The $RO(S^1)$-Graded Equivariant Homotopy of $\mathrm{THH}(\mathbb{F}_p)$

by

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Abstract

The main result of this thesis is the computation of $\mathrm{TR}^n_\alpha(\mathbb{F}_p; p)$ for $\alpha \in RO(S^1)$. These $RO(S^1)$-graded TR-groups are the equivariant homotopy groups naturally associated to the S^1-spectrum $\mathrm{THH}(\mathbb{F}_p)$, the topological Hochschild S^1-spectrum. This computation, which extends a partial result of Hesselholt and Madsen, provides the first example of the $RO(S^1)$-graded TR-groups of a ring. In particular, we compute the groups $\mathrm{TR}^n_\alpha(\mathbb{F}_p; p)$ for all even dimensional representations α, and the order of these groups for odd dimensional α. These groups arise in algebraic K-theory computations, and are particularly important to the understanding of the algebraic K-theory of non-regular schemes.

We also study $RO(S^1)$-graded TR-theory as an $RO(S^1)$-graded Mackey functor. Using Lewis and Mandell's homological algebra tools for graded Mackey functors, we provide examples of how Kunneth spectral sequences can be used to understand $RO(S^1)$-graded TR.

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Chapter 1

Introduction

Algebraic K-theory groups are generally very difficult to compute, but Bökstedt, Hsiang, and Madsen [7] developed a homotopy-theoretic approach to algebraic K-theory computations that has been quite fruitful. For every ring A, they defined a cyclotomic trace map

$$\mathrm{trc} : K_q(A) \to \mathrm{TC}_q(A)$$

relating algebraic K-theory to topological cyclic homology. This map is highly nontrivial, and thus one can often understand algebraic K-theory by understanding topological cyclic homology (see, for instance, McCarthy [20]).

As an approach to computing topological cyclic homology, Hesselholt and Madsen studied TR-theory [12, 13]. For a ring A and a fixed prime p,

$$\mathrm{TR}^n_q(A;p) := \pi_q(\mathrm{T}(A)^{C_{p^{n-1}}}) = [S^q \wedge S^1/C_{p^{n-1}+}, \mathrm{T}(A)]_{S^1}$$

where $\mathrm{T}(A)$ denotes the topological Hochschild S^1-spectrum of A and $\mathrm{T}(A)^{C_{p^{n-1}}}$ denotes the $C_{p^{n-1}}$ fixed point spectrum. These TR-groups come equipped with several operators and relations which provide a rigid algebraic structure, making computations possible. Topological cyclic homology is defined by a homotopy limit construction that involves these operators. Thus understanding the TR-groups of a ring helps us to understand its topological cyclic homology, and hence its algebraic K-theory.

The topological Hochschild S^1-spectrum also has naturally associated equivariant homotopy groups which give a TR-theory graded by the real representation ring of the circle, $RO(S^1)$. These groups arise naturally from the computational method outlined above.

Elements in the representation ring are given by formal differences of isomorphism classes of representations. For every $\alpha \in RO(S^1)$ we choose representatives β and γ such that $\alpha = [\beta] - [\gamma]$. Let S^β denotes the one-point compactification of the representation β. Then the $RO(S^1)$-graded TR-groups are defined as

$$\mathrm{TR}^n_\alpha(A;p) = [S^\beta \wedge S^1/C_{p^{n-1}+}, S^\gamma \wedge T(A)]_{S^1}.$$

Lewis and Mandell [18] have proven that this definition gives a well-behaved theory of $RO(S^1)$-graded homotopy groups.

These $RO(S^1)$-graded groups first arose in computations of the algebraic K-theory of non-regular schemes. For instance, for an $\mathbb{F}_p$-algebra A, Hesselholt and Madsen [9] expressed the K-theory of $A[x]/(x^e)$ in terms of the $RO(S^1)$-graded TR-groups of A. While $RO(S^1)$-graded TR-groups have been used in computations, there are no fully computed examples for all $\alpha \in RO(S^1)$. The first computation to be done is that of $\mathrm{TR}^n_\alpha(\mathbb{F}_p;p)$. This is the main result of this dissertation.

Before stating the result of this computation, we introduce some notation. Let

$$\rho_p : S^1 \to S^1/C_p$$

be the isomorphism given by the pth root. Then we define a prime operation as follows: for $\alpha \in RO(S^1)$,

$$\alpha' = \rho_p^*(\alpha^{C_p}).$$

Recall the ring structure of $RO(S^1)$.

$$RO(S^1) = \mathbb{Z}[t_1, t_2, \ldots]/(t_i t_j - t_{i+j} - t_{i-j})$$

where $t_0 = 2$. The generator t_i corresponds to the representation λ_i given by the i-fold power map. When we use $RO(S^1)$ for grading TR-groups, only the additive structure is used. If p divides i, then $t_i' = t_{i/p}$. If p does not divide i, then $t_i' = 0$. This prime operation is additive. We use the notation $\alpha^{(k)}$ to denote the prime operation applied k times to α. We denote the dimension of α as a real vector space by $|\alpha|$. Note that every sequence of integers can be realized as the dimensions $|\alpha|, |\alpha'|, \ldots, |\alpha^{(n-1)}|, \ldots$ for some virtual representation $\alpha \in RO(S^1)$.

Our main result shows $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p)$, is isomorphic to a direct product of groups $\mathbb{Z}/p^{l_{i,n}}$, $1 \le i \le n$, where the exponents $l_{i,n}$ are explicit recursively defined functions of the integers $q, |\alpha|, |\alpha'|, \ldots, |\alpha^{(n-1)}|$.

These recursively defined functions are based on an inductive argument which computes the group $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p)$ from the group $\mathrm{TR}^{n-1}_{q+\alpha'}(\mathbb{F}_p;p)$ using the fundamental long exact sequence of TR-theory:

$$\cdots \to \mathbb{H}_q(C_{p^{n-1}}, T(\mathbb{F}_p) \wedge S^{-\alpha}) \to \mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p) \to \mathrm{TR}^{n-1}_{q+\alpha'}(\mathbb{F}_p;p) \to \cdots$$

Here, $\mathbb{H}_*(C_{p^{n-1}}, T(\mathbb{F}_p) \wedge S^{-\alpha})$ denotes the group homology spectrum. In order to determine this extension we use information about a map

$$\hat{\Gamma}_{n-1} : \mathrm{TR}^{n-1}_{q+\alpha'}(\mathbb{F}_p;p) \to \hat{\mathbb{H}}^{-q}((C_{p^{n-1}}, T(\mathbb{F}_p) \wedge S^{-\alpha})$$

fitting into a diagram of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{H}_q(C_{p^{n-1}}, T^{-\alpha}) \arrow[r] \arrow[d, equal] & \mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p) \arrow[r] \arrow[d, "\Gamma_{n-1}"] & \mathrm{TR}^{n-1}_{q+\alpha'}(\mathbb{F}_p;p) \arrow[r] \arrow[d, "\hat{\Gamma}_{n-1}"] & \cdots \\
\cdots \arrow[r] & \mathbb{H}_q(C_{p^{n-1}}, T^{-\alpha}) \arrow[r] & \mathbb{H}^{-q}(C_{p^{n-1}}, T^{-\alpha}) \arrow[r] & \hat{\mathbb{H}}^{-q}(C_{p^{n-1}}, T^{-\alpha}) \arrow[r] & \cdots
\end{tikzcd}$$

We use the notation $T^{-\alpha}$ to denote $T(\mathbb{F}_p) \wedge S^{-\alpha}$. The spectra $\hat{\mathbb{H}}^*((C_{p^{n-1}}, T^{-\alpha})$, and $\mathbb{H}^*(C_{p^{n-1}}, T^{-\alpha})$ are the Tate and group cohomology spectra respectively. We have spectral sequences which allow us to compute the groups in the bottom row of this diagram. If we compute the map $\hat{\Gamma}_{n-1}$, then we can identify $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p)$ as the pullback of the diagram

$$\begin{tikzcd} & \mathrm{TR}^{n-1}_{q+\alpha'}(\mathbb{F}_p;p) \arrow[d, "\hat{\Gamma}_{,n-1}"] \\ \mathbb{H}^{-q}(C_{p^{n-1}}, T^{-\alpha}) \arrow[r] & \hat{\mathbb{H}}^{-q}(C_{p^{n-1}}, T^{-\alpha}) \end{tikzcd}$$

In our argument, we induct down on the number of prime operations applied to α. Thus we compute $\mathrm{TR}^j_{q+\alpha^{(n-j)}}(\mathbb{F}_p;p)$ from $\mathrm{TR}^{j-1}_{q+\alpha^{(n-j+1)}}(\mathbb{F}_p;p)$ using the method outlined above. When $j = n$ this computes $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p)$. At each step of the induction we need to compute the group $\mathrm{TR}^j_{q+\alpha^{(n-j)}}(\mathbb{F}_p;p)$ and the map $\hat{\Gamma}_j$ on that group.

To do this we recursively define integers $l_{m,j} = l_{m,j}(q,\alpha)$ indexed by pairs of integers $1 \leq j \leq n$ and $1 \leq m \leq j$. These values $l_{m,j}$ give the orders of the summands of the group $\mathrm{TR}^j_{q+\alpha^{(n-j)}}(\mathbb{F}_p;p)$. The recursive definition also uses auxiliary integers $k_{m,j} = k_{m,j}(q,\alpha)$, $r_j = r_j(q,\alpha)$ and $g_{m,j} = g_{m,j}(q,\alpha)$ indexed by integers $1 \leq j \leq n$ and $1 \leq m \leq j$. The values $k_{m,j}$ store information about the map $\hat{\Gamma}_j$. The recursive definition is as follows.

We begin by setting our initial conditions. Let

$$l_{1,1} = \begin{cases} 1 & \text{if } q \geq -|\alpha^{(n-1)}| \\ 0 & \text{else} \end{cases}$$

and set $k_{1,1} = 0$.

Let $j = 2$. We compute the values $l_{m,j}$ as follows, incrementing j by 1 after each iteration and stopping after the iteration where $j = n$.

Define $r_j = \min(j, \frac{q+|\alpha^{(n-j)}|}{2} + 1)$ and $g_{i,j-1} = l_{i,j-1} - k_{i,j-1}$ for integers $1 \leq i \leq j-1$. Let $w = \#\{1 \leq i \leq j-1 | k_{i,j-1} \geq r_j\}$. We choose a permutation $\tau \in \Sigma_{j-1}$ satisfying the conditions that

$$r_j \leq k_{\tau(1),j-1} \leq k_{\tau(2),j-1} \leq \ldots \leq k_{\tau(w),j-1},$$

$$r_j > k_{\tau(w+1),j-1}, k_{\tau(w+2),j-1}, \ldots, k_{\tau(j-1),j-1},$$

$$g_{\tau(w+1),j-1} \geq g_{\tau(w+2),j-1} \geq \ldots \geq g_{\tau(j-1),j-1}.$$

We can choose any permutation $\tau \in \Sigma_{j-1}$ satisfying these conditions. If $w = j-1$ we define $g_{\tau(w+1),j-1} = 0$.

We now give a recursive definition of the values $l_{i,j}$ and $k_{i,j}$. We set

$$l_{1,j} = \begin{cases} \max(0, r_j) & \text{if } w = j-1 \\ \min(j, r_j + g_{\tau(w+1),j-1}) & \text{else} \end{cases}$$

and $k_{1,j} = 0$. For each integer $1 \leq m \leq w$, set $l_{m+1,j} = l_{\tau(m),j-1}$.

$$k_{m+1,j} = \begin{cases} \min(l_{m+1,j}, k_{\tau(m),j-1} - r_j) & \text{if } g_{\tau(m),j-1} > g_{\tau(w+1),j-1} \\ l_{m+1,j} & \text{else} \end{cases}$$

For each integer $w + 2 \leq m \leq j - 1$, let $\tau(z)$ be the smallest number such that $w + 1 \leq z < m$. Then set:

$$l_{m,j} = \begin{cases} l_{\tau(m),j-1} & \text{if } \tau(v) < \tau(m) \text{ for some } w + 1 \leq v < m \\ g_{\tau(m),j-1} + k_{\tau(z),j-1} & \text{else} \end{cases}$$

Then we set $k_{m,j} = l_{m,j}$. Finally, $l_{j,j} = k_{j,j} = 0$ if $w < j - 1$. If $j < n$ we increment j by 1 and iterate.

We now state our main result

Theorem 1.0.1. *For q even there is an isomorphism of abelian groups*

$$\mathbb{Z}/p^{l_{1,n}} \oplus \mathbb{Z}/p^{l_{2,n}} \oplus \ldots \mathbb{Z}/p^{l_{n,n}} \xrightarrow[\sim]{\iota_n} \mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p; p)$$

where the $l_{i,n}$ are the recursively defined numbers above.

This extends the result of Hesselholt and Madsen ([12], Propostion 8.1) for representations of the form $\alpha = q - \gamma$, γ a real representation of S^1 and $q \in \mathbb{Z}$. We also compute the orders of the groups indexed by odd dimensional representations:

Theorem 1.0.2. *For q even, the order of the group $\mathrm{TR}^n_{q-1+\alpha}(\mathbb{F}_p; p)$ is p^{d_n} where d_n is given as follows. Let $d_1 = 0$. For $2 \leq j \leq n$, define $r_j = \min(j, \frac{q+|\alpha^{(n-j)}|}{2} + 1)$. Then:*

$$d_j = d_{j-1} + \begin{cases} \max(0, r_j - 1) & \text{if } r_j + g_{\tau(w+1),j-1} \leq j \\ r_j + g_{\tau(w+1),j-1} - j & \text{else} \end{cases}$$

We make a few observations about the results of these computations. If $q < \min(-|\alpha|, -|\alpha'|, \ldots - |\alpha^{(n-1)}|)$, then $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p; p) = 0$. For $q \geq \max(-|\alpha|, -|\alpha'|, \ldots - |\alpha^{(n-1)}|)$, $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p; p) \cong \mathbb{Z}/p^n$. Thus the TR-groups are easily understood outside a finite range. For any even q, each of the values $l_{i,n}$ given in the computation of $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p; p)$ satisfies $l_{i,n} \leq n$ since $TR^n_{*+\alpha}(\mathbb{F}_p; p)$ is a $TR^n_*(\mathbb{F}_p; p)$-module. However, the sum $l(n) = l_{1,n} + l_{2,n} + \cdots + l_{n,n}$ can be greater than n. Note also that the computation of the $l_{i,n}$ is independent of the prime p.

If the virtual representation α is of the form λ or $0 - \lambda$ for an actual real S^1-representation λ, then the groups $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p; p)$ are cyclic for q even. The computations of the TR-groups in these special cases is presented in Chapter 7.

1.1 Organization

In Chapter 2 we recall G-spectra and review the basic definitions of equivariant homotopy theory. In Chapter 3 we discuss a homotopy-theoretic approach to algebraic

K-theory computations. In particular, we recall the definition of TR-theory, and the operators and relations on it. We then define the $RO(S^1)$-graded TR-groups of a ring A. In Chapter 4 we study the fundamental long exact sequence of TR-theory, and its relation to the norm-restriction sequence. Chapter 5 focuses on applying this relationship to execute the first half of the induction step needed to prove Theorem 1.0.1. In Chapter 6 we finish the induction step by studying the map $\hat{\Gamma}$. In Chapter 7 we put our earlier results together to prove Theorem 1.0.1 and several corollaries. Finally, in Chapter 8 we study TR-theory as a Mackey functor using the homological algebra tools developed by Lewis and Mandell [18] for Mackey functors.

1.2 Notation and conventions

Throughout this dissertation, A denotes a commutative ring, and p a fixed prime.

The symbol α always denotes a virtual representation in $RO(S^1)$. Every virtual representation can be written uniquely as a sum of a trivial representation q and a virtual representation α with no trivial summands. We use $q+\alpha$ to denote this unique decomposition.

We use the convention $a \doteq b$ for $a = \mu b$, where μ is a unit.

Chapter 2

Equivariant Homotopy Theory

2.1 G-spectra

We begin by recalling some of the basic definitions of equivariant stable homotopy theory. Let G be a compact Lie group.

Definition 2.1.1. A G-universe U is a countably infinite-dimensional G representation containing the trivial representation, which also contains infinitely many copies of each of its finite-dimensional subrepresentations. The G-universe U is said to be complete if all irreducible G-representations occur in U.

