})\, \zeta(\mathcal{S},\mathcal{Q})\, u_{\mathcal{Q}}(\{\sigma_i\}) \ ; \tag{17}$$

since

$$\zeta(\mathcal{S},\mathcal{P})\zeta(\mathcal{S},\mathcal{Q}) = \zeta(\mathcal{S},\mathcal{P}\vee\mathcal{Q}) \tag{18}$$

we have

$$\begin{aligned} u_{|\mathcal{P}|}(\{\sigma_P\}) &= \sum_{\mathcal{S},\mathcal{Q}} \mu(\mathbb{1},\mathcal{S})\zeta(\mathcal{S},\mathcal{P}\vee\mathcal{Q})\, u_{\mathcal{Q}}(\{\sigma_i\}) \\ &= \sum_{\mathcal{Q}} \delta_{\mathbf{1},\mathcal{P}\vee\mathcal{Q}}\, u_{\mathcal{Q}}(\{\sigma_i\}) \\ &= \sum_{\mathcal{Q}:\,\mathcal{P}\vee\mathcal{Q}=\mathbb{1}} u_{\mathcal{Q}}(\{\sigma_i\}) \end{aligned} \tag{19}$$

as desired.

$\overline{\underline{\text{QED}}}$

Chapter V: Infinite Ising Models

Section 1: Introduction

In the Ising models we have dealt with so far, the set of sites has been finite. These models are mathematically very regular: the thermal expectations of products of spins (moments of the Gibbs measure) are real analytic in the parameters of the Hamiltonian. They exhibit none of the interesting physical properties, such as phase transitions, which are observed in nature. To create a structure which is mathematically more interesting and physically more realistic, we introduce and analyze in this chapter models with an infinite set of sites. The inequalities proved in the preceding chapters are important tools in the construction and investigation of these infinite models.

In Section 2 we present the basic definitions of infinite Ising ferromagnets. We construct the infinite-volume Gibbs measure (with the free boundary condition) for extremely general models, realizing it as a measure on the spectrum of a certain naturally-arising commutative C^*-algebra. Unfortunately, a price must be paid for this generality: the spectrum of the algebra is slightly larger than the configuration space on which we would like to have the measure. (The configuration space is a dense G_δ in the spectrum.) However, if the (second) moments of the Gibbs measure are finite, as we show they are in most models of interest, the Gibbs measure is actually carried by the configuration space. We conclude with a brief discussion of the equilibrium equations, which give intrinsic meaning to the notion that a measure is an equilibrium state of a model at a specified temperature, and a short description of boundary conditions other than the free boundary condition.

Section 3 concerns the decay of spin expectations $\langle \sigma_A \sigma_B \rangle$ in translation-invariant models when the distance between the two families of sites A,B becomes large. We define the correlation length χ and, after remarking that it is controlled by the decay of the two-point function, point out that it is monotone decreasing in the external field h when the G.H.S. inequality (Corollary II.2.8) holds. We show that the moments of the Gibbs measure are jointly C^{∞} in any finite set of field parameters h_i at sites i. We prove that the two-point function must cluster except possibly for a set of values of the external field h of measure 0, and that this set of exceptional points is reduced to the single point h=0 when the G.H.S. inequality is satisfied. In fact, as we next establish, any connected nearest-neighbor ferromagnet in two or more dimensions with zero external field pair Hamiltonian whose single-spin measure is not δ is long-range ordered at sufficiently low temperature. This is one of our main results. We finish by giving an elementary definition of the infinite-volume transfer matrix $\mathfrak{J}$, and characterizing the cluster properties of an Ising model in terms of spectral properties of its transfer matrix. The definition we give of $\mathfrak{J}$ permits the simple derivation of many interesting results. This definition is well-known in quantum field theory [38] but appears to be less familiar in statistical mechanics.

In Section 4 we study spontaneous magnetization. We find that a model which is long-range ordered is necessarily spontaneously magnetized. Proposition II.4.2 may be used to compare the critical temperature of a model whose single-spin measure is absolutely continuous near zero with the critical temperature of a two-dimensional Ising model with spin $\frac{1}{2}$ spins (which is known by explicit calculation [37]). As a corollary of our main result, we prove that ferromagnetic anisotropic plane rotors on a lattice of dimension at least two are spontaneously magnetized by a method of Kunz [25].

The fifth section treats non-translation-invariant equilibrium states of translation-invariant models. We show that any isotropic nearest-neighbor ferromagnet on a lattice of dimension at least three with single-spin measure $\nu \neq \delta$ has at sufficiently low temperature an equilibrium state with a sharp phase interface. In particular, the equilibrium state is not translation-invariant.

The final section deals with applications of correlation inequalities in models of scalar quantum fields. Guerra, Rosen, and Simon have shown [20] that Euclidean $P(\Phi)_2$ models are well-approximated by ferromagnetic nearest-neighbor Ising models with continuous spins (lattice approximation). Thus, inequalities known for Ising models may be carried over directly to give inequalities for Euclidean scalar quantum fields. These inequalities in field theory serve many of the same purposes as the corresponding inequalities for spin systems (construction of the infinite-volume limit, domination of the n-point function by sums of products of two-point functions, control of the mass by the two-point function and monotonicity of the mass in the external field), and have also been applied to problems unmotivated by statistical mechanics (absence of bound states, absolute bounds on couplings and vertices).

Section 2: The Infinite-Volume Limit

In this section we construct Gibbs measures for Ising models with infinite sets of sites. We also introduce the Dobrushin-Lanford-Ruelle [7,27] equilibrium equation, which gives a non-constructive criterion for a measure to be a Gibbs equilibrium state of an infinite Ising model. The Hamiltonian for an infinite model is a formal power series in the spins which makes sense as a function only when restricted in some way to a finite set of sites. An expectation in the infinite model is defined by restricting to a finite set of sites Λ, calculating the expectation in the restricted model, and using the monotonicity property Corollary II.3.2 to take the limit as Λ becomes arbitrarily large. Using this limiting procedure we may construct the Gibbs measure of the infinite model in quite general circumstances.

We introduce some definitions and notation pertinent to infinite models, generalizing the definitions given at the end of Chapter I for finite models. An infinite ferromagnetic Ising model is a triple $(\mathcal{L}, H, \nu)$ where:

(1) The set of sites $\mathcal{L}$ is an infinite set, which for measure-theoretic reasons we take to be denumerable. We associate with each site $i \in \mathcal{L}$ a spin variable $\sigma_i \in \mathbb{R}$, and the product $\mathbb{R}^{\mathcal{L}} = \prod_{\mathcal{L}} \mathbb{R}$ is called the configuration space.

(2) The Hamiltonian H is a formal power series in the spins of locally finite degree:

$$H = -\sum_{K \in \mathcal{J}_0(\mathcal{L})} J_K \sigma_K \quad , \qquad J_K \geq 0, \tag{1}$$

where the couplings J_K obey the restriction that for all finite $\Lambda \subset \mathcal{L}$

$$|\{K \in \mathcal{F}_0(\Lambda) : J_K \neq 0\}| < \infty . \tag{2}$$

The degree d of H is

$$d = \sup \{|K| : K \in \mathcal{F}_0(\Lambda) \& J_K \neq 0\} . \tag{3}$$

(3) The single-spin measure ν is an even Borel probability measure on $\mathbb{R}$ which decays sufficiently rapidly that if d is the degree of H then

$$\int_{\mathbb{R}} \exp(a|\sigma|^d)\, d\nu(\sigma) < \infty \qquad \forall a \in \mathbb{R} ; \tag{4}$$

that is, $\nu \in \mathcal{N}_d$.

As with finite models, the linear term $-\sum_{i \in \mathcal{L}} J_i \sigma_i$ in H is usually thought of as an external magnetic field $-\sum h_i \sigma_i$. The model is called connected if any pair of sites $i,j \in \mathcal{L}$ is connected by a finite chain $K_1, K_2, \ldots, K_n$ of families with $J_{K_1}, \ldots, J_{K_n} \neq 0$, $i \in K_1$, $j \in K_n$, and for all ℓ, $K_\ell \cap K_{\ell+1} \neq \phi$. It has bounded couplings if

$$\sup_{K \in \mathcal{P}_0(\mathcal{L})} J_K < \infty . \tag{5}$$

Recall that the collection $\mathcal{P}_0(\mathcal{L})$ of finite subsets of $\mathcal{L}$ is directed by containment: $\Lambda_1 \leq \Lambda_2 \Leftrightarrow \Lambda_1 \subset \Lambda_2$. We say that a real-valued net x_Λ indexed by $\mathcal{P}_0(\mathcal{L})$ converges to x as $\Lambda \to \infty$ if $\forall \epsilon > 0\ \exists \Lambda_0$ such that $\Lambda_0 \leq \Lambda \Rightarrow |x - x_\Lambda| < \epsilon$.

The restriction of the Hamiltonian H to a Hamiltonian H_Λ for a finite region $\Lambda \subset \mathcal{L}$ is performed by just keeping the terms in (1) involving only spins in Λ :

$$H_\Lambda = -\sum_{K \in \mathcal{G}_0(\Lambda)} J_K \sigma_K \quad . \tag{6}$$

By (2) this expression is a polynomial, and so is well defined as a function on the restricted space $\mathbb{R}^\Lambda$ of configurations. Definition (6) of H_Λ amounts to setting all spins not in Λ to zero. This choice of spins in $\tilde{\Lambda}$ is called the free boundary condition. Other boundary conditions are briefly discussed in connection with the equilibrium equation at the close of this section. Denote by μ_Λ the Gibbs measure of the finite model $(\Lambda, H_\Lambda, \nu)$. We often use subscripted brackets $\langle\ \rangle_\Lambda$ to indicate expectations with respect to μ_Λ , though sometimes the more explicit notation $\langle\ ; H_\Lambda, \nu, \beta\rangle$ is convenient:

$$\langle f; H_\Lambda, \nu, \beta\rangle = \langle f\rangle_\Lambda = \int_{\mathbb{R}^\Lambda} f\, d\mu_\Lambda = Z_\Lambda^{-1} \int_{\mathbb{R}^\Lambda} f e^{-\beta H_\Lambda}\, d\nu \, . \tag{7}$$

The infinite Ising ferromagnets we have defined have as yet no geometric structure. In contrast, the models of principal physical and mathematical interest are those in which the set of sites $\mathcal{L}$ is an n-dimensional lattice $\mathbb{Z}^n$ and the Hamiltonian has properties somehow connected with the geometrical nature of $\mathbb{Z}^n$. (When we refer to $\mathbb{Z}^n$ as a lattice we mean a lattice in the physical sense of a regularly spaced grid of points. The mathematical connotation is unintended, though of course $\mathbb{Z}^n$ is a lattice in this sense also.) We describe some typical geometrical properties of the Hamiltonian. A Hamiltonian $H = -\sum_{K\in\mathcal{G}_0(\mathcal{L})} J_K\sigma_K$ on a lattice $\mathbb{Z}^n$ is called translation-invariant if

$$J_K = J_{K+i} \qquad\qquad \forall K\in\mathcal{G}_0(\mathcal{L}),\ \forall i\in\mathbb{Z}^n, \tag{8}$$

where of course if $K = \{k_\alpha\}$ then $K + i = \{k_\alpha + i\}$. The range of H is

$$\mathrm{ran}(H) = \sup_{\{K\in\mathcal{G}_0(\mathbb{Z}^n):J_K\neq 0\}} \operatorname{diam} K \quad . \tag{9}$$

(The diameter of a set K in a metric space with metric d is diam $K = \sup_{k_1,k_2\in K} d(k_1,k_2)$; in (9) we employ the usual Euclidean metric on $\mathbb{Z}^n$.) A finite-range interaction has ran(H) $< \infty$. A nearest-neighbor Hamiltonian is one with range 1 and degree 2 in which no quadratic self-interaction terms $J_{ii}\cdot(\sigma_i)^2$ appear. Often nearest-neighbor interactions are assumed to be translation-invariant as well. Thus a (translation-invariant) nearest-neighbor Hamiltonian on $\mathbb{Z}^2$ has the form

$$H = -J_1 \sum_{(i_1,i_2)\in\mathbb{Z}^2} \sigma_{(i_1,i_2)}\,\sigma_{(i_1+1,i_2)} - J_2 \sum_{(i_1,i_2)\in\mathbb{Z}^2} \sigma_{(i_1,i_2)}\sigma_{(i_1,i_2+1)} - h\sum_{i\in\mathbb{Z}^2}\sigma_i \ . \tag{10}$$

Armed with these definitions, we turn to the construction of the infinite-volume Gibbs measure. Let us first suppose that the single-spin measure ν has compact support S. Then we may take the configuration space of the infinite model to be $S^{\mathcal{L}}$, which is compact in the product topology. If $P(\sigma)$ is a polynomial in the spins with positive coefficients, by the monotonicity property (II.3.11) the expectations $\langle P(\sigma)\rangle_\Lambda$ form an increasing net on $\mathcal{P}_0(\mathcal{L})$. This net is bounded above by $\|P\|_\infty = \sup_{\sigma \in S^{\mathcal{L}}} |P(\sigma)|$ and so has a unique limit, the infinite-volume expectation $\langle P(\sigma)\rangle = \lim_{\Lambda \to \infty} \langle P(\sigma)\rangle_\Lambda$. Invoking the Stone-Weierstrass Theorem to show density of the polynomials in $C(S^{\mathcal{L}})$, we extend the infinite-volume expectation to a state on $C(S^{\mathcal{L}})$ by linearity and continuity. By the Riesz Representation Theorem this state is given by integration against a unique Baire probability measure μ , which we call the (infinite-volume) Gibbs measure of the model $(\mathcal{L}, H, \nu)$ (with free boundary conditions).

If we try to duplicate the construction of the previous paragraph for general single-spin measures ν we meet two problems: the polynomials $P(\sigma)$ on the configuration space $\mathbb{R}^{\mathcal{L}}$ are not bounded functions, so that the increasing net $\langle P(\sigma)\rangle_\Lambda$ need not be bounded above, and $\mathbb{R}^{\mathcal{L}}$ is not compact. We partially overcome both these difficulties by replacing the polynomials $P(\sigma)$ with polynomials $P(F_a(\sigma_a), F_b(\sigma_b), \ldots)$ with positive coefficients in bounded continuous functions F_a, $F_b, \ldots$ of the spins which are of definite parity, nonnegative on $[0,\infty)$, and increasing there. Applying the extended monotonicity result (II.3.10), we find $\langle P(F_a(\sigma_a), F_b(\sigma_b), \cdots)\rangle = \lim_{\Lambda \to \infty} \langle P(F_a, F_b, \cdots)\rangle_\Lambda$ exists for such polynomials, and we may extend this infinite-volume thermal expectation by linearity and continuity to a state on the C*-algebra $\mathfrak{A}$ (trivial involution, sup norm) these polynomials generate. The C*-algebra on $\mathbb{R}$ generated by the bounded continuous functions $F: \mathbb{R} \to \mathbb{R}$ of definite parity which are nonnegative on $[0,\infty)$ and increasing there is $C(\ddot{\mathbb{R}})$, the space of bounded continuous functions on $\mathbb{R}$ with limits at $\pm\infty$. ($\ddot{\mathbb{R}}$ is the two-point compactification of $\mathbb{R}$ obtained by including the two endpoints $\pm\infty$.) Thus the algebra $\mathfrak{A}$ on which we have defined the infinite-volume thermal expectation $\langle\ \rangle$ is the tensor product $\otimes_{\mathcal{L}} C(\ddot{\mathbb{R}}) = C(\prod_{\mathcal{L}} \ddot{\mathbb{R}})$, and by the Riesz Representation Theorem this state is given as a measure on the spectrum spec ($\otimes_{\mathcal{L}} C(\ddot{\mathbb{R}})) = \prod_{\mathcal{L}} \ddot{\mathbb{R}}$. We identify this measure as the infinite-volume Gibbs measure μ (with the free boundary condition).

Ideally we would like the Gibbs measure μ to be a measure on the configuration space $\prod_{\mathcal{L}} \mathbb{R}$, but instead we have it on the slightly larger space $\prod_{\mathcal{L}} \ddot{\mathbb{R}}$ in which $\prod_{\mathcal{L}} \mathbb{R}$ is a dense G_δ. We may hope that in reasonable situations the Gibbs measure μ is carried by $\mathbb{R}^{\mathcal{L}}$. Theorem 1 below, which summarizes our arguments to this point, shows when combined with Proposition 2 that this is so.

Theorem 1: Let $(\mathcal{L}, H, \nu)$ be an infinite Ising model. Then there exists a unique state $\langle\ ; H, \nu, \beta\rangle$ on the C*-algebra $\otimes_{\mathcal{L}} C(\ddot{\mathbb{R}})$, and a unique Baire probability measure μ on the spectrum $\prod_{\mathcal{L}} \ddot{\mathbb{R}}$ of $\otimes_{\mathcal{L}} C(\ddot{\mathbb{R}})$, such that $\forall$ finite $\Lambda' \subset \mathcal{L}$, $\forall\, f \in \otimes_{\Lambda'} C(\ddot{\mathbb{R}})$

$$\langle f; H, \nu, \beta\rangle = \int_{\ddot{\mathbb{R}}^{\mathcal{L}}} f\, d\mu = \lim_{\Lambda\to\infty} \langle f; H_\Lambda, \nu, \beta\rangle_\Lambda \quad . \tag{11}$$

The configuration space $\prod_{\mathcal{L}} \mathbb{R}$ is a dense G_δ in $\prod_{\mathcal{L}} \ddot{\mathbb{R}}$. If $\forall\ i \in \mathcal{L}$ $\exists\ F_i \in C(\mathbb{R})$ which is even, monotone increasing and nonnegative on $[0,\infty)$, and which tends to ∞ at $\pm\infty$, such that the net $\langle F_i(\sigma_i); H_\Lambda, \nu, \beta\rangle_\Lambda$

is bounded above, then μ is carried on $\mathbb{R}^{\mathcal{L}}$: $\mu(\mathbb{R}^{\mathcal{L}})=1$.

Proof:

The only part of Theorem 1 we have not yet established is the sufficient condition for μ to be carried on $\mathbb{R}^{\mathcal{L}}$. For $i\in\mathcal{L}$, define $E_i=\{x\in\ddot{\mathbb{R}}^{\mathcal{L}}: x_i\in\mathbb{R}\}$, where x_i is the ith component of $x\in\prod_{\mathcal{L}}\ddot{\mathbb{R}}$. It suffices to show that $\mu(E_i)=1$, because $\mathbb{R}^{\mathcal{L}}=\bigcap_{i\in\mathcal{L}}E_i$, and the intersection of a countable family of sets of measure 1 in a probability space again has measure 1. For $n\in\mathbb{Z}^+$ define F_{in} by

$$F_{in}(t)=\begin{Bmatrix} F_i(t), & |t|\leq n \\ F_i(n), & |t|>n \end{Bmatrix} , \tag{12}$$

and set E_{in} = $\{x\in\prod_{\mathcal{L}}\ddot{\mathbb{R}}: x_i\in\mathbb{R}\ \&\ |x_i|\leq n\}$. If the expectations $\langle F_i(\sigma_i)\rangle_\Lambda$ are bounded above by c, so are the expectations $\langle F_{in}(\sigma_i)\rangle_\Lambda$. Taking $\Lambda\to\infty$,

$$\frac{c}{F_i(n)} \geq \frac{1}{F_i(n)}\int F_{in}(\sigma_i)d\mu \geq 1\cdot\mu(\tilde{E}_{in}) \geq \mu(\tilde{E}_i), \quad \tilde{E}_i=\prod_{\mathcal{L}}\ddot{\mathbb{R}}-E_i .$$

As $n\to\infty$, $c/F_i(n)\to 0$ so that $0\geq\mu(\tilde{E}_i)$ and $\mu(E_i)=1$ as desired.

QED

Of course, when properly formulated all the inequalities proved in Chapters II - IV for finite Ising models also hold in the infinite-volume limit, and so may be used in the analysis of infinite models.

The example we have in mind for the functions F_i in Theorem 1 is $F_i(\sigma_i)=\sigma_i^2$, so that roughly speaking the infinite-volume Gibbs measure μ is carried by the configuration space $\mathbb{R}^{\mathcal{L}}$ if its second moments are finite. Under reasonable geometric assumptions, we show that this is the case by following an idea in [30]. If the single-spin measure is such that Corollary II.2.7

or Corollary III.2.2 holds, finiteness of the second moments implies finiteness of all moments.

Proposition 2: Let $(\mathbb{Z}^n, H, \nu)$ be an Ising ferromagnet with bounded couplings whose Hamiltonian H is finite-range and of finite degree d. Then the second moments of the infinite-volume Gibbs measure are finite:

$$\langle \sigma_i \sigma_j ; H, \nu, \beta \rangle = \lim_{\Lambda \to \infty} \langle \sigma_i \sigma_j ; H_\Lambda, \nu, \beta \rangle_\Lambda < \infty \qquad \forall\, i, j \in \mathbb{Z}^n. \tag{13}$$

Moreover, if H is a pair interaction and the single-spin measure ν is such that Corollary II.2.7 or Corollary III.2.2 holds, all moments $\langle \sigma_A ; H, \nu, \beta \rangle$ are finite.

Proof:

To show $\langle \sigma_i \sigma_j \rangle < \infty$ we need only consider the case i=j. We invoke the technical device of periodic boundary conditions. Let Λ be a large cube $\Lambda = \prod_{\alpha=1}^{n} \{-m, -m+1, \cdots, m\}$, which we identify with the group $\prod_{\alpha=1}^{n} \mathbb{Z}_{2m+1}$. If $H = -\sum J_K \sigma_K$, define

$$J'_K = \sup_{j \in \Lambda} J_{K+j} \qquad , \quad K \in \mathcal{G}_0(\Lambda) \tag{14}$$

$$H'_\Lambda = -\sum_{K \in \mathcal{G}_0(\Lambda)} J'_K \sigma_K \quad , \tag{15}$$

where the translation K + j is performed with respect to the group action in $\Lambda = \prod_{1}^{n} \mathbb{Z}_{2m+1}$. Notice that H'_Λ is translation-invariant with respect to this group. By translation-invariance and the second Griffiths inequality

(Corollary II.2.3),

$$\langle \sigma_i^2 ; H_\Lambda \rangle_\Lambda \leqslant \langle \sigma_i^2 ; H'_\Lambda \rangle_\Lambda = \frac{1}{|\Lambda|} \sum_{j \in \Lambda} \langle \sigma_j^2 ; H'_\Lambda \rangle_\Lambda \quad . \tag{16}$$

Now by the Jensen inequality,

$$\exp\left[\int \left(\sum_\Lambda \sigma_j^2 \right) d\mu'_\Lambda \right] \leqslant \int \exp\left(\sum_\Lambda \sigma_j^2 \right) d\mu'_\Lambda \quad , \tag{17}$$

so that

$$\langle \sigma_i^2 ; H_\Lambda \rangle_\Lambda \leqslant \frac{1}{|\Lambda|} \int \left(\sum_\Lambda \sigma_j^2 \right) d\mu'_\Lambda \leqslant \frac{1}{|\Lambda|} \log \left[\int \exp\left(\sum_\Lambda \sigma_j^2 \right) d\mu'_\Lambda \right] . \tag{18}$$

Since H is finite-range and has bounded couplings, there are Λ -independent constants $J, A \in [0,\infty)$ such that

$$| H'_\Lambda(\sigma) | \leqslant \sum_{j \in \Lambda} (J |\sigma_j|^d + A) \quad . \tag{19}$$

Using this estimate in (18), we find

$$\langle \sigma_i^2 ; H_\Lambda \rangle_\Lambda \leqslant \frac{1}{|\Lambda|} \log \prod_{j \in \Lambda} \left[\frac{\int_{\mathbb{R}} e^{J|\sigma|^d + \sigma^2 + A} \, d\nu(\sigma)}{\int_{\mathbb{R}} e^{-(J|\sigma|^d + A)} \, d\nu(\sigma)} \right] = \log \left[\frac{\int e^{J|\sigma|^d + \sigma^2 + A} \, d\nu(\sigma)}{\int e^{-(J|\sigma|^d + A)} \, d\nu(\sigma)} \right] . \tag{20}$$

Since $\nu \in \mathcal{N}_d$ the right-hand side is a finite constant independent of Λ . Any finite region lies in some large cube , so by monotonicity the first part of the proposition is true.

If Corollary III.2.2 holds, then a bound $\langle \sigma_i \sigma_j \rangle \leqslant C$ immediately gives a bound on the even moments, and hence on all of them. As shown in [13], Corollary II.2.7 may be used in the same way.

QED

We remark that if we require the interaction to be translation-invariant and of finite degree, the same proof shows that long-range forces may be allowed in Proposition 2 provided the couplings J_K obey

$$\sum_{\{K \in \mathcal{F}_0(\mathbb{Z}^n) : i \in K\}} J_K < \infty \tag{21}$$

for some (and hence all) i.