Definition 2.1.2. A G-prespectrum Y indexed on a complete G-universe U associates to each finite dimensional sub-inner product space $V \subset U$ a based G-space $Y(V)$, together with G-equivariant structure maps

$$\sigma : Y(V) \to \Omega^{W-V} Y(W)$$

for $V \subset W$. Here $W - V$ denotes the orthogonal complement of V in W and $\Omega^V X = F(S^V, X)$. The prespectrum Y is a G-spectrum if the maps σ are all G-homeomorphisms.

These G-spectra are the objects of the G-stable category. The set of homotopy classes of maps between two G-spectra Y and Y' is denoted by $[Y, Y']_G$. Let β be a finite dimensional orthogonal G-representation. The G-stable category is stable in the sense that the suspension homomorphism

$$[Y, Y']_G \to [Y \wedge S^\beta, Y' \wedge S^\beta]_G$$

is an isomorphism.

We introduce two notions of fixed point spectra associated to a G-spectra. Let C be a subgroup of G. Let Y be a G spectrum. We first define the G/C-spectrum Y^C. For each $V \subset U^C$ we define

$$Y^C(V) = Y(V)^C.$$

We refer to this as the fixed points of Y. We now define the second notion of a fixed point spectrum, written $\Phi^C Y$ and referred to as the geometric fixed points. For each

$V \subset U^C$ we choose a $W \subset U$ such that $W^C = V$ and as V runs through the finite dimensional sub-inner product spaces of U^C, the union of the associated W's is all of U. Then

$$(\Phi^C(Y))(V) = Y(W)^C.$$

While the first fixed point construction gives a G/C-spectrum, the geometric fixed point construction gives only a G/C-prespectrum, and thus to get the spectrum $\Phi^C(Y)$ the construction above is followed by spectrification.

For the rest of this discussion, we consider the group $G = S^1$ and fix a universe U: Let $\mathbb{C}(n) = \mathbb{C}$ with S^1 acting by $g \cdot z = g^n z$, $g \in S^1$. These 1-dimensional complex representations give 2-dimensional real representations of S^1, which generate the real representation ring. Let

$$U = \bigoplus_{n \in \mathbb{Z}, a \in \mathbb{N}} \mathbb{C}(n)_a.$$

In other words, we have countably many copies of each representation $\mathbb{C}(n)$ in U.

Let $C_m \subset S^1$ be the cyclic group of order m in S^1. We have an isomorphism of groups

$$\rho_m : S^1 \to S^1/C_m$$

given by the m'th root. Let X be an S^1/C_m-space. Then we write $\rho_m^* X$ for the S^1-space associated to X via ρ_m. If we have an S^1/C_m-spectrum Y, indexed on U^{C_m}, then we can associate to it an S^1-spectrum indexed on U, denoted $\rho_{C_m}^{\#} Y$. We do this as follows: Consider the S^1-spaces

$$\rho_m^* Y((\rho_m^{-1})^*(V))$$

where $V \subset \rho_m^* U^{C_m}$. These spaces form an S^1-spectrum indexed on $\rho_m^* U^{C_m}$. The identification

$$\rho_m^* U^{C_m} = \bigoplus_{n \in m\mathbb{Z}, a \in \mathbb{N}} \mathbb{C}(n/m)_a \cong \bigoplus_{n \in \mathbb{Z}, a \in \mathbb{N}} \mathbb{C}(n)_a = U$$

gives us an S^1-spectrum indexed on U. We write $\rho_{C_m}^{\#} Y$ for this spectrum.

We also need the definition of a cyclotomic spectrum.

Definition 2.1.3. An S^1-spectrum T, indexed on U, is a cyclotomic spectrum if it comes with a S^1-equivalence

$$r_{C_l} : \rho_{C_l}^{\#} \Phi^{C_l} T \to T$$

for all finite $C_l \subset S^1$, such that for all pairs of finite subgroups, C_l, C_m the following diagram commutes:

$$\begin{tikzcd}
\rho_{C_l}^{\#} \Phi^{C_l} \rho_{C_m}^{\#} \Phi^{C_m} T \arrow[r, equal] \arrow[d, "\rho_{C_l}^{\#} \Phi^{C_l} r_{C_m}"'] & \rho_{C_{lm}}^{\#} \Phi^{C_{lm}} T \arrow[d, "r_{C_{lm}}"] \\
\rho_{C_l}^{\#} \Phi^{C_l} T \arrow[r, "r_{C_l}"] & T
\end{tikzcd}$$

In particular, a cyclotomic spectrum T comes equipped with an equivalence of

S^1-spectra
$$r : \rho_p^*((\tilde{E} \wedge T)^{C_p} \to T.$$
Here E denotes a free contractible S^1-space, and $\tilde{E}$ is the cofiber of the map $E_+ \to S^0$ given by projection onto the non-basepoint.

The topological Hochschild S^1-spectrum $T(A)$ is a cyclotomic spectrum. This fact is essential to the construction of the long exact sequence of TR-groups. For a more detailed exposition on cyclotomic spectra, see [12].

2.2 $RO(G)$-graded homotopy

Let $RO(G)$ denote the real representation ring of G. Elements $\alpha \in RO(G)$ are virtual real representations of G and hence can be written in the form
$$\alpha = [\beta] - [\gamma],$$
a formal difference of isomorphism classes of G-representations. The real virtual dimension of a virtual representation $\alpha \in RO(G)$ is denoted $|\alpha|$. The trivial representation of real dimension q is denoted by q, where q is a nonnegative integer.

For every $\alpha \in RO(G)$ we choose representatives β and γ such that $\alpha = [\beta] - [\gamma]$. Then to any G-spectrum Y, we have naturally associated $RO(G)$-graded equivariant homotopy groups defined by
$$\pi_\alpha(Y) = [S^\beta, Y \wedge S^\gamma]_G.$$
This definition gives a well-behaved theory of $RO(G)$-graded homotopy groups [18].

Chapter 3

Introduction to TR-Groups

3.1 Overview of the approach

Let p be a fixed prime and A a commutative ring. In this section we outline a homotopy-theoretic approach to algebraic K-theory computations. In particular, we recall the definitions and properties of TR-groups, studied by Hesselholt and Madsen (see [12, 13]).

The Hochschild homology of a ring A is given by a cyclic abelian group $\mathrm{HH}_{.}(A)$, with k-simplices

$$\mathrm{HH}(A)_k = A \otimes \ldots \otimes A$$

with $k+1$ tensor factors, and structure maps:

$$\begin{array}{llll} d_r(a_0 \otimes \ldots \otimes a_k) & = a_0 \otimes \ldots a_r a_{r+1} \otimes \ldots \otimes a_k & 0 \leq r < k \\ & = a_k a_0 \otimes a_1 \otimes \ldots \otimes a_{k-1} & r = k \\ s_r(a_0 \otimes \ldots \otimes a_k) & = a_0 \otimes a_r \otimes 1 \otimes a_{r+1} \otimes \ldots \otimes a_k & 0 \leq r \leq k \\ t_k(a_0 \otimes \ldots \otimes a_k) & = a_k \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_{k-1} & \end{array}$$

Then the Hochschild homology groups can be defined as the homotopy groups of the geometric realization of the underlying simplicial set

$$\mathrm{HH}(A) = |[k] \mapsto \mathrm{HH}(A)_k|.$$

The topological Hochschild homology space THH(A) is defined in an analogous manner by replacing the ring A in the Hochschild complex with an Eilenberg-MacLane spectrum for A, and the tensor products with smash products over the sphere spectrum. This was carried out by Bökstedt [5]. Bökstedt's topological Hochschild homology is naturally an S^1-space. However, in order to use topological Hochschild homology to study topological cyclic homology, we need a topological Hochschild S^1-spectrum, $T(A)$. Furthermore we want $T(A)$ to have the structure of a cyclotomic spectrum. For a detailed construction of this spectrum, see [10].

The S^1-spectrum structure of $T(A)$ allows us to take C_{p^n}-fixed point spectra of $T(A)$ for $C_{p^n} \subset S^1$ the cyclic group of order p^n. Then inclusion of fixed points induces a map

$$F : T(A)^{C_{p^n}} \to T(A)^{C_{p^{n-1}}}$$

called the Frobenius. We also get a map

$$R : T(A)^{C_{p^n}} \to T(A)^{C_{p^{n-1}}}$$

which we define in Section 4. Using these operators, we can define the topological cyclic homology spectrum of A. Let

$$\mathrm{TC}(A;p) = [\underleftarrow{\mathrm{holim}} T(A)^{C_{p^n}}]^{hF}$$

where the homotopy limit is take across the maps R above, and X^{hF} denotes the F-homotopy fixed points of X, i.e. the homotopy fiber of id $-F$.

It is possible to repeat the above, working with all natural numbers rather than just powers of one prime. The resulting spectrum $\mathrm{TC}(A)$, however, does not carry much more information than the product of the $\mathrm{TC}(A;p)$ spectra. In particular, after profinite completion

$$\mathrm{TC}(A)^{\wedge} \simeq \prod \mathrm{TC}(A;p)^{\wedge}.$$

For every ring A, we have a cyclotomic trace map [7] from algebraic K-theory of A to this topological cyclic homology spectrum of A

$$\mathrm{trc} : K(A) \to \mathrm{TC}(A).$$

We can often understand algebraic K-theory by understanding topological cyclic homology and this cyclotomic trace map. For example, for $I \subset A$ a nilpotent ideal, McCarthy's theorem [20] says that after profinite completion, there is an equivalence

$$\mathrm{trc} : K(A, I) \xrightarrow{\sim} \mathrm{TC}(A, I),$$

where $K(A, I)$ and $\mathrm{TC}(A, I)$ denote relative K-theory and relative TC respectively.

3.2 $\mathbb{Z}$-graded TR-groups

As an approach to understanding topological cyclic homology, Hesselholt and Madsen studied TR-groups. For a ring A and a fixed prime p, let

$$\mathrm{TR}^n(A;p) = T(A)^{C_{p^{n-1}}}$$

and

$$\mathrm{TR}^n_q(A;p) = \pi_q(T(A)^{C_{p^{n-1}}}).$$

These TR-groups come equipped with several operators. Again, inclusion of fixed points induces the Frobenius map

$$F : \mathrm{TR}^n_q(A;p) \to \mathrm{TR}^{n-1}_q(A;p).$$

This map has an associated transfer, the Verschiebung

$$V : \mathrm{TR}_q^{n-1}(A;p) \to \mathrm{TR}_q^n(A;p).$$

There is also a derivation $d : \mathrm{TR}_q^n(A;p) \to \mathrm{TR}_{q+1}^n(A;p)$ induced from the circle action on $T(A)$. Lastly, we have a restriction map

$$R : \mathrm{TR}_q^n(A;p) \to \mathrm{TR}_q^{n-1}(A;p)$$

which we define in Section 4. These maps satisfy the relations $FV = p$, $FdV = d$, and $VF = V(1)$, where 1 is the multiplicative unit $[1]_n$ with Witt coordinates $(1, 0, \dots 0) \in W_n(A) \cong \mathrm{TR}_0^n(A;p)$. For a more detailed introduction to TR-theory, see for instance [12, 13].

We can define $\mathrm{TC}^n(A;p)$ as the homotopy equalizer of the maps

$$R, F : \mathrm{TR}^n(A;p) \to \mathrm{TR}^{n-1}(A;p).$$

Then $\mathrm{TC}(A;p)$ is the homotopy limit of the spectra $\mathrm{TC}^n(A;p)$. Thus understanding the TR-groups of a ring helps us to understand its topological cyclic homology, and hence its algebraic K-theory.

Computations are possible on the level of TR because the operators on TR-groups and the relations between them give TR-groups a rigid algebraic structure. Indeed they have the structure of a Witt complex over A. Let A be an $\mathbb{F}_p$-algebra.

Definition 3.2.1. A Witt complex over A is

(i) A pro-differential graded ring $E_\cdot^*$ and a strict map of pro-rings

$$\lambda : W_\cdot(A) \to E_\cdot^0$$

from the pro-ring of Witt vectors on A.

(ii) A strict map of pro-graded rings

$$F : E_\cdot^* \to E_{\cdot-1}^*$$

such that $\lambda F = F\lambda$ and such that for all $a \in A$

$$Fd\lambda([a]_n) = \lambda([a]_{n-1})^{p-1} d\lambda([a]_{n-1})$$

where $[a]_n = (a, 0, \dots 0) \in W_n(A)$ is the multiplicative representative.

(iii) A strict map of graded $E_\cdot^*$-modules

$$V : F_* E_{\cdot-1}^* \to E_\cdot^*$$

such that $\lambda V = V\lambda$, $FdV = d$, and $FV = p$.

The pro-differential graded ring $E_\cdot^* = \mathrm{TR}_*^\cdot(A;p)$ is an example of such a Witt complex. In this example, the restriction map R is the structure map in the prosystem. We have a map

$$\lambda : W_n(A) \to \mathrm{TR}^n_0(A;p)$$

as in part (i) of the definition of a Witt complex. In the case of TR this map is an isomorphism. For more about Witt complexes, and TR as a Witt complex, see [11].

Via standard category theoretic arguments, one can show that the category of Witt complexes over A has an initial object, which we write as $W.\Omega^*_A$. This is the de Rham-Witt complex of A. Hesselholt and Madsen [11], [13] have given a construction of the de Rham-Witt complex for $\mathbb{Z}_{(p)}$-algebras which extends the Bloch-Deligne-Illusie [3], [16] construction for $\mathbb{F}_p$- algebras. Since $W.\Omega^*_A$ is initial, we have a map

$$W.\Omega^*_A \to \mathrm{TR}^{\cdot}_*(A;p).$$

This can then help us understand TR in terms of the de Rham-Witt complex of A.

3.3 $RO(S^1)$-graded TR-groups

Recall that TR-groups is defined as

$$\mathrm{TR}^n_q(A;p) = \pi_q(T(A)^{C_{p^{n-1}}})$$

or equivalently

$$\mathrm{TR}^n_q(A;p) \cong [S^q \wedge S^1/C_{p^{n-1}}{}_+, T(A)]_{S^1}.$$

To any S^1-spectrum there are naturally associated equivariant homotopy groups graded by the real representation ring of the circle, $RO(S^1)$. To further understand the S^1-equivariant structure of $T(A)$, we look at these groups.

Recall that elements in the representation ring are given by formal differences of isomorphism classes of representations. For every $\alpha \in RO(S^1)$ we choose representatives β and γ such that $\alpha = [\beta] - [\gamma]$. Then the $RO(S^1)$-graded TR groups are defined as

$$\mathrm{TR}^n_\alpha(A;p) = [S^\beta \wedge S^1/C_{p^{n-1}}{}_+, S^\gamma \wedge T(A)]_{S^1}.$$

For each positive n these groups form an $RO(S^1)$-graded ring. These $RO(S^1)$-graded TR-groups arise naturally in computations. For example, suppose we wanted to study the algebraic K-theory of a pointed monoid algebra $A(\Pi)$. To use the method outlined above, the first step is understanding the topological Hochschild homology of $A(\Pi)$. There is an equivalence of S^1-spectra [10]

$$T(A) \wedge N^{cy}(\Pi) \xrightarrow{\sim} T(A(\Pi)).$$

where $N^{cy}(\Pi)$ denotes the cyclic bar construction on Π. For instance, in their study of the algebraic K-theory of truncated polynomial algebras, Hesselholt and Madsen [9] used this equivalence to study $T(A[x]/(x^e))$ for A an $\mathbb{F}_p$-algebra. Let Π_e denotes the pointed multiplicative monoid $\{0, 1, x, \dots x^{e-1}\}$ where $x^e = 0$. Then

$$T(A) \wedge N^{cy}(\Pi_e) \simeq T(A(\Pi_e)) = T(A[x]/(x^e)).$$

So, to study the TR-theory of $A[x]/(x^e)$, one should look at

$$\mathrm{TR}_q^n(A[x]/(x^e), p) = [S^q \wedge S^1/C_{p^{n-1}}{}_+, T(A[x]/(x^e))]_{S^1}$$

which by the equivalence above, we can rewrite as

$$\mathrm{TR}_q^n(A[x]/(x^e), p) = [S^q \wedge S^1/C_{p^{n-1}}{}_+, T(A) \wedge N^{cy}(\Pi_e)]_{S^1}.$$

Hesselholt and Madsen then demonstrated how the S^1-equivariant homotopy type of $N^{cy}(\Pi_e)$ can be built from representation spheres. Replacing $N^{cy}(\Pi_e)$ by these representation spheres in our definition of the TR-groups above, we end up in the $RO(S^1)$-graded TR-groups of the ring A. So this expresses $K(A[x]/(x^e))$ in terms of the $RO(S^1)$-graded TR-groups of the ring A.

By considering different pointed monoids Π, one can use the $RO(S^1)$-graded TR-groups of A to understand the algebraic K-theory of a variety of rings $A(\Pi)$. Using the pointed monoid $\Pi = \{0, 1, x, x^2, \ldots y, y^2, \ldots\}$ with $xy = 0$, Hesselholt [14] has similarly computed the algebraic K-theory of the coordinate axes $k[x, y]/(xy)$ in terms of the algebraic K-theory of k.

In the non-equivariant case, the initial example completely describes the algebraic structure of the integral graded TR-groups. We would like to define a new algebraic structure embodying the structure of $RO(S^1)$-graded TR. The initial object in such a category of $RO(S^1)$-graded Witt complexes would serve as an $RO(S^1)$-graded de Rham-Witt complex.

The first step in developing such an algebraic structure is completely understanding an example of $\mathrm{TR}_\alpha^n(A;p)$ for some ring A and all $\alpha \in RO(S^1)$. Then it should be possible to formalize these structures to define the $RO(S^1)$-graded Witt complex. In this dissertation we compute $\mathrm{TR}_\alpha^n(\mathbb{F}_p;p)$ to provide such an example.

To study these $RO(S^1)$-graded TR-groups, we first consider the operators and relations that we have in this $RO(S^1)$-graded version of TR (see [12]). Again, inclusion of fixed points induces the Frobenius map, a map of $RO(S^1)$-graded rings

$$F : \mathrm{TR}_\alpha^n(A;p) \to \mathrm{TR}_\alpha^{n-1}(A;p).$$

As in the integer graded case, this map has an associated transfer, the Verschiebung

$$V : \mathrm{TR}_\alpha^{n-1}(A;p) \to \mathrm{TR}_\alpha^n(A;p)$$

which is a map of $RO(S^1)$-graded abelian groups. There is also a derivation

$$d : \mathrm{TR}_\alpha^n(A;p) \to \mathrm{TR}_{\alpha+1}^n(A;p)$$

These maps again satisfy the relations $FV = p, VF = V(1)$, and $FdV = d$. In fact, they satisfy the projection formula. For $\omega \in \mathrm{TR}_\alpha^n(A;p), \gamma \in \mathrm{TR}_\beta^{n-1}(A;p)$:

$$\omega \cdot V(\gamma) = V(F(\omega) \cdot \gamma).$$

While these maps F, V and d look similar to the $\mathbb{Z}$-graded case, the restriction map is different. In the $RO(S^1)$-graded setting we have

$$R : \mathrm{TR}^n_\alpha(A;p) \to \mathrm{TR}^{n-1}_{\alpha'}(A;p).$$

We explain in Section 4 how this restriction map is defined.

For a fixed representation α, $\mathrm{TR}^n_{*+\alpha}(A;p)$ is a module over $\mathrm{TR}^n_*(A;p)$, where $*$ is an integer-grading. In Section 6 we see that understanding this module structure is essential to our main result.

Chapter 4

Norm-Restriction Sequence

4.1 Construction of norm-restriction diagram

Throughout this section, we use the notation $T^{-\alpha}$ for $T(A) \wedge S^{-\alpha}$. Our main tool in doing computations is the fundamental long exact sequence of TR-groups and its relation to the norm-restriction cofiber sequence. Consider the cofibration sequence

$$E_+ \to S^0 \to \tilde{E}$$

where E denotes a free contractible S^1 space and the first map is given by projection onto the non-basepoint of S^0. We can smash this cofibration sequence with $T^{-\alpha}$ to get

$$E_+ \wedge T^{-\alpha} \to T^{-\alpha} \to \tilde{E} \wedge T^{-\alpha}.$$

If we then consider the map from $T^{-\alpha}$ to the function spectrum $F(E_+, T^{-\alpha})$ induced by $E_+ \to S^0$, we get a diagram of cofiber sequences

$$\begin{tikzcd}
E_+ \wedge T^{-\alpha} \arrow[r] \arrow[d] & T^{-\alpha} \arrow[r] \arrow[d] & \tilde{E} \wedge T^{-\alpha} \arrow[d] \\
E_+ \wedge F(E_+, T^{-\alpha}) \arrow[r] & F(E_+, T^{-\alpha}) \arrow[r] & \tilde{E} \wedge F(E_+, T^{-\alpha})
\end{tikzcd}$$

where for typographical reasons we have written $T^{-\alpha}$ for $T(A) \wedge S^{-\alpha}$. We can then take the homotopy groups of the $C_{p^{n-1}}$ fixed point spectra of these spectra to get a diagram

$$\begin{tikzcd}
\pi_q((E_+ \wedge T^{-\alpha})^{C_{p^{n-1}}}) \arrow[r] \arrow[d] & \pi_q((T^{-\alpha})^{C_{p^{n-1}}}) \arrow[r] \arrow[d] & \pi_q((\tilde{E} \wedge T^{-\alpha})^{C_{p^{n-1}}}) \arrow[d] \\
\pi_q((E_+ \wedge F(E_+, T^{-\alpha}))^{C_{p^{n-1}}}) \arrow[r] & \pi_q((F(E_+, T^{-\alpha}))^{C_{p^{n-1}}}) \arrow[r] & \pi_q((\tilde{E} \wedge F(E_+, T^{-\alpha}))^{C_{p^{n-1}}})
\end{tikzcd}$$

We see that the group in the center of the top line of our diagram is the definition of $\mathrm{TR}^n_{q+\alpha}(A;p)$. We would also like to identify the top right group as an $RO(S^1)$-graded TR-group. The spectrum $T(A)$ is a cyclotomic spectrum, which means, in particular, that we have an equivalence of S^1-spectra

$$r : \rho_p^*((\tilde{E} \wedge T^{-\alpha})^{C_p} \to T^{-\alpha'}.$$

Using this, we can identify the term $\pi_q((\tilde{E} \wedge T^{-\alpha})^{C_{p^{n-1}}})$ as $\mathrm{TR}^{n-1}_{q+\alpha'}(A;p)$. The map

$$R : \mathrm{TR}^n_{q+\alpha}(A;p) \to \mathrm{TR}^{n-1}_{q+\alpha'}(A;p)$$

in the top row of our diagram is the restriction map. We now turn our attention to the bottom row of our diagram. We define

$$\mathbb{H}^{-q}(C_{p^{n-1}}, T^{-\alpha}) := \pi_q((F(E_+, T^{-\alpha}))^{C_{p^{n-1}}}).$$

We refer to this spectrum $\mathbb{H}^*$ as the group cohomology spectrum of $C_{p^{n-1}}$. It is also known as the homotopy fixed point spectrum. Similarly, we define

$$\hat{\mathbb{H}}^{-q}(C_{p^{n-1}}, T^{-\alpha}) := \pi_q((\tilde{E} \wedge F(E_+, T^{-\alpha}))^{C_{p^{n-1}}}).$$

We refer to this spectrum as the Tate spectrum. From [12], Proposition 1.1,

$$\pi_q((E_+ \wedge T^{-\alpha})^{C_{p^{n-1}}}) \cong \pi_q((E_+ \wedge F(E_+, T^{-\alpha}))^{C_{p^{n-1}}})$$

and we write $\mathbb{H}_q(C_{p^{n-1}}, T^{-\alpha})$ for both of these. This spectrum $\mathbb{H}_*$ is referred to as the group homology spectrum, or the homotopy orbit spectrum. Returning to our diagram above, we have a diagram of long exact sequences:

$$\begin{tikzcd}
\arrow[r] & \mathbb{H}_q(C_{p^{n-1}}, T^{-\alpha}) \arrow[r] \arrow[d, equal] & \mathrm{TR}^n_{q+\alpha}(A;p) \arrow[r] \arrow[d, "\Gamma_{n-1}"] & \mathrm{TR}^{n-1}_{q+\alpha'}(A;p) \arrow[r] \arrow[d, "\hat{\Gamma}_{n-1}"] & {} \\
\arrow[r] & \mathbb{H}_q(C_{p^{n-1}}, T^{-\alpha}) \arrow[r] & \mathbb{H}^{-q}(C_{p^{n-1}}, T^{-\alpha}) \arrow[r] & \hat{\mathbb{H}}^{-q}(C_{p^{n-1}}, T^{-\alpha}) \arrow[r] & {}
\end{tikzcd}$$

The top row of this diagram is the fundamental long exact sequence of TR-groups. The bottom row is the norm-restrcition sequence. We see that the bottom row depends only on the dimension of α and not on the representation itself, and is thus easier to compute. In particular, there are spectral sequences that allow us to compute the groups on the bottom row. We have spectral sequences

$$\hat{E}^2_{s,t} = \hat{H}^{-s}(C_{p^{n-1}}, \pi_t(T^{-\alpha})) \Rightarrow \hat{\mathbb{H}}^{-s-t}(C_{p^{n-1}}, T^{-\alpha}),$$

$$E^2_{s,t} = H_s(C_{p^{n-1}}, \pi_t(T^{-\alpha})) \Rightarrow \mathbb{H}_{s+t}(C_{p^{n-1}}, T^{-\alpha}),$$

and

$$E^2_{s,t} = H^{-s}(C_{p^{n-1}}, \pi_t(T^{-\alpha})) \Rightarrow \mathbb{H}^{-s-t}(C_{p^{n-1}}, T^{-\alpha}).$$

4.2 Computations for $\mathbb{F}_p$

We now turn our attention to the case $A = \mathbb{F}_p$.

Proposition 4.2.1. *For q even, the norm-restriction diagram of long exact sequences is of one of two forms. If $q \geq -|\alpha|$, the diagram is of the form*

$$\begin{tikzcd}
{} \arrow[r, "0"] & \mathbb{Z}/p^{r_n} \arrow[r, "N"] \arrow[d, equal] & \mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p) \arrow[r, "R"] \arrow[d, "\Gamma_{n-1}"] & \mathrm{TR}^{n-1}_{q+\alpha'}(\mathbb{F}_p;p) \arrow[r] \arrow[d, "\hat{\Gamma}_{n-1}"] & \mathbb{Z}/p^{r_n-1} \arrow[r] \arrow[d, equal] & {} \\
{} \arrow[r, "0"] & \mathbb{Z}/p^{r_n} \arrow[r] & \mathbb{Z}/p^n \arrow[r] & \mathbb{Z}/p^{n-1} \arrow[r] & \mathbb{Z}/p^{r_n-1} \arrow[r] & {}
\end{tikzcd}$$

where $r_n = \min(n, \frac{q+|\alpha|}{2} + 1)$. If $q < -|\alpha|$, the diagram is of the form

$$\begin{tikzcd}
{} \arrow[r, "0"] & 0 \arrow[r, "N"] \arrow[d, equal] & \mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p) \arrow[r, "R"] \arrow[d, "\Gamma_{n-1}"] & \mathrm{TR}^{n-1}_{q+\alpha'}(\mathbb{F}_p;p) \arrow[r] \arrow[d, "\hat{\Gamma}_{n-1}"] & 0 \arrow[r] \arrow[d, equal] & {} \\
{} \arrow[r, "0"] & 0 \arrow[r] & \mathbb{Z}/p^n \arrow[r] & \mathbb{Z}/p^n \arrow[r] & 0 \arrow[r] & {}
\end{tikzcd}$$

We prove this proposition by using the above spectral sequences to evaluate the homotopy groups of the Tate spectrum, the group homology spectrum, and the group cohomology spectrum. We start with the Tate spectrum.

4.2.1 The Tate spectrum

Lemma 4.2.2. *Letting $[S^{-\alpha}]$ denote a shift in degree by $-|\alpha|$,*

$$\hat{\mathbb{H}}^*(C_{p^n}, T(\mathbb{F}_p) \wedge S^{-\alpha}) \cong S_{\mathbb{Z}/p^n}\{\hat{\sigma}, \hat{\sigma}^{-1}\}[S^{-\alpha}], |\hat{\sigma}| = 2.$$

To prove this lemma, we recall some results of Hesselholt and Madsen. Hesselholt and Madsen ([12], Section 4) have computed that the spectral sequence

$$\hat{E}^2_{s,t} = \hat{H}^{-s}(C_{p^n}, \pi_t(T(\mathbb{F}_p))) \Rightarrow \hat{\mathbb{H}}^{-s-t}(C_{p^n}, T(\mathbb{F}_p))$$

has $\hat{E}^2$ term

$$\hat{E}^2 = \Lambda_{\mathbb{F}_p}\{u_n\} \otimes S_{\mathbb{F}_p}\{t, t^{-1}\} \otimes S_{\mathbb{F}_p}\{\sigma\}$$

where the classes u_n, t, σ are in bidegrees $(-1,0), (-2,0), (0,2)$ respectively. The notation $S_{\mathbb{F}_p}\{t, t^{-1}\}$ denotes a polynomial algebra over $\mathbb{F}_p$ with generators t and t^{-1}. Hesselholt and Madsen also computed that the non-zero differentials are given by

$$d^{2n+1}u_n = t^{n+1}\sigma^n$$

and the spectral sequence converges to

$$\hat{\mathbb{H}}^*(C_{p^n}, T(\mathbb{F}_p)) \cong S_{\mathbb{Z}/p^n}\{\hat{\sigma}, \hat{\sigma}^{-1}\}, |\hat{\sigma}| = 2.$$

In our case we would like to look at the spectral sequence

$$\hat{E}^2_{s,t} = \hat{H}^{-s}(C_{p^n}, \pi_t(T^{-\alpha})) \Rightarrow \hat{\mathbb{H}}^{-s-t}(C_{p^n}, T^{-\alpha}).$$

We now verify that the E^2-term and differentials in this spectral sequence depend only on the dimension of α, and not on which virtual representation it is of that dimension. We first observe that we can write the E^2-term as

$$\hat{E}^2_{s,t} = \hat{H}^{-s}(C_{p^n}, \pi_t(T(\mathbb{F}_p) \wedge S^{-\alpha})) \cong \hat{H}^{-s}(C_{p^n}, \pi_{t+|\alpha|}(T(\mathbb{F}_p))).$$

Since the C_{p^n}-action on $T(\mathbb{F}_p)$ comes from an S^1 action, it is trivial on homotopy, and hence this E^2-term doesn't depend on any information about the representation α other than its dimension. Thus we write

$$\hat{E}^2(T(\mathbb{F}_p) \wedge S^{-\alpha}) = \hat{E}^2(T(\mathbb{F}_p))[S^{-\alpha}] = \big(\Lambda_{\mathbb{F}_p}\{u_n\} \otimes S_{\mathbb{F}_p}\{t, t^{-1}\} \otimes S_{\mathbb{F}_p}\{\sigma\}\big)[S^{-\alpha}]$$

where $[S^{-\alpha}]$ denotes a shift in bidegree by $(0, |-\alpha|)$. We now look at the differentials in this spectral sequence.