Although Proposition 2 is adequate for our present purposes, one expects that much better results hold. In particular, with the hypotheses we have made one hopes that all moments are finite regardless of any special properties of ν beyond its decay rate. This can be shown if the Hamiltonian is a pair interaction (degree 2) by transcribing to lattice models results of Ruelle derived for systems of classical particles interacting by superstable pair potentials [41]. These results probably generalize to Hamiltonians of degree higher than two, but the technical estimates involved are formidable and a general proof does not seem to be available. Various special cases are amenable to simpler techniques. In Appendix C we use the method of transfer matrices to prove finiteness of the moments if the Hamiltonian is nearest-neighbor. Also, for certain single-spin measures Theorem II.4.1 may be used to compare the moments of a model with single-spin measure ν with those of a model having the same Hamiltonian and a Gaussian single-spin measure. For pair interactions (the only kind that make sense with a Gaussian single-spin measure) these moments are explicitly computable. As an example, if $d\nu(\sigma)=\exp(-P(\sigma))d\sigma$, where P is an even polynomial bounded from below, then by sufficiently increasing the dispersion of the Gaussian $d\nu_G = [\exp(-\frac{1}{2}\frac{\sigma^2}{a^2})/a\sqrt{2\pi}]d\sigma$ we may arrange matters as in Figure 1 below:

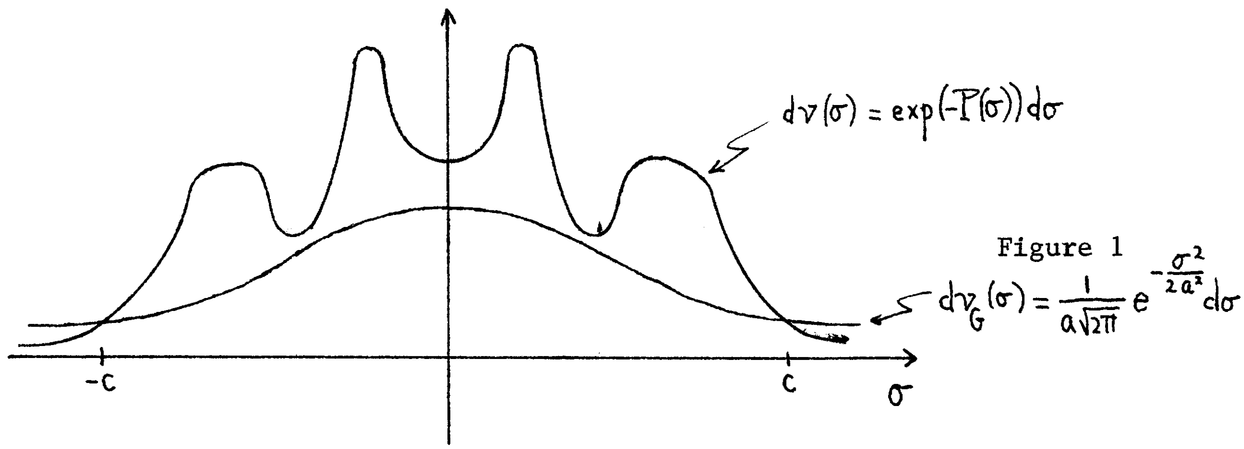


Figure 1

In this situation the measure ν may be obtained from ν_G by multiplying by an even function monotonically decreasing outside the interval [-c,c] and redistributing the lost probability mass inside [-c,c] . Theorem II.4.1 tells us this procedure decreases the moments. Of course, this comparison is of little value at low temperature since even in a finite volume the Gibbs measure of a low-temperature model with Gaussian single-spin measure is ill-defined.

In Theorem 1 we constructed the infinite-volume Gibbs measure with the free boundary condition as a state on the C*-algebra $C(\Pi_{\mathcal{L}}\ddot{\mathbb{R}})$. By its very construction, $\langle\ ;H,\nu,\beta\rangle$ should be called an equilibrium state of the infinite model $(\mathcal{L},H,\nu)$ at inverse temperature β . We now briefly discuss how to decide independently of any constructive procedure whether a state on $C(\prod_{\mathcal{L}} \ddot{\mathbb{R}})$ is an equilibrium state of a model $(\mathcal{L}, H, \nu)$. We restrict our attention to finite range Hamiltonians on a lattice $\mathbb{Z}^n$. Consider two finite regions $\Lambda \subset \Lambda' \subset \mathbb{Z}^n$, where Λ' is much larger than Λ. Denote spins in Λ by $\sigma_i, \sigma_j, \cdots$ and spins in $\Lambda'-\Lambda$ by $\tau_a, \tau_b, \cdots$. Since H is finite range, if Λ' is sufficiently larger than Λ $(\text{dist}(\partial\Lambda', \Lambda) > \text{ran}(H))$ then $H_{\Lambda'}$ may be written in the form

$$H_{\Lambda'}(\sigma,\tau) = H_{\Lambda}(\sigma) + H_{\Lambda'-\Lambda}(\tau) + W_{\Lambda}(\sigma,\tau), \tag{22}$$

where because of the large size of Λ' the interaction $W_{\Lambda'}(\sigma,\tau)$ between Λ and Λ' is independent of Λ'. Consequently, the conditional expectation with respect to the Gibbs measure $\mu_{\Lambda'}$ of $H_{\Lambda'}$ of a function $f(\sigma)$ of the spins σ in Λ, conditioned on the spins τ in $\Lambda'-\Lambda$, is

$$\begin{aligned} \langle f(\sigma) | \Lambda'-\Lambda \rangle_{\Lambda'} &= \frac{\int f(\sigma) \exp\left[-\beta\left(H_{\Lambda}(\sigma) + W_{\Lambda}(\sigma,\tau) + H_{\Lambda'-\Lambda}(\tau)\right)\right] \prod_{\Lambda} d\nu(\sigma_i)}{\int \exp\left[-\beta\left(H_{\Lambda}(\sigma) + W_{\Lambda}(\sigma,\tau) + H_{\Lambda'-\Lambda}(\tau)\right)\right] \prod_{\Lambda} d\nu(\sigma_i)} \\ &= \frac{\int f(\sigma) e^{-\beta[H_{\Lambda}(\sigma) + W_{\Lambda}(\sigma,\tau)]} \prod_{\Lambda} d\nu(\sigma_i)}{\int e^{-\beta[H_{\Lambda}(\sigma) + W_{\Lambda}(\sigma,\tau)]} \prod_{\Lambda} d\nu(\sigma_i)} . \end{aligned} \tag{23}$$

Observe that the right-hand side, which may be thought of as the thermal expectation of $f(\sigma)$ in the region Λ with Hamiltonian H_{Λ} and boundary conditions $W_{\Lambda}(\tau)$, is independent of the region Λ'. Any equilibrium state of the model $(\mathbb{Z}^n, H, \nu)$ should certainly satisfy this equation for conditional expectations, and we take it as the definition of an equilibrium state. Translating into the language of C*-algebras, we find that a state φ of $C(\prod_{\mathbb{Z}^n}\ddot{\mathbb{R}})$ is an equilibrium state of $(\mathbb{Z}^n, H, \nu)$ at inverse temperature β if $\forall$ finite $\Lambda \subset \mathbb{Z}^n$, $\forall$ $f \in C(\prod_{\Lambda}\ddot{\mathbb{R}})$, $\forall$ $G \in C(\prod_{\mathbb{Z}^n-\Lambda}\ddot{\mathbb{R}})$

$$\varphi\left(\left[f(\sigma) - \frac{\int f(\sigma) e^{-\beta[H_\Lambda(\sigma)+W_\Lambda(\sigma,\tau)]} \prod_\Lambda d\nu(\sigma_i)}{\int e^{-\beta[H_\Lambda(\sigma)+W_\Lambda(\sigma,\tau)]} \prod_\Lambda d\nu(\sigma_i)}\right] \cdot G(\tau)\right) = 0 . \tag{24}$$

This equation for an equilibrium state is called the equilibrium, or D.L.R. equation [7,27]. There are several technicalities associated with its interpretation, but we shall not discuss them here.

It is clear from the definitions that the infinite-volume Gibbs measure with the free boundary condition is a solution of the equilibrium equation (24). In general this solution is not unique, the lack of uniqueness being closely connected with the presence of multiple phases in the model. We now mention other ways of taking the infinite-volume limit which sometimes give rise to different equilibrium states. For each $\Lambda \in \mathcal{G}_0(\mathcal{L})$ let $\eta_{\tilde{\Lambda}} \in \mathbb{R}^{\tilde{\Lambda}}$ be a configuration of spins outside Λ. The energy $H_{\Lambda,\eta}(\sigma)$ of a configuration of spins $\sigma \in \mathbb{R}^\Lambda$ surrounded by spins in $\tilde{\Lambda}$ fixed at values given by $\eta_{\tilde{\Lambda}}$ is

$$H_{\Lambda,\eta}(\sigma) = H_\Lambda(\sigma) + W_\Lambda(\sigma,\eta) + C \quad , \tag{25}$$

where as before $H_\Lambda(\sigma)$ represents the mutual interaction of spins in Λ, $W_\Lambda(\sigma,\eta)$ represents the interaction between the spins σ in Λ and the fixed spins $\psi_{\tilde{\Lambda}}$ in $\tilde{\Lambda}$, and C is an infinite constant arising from the mutual interactions of the fixed spins in $\tilde{\Lambda}$. We drop C, and call $H_{\Lambda,\psi}(\sigma) = H_\Lambda(\sigma) + W_\Lambda(\sigma,\psi)$ the Hamiltonian of the region Λ with boundary condition $\psi_{\tilde{\Lambda}}$. The free boundary condition is $\psi_{\tilde{\Lambda}} = 0$.

Let $\Lambda' \supset \Lambda$ be a region large enough so that dist $(\Lambda, \partial\Lambda') > \mathrm{ran}(H)$. Then the conditional expectation with respect to the Gibbs measure in Λ' of $H_{\Lambda',\psi}$ with boundary condition $\psi_{\tilde{\Lambda}'}$ in $\tilde{\Lambda}'$ of a function $f(\sigma)$ of the spins σ in Λ, conditioned on the spins τ in $\Lambda'-\Lambda$, is

$$\begin{aligned}\langle f(\sigma)|\Lambda'-\Lambda\rangle_{\Lambda',\psi} &= \frac{\int f(\sigma)\exp[-\beta(H_\Lambda(\sigma)+W_\Lambda(\sigma,\tau)+H_{\Lambda'-\Lambda}(\tau)+W_{\Lambda'}(\tau,\psi))]\prod_\Lambda d\nu(\sigma_i)}{\int\exp[-\beta(H_\Lambda(\sigma)+W_\Lambda(\sigma,\tau)+H_{\Lambda'-\Lambda}(\tau)+W_{\Lambda'}(\tau,\psi))]\prod_\Lambda d\nu(\sigma_i)} \\ &= \frac{\int f(\sigma)\exp[-\beta(H_\Lambda(\sigma)+W_\Lambda(\sigma,\tau))]\prod_\Lambda d\nu(\sigma_i)}{\int\exp[-\beta(H_\Lambda(\sigma)+W_\Lambda(\sigma,\tau))]\prod_\Lambda d\nu(\sigma_i)} \;. \end{aligned} \tag{26}$$

Here as before the interaction $W_\Lambda(\sigma,\tau)$ between the spins σ in Λ and the spins τ in $\Lambda'-\Lambda$ is independent of the region Λ' because of its large size, and the interaction $W_{\Lambda'}(\tau,\psi)$ between the spins in Λ' and the fixed spins $\psi_{\tilde{\Lambda}'}$ outside Λ' involves only the spins τ in $\Lambda'-\Lambda$ for the same reason. We see from (26) that if we can control the infinite-volume limit of the Gibbs measure associated with the Hamiltonians $H_{\Lambda,\psi}$ having boundary condition $\psi_{\tilde{\Lambda}}$, then the resulting state - the Gibbs measure with boundary condition ψ - will be an equilibrium state. This state is not necessarily the same as the state with the free boundary condition, and in Section 4 we shall see a specific example when it is not.

Section 3: Clustering, Correlation Length, and Long-Range Order

In this section we concentrate our attention on ferromagnetic Ising models $(\mathbb{Z}^n, H, \nu)$ on a lattice which have translation-invariant connected finite-range pair Hamiltonians H. Explicitly, H must be of the form

$$H=-\sum_{i,j} J_{(i-j)}\sigma_i\sigma_j - h\sum_i \sigma_i \quad ; \qquad h\geq 0,\ J_k=J_{-k}\geq 0,\ k\in\mathbb{Z}^n. \quad (1)$$

Sometimes we require further that H be nearest-neighbor:

$$H=-\sum_{i\in\mathbb{Z}^n}\sum_{\alpha=1}^{n} J_\alpha \sigma_i\sigma_{i+1_\alpha} - h\sum_{i\in\mathbb{Z}^n}\sigma_i \quad ; \ h\geq 0,\ J_\alpha>0,\ \text{where } 1_\alpha=(\overbrace{0,\cdots,0}^{\alpha},1,\underbrace{0,\cdots,0}_{n-\alpha}). \quad (2)$$

As we mentioned in Section 2, all moments $\langle \sigma_A ; H,\nu,\beta\rangle$ of the Gibbs measure with free boundary condition of a model having translation-invariant finite range pair Hamiltonian (1) are finite. Here we are primarily concerned with the dependence of the infinite-volume Gibbs state on inverse temperature and external field parameters β,h; for this reason we often use the notation $\langle\ ; h,\beta\rangle$ in place of $\langle\ ; H,\nu,\beta\rangle$.

Let $A,B\in \mathfrak{F}_0(\mathbb{Z}^n)$ be families of sites. Since the Hamiltonian (1) is finite-range, it is reasonable that if B is translated by some $i\in\mathbb{Z}^n$, then the random variables σ_A,σ_{B+i} become asymptotically independent as $|i|\to\infty$:

$$\lim_{|i|\to\infty}\left(\langle\sigma_A\sigma_{B+i}\rangle-\langle\sigma_A\rangle\langle\sigma_{B+i}\rangle\right)=\lim_{|i|\to\infty}\left(\langle\sigma_A\sigma_{B+i}\rangle-\langle\sigma_A\rangle\langle\sigma_B\rangle\right)=0. \quad (3)$$

If (3) holds for all families of sites A,B we say the state $\langle\ ;H,\nu,\beta\rangle$ clusters.

The decay of $(\langle\sigma_A\sigma_{B+i}\rangle-\langle\sigma_A\rangle\langle\sigma_B\rangle)$ is often exponential in the separation, for example at low inverse temperature β or large external field h [22]. For this reason we define the correlation length in the 1-direction by

$$1/\chi_1 = \inf_{A,B\in\mathcal{G}_0(\mathbb{Z}^n)}\left(\log\left[\liminf_{i\in\mathbb{Z}^+}\left(\langle\sigma_A\sigma_{B+i_1}\rangle-\langle\sigma_A\rangle\langle\sigma_B\rangle\right)^{-\frac{1}{i}}\right]\right), \qquad (4)$$

where $i_1 = (i,0,0\cdots)\in\mathbb{Z}^n$. The argument of the logarithm lies in $[1,\infty]$ because $0\leqslant\langle\sigma_A\sigma_{B+i_1}\rangle-\langle\sigma_A\rangle\langle\sigma_B\rangle\leqslant 2\langle\sigma^{|A|+|B|}\rangle$, where the site subscript on σ is unnecessary because a translation-invariant model has translation-invariant spin expectations. Loosely speaking, $1/\chi_1$ is the smallest asymptotic rate of exponential decay of any correlation $\langle\sigma_A\sigma_{B+i_1}\rangle-\langle\sigma_A\rangle\langle\sigma_B\rangle$. It is the least number such that $\forall\ A,B\in\mathcal{G}_0(\mathbb{Z}^n)$, $\forall\ \epsilon>0$,

$$\langle\sigma_A\sigma_{B+i_1}\rangle-\langle\sigma_A\rangle\langle\sigma_B\rangle\leqslant \text{const. } e^{-(1/\chi_1-\epsilon)i} \qquad (5)$$

for large i. We emphasize that an infinite correlation length does not necessarily mean that the model fails to cluster, but only that it fails to cluster exponentially. If $(\mathbb{Z}^n,H,\nu)$ has bounded spins (suppν compact), one may apply the F.K.G. inequality in the manner of [29] to show that an arbitrary correlation $\langle\sigma_A\sigma_B\rangle-\langle\sigma_A\rangle\langle\sigma_B\rangle$ is bounded by sums of two-point correlations $\langle\sigma_k\sigma_\ell\rangle-\langle\sigma_k\rangle\langle\sigma_\ell\rangle$, so that the infimum in (4) over families of sites $A,B\in\mathcal{Y}_0(\mathbb{Z}^n)$ may be replaced by an infimum over pairs of sites $k,\ell\in\mathbb{Z}^n$:

$$1/\chi_1 = \inf_{k,\ell\in\mathbb{Z}^n}\left[\log\left(\liminf_{i\in\mathbb{Z}^+}\left[\langle\sigma_k\sigma_{\ell+i_1}\rangle-\langle\sigma\rangle^2\right]^{-\frac{1}{i}}\right)\right]. \tag{6}$$

It follows from (6) that χ_1 is monotone decreasing in the external field h for those models $(\mathbb{Z}^n,H,\nu)$ whose single-spin measure ν is such that Corollary II.2.8, the G.H.S. inequality, is satisfied. This is because the G.H.S. inequality says that $\langle\sigma_k\sigma_\ell\rangle-\langle\sigma_k\rangle\langle\sigma_\ell\rangle$ is decreasing in h. If the spins are not bounded, the situation is substantially the same [43].

We now investigate in greater detail the clustering properties of the two-point function. We first show that if we fix the inverse temperature β and couplings J_k in (1) while regarding the external field h as an adjustable parameter, then the two-point function clusters except possibly for a set of values of h of (Lebesgue) measure 0:

$$\lim_{|i|\to\infty}\left(\langle\sigma_k\sigma_\ell;h\rangle-\langle\sigma_j h\rangle^2\right)=0 \tag{7}$$

except for a set of h's of measure 0. If the G.H.S. inequality holds, this set of measure 0 decreases to the single point h=0. Next we show that at sufficiently low temperature β^{-1}, if the single-spin measure is not the delta-function δ clustering must fail for zero external field.

Indeed, there is a constant $L > 0$ such that

$$\langle \sigma_i \sigma_j ; h=0,\beta \rangle - \langle \sigma_i ; 0,\beta \rangle \langle \sigma_j ; 0,\beta \rangle = \langle \sigma_i \sigma_j ; 0,\beta \rangle \geq L \qquad \forall i,j \in \mathbb{Z}^n . \tag{8}$$

In this case the model is said to exhibit long-range order. In the following section we shall show long-range order implies spontaneous magnetization. The remainder of this section deals with the infinite-volume transfer matrix $\mathcal{J}$. We define $\mathcal{J}$ via the Osterwalder-Schrader reconstruction technique, and state a theorem proved in Appendix C relating the cluster properties of an Ising model with the spectral properties of its transfer matrix.

Lemma 1: Let $(\mathbb{Z}^n, H, \nu)$ be an Ising ferromagnet with bounded couplings having finite-range pair Hamiltonian with (nonuniform) bounded external field:

$$H = -\sum J_{ij} \sigma_i \sigma_j - \sum h_i \sigma_i \qquad , \quad 0 \leq J_{ij} \leq J^* \quad , 0 \leq h_i \leq h^* . \tag{9}$$

Then for any family of sites $A \in \mathcal{F}_0(\mathbb{Z}^n)$ and any $j \in \mathbb{Z}^n$, $\langle \sigma_A ; H \rangle$ is differentiable with respect to h_j and

$$\frac{\partial}{\partial h_j} \langle \sigma_A ; H \rangle = \langle \sigma_A \sigma_j ; H \rangle - \langle \sigma_A ; H \rangle \langle \sigma_j ; H \rangle . \tag{10}$$

Thus the Gibbs state has moments jointly C^{∞} in any finite collection of external fields h_j.

Proof:

Notationally we suppress all fields but h_j, so that

Then

$$\langle \sigma_A ; h_j \rangle - \langle \sigma_A ; h_j = 0 \rangle = \lim_{\Lambda \to \infty} \left(\langle \sigma_A ; h_j \rangle_\Lambda - \langle \sigma_A ; 0 \rangle_\Lambda \right)$$

$$= \lim_{\Lambda \to \infty} \int_0^{h_j} \frac{d}{dt} \langle \sigma_A ; t \rangle_\Lambda \, dt = \lim_{\Lambda \to \infty} \int_0^{h_j} \left(\langle \sigma_A \sigma_j ; t \rangle_\Lambda - \langle \sigma_A ; t \rangle_\Lambda \langle \sigma_j ; t \rangle_\Lambda \right)$$

$$= \int_0^{h_j} \lim_{\Lambda \to \infty} \left(\langle \sigma_A \sigma_j ; t \rangle_\Lambda - \langle \sigma_A ; t \rangle_\Lambda \langle \sigma_j ; t \rangle_\Lambda \right) dt \quad \text{(Monotone Convergence)}$$

$$= \int_0^{h_j} \left(\langle \sigma_A \sigma_j ; t \rangle - \langle \sigma_V ; t \rangle \langle \sigma_j ; t \rangle \right) dt. \tag{11}$$

It follows from this integral representation that $\langle \sigma_A ; h_j \rangle$ is continuous in h_j. Therefore the integrand of (11) is continuous in t, so we may differentiate to obtain (10).

QED

We remark that Lemma 1 holds in much more general models.

Proposition 2: Let $(\mathbb{Z}^n, H, \nu)$ be an Ising ferromagnet with bounded couplings having the finite-range pair Hamiltonian

$$H = -\sum_{i,j} J_{ij} \sigma_i \sigma_j - h \sum_i \sigma_i \qquad 0 \leq J_{ij} \leq J^* \tag{12}$$

with uniform external field h, and let $A \in \mathcal{G}_0(\mathbb{Z}^n)$. If $\langle \sigma_A ; H \rangle$ is differentiable at h (which it must be except on a set of measure 0 since it increases in h) then

$$\frac{d}{dh}\langle\sigma_A;h\rangle \geq \sum_{j\in\mathbb{Z}^n}\left(\langle\sigma_A\sigma_j;h\rangle-\langle\sigma_A;h\rangle\langle\sigma_j;h\rangle\right). \tag{13}$$

Consequently,

$$\lim_{\mathrm{dist}(j,A)\to\infty}\left(\langle\sigma_A\sigma_j;h\rangle-\langle\sigma_A;h\rangle\langle\sigma_j;h\rangle\right)=0 \tag{14}$$

except possibly for a set of values of h of measure 0.

Proof:

By monotonicity in the external field,

$$\Big\langle\sigma_A;\ H=-\sum_{i,j\in\mathbb{Z}^n}J_{ij}\sigma_i\sigma_j-(h+\Delta h)\sum_{\mathbb{Z}^n}\sigma_j\Big\rangle \geq \Big\langle\sigma_A;\ H'=-\sum_{i,j\in\mathbb{Z}^n}J_{ij}\sigma_i\sigma_j-h\sum_{j\in\mathbb{Z}^n}\sigma_j-\Delta h\sum_{j\in\Lambda}\sigma_j\Big\rangle \tag{15}$$

for any finite region Λ. But by Lemma 1 this means

$$\frac{d}{dh}\langle\sigma_A;H\rangle \geq \sum_{j\in\Lambda}\frac{\partial}{\partial h_j}\langle\sigma_A;H\rangle=\sum_{j\in\Lambda}\left(\langle\sigma_A\sigma_j;H\rangle-\langle\sigma_A;H\rangle\langle\sigma_j;H\rangle\right). \tag{16}$$

Each term in the sum on the right is nonnegative by Corollary II.2.3 so sending $\Lambda\to\infty$ the proposition follows.

$\overline{\text{QED}}$

Like Lemma 1, Proposition 2 holds in more general circumstances. Taking the special case when the family A is a single site, we obtain clustering of the two-point function except possibly for some set of values h of measure 0. If the single-spin measure ν is such that Corollary II.2.8, the G.H.S. inequality, is obeyed, then the set of measure 0 of values of h where clustering may break down reduces to the single point h=0.

Corollary 3: Let $(\mathbb{Z}^n, H, \nu)$ be a ferromagnet with Hamiltonian (12) whose single-spin measure ν is such that the G.H.S. inequality (Corollary II.2.8) holds. Then the two-point function clusters except possibly at h=0:

$$\lim_{|i-j|\to\infty} \left(\langle\sigma_i\sigma_j; h\rangle - \langle\sigma_i; h\rangle\langle\sigma_j; h\rangle\right) = 0 \tag{17}$$

except possibly at h=0.

Proof:

The G.H.S. inequality implies that $(\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle)$ decreases in h. Thus if clustering fails at some $h > 0$ it fails in the entire interval $[0,h]$. This interval of no clustering, which has positive measure, violates Proposition 2.