Lemma 4.2.3. *For $i, j \in \mathbb{Z}, j \geq 0$, the classes $t^i\sigma^j[S^{-\alpha}] \in \hat{E}^2(C_{p^n}, T^{-\alpha})$ are permanent cycles.*

Proof. By [13] (Section 4),

$$\hat{E}^2(S^1, T(\mathbb{F}_p)) \cong S_{\mathbb{F}_p}\{t, t^{-1}\} \otimes S_{\mathbb{F}_p}(\sigma).$$

Thus,

$$\hat{E}^2(S^1, T(\mathbb{F}_p) \wedge S^{-\alpha}) = \hat{E}^2(S^1, T(\mathbb{F}_p))[S^{-\alpha}]$$

is also concentrated in even total degree. Hence there are no differentials in this spectral sequence. The canonical inclusion induces a map of spectral sequences

$$\hat{E}^2(S^1, T^{-\alpha}) \to \hat{E}^2(C_{p^n}, T^{-\alpha}).$$

By [13] (Section 4) this map has image

$$\big(S_{\mathbb{F}_p}\{t, t^{-1}\} \otimes S_{\mathbb{F}_p}(\sigma)\big)[S^{-\alpha}] \in \hat{E}^2(C_{p^n}, T^{-\alpha})$$

and hence these classes are permanent cycles. □

So we conclude that either

$$d^r(u_n t^i \sigma^j [S^{-\alpha}]) \neq 0$$

for some $r \geq 2$, or all differentials are zero. Suppose for some r, i, j this differential is nonzero. Note that

$$\hat{E}^2_{s,t}(C_{p^n}, T(\mathbb{F}_p) \wedge S^{-\alpha}) = \hat{H}^{-s}(C_{p^n}, \pi_{t+|\alpha|}(T(\mathbb{F}_p))) = 0$$

if t is odd. Hence r must be odd. So, we rewrite our non-zero differential as

$$d^{2r+1}(u_n t^i \sigma^j [S^{-\alpha}]) = d^{2r+1}(u_n) t^i \sigma^j [S^{-\alpha}].$$

Hesselholt and Madsen prove in [12] Lemma 4.4 that

$$d^{2r+1}(u_n) \neq 0$$

only when $r = n$ and in that case

$$d^{2n+1}(u_n) = t^{n+1}\sigma^n.$$

Thus the only nonzero differentials in the spectral sequence we are studying are generated by

$$d^{2n+1}(u_n[S^{-\alpha}]) = t^{n+1}\sigma^n[S^{-\alpha}]$$

and hence from [12], Section 4, this spectral sequence converges to

$$\hat{\mathbb{H}}^*(C_{p^n}, T(\mathbb{F}_p) \wedge S^{-\alpha}) \cong S_{\mathbb{Z}/p^n}\{\hat{\sigma}, \hat{\sigma}^{-1}\}[S^{-\alpha}], |\hat{\sigma}| = 2.$$

This proves Lemma 4.2.2.

4.2.2 The group homology spectrum

We now turn our attention to the group homology spectrum.

Lemma 4.2.4. *For q even*

$$\mathbb{H}_q(C_{p^{n-1}}, T^{-\alpha}) \cong \begin{cases} \mathbb{Z}/p^r & \text{if } q \geq -|\alpha| \\ 0 & \text{if } q < -|\alpha| \end{cases}$$

and

$$\mathbb{H}_{q-1}(C_{p^{n-1}}, T^{-\alpha}) \cong \begin{cases} \mathbb{Z}/p^{r-1} & \text{if } q \geq -|\alpha| \\ 0 & \text{if } q < -|\alpha| \end{cases}$$

where $r_n = \min(n, \frac{q+|\alpha|}{2} + 1)$.

Proof. Recall from above that we have a spectral sequence

$$E^2_{s,t} = H_s(C_{p^{n-1}}, \pi_t(T(\mathbb{F}_p) \wedge S^{-\alpha})) \Rightarrow \mathbb{H}_{s+t}(C_{p^{n-1}}, T(\mathbb{F}_p) \wedge S^{-\alpha}).$$

We can see easily from the spectral sequence for $\mathbb{H}_q$ that

$$\mathbb{H}_q(C_{p^{n-1}}, T(\mathbb{F}_p) \wedge S^{-\alpha}) = 0 \text{ for } q < -|\alpha|.$$

We now compute these groups for $q \geq -|\alpha|$. Note that

$$E^2_{s,t} = H_s(C_{p^{n-1}}, \pi_t(T(\mathbb{F}_p) \wedge S^{-\alpha})) \cong H_s(C_{p^{n-1}}, \pi_{t+|\alpha|}(T(\mathbb{F}_p)).$$

Bökstedt and Breen ([6], [4]) calculated that $\pi_*(T(\mathbb{F}_p)) \cong S_{\mathbb{F}_p}(\sigma_1), |\sigma_1| = 2$. So our E^2 term is

$$E^2_{s,t} = \begin{cases} H_s(\mathbb{Z}/p^{n-1}, \mathbb{Z}/p) = \mathbb{Z}/p & \text{if } t + |\alpha| \geq 0, \text{even}, s \geq 0 \\ 0 & \text{else} \end{cases}$$

Viewing this E^2 term as the shifted first quadrant part of the $\hat{E}^2$ term of the spectral sequence computing the Tate cohomology, we have names for these classes. We have

$$E^2_{s,t}(C_{p^n}, T(\mathbb{F}_p) \wedge S^{-\alpha}) = \big(S_{\mathbb{F}_p}(t^{-1}) \otimes S_{\mathbb{F}_p}(\sigma) \otimes \Lambda_{\mathbb{F}_p}(t^{-1}u_n)\big)[S^{-\alpha}].$$

In our discussion of the Tate spectrum above, we saw that the differentials in the spectral sequence are generated by

$$d^{2n+1}u_n = t^{n+1}\sigma^n.$$

Using this and the fact that the extensions in the passage from E^∞ to homotopy groups are maximally nontrivial, we can compute the groups we need for our long exact sequence above. They look like:

q	$\mathbb{H}_q(C_p, T^{-\alpha})$	$\mathbb{H}_q(C_{p^2}, T^{-\alpha})$	$\mathbb{H}_q(C_{p^3}, T^{-\alpha})$	$\dots$
$-\|\alpha\|+0$	$\mathbb{Z}/p$	$\mathbb{Z}/p$	$\mathbb{Z}/p$	$\dots$
$-\|\alpha\|+1$	$\mathbb{Z}/p$	$\mathbb{Z}/p$	$\mathbb{Z}/p$	$\dots$
$-\|\alpha\|+2$	$\mathbb{Z}/p^2$	$\mathbb{Z}/p^2$	$\mathbb{Z}/p^2$	$\dots$
$-\|\alpha\|+3$	$\mathbb{Z}/p$	$\mathbb{Z}/p^2$	$\mathbb{Z}/p^2$	$\dots$
$-\|\alpha\|+4$	$\mathbb{Z}/p^2$	$\mathbb{Z}/p^3$	$\mathbb{Z}/p^3$	$\dots$
$-\|\alpha\|+5$	$\mathbb{Z}/p$	$\mathbb{Z}/p^2$	$\mathbb{Z}/p^3$	$\dots$
$-\|\alpha\|+6$	$\mathbb{Z}/p^2$	$\mathbb{Z}/p^3$	$\mathbb{Z}/p^4$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

This gives the stated result. □

4.2.3 The group cohomology spectrum

Finally we look at the homotopy groups of the group cohomology spectrum.

Lemma 4.2.5. *The homotopy groups of the group cohomology spectrum are given by*

$$\mathbb{H}^{-q}(C_{p^n}, T^{-\alpha}) = \begin{cases} \mathbb{Z}/p^{n+1} & q \geq |-\alpha|, \text{ even} \\ \mathbb{Z}/p^n & q < |-\alpha|, \text{ even} \\ 0 & \text{else} \end{cases}$$

Proof. Recall that we have a spectral sequence

$$E^2_{s,t} = H^{-s}(C_{p^n}, \pi_t(T^{-\alpha})) \Rightarrow \mathbb{H}^{-s-t}(C_{p^n}, T^{-\alpha}).$$

If we look at the spectral sequence

$$E^2_{s,t} = H^{-s}(C_{p^n}, \pi_t(T(\mathbb{F}_p)) \Rightarrow \mathbb{H}^{-s-t}(C_{p^n}, T(\mathbb{F}_p))$$

we see that the E^2 term of this spectral sequence is the second quadrant part of the $\hat{E}^2$-term for our Tate spectral sequence. In other words, the E^2-term for this group cohomology spectral sequence is

$$E^2 = \Lambda_{\mathbb{F}_p}(u_n) \otimes S_{\mathbb{F}_p}(t) \otimes S_{\mathbb{F}_p}(\sigma).$$

Again, the differentials in this spectral sequence are generated by $d^{2n+1}u_n = t^{n+1}\sigma^n$. Hence we have

$$E^\infty = S_{\mathbb{F}_p}(t) \otimes S_{\mathbb{F}_p}(\sigma)/t^{n+1}\sigma^n.$$

As above, in our case this is shifted by the dimension of our representation α. It follows by [12] Section 4 that

$$\mathbb{H}^{-q}((C_{p^n}, T(\mathbb{F}_p) \wedge S^{-\alpha}) = \begin{cases} \mathbb{Z}/p^{n+1} & q \geq |-\alpha|, \text{ even} \\ \mathbb{Z}/p^n & q < |-\alpha|, \text{ even} \\ 0 & \text{else} \end{cases}$$

□

The results of Lemmas 4.2.2, 4.2.4, and 4.2.5, prove Proposition 4.2.1.

Chapter 5

Inductive method

The main result of this dissertation is the computation of $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p)$, for all $n \in \mathbb{N}$, $q \in \mathbb{Z}$ even, and $\alpha \in RO(S^1)$. For a fixed virtual representation α, we make this computation inductively on n.

5.1 Base case

The base case of our induction is the computation of $\mathrm{TR}^1_{q+\alpha^{(n-1)}}(\mathbb{F}_p;p)$. Note that

$$\mathrm{TR}^1_{q+\alpha^{(n-1)}}(\mathbb{F}_p;p) \cong \pi_{q+|\alpha^{(n-1)}|}(T(\mathbb{F}_p)) \cong \begin{cases} \mathbb{Z}/p & q \geq -|\alpha^{(n-1)}|, \text{ even} \\ 0 & \text{else} \end{cases}$$

This is a shift of the homotopy groups

$$\mathrm{TR}^1_*(\mathbb{F}_p;p) \cong \pi_*(T(\mathbb{F}_p)) \cong S_{\mathbb{Z}/p}\{\sigma_1\}$$

where $|\sigma_1| = 2$. For a fixed representation β, we can consider $\mathrm{TR}^1_{*+\beta}(\mathbb{F}_p;p)$ as a $\mathrm{TR}^1_*(\mathbb{F}_p;p)$ module, where $*$ denotes an integer grading. The module $\mathrm{TR}^1_{*+\beta}(\mathbb{F}_p;p)$ is a free $\mathrm{TR}^1_*(\mathbb{F}_p;p)$-module of rank one with a generator in degree $-|\beta|$. Multiplication by σ_1 gives an isomorphism

$$\sigma_1 : \mathrm{TR}^1_{q+\beta}(\mathbb{F}_p;p) \xrightarrow{\sim} \mathrm{TR}^1_{q+2+\beta}(\mathbb{F}_p;p), \text{ for } q \geq -|\beta|, \text{ even.}$$

Understanding the structure of the $RO(S^1)$-graded TR-groups as modules over $\mathbb{Z}$-graded TR is essential to our computations.

Note that the groups $\mathrm{TR}^1_{q+\beta}$ depend only on the dimensions of β. Similarly, $\mathrm{TR}^n_{q+\beta}$ depends only on the dimensions of $\beta, \beta', \dots \beta^{(n-1)}$.

5.2 Method for inductive step

In order to compute $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p)$ we use the norm-restriction diagram of long exact sequences to work inductively. Given the group $\mathrm{TR}^{n-1}_{q+\alpha'}(\mathbb{F}_p;p)$ and the map

$$\hat{\Gamma}_{n-1} : \mathrm{TR}^{n-1}_{q+\alpha'}(\mathbb{F}_p;p) \to \hat{\mathbb{H}}^{-q}(C_{p^{n-1}}, T(\mathbb{F}_p) \wedge S^{-\alpha}),$$

in the inductive step we compute the group $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p)$ and the map

$$\hat{\Gamma}_n : \mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p) \to \hat{\mathbb{H}}^{-q}(C_{p^n}, T(\mathbb{F}_p) \wedge S^{-\beta}).$$

Here β is any virtual representation in $RO(S^1)$ such that $\beta' = \alpha$.

We first discuss how to compute the group $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p)$ and then how to compute the map $\hat{\Gamma}_n$.

Recall $r_n = \min(n, \frac{q+|\alpha|}{2})$. Then for $q + |\alpha| \geq 0$ the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "0"] & \mathbb{H}_q \arrow[r, "N"] \arrow[d, equal] & \mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p) \arrow[r, "R"] \arrow[d, "\Gamma_{n-1}"] & \mathrm{TR}^{n-1}_{q+\alpha'}(\mathbb{F}_p;p) \arrow[r] \arrow[d, "\hat{\Gamma}_{n-1}"] & \mathbb{H}_{q-1} \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r, "0"] & \mathbb{H}_q \arrow[r] & \mathbb{H}^{-q} \arrow[r] & \hat{\mathbb{H}}^{-q} \arrow[r] & \mathbb{H}_{q-1} \arrow[r] & \cdots
\end{tikzcd}$$

is of the form

$$\begin{tikzcd}
\arrow[r, "0"] & \mathbb{Z}/p^{r_n} \arrow[r, "N"] \arrow[d, equal] & \mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p) \arrow[r, "R"] \arrow[d, "\Gamma_{n-1}"] & \mathbb{Z}/p^{l_{1,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n-1,n-1}} \arrow[r] \arrow[d, "\hat{\Gamma}_{1,n-1}+\ldots+\hat{\Gamma}_{n-1,n-1}"] & \mathbb{Z}/p^{r_n-1} \arrow[r] \arrow[d, equal] & {} \\
\arrow[r, "0"] & \mathbb{Z}/p^{r_n} \arrow[r] & \mathbb{Z}/p^n \arrow[r] & \mathbb{Z}/p^{n-1} \arrow[r] & \mathbb{Z}/p^{r_n-1} \arrow[r] & {}
\end{tikzcd}$$

where the map $\hat{\Gamma}_{i,n-1}$ maps

$$\hat{\Gamma}_{i,n-1} : \mathbb{Z}/p^{l_{i,n-1}} \to \mathbb{Z}/p^{n-1}$$

We let

$$k_{i,n-1} = \mathrm{length}_{\mathbb{Z}_p}(\ker \hat{\Gamma}_{i,n-1}).$$

Without loss of generality, we suppose the summands are ordered such that $k_{1,n-1} \leq k_{2,n-1} \leq \ldots \leq k_{n-1,n-1}$.

The following proposition computes $l_{i,n} = \mathrm{length}_{\mathbb{Z}_p}(\mathbb{Z}/p^{l_{i,n}})$, the lengths of the summands as $\mathbb{Z}_p$-modules. This computation gives half of the induction step for proving Theorem 1.0.1.

Proposition 5.2.1. *For q even, there is an isomorphism of abelian groups*

$$\mathbb{Z}/p^{l_{1,n}} \oplus \mathbb{Z}/p^{l_{2,n}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n,n}} \xrightarrow[\sim]{\iota_n} \mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p)$$

where the lengths $l_{m,n}$ are given as follows: Let $g_{i,n-1} = l_{i,n-1} - k_{i,n-1}$ and $w = \#\{1 \leq$

$i \leq n-1 | k_{i,n-1} \geq r_n\}$. We choose a permutation $\tau \in \Sigma_{n-1}$ satisfying the conditions

$$r_n \leq k_{\tau(1),n-1} \leq k_{\tau(2),n-1} \leq \ldots \leq k_{\tau(w),n-1},$$

$$r_n > k_{\tau(w+1),n-1}, k_{\tau(w+2),n-1}, \ldots k_{\tau(n-1),n-1},$$

$$g_{\tau(w+1),n-1} \geq g_{\tau(w+2),n-1} \geq \cdots \geq g_{\tau(n-1),n-1}.$$

We can choose any permutation τ meeting these conditions. Then

$$l_{1,n} = \begin{cases} r & \text{if } w = n-1 \\ \min(n, r_n + g_{\tau(w+1),n-1}) & \text{else} \end{cases}$$

For $1 \leq m \leq w$, $l_{m+1,n} = l_{\tau(m),n-1}$. For $w+2 \leq m \leq n-1$:

$$l_{m,n} = \begin{cases} l_{\tau(m),n-1} & \text{if } \tau(v) < \tau(m) \text{ for some } w+1 \leq v < m \\ g_{\tau(m),n-1} + k_{\tau(z),n-1} & \text{else} \end{cases}$$

Here $\tau(z)$ is the smallest number such that $w+1 \leq z < m$. Finally, $l_{n,n} = k_{1,n-1}$ if $w < n-1$.