QED

We have seen that the two-point function of a nearest-neighbor ferromagnet must cluster except possibly at a set of values of the external field of measure 0, and if the G.H.S. inequality holds it must cluster except possibly at h=0. We now show that if the temperature β^{-1} is sufficiently low and the single-spin measure ν is not the delta-function δ, then clustering must indeed fail at h=0. In fact, the model is long-range ordered. The proof proceeds in several steps. First we use the second

Griffiths inequality (Corollary II.2.3) to reduce the problem to a nearest-neighbor interaction on $\mathbb{Z}^2$. We next establish long-range order in models on $\mathbb{Z}^2$ for a restricted class of single-spin measures by extending an argument of Bortz and Griffiths [3], which in turn generalizes an idea of Peierls. The general result follows by applying Theorem II.4.1 to conclude that the two-point function $\langle \sigma_i \sigma_j \rangle$ decreases when the single-spin measure is altered to bring it into the class covered by the Bortz-Griffiths method.

As a preliminary, we define an isotropic nearest-neighbor Ising ferromagnet to be one all of whose (nonzero) couplings are equal: $J_\alpha = J > 0 \quad \forall \alpha$. In proving long-range order it is sufficient to consider isotropic models, because by decreasing some couplings J_α - which decreases the moments of the Gibbs measure - we may make any model isotropic.

Lemma 4: Let $(\mathbb{Z}^n, H_n(J,h), \nu)$ be the isotropic nearest-neighbor Ising ferromagnet on $\mathbb{Z}^n$ with coupling J, external field h, and single-spin measure ν. If $\exists$ $L > 0$ such that

$$\langle \sigma_i \sigma_j ; H_2(J,h), \nu, \beta \rangle \geq L \qquad \forall\, i,j \in \mathbb{Z}^2 \tag{18}$$

then for any $n \geq 2$

$$\langle \sigma_i \sigma_j ; H_n(J,h), \nu, \beta \rangle \geq L \qquad \forall\, i,j \in \mathbb{Z}^n . \tag{19}$$

Proof:

We use induction, and show that if $\langle \sigma_i \sigma_j ; H_n \rangle$ is bounded below by L, so is $\langle \sigma_i \sigma_j ; H_{n+1} \rangle$. Let $i,j \in \mathbb{Z}^{n+1}$. By translation-invariance and isotropy we may assume i=0 and $j = (j_1, j_2, \cdots, j_{n+1})$ with $j_1 \geq 0$, $j_{n+1} \geq 0$. Define

$$\begin{aligned} V_1 &= \{(k_1, \cdots, k_{n+1}) \in \mathbb{Z}^{n+1} : k_1 \leq 0 \ \& \ k_{n+1} = 0\} \\ V_2 &= \{(k_1, \cdots, k_{n+1}) \in \mathbb{Z}^{n+1} : k_1 = 0 \ \& \ 0 \leq k_{n+1} \leq j_{n+1}\} \\ V_3 &= \{(k_1, \cdots, k_{n+1}) \in \mathbb{Z}^{n+1} : k_1 \geq 0 \ \& \ k_{n+1} = j_{n+1}\} \end{aligned} \tag{20}$$,

and let $V = V_1 \cup V_2 \cup V_3$. If we reduce to 0 all couplings not between two sites in V, which decreases $\langle \sigma_i \sigma_j \rangle$, V becomes a sublattice disconnected from the remainder of $\mathbb{Z}^{n+1}$ that is isomorphic with $(\mathbb{Z}^n, H_n(J,h), \mathcal{V})$. In the case n=2 the set V is illustrated in Figure 1.

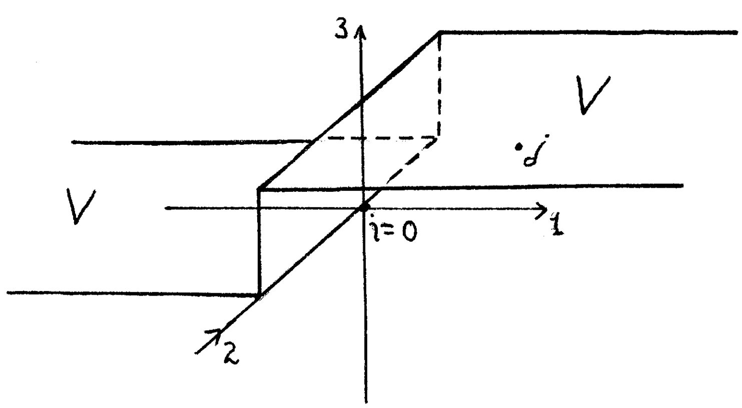


Figure 1

QED

Lemma 5: Let $(\mathbb{Z}^2, H, \nu)$ be a nearest-neighbor Ising ferromagnet with Hamiltonian

$$H = -J \sum_{(\ell,m)\in\mathbb{Z}^2} \left[\sigma_{(\ell,m)}\sigma_{(\ell+1,m)} + \sigma_{(\ell,m)}\sigma_{(\ell,m+1)}\right] \qquad J>0. \tag{21}$$

If there exists a constant $c \in (0,\infty)$ such that $\operatorname{supp}\nu \subset [-c,c]$, and if there exists $\eta>0$ such that for all measurable $E \subset \left(-\frac{c}{3}, \frac{c}{3}\right)$

$$\nu\left(E+\tfrac{2}{3}c\right) \geqslant \eta \cdot \nu(E) \quad , \tag{22}$$

and if $\nu\left[\frac{3}{4}c, c\right] \neq 0$, then for sufficiently low temperature β^{-1} there exists $L>0$ such that $\forall\ i,j \in \mathbb{Z}^2$,

$$\langle \sigma_i \sigma_j ; H, \nu, \beta \rangle \geqslant L \quad . \tag{23}$$

(The infinite-volume limit is taken with the free boundary condition.)

Proof:

As much of this proof follows standard reasoning, we shall give the details in a condensed manner. We may assume without loss of generality that $c=1$ and $\eta<1$. Let χ_n be the characteristic function of the interval

$$\begin{aligned} n=1: &\quad \left[\tfrac{1}{3}, 1\right] \\ n=2: &\quad \left(-\tfrac{1}{3}, \tfrac{1}{3}\right) \\ n=3: &\quad \left[-1, -\tfrac{1}{3}\right] \quad , \end{aligned} \tag{24}$$

and estimate $\langle\sigma_i\sigma_j\rangle$ using these characteristic functions:

$$\langle\sigma_i\sigma_j\rangle = \sum_{n,m}\langle\sigma_i\sigma_j\chi_m(\sigma_i)\chi_n(\sigma_j)\rangle = \sum_n\langle\sigma_i\sigma_j\chi_n(\sigma_i)\chi_n(\sigma_j)\rangle + \sum_{m\neq n}\langle\sigma_i\sigma_j\chi_m(\sigma_i)\chi_n(\sigma_j)\rangle$$

$$\geq \frac{1}{9}\left(\langle\chi_1(\sigma_i)\chi_1(\sigma_j)\rangle + \langle\chi_3(\sigma_i)\chi_3(\sigma_j)\rangle\right) - \frac{1}{3}\langle\chi_2(\sigma_i)\chi_2(\sigma_j)\rangle - \sum_{m\neq n}\langle\chi_m(\sigma_i)\chi_n(\sigma_j)\rangle . \tag{25}$$

We show that by choosing β sufficiently large (independently of i,j) the two negative terms in (25) may be made as close to zero as desired, so that the first must approach 2/9 and thus give the lower bound.

The term $\langle\chi_2(\sigma_i)\chi_2(\sigma_j)\rangle \leq \langle\chi_2(\sigma_i)\rangle$ is easily disposed of. By Proposition II.3.1, $\langle\chi_2(\sigma_i)\rangle$ increases when any coupling between two sites is decreased. Thus if we consider a model with just two sites i, i' at inverse temperature β with coupling J and single-spin measure ν ,

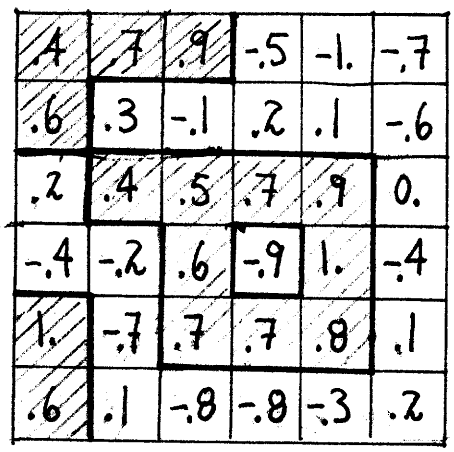
$$\langle\chi_2(\sigma_i)\rangle \leq \langle\chi_2(\sigma_i)\rangle' , \tag{26}$$

where the prime on the right indicates the expectation in the two-site system. But the condition that $\nu[\frac{3}{4},1] > 0$ in the hypothesis assures $\lim_{\beta\to\infty}\langle\chi_2(\sigma_i)\rangle' = 0$, so that $\langle\chi_2(\sigma_i)\chi_2(\sigma_j)\rangle \leq \langle\chi_2(\sigma_i)\rangle'$ approaches zero as β is increased in the original model.

The term $\sum_{m\neq n}\langle\chi_m(\sigma_i)\chi_n(\sigma_j)\rangle$ is controlled by an extension of the ideas of Bortz and Griffiths [3], who considered in a somewhat different context the case when ν was Lebesgue measure restricted to $[-1,1]$. By the spin-reversal symmetry of the Gibbs measure it suffices to show that $\langle\chi_1(\sigma_i)[\chi_2(\sigma_j)+\chi_3(\sigma_j)]\rangle$ becomes small for large β . To accomplish this, we shall prove that if $\Lambda \ni i,j$ is sufficiently large, then $\forall \epsilon > 0\ \exists \beta_\epsilon$ independent of i,j,Λ such that

$$\beta > \beta_\epsilon \Rightarrow \langle \chi_1(\sigma_i)[\chi_2(\sigma_j) + \chi_3(\sigma_j)] \rangle < \epsilon \ . \tag{27}$$

Regard $\mathbb{Z}^2$ as a subset of $\mathbb{R}^2$, and associate with each $i \in \mathbb{Z}^2$ a closed unit square $\Delta_i \subset \mathbb{R}^2$ centered at i. If $\Lambda \subset \mathbb{Z}^2$, define $\underset{\sim}{\Lambda} \subset \mathbb{R}^2$ by $\underset{\sim}{\Lambda} = \bigcup_{i\in\Lambda} \Delta_i$. Given a configuration $\sigma \in [-1,1]^\Lambda$, we call the spin at site $k \in \Lambda$ plus (+) if $\sigma_k \in [\frac{1}{3}, 1]$ and minus (-) if $\sigma_k \in [-1, \frac{1}{3})$. Break up $\underset{\sim}{\Lambda}$ into + and - connected components by saying that two squares Δ_k, Δ_ℓ are in the same + (-) connected component if their spins are both + (-) and they are connected by a chain of nearest-neighbor squares with all + (-) spins. A border B associated with the configuration σ is defined as a connected component of the boundary taken in the interior of $\underset{\sim}{\Lambda}$ of a $\pm$ connected component. Note that a border must either be a closed polygon or have both ends on $\partial\underset{\sim}{\Lambda}$. Thus B separates $\underset{\sim}{\Lambda}$ into two connected components. A site k is called a circumference site if its unit square Δ_k has a side in B. If b is the length of B there are at most b circumference sites in each component. The circumference sites in one of the connected components must be either all + or all - , and in the other all - or all + . We call the + (-) component the one in which all sites are + (-). An example is shown in Figure 2.



This example shows three + components (lightly hatched ////) and three − components. There are four borders, drawn with heavy black lines ——.

Figure 2

Let $\Lambda \subset \mathbb{Z}^2$ be a square containing i,j which is so large that the inequality

$$\operatorname{dist}(\{i,j\}, \partial\underset{\sim}{\Lambda}) \geq |i_1 - j_1| + |i_2 - j_2| \tag{28}$$

is satisfied by the corresponding square $\underset{\sim}{\Lambda} \subset \mathbb{R}^2$. We shall show that if B is a border in $\underset{\sim}{\Lambda}$ and $\mathfrak{B} \subset [-1,1]^{\Lambda}$ is the set of all configurations σ which have B as a border, then the Gibbs measure $P_B = Z^{-1}\int_{\mathfrak{B}} e^{-\beta H} \prod_{\Lambda} d\nu$ of $\mathfrak{B}$ decays exponentially in the length b of the border B:

$$P_B \leq 4\left(\frac{2}{\eta}\right)^b e^{-\beta J b/9} . \tag{29}$$

Let us see why this estimate gives long-range order.

If σ_i is + and σ_j is - in some configuration, then one of the borders B bounding the + connected component containing i must separate i from j in $\underset{\sim}{\Lambda}$. Thus, if B(i,j) is the set of all borders separating i from j in $\underset{\sim}{\Lambda}$, we have the inequality

$$\langle \chi_1(\sigma_i)[\chi_2(\sigma_j)+\chi_3(\sigma_j)]\rangle \leqslant \sum_{B\in B(i,j)} P_B \ . \tag{30}$$

A border may separate i from j in one of three ways: it may be a closed polygon with i in its interior and j in its exterior, it may be a closed polygon with j in its interior and i in its exterior, or it may have both endpoints on $\partial\underset{\sim}{\Lambda}$ and pass between i and j. The number of borders of length b enclosing either i or j is at most $b3^b$. Also, since the number of borders of length b containing a particular side of a particular square Δ_k is bounded by $b3^b$, and since any border separating i from j must pass through one of the $|i_1-j_1| + |i_2-j_2|$ intervening sides pointed out in Figure 3, the number of borders of length b separating i from j with endpoints on $\partial\underset{\sim}{\Lambda}$ is at most $(|i_1-j_1| + |i_2-j_2|)b3^b$.

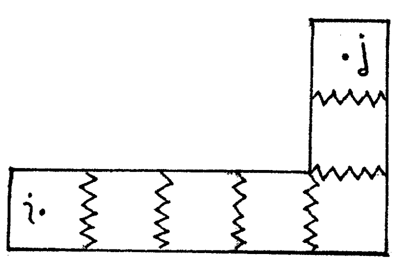


Figure 3

Any border separating i,j must pass through one of the jagged sides.

However, if the border B is long enough to separate i,j and to extend to $\partial\underset{\sim}{\Lambda}$, then by (28) we must have

$$b \geq \mathrm{dist}(\{i,j\}, \partial\underset{\sim}{\Lambda}) \geq |i_1 - j_1| + |i_2 - j_2| \; . \tag{31}$$

Combining these estimates, we find that the number #(b) of borders of length b separating i from j is at most

$$\#(b) \leq 2b^2 3^b \; . \tag{32}$$

It now follows from (29),(30), and (32) that

$$\langle \chi_1(\sigma_i)[\chi_2(\sigma_j) + \chi_3(\sigma_j)] \rangle \leq \sum_{b=4}^{\infty} 4b^2 \left(\frac{6}{7}\right)^b e^{-\beta J b/9} \; . \tag{33}$$

Since the right-hand side of (33) becomes arbitrarily small for large β independent of i,j, and Λ , we will have long-range order once the exponential decay of P_B in b is established.

We shall say that the spin σ_k at site k is in class n, n=1,2,3, if

$$\begin{aligned} n=1 &: \; \sigma_k \in [\tfrac{1}{3}, 1] \\ n=2 &: \; \sigma_k \in (-\tfrac{1}{3}, \tfrac{1}{3}) \\ n=3 &: \; \sigma_k \in [-1, -\tfrac{1}{3}] \; . \end{aligned} \tag{34}$$

Fix a particular border B. It separates $\underset{\sim}{\Lambda}$ into two components $\underset{\sim}{\Lambda}', \underset{\sim}{\Lambda}''$. Let $\mathcal{B}' \subset [-1,1]^{\Lambda}$ be the set of all configurations in which B appears as a border and $\underset{\sim}{\Lambda}'$ is the - component. Let C_B be the set of circumference sites of B in $\underset{\sim}{\Lambda}'$, let $n \in \prod_{k \in C_B} \{2,3\}$ be a multi-index, and let $\mathcal{B}_n \subset \mathcal{B}'$ be the set of configurations for which the spin at site $k \in C_B$ is in class n_k. Define the transformation $\tau_1 : \mathcal{B}_n \to [-1,1]^{\Lambda}$ by

$$(\tau_1 \sigma)_k = \begin{Bmatrix} \sigma_k & \text{if } k \text{ is in the + component } \underset{\sim}{\Lambda}'' \\ -\sigma_k + \frac{2}{3} & \text{if } k \in C_B \ \& \ n_k = 2 \\ -\sigma_k & \text{otherwise} \end{Bmatrix} . \tag{35}$$

Define the transformation $\tau_2 : \mathcal{B}_n \to [-1,1]^{\Lambda}$ by

$$(\tau_2 \sigma)_k = \begin{Bmatrix} \sigma_k + \frac{2}{3} & \text{if } k \in C_B \ \& \ n_k = 2 \\ \sigma_k & \text{otherwise} \end{Bmatrix} . \tag{36}$$

Note that both τ_1 and τ_2 factor:

$$(\tau_\alpha \sigma)_k = \tau_{\alpha k}(\sigma_k) \ , \qquad \alpha = 1,2 \ , \tag{37}$$

where $\tau_{\alpha k} : [-1,1] \to [-1,1]$ is determined from definitions (35), (36). Also, they are both injective, so we may define the measures $\tau_\alpha^* \nu$ on $\mathcal{B}_n$ by

$$(\tau_\alpha^* \nu)(E) = \nu(\tau_\alpha E) \ , \quad E \subset \mathcal{B}_n \ , \ \alpha = 1,2 \ . \tag{38}$$

We claim that

$$\tau_\alpha^* \nu(E) \geq \eta^b \nu(E) \qquad E \subset \mathcal{B}_n, \alpha = 1,2 . \tag{39}$$

It suffices to verify this for rectangles $E = \prod_{k \in \Lambda}(E_k)$, $E_k \subset [-1,1]$. If $k \in \underset{\sim}{\Lambda}''$, $\tau_{\alpha k} E_k = E_k$ so $\nu(\tau_{\alpha k} E_k) = \nu(E_k)$. If $k \in \underset{\sim}{\Lambda}'$ but is not a circumference site of class 2, $\tau_{\alpha k} E_k = \pm E_k$; by the evenness of ν, $\nu(\tau_{\alpha k} E_k) = \nu(E_k)$. If k is a circumference site of class 2 in $\underset{\sim}{\Lambda}'$, $\tau_{\alpha k} E_k = 2/3 \pm E_k$; by the hypothesis of the lemma $\nu(\frac{2}{3} \pm E_k) \geq \eta \nu(\pm E_k) = \eta \nu(E_k)$. Since there are at most b circumference sites in $\underset{\sim}{\Lambda}'$, inequality (39) must hold as claimed.

We finish the argument by following [3] in estimating $\int_{\mathcal{B}'} e^{-\beta H} d\nu$. They show that either

$$H_\Lambda(\tau_1 \sigma) \leq H_\Lambda(\sigma) - Jb/9 \tag{40a}$$

or

$$H_\Lambda(\tau_2 \sigma) \leq H_\Lambda(\sigma) - Jb/9 . \tag{40b}$$

Let $\mathcal{B}_n^1 \subset \mathcal{B}_n$ be the set of all configurations in $\mathcal{B}_n$ such that (40a) holds, and let $\mathcal{B}_n^2 = \mathcal{B}_n - \mathcal{B}_n^1$. Then

$$\int_{\mathcal{B}'} e^{-\beta H(\sigma)} d\nu(\sigma) = \sum_{\alpha=1,2} \sum_{n \in \prod_{C_B}\{2,3\}} \int_{\mathcal{B}_n^\alpha} e^{-\beta H(\sigma)} d\nu(\sigma). \tag{41}$$

But

$$\int_{\mathcal{B}_n^\alpha} e^{-\beta H(\sigma)} d\nu(\sigma) \leqslant \eta^{-b} e^{-\frac{1}{9}\beta J b} \cdot Z \tag{42}$$

since

$$Z \geqslant \int_{\tau_\alpha \mathcal{B}_n^\alpha} e^{-\beta H(s)} d\nu(s) = \int_{\mathcal{B}_n^\alpha} e^{-\beta H(\tau_\alpha \sigma)} d(\tau_\alpha^* \nu)(\sigma) \geqslant \eta^b e^{\beta J b/9} \int_{\mathcal{B}_n^\alpha} e^{-\beta H(\sigma)} d\nu(\sigma). \tag{43}$$

Using estimate (42) in (41) and summing over α and n we find

$$Z^{-1} \int_{\mathcal{B}'} e^{-\beta H(\sigma)} d\nu(\sigma) \leqslant 2 \left(\frac{2}{\eta}\right)^b e^{-\beta J b/9} \quad . \tag{44}$$

If we take into account the fact that when the border B appears either of the components Λ', Λ'' may be the -component then we obtain estimate (29) for P_B.

QED

<u>Theorem 6</u>: Let $(\mathbb{Z}^n, H, \nu)$ be a nearest-neighbor Ising ferromagnet on a lattice of dimension $n \geqslant 2$ with Hamiltonian

$$H = -\sum_{k \in \mathbb{Z}^n} \sum_{\alpha=1}^{n} J_\alpha \sigma_k \sigma_{k+1_\alpha} \qquad J_\alpha > 0, \; 1_\alpha = (\overbrace{0, \cdots, 0, 1}^{\alpha}, \underbrace{0, \cdots, 0}_{n-\alpha}) \tag{45}$$

whose single-spin measure ν is not the delta-function: $\nu \neq \delta$. If the temperature β^{-1} is sufficiently low, there exists $L > 0$ such that

$$\langle \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_{2p}} ; H, \nu, \beta \rangle \geqslant L^p \qquad \forall\, i_1, \cdots, i_{2p} \in \mathbb{Z}^n , \tag{46}$$

where the infinite-volume limit is taken with the free boundary condition. Thus the model is long-range ordered at zero external field and low temperature.

Proof:

By Lemma 4 and the remarks preceding it, it is sufficient to consider isotropic two-dimensional models. Since the single-spin measure ν is not entirely concentrated at zero, there exists $c>0$ such that $0<\nu[-c,c]<1$. If $\nu(-\frac{c}{3},\frac{c}{3})=0$ define the measure $\Delta\nu$ to be

$$\Delta\nu = \frac{1}{2}\left(\frac{1}{\nu[-c,c]}-1\right)\chi_{[-c,c]}\cdot\nu + \frac{1}{2}(1-\nu[-c,c])\,\frac{\delta_{-c}+\delta_c}{2}\;, \tag{47}$$

where $\chi_{[-c,c]}$ is the characteristic function of the interval $[-c,c]$ and $\delta_{\pm c}$ is the delta-function at $\pm c$. If $\nu(-\frac{c}{3},\frac{c}{3})\neq 0$ define the measure $\Delta\nu$ by

$$\Delta\nu(E) = \frac{1-\nu[-c,c]}{4\nu(-\frac{c}{3},\frac{c}{3})}\cdot\left[\nu\left(\left[E\cap(-c,-\tfrac{c}{3})\right]+\tfrac{2c}{3}\right)+\nu\left(\left[E\cap(\tfrac{c}{3},c)\right]-\tfrac{2c}{3}\right)\right]+\frac{1}{2}(1-\nu[-c,c])\,\frac{\delta_{-c}+\delta_c}{2}. \tag{48}$$

Formula (48) shifts a small multiple of the measure in the central third to the left and right thirds and adds delta-functions at $\pm c$.

Set $\nu_c=\chi_{[-c,c]}\cdot\nu+\Delta\nu$. By Theorem II.4.1, replacing ν by ν_c causes the moments $\langle\sigma_A\rangle$ to decrease. But ν_c obeys the hypothesis of Lemma 5, so by its conclusion at low temperature $\langle\sigma_i\sigma_j\rangle\geq L\;\;\forall i,j\in\mathbb{Z}^n$. The rest of the theorem follows by repeated use of the second Griffiths inequality, Corollary II.2.3.

QED

If the single-spin measure $\nu=\delta$, there is obviously no long-range order because all nontrivial moments of the Gibbs measure are zero. Also, we show in Appendix C that one-dimensional nearest-neighbor models always cluster exponentially (have finite correlation length). Thus, our qualitative information on long-range order in nearest-neighbor ferromagnets is fairly complete.

We conclude this section with a short discussion of the infinite-volume transfer matrix $\mathcal{J}$ of a nearest-neighbor ferromagnet $(\mathbb{Z}^n, H, \nu)$. (The external field h need not be zero.) We define this operator using a method of Osterwalder and Schrader [38] devised in the study of quantum field theory. Other definitions of $\mathcal{J}$ of a more probabilistic [34] or algebraic [48] nature have been given, but we feel that the approach followed here is the simplest one currently available. We end our remarks on the transfer matrix by stating a theorem that characterizes the cluster properties of $(\mathbb{Z}^n, H, \nu)$ in terms of spectral properties of $\mathcal{J}$.

The transfer matrix is associated with the decay properties of a model in a definite direction, which we assume to be the 1-direction. Let $\mathbb{Z}^n_+$ be the half-space $\mathbb{Z}^n_+ = \{(k_1, \ldots, k_n) \in \mathbb{Z}^n : k_1 \geq 0\}$. Associate with each $i \in \mathbb{Z}^n_+$ a commutative indeterminate s_i, and let $\mathcal{S}_+$ be the vector space of formal polynomials $P(s_{i_1}, s_{i_2}, \cdots)$ with real coefficients in these commuting indeterminates. Define the bilinear form $(\, , \,)_+$ on $\mathcal{S}_+$ by

$$\left(P(s_{i_1}, \ldots, s_{i_a}), Q(s_{j_1}, \cdots, s_{j_b})\right)_+ = \langle P(\sigma_{\theta(i_1)}, \ldots, \sigma_{\theta(i_a)}) \cdot Q(\sigma_{j_1}, \ldots, \sigma_{j_b}); H, \nu, \beta \rangle . \quad (49)$$

Here θ reverses the first component of $k\in\mathbb{Z}_+^n$: if $k = (k_1,\cdots,k_n)$ then $\theta(k) = (-k_1,k_2,\cdots,k_n)$. In Appendix C we show that $(\ ,\ )_+$ is a positive semidefinite scalar product. Define $\mathcal{T}:\mathcal{S}_+\to\mathcal{S}_+$ by

$$(\mathcal{T}P)(s_{i_1},\cdots,s_{i_a}) = P(s_{i_1+1_1},\cdots,s_{i_a+1_1}), \tag{50}$$

where as usual $1_1\in\mathbb{Z}^n$ is $(1,0,\cdots,0)$. Note that the polynomial 1 is an eigenvector of $\mathcal{T}$ with eigenvalue 1; we call this eigenvector the ground state of $\mathcal{T}$. By translation-invariance, $\mathcal{T}$ is symmetric with respect to $(\,,)_+$: $(\mathcal{T}P,Q)_+=(P,\mathcal{T}Q)_+$. We show in Appendix C that for all $P\in\mathcal{S}_+$

$$0\leqslant(P,\mathcal{T}P)\leqslant(P,P)_+ . \tag{51}$$

In particular, $\mathcal{T}$ annihilates the null-space $\mathcal{N}=\{Q\in\mathcal{S}_+:(Q,Q)_+=0\}$, and so extends by continuity to a nonnegative self-adjoint contraction on the Hilbert space completion of the quotient $\mathcal{S}_+/\mathcal{N}$. This contraction, which we also denote by $\mathcal{T}$, is the transfer matrix at inverse temperature β of the model $(\mathbb{Z}^n,H,\nu)$. As the following theorem shows, cluster properties of $(\mathbb{Z}^n,H,\nu)$ in the 1-direction may be characterized in terms of the behavior of the spectrum of $\mathcal{T}$ near 1. We recall that the geometric multiplicity of an eigenvalue λ of an operator $\mathcal{T}$ is $\dim\{\psi:\mathcal{T}\psi=\lambda\psi\}$.

<u>Theorem 7</u>: Let $\mathcal{T}$ be the infinite-volume transfer matrix in the 1-direction of the nearest-neighbor ferromagnet $(\mathbb{Z}^n,H,\nu)$ at inverse temperature β .

Then:

(1) The model is long-range ordered $\Leftrightarrow$ the geometric multiplicity of $1 \in \mathrm{spec}(\mathcal{J})$ is greater than 1.

(2) The model clusters in the 1-direction $\Leftrightarrow$ the geometric multiplicity of $1 \in \mathrm{spec}(\mathcal{J})$ is 1.

(3) The model clusters exponentially in the 1-direction $\Leftrightarrow$ 1 is an isolated eigenvalue of $\mathcal{J}$. Let $\lambda_1 \in \mathrm{spec}(\mathcal{J})$ be $\sup\{\lambda \in \mathrm{spec}(\mathcal{J}) : \lambda < 1\}$. Then the correlation length in the 1-direction χ_1 is given by the formula

$$\chi_1 = 1/\ln\left(\frac{1}{\lambda_1}\right) . \tag{52}$$

The proof of this result is given in Appendix C. Note that by the first equivalence, Theorem 7 may be reinterpreted to say that the ground state of the infinite-volume transfer matrix is degenerate at low temperatures. The transfer matrix of nearest-neighbor models with spin ½ spins is analyzed in detail in [34].

Section 4: Spontaneous Magnetization

Let $(\mathcal{L}, H, \nu)$ be an Ising ferromagnet such that at inverse temperature β all moments of the infinite-volume Gibbs measure with the free boundary condition remain finite in the presence of an arbitrary additional uniform external field h:

$$\langle \sigma_A ; H - h\sum_{\mathcal{L}}\sigma_j, \nu, \beta\rangle < \infty \qquad \forall h \in [0,\infty), \forall A \in \mathcal{F}_0(\mathcal{L}). \tag{1}$$

The moments $\langle \sigma_A ; h\rangle = \langle \sigma_A ; H - h\sum_{\mathcal{L}}\sigma_j, \nu, \beta\rangle$ all increase in $h \in [0,\infty)$, and so are continuous functions of h except possibly for a countable set of points. Moreover, $\langle \sigma_A ; h\rangle$ is always continuous from the left (in $[0,\infty)$) by a monotonicity argument: if $h_n \uparrow h$ then

$$\begin{aligned}\lim_{n\to\infty}\langle \sigma_A ; h_n\rangle = \sup_n \langle \sigma_A ; h_n\rangle = \sup_n \sup_\Lambda \langle \sigma_A ; h_n\rangle_\Lambda = \sup_\Lambda \sup_n \langle \sigma_A ; h_n\rangle_\Lambda \\ = \langle \sigma_A ; h\rangle .\end{aligned} \tag{2}$$

The magnetization at site i of $(\mathcal{L}, H, \nu)$ with inverse temperature β is by definition the first moment $\langle \sigma_i ; H, \nu, \beta\rangle$. If $\langle \sigma_i ; h\rangle$ is discontinuous from the right at h=0 we say $(\mathcal{L}, H, \nu)$ is spontaneously magnetized at site i, and we define the spontaneous magnetization at i to be

$$m_s(i) = \lim_{h\downarrow 0}\langle \sigma_i ; H - h\sum_{\mathcal{L}}\sigma_j, \nu, \beta\rangle - \langle \sigma_i ; H, \nu, \beta\rangle . \tag{3}$$

Suppose that the single-spin measure ν is such that the G.H.S. inequality (Corollary II.2.8) holds, and that the external field of the Hamiltonian H is uniformly bounded below:

$$H = -\sum_{|K|\geq 2} J_K \sigma_K - \sum_{j\in\mathcal{L}} h_j \sigma_j \qquad , \; 0 < h_* \leq h_j \;\; \forall j . \tag{4}$$

Then $\langle \sigma_i ; H - h\sum_j \sigma_j \rangle$ extends to a monotone increasing function of h on the enlarged interval $[-h_*, \infty)$ which by the G.H.S. inequality is concave, and in particular continuous on $(-h_*, \infty)$. Thus if the G.H.S. inequality holds and the external field of H is bounded away from zero, there can be no spontaneous magnetization.

If the Hamiltonian $H = -\sum J_K \sigma_K$ is invariant under simultaneous reversal of all spins, so that $J_K = 0$ unless $|K|$ is even, then all odd moments $\langle \prod_{a=1}^{2n+1} \sigma_{i_a} ; H \rangle$ must vanish. In this case the appearance of spontaneous magnetization may be viewed as a spontaneous breaking of the spin reversal symmetry. We note that by the second Griffiths inequality (Corollary II.2.3), if all sites are spontaneously magnetized then the discontinuity in the magnetization shows up in the higher odd moments:

$$\lim_{h \downarrow 0} \langle \prod_{a=1}^{2n+1} \sigma_{i_a} ; h \rangle \geq \prod_{a=1}^{2n+1} \lim_{h \downarrow 0} \langle \sigma_{i_a} ; h \rangle > 0. \tag{5}$$

Also by the second Griffiths inequality the spontaneous magnetization $m_s(i)$ for a Hamiltonian invariant under spin-reversal symmetry increases in the couplings J_K and the inverse temperature β.

We restrict our further analysis to Ising ferromagnets $(\mathbb{Z}^n, H, \nu)$ with connected translation-invariant finite-range pair interactions, which for simplicity we call translation-invariant ferromagnets in this section. We show that if a translation-invariant model is long-range ordered in some direction, then it is spontaneously magnetized. Since we have shown in Section 3 that the nearest-neighbor ferromagnets in at least two dimensions with zero external field are long-range ordered at low temperature, we conclude that they are also spontaneously magnetized. By a small geometric argument we may eliminate the nearest-neighbor restriction and obtain spontaneous magnetization for all translation-invariant models. It is also possible to modify the proof of Theorem 3.6 and obtain this result directly. We finish the section by deriving spontaneous magnetization of anisotropic plane rotators on lattices of dimension at least two as a corollary of our main result.

Proposition 1: Let $(\mathbb{Z}^n, H, \nu)$ be a translation-invariant ferromagnet. If $\exists\ L>0$ such that

$$\langle \sigma_i\sigma_j ; H,\nu,\beta\rangle - \langle \sigma_i ; H,\nu,\beta\rangle\langle \sigma_j ; H,\nu,\beta\rangle \geq L \qquad \forall i,j\in\mathbb{Z}^n, \tag{6}$$

then $(\mathbb{Z}^n, H, \nu)$ is spontaneously magnetized, and

$$m_s = \lim_{h\downarrow 0}\langle \sigma_i ; H-h\sum_{k\in\mathbb{Z}^n}\sigma_k,\nu,\beta\rangle - \langle\sigma_i ; H,\nu,\beta\rangle \geq \sqrt{L+\langle\sigma_i ; H\rangle^2} - \langle\sigma_i ; H\rangle > 0. \tag{7}$$

Proof:

Let h be the uniform external field added to the field already present in H, and for $A\in\mathcal{Y}_0(\mathbb{Z}^n)$ set $\langle\sigma_A ; H-h\sum_{\mathbb{Z}^n}\sigma_k,\nu,\beta\rangle = \langle\sigma_A ; h\rangle$. By Proposition 3.2 the two-point function of $(\mathbb{Z}^n, H-h\sum_{\mathbb{Z}^n}\sigma_k, \nu)$ must cluster except for a countable set of h's, so we may find a decreasing sequence $h_m\downarrow 0$ such that $\langle\sigma_i\sigma_j ; h_m\rangle\to\langle\sigma_i ; h_m\rangle\langle\sigma_j ; h_m\rangle$ as $|i-j|\to\infty$ $\forall m$. By the long-range order (6) and the second Griffiths inequality,

$$\langle\sigma_i\sigma_j ; h_m\rangle \geq \langle\sigma_i ; 0\rangle^2 + L\ . \tag{8}$$

Taking first $|i-j|\to\infty$ and then $h_m\downarrow 0$ gives the desired result.

QED

With Proposition 1 in hand, it follows from Theorem 3.6 that translation-invariant models in dimension at least two are spontaneously magnetized. We generalize this slightly to obtain

Theorem 2: Let $(\mathbb{Z}^n, H, \nu)$ be an Ising ferromagnet on a lattice of dimension $n \geqslant 2$ with the connected translation-invariant pair interaction

$$H = -\sum_{i,j\in\mathbb{Z}^n} J_{(i-j)}\sigma_i\sigma_j \qquad J_{-k} = J_k \geqslant 0 \quad \forall\, k\in\mathbb{Z}^n \tag{9}$$

whose (even) single-spin measure ν is not the delta-function δ. If the temperature β^{-1} is sufficiently low the model is spontaneously magnetized:

$$m_s = \lim_{h\downarrow 0} \langle \sigma_i ; H - h\sum_{\mathbb{Z}^n}\sigma_k, \nu, \beta\rangle > 0. \tag{10}$$

Proof:

Take $i=0$. By connectedness and translation invariance $\exists\, k,\ell\in\mathbb{Z}^n$ linearly independent such that $J_k, J_\ell \neq 0$. Let $V = \{ak+b\ell : a,b\in\mathbb{Z}\}$. Reducing to zero all couplings in H except for J_k, J_ℓ makes V a sublattice disconnected from the rest of $\mathbb{Z}^n$ isomorphic to a nearest-neighbor model on $\mathbb{Z}^2$. The theorem now follows from Theorem 3.6 and Proposition 1. QED

This theorem may also be proved directly. Indeed, the proof of Theorem 3.6 may be modified to show that putting an arbitrarily small (volume-independent) external field on the boundary of a sequence of regions growing to infinity causes a spontaneous magnetization. In the language of equilibrium states (see equations (2.23, 2.24) and the accompanying discussion), we say that the equilibrium state $\langle\, ; H, \nu, \beta\rangle_h$ of $(\mathbb{Z}^n, H, \nu)$ with boundary condition

$$(\mathcal{Y}_{\tilde{\Lambda}})_i = h > 0 \qquad\qquad \forall \Lambda \in \mathcal{P}_0(\mathbb{Z}^n),\ \forall i \in \tilde{\Lambda} \tag{11}$$

is spontaneously magnetized: $\langle \sigma; H, \nu, \beta \rangle_h > 0$. For spin ½ models, the state $\langle\ ; H, \nu, \beta \rangle_h$ is independent of h for $h > 0$:

$$\langle\ ; H, \nu, \beta \rangle_h = \langle\ ; H, \nu, \beta \rangle_+ \qquad \forall\ h > 0. \quad [30] \tag{12}$$

One expects (12) is true for any single-spin measure, but this is not presently known. By taking the negative boundary condition $-h < 0$ one may also construct the equilibrium state $\langle\ ; H, \nu, \beta \rangle_{-h}$, which differs from $\langle\ ; H, \nu, \beta \rangle_h$ only in the sign of the expectation of an odd number of spins by virtue of the spin-reversal transformation $\sigma \to -\sigma$. The average

$$\langle\ ; H, \nu, \beta \rangle_{av} = \tfrac{1}{2}\left(\langle\ \rangle_{-h} + \langle\ \rangle_h \right) \tag{13}$$

thus has vanishing odd moments like the state $\langle\ ; H, \nu, \beta \rangle$ with the free boundary condition. By the second Griffiths inequality (Corollary II.2.3) and the spontaneous magnetization of $\langle\ \rangle_{\pm h}$, the average state is long-range ordered:

$$\langle \sigma_i \sigma_j \rangle_{av} \geq \tfrac{1}{2}\left(\langle \sigma_i \rangle_h \langle \sigma_j \rangle_h + \langle \sigma_i \rangle_{-h} \langle \sigma_j \rangle_{-h} \right) > 0 \ . \tag{14}$$

If we could show

$$\langle\ ; H, \nu, \beta \rangle = \langle\ ; H, \nu, \beta \rangle_{av} \qquad \text{(conjecture)} \tag{15}$$

for connected models ((15) need not hold for disconnected models), then taking into account Proposition 1 we could conclude that spontaneous magnetization and long-range order are simply different manifestations of the same basic phenomenon. Unfortunately, we cannot prove (15) at present.

The critical (inverse) temperature β_c for spontaneous magnetization is the largest inverse temperature for which spontaneous magnetization does not appear:

$$\beta_c = \sup\{\beta : \lim_{h\downarrow 0}\langle\sigma; H-h\Sigma\sigma_k, \nu, \beta\rangle = 0\}. \tag{16}$$

For the two-dimensional spin ½ nearest-neighbor Ising model with Hamiltonian

$$H = -J_1 \sum_{(i_1,i_2)\in\mathbb{Z}^2} \sigma_{(i_1,i_2)}\sigma_{(i_1+1,i_2)} - J_2 \sum_{(i_1,i_2)\in\mathbb{Z}^2} \sigma_{(i_1,i_2)}\sigma_{(i_1,i_2+1)}, \tag{17}$$

β_c is well-known [3 and references therein] to be given by

$$\sinh(2\beta_c J_1)\cdot\sinh(2\beta_c J_2) = 1 \quad ; \tag{18}$$

when $J_1=J_2=J$ this yields

$$\beta_c = \frac{1}{2J}\log(1+\sqrt{2}) \quad . \tag{19}$$

If we combine (19) with Proposition II.4.2 we obtain upper bounds on the critical (inverse) temperatures of models whose single-spin measures are absolutely continuous near zero. For example, in the model $(\mathbb{Z}^2, H, \nu)$ with Hamiltonian (17), $J_1=J_2=J$, and single-spin measure

$$d\nu(\sigma) = \left[\frac{2}{\Gamma(\frac{1}{4})}\right] \exp(-\sigma^4)\, d\sigma \quad ,$$

(20)

$T = \sup \left\{ t \in \mathbb{R} : 2t \cdot \operatorname{ess\,sup}_{[-t,t]} \left[\frac{d\nu(\sigma)}{d\sigma}\right] \leqslant 1 \right\}$ is given by

$$T = \Gamma(5/4), \tag{21}$$

so that the critical temperature β_c of this model has the upper bound

$$\beta_c \leqslant \frac{2}{J(\Gamma(\frac{5}{4}))^2} \log(1+\sqrt{2}). \tag{22}$$

A more interesting single-spin measure to which this method applies is $d\nu(\sigma) = \exp(-\sigma^4 + b\sigma^2)\,d\sigma \Big/ \int_{\mathbb{R}} \exp(-s^4 + bs^2)\,ds$, though we have not worked out the details.

We conclude with a corollary exhibiting spontaneous magnetization for anisotropic plane rotors by a comparison with Ising ferromagnets.

Corollary 3: Let $(\mathbb{Z}^n, H, d\rho(r)\,d\theta)$ be an anisotropic ferromagnet plane rotor on a lattice of dimension $n \geqslant 2$ whose Hamiltonian H is a connected translation-invariant pair interaction of the form

$$H = -\sum_{i,j \in \mathbb{Z}^n} \left[J^x_{(i-j)} \sigma^x_i \sigma^x_j + \gamma_{(i-j)} J^x_{(i-j)} \sigma^y_i \sigma^y_j \right], \quad \begin{array}{l} J^x_k = J^x_{-k} \geqslant 0 \\ 0 \leqslant \gamma_k = \gamma_{-k} < 1 . \end{array} \tag{23}$$

If the radial measure ρ is not the delta-function δ, then for sufficiently low temperature β^{-1} the model is spontaneously magnetized in the x-direction.

Proof:

An argument of H. Kunz [25] employing the Ginibre inequality for plane rotors [12] shows that the spontaneous magnetization m_{Rot} in the x-direction of the model $(\mathbb{Z}^n, H, d\rho(r)d\theta)$ is bounded below by the spontaneous magnetization m_{Ising} of the Ising ferromagnet $(\mathbb{Z}^n, H_{Ising}, \nu)$, where

$$H_{Ising} = -\sum_{i,j\in\mathbb{Z}^n} J^x_{(i-j)} \left(1-\gamma_{(i-j)}\right)\sigma_i\sigma_j \tag{24}$$

and ν is defined by

$$\nu(E) = \int_{E\times\mathbb{R}} d\rho(r)d\theta \ . \tag{25}$$

Since ρ is not the delta-function δ, neither is ν, so by Theorem 2 $0 < m_{Ising} \leq m_{Rot}$ for sufficiently low temperature.

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We emphasize that this is a result about anisotropic rotors. It is known that isotropic plane rotors $(\gamma_{(i-j)}=1)$ on a lattice $\mathbb{Z}^n$ of dimension $n \leq 2$ are not spontaneously magnetized [33]. The likelihood that isotropic plane rotors are magnetized on a lattice of dimension at least three is a subject of considerable current interest.

Section 5: Phase Separation and Breakdown of Translational Symmetry

In this section we investigate the breaking of translational symmetry at low temperature. It is clear from the construction of Section 2 that if a ferromagnet $(\mathbb{Z}^n, H, \nu)$ has a translation-invariant Hamiltonian then the infinite-volume Gibbs state with the free boundary condition is translation-invariant:

$$\langle f(\sigma_{k_1},\cdots,\sigma_{k_m})\rangle = \langle f(\sigma_{k_1+i},\cdots,\sigma_{k_m+i})\rangle \quad \forall\, i\in\mathbb{Z}^n,\ \forall f\in\otimes^m C(\ddot{\mathbb{R}}). \tag{1}$$

However, as with the spin-reversal symmetry discussed in the previous section, we shall find that at low temperature a model may have non-translationally-invariant equilibrium states. For certain models with disconnected Hamiltonians this is readily apparent. As an example, consider a model on $\mathbb{Z}^2$ with spin measure $\nu\neq\delta$ and a pure second-neighbor Hamiltonian:

$$H = -J\sum_{(i_1,i_2)\in\mathbb{Z}^2}\left[\sigma_{(i_1,i_2)}\sigma_{(i_1+2,i_2)} + \sigma_{(i_1,i_2)}\sigma_{(i_1,i_2+2)}\right] \quad . \tag{2}$$

The lattice has four (non-interacting) components connected with respect to H, each of which is equivalent to a two-dimensional nearest-neighbor interaction. Apply a uniform external field h to the sites of one of the components, which for specificity we take to be the even component $\{(i_1,i_2)\in\mathbb{Z}^2 : i_1, i_2 \text{ are even}\}$. Upon decreasing h to zero we are left in an equilibrium state $\langle\ \rangle_{NTI}$ of $(\mathbb{Z}^2, H, \nu)$ which at sufficiently low temperature

β^{-1} breaks translational symmetry because

$$\langle \sigma_{(i_1,i_2)} \rangle_{NTI} = \begin{Bmatrix} m_s \neq 0 \text{ if } i_1, i_2 \text{ are both even} \\ 0 \text{ otherwise} \end{Bmatrix}. \tag{3}$$

An example of a connected model with non-translation-invariant equilibrium states was given by Slawny [44]. The following theorem shows that the appearance at low temperature of equilibrium states which are not translation-invariant is a fairly general phenomenon.

Theorem 1: Let $(\mathbb{Z}^n, H, \nu)$ be the nearest-neighbor Ising ferromagnet in dimension $n \geq 3$ with Hamiltonian

$$H = -J \sum_{i \in \mathbb{Z}^n} \sum_{\alpha=1}^{n} \sigma_i \sigma_{i+1} \qquad J > 0 \qquad 1_\alpha = (\overbrace{0,\cdots,0,1}^{\alpha},\underbrace{0,\cdots,0}_{n-\alpha}) \tag{4}$$

and $\nu \neq \delta$. Let m_s be the spontaneous magnetization of the nearest-neighbor ferromagnet $(\mathbb{Z}^{n-1}, H', \nu)$ in dimension n-1 with the same single-spin measure ν and coupling J:

$$H = -J \sum_{i' \in \mathbb{Z}^{n-1}} \sum_{\alpha'=1}^{n-1} \sigma'_{i'} \sigma'_{i'+1_{\alpha'}}. \tag{5}$$

Then for any inverse temperature β there exists an equilibrium state $\langle\ \rangle_{NTI}$ of $(\mathbb{Z}^n, H, \nu)$ such that

$$\begin{aligned} \langle \sigma_i \rangle_{NTI} &\geq m_s \qquad \forall\ i = (i_1, \cdots, i_n) \in \mathbb{Z}^n :\ i_1 \geq 0 \\ \langle \sigma_i \rangle_{NTI} &\leq -m_s \qquad \forall\ i = (i_1, \cdots i_n) \in \mathbb{Z}^n :\ i_1 < 0. \end{aligned} \tag{6}$$

Since for low enough temperature the spontaneous magnetization $m_s > 0$, the state $\langle\ \rangle_{NTI}$ is not translation-invariant.

Remark: The phenomenon elucidated by this theorem - an equilibrium state where one half of the model is positively magnetized and the other half negatively magnetized - is called phase separation or sharp phase interface. Other work on phase separation has been done by Dobrushin [6].

Proof:

The proof, taken largely from [46], generalizes to arbitrary single-spin measures an argument given by van Beijeren [1] for spin ½ spins. Let $\Lambda_M \subset \mathbb{Z}^{n-1}$ be the square $\{(i_1,\dots,i_{n-1}) \in \mathbb{Z}^{n-1} : |i_k| \leq M\ \forall k\}$ and let $\Lambda_{M,N} \subset \mathbb{Z}^n$ be the box $\{-N,-N+1,\dots,N\} \times \Lambda_M = \{(i_1,\dots,i_n) \in \mathbb{Z}^n : |i_1| \leq N,\ |i_k| \leq M,\ k \geq 2\}$.

Define

$$\Lambda^+_{M,N} = \{i \in \Lambda_{M,N} : i_1 > 0\} \tag{7a}$$

$$\Lambda^0_{M,N} = \{i \in \Lambda_{M,N} : i_1 = 0\} \tag{7b}$$

$$\Lambda^-_{M,N} = \{i \in \Lambda_{M,N} : i_1 < 0\}\ . \tag{7c}$$

Add a uniform external field h to all spins in $\Lambda^+_{M,N} \cup \Lambda^0_{M,N}$, a field -h to all spins in $\Lambda^-_{M,N}$, and a field h to the spins in $\Lambda_M \subset \mathbb{Z}^{n-1}$. Figure 1 aids in visualizing the situation.