Before proving the proposition, we recall some general facts about diagrams of exact sequences. Suppose we have a diagram of the form

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[d, equal] & & C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d, equal] & {} \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & {} \end{tikzcd}$$

The pullback

$$\begin{tikzcd} R \arrow[r] \arrow[d] & C \arrow[d] \\ B' \arrow[r] & C' \end{tikzcd}$$

makes the above diagram commute, and the sequence

$$0 \longrightarrow A \longrightarrow R \longrightarrow C \longrightarrow D \longrightarrow$$

is exact. Any group B making the diagram of exact sequences commute factors through the pullback, giving a diagram of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & D \arrow[r] \arrow[d, equal] & {} \\ 0 \arrow[r] & A \arrow[r] & R \arrow[r] & C \arrow[r] & D \arrow[r] & {} \end{tikzcd}$$

Then, by the five lemma, $R \cong B$. So, up to isomorphism there is only one group fitting into such a diagram of long exact sequences.

To calculate $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p)$ we use the diagram of long exact sequences

$$\begin{tikzcd}
{} \arrow[r, "0"] & \mathbb{Z}/p^r \arrow[r, "N"] \arrow[d, equal] & \mathrm{TR}^n_{q+\alpha} \arrow[r, "R"] \arrow[d, "\Gamma_{n-1}"] & \mathbb{Z}/p^{l_{1,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n-1,n-1}} \arrow[r, "\partial"] \arrow[d, "\hat{\Gamma}_{1,n-1}+\ldots+\hat{\Gamma}_{n-1,n-1}"] & \mathbb{Z}/p^{r-1} \arrow[r] \arrow[d, equal] & {} \\
{} \arrow[r, "0"] & \mathbb{Z}/p^r \arrow[r, "N^h"] & \mathbb{Z}/p^n \arrow[r, "R^h"] & \mathbb{Z}/p^{n-1} \arrow[r, "\partial^h"] & \mathbb{Z}/p^{r-1} \arrow[r] & {}
\end{tikzcd}$$

The maps in the bottom row are given by $N^h(1) \doteq p^{n-r}$, $R^h(1) \doteq p^{r-1}$ and $\partial(1) \doteq 1$. From above, the group $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p)$ is the pullback of the diagram

$$\begin{tikzcd}
 & \mathbb{Z}/p^{l_{1,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n-1,n-1}} \arrow[d, "\hat{\Gamma}_{1,n-1}+\ldots+\hat{\Gamma}_{n-1,n-1}"] \\
\mathbb{Z}/p^n \arrow[r, "R^h"] & \mathbb{Z}/p^{n-1}
\end{tikzcd}$$

The following proposition evaluates this pullback.

Proposition 5.2.2. *Let $\hat{\Gamma}_{i,n-1}$ be a map of abelian groups*

$$\hat{\Gamma}_{i,n-1} : \mathbb{Z}/p^{l_{i,n-1}} \to \mathbb{Z}/p^{n-1}.$$

Suppose $k_{i,n-1} = \mathrm{length}_{\mathbb{Z}_p}(\ker \hat{\Gamma}_{i,n-1})$, and $k_{1,n-1} \leq k_{2,n-1} \leq \ldots \leq k_{n-1,n-1}$. Then the following square is a pullback diagram

$$\begin{tikzcd}
\mathbb{Z}/p^{l_{1,n}} \oplus \mathbb{Z}/p^{l_{2,n}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n,n}} \arrow[r, "R"] \arrow[d] & \mathbb{Z}/p^{l_{1,n-1}} \oplus \mathbb{Z}/p^{l_{2,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n-1,n-1}} \arrow[d, "\hat{\Gamma}_{1,n-1}+\ldots+\hat{\Gamma}_{n-1,n-1}"] \\
\mathbb{Z}/p^n \arrow[r, "R^h"] & \mathbb{Z}/p^{n-1}
\end{tikzcd}$$

where the map R^h is given by $R^h(1) \doteq p^{r-1}$ and the integers $l_{i,n}$, $1 \leq i \leq n$ are defined as follows: Let $g_{i,n-1} = l_{i,n-1} - k_{i,n-1}$. We choose a permutation $\tau \in \Sigma_{n-1}$ satisfying the conditions

$$r \leq k_{\tau(1),n-1} \leq k_{\tau(2),n-1} \leq \ldots k_{\tau(w),n-1},$$

$$r > k_{\tau(w+1),n-1}, k_{\tau(w+2),n-1}, \ldots k_{\tau(n-1),n-1},$$

$$g_{\tau(w+1),n-1} \geq g_{\tau(w+2),n-1} \geq \ldots \geq g_{\tau(n-1),n-1},$$

Then

$$l_{1,n} = \begin{cases} r & \text{if } w = n-1 \\ \min(n, r_n + g_{\tau(w+1),n-1}) & \text{else} \end{cases}$$

For $1 \leq m \leq w$, $l_{m+1,n} = l_{\tau(m),n-1}$. For $w+2 \leq m \leq n-1$:

$$l_{m,n} = \begin{cases} l_{\tau(m),n-1} & \text{if } \tau(v) < \tau(m) \text{ for some } w+1 \leq v < m \\ g_{\tau(m),n-1} + k_{\tau(z),n-1} & \text{else} \end{cases}$$

Here $\tau(z)$ is the smallest number such that $w+1 \leq z < m$. Finally, $l_{n,n} = k_{1,n-1}$ if

$w < n-1$.

Proof. Let G denote the pullback of the above diagram. We would like to prove that

$$G \cong \mathbb{Z}/p^{l_{1,n}} \oplus \mathbb{Z}/p^{l_{2,n}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n,n}}.$$

By definition

$$G \cong (x, y_1, y_2, \ldots y_{n-1}) \in \mathbb{Z}/p^n \oplus \mathbb{Z}/p^{l_{1,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n-1,n-1}}$$

such that

$$p^{r-1}x = p^{n-1-g_{1,n-1}}y_1 + p^{n-1-g_{2,n-1}}y_2 + \ldots + p^{n-1-g_{n-1,n-1}}y_{n-1}.$$

We let $g_{i,n-1} = l_{i,n-1} - k_{i,n-1}$. We choose a permutation $\tau \in \Sigma_{n-1}$ such that

$$r \leq k_{\tau(1),n-1} \leq k_{\tau(2),n-1} \leq \ldots \leq k_{\tau(w),n-1},$$

$$r > k_{\tau(w+1),n-1} \leq k_{\tau(w+2),n-1} \leq \ldots k_{\tau(n-1),n-1},$$

$$g_{\tau(w+1),n-1} \geq g_{\tau(w+2),n-1} \geq \ldots \geq g_{\tau(n-1),n-1},$$

For each $1 \leq m \leq w$ the element

$$\zeta_{\tau(m)} = (p^{n-g_{\tau(m),n-1}-r_n}, 0, \ldots 1, 0, \ldots 0) \in \mathbb{Z}/p^n \oplus \mathbb{Z}/p^{l_{1,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n-1,n-1}}$$

where the 1 is in the $y_{\tau(m)}$ coordinate, generates a subgroup of G of order $p^{l_{\tau(m),n-1}}$. Note that the different m give linearly independent elements of G. The element

$$(p^{n-r_n}, 0, \ldots 0) \in \mathbb{Z}/p^n \oplus \mathbb{Z}/p^{l_{1,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n-1,n-1}}$$

generates a subgroup of G of order p^r. If $w = n-1$, $k_{i,n-1} \geq r_n$ for all $1 \leq i \leq n-1$, so this element is linearly independent from the elements ζ_i. So in the case $w = n-1$ we have computed

$$G \cong \mathbb{Z}/p^r \oplus \mathbb{Z}/p^{l_{1,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n-1,n-1}}.$$

Suppose then that $w \neq n-1$. For $w+1 \leq m \leq t$, if $k_{\tau(m),n-1} \geq l_{\tau(m),n-1}$, then the element

$$\eta_{\tau(m)} = (0, 0 \ldots 1, 0 \ldots 0) \in \mathbb{Z}/p^n \oplus \mathbb{Z}/p^{l_{1,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n-1,n-1}}$$

which is non-zero only in the $y_{\tau(m)}$-coordinate, generates a subgroup of length $l_{\tau(m),n-1}$. For $k_{\tau(m),n-1} < l_{\tau(m),n-1}$, $w+1 \leq m \leq n-1$, if $l_{\tau(m),n-1} - k_{\tau(m),n-1} + r_n \leq n$, we consider the element

$$\psi_{\tau(m)} = (p^{n-g_{\tau(m),n-1}-r_n}, 0, \ldots 1, 0, \ldots 0) \in \mathbb{Z}/p^n \oplus \mathbb{Z}/p^{l_{1,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n-1,n-1}}$$

where the 1 is in the $y_{\tau(m)}$-coordinate. This generates an order $p^{r+g_{\tau(m),n-1}}$ subgroup of G. If instead $g_{\tau(m),n-1} + r_n \geq n$, we consider the element

$$\psi_{\tau(m)} = (1, 0, \dots p^{g_{\tau(m),n-1}+r_n-n}, 0 \dots 0) \in \mathbb{Z}/p^n \oplus \mathbb{Z}/p^{l_{1,n-1}} \oplus \dots \oplus \mathbb{Z}/p^{l_{n-1,n-1}}$$

where the second nonzero entry is in the $y_{\tau(m)}$-coordinate. This generates a subgroup of length n. For $w+2 \leq m \leq n-1$, $w+1 \leq m' < m$, consider also the elements

$$\nu_{\tau(m'),\tau(m)} = (0, 0, \dots, p^{g_{\tau(m'),n-1}-g_{\tau(m),n-1}}, 0 \dots -1, 0, \dots 0).$$

Here, the first nonzero term is in the $y_{\tau(m')}$-coordinate, and the second nonzero term is in the $y_{\tau(m)}$-coordinate. These generate subgroups of length $\max(l_{\tau(m),n-1}, g_{\tau(m),n-1} + k_{\tau(m'),n-1})$. Since we have ordered the summands such that $k_{\tau(m),n-1} \leq k_{\tau(m'),n-1}$ exactly when $\tau(m) \leq \tau(m')$, we conclude that the length of this group is $l_{\tau(m),n-1}$ if $\tau(m') \leq \tau(m)$, and $g_{\tau(m),n-1} + k_{\tau(m'),n-1}$ otherwise. Finally, consider the element

$$\nu_{n-1} = (0, p^{g_{1,n-1}}, 0, 0 \dots 0).$$

which generates a subgroup of order $p^{k_{1,n-1}}$.

Together, the elements above generate the group G. In order to identify this group, we need to find a linearly independent set of generators that spans. For a fixed $w+2 \leq m \leq n-1$ let m' be given by $\tau(m') = \min(\tau(i) | w+1 \leq i < m)$. Our set of generators is the following:

$$\{\psi_{\tau(w+1)}\}, \{\zeta_{\tau(m)} | 1 \leq m \leq w\},$$

$$\{\eta_{\tau(m)} | w+2 \leq m \leq n-1, k_{\tau(m),n-1} \geq l_{\tau(m),n-1}\},$$
$$\{\nu_{\tau(m),\tau(m')} | w+2 \leq m \leq n-1, k_{\tau(m),n-1} < l_{\tau(m),n-1}\}, \{\nu_{n-1}\}.$$

The summands in the proposition correspond to these generators in the following way:

$$\psi_{\tau(w+1)} \mapsto \mathbb{Z}/p^{l_{1,n}},\ \zeta_{\tau(m)} \mapsto \mathbb{Z}/p^{l_{m+1,n}},\ \eta_{\tau(m)} \mapsto \mathbb{Z}/p^{l_{m,n}},$$
$$\nu_{\tau(m),\tau(m')} \mapsto \mathbb{Z}/p^{l_{m,n}},\ \nu_{n-1} \mapsto \mathbb{Z}/p^{l_{n,n}}.$$

Then the values of the lengths $l_{i,n}$ follow directly from our analysis above. $\square$

Proposition 5.2.1 then follows from Proposition 5.2.2. The proof of Proposition 5.2.2 above also determines the restriction map:

$$\begin{tikzcd}
\mathbb{Z}/p^{l_{1,n}} \oplus \dots \oplus \mathbb{Z}/p^{l_{n,n}} \arrow[r, "\iota_n", "\sim"'] \arrow[d, "R^\iota"] & \mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p; p) \arrow[d, "R"] \\
\mathbb{Z}/p^{l_{1,n-1}} \oplus \dots \oplus \mathbb{Z}/p^{l_{n-1,n-1}} \arrow[r, "\iota_{n-1}", "\sim"'] & \mathrm{TR}^{n-1}_{q+\alpha'}(\mathbb{F}_p; p)
\end{tikzcd}$$

Note that we can write the isomorphism ι_n as

$$\iota_n = \iota_{1,n} + \dots + \iota_{n,n}$$

where

$$\iota_{j,n} : \mathbb{Z}/p^{l_{j,n}} \longrightarrow \mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p).$$

Then we can express the composite $R\iota_{j,n}$ as

$$R\iota_{j,n}(a) = \sum_{1\leq i\leq n-1} \iota_{i,n-1}(R^{\iota}_{i,j,n}(a))$$

where

$$R^{\iota}_{i,j,n} : \mathbb{Z}/p^{l_{j,n}} \longrightarrow \mathbb{Z}/p^{l_{i,n-1}}.$$

We now determine the map R by specifying the maps $R^{\iota}_{i,j,n}$.

Theorem 5.2.3. *The maps $R^{\iota}_{i,j,n}$ are given by the following. If $i = \tau(w+1)$:*

$$R^{\iota}_{i,1,n}(1) \doteq \begin{cases} 1 & \text{if } r_n + g_{\tau(w+1),n-1} \leq n \\ p^{r+g_{\tau(w+1),n-1}-n} & \text{if } r_n + g_{\tau(w+1),n-1} > n \end{cases}$$

For $i \neq \tau(w+1)$, $R^{\iota}_{i,1,n}(1) = 0$. For $1 < m \leq n$, if $m-1 \leq w$:

$$R^{\iota}_{i,m,n}(1) \doteq \begin{cases} 1 & \text{if } i = \tau(m-1) \\ 0 & \text{else} \end{cases}$$

For $w < m-1 < n$:

$$R^{\iota}_{i,m,n}(1) \doteq \begin{cases} 1 & \text{if } i = \tau(m), k_{\tau(m),n-1} \geq l_{\tau(m),n-1} \\ -1 & \text{if } i = \tau(m), k_{\tau(m),n-1} < l_{\tau(m),n-1} \\ p^{g_{\tau(m'),n-1}-g_{\tau(m),n-1}} & \text{if } i = \tau(m'), k_{\tau(m),n-1} < l_{\tau(m),n-1} \\ 0 & \text{else} \end{cases}$$

If $w < n-1$:

$$R^{\iota}_{i,n,n}(1) \doteq \begin{cases} p^{g_{1,n-1}} & \text{if } i = 1 \\ 0 & \text{else} \end{cases}$$

Proof. This follows directly from the proof of Proposition 5.2.2. □

Given the group $\mathrm{TR}^{n-1}_{q+\alpha'}(\mathbb{F}_p;p)$ and the map $\hat{\Gamma}_{n-1}$ on this group, we have now calculated the group $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p)$. In order to complete the induction step we also need to compute the map $\hat{\Gamma}_n$. We study this map in the next section.

Chapter 6

The Map $\hat{\Gamma}$

In this section we study the map

$$\hat{\Gamma}_n : \mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p) \to \hat{\mathbb{H}}^{-q}(C_{p^n}, T(\mathbb{F}_p) \wedge S^{-\beta})$$

for $\beta \in RO(S^1)$ with $\beta' = \alpha$. Hesselholt and Madsen have proven the following proposition about this map ([12], Addendum 8.1).

Proposition 6.0.4. *The map $\hat{\Gamma}_n$ above induces isomorphisms on homotopy groups in dimensions $q \geq -\min(|\alpha|, |\alpha''|, \ldots |\alpha^{(n-1)}|)$.*

Thus we are left to understand the map $\hat{\Gamma}_n$ for values of

$$q < -\min(|\alpha|, |\alpha''|, \ldots |\alpha^{(n-1)}|).$$

We need to understand the module structure of $\mathrm{TR}^n_{*+\alpha}(\mathbb{F}_p;p)$ as a $\mathrm{TR}^n_*(\mathbb{F}_p;p)$-module. We first recall from [12], Proposition 4.4, that

$$\mathrm{TR}^{n-1}_*(\mathbb{F}_p;p) = S_{\mathbb{Z}/p^{n-1}}\{\sigma_{n-1}\}$$

where $|\sigma_{n-1}| = 2, F(\sigma_{n-1}) = \sigma_{n-2}$, $V(\sigma_{n-2}) = p\sigma_{n-1}$, and $R(\sigma_{n-1}) = p\lambda_{n-1}\sigma_{n-2}$, where $\lambda_{n-1} \in \mathbb{Z}/p^{n-1}$ is a unit. It follows from [15], Theorem B (iii), that this unit $\lambda_{n-1} = 1$.