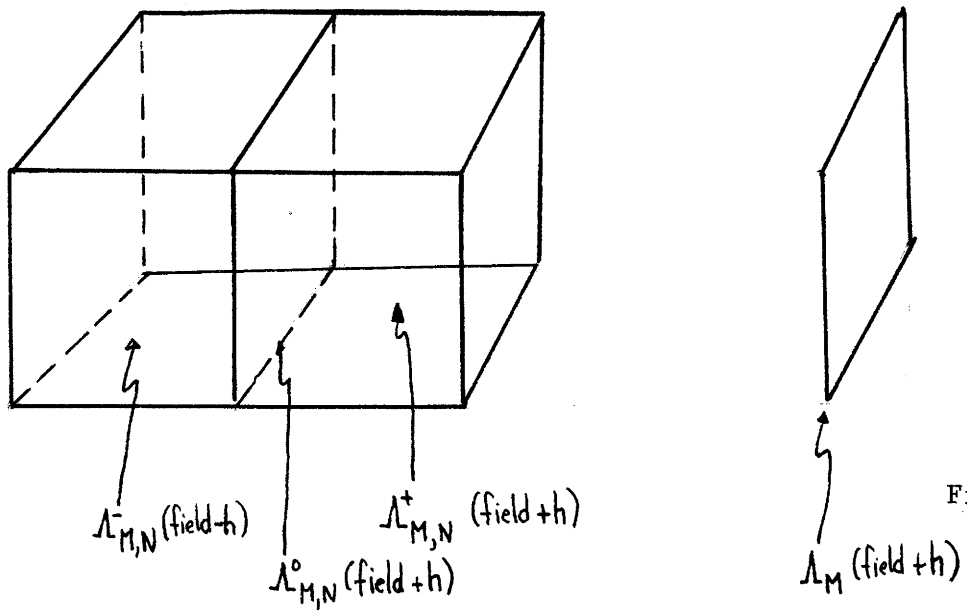


Figure 1

This alters the Hamiltonians $H_{\Lambda_{M,N}}$ and H'_{Λ_M} obtained from $(\mathbb{Z}^n, H, \nu)$ and $(\mathbb{Z}^{n-1}, H', \nu)$ to

$$H_{\Lambda_{M,N}}(h) = -J\Big[\sum_{[ij]\subset\Lambda^{+}_{M,N}} (\sigma_i\sigma_j + \sigma_{-i}\sigma_{-j}) + \sum_{[ab]\subset\Lambda^{o}_{M,N}} \sigma_a\sigma_b + \sum_{[ia]\left\{\begin{smallmatrix} i\in\Lambda^{+}_{M,N} \\ a\in\Lambda^{o}_{M,N}\end{smallmatrix}\right.} \sigma_a(\sigma_i+\sigma_{-i})\Big] - h\Big[\sum_{\Lambda^{+}_{M,N}} (\sigma_i - \sigma_{-i}) + \sum_{\Lambda^{o}_{M,N}} \sigma_a\Big]$$

$$H'_{\Lambda_M}(h) = -J\sum_{[a'b']\subset\Lambda_M} \sigma'_{a'}\sigma'_{b'} - h\sum_{a'\in\Lambda_M} \sigma'_{a'} , \tag{8a}$$

(8b)

where by site $-i$ we mean the reflection $(-i_1, i_2, \cdots, i_n)$ of i through $\Lambda^{o}_{M,N}$ and square brackets $[ij]$ indicate nearest neighbors. We claim that

$$\langle \sigma_a ; H_{\Lambda_{M,N}}(h), \nu, \beta\rangle \geq \langle \sigma'_{a'} ; H'_{\Lambda_M}(h), \nu, \beta\rangle \qquad \forall\, a \in \Lambda^{o}_{M,N}, \tag{9}$$

where if $a = (0, a_2, \cdots, a_n)$ then $a' = (a_2, \cdots, a_n)$. This says that by burying the square Λ_M in the box $\Lambda_{M,N}$ as we have done we increase the magnetization. To verify (9), regard the two non-interacting systems $(\mathbb{Z}^n, H, \nu)$,

$(\mathbb{Z}^{n-1}, H', \nu)$ as a single combined system, and introduce the sum and difference variables (II.2.1):

$$t_i = \frac{1}{\sqrt{2}}(\sigma_i + \sigma_{-i}) \qquad q_i = \frac{1}{\sqrt{2}}(\sigma_i - \sigma_{-i}) \qquad i \in \Lambda^+_{M,N} \tag{10a}$$

$$t_a = \frac{1}{\sqrt{2}}(\sigma_a + \sigma'_{a'}) \qquad q_a = \frac{1}{\sqrt{2}}(\sigma_a - \sigma'_{a'}) \qquad a \in \Lambda^0_{M,N} . \tag{10b}$$

The total Hamiltonian in the transformed variables $H(\sigma) + H'(\sigma')$ is

$$-J\Big[\sum_{[ij]}(t_i t_j + q_i q_j) + \sum_{[ab]}(t_a t_b + q_a q_b) + \sum_{[ia]}(t_a + q_a)t_i\Big] - h\sqrt{2}\Big[\sum_i q_i + \sum_a t_a\Big]. \tag{11}$$

Since this is a polynomial in the q's and t's with negative coefficients, by the method of proof of Theorem II.2.2 we find that for any families of sites $A, B \in \mathcal{Y}_0(\Lambda^+_{M,N} \cup \Lambda^0_{M,N})$,

$$\langle q_A t_B \rangle \geq 0 . \tag{12}$$

In particular, for any $a \in \Lambda^0_{M,N}$

$$\langle q_a \rangle \geq 0 \Rightarrow \langle \sigma_a \rangle_{\Lambda_{M,N}} \geq \langle \sigma'_{a'} \rangle_{\Lambda_M} \tag{13}$$

as claimed.

Now let $N \to \infty$, so that the box $\Lambda_{M,N}$ increases to a box $\Lambda_{M,\infty}$ which is infinite in one direction. Using the transfer matrix method described in Appendix C we can control this limit to obtain a state $\langle\ ; H_{\Lambda_{M,\infty}}(h), \nu, \beta \rangle$

on $C(\Pi_{\Lambda_{M,\infty}} \ddot{\mathbb{R}})$ such that

$$\langle \sigma_a ; H_{\Lambda_{M,\infty}}(h), \nu, \beta \rangle \geq \langle \sigma_{a'} ; H'_{\Lambda_M}(h), \nu, \beta \rangle \qquad \forall a \in \overset{\circ}{\Lambda}_{M,\infty}. \tag{14}$$

Moreover for any $i \in \Lambda^{+}_{M,\infty}$ if we define $i' = (i_2, \cdots, i_n)$ to be the projection on $\overset{\circ}{\Lambda}_M$ then

$$\langle \sigma_i ; H_{\Lambda_{M,\infty}}(h) \rangle \geq \langle \sigma_{i'} ; H_{\Lambda_M}(h) \rangle \quad . \tag{15}$$

This is because by increasing to $+h$ all fields on spins $-j \in \Lambda^{-}_{M,\infty}$ such that $|j_1| \leq i_1$ we bring the spin $(0, i_2, \ , i_n)$ to the position previously occupied by spin i. But by Corollary II.2.5, which holds for arbitrary external field, increasing the field can only increase the magnetization.

Each state $\langle \ ; H_{\Lambda_{M,\infty}}(h) \rangle$ on $C(\Pi_{\Lambda_{M,\infty}} \ddot{\mathbb{R}})$ may be extended to a state on $C(\Pi_{\mathbb{Z}^n} \ddot{\mathbb{R}})$, for which we use the same notation, by abstract considerations. This gives a sequence of states on $C(\Pi_{\mathbb{Z}^n} \ddot{\mathbb{R}})$ indexed by the size M of Λ_M; since the set of states is compact in the weak* topology some subnet of this sequence converges and the limit $\langle \ ; H(h), \nu, \beta \rangle$ is an equilibrium state of the (non-ferromagnetic) model $(\mathbb{Z}^n, H(h), \nu)$ with the property that

$$\langle \sigma_i ; H(h), \nu, \beta \rangle_{NTI} \geq \langle \sigma_{i'} ; H'(h), \nu, \beta \rangle \quad \forall \ i : i_1 \geq 0 \tag{16a}$$

$$\langle \sigma_i ; H(h), \nu, \beta \rangle_{NTI} \leq - \langle \sigma_{i'} ; H'(h), \nu, \beta \rangle \quad \forall \ i : i_1 < 0. \tag{16b}$$

If we let $h \to 0$ and again use the method of compactness and subnets we obtain an equilibrium state $\langle\ ; H, \nu, \beta\rangle_{NTI}$ of the ferromagnet $(\mathbb{Z}^n, H, \nu)$ with

$$\langle\sigma_i ; H, \nu, \beta\rangle \geq m_s \qquad , \; i_1 \geq 0 \tag{17a}$$

$$\langle\sigma_i ; H, \nu, \beta\rangle \leq -m_s \qquad , \; i_1 < 0 \; . \tag{17b}$$

$\overline{\underline{\text{QED}}}$

Section 6: Applications to Quantum Field Theory

In this section we comment on applications of correlation inequalities in quantum field theory. Since an adequate description of the formalism of quantum field theory is lengthy, and since the connection between statistical mechanics and field theory is discussed in detail elsewhere (e.g. [20], [43], and further references therein), we shall only make very brief remarks and avoid technical details.

There is a strong similarity between the formal expression [43]

$$\langle \Omega, \Phi(x_1)\cdots\Phi(x_n)\Omega\rangle = \tag{1}$$

$$= \frac{\int_{\Pi_{\mathbb{R}^2}(\mathbb{R})} \Phi(x_1)\cdots\Phi(x_n)\exp\left[-\int_{\mathbb{R}^2}\left\{\frac{1}{2}\Phi(x)(-\Delta+m^2)\Phi(x)+:P(\Phi(x)):\right\}d^2x\right]\mathcal{D}\Phi}{\int_{\Pi_{\mathbb{R}^2}(\mathbb{R})} \exp\left[-\int_{\mathbb{R}^2}\left\{\frac{1}{2}\Phi(x)(-\Delta+m^2)\Phi(x)+:P(\Phi(x)):\right\}d^2x\right]\mathcal{D}\Phi}$$

for the Schwinger functions of a Euclidean $P(\phi)_2$ quantum field theory and the formal expression

$$\langle \sigma_{i_1}\cdots\sigma_{i_n}\rangle = \frac{\int_{\Pi_{\mathbb{Z}^2}(\mathbb{R})} \sigma_{i_1}\cdots\sigma_{i_n}\exp[-\beta H(\sigma)]\prod d\nu(\sigma_i)}{\int_{\Pi_{\mathbb{Z}^2}(\mathbb{R})} \exp[-\beta H(\sigma)]\prod d\nu(\sigma_i)} \tag{2}$$

for the moments of the Gibbs measure of an Ising model ($\mathbb{Z}^2$,H,ν). If we approximate the plane $\mathbb{R}^2$ by a lattice $\epsilon\mathbb{Z}^2$ with grid spacing ϵ in formula (1) we find

$$\langle \Omega, \Phi(x_1)\cdots\Phi(x_n)\Omega\rangle \cong$$

$$\frac{\displaystyle\int_{\Pi_{\epsilon\mathbb{Z}^2}(\mathbb{R})} \Phi(i_1)\cdots\Phi(i_n)\exp\Big[\lambda\sum_{[ij]}\Phi(i)\Phi(j)\Big]\prod_{\epsilon\mathbb{Z}^2}\exp\Big[-\tfrac{1}{2}(4+\epsilon^2m^2)\Phi(i)^2-\epsilon^2{:}P(\Phi(i)){:}\Big]d\Phi(i)}{\displaystyle\int_{\Pi_{\epsilon\mathbb{Z}^2}(\mathbb{R})} \exp\Big[\lambda\sum_{[ij]}\Phi(i)\Phi(j)\Big]\prod_{\epsilon\mathbb{Z}^2}\exp\Big[-\tfrac{1}{2}(4+\epsilon^2m^2)\Phi(i)^2-\epsilon^2{:}P(\Phi(i)){:}\Big]d\Phi(i)}, \tag{3}$$

where $i_1, \cdots, i_n$ are the closest lattice points to $x_1, \cdots, x_n$ and $[ij]$ as usual indicates nearest-neighbors in $\epsilon\mathbb{Z}^2$. This is the formula for the moments of the Gibbs measure of ferromagnetic Ising model with nearest-neighbor Hamiltonian

$$H_{\text{Ising}} = -\lambda\sum_{[ij]}\sigma_i\sigma_j \tag{4}$$

and single-spin measure

$$d\nu = \exp\Big[-\tfrac{1}{2}(4+\epsilon^2m^2)\sigma^2-\epsilon^2{:}P(\sigma){:}\Big]d\sigma \tag{5}$$

Observe that the (off-diagonal part of the) free quantum Hamiltonian gives rise to the Ising Hamiltonian coupling the spins at different sites, while the interaction part, and on-diagonal free part, of the quantum Hamiltonian appear only in the single-spin measure of the Ising model. In [20], Guerra, Rosen, and Simon show by means of a special ultraviolet cutoff that the heuristic approximation (3), known as the lattice approximation, in fact converges rigorously as the grid spacing $\epsilon \to 0$. This is the primary physical motivation for the study of continuous-spin Ising models.

By means of the lattice approximation, inequalities for Ising models may be taken over directly to give inequalities for Euclidean $P(\phi)_2$ models. Theorems II.2.1, II.2.2, II.2.4 and the F.K.G. inequality [11], which hold for Ising models with arbitrary single-spin measures, are also valid for all $P(\phi)_2$ models. Theorems II.2.6, III.2.1, and IV.3.1, which hold for Ising models with a restricted class of single-spin measures, are known to be valid only for ϕ^4 models. (Of course, these are the models of principal interest in four dimensions.) Also, certain objects we have dealt with in Ising models have direct counterparts in field theories. For example, the inverse correlation length in a spin system corresponds with the mass in a field theory, and the transfer matrix $\mathfrak{J}$ with the exponential of the Hamiltonian e^{-H}.

Many of the applications of correlation inequalities in field theory are motivated by similar applications in spin systems. For example, the second Griffiths inequality (Corollary II.2.3) is used to control the infinite-volume limit [43]. The F.K.G. inequality (which we have not yet described) shows that the mass is determined by the decay of the two-point function. The G.H.S. inequality (Corollary II.2.8) yields monotonicity of the mass in the external field, and either the Lebowitz or the Gaussian inequalities (Corollary II.2.7; Corollary III.2.2) prove that the n-point function is bounded above by sums of products of two-point functions.

The results we have presented on long-range order and spontaneous magnetization also have analogs in field theory, though the methods are more complex. Glimm, Jaffe, and Spencer show in [16] that the model is long-range ordered for large dimensionless coupling constant. Finally, correlation inequalities have been used in some field theory problems with no immediate antecedents in statistical mechanics. For example, Spencer [45] (see Feldman [10]) employs the Lebowitz inequalities to show weakly coupled ϕ_2^4 theories have no even two-particle bound states, and Glimm and Jaffe [14] invoke the Griffiths, Lebowitz, and u_6 inequalities to prove absolute bounds on vertices and couplings.

Correlation inequalities, and the Euclidean methods on which they are predicated, are of major importance in the recent progress of the constructive program in quantum field theory.

Chapter VI: Unsolved Problems and Concluding Remarks

In this final chapter we present four unsolved problems, and make some concluding remarks. The problems we shall comment on are: the conjectured Γ_6 inequality, long-range order and spontaneous magnetization for plane rotors on lattices of dimension at least three, Griffiths inequalities for vector spin models, and the conjectured u_n inequality.

Let $(\mathbb{Z}^2, H, \nu)$ be a translation-invariant nearest-neighbor ferromagnet with zero external field and single-spin measure of the form $d\nu(\sigma) = \exp(-\sigma^4 + a\sigma^2)d\sigma$. Let C be the covariance matrix

$$C_{ij} = \langle \sigma_i \sigma_j \rangle \tag{1}$$

and let C^{-1} be its formal inverse. Given sites $i_1, \cdots, i_6 \in \mathbb{Z}^2$ we define

$$\Gamma_6(i_1,\cdots,i_6) = u_6(i_1,\cdots,i_6) + \sum_{\{a,b,c\}\subset\{1,\cdots,6\}} \sum_{k,\ell\in\mathbb{Z}^2} u_4(i_a,i_b,i_c,k)\,[C^{-1}]_{k\ell}\, u_4(\ell,i_d,i_e,i_f), \tag{2}$$

where u_4, u_6 are Ursell functions and the sum $\sum_{\{a,b,c\}\subset\{1,\cdots,6\}}$ is over all ten partitions of $\{1,\cdots,6\}$ into two disjoint sets $\{a,b,c\}$, $\{d,e,f\}$. The conjecture to be proved (or disproved), at least for the minimal class of models indicated above, is that

$$\Gamma_6(i_1,\cdots,i_6) \leqslant 0 . \tag{3}$$

A proof of (3) would have many important consequences in quantum field theory [15]. For spin ½ spins on a one-dimensional lattice (3) has been established by explicit calculation (J. Rosen: private communication), but in the continuous-spin case little seems to be known. Actually, knowledge of the special case

$$T_6(i,i,i,j,j,j) \leq 0 \tag{4}$$

would be very useful. This inequality has been established numerically (in a computer study) for the anharmonic oscillator [32], which is a continuum limit of our one-dimensional lattice theory. Also, for spin ½ spins a straightforward computation yields

$$T_6(i,i,i,j,j,j) = -24\langle\sigma_i\sigma_j\rangle^3 \leq 0, \tag{5}$$

essentially independently of any geometric restrictions on the Hamiltonian. But again, for continuous spins on a lattice of dimension at least two, very little is known.

Let us turn now to the question of spontaneous magnetization for plane rotors. In Chapter V we proved that in two or more dimensions a wide class of Ising ferromagnets is spontaneously magnetized at low temperature. This magnetization may be viewed as a spontaneous breakdown of the discrete internal symmetry group $\mathbb{Z}_2$ associated with spin reversal. If we replace the linear spin $\sigma \in \mathbb{R}$ by a vector spin $\sigma \in S^1$ and consider plane rotors, the internal symmetry group becomes the continuous group O(2). As shown in [33], this larger internal symmetry group precludes any spontaneous magnetization in two or fewer dimensions, though if we break the O(2) symmetry by making the Hamiltonian anisotropic a magnetization appears

(Corollary V.4.3). Thus, the natural problem is to determine whether plane rotors on three-dimensional lattices are spontaneously magnetized. Indications are that this is so, but it has not yet been proved. One approach might be to replace the circle S^1 on which the spin is distributed by the set of n^{th} roots of unity, show there is a magnetization in this simpler case, and control the limit as $n \to \infty$. We note that once one has spontaneous magnetization for this classical plane rotor (single-spin measure the uniform distribution on S^1), the rotor analog of Theorem II.4.1 and other techniques permit the extension of this result to a much larger class of single-spin measures.

Continuing our discussion of vector spin models, let us now consider the problem of Griffiths inequalities. As we easily show in Chapter II, the first Griffiths inequality (Theorem II.2.1) holds for all O(n)-symmetric vector spin models with spin $\sigma \in \mathbb{R}^n$. (Indeed, it holds even more generally.) Unfortunately, for models with the usual $\vec{\sigma}_i \cdot \vec{\sigma}_j$ interactions, our information about the second Griffiths inequality is much more limited: it is only proved for n=1,2 [12] , or, very recently, n=3 [26] . A general result in this direction (or knowledge that there is no such result) would be valuable in the construction and understanding of vector spin models.

We finish our remarks on unsolved problems with a reformulation of the conjecture

$$(-1)^{\frac{n}{2}+1} u_n \geq 0 \qquad n \text{ even} \qquad (6)$$

on the signs of the Ursell functions of spin ½ Ising ferromagnets with pair interactions and zero external field. Chapter IV was devoted to a study of this problem. Our method, which we could only carry through for n=2,4, and 6, was to expand $Z^{\frac{n}{2}}u_n$ as a Maclaurin series in the couplings J_{ij} and show that each Maclaurin coefficient $\left.\frac{\partial^m}{\partial J_{i_1j_1}\cdots\partial J_{i_mj_m}} Z^{\frac{n}{2}}u_n\right|_{J=0}$ had sign $(-1)^{\frac{n}{2}+1}$. In Appendix B we transform this problem about derivatives into an abstract combinatoric question about the topological nature of certain linear graphs. We now give a brief description of this reformulation; for greater detail see the Appendix.

For Ursell functions of order n, call a graph G a nontrivial graph of order n if it is connected, if exactly n vertices ("dummy vertices") have a single incident edge (called "argument edges"), and if all other vertices have an even number of incident edges. An even partition of G is a graph which is formed by partitioning the edges incident at each non-dummy vertex into sets of even cardinality, and tying together the edges of each set in the partition at a newly-created vertex. For example, taking n=4, if G is the nontrivial graph

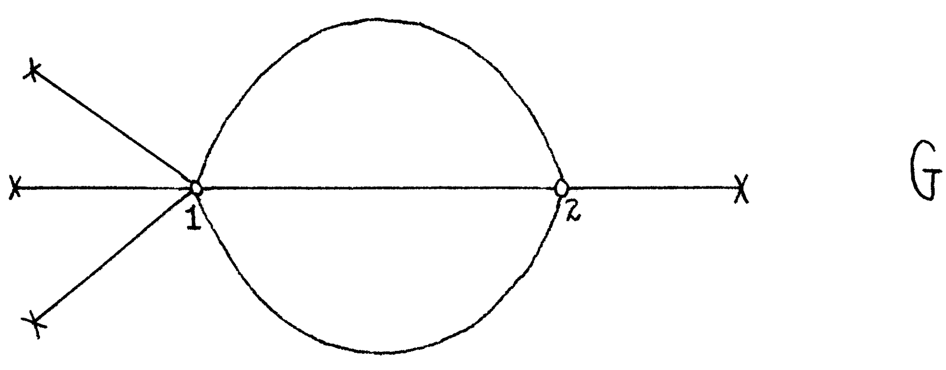
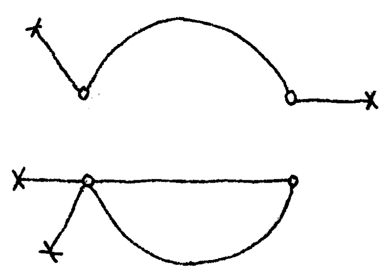


Figure 1

and we partition the edges at non-dummy vertices 1 and 2 as in Figure 2,

Figure 2

the resulting even partition $\mathcal{G}$ of G is the (disconnected) graph



$\mathcal{G}$. Figure 3

Let $\amalg_e(G)$ be the set of even partitions of G, and partially order $\amalg_e(G)$ by refinement: $\mathcal{G} \leq \mathcal{H} \Leftrightarrow \mathcal{G}$ refines $\mathcal{H}$. Then $\amalg_e(G)$ has no least element, so we adjoin to $\amalg_e(G)$ a least element 0 and call the enlarged set $\amalg_e^*(G)$. For $\mathcal{G} \in \amalg_e^*(G)$ set

$$c_0(\mathcal{G}) = \begin{cases} -\infty \text{ if } \mathcal{G}=0 \text{ or all argument edges are not in same connected component of } \mathcal{G} \\ \text{one less than number of connected components if all argument edges in same component} \end{cases} \tag{7}$$

Thus for the even partition $\mathcal{G}$ of Figure 3, $c_0(\mathcal{G}) = -\infty$. If $\mathcal{G} \in \amalg_e^*(G)$ we define $u(\mathcal{G})$ recursively by

$$-\left(\frac{n}{2}\right)^{c_0(\mathcal{G})} = \sum_{\mathcal{G} \leq \mathcal{H} \in \amalg_e^*(G)} u(\mathcal{H}). \tag{8}$$

Then, as shown in Appendix B, all derivatives $\left.\frac{\partial^m}{\partial J_{i_1 j_1} \cdots \partial J_{i_m j_m}} Z^{\frac{n}{2}} u_n \right|_{J=0}$ have sign $(-1)^{\frac{n}{2}+1}$ if and only if

$$(-1)^{\frac{n}{2}+1} u(0) \geq 0 \tag{9}$$

for all nontrivial graphs G of order n. Proposition IV.4.1 implies this inequality if the cyclomatic number of G is at most four. The general case is not known. This completes our discussion of unsolved problems.

In this thesis we have given new proofs, extended prior results, and derived entirely new theorems, of which the most elegant are probably Theorem V.3.6 and Theorem V.4.2. These theorems show that at low temperature ferromagnetic Ising models in two or more dimensions are long-range ordered and spontaneously magnetized for arbitrary single-spin measure $\nu \neq \delta$. Thus in mathematics as in nature, phase transitions are not pathological but ubiquitous.

Acknowledgements

I would like to thank my advisor, Professor Arthur Jaffe, for his extraordinarily accurate suggestions and helpful guidance, for his frequent encouragement, and for his patience. I would like to thank Henk van Beijeren, Richard Ellis, Joel Feldman, Professor Joel Lebowitz, and Professor Richard Stanley for many useful conversations. Finally, I would like to thank a number of people who, though they did not contribute directly to the writing of this thesis, made it possible for it to be written: Anthony F. Davidowski, Sue Ann Garwood, Harvey P. Greenspan, Henry B. Laufer, Konrad Osterwalder, Barry Simon, Thomas Spencer, Charles A. Strack, and, most of all, my parents Claire and Herbert Sylvester.

To all of these people I am grateful.

Garrett Smith Sylvester
January, 1976

Appendix A: Extensions of Theorem II.2.6

In this appendix we weaken the hypotheses of Theorem II.2.6 and its corollaries, the Lebowitz correlation inequality and the G.H.S. inequality. In Section II.2 we proved this theorem for continuous single-spin distributions of the form

$$d\nu(\sigma) = \exp(-P(\sigma))\,d\sigma \Big/ \int_{\mathbb{R}} \exp(-P(s))\,ds, \tag{1}$$

where P is an even polynomial with arbitrary quadratic (and constant) and nonnegative higher coefficients:

$$P(\sigma) = \sum_{i=0}^{p} c_{2i}\sigma^{2i} \quad , \quad c_{2i} \geq 0 \text{ for } i \geq 2;\ c_2, c_0 \text{ arbitrary}. \tag{2}$$

Here we show it holds for arbitrary c_2, c_0 when just $c_4, c_{2p} > 0$, provided $c_6, c_8, \cdots, c_{2p-2}$ are not too negative. Recent results of a similar nature may be found in [9] . We prove additionally that one may even have $c_4 < 0$, provided it is not too negative, though in this case the range of c_2 must be restricted.

The proof in Section II.2 reduced to showing

$$\int_{\mathbb{R}^4} \alpha^k \beta^\ell \gamma^m \delta^n \sinh[\alpha\beta\gamma\delta \cdot R(\alpha^2, \cdots, \delta^2)] \exp[-Q(\alpha^2, \cdots, \delta^2)]\,d\alpha \cdots d\delta \geq 0 \tag{3}$$

for k, ℓ, m, n all odd. Here Q and R are polynomials related to P by

$$P(\sigma) + P(\tau) + P(\sigma') + P(\tau') = Q(\alpha^2, \cdots, \delta^2) - \alpha\beta\gamma\delta\, R(\alpha^2, \cdots, \delta^2). \tag{4}$$

They are given explicitly by

$$Q(W,X,Y,Z) = \sum_{i=0}^{p} 4^{-i+1} c_{2i}\, Q_i(W,X,Y,Z) \tag{5a}$$

$$R(W,X,Y,Z) = \sum_{i=2}^{p} 4^{-i+1} c_{2i}\, R_{i-2}(W,X,Y,Z), \tag{5b}$$

Q_i and R_i being the symmetric homogeneous polynomials of degree i with positive coefficients defined by

$$Q_i(W,X,Y,Z) = \sum_{a+b+c+d=i} \frac{(2i)!}{(2a)!\cdots(2d)!} W^a X^b Y^c Z^d \tag{6a}$$

$$R_i(W,X,Y,Z) = \sum_{a+b+c+d=i} \frac{(2i+4)!}{(2a+1)!\cdots(2d+1)!} W^a X^b Y^c Z^d . \tag{6b}$$

(Note that c_2 and c_0 do not appear in R. This is the reason that Theorem II.2.6 holds for arbitrary c_2, c_0.)

As the exponents k, ℓ, m, n in (3) are all odd, the integrand has the same sign as $R(\alpha^2, \cdots, \delta^2)$. Thus, if $R(\alpha^2, \cdots, \delta^2)$ is nonnegative, (3) will hold, and Theorem II.2.6 will be valid. We state this as a proposition:

Proposition 1: Let A,B,C,D be families of sites in a ferromagnetic Ising model with Hamiltonian

$$H = -\sum_{i,j} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i \qquad J_{ij} \geq 0,\ h_i \geq 0 \tag{7}$$

and single-spin distribution (1). If the polynomial $R(\alpha^2, \cdots, \delta^2)$ defined by (5b), (6b) is nonnegative, then

$$\langle \alpha_A \beta_B \gamma_C \delta_D \rangle \geq 0. \tag{8}$$

From (5b) and (6b) it is clear that a sufficient condition for R to be nonnegative is that employed in Theorem II.2.6, namely, that all coefficients of P higher than the quadratic be nonnegative. However, this condition is not necessary: only the trailing and leading coefficients of R, which control the behavior at zero and infinity, must be positive; intervening coefficients may be negative, provided they are not too large. Unfortunately, the author as yet has been unable to locate or determine necessary and sufficient conditions on the coefficients of an even symmetric polynomial in four variables which will make it nonnegative; indeed, he has been unable to find useful such conditions for polynomials in just one variable, though sufficient conditions stronger than nonnegativity of all coefficients are not difficult to manufacture in this simpler instance. The one-variable case is of interest because we may decouple the four variables in R, reducing it to a sum of four polynomials in a single variable each, thereby allowing one-variable criteria to be applied. Information is lost in process, however.

<u>Proposition 2</u>: Assume that the coefficients c_4, c_{2p} in (2) are positive, while all remaining coefficients are arbitrary. Let

$I_+ = \{p\} \cup \{i \in \{3,4,\cdots,p-1\} : c_{2i} \geq 0\}$, $I_- = \{i \in \{3,4,\cdots,p\} : c_{2i} < 0\}$.

Define the polynomial

$$T(X) = \sum_{i \in I_+} (2i)(2i-1)(2i-2)\cdot 4^{-i+1}\cdot c_{2i}X^{i-2} + \sum_{i \in I_-} \frac{(4^{i-1}-1)}{2} c_{2i}X^{i-2} + \frac{3}{2} c_4. \quad (9)$$

Then

$$R(\alpha^2,\beta^2,\gamma^2,\delta^2) \geqslant T(\alpha^2) + T(\beta^2) + T(\gamma^2) + T(\delta^2), \tag{10}$$

so that the hypothesis of Proposition 1 will be satisfied if $T(X) \geqslant 0$ for $X \geqslant 0$.

Proof:

To obtain (10) from the definition (5b) of R, we must give a lower bound on those R_i associated with positive coefficients and an upper bound on those associated with negative coefficients.

The lower bound is immediate: just drop all cross-terms in definition (6b) to conclude

$$R_i(W,X,Y,Z) \geqslant (2i+4)(2i+3)(2i+2)(W^i + X^i + Y^i + Z^i), \quad W,X,Y,Z \geq 0. \tag{11}$$

The upper bound requires a little more effort. Exploiting the inequality

$$W^a X^b Y^c Z^d \leqslant \frac{a}{i}W^i + \frac{b}{i}X^i + \frac{c}{i}Y^i + \frac{d}{i}Z^i \ , \quad i=a+b+c+d \ , \tag{12}$$

which holds whenever all quantities involved are nonnegative, we find

$$R_i(W,X,Y,Z) \leqslant R_i(1,1,1,1)(W^i + X^i + Y^i + Z^i)/4, \quad W,X,Y,Z \geqslant 0. \tag{13}$$

We evaluate $R_i(1,1,1,1)$, finding

$$R_i(1,1,1,1) = 2\cdot(4^{2i+2} - 4^{i+1}), \tag{14}$$

so that (13) becomes

$$R_i(W,X,Y,Z) \leqslant \tfrac{1}{2}(4^{2i+2} - 4^{i+1})(W^i + \cdots + Z^i), \quad W,X,Y,Z \geqslant 0. \tag{15}$$

From (11) and (15) the proposition is immediate.

QED

Thus far we have proved (8), the Ellis-Monroe inequality, when $R(\alpha^2, \beta^2, \gamma^2, \delta^2) \geqslant 0$, obtaining results valid for all quadratic coefficients c_2. One may also allow R to become slightly negative and still show that inequality (3), and hence inequality (8), is valid. However, the range of allowed negative coefficients must depend on c_2, because as $c_2 \to \infty$, the integrand in (3) becomes concentrated at zero, with $\pi^{-2} c_2^2 \exp[-c_2(\alpha^2 + \cdots + \delta^2)]$ converging to $\delta(\alpha)\delta(\beta)\delta(\gamma)\delta(\delta)$, in somewhat unfortunate notation. In particular, if the trailing coefficient of R is negative, inequality (3) will always be violated for sufficiently large c_2, even though it holds in the limit because the δ-function forces the integral to zero. This is because the contribution to the integral where $R \leqslant 0$ decays like an inverse power of c_2, while the remainder of the integral decays exponentially in c_2.

We indicate one set of constraints on the coefficients under which the Ellis-Monroe inequality (8) may be proved.

Proposition 3: Let A,B,C,D be families of sites in a ferromagnetic Ising model with single-spin distribution (1) and Hamiltonian (7). Let the leading coefficient $c_{2p} > 0$ of P be specified. For all $M > 0$, there exists $-\epsilon < 0$, depending only on c_{2p} and M, such that if

$$|c_2| \leq M \text{ and } -\epsilon \leq c_{2i} \leq M, \quad 2 \leq i \leq p-1 \tag{16}$$

then

$$\langle \alpha_A \beta_B \gamma_C \delta_D \rangle \geq 0 .$$

Proof:

The first part of the proof proceeds as in Section II.2, and we reduce the proposition to proving inequality (3) for all odd k, ℓ, m, n. Denote by $B(\epsilon, M)$ the compact region $[-M,M] \times \left(\bigtimes_{i=2}^{p-1} [-\epsilon, M] \right)$ in which the coefficients c_{2i}, $1 \leq i \leq p-1$, are allowed to vary. Inequality (3) is strict when $\epsilon = 0$, because the integrand is a nonnegative continuous function not identically zero. Since $B(0,M)$ is compact, we see that by taking ϵ small enough (3) may be guaranteed by a continuity argument for any finite set of exponents k, ℓ, m, n. The proof will be complete if we can also show there exist $-\epsilon < 0$ and exponents k_0, ℓ_0, m_0, n_0 such that if the coefficients c_{2i} lie in $B(\epsilon, M)$, then (3) holds for all $k \geq k_0, \cdots, n \geq n_0$. For this purpose we need the following lemma, which is proved by the method of Proposition 2.

Lemma 4: There exist $c, \epsilon > 0$ such that if the coefficients c_{2i} lie in $B(\epsilon, M)$ then $R(\alpha^2, \cdots, \delta^2) \geq c$ for $(\alpha, \beta, \gamma, \delta)$ outside the cube $K = [-1,1] \times [-1,1] \times [-1,1] \times [-1,1]$.

Take c, ϵ from Lemma 4. Since $|\sinh(\alpha\beta\gamma\delta \cdot R(\alpha^2, \cdots, \delta^2)) \exp(-Q(\alpha^2, \cdots, \delta^2))|$ is uniformly bounded above by some constant C as $(\alpha, \beta, \gamma, \delta)$ varies over K and the coefficients c_{2i} vary over $B(\epsilon, M)$, we have

$$\left| \int_K \alpha^k \cdots \delta^n \sinh(\alpha\beta\gamma\delta \cdot R) \exp(-Q)\, d\alpha \cdots d\delta \right| \leqslant 16C \frac{1}{(k+1)\cdots(n+1)} \quad . \tag{17}$$

On the other hand, as the coefficients c_{2i} vary over $B(\epsilon, M)$, $\exp(-Q(\alpha^2, \cdots, \delta^2))$ is bounded below by some positive function $e(\alpha^2, \cdots, \delta^2)$. Also, by the lemma, $R(\alpha^2, \cdots, \delta^2) \geqslant c$ for $(\alpha, \beta, \gamma, \delta) \notin K$; thus,

$$\alpha^k \cdots \delta^n \sinh(\alpha\beta\gamma\delta \cdot R(\alpha^2, \cdots, \delta^2)) \geqslant c\alpha^{k+1} \cdots \delta^{n+1} \tag{18}$$

for $(\alpha, \beta, \gamma, \delta) \notin K$. Consequently,

$$\begin{aligned} \int_{\tilde{K}} \alpha^k \cdots \delta^n \sinh(\alpha\beta\gamma\delta \cdot R) \exp(-Q)\, d\alpha \cdots d\delta &\geqslant \int_{\tilde{K}} c\alpha^{k+1} \cdots \delta^{n+1} e(\alpha^2, \cdots, \delta^2)\, d\alpha \cdots d\delta \\ &\geqslant \int_{|\alpha| \geqslant 2, \cdots, |\delta| \geqslant 2} c\alpha^{k+1} \cdots \delta^{n+1} e(\alpha^2, \cdots, \delta^2)\, d\alpha \cdots d\delta \\ &\geqslant \left[16c \int_{|\alpha| \geqslant 2, \cdots, |\delta| \geqslant 2} e(\alpha^2, \cdots, \delta^2)\, d\alpha \cdots d\delta \right] 2^{k+\ell+m+n} \quad . \end{aligned} \tag{19}$$

Taken together, (17) and (19) show that indeed (3) holds for large k, ℓ, m, n uniformly as the coefficients c_{2i} lie in $B(\epsilon, M)$.

$\overline{\text{QED}}$

Appendix B: Computational Algorithms for Ursell Functions

In this appendix we give algorithms to evaluate the networks and graphs of arbitrary order introduced in Section IV.4. We point out an interesting combinatoric interpretation of the algorithm for graphs. Finally, we present the results of a computer study of graphs whose signs were not determined by the methods of Chapter IV. In every calculated example. the sign was $(-1)^{\frac{n}{2}+1}$ as conjectured.

In evaluating networks and graphs, the trace factors over the sites of the model, so our primary problem is to compute $\mathrm{Tr}(\sigma^{q_1}\cdots\sigma^{q_v})$. Recall that the superscripts q_i are copy indices with the superscript vector $\vec{q}\in \times_{i=1}^{v}\{0,1,\cdots,\frac{n}{2}-1\}$; the site subscript common to all the σ's is omitted. We may suppose v is even, for if it is odd the trace vanishes. Let $\amalg_e(\{1,\cdots,v\})$ be the collection of all even partitions of $\{1,\cdots,v\}$; that is, all partitions $\mathcal{P}$ of $\{1,\cdots,v\}$ each of whose elements $P\in\mathcal{P}$ has even cardinality. Given a vector $\vec{q}\in\times_{i=1}^{v}\{0,1,\cdots,\frac{n}{2}-1\}$ of superscripts, the partition $\mathcal{P}(\vec{q})$ of $\{1,\cdots,v\}$ is defined by the equivalence relation $i\equiv j \Leftrightarrow q_i=q_j$. With this notation we find

$$\mathrm{Tr}(\sigma^{q_1}\cdots\sigma^{q_v}) = \begin{Bmatrix} 1, & \mathcal{P}(\vec{q})\in\amalg_e(\{1,\cdots,v\}) \\ 0 & \text{otherwise} \end{Bmatrix}. \tag{1}$$

We next express equation (1) in a way convenient for calculation. Given $\mathcal{P}\in\amalg_e(\{1,\cdots,v\})$ define $\delta_{\mathcal{P}}: \times_1^{v}\{0,\cdots,\frac{n}{2}-1\}\to\{0,1\}$ by

$$\delta_{\mathcal{P}}(\vec{y}) = \begin{Bmatrix} 1 \text{ if } \forall P\in\mathcal{P},\ \forall i,j\in P,\ y_i = y_j \\ 0 \text{ otherwise} \end{Bmatrix}. \tag{2}$$

Partially order the set $\coprod_e(\{1,\cdots,v\})$ by refinement: $\mathcal{P}\leq\mathcal{Q}$ if and only if $\mathcal{P}$ refines $\mathcal{Q}$. Define the numbers $\mu_{\mathcal{P}}$, $\mathcal{P}\in\coprod_e(\{1,\cdots,v\})$, by recursion:

$$\sum_{\mathcal{P}'\leq\mathcal{P}} \mu_{\mathcal{P}'} = 1 \ . \tag{3}$$

It may be shown that

$$\mu_{\mathcal{P}} = \prod_{P\in\mathcal{P}} \frac{d^{|P|-1}}{d\lambda^{|P|-1}} \tanh\lambda \Big|_{\lambda=0.} \tag{4}$$

With these definitions, the formula

$$\mathrm{Tr}(\sigma^{y_1}\cdots\sigma^{y_v}) = \sum_{\mathcal{P}\in\coprod_e(\{1,\cdots,v\})} \mu_{\mathcal{P}}\,\delta_{\mathcal{P}}(\vec{y}) \tag{5}$$

is apparent. This is our fundamental identity.

Let us exploit formula (5) to evaluate networks. By the factorization of the trace, we need only evaluate a single vertex $\mathrm{Tr}(f^{\alpha_1}\cdots f^{\alpha_v})$. Now

$$\begin{aligned} \mathrm{Tr}(f^{\alpha_1}\cdots f^{\alpha_v}) &= \sum_{\vec{y}\in\times_1^{v}\{0,\cdots,\frac{n}{2}-1\}} \omega^{\vec{\alpha}\cdot\vec{y}}\ \mathrm{Tr}(\sigma^{y_1}\cdots\sigma^{y_v}) \\ &= \sum_{\vec{y},\mathcal{P}} \omega^{\vec{\alpha}\cdot\vec{y}} \mu_{\mathcal{P}}\,\delta_{\mathcal{P}}(\vec{y})\ . \end{aligned} \tag{6}$$

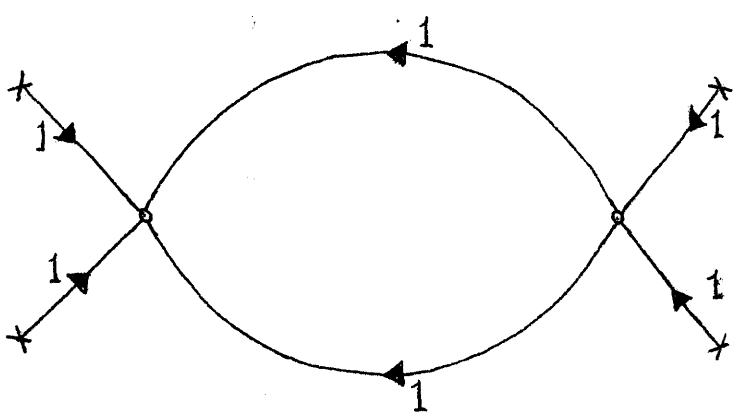
But

$$\sum_{\vec{\eta}} \omega^{\vec{\alpha}\cdot\vec{\eta}} \delta_{\mathcal{P}}(\vec{\eta}) = \prod_{P\in\mathcal{P}} \left(\sum_{k=0}^{\frac{n}{2}-1} \omega^{k\cdot\sum_{i\in P}\alpha_i} \right)$$
$$= \begin{cases} \left(\frac{n}{2}\right)^{|\mathcal{P}|} \text{ if } \sum_{i\in P}\alpha_i \equiv 0 \bmod (n)\ \forall P\in\mathcal{P} \\ 0 \quad \text{otherwise} \end{cases} \tag{7}$$

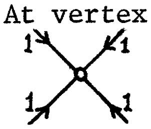
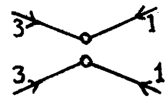
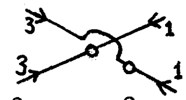
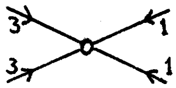
Call a partition of the incident edges at a vertex matched if the sum over each set of edges in the partition of the incoming currents is zero in $\mathbb{Z}_n$. Equation (7) says that the value of a vertex is the sum over all matched partitions $\mathcal{P}$ of $\mu_{\mathcal{P}}\left(\frac{n}{2}\right)^{|\mathcal{P}|}$. We state this formally:

Algorithm 1: With notation as above, the value of a nontrivial network of order n is the product of the values of its vertices. The value of a vertex is the sum over all matched partitions $\mathcal{P}$ of the incident edges of $\left(\frac{n}{2}\right)^{|\mathcal{P}|} \prod_{P\in\mathcal{P}} \frac{d^{|P|-1}}{d\lambda^{|P|-1}} \tanh\lambda \Big|_{\lambda=0}$. The value of a graph may be found by summing over its nontrivial networks.

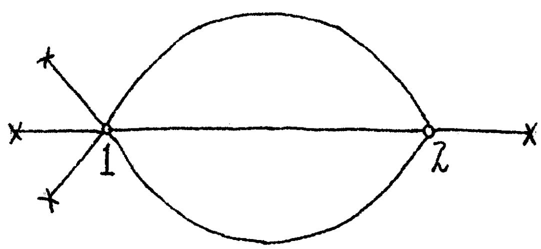
To illustrate the evaluation of networks, consider the example shown in Figure IV.4.2, reproduced below for convenience.



(Figure IV.4.2)

At vertex 1, there is a single matched partition, the one-element partition  Vertex 1 therefore has the value $2^1\cdot(-2) = -4$. At vertex 2 there are three matched partitions:  ,  , and  . Vertex 2 therefore has the value $2^2\cdot 1 + 2^2\cdot 1 + 2^1(-2) = 4$. Multiplying, we find the entire network has value -16.

We may also use formula (5) to evaluate graphs directly, without the necessity of summing over networks. To explain the algorithm we must make a preliminary definition. Let G be a graph which is nontrivial in the sense of Section IV.4. A partition $\mathcal{G}$ of G is a graph formed by partitioning in some way the incident edges at each vertex and tying together the edges of each set in the partition at a newly created vertex. For example, if G is the graph



G

Figure 1

and we partition the edges at vertices 1 and 2 as in Figure 2,

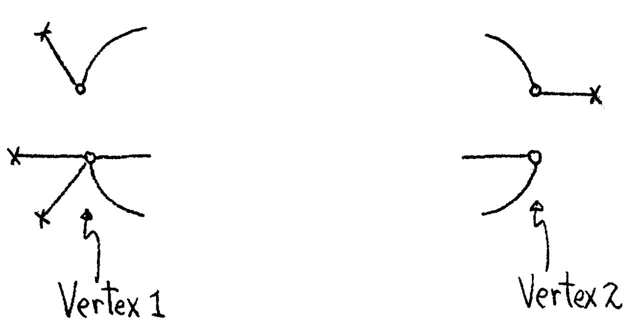
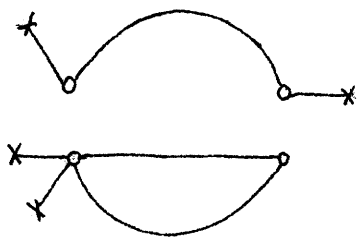


Figure 2

the resulting partition $\mathcal{G}$ of G is the (disconnected) graph



$\mathcal{G}.$ Figure 3

A partition of G is called even if the partition at each (non-dummy) vertex is even, and we denote by $\amalg_e(G)$ the set of all even partitions of G. The graph in Figure 3 is an even partition of the graph in Figure 1.