There is a commutative diagram

$$\begin{tikzcd}
\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p) \arrow[r, "\sigma_n"] \arrow[d, "\hat{\Gamma}_n"] & \mathrm{TR}^n_{q+2+\alpha}(\mathbb{F}_p;p) \arrow[d, "\hat{\Gamma}_n"] \\
\hat{\mathbb{H}}^{-q}(C_{p^n}; T(\mathbb{F}_p) \wedge S^{-\beta}) \arrow[r, "\hat{\Gamma}_n(\sigma_n)"] & \hat{\mathbb{H}}^{-q-2}(C_{p^n}; T(\mathbb{F}_p) \wedge S^{-\beta})
\end{tikzcd}$$

If we understand the maps given by multiplication by σ_n we can use this commutative diagram to understand the map $\hat{\Gamma}_n$ by inducting down on q.

To study multiplication by σ_n we consider several commutative diagrams. Since

F is a map of $RO(S^1)$-graded rings, we have a commutative diagram

$$\begin{tikzcd} \mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p) \arrow[r, "\sigma_n"] \arrow[d, "F"] & \mathrm{TR}^n_{q+2+\alpha}(\mathbb{F}_p;p) \arrow[d, "F"] \\ \mathrm{TR}^{n-1}_{q+\alpha}(\mathbb{F}_p;p) \arrow[r, "\sigma_{n-1}"] & \mathrm{TR}^{n-1}_{q+2+\alpha}(\mathbb{F}_p;p) \end{tikzcd}$$

Using the identity $R(\sigma_n) = p\sigma_{n-1}$, we get a commutative diagram

$$\begin{tikzcd} \mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p) \arrow[r, "\sigma_n"] \arrow[d, "R"] & \mathrm{TR}^n_{q+2+\alpha}(\mathbb{F}_p;p) \arrow[d, "R"] \\ \mathrm{TR}^{n-1}_{q+\alpha'}(\mathbb{F}_p;p) \arrow[r, "p\sigma_{n-1}"] & \mathrm{TR}^{n-1}_{q+2+\alpha'}(\mathbb{F}_p;p) \end{tikzcd}$$

Finally, the projection formula

$$\omega \cdot V(\gamma) = V(F(\omega) \cdot \gamma).$$

applied to $\omega = \sigma_n \in \mathrm{TR}^n_2(\mathbb{F}_p;p)$, and $\gamma \in \mathrm{TR}^{n-1}_{q+\alpha}(\mathbb{F}_p;p)$ gives a commutative diagram

$$\begin{tikzcd} \mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p) \arrow[r, "\sigma_n"] & \mathrm{TR}^n_{q+2+\alpha}(\mathbb{F}_p;p) \\ \mathrm{TR}^{n-1}_{q+\alpha}(\mathbb{F}_p;p) \arrow[r, "\sigma_{n-1}"] \arrow[u, "V"] & \mathrm{TR}^{n-1}_{q+2+\alpha}(\mathbb{F}_p;p) \arrow[u, "V"] \end{tikzcd}$$

We use these commutative diagrams to aid in our computation of the maps σ_n, and hence the map $\hat{\Gamma}_n$.

6.1 Completion of the inductive step

We divide our study of the map $\hat{\Gamma}_n$ into four lemmas, which we then use to prove Theorem 1.0.1. In the remainder of this section we state and prove these four lemmas. Let $\beta \in RO(S^1)$ be a representation such that $\beta' = \alpha$. Recall that for a summand $\mathbb{Z}/p^{l_{i,n}}$ of $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p)$, $q \geq -|\alpha|$ even, we have

$$\hat{\Gamma}_{i,n} : \mathbb{Z}/p^{l_{i,n}} \to \hat{\mathbb{H}}^{-q}(C_{p^{n-1}}, T(\mathbb{F}_p) \wedge S^{-\beta}) \cong \mathbb{Z}/p^n$$

given by composite $\hat{\Gamma}_n \iota_{i,n}$. Recall also that we defined $k_{i,n} = \mathrm{length}_{\mathbb{Z}_p}(\ker \hat{\Gamma}_{i,n})$. All four lemmas focus on the computation of these values $k_{i,n}$. The first lemma computes $k_{1,n}$

6.1.1 Computation of $k_{1,n}$

Lemma 6.1.1. *Let $q \geq -|\alpha|$ even. Then the map*

$$\hat{\Gamma}_{1,n} : \mathbb{Z}/p^{l_{1,n}} \to \mathbb{Z}/p^n$$

is injective, and hence $k_{1,n} = 0$.

We recall by Proposition 6.0.4 that there exists a $v \in \mathbb{Z}$ such that

$$\mathrm{TR}^n_{s+\alpha}(\mathbb{F}_p;p) \cong \hat{\mathbb{H}}^{-q}(C_{p^n}, T(\mathbb{F}_p) \wedge S^{-\beta}) \cong \mathbb{Z}/p^n$$

for all $s \geq v$ even. Thus if we iterate the map σ_n sufficiently many times, we get a map

$$(\sigma_n)^t : \mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p) \to \mathbb{Z}/p^n.$$

We would like to study this map.

Proposition 6.1.2. *For $q \geq -|\alpha|$ even, the map*

$$(\sigma_n)^t_{1,n} : \mathbb{Z}/p^{l_{1,n}} \to \mathbb{Z}/p^n$$

given by the composite $(\sigma_n)^t \iota_{1,n}$ is injective.

Before proving this proposition, we demonstrate that if the proposition holds, Lemma 6.1.1 follows easily. To see this, we assume the result of Proposition 6.1.2 and consider the commutative diagram

$$\begin{tikzcd}
\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p) \arrow[r, "\sigma_n"] \arrow[d, "\hat{\Gamma}_n"] & \mathrm{TR}^n_{q+2+\alpha}(\mathbb{F}_p;p) \arrow[d, "\hat{\Gamma}_n"] \\
\hat{\mathbb{H}}^{-q}(C_{p^n}; T(\mathbb{F}_p) \wedge S^{-\beta}) \arrow[r, "\hat{\Gamma}_n(\sigma_n)"] & \hat{\mathbb{H}}^{-q-2}(C_{p^n}; T(\mathbb{F}_p) \wedge S^{-\beta})
\end{tikzcd}$$

Iterating the map σ_n as in the above proposition, we get the commutative diagram

$$\begin{tikzcd}
\mathbb{Z}/p^{l_{1,n}} \oplus \ldots \mathbb{Z}/p^{l_{n,n}} \arrow[r, "(\sigma_n)^t"] \arrow[d, "\hat{\Gamma}_{1,n}+\ldots\hat{\Gamma}_{n,n}"] & \mathbb{Z}/p^n \arrow[d, "\hat{\Gamma}_n"] \\
\mathbb{Z}/p^n \arrow[r, "(\hat{\Gamma}_n(\sigma_n))^t"] & \mathbb{Z}/p^n
\end{tikzcd}$$

Since each map $\hat{\Gamma}_n(\sigma_n)$ is an isomorphism, the bottom composite $(\hat{\Gamma}_n(\sigma_n))^t$ is an isomorphism. By Proposition 6.0.4 we can choose t large enough that the map $\hat{\Gamma}_n$ on the right is an isomorphism. The map $(\sigma_n^t)_{1,n}$ is injective by the above proposition, and hence the composite $\hat{\Gamma}_n(\sigma_n)^t_{1,n}$ is also injective. By commutativity, it then follows that the map $\hat{\Gamma}_{1,n}$ is injective. We now prove Proposition 6.1.2.

Proof. We induct on n. We have seen that for any α and $q \geq -|\alpha|$ even, the map

$$\sigma_1 : \mathrm{TR}^1_{q+\alpha}(\mathbb{F}_p;p) \to \mathrm{TR}^1_{q+2+\alpha}(\mathbb{F}_p;p)$$

is an isomorphism, and hence the composite $(\sigma_1)^m$ for any positive m is also an isomorphism. This is the base case for our induction. Suppose we have chosen an isomorphism

$$\mathbb{Z}/p^{l'_{1,n-1}} \oplus \ldots \mathbb{Z}/p^{l'_{n-1,n-1}} \xrightarrow[\sim]{\iota'_{n-1}} \mathrm{TR}^{n-1}_{q+\alpha}(\mathbb{F}_p;p)$$

We assume by induction that for any α, $q \geq -|\alpha|$ even, and appropriately chosen t', the map

$$(\sigma_{n-1})^{t'}_{1,n-1} : \mathbb{Z}/p^{l'_{1,n-1}} \to \mathbb{Z}/p^{n-1}$$

is injective. We would like to see that the map

$$(\sigma_n)^t_{1,n} : \mathbb{Z}/p^{l_{1,n}} \to \mathbb{Z}/p^n$$

is injective. We can choose t large enough such that we have a map

$$(\sigma_{n-1})^t : \mathbb{Z}/p^{l'_{1,n-1}} \oplus \ldots \mathbb{Z}/p^{l'_{n-1,n-1}} \to \mathbb{Z}/p^{n-1}$$

Then we have the commutative diagrams

$$\begin{tikzcd}
\mathbb{Z}/p^{l_{1,n}} \oplus \ldots \mathbb{Z}/p^{l_{n,n}} \arrow[r, "(\sigma_n)^t"] \arrow[d, "F^\iota"] & \mathbb{Z}/p^n \arrow[d, "F^\iota"] \\
\mathbb{Z}/p^{l'_{1,n-1}} \oplus \ldots \mathbb{Z}/p^{l'_{n-1,n-1}} \arrow[r, "(\sigma_{n-1})^t"] & \mathbb{Z}/p^{n-1}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbb{Z}/p^{l_{1,n}} \oplus \ldots \mathbb{Z}/p^{l_{n,n}} \arrow[r, "(\sigma_n)^t"] & \mathbb{Z}/p^n \\
\mathbb{Z}/p^{l'_{1,n-1}} \oplus \ldots \mathbb{Z}/p^{l'_{n-1,n-1}} \arrow[r, "(\sigma_{n-1})^t"] \arrow[u, "V^\iota"] & \mathbb{Z}/p^{n-1} \arrow[u, "V^\iota"]
\end{tikzcd}$$

The relations $FV = p$ and $VF = V(1)$ imply that the Verschiebung map on the right, $V^\iota : \mathbb{Z}/p^{n-1} \to \mathbb{Z}/p^n$ is injective. We have assumed by induction that on the bottom the map $(\sigma_{n-1})^t_{1,n-1}$ injects. Since the map V^ι is injective on the right side, it follows that the composite

$$V^\iota(\sigma_{n-1})^t_{1,n-1} : \mathbb{Z}/p^{l'_{1,n-1}} \to \mathbb{Z}/p^n$$

is injective.

By commutativity, this implies that the composite $(\sigma_n)^t V^\iota$ applied to $\mathbb{Z}/p^{l'_{1,n-1}}$ is injective. This map factors through at least one summand of $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p)$. We want the following simple lemma:

Lemma 6.1.3. *If the composite map*

$$\mathbb{Z}/p^a \xrightarrow{f} \mathbb{Z}/p^b \xrightarrow{g} \mathbb{Z}/p^c$$

is injective, then the maps f and g are both injective as well.

Proof. It is clear that the map f is injective, so $f(1) \doteq p^{b-a}$. Suppose g is not injective. Then $g(1) = \nu p^{c-b+1}$ for some $\nu \in \mathbb{Z}/p^c$. Then the composite gf is not injective, which is a contradiction. $\square$

We now prove that the map $(\sigma_n)^t V^\iota$ must factor through the first summand of $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p)$, $\mathbb{Z}/p^{l_{1,n}}$, and hence by Lemma 6.1.3, the map

$$(\sigma^n)^t_{1,n} : \mathbb{Z}/p^{l_{1,n}} \longrightarrow \mathbb{Z}/p^n$$

is injective. Suppose we have specified the isomorphisms

$$\mathbb{Z}/p^{l_{1,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n-1,n-1}} \xrightarrow{\iota_{n-1}} \mathrm{TR}^{n-1}_{q+\alpha'}(\mathbb{F}_p;p)$$

and

$$\mathbb{Z}/p^{l'_{1,n-2}} \oplus \ldots \oplus \mathbb{Z}/p^{l'_{n-2,n-2}} \xrightarrow{\iota'_{n-2}} \mathrm{TR}^{n-2}_{q+\alpha'}(\mathbb{F}_p;p).$$

The Verschiebung and restriction maps give a commutative diagram

$$\begin{tikzcd}
\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p) \arrow[r, "R"] & \mathrm{TR}^{n-1}_{q+\alpha'}(\mathbb{F}_p;p) \\
\mathrm{TR}^{n-1}_{q+\alpha}(\mathbb{F}_p;p) \arrow[r, "R"] \arrow[u, "V"] & \mathrm{TR}^{n-2}_{q+\alpha'}(\mathbb{F}_p;p) \arrow[u, "V"]
\end{tikzcd}$$

We can rewrite this diagram as

$$\begin{tikzcd}
\mathbb{Z}/p^{l_{1,n}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n,n}} \arrow[r, "R^\iota"] & \mathbb{Z}/p^{l_{1,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n-1,n-1}} \\
\mathbb{Z}/p^{l'_{1,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l'_{n-1,n-1}} \arrow[r, "R^\iota"] \arrow[u, "V^\iota"] & \mathbb{Z}/p^{l'_{1,n-2}} \oplus \ldots \oplus \mathbb{Z}/p^{l'_{n-2,n-2}} \arrow[u, "V^\iota"]
\end{tikzcd}$$

We express the map

$$V^\iota : \mathbb{Z}/p^{l'_{j,n-1}} \longrightarrow \mathbb{Z}/p^{l_{1,n}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n,n}}$$

as

$$V^\iota_{j,n-1}(a) = \sum_{1 \le i \le n} V^\iota_{i,j,n-1}(a)$$

where

$$V^\iota_{i,j,n-1} : \mathbb{Z}/p^{l'_{j,n-1}} \longrightarrow \mathbb{Z}/p^{l_{i,n}}.$$

By Lemma 6.1.3, since the composite $(\sigma_n)^t V^\iota$ is injective on the summand $\mathbb{Z}/p^{l'_{1,n-1}}$, the map $V^\iota_{i,1,n-1}$ must be injective for some $1 \le i \le n$, and for that i, the map $(\sigma_n)^t_{i,n}$ must also inject. We now show that this value of i is 1.

Suppose first that the extension forming $\mathrm{TR}^{n-1}_{q+\alpha}(\mathbb{F}_p;p)$ is non-trivial. We look at the element

$$1 \in \mathbb{Z}/p^{l'_{1,n-1}} \xrightarrow{\iota'_{1,n-1}} \mathrm{TR}^{n-1}_{q+\alpha}(\mathbb{F}_p;p).$$

Let L denote $r'_{n-1} + l'_{\tau(w'+1),n-2} - k'_{\tau(w'+1),n-2}$. By Theorem 5.2.3,

$$R^{\iota}_{i,1,n-1}(1) \doteq \begin{cases} 1 & \text{if } i = \tau(w'+1), L \leq n-1 \\ p^{L-n+1} & \text{if } \tau(w'+1), L > n-1 \\ 0 & \text{else} \end{cases}$$

Suppose the map $V^{\iota}_{j,1,n-1}$ is injective for some $j > 1$. Using Proposition 5.2.1 we can categorize the summands of $\mathrm{TR}^{n}_{q+\alpha}(\mathbb{F}_p;p)$ into several categories.

First suppose that for this j, $j-1 \leq w$. Then, by Proposition 5.2.1 and Theorem 5.2.3, we have an isomorphism

$$R^{\iota}_{\tau(j-1),j,n} : \mathbb{Z}/p^{l_{j,n}} \to \mathbb{Z}/p^{l_{\tau(j-1),n-1}}.$$

Note also that no other summand of $\mathrm{TR}^{n}_{q+\alpha}(\mathbb{F}_p;p)$ hits the summand $\mathbb{Z}/p^{l_{\tau(j-1),n-1}}$ via the restriction map. So, we conclude that the composite $R^{\iota}V^{\iota}_{i,1,n-1}$ maps

$$R^{\iota}V^{\iota}_{i,1,n-1}(1) \doteq \begin{cases} p^{l_{\tau(j-1),n-1}-L} & \text{if } i = j, L < n-1 \\ p^{l_{\tau(j-1),n-1}-n+1} & \text{if } i = j, L \geq n-1 \\ 0 & \text{else} \end{cases}$$

By commutativity,

$$V^{\iota}_{j,\tau(w'+1),n-2}(1) \doteq p^{l_{\tau(j-1),n-1}-L}.$$

However, since $k'_{\tau(w'+1),n-2} < r'_{n-1}$ there is no such map. Hence the map $V^{\iota}_{j,1,n-1}$ cannot be injective for $j > 1$ and $j-1 \leq w$. So we now assume $j - 1 > w$.