Let us now focus our attention on the derivative

$$
\begin{aligned}
[G] &= \frac{\partial^m}{\partial J_{i_1 j_1}\cdots \partial J_{i_m j_m}} Z^{\frac{n}{2}} u_n(\sigma_{k_1},\dots,\sigma_{k_n})\Big|_{J=0} \\
&= \left(\frac{2}{n}\right) \sum_{\vec{\beta}\in \times_1^m \{0,\dots,\frac{n}{2}-1\}} \mathrm{Tr}\left(t_{k_1}\cdots t_{k_n}\,\sigma_{i_1}^{\beta_1}\sigma_{j_1}^{\beta_1}\cdots \sigma_{i_m}^{\beta_m}\sigma_{j_m}^{\beta_m}\right) \\
&= \left(\frac{2}{n}\right) \sum_{\substack{\vec{\beta}\in \times_1^m \{0,\dots,\frac{n}{2}-1\} \\ \vec{\alpha}\in \times_1^n \{0,\dots,\frac{n}{2}-1\}}} \omega^{\alpha_1+\cdots+\alpha_n}\, \mathrm{Tr}\left(\sigma_{k_1}^{\alpha_1}\cdots\sigma_{k_n}^{\alpha_n}\sigma_{i_1}^{\beta_1}\cdots\sigma_{j_m}^{\beta_m}\right)
\end{aligned}
\tag{8}
$$

with graph G and (non-dummy) vertices $\mathcal{V}'(G)$. Using (5) to evaluate the trace in (8), we find

$$
[G] = \left(\frac{2}{n}\right) \sum_{\mathcal{P}_v \in \amalg_e(\{1,\dots,|v|\})} \left[\left(\prod_{v\in\mathcal{V}'} \mu_{\mathcal{P}_v}\right) \cdot \sum_{\vec{\alpha}} \left(\omega^{\alpha_1+\cdots+\alpha_n}\left[\sum_{\vec{\beta}} \prod_{v\in\mathcal{V}'} \delta_{\mathcal{P}_v}(\vec{\alpha},\vec{\beta})\right]\right)\right] , \tag{9}
$$

where the number of edges incident on a vertex $v\in\mathcal{V}'(G)$ is $|v|$. We may identify in a natural way each choice of an even partition $\mathcal{P}_v$ at every vertex v with an even partition $\mathcal{G}$ of G, and regard the outer sum $\sum_{\mathcal{P}_v \in \amalg_e(\{1,\cdots,|v|\})}$ as a sum over $\amalg_e(G)$ instead. For fixed $\mathcal{G} \in \amalg_e(G)$ and $\vec{\alpha} \in \times_1^n \{0,\cdots,\frac{n}{2}-1\}$ the innermost sum $\sum_{\vec{\beta}} \prod_{v\in\mathcal{V}'} \delta_{\mathcal{P}_v}(\vec{\alpha},\vec{\beta})$ may be performed explicitly. Define the partition $\mathcal{C}$ of the index set $\{1,\cdots,n\}$ to have as elements the sets C composed of all indices whose argument edges lie in the same connected component of the partition $\mathcal{G}$. $\mathcal{C}$ is an even partition, because the number of odd vertices in a connected graph is even. Let $c_0(\mathcal{G})$ be the number of connected components of $\mathcal{G}$ without argument edges. Then

$$\sum_{\vec{\beta}} \prod_{v\in\mathcal{V}'} \delta_{\mathcal{P}_v}(\vec{\alpha},\vec{\beta}) = \left(\frac{n}{2}\right)^{c_0(\mathcal{G})} \delta_{\mathcal{C}}(\vec{\alpha}) \ . \tag{10}$$

With this formula in hand, the sum over $\vec{\alpha}$ now may be performed:

$$\begin{aligned} \sum_{\vec{\alpha}} \omega^{\alpha_1+\cdots+\alpha_n} \delta_{\mathcal{C}}(\vec{\alpha}) &= \prod_{C\in\mathcal{C}} \left(\sum_{k=0}^{\frac{n}{2}-1} \omega^{k|C|}\right) \\ &= \begin{cases} \frac{n}{2} \ , & \mathcal{C} = \{\{1,\cdots,n\}\} \\ 0 & \text{otherwise} \end{cases} \ . \end{aligned} \tag{11}$$

Thus the term in the sum (9) associated with the even partition vanishes unless all argument edges lie in the same connected component of $\mathcal{G}$. Let $\amalg_{e,c}(G)$ denote the set of even partitions of G having all argument edges in a single connected component. Then we have

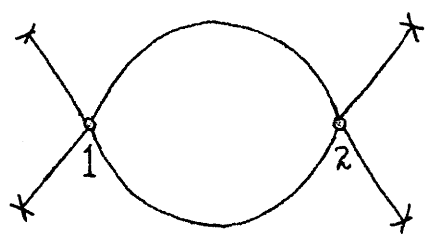
Algorithm 2: With notation as above, the derivative (8) has value

$$[G] = \sum_{\mathcal{G} \in \amalg_{e,c}(G)} \left(\frac{n}{2}\right)^{c_0(\mathcal{G})} \cdot \prod_{v \in \mathcal{V}'(\mathcal{G})} \frac{d^{|v|-1}}{d\lambda^{|v|-1}} \tanh(\lambda) \Big|_{\lambda=0.} \tag{12}$$

It is immediate from this formula that if G is a tree (and hence $\amalg_{e,c}(G)$ has only one element) then

$$[G] = \prod_{v \in \mathcal{V}'(G)} \frac{d^{|v|-1}}{d\lambda^{|v|-1}} \tanh(\lambda) \Big|_{\lambda=0.} \tag{13}$$

As another illustration, consider the graph G of Figure IV.4.1, reproduced below for convenience.



G (Figure IV.1)

The five elements of $\amalg_{e,c}(G)$, and their contributions to [G], are shown in Figure 4.

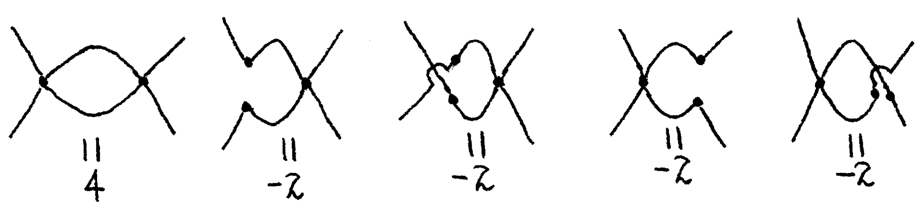


Figure 4

Summing, we find [G]= -4.

Algorithm 2 has an interesting combinatoric interpretation. Fix a nontrivial graph G. The partially ordered set $\coprod_e(G)$ has no least element, so we adjoin to $\coprod_e(G)$ a least element 0 and call the enlarged set $\coprod_e^*(G)$. From the definition (3) and explicit formula (4) for $\mu_{\mathcal{P}}$, we see that for $\mathcal{G}\in\coprod_e(G)$,

$$\prod_{v\in\mathcal{V}(G)} \frac{d^{|v|-1}}{d\lambda^{|v|-1}} \tanh(\lambda)\Big|_{\lambda=0} = -\mu(\mathcal{G},0), \tag{14}$$

where μ is the Möbius function of $\coprod_e^*(G)$. (The definition and some elementary properties of Möbius functions are given in the Technical Appendix to Chapter IV.) Let us alter slightly the definition of $c_0(\mathcal{G})$. For $\mathcal{G}\in\coprod_e^*(G)$ set

$$c_0(\mathcal{G}) = \begin{cases} -\infty \text{ if } \mathcal{G}=0 \text{ or all argument edges fail to lie in same connected component of } \mathcal{G} \\ \text{the number of connected components devoid of argument edges otherwise} \end{cases}. \tag{15}$$

Using (14) and (15), the formula (12) of Algorithm 2 takes the more perspicuous form

$$[G] = -\sum_{\coprod_e^*(G)} \mu(\mathcal{G},0)\left(\tfrac{n}{2}\right)^{c_0(\mathcal{G})} . \tag{16}$$

We may invert this to obtain an expression for $[G]$ strongly analogous to the definition (IV.1.2) of u_n. If $\mathcal{G}\in\coprod_e^*(G)$, define $u(\mathcal{G})$ recursively by

$$-\left(\frac{n}{2}\right)^{c_0(\mathcal{G})} = \sum_{\mathcal{G}\leq\mathcal{H}\in\coprod_e^*(G)} u(\mathcal{H}). \tag{17}$$

By the Möbius inversion formula,

$$[G] = u(0) \, , \tag{18}$$

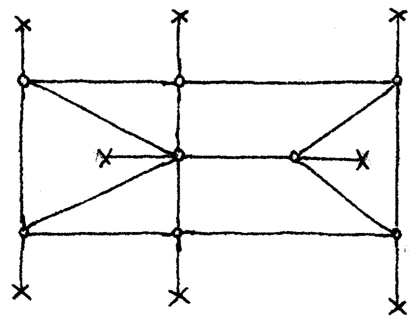
and in this language the conjecture that all derivatives with respect to couplings of $Z^{\frac{n}{2}} u_n$ have sign $(-1)^{\frac{n}{2}+1}$ becomes

$$(-1)^{\frac{n}{2}+1} u(0) \geq 0 \tag{19}$$

for all nontrivial graphs G. Thus we have transformed our conjecture about Ising models into an entirely abstract question about the topological nature of certain linear graphs.

We close with a table of the values of graphs calculated in a computer study. To make the programming problem more tractable, the graphs were evaluated by summing over their nontrivial networks. Networks for graphs 1,2,3,8,and 9 were evaluated in the integer mode with Algorithm 1. Networks for graphs 4-7 were evaluated by using equation (IV.4.11) in the floating point mode. Comparison of identical graphs calculated by the differing methods suggests the error from employing floating point arithmetic was a few parts in 10^5. The primary factor inhibiting a more complete investigation was execution time; Graph 2 ran for 18.2 minutes and Graph 7 for 42.5 minutes.

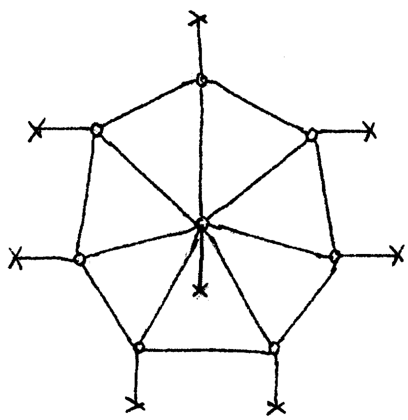
Derivatives of $Z^{n/2}u_n$: Evaluated Graphs



Graph 1: n=8

8 vertices, 6 loops

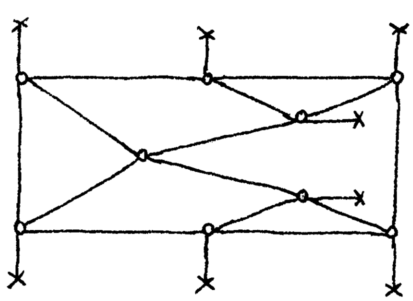
Value: -18,880



Graph 2: n=8

8 vertices, 7 loops

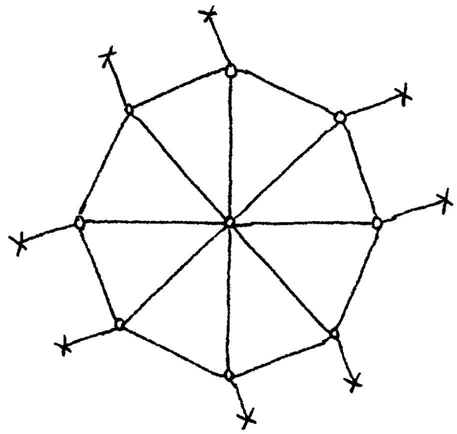
Value: -63,312



Graph 3: n=8

9 vertices, 6 loops

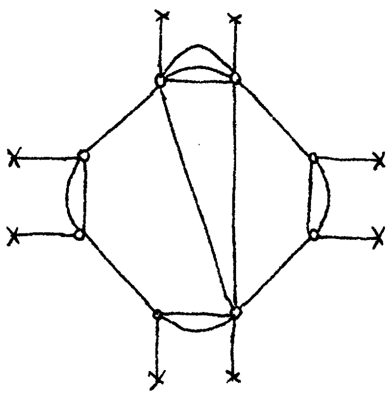
Value: -10,640



Graph 4: n=8

9 vertices, 8 loops

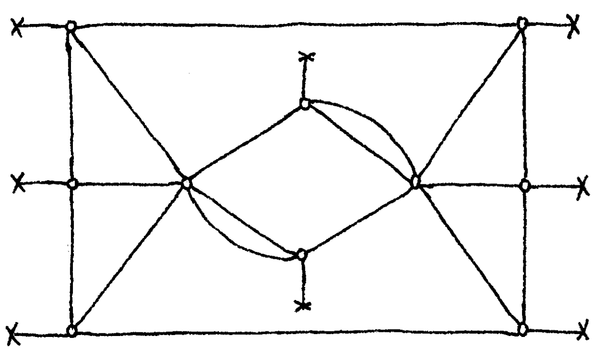
Value: -239,955



Graph 5: n=8

8 vertices, 8 loops

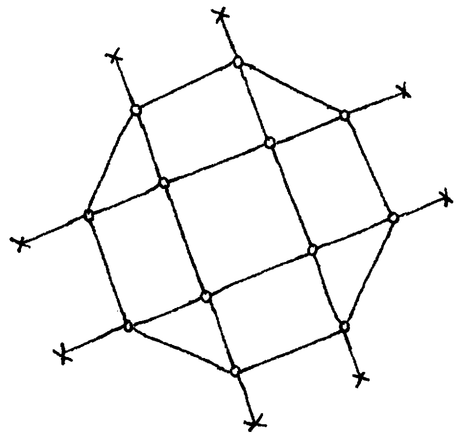
Value: -223,834



Graph 6: n=8

10 vertices, 9 loops

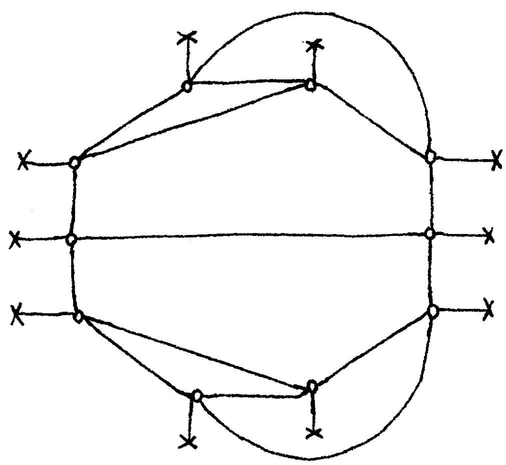
Value: -461,069



Graph 7: n=8

12 vertices, 9 loops

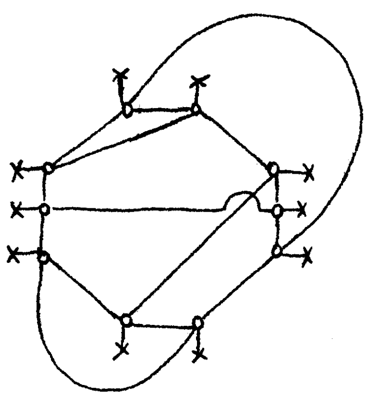
Value: -184,317



Graph 8: n=10

10 vertices, 6 loops

Value: 30,400



Graph 9: n=10

10 vertices, 6 loops

Value: 66,976

Appendix C: Transfer Matrices

In this appendix we define transfer matrices for nearest-neighbor ferromagnets, describe some of their elementary properties, and derive the results alluded to in Chapter V. Although we shall make no use of it, we point out that there is an intimate connection between the theory of transfer matrices and the Markov property of nearest-neighbor Ising models. If the model is not nearest-neighbor one may still write down a transfer matrix, but as it need not be self-adjoint the analysis is more complicated.

We begin by introducing some notation. We analyze ferromagnetic nearest-neighbor models (Λ_g, H, ν), where:

(1) The index vector $g \in \mathbb{Z}^n$ has positive components, and

$$\Lambda_g = \{(i_1, \cdots, i_n) \in \mathbb{Z}^n : |i_\alpha| \leq g_\alpha,\ \alpha = 1, \cdots, n\}\ . \tag{1}$$

With each site $i \in \Lambda_g$ we associate a spin $\sigma_i \in \mathbb{R}$.

(2) The Hamiltonian H is

$$H = -\sum_{[ij] \subset \Lambda_g} J\sigma_i\sigma_j - \sum_{i \in \Lambda_g} h\sigma_i \qquad J, h \geq 0\ . \tag{2}$$

Here we use square brackets $[ij]$ to indicate nearest-neighbor pairs.

(3) The single-spin measure ν is an even Borel probability measure on $\mathbb{R}$ which decays sufficiently rapidly that

$$\int_{\mathbb{R}} \exp(a\sigma^2)\, d\nu(\sigma) < \infty \qquad \forall a \in \mathbb{R}. \tag{3}$$

We absorb the inverse temperature β into J and h.

If we drop the $\alpha^{\underline{th}}$ component of $g \in \mathbb{Z}^n$, we denote the vector in $\mathbb{Z}^{n-1}$ formed from the remaining components by $\hat{g}_\alpha$. Thus, for example, if $g = (g_1, \cdots, g_n)$, then $\hat{g}_1 = (g_2, \cdots, g_n)$, and $\Lambda_{\hat{g}_1}$ is the cross-section of Λ_g orthogonal to the 1-direction. To ease the notation we often write Λ for Λ_g and Λ_1 for $\Lambda_{\hat{g}_1}$. Since we frequently encounter configurations of all spins $\sigma_a \in \mathbb{R}$ with $a \in \Lambda_1$, we group them into a vector: $\vec{\sigma} \in \prod_{\Lambda_1} \mathbb{R}$ has components $\sigma_a \in \mathbb{R}$.

The unnormalized transfer matrix T (in the 1-direction) of the model (Λ_g, H, ν) is the linear map $T: L^2(\vec{\nu}) \rightarrow L^2(\vec{\nu})$ defined by

$$(Tf)(\vec{\sigma}) = \int_{\mathbb{R}^{\Lambda_1}} \exp\Big[J\Big(\vec{\sigma}\cdot\vec{\tau} + \frac{1}{2}\sum_{[ab]\subset\Lambda_1} \sigma_a\sigma_b + \frac{1}{2}\sum_{[cd]\subset\Lambda_1} \tau_c\tau_d\Big) + \frac{\vec{h}}{2}\cdot(\vec{\sigma}+\vec{\tau})\Big] f(\vec{\tau}) \prod_{\Lambda_1} d\nu(\tau_a), \tag{4}$$

where we write $L^2(\vec{\nu})$ for $L^2(\prod_{\Lambda_1}\mathbb{R}, \prod_{\Lambda_1}\nu)$. (We use real Hilbert spaces.) The following proposition collects some elementary mathematical facts about transfer matrices.

<u>Proposition 1</u>: The (unnormalized) transfer matrix T of a nearest-neighbor ferromagnet (Λ_g, H, ν) is a Hilbert-Schmidt self-adjoint nonnegative operator. The largest eigenvalue λ_{max} has multiplicity 1, and the associated eigenvector $\Omega_{\hat{g}_1} \in L^2(\vec{\nu})$, which we take to have norm 1, is a continuous function which may be chosen strictly positive. T has a unique "extension"

to a bounded linear map T: $L^p(\vec{\nu}) \longrightarrow L^{p'}(\vec{\nu})$ for any $p \in (1,\infty]$, $p' \in [1,\infty)$, and with $p^{-1} + q^{-1} = 1$,

$$\|T\|_{L^p(\mathbb{R}^{\Lambda_1},\nu),\, L^{p'}(\mathbb{R}^{\Lambda_1},\nu)} \leqslant \left(\left\| e^{\frac{nJ}{2}\sigma^2 + \frac{h}{2}\sigma} \right\|_{L^q(\mathbb{R},\nu)} \cdot \left\| e^{\frac{nJ}{2}\sigma^2 + \frac{h}{2}\sigma} \right\|_{L^{p'}(\mathbb{R},\nu)} \right)^{|\Lambda_1|} \tag{5}$$

<u>Proof</u>:

The kernel of T is obviously symmetric, and since

$$\Big| \sum_{[ab]\subset\Lambda_1} \sigma_a \sigma_b \Big| \leqslant (n-1)\, \vec{\sigma}^2 \tag{6}$$

we have

$$\int_{\mathbb{R}^{\Lambda_1}\times\mathbb{R}^{\Lambda_1}} \exp\Big[2J\Big(\vec{\sigma}\cdot\vec{\tau} + \frac{1}{2}\sum_{[ab]\subset\Lambda_1}\sigma_a\sigma_b + \frac{1}{2}\sum_{[cd]\subset\Lambda_1}\tau_c\tau_d\Big) + \vec{h}\cdot(\vec{\sigma}+\vec{\tau})\Big]\, d\nu(\vec{\tau})\, d\nu(\vec{\sigma}) \leqslant$$

$$\leqslant \left(\int_{\mathbb{R}} e^{nJ\sigma^2 + h\sigma}\, d\nu(\sigma) \right)^{2|\Lambda_1|}, \tag{7}$$

which is finite by (3). Thus T is Hilbert-Schmidt and symmetric. Also,

$$(f,Tf) = \int_{\mathbb{R}^{\Lambda_1}\times\mathbb{R}^{\Lambda_1}} \Big[f(\vec{\sigma})\, e^{\frac{1}{2}J\sum_{[ab]\subset\Lambda_1}\sigma_a\sigma_b + \frac{\vec{h}}{2}\cdot\vec{\sigma}} \Big] e^{J\vec{\sigma}\cdot\vec{\tau}} \Big[f(\vec{\tau})\, e^{\frac{1}{2}J\sum_{[cd]\subset\Lambda_1}\tau_c\tau_d + \frac{\vec{h}}{2}\cdot\vec{\tau}} \Big] d\nu(\vec{\sigma})\, d\nu(\vec{\tau})$$

$$= \sum_{m=0}^{\infty} \int F(\vec{\sigma})\, \frac{J^m}{m!} (\vec{\sigma}\cdot\vec{\tau})^m\, F(\vec{\tau})\, d\nu(\vec{\sigma})\, d\nu(\vec{\tau})$$

$$= \sum_{m=0}^{\infty} \frac{J^m}{m!} \sum_{a_1,\cdots,a_m\in\Lambda_1} \Big[\int F(\vec{\sigma})\, \sigma_{a_1}\cdots\sigma_{a_m}\, d\nu(\vec{\sigma}) \Big]^2$$

$$\geqslant 0, \tag{8}$$

so $T \geqslant 0$.

Since T is Hilbert-Schmidt it has a complete set of eigenvectors; let λ_{max} be the largest eigenvalue and Ω be any eigenvector with eigenvalue λ_{max}. Note that since the integral kernel of T is strictly positive, if $f \not\equiv 0$ and $f \geq 0$ then $Tf > 0$ almost everywhere. By positivity of the kernel of T,

$$(|\Omega|, T|\Omega|) \geq (\Omega, T\Omega) = \lambda_{max}(|\Omega|, |\Omega|), \tag{9}$$

so $|\Omega|$ is also an eigenvector with eigenvalue λ_{max}. We claim Ω is either strictly positive or strictly negative. Since one of $|\Omega| \pm \Omega \geq 0$ cannot be identically zero, one of $\lambda_{max}^{-1} T(|\Omega| \pm \Omega) = |\Omega| \pm \Omega$ must be strictly positive so that $|\Omega| = \pm\Omega$ as claimed. Let $\Omega_{\hat{g}_1} = |\Omega|$. Since every eigenvector with eigenvalue λ_{max} must be of definite sign, no two can be orthogonal, so $\Omega_{\hat{g}_1}$ is unique. (This argument is taken from [43], though the result also follows from the Perron-Frobenius Theorem.) Continuity of $\Omega_{\hat{g}_1}$ is a straightforward application of the Lebesgue dominated convergence theorem.

The estimate for $\|T\|_{p,p'}$ follows in the same way as (7):

$$\begin{aligned} \|T\|_{p,p'} &\leq \left[\int_{\mathbb{R}^{\Lambda_1}} \left(\int_{\mathbb{R}^{\Lambda_1}} \exp\left[q\cdot\left(J\cdot\left[\vec{\sigma}\cdot\vec{\tau} + \frac{1}{2}\sum_{[ab]\subset\Lambda_1}\sigma_a\sigma_b + \frac{1}{2}\sum_{[cd]\subset\Lambda_1}\tau_c\tau_d\right] + \frac{\vec{h}}{2}\cdot[\vec{\sigma}+\vec{\tau}]\right)\right] d\nu(\vec{\tau})\right)^{\frac{p'}{q}} d\nu(\vec{\sigma})\right]^{\frac{1}{p'}} \\ &\leq \left(\left\|e^{\frac{n}{2}J\sigma^2 + \frac{h}{2}\sigma}\right\|_q \left\|e^{\frac{nJ}{2}\sigma^2 + \frac{h}{2}\sigma}\right\|_{p'}\right)^{|\Lambda_1|} . \end{aligned} \tag{10}$$

QED

Notice that since the image $\mathrm{im}(T) \subset \bigcap_{p<\infty} L^p(\vec{\nu})$ and $\Omega_{\hat{g}_1}$ is an eigenvector, we have $\Omega_{\hat{g}_1} \in \bigcap_{p<\infty} L^p(\vec{\nu})$.

We next relate T to statistical quantities of the model (Λ_g, H, ν). The partition function Z is given by the equation

$$Z = (E, T^{2g_1} E) , \tag{11}$$

where $E \in \bigcap_{p<\infty} L^p(\vec{\nu})$ is the positive function

$$E(\vec{\sigma}) = \exp\Big(\frac{J}{2} \sum_{[ab]\subset\Lambda_1} \sigma_a \sigma_b + \frac{\vec{h}}{2}\cdot\vec{\sigma} \Big) . \tag{12}$$

If $f_{-L}(\vec{\sigma}_{-L}), \cdots, f_L(\vec{\sigma}_L)$ are functions f_ℓ of the spins $\vec{\sigma}_\ell$ having first coordinate $\ell \in \{-L, \cdots, L\}$, then formally

$$\Big\langle \prod_{\ell=-L}^{L} f_\ell(\vec{\sigma}_\ell); H, \nu \Big\rangle_{\Lambda_g} = \frac{\left(T^{g_1 - L} E, f_{-L} T f_{-L+1} T \cdots f_L T^{g_1 - L} E\right)_{L^2(\vec{\nu})}}{\left(E, T^{2g_1} E\right)_{L^2(\vec{\nu})}} , \tag{13}$$

where the f_ℓ act on $L^2(\vec{\nu})$ by multiplication. If the functions f_ℓ decay sufficiently rapidly that the thermal expectation $\langle \prod_{-L}^{L} f_\ell \rangle$ is finite, e.g. $f_\ell \in \bigcap_{p<\infty} L^p(\vec{\nu}) \ \forall \ell$, then (13) is rigorously correct. This condition is satisfied by all polynomials in the spins, and also by the more rapidly diverging functions $\exp(\vec{\sigma}_\ell^{\,2})$.

Define the normalized transfer matrix

$$\mathcal{J} = T / \|T\| . \tag{14}$$

In the following proposition we take the $g_1 \to \infty$ limit.

Proposition 2: Let $f_\ell \in \bigcap_{p<\infty} L^p(\vec{\nu})$, $-L \leq \ell \leq L$. Then $\lim_{g_1\to\infty} \langle \prod_{\ell=-L}^{L} f_\ell(\vec{\sigma}_\ell); H, \nu \rangle_{\Lambda_g}$ exists and is given by the formula

$$\lim_{g_1\to\infty} \left\langle \prod_{\ell=-L}^{L} f_\ell(\vec{\sigma}_\ell) \right\rangle_{\Lambda_g} = \left(\Omega_{\hat{g}_1}, f_{-L} \cdot \mathcal{J} \cdot f_{-L+1} \cdots \mathcal{J} f_L \cdot \Omega_{\hat{g}_1}\right)_{L^2(\vec{\nu})}. \tag{15}$$

Proof:

This is just a calculation:

$$\left\langle \prod_{-L}^{L} f_\ell(\vec{\sigma}_\ell) \right\rangle_{\Lambda_g} = \frac{\left(\mathcal{J}^{g_1-L-1}E, [\mathcal{J}f_{-L}\mathcal{J}f_{-L+1}\cdots\mathcal{J}f_L]\mathcal{J}^{g_1-L}E\right)}{\left(\mathcal{J}^{g_1}E, \mathcal{J}^{g_1}E\right)} \quad . \tag{16}$$

Since $\mathcal{J}$ is a self-adjoint nonnegative operator with norm 1, $\mathcal{J}^{g_1}$ converges strongly to the projection onto $\ker(\mathcal{J} - 1)$; by Proposition 1 this projection is just $\Omega_{\hat{g}_1}(\Omega_{\hat{g}_1}, \cdot\,)$. The denominator of (16) tends to $(\Omega_{\hat{g}_1}, E)^2$, which is nonzero since $\Omega_{\hat{g}_1}$ is strictly positive. If we note that for $f \in \bigcap_{p<\infty} L^p(\vec{\nu})$, $\mathcal{J}f$ is a bounded operator, then we may take the limit in the numerator to obtain a similar formula; (15) follows upon cancelling the common factor $(\Omega_{\hat{g}_1}, E)^2$.

QED

Formula (15), which controls the infinite-volume limit without the use of correlation inequalities, obviously still holds if we allow the two faces of the box to move out to infinity in the 1-direction independently. Sending them simultaneously as we have done is merely a notational convenience. Since the spectrum of $\mathcal{J}$ is discrete and the largest eigenvalue 1 is nondegenerate, exponential clustering follows from (15).

In Section V.5 we needed to control the infinite-volume limit in the 1-direction of a model having field -h on sites with negative 1-component and +h on sites with nonnegative 1-component. This may be done in exactly the same manner, by introducing two transfer matrices $\mathcal{J}_{\pm} : L^2(\vec{\nu}) \to L^2(\vec{\nu})$, one for each half:

$$(T_{\pm}f)(\vec{\sigma}) = \int \exp\left[J\left(\vec{\sigma}\cdot\vec{\tau} + \frac{1}{2}\sum_{[ab]\subset\Lambda_1} \sigma_a\sigma_b + \frac{1}{2}\sum_{[cd]\subset\Lambda_1} \tau_c\tau_d\right) \pm \frac{\vec{h}}{2}\cdot(\vec{\sigma}+\vec{\tau})\right] f(\vec{\tau})\, d\vec{\nu}(\vec{\tau}) \tag{17}$$

$$\mathcal{J}_{\pm} = T_{\pm} / \| T_{\pm} \| . \tag{18}$$

For a finite model with field -h on sites with negative 1-component and +h on sites with nonnegative 1-component, if $f_\ell \in \bigcap_{p<\infty} L^p(\vec{\nu})$ we have

$$\left\langle \prod_{\ell=-L}^{L} f_\ell(\vec{\sigma}_\ell) \right\rangle_{\Lambda_g} = \frac{\left(\mathcal{J}_-^{g_1-L} E_-, f_{-L}\mathcal{J}_- \cdots f_{-1}\mathcal{J}_- e^{\vec{h}\cdot\vec{\sigma}} f_0 \mathcal{J}_+ f_1 \mathcal{J}_+ \cdots f_L \mathcal{J}_+^{g_1-L} E_+\right)_{L^2(\vec{\nu})}}{\left(\mathcal{J}_-^{g_1} E_-, e^{\vec{h}\cdot\vec{\sigma}} \mathcal{J}_+^{g_1} E_+\right)_{L^2(\vec{\nu})}} . \tag{19}$$

Taking the limit $g_1 \to \infty$ in the denominator yields $(\Omega^-_{\hat{g}_1}, E_-)(\Omega^-_{\hat{g}_1}, e^{\vec{h}\cdot\vec{\sigma}}\Omega^+_{\hat{g}_1})(\Omega^+_{\hat{g}_1}, E_+)$ which is again nonzero since $\Omega^{\pm}_{\hat{g}_1}$ is strictly positive. If we take the limit in the numerator as well and cancel common factors, we find

$$\lim_{g_1\to\infty} \left\langle \prod_{-L}^{L} f_\ell(\vec{\sigma}_\ell) \right\rangle_{\Lambda_g} = \left(\Omega^-_{\hat{g}_1}, f_{-L}\mathcal{J}_- \cdots f_{-1}\mathcal{J}_- e^{\vec{h}\cdot\vec{\sigma}} f_0 \mathcal{J}_+ \cdots f_L \Omega^+_{\hat{g}_1}\right) / \left(\Omega^-_{\hat{g}_1}, e^{\vec{h}\cdot\vec{\sigma}} \Omega^+_{\hat{g}_1}\right) . \tag{20}$$

As with formula (15), we arrive at the same limit if we send the two faces of the box Λ_g to infinity independently.

We next establish the results alluded to in Section V.3 during our discussion of the infinite-volume transfer matrix. We conclude the appendix with a proof that all moments of a nearest-neighbor ferromagnet are bounded above. We begin by constructing a transfer matrix $\mathcal{J}'$ for the region $\mathbb{Z}\times\Lambda_1$ in the manner of Section V.3 and identifying it with the operator $\mathcal{J}$ we have already defined.

Let $\mathbb{Z}^n_{1+} = \{(k_1,\cdots,k_n)\in\mathbb{Z}^n : k_1 \geq 0\}$, associate with each $i\in\mathbb{Z}^n_{1+}$ a (real) indeterminate s_i, and let $\mathcal{S}_+$ be the vector space of formal polynomials $P(s_{i_1}, s_{i_2}, \cdots)$, $i_j \in \mathbb{Z}^n_{1+}$, in these indeterminates. It is convenient to group together the indeterminates whose site has first component $\ell\in\mathbb{Z}_+$ into a vector $\vec{s}_\ell$ with components $s_{(\ell,a)}$, $a\in\mathbb{Z}^{n-1}$. If we introduce multi-index exponents $\vec{m}_\ell \in \prod_{\Lambda_1}\mathbb{Z}_+$, then the monomials $\prod_{\ell=0}^{L} \vec{s}_\ell^{\,\vec{m}_\ell} = \prod_{\ell=0}^{L}\prod_{a\in\Lambda_1}(s_{(\ell,a)})^{m_{(\ell,a)}}$ are a basis for $\mathcal{S}_+$. Define the map $u:\mathcal{S}_+\to L^2(\vec{\nu})$ on basis elements $\prod_0^L \vec{s}_\ell^{\,\vec{m}_\ell}$ by

$$u\Big(\prod_0^L \vec{s}_\ell^{\,\vec{m}_\ell}\Big) = \vec{\sigma}^{\,\vec{m}_0}\,\mathcal{J}\,\vec{\sigma}^{\,\vec{m}_1}\,\mathcal{J}\cdots\vec{\sigma}^{\,\vec{m}_L}\cdot\Omega_{\hat{g}_1} . \tag{21}$$

We claim the image of u is dense in $L^2(\vec{\nu})$. To see this, note first that any product

$$P(\vec{\sigma})\exp\Big(-\sum_{a\in\Lambda_1} c_a(\sigma_a)^2\Big)\Omega_{\hat{g}_1}, \quad P \text{ a polynomial}, \quad c_a \geq 0 \tag{22}$$

lies in the closure $\overline{\mathrm{im}(u)}$ because the exponential may be expanded in a convergent Taylor series. By the Stone-Weierstrass Theorem, the linear span of the functions $P(\vec{\sigma})\exp(-\sum_{a\in\Lambda_1} c_a(\sigma_a)^2)$ is uniformly dense in $C_\infty(\mathbb{R}^{\Lambda_1})$. Since ν is finite and $\Omega_{\hat{g}_1}$ strictly positive and continuous, this implies $\overline{\mathrm{im}(u)} \supset C_\infty(\mathbb{R}^{\Lambda_1})$. As a finite Baire measure is regular, we now conclude that

$$\overline{\mathrm{im}(u)} = L^2(\vec{\nu}) \tag{23}$$

as claimed.

Define the bilinear form $(\ ,\ )_+$ on $\mathcal{S}_+$ by

$$\big(P(s_{i_1},\cdots,s_{i_L}),\,Q(s_{j_1},\cdots,s_{j_M})\big)_+ = \big\langle P(\sigma_{\theta(i_1)},\cdots,\sigma_{\theta(i_L)})\cdot Q(\sigma_{j_1},\cdots,\sigma_{j_M});\,H,\nu\big\rangle_{\mathbb{Z}\times\Lambda_1}, \tag{24}$$

where as usual θ reverses the sign of the first component of i. We find immediately from (21) and (24) that

$$\left(P(s_{i_1},\cdots,s_{i_L}),Q(s_{j_1},\cdots,s_{j_M})\right)_+ = \left(u(P),u(Q)\right)_{L^2(\vec{\nu})} \; . \tag{25}$$

Consequently $(\;,\;)_+$ is a positive semidefinite scalar product, and it remains so in the limit $\Lambda_1 \to \mathbb{Z}^{n-1}$. Define $\mathcal{J}': \mathcal{S}_+ \to \mathcal{S}_+$ on basis elements by

$$\mathcal{J}'\Big(\prod_0^L \vec{s}_\ell^{\,\vec{m}_\ell}\Big) = \prod_0^L \vec{s}_{\ell+1}^{\,\vec{m}_\ell} \tag{26}$$

and extend by linearity. (This is the construction of Section V.3) Then

$$u\Big(\mathcal{J}'\prod_0^L \vec{s}_\ell^{\,\vec{m}_\ell}\Big) = u\Big(\vec{s}_0^{\,\vec{0}}\prod_1^{L+1} \vec{s}_\ell^{\,\vec{m}_{\ell-1}}\Big) = \mathcal{J}\vec{\sigma}^{\,\vec{m}_0}\mathcal{J}\cdots\vec{\sigma}^{\,\vec{m}_\ell}\Omega_{\hat{g}_1} = \mathcal{J}u\Big(\prod_0^L \vec{s}_\ell^{\,\vec{m}_\ell}\Big)$$

so

$$u\mathcal{J}' = \mathcal{J}u \, . \tag{27}$$

In particular,

$$0 \leqslant (uP,\mathcal{J}uP)_{L^2(\vec{\nu})} = (P,\mathcal{J}'P)_+$$

$$(P,\mathcal{J}'P)_+ = (uP,\mathcal{J}uP)_{L^2(\vec{\nu})} \leqslant (uP,uP)_{L^2(\vec{\nu})} = (P,P)_+ \, . \tag{28}$$

Again, this holds in the $\Lambda_1 \to \mathbb{Z}^{n-1}$ limit.

It follows from (25) that the kernel of u is the null space $\mathcal{N}$ in $\mathcal{S}_+$ of $(\;,\;)_+$; as we have already shown, the image of u is dense in $L^2(\vec{\nu})$. Thus, u is well defined on the quotient $\mathcal{S}_+/\mathcal{N}$ and extends by continuity to a unitary U from the Hilbert space completion $\mathcal{H}$ of $\mathcal{S}_+/\mathcal{N}$ to $L^2(\vec{\nu})$. By (27) we have

$$U\mathcal{J}'U^{-1} = \mathcal{J}, \tag{29}$$

identifying $\mathcal{J}$ with $\mathcal{J}'$. We remark that the analysis we have performed on the transfer matrix of a nearest-neighbor model with real spins extends easily to nearest-neighbor models with vector spins. Propositions 1 and 2 go through in straightforward fashion.

It remains to prove Theorem V.3.7, which we restate here for ease of reference.

Theorem V.3.7: Let $\mathcal{J}$ be the infinite-volume transfer matrix (in the 1-direction) of the nearest-neighbor ferromagnet ($\mathbb{Z}^n$,H,ν) at inverse temperature β. Then:

(1) The model is long-range ordered $\Leftrightarrow$ the geometric multiplicity of 1 in the spectrum of $\mathcal{J}$ is greater than 1.

(2) The model clusters in the 1-direction $\Leftrightarrow$ the geometric multiplicity of $1 \in \operatorname{spec}(\mathcal{J})$ is 1.

(3) The model clusters exponentially $\Leftrightarrow$ 1 is an isolated eigenvalue of $\mathcal{J}$. Let $\lambda_1 \in \mathrm{spec}(\mathcal{J})$ be $\sup\{\lambda \in \mathrm{spec}(\mathcal{J}) : \lambda < 1\}$. Then the correlation length χ_1 in the 1-direction is given by the formula

$$\chi_1 = 1 / \log\left(\frac{1}{\lambda_1}\right). \tag{30}$$

Proof:

As the ideas involved in this theorem are fairly standard, we give the proof in a condensed manner.

(1): The $\Rightarrow$ direction follows from the observation that if P is the projection onto the geometric eigensubspace $\{\psi : \mathcal{J}\psi = \psi\}$ then $P = \underset{m\to\infty}{s\text{-}\lim}\, \mathcal{J}^m$. Write Ω for the element of $\mathcal{H}$ corresponding to $1 \in \mathcal{S}_+$ and P_Ω for the projection onto Ω. Then

$$\begin{aligned} \lim_{m\to\infty} \left(\langle \sigma_0 \sigma_m \rangle - \langle \sigma_0 \rangle^2\right) &= \lim_{m\to\infty} (s_0, \mathcal{J}^m s_0)_+ - (s_0, P_\Omega s_0)_+ \\ &= (s_0, [P - P_\Omega] s_0)_+ . \end{aligned} \tag{31}$$

If the model is long-range ordered then this limit is nonzero, so $P \neq P_\Omega$ and 1 is a degenerate eigenvalue.

Conversely, suppose $P \neq P_\Omega$. Then $\exists \psi \in \mathcal{H}$ such that $P\psi \neq P_\Omega \psi$; since the linear span of the monomials $M = \prod_0^L \vec{s}_\ell^{\,\vec{m}_\ell}$ is dense in $\mathcal{H}$, we may suppose that

$$\exists\, M : \quad \lim_{m\to\infty} \left(\langle \theta(M) \mathcal{J}^m M \rangle - \langle M \rangle^2\right) > 0 . \tag{32}$$

For $c \in (0, \infty)$, define $\sigma_c = \begin{cases} \sigma, & |\sigma| \leq c \\ c, & |\sigma| \geq c \end{cases}$, and let M_c be the monomial obtained from M by replacing all spins σ by their cut-off approximation. We may apply the Lebesgue dominated convergence theorem to conclude $\|M-M_c\|_+\to 0$ as $c\to\infty$. Now by using the F.K.G. inequality [11] one may show [43] that $(\langle\theta(M_c)J^m M_c\rangle-\langle M_c^2\rangle)$ decays to 0 if the two-point correlation does. But since $\|M-M_c\|_+\sim 0$, the two-point function cannot decay to zero because this would contradict (32). Thus the model must be long-range ordered.

(2): The $\Rightarrow$ direction follows from part (1) above, and the converse from (31).

(3): This is just a calculation with the spectral theorem.

QED

It remains only to prove that the moments of the infinite-volume Gibbs measure are finite. We can do even better: $\exp(\sum a_i\sigma_i^2)$ has finite thermal expectation for a finite sum $\sum a_i\sigma_i^2$. This is reasonable because $\nu\in\mathcal{N}_2$. The proof we give is the translation to lattice models of a standard argument in field theory [43].

<u>Proposition 3</u>: Let $(\mathbb{Z}^n,H,\nu)$ be a nearest-neighbor Ising ferromagnet with Hamiltonian

$$H=-\sum_{i\in\mathbb{Z}^n}\sum_{\alpha=1}^{n}J_\alpha\sigma_i\sigma_{i+1_\alpha}\ -h\sum_{i\in\mathbb{Z}^n}\sigma_i\ . \tag{33}$$

Then given any finite sum $\sum a_i\sigma_i^2$ there exists a constant $C<\infty$ such that

$$\langle\exp(\sum a_i\sigma_i^2);H_\Lambda,\nu,\beta\rangle_\Lambda\leq C \qquad \forall\,\Lambda\in\mathcal{P}_0(\mathbb{Z}^n). \tag{34}$$

Thus, if μ is the infinite-volume Gibbs measure of $(\mathbb{Z}^n,H,\nu)$,

$$\exp(\sum a_i\sigma_i^2)\in\bigcap_{p<\infty}\mathbb{L}^p(\mu)\ . \tag{35}$$

Proof:

It suffices to consider a single term $a\sigma_i^2$, $a \in [0,\infty)$, in the sum $\sum a_i \sigma_i^2$; for convenience we suppose i=0. If n=1, then since $\exp(a\sigma^2) \in \bigcap_{p<\infty} L^p(\vec{\nu})$, by our explicit formula (15) the result is immediate. We now suppose $n \geq 2$.

Let Λ_g be a rectangle; for technical convenience we suppose all components of $g \in \mathbb{Z}_+^n$ are odd. For $\alpha \in \{1,\cdots n\}$ define the function $F_\alpha : \mathbb{R}^{\Lambda_g} \to \mathbb{R}$ by $F_\alpha(\sigma) = \prod_{i \in \Lambda_g} F_{\alpha i}(\sigma_i)$, where

$$F_{\alpha i}(\sigma_i) = \begin{cases} e^{a\sigma_i^2} & \text{if first } (\alpha-1) \text{ components of } i \text{ even \& last } (n-\alpha+1) \text{ vanish} \\ 1 & \text{otherwise} \end{cases}. \tag{36}$$

In particular $F_1(\sigma) = \exp(a\sigma_0^2)$. Note that F_α depends only on the spins in $\Lambda_{\hat{g}_\alpha}$, so we may regard it as a function on $\mathbb{R}^{\Lambda_\alpha}$. We shall make frequent use of the fact that if A is a nonnegative self-adjoint bounded operator on some Hilbert space $\mathcal{H}$ whose largest spectral value λ_{max} has geometric multiplicity 1, then if Ω is the eigenvector associated with λ_{max} and $\psi \in \mathcal{H}$ is not orthogonal to Ω ,

$$\lambda_{max} = \|A\| = \lim_{g\to\infty} (\psi, A^g \psi)^{\frac{1}{g}} \quad . \tag{37}$$

As a preliminary application, we note that if T_n is the (unnormalized) transfer matrix for Λ_g in the n-direction, then

$$\|T_n\| = \lim_{g_n\to\infty} \left(E_n, T^{2g_n} E_n\right)_{L^2(\vec{\nu}_n)}^{1/2g_n} = \lim_{g_n\to\infty} \left(Z_{\Lambda_g}\right)^{1/2g_n}, \tag{38}$$

where of course

$$E_n = \exp\left[\frac{J}{2} \sum_{[ab]\subset\Lambda_n} \sigma_a\sigma_b + \frac{h}{2} \sum_{a\in\Lambda_n} \sigma_a \right]$$

$$L^2(\vec{\nu}_n) = L^2\left(\prod_{\Lambda_n} \mathbb{R}, \prod_{\Lambda_n} \nu\right) .$$

By the method of Proposition 1, we estimate

$$Z_{\Lambda_g} \geq \prod_{i\in\Lambda_g} \int_{\mathbb{R}} \exp\left[-nJ\sigma^2 + h\sigma\right] d\nu(\sigma), \tag{39}$$

so that

$$\|T_n\|_{L^2(\vec{\nu}_n),L^2(\vec{\nu}_n)} \geq \gamma^{|\Lambda_n|} \quad , \quad \gamma>0 \text{ indep. of } \Lambda_g. \tag{40}$$

More generally, we shall apply formula (37) to estimate $\|\mathcal{T}_\alpha F_\alpha \mathcal{T}_\alpha\|$, where $\mathcal{T}_\alpha$ is the normalized transfer matrix in the α-direction and we regard $F_\alpha(\sigma)$ as a multiplication operator. Note that although $F_\alpha(\sigma)$ is unbounded, the composition $\mathcal{T}_\alpha F_\alpha(\sigma)\mathcal{T}_\alpha$ is a compact nonnegative self-adjoint operator with positive integral kernel. Thus, the largest eigenvalue is nondegenerate and has a strictly positive eigenvector, which necessarily is not orthogonal to E . We need to estimate the norm $\|T_n F_n T_n\|$, and we do this as in Proposition 1 to obtain

$$\|T_n\|_{L^2(\vec{\nu}_n),L^2(\vec{\nu}_n)} \leq \Gamma^{|\Lambda_n|} \quad , \quad \Gamma>0 \text{ indep. of } \Lambda_g. \tag{41}$$

Combining this with (40), we conclude

$$\|\mathcal{T}_n F_n \mathcal{T}_n\| \leq D^{|\Lambda_n|} \quad , \quad D>0 \text{ indep. of } \Lambda_g. \tag{42}$$

We want to show

$$\langle \exp(a\sigma_0^2); H_{\Lambda_g}\rangle = \frac{(E_1, \mathcal{T}_1^{g_1} F_1(\vec{\sigma}) \mathcal{T}_1^{g_1} E_1)}{(E_1, \mathcal{T}_1^{2g_1} E_1)} \leq C \quad , \tag{43}$$

$$C \text{ independent of } \Lambda_g.$$

Now

$$\frac{(E_1, \mathcal{J}_1^{g_1} F_1 \mathcal{J}_1^{g_1} E_1)}{(E_1, \mathcal{J}_1^{2g_1} E_1)} \leqslant \|\mathcal{J}_1 F_1 \mathcal{J}_1\|_{L^2(\vec{\nu}_1), L^2(\vec{\nu}_1)} \cdot \frac{(E_1, \mathcal{J}_1^{2g_1-2} E_1)}{(E_1, \mathcal{J}_1^{2g_1} E_1)} , \tag{44}$$

and by the second Griffiths inequality we may send $g_1 \to \infty$ on the right to eliminate the second factor, so it suffices to bound $\|\mathcal{J}_1 F_1 \mathcal{J}_1\|$ independently of Λ_g . Apply formula (38):

$$\|\mathcal{J}_1 F_1 \mathcal{J}_1\| = \lim_{g_1 \to \infty} \left[\frac{(E_1, [\mathcal{J}_1 F_1 \mathcal{J}_1]^{g_1} E_1)}{(E_1, \mathcal{J}_1^{2g_1} E_1)} \right]^{\frac{1}{g_1}} \tag{45}$$

since the denominator goes to $\|\mathcal{J}_1\|^2 = 1$. But looking from the 2-direction, this is just

$$\lim_{g_1 \to \infty} \left[\frac{(E_2, \mathcal{J}_2^{g_2-1} [\mathcal{J}_2 F_2 \mathcal{J}_2] \mathcal{J}_2^{g_2-1} E_2)}{(E_2, \mathcal{J}_2^{2g_2} E_2)} \right]^{\frac{1}{g_1}} \leqslant \lim_{g_1 \to \infty} \|\mathcal{J}_2 F_2 \mathcal{J}_2\|^{1/g_1} , \tag{46}$$

where we have estimated as in (45). Continuing this process we find

$$\langle \exp(a\sigma_0^{2}) ; H_{\Lambda_g} \rangle \leqslant \lim_{g_1 \to \infty} \lim_{g_2 \to \infty} \cdots \lim_{g_{n-1} \to \infty} \|\mathcal{J}_n F_n \mathcal{J}_n\|^{1/g_1 g_2 \cdots g_{n-1}} . \tag{47}$$

By our estimate (42), this yields

$$\langle \exp(a\sigma_0^{2}) ; H_{\Lambda_g} \rangle \leqslant D^{|\Lambda_n| / g_1 \cdots g_{n-1}} \leqslant C , \tag{48}$$

with C independent of Λ_g .

QED

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