The next case we consider is $k_{\tau(j),n-1} \geq l_{\tau(j),n-1}$. Then by Proposition 5.2.1 and Theorem 5.2.3 we have an isomorphism

$$R^{\iota}_{\tau(j),j,n} : \mathbb{Z}/p^{l_{j,n}} \to \mathbb{Z}/p^{l_{\tau(j),n-1}}.$$

Applying the same argument we used in the case above, it follows that the map $V^{\iota}_{j,1,n-1}$ cannot be injective for $j > 1$ such that $k_{\tau(j),n-1} \geq l_{\tau(j),n-1}$.

The last case to consider is $j - 1 > w$ and $k_{\tau(j),n-1} < l_{\tau(j),n-1}$. By Proposition 5.2.1 either

$$\mathbb{Z}/p^{l_{j,n}} \cong \mathbb{Z}/p^{l_{\tau(j),n-1}}$$

or

$$\mathbb{Z}/p^{l_{j,n}} \cong \mathbb{Z}/p^{g_{\tau(j),n-1}+k_{\tau(z),n-1}}$$

where $\tau(z)$ is the smallest number such that $w + 1 \leq z < m$. In both cases we have

$$R^{\iota}_{i,j,n}(1) \doteq \begin{cases} -1 & \text{if } i = \tau(j) \\ p^{g_{\tau(z),n-1}-g_{\tau(j),n-1}} & \text{if } i = \tau(z) \\ 0 & \text{else} \end{cases}$$

In the situation where $\mathbb{Z}/p^{l_{j,n}} \cong \mathbb{Z}/p^{l_{\tau(j),n-1}}$, the argument used in the earlier cases applies directly, since no other summand hits the summand $\mathbb{Z}/p^{l_{\tau(j),n-1}}$ via the restriction map. So, we are left to consider when $\mathbb{Z}/p^{l_{j,n}} \cong \mathbb{Z}/p^{g_{\tau(j),n-1}+k_{\tau(z),n-1}}$.

We note from Proposition 5.2.1 that if $l_{j,n} = g_{\tau(j),n-1} + k_{\tau(z),n-1}$, then either $l_{1,n} = r_n + g_{\tau(z),n-1}$ or there exists some $1 < m < j$ such that $l_{m,n} = g_{\tau(z),n-1} + k_{\tau(z'),n-1}$ for some z'. Thus $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p)$ has a subsequence of summands of the form

$$\mathbb{Z}/p^{g_{\tau(z_1)},n-1+k_{\tau(w+1),n-1}} \oplus \mathbb{Z}/p^{g_{\tau(z_2)},n-1+k_{\tau(z_1),n-1}} \oplus \dots$$

$$\oplus \mathbb{Z}/p^{g_{\tau(z_s)},n-1+k_{z_s-1}} \oplus \mathbb{Z}/p^{g_{\tau(z),n-1}+k_{\tau(z_s),n-1}} \oplus \mathbb{Z}/p^{g_{\tau(j),n-1}+k_{\tau(z),n-1}}$$

for some s.

Suppose the map

$$V^{\iota}_{j,1,n-1} : \mathbb{Z}/p^{l'_{1,n-1}} \longrightarrow \mathbb{Z}/p^{l_{j,n}} \cong \mathbb{Z}/p^{g_{\tau(j),n-1}+k_{\tau(z),n-1}}$$

is injective. Then the composite $(R^{\iota}(V^{\iota}_{j,1,n-1}))_{i,1,n-1}$ is given by

$$(R^{\iota}(V^{\iota}_{j,1,n-1}))_{i,1,n-1}(1) \doteq \begin{cases} p^{g_{\tau(j),n-1}+k_{\tau(z),n-1}-l'_{1,n-1}} & \text{if } i = \tau(j) \\ p^{l_{\tau(z),n-1}-l'_{1,n-1}} & \text{if } i = \tau(z) \\ 0 & \text{else} \end{cases}$$

We produce a contradiction by studying the $\mathbb{Z}/p^{l_{\tau(z),n-1}}$ summand of $\mathrm{TR}^{n-1}_{q+\alpha'}(\mathbb{F}_p;p)$. Our argument for the earlier cases relied on the fact that only one summand of $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p)$ mapped to the summand of $\mathrm{TR}^{n-1}_{q+\alpha'}(\mathbb{F}_p;p)$ we wanted to study, but that is not true in this case. The summand

$$\mathbb{Z}/p^{l_{z,n}} \cong \mathbb{Z}/p^{g_{\tau(z),n-1}+k_{\tau(z_s),n-1}}$$

surjects onto the summand

$$\mathbb{Z}/p^{l_{\tau(z),n-1}} \xrightarrow{\iota_{\tau(z),n-1}} \mathrm{TR}_{q+\alpha'}(\mathbb{F}_p;p)$$

via the restriction map. So if

$$V^{\iota}_{z,1,n-1}(1) \doteq -p^{l_{\tau(z),n-1}-l'_{1,n-1}},$$

the composite $(R^{\iota}(V^{\iota}_{z,1,n-1} + V^{\iota}_{j,1,n-1}))_{\tau(z),1,n-1}(1) = 0$. Indeed

$$(R^{\iota}(V^{\iota}_{z,1,n-1} + V^{\iota}_{j,1,n-1}))_{i,1,n-1}(1) \doteq \begin{cases} -p^{l_{\tau(z_s),n-1}-l'_{1,n-1}} & \text{if } i = \tau(z_s) \\ p^{g_{\tau(j),n-1}+k_{\tau(z),n-1}-l'_{1,n-1}} & \text{if } i = \tau(j) \\ 0 & \text{else} \end{cases}$$

Again, we are in the situation where either the argument we used from the earlier cases is now applicable, or else another summand maps to the $\mathbb{Z}/p^{l_{\tau(z_s),n-1}}$ summand via the restriction. If we are in the latter case, we continue as above. Iterating this argument, either the argument we used for the earlier cases applies at some stage, or else the composite of maps

$$\phi = R^\iota(V^\iota_{z1,1,n-1} + V^\iota_{z2,1,n-1} + \ldots + V^\iota_{z,1,n-1} + V^\iota_{j,1,n-1})$$

maps as

$$\phi_{i,1,n-1}(1) \doteq \begin{cases} -p^{l_{\tau(w+1),n-1}-l'_{1,n-1}} & \text{if } i = \tau(w+1) \\ p^{g_{\tau(j),n-1}+k_{\tau(z),n-1}-l'_{1,n-1}} & \text{if } i = \tau(j) \\ 0 & \text{else} \end{cases}$$

Now, we can either apply the argument used in the earlier cases, or else there is another term mapping to the $\mathbb{Z}/p^{l_{\tau(w+1),n-1}}$-coordinate to cancel the $-p^{l_{\tau(w+1),n-1}-l'_{1,n-1}}$. By Theorem 5.2.3, the only other term mapping to that summand is the first term,

$$\mathbb{Z}/p^{l_{1,n}} \cong \mathbb{Z}/p^{r+g_{\tau(w+1),n-1}}.$$

But noting that

$$l_{\tau(w+1),n-1} - l'_{1,n-1} < r_n + g_{\tau(w+1),n-1} - l'_{1,n-1},$$

it follows that $\mathbb{Z}/p^{l'_{1,n-1}}$ cannot map to the summand $\mathbb{Z}/p^{r+g_{\tau(w+1),n-1}}$ in a way that would cancel the $-p^{l_{\tau(w+1),n-1}-l'_{1,n-1}}$ in the $\tau(w+1)$-coordinate. Thus, we apply the argument we used in the other cases to show that the diagram cannot commute, and hence $\mathbb{Z}/p^{l'_{1,n-1}}$ cannot map injectively to a summand of the form $\mathbb{Z}/p^{g_{\tau(j),n-1}+k_{\tau(z),n-1}}$.

If $w < n-1$ we need to check the case $j = n$ separately. In this case the map $V^\iota_{n,1,n-1}$ cannot inject because $p^{r'_{n-1}+g'_{\tau(w'+1),n-2}}$, the order of the summand $\mathbb{Z}/p^{l'_{1,n-1}}$, is greater than the order of the summand $\mathbb{Z}/p^{l_{n,n}}$, $p^{k_{1,n-1}}$.

We have verified that the map $V^\iota_{j,1,n-1}$ can only be injective if $j = 1$. Thus the injective map

$$(\sigma_n)^t(V^\iota_{1,n-1}) : \mathbb{Z}/p^{l'_{1,n-1}} \to \mathbb{Z}/p^n$$

can be written as $(\sigma_n)^t(V^\iota_{1,n-1})(1) \doteq ((\sigma_n)^t_{1,n})(V^\iota_{1,1,n-1})(1)$ and the map $(\sigma_n)^t_{1,n}$ is injective.

We now consider the case where the extension

$$\xrightarrow{0} \mathbb{Z}/p^{r'_{n-1}} \longrightarrow \mathbb{Z}/p^{l'_{1,n-1}} \oplus \ldots \mathbb{Z}/p^{l'_{n-1,n-1}} \xrightarrow{R^\iota} \mathbb{Z}/p^{l'_{1,n-2}} \oplus \ldots \mathbb{Z}/p^{l'_{n-2,n-2}} \longrightarrow$$

forming $\mathrm{TR}^{n-1}_{q+\alpha}(\mathbb{F}_p;p)$ is trivial. Then $l'_{1,n-1} = r'_{n-1}$ where $r'_{n-1} = \min(n-1, \frac{q+|\alpha|}{2}+1)$. We again use the diagram relating V^ι and R^ι to compute the maps

$$V^\iota_{j,1,n-1} : \mathbb{Z}/p^{r'_{n-1}} \cong \mathbb{Z}/p^{l'_{1,n-1}} \to \mathbb{Z}/p^{l_{j,n}}.$$

Suppose the map $V^\iota_{j,1,n-1}$ is injective for some $1 < j < n$. We consider the case $j = n$ separately. By Theorem 5.2.3, the map $R^\iota_{1,n-1}$ is the zero map. Thus the composite $(V^\iota R^\iota)_{1,n-1} = 0$. Hence, by commutativity the composite

$$R^\iota V^\iota_{1,n-1} : \mathbb{Z}/p^{r'_{n-1}} \cong \mathbb{Z}/p^{l'_{1,n-1}} \to \mathbb{Z}/p^{l_{1,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n-1,n-1}}$$

is zero. If $j-1 \leq w$, we have an isomorphism

$$R^{\iota}_{\tau(j-1),j,n} : \mathbb{Z}/p^{l_{j,n}} \longrightarrow \mathbb{Z}/p^{l_{\tau(j-1),n-1}}.$$

Since no other summand hits the $\mathbb{Z}/p^{l_{\tau(j-1),n-1}}$ summand under the restriction map, this implies that the composite $R^{\iota}V^{\iota}_{1,n-1}$ is non-zero. This is a contradiction, and hence the map $V^{\iota}_{j,1,n-1}$ cannot be injective for such a j. We consider next the case $w < j-1$ and $k_{\tau(j),n-1} = l_{\tau(j),n-1}$. In this case the restriction map

$$R^{\iota}_{\tau(j),j,n} : \mathbb{Z}/p^{l_{j,n}} \longrightarrow \mathbb{Z}/p^{l_{\tau(j),n-1}}$$

is again an isomorphism. As above, this gives a contradiction, so the map $V^{\iota}_{j,1,n-1}$ cannot be injective for such a value of j.

Suppose $j-1 > w$ and $k_{\tau(j),n-1} < l_{\tau(j),n-1}$. Then the composite

$$(R^{\iota}(V^{\iota}_{j,1,n-1}))_{\tau(j),1,n-1} : \mathbb{Z}/p^{r'_{n-1}} \cong \mathbb{Z}/p^{l'_{1,n-1}} \longrightarrow \mathbb{Z}/p^{l_{\tau(j),n-1}}$$

maps by $(R^{\iota}(V^{\iota}_{j,1,n-1}))_{\tau(j),1,n-1}(1) \doteq -p^{l_{j,n}-r'_{n-1}}$. No other summand maps to the $\mathbb{Z}/p^{l_{\tau(j),n-1}}$ summand under the restriction map, so we must have

$$-p^{l_{j,n}-r'_{n-1}} = 0 \in \mathbb{Z}/p^{l_{\tau(j),n-1}}.$$

Hence $l_{j,n} - r'_{n-1} \geq l_{\tau(j),n-1}$. However,

$$l_{j,n} - r'_{n-1} = l_{\tau(j),n-1} - k_{\tau(j),n-1} + k_{\tau(z),n-1} - r'_{n-1}.$$

Since $k_{\tau(z),n-1} < r_n$ it follows that $k_{\tau(z),n-1} < r'_{n-1}$, and hence the left side of the equality above is less than $l_{\tau(j),n-1}$. This is a contradiction, and hence the map $V^{\iota}_{j,1,n-1}$ is not injective.

As above, if $w < n-1$ we need to check the case $j = n$ separately. Again, the map $V^{\iota}_{n,1,n-1}$ cannot inject because $p^{r'_{n-1}+g'_{\tau(w'+1),n-2}}$, the order of the summand $\mathbb{Z}/p^{l'_{1,n-1}}$, is greater than the order of the summand $\mathbb{Z}/p^{l_{n,n}}$, $p^{k_{1,n-1}}$.

We have verified that the map $V^{\iota}_{j,1,n-1}$ can only be injective if $j = 1$, and hence the map $(\sigma_n)^t_{1,n}$ is injective.

This completes the proof. □

6.1.2 Computation of $k_{m,n}$ for $k_{\tau(m),n-1} < r_n$

Lemma 6.1.4. *Let $r_n = \min(n, \frac{q+|\alpha|}{2}+1)$. Let $2 \leq m \leq n-1$. If $k_{\tau(m),n-1} < r_n$ then $k_{m,n} \geq l_{m,n} - l_{1,n}$.*

Proof. From Proposition 5.2.1, if $k_{\tau(m),n-1} < r$, the restriction map

$$R^{\iota}_{\tau(m),m,n} : \mathbb{Z}/p^{l_{m,n}} \longrightarrow \mathbb{Z}/p^{l_{\tau(m),n-1}}$$

is surjective. We would like to compute $k_{m,n}$. It follows by the definition of $\tau(w+1)$ that $g_{\tau(m),n-1} \leq g_{\tau(w+1),n-1}$. By Proposition 5.2.1 $l_{m,n} = l_{\tau(m),n-1}$ or $l_{m,n} = g_{\tau(m),n-1}+$

$k_{\tau(z),n-1}$ for some z with $k_{\tau(z),n-1} \leq k_{\tau(m),n-1}$. In either case

$$l_{1,n} = \min(n, r_m + g_{\tau(w+1),n-1}) \geq l_{m,n}$$

since $k_{\tau(z),n-1} < k_{\tau(m),n-1} < r_n$ implies that

$$g_{\tau(m),n-1} + k_{\tau(z),n-1} < r_n + g_{\tau(w+1),n-1}.$$

It follows that the nonnegative integer $k_{m,n} \geq l_{m,n} - l_{1,n}$.

□

6.1.3 Computation of $k_{m+1,n}$ for $k_{\tau(m),n-1} \geq r_n$

We now look at the case where $k_{\tau(m),n-1} \geq r_n$.

Lemma 6.1.5. *If $k_{\tau(m),n-1} \geq r_n$ and $g_{\tau(m),n-1} \leq g_{\tau(w+1),n-1}$, then $k_{m+1,n} \geq l_{m+1,n} - l_{1,n}$.*

Proof. It follows from the commutative diagram relating the maps R and σ that $k_{m+1,n} \geq k_{\tau(m),n-1} - r_n$. The hypothesis $g_{\tau(m),n-1} \leq g_{\tau(w+1),n-1}$ implies that

$$k_{\tau(m),n-1} - r_n \geq l_{\tau(m),n-1} - r_n - g_{\tau(w+1),n-1}.$$

Since $l_{1,n} = r_n + g_{\tau(w+1),n-1}$, the inequality $k_{m+1,n} \geq l_{m+1,n} - l_{1,n}$ follows.

□

Lemma 6.1.6. *If $k_{\tau(m),n-1} \geq r_n$ and $g_{\tau(m),n-1} > g_{\tau(w+1),n-1}$, then $k_{m+1,n} = k_{\tau(m),n-1} - r_n$.*

Proof. We compute $k_{m+1,n}$ inductively using the commutative diagrams

$$\begin{tikzcd}
\mathbb{Z}/p^{l_{1,n}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n,n}} \arrow[r, "(\sigma_n)^t"] \arrow[d, "F^\iota"] & \mathbb{Z}/p^n \arrow[d, "F^\iota"] \\
\mathbb{Z}/p^{l'_{1,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l'_{n-1,n-1}} \arrow[r, "(\sigma_{n-1})^t"] & \mathbb{Z}/p^{n-1}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{Z}/p^{l_{1,n}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n,n}} \arrow[r, "(\sigma_n)^t"] & \mathbb{Z}/p^n \\
\mathbb{Z}/p^{l'_{1,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l'_{n-1,n-1}} \arrow[r, "(\sigma_{n-1})^t"] \arrow[u, "V^\iota"] & \mathbb{Z}/p^{n-1} \arrow[u, "V^\iota"]
\end{tikzcd}$$

We induct on the value of n and within each n we induct on the value of $g_{\tau(m),n-1} - g_{\tau(w+1),n-1}$. Thus, we need as base cases, the cases where

$$g_{\tau(m),n-1} - g_{\tau(w+1),n-1} = 1.$$

for all n. We also need to verify the lemma for the smallest n such that the hypotheses of the lemma can be met. Recall that we have chosen an isomorphism

$$\mathbb{Z}/p^{l_{1,1}} \xrightarrow{\iota_1} \mathrm{TR}^1_{q+\alpha}(\mathbb{F}_p;p)$$

where $l_{1,1} = 0$ or 1. The conditions of the lemma can only be met if $r_2 = 0$, $k_{1,1} = 0$, and $l_{1,1} = 1$. In this case $g_{1,1} - g_{\tau(w+1),1} = 1$, so we handle this case with the rest of the base cases, which we now address.

Suppose $k_{\tau(m),n-1} \geq r$ and $g_{\tau(m),n-1} - g_{\tau(w+1),n-1} = 1$. We want to prove that $k_{m+1,n} = k_{\tau(m),n-1} - r_n$. We consider the commutative diagram

$$\begin{tikzcd}
\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p) \arrow[r, "\sigma_n"] \arrow[d, "R"] & \mathrm{TR}^n_{q+2+\alpha}(\mathbb{F}_p;p) \arrow[d, "R"] \\
\mathrm{TR}^{n-1}_{q+\alpha'}(\mathbb{F}_p;p) \arrow[r, "p\sigma_{n-1}"] & \mathrm{TR}^{n-1}_{q+2+\alpha'}(\mathbb{F}_p;p)
\end{tikzcd}$$

Using the chosen isomorphisms, we rewrite this diagram as

$$\begin{tikzcd}
\mathbb{Z}/p^{l_{1,n}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n,n}} \arrow[r, "\sigma_n"] \arrow[d, "R^\iota"] & \mathbb{Z}/p^{d_{1,n}} \oplus \ldots \oplus \mathbb{Z}/p^{d_{n,n}} \arrow[d, "R^\iota"] \\
\mathbb{Z}/p^{l_{1,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n-1,n-1}} \arrow[r, "p\sigma_{n-1}"] & \mathbb{Z}/p^{d_{1,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{d_{n-1,n-1}}
\end{tikzcd}$$

where to minimize confusion we use $d_{i,n}$ to denote the orders of the summands of $\mathrm{TR}^n_{q+2+\alpha}(\mathbb{F}_p;p)$ in the same way we used $l_{i,n}$ for $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p)$.

We compute the map R^ι in Theorem 5.2.3. It follows from this explicit computation, and the commutativity of the diagram above, that $k_{m+1,n} = k_{\tau(m),n-1} - r_n$.

Now, assume that the lemma holds for all TR^{n-1}-groups and the TR^n-groups and values of v where

$$g_{\tau(v),n-1} - g_{\tau(w+1),n-1} \leq j - 1.$$

We demonstrate that it holds for TR^n-groups and values of m where

$$g_{\tau(m),n-1} - g_{\tau(w+1),n-1} = j.$$

We do this using the commutative diagram relating maps F and σ_n. We need to first understand the relationship between the structure of $\mathrm{TR}^{n-1}_{q+\alpha}(\mathbb{F}_p;p)$ and the structure of $\mathrm{TR}^{n-2}_{q+\alpha}(\mathbb{F}_p;p)$. We summarize this in the following proposition.

Proposition 6.1.7. *Suppose we have chosen isomorphisms*

$$\mathbb{Z}/p^{l'_{1,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l'_{n-1,n-1}} \xrightarrow{\iota'_{n-1}} \mathrm{TR}^{n-1}_{q+\alpha}(\mathbb{F}_p;p)$$

$$\mathbb{Z}/p^{l''_{1,n-2}} \oplus \ldots \oplus \mathbb{Z}/p^{l''_{n-2,n-2}} \xrightarrow{\iota''_{n-2}} \mathrm{TR}^{n-2}_{q+\alpha}(\mathbb{F}_p;p)$$

Then there is a permutation $\gamma_{n-2} \in \Sigma_{n-2}$ such that $l''_{\gamma_{n-2}(i),n-2} = l'_{i,n-1}$ or $l''_{\gamma_{n-2}(i),n-2} = l'_{i,n-1} - 1$ for $1 \leq i \leq n-2$. Further $k''_{\gamma_{n-2}(i),n-2} = \min(k'_{i,n-1}, l''_{\gamma_{n-2}(i),n-2})$.

Proof. By our induction hypothesis, Lemma 6.1.6 holds for all TR^{n-1}-groups. Thus the recursive definition in the main theorem has been proven through TR^{n-1}. So we can assume that the lengths $l'_{i,n-1}$ and $l''_{i,n-2}$ above are given by this recursive definition. We then show inductively that the claim of the proposition holds. Recall that

$$\mathrm{TR}^1_{q+\alpha^{(n-2)}}(\mathbb{F}_p;p) \cong \begin{cases} \mathbb{Z}/p & \text{if } q \geq -|\alpha^{(n-2)}| \\ 0 & \text{else} \end{cases}$$

We also computed earlier that $k''_{1,1} = 0$. Applying the recusive definition we find that if $-|\alpha^{(n-2)}| \geq -|\alpha^{(n-1)}|$:

$$\mathrm{TR}^2_{q+\alpha^{(n-2)}}(\mathbb{F}_p;p) \cong \begin{cases} \mathbb{Z}/p^2 \oplus 0 & \text{if } q \geq -|\alpha^{(n-2)}| \\ 0 \oplus \mathbb{Z}/p & \text{if } -|\alpha^{(n-2)}| > q \geq -|\alpha^{(n-1)}| \\ 0 \oplus 0 & \text{else} \end{cases}$$

If $-|\alpha^{(n-2)}| < -|\alpha^{(n-1)}|$:

$$\mathrm{TR}^2_{q+\alpha^{(n-2)}}(\mathbb{F}_p;p) \cong \begin{cases} \mathbb{Z}/p^2 \oplus 0 & \text{if } q \geq -|\alpha^{(n-2)}| + 2 \\ \mathbb{Z}/p \oplus 0 & \text{if } q = -|\alpha^{(n-2)}| \\ 0 \oplus 0 & \text{else} \end{cases}$$

In the case where $-|\alpha^{(n-1)}| \leq q < -|\alpha^{(n-2)}| - 2$, $k'_{1,2} = 1$. Otherwise $k'_{1,2} = k'_{2,2} = 0$.

So we have computed $l'_{1,2}, k'_{1,2}, l''_{1,1}$, and $k''_{1,1}$, and observed that $l''_{1,1} = l'_{1,2}$ or $l''_{1,1} = l'_{1,2} - 1$. Further $k''_{1,1} = k'_{1,2}$. So the proposition holds in this case when γ_1 is the identity permutation. We now assume that the conclusion of the proposition holds for

$$\mathrm{TR}^{j-1}_{q+\alpha^{(n-j+1)}}(\mathbb{F}_p;p) \cong \mathbb{Z}/p^{l'_{1,j-1}} \oplus \ldots \mathbb{Z}/p^{l'_{j-1,j-1}}$$

and

$$\mathrm{TR}^{j-2}_{q+\alpha^{(n-j+1)}}(\mathbb{F}_p;p) \cong \mathbb{Z}/p^{l''_{1,j-2}} \oplus \ldots \mathbb{Z}/p^{l''_{j-2,j-2}}$$

and show that there exists $\gamma_{j-1} \in \Sigma_{j-1}$ such that $l''_{\gamma_{j-1}(i),j-1} = l'_{i,j}$ or $l''_{\gamma_{j-1}(i),j-1} = l'_{i,j} - 1$ for $1 \leq i \leq j-1$, and $k''_{\gamma_{j-1}(i),j-1} = \min(k'_{i,j}, l''_{\gamma_{j-1}(i),j-1})$.

Let $r'_j = \min(j, \frac{q+|\alpha^{(n-j)}|}{2} + 1)$ and $r''_{j-1} = \min(j-1, \frac{q+|\alpha^{(n-j)}|}{2} + 1)$. Observe that $r''_{j-1} = r'_j$ or $r''_{j-1} = j-1$ and $r'_j = j$. In the latter case, the value $w' = \#\{1 \leq i \leq j-1 | k'_{i,j-1} \geq r_j\} = 0$ and $w'' = \#\{1 \leq i \leq j-2 | k''_{i,j-2} \geq r_{j-1}\} = 0$ since the values $k'_{i,j-1}$ and $k''_{i,j-2}$ are bounded by $j-1$ and $j-2$ respectively. We continue to analyze this case and then return to the $r''_{j-1} = r'_j$ case. By the recursive definition

$$l'_{1,j} = \min(j, r'_j + g'_{\tau'(1),j-1})$$

and

$$l''_{1,j-1} = \min(j, r''_{j-1} + g''_{\tau''(1),j-2}).$$

By hypothesis, $g''_{i,j-2} = g'_{i,j-1}$ or $g''_{i,j-2} = g'_{i,j-1} - 1$. In either case it follows that $l''_{1,j-1} = l'_{1,j}$ or $l''_{1,j-1} = l'_{1,j} - 1$, and in both cases $k''_{1,j-1} = k'_{1,j} = 0$. So we let $\gamma_{j-1}(1) = 1$. For $2 \leq m \leq j-1$, let $\tau'(z')$ be the smallest number such that $1 \leq z' < m$. Then

$$l'_{m,j} = \begin{cases} l'_{\tau'(m),j-1} & \text{if for some } 1 \leq v < m, \tau'(v) < \tau'(m) \\ g_{\tau'(m),j-1} + k_{\tau'(z'),j-1} & \text{else} \end{cases}$$

We choose $\gamma_{j-1}(m) = \gamma_{j-2}(\tau'(m))$. It follows that $l''_{\gamma_{j-1}(m),j-1} = l'_{m,j}$ or $l''_{\gamma_{j-1}(m),j-1} = l'_{m,j} - 1$ and $k''_{\gamma_{j-1}(m),j-1} = \min(k'_{m,j}, l''_{\gamma_{j-1}(m),j-1})$. Finally $l''_{j-1,j-1} = k''_{j-1,j-1} = l'_{j,j} = k'_{j,j} = 0$.

We now look at the case $r''_{j-1} = r'_j$. The arguments for $m = 1$ and $w' + 2 \leq m \leq j-1$ are the same as above. So we want to look at $2 \leq m \leq w'+1$. Since $l'_{m,j} = l'_{\tau'(m-1),j-1}$, we choose $\gamma_{j-1}(m) = \gamma_{j-2}(\tau'(m-1))$. Then $l''_{m,j-1} = l'_{m,j}$ or $l''_{m,j-1} = l'_{m,j} - 1$. We now look the values $k''_{\tau_{j-1}(m),j-1}$ and $k'_{m,j}$. By the recursive definition

$$k'_{m,j} = \begin{cases} \min(l'_{m,j}, k'_{\tau'(m-1),j-1} - r'_j) & \text{if } g'_{\tau'(m-1),j-1} > g'_{\tau'(w'+1),j-1} \\ l'_{m,j} & \text{else} \end{cases}$$

We are either in the case where $2 \leq \gamma_{j-1}(m) \leq w''+1$ or we are in the case where $w''+2 \leq \gamma_{j-1}(m) \leq j-1$. Since $r'_j = r''_{j-1}$ we only would arrive in the later case if $k'_{\tau'(m-1),j-1} = k''_{\tau''(\gamma_{j-1}(m)),j-2} + 1$. By our induction hypothesis, this only occurs if

$$g'_{\tau(m-1),j-1} = g''_{\tau''(\gamma_{j-1}(m)),j-2} = 0.$$

In this case $k'_{m,j} = l'_{m,j}$ and $k''_{\gamma_{j-1}(m),j-1} = l''_{\gamma_{j-1}(m),j-1}$. In the case $2 \leq \gamma_{j-1}(m) \leq w''+1$, we have

$$k''_{\gamma_{j-1}(m),j-1} = \min(l''_{\gamma_{j-1}(m),j-1}, k''_{\tau''(\gamma_{j-2}(m)-1),j-2} - r''_{j-1})$$

if $g''_{\tau''(\gamma_{j-2}(m)-1),j-2} > g''_{\tau''(w''+1),j-2}$. Otherwise, $k''_{\gamma_{j-1}(m),j-1} = l''_{\gamma_{j-1}(m),j-1}$. Thus, if $k'_{m,j} = l'_{m,j}$ then $k''_{\gamma_{j-1}(m),j-1} = l''_{\gamma_{j-1}(m),j-1}$. Otherwise, $k'_{m,j} = k''_{\gamma_{j-1}(m),j-1}$ since $r'_j = r''_{j-1}$ and τ' and τ'' are chosen in such a way that $k'_{\tau'(m-1),j-1} = k''_{\tau''(\gamma_{j-2}(m)-1),j-2}$. This completes the proof. $\square$

Throughout the rest of the discussion, we drop the permutation γ (from Proposition 6.1.7) from our notation, but we should keep in mind that the ordering of the summands of $\mathrm{TR}^{n-1}_{q+\alpha}(\mathbb{F}_p;p)$ and $\mathrm{TR}^{n-2}_{q+\alpha}(\mathbb{F}_p;p)$ may not correspond exactly.

Now suppose we have $k_{\tau(m),n-1} \geq r_n$ and $g_{\tau(m),n-1} - g_{\tau(w+1),n-1} = j$. We are interested in the map

$$((\sigma_n)^t)_{m+1,n} : \mathbb{Z}/p^{l_{\tau(m),n}} \longrightarrow \mathbb{Z}/p^n.$$

We prove below that $((\sigma_n)^t)_{\tau(m),n}(1) \doteq p^{n-g_{\tau(m),n-1}-r_n}$, and hence $k_{m,n} = k_{\tau(m),n-1} - r_n$.

If $g_{\tau(m),n-1} - g_{\tau(w+1),n-1} \geq 2$ then $g'_{\tau(m),n-2} - g'_{\tau(w'+1),n-2} \geq 1$. So we can proceed inductively.

By the induction hypothesis

$$(\sigma_{n-1})^t_{m+1,n-1}(1) \doteq p^{n-1-g'_{\tau(m),n-2}-r'_{n-1}}.$$

We argue below that it follows from this that

$$(\sigma_n)^t_{m+1,n}(1) \doteq p^{n-g_{\tau(m),n-1}-r_n}.$$

We analyze the map $(\sigma_n)^t_{m+1,n}$ by carefully studying the maps F^ι and V^ι. Recall the commutative diagram

$$\begin{tikzcd}
\mathbb{Z}/p^{l_{1,n}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n,n}} \arrow[r, "(\sigma_n)^t"] & \mathbb{Z}/p^n \\
\mathbb{Z}/p^{l'_{1,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l'_{n-1,n-1}} \arrow[r, "(\sigma_{n-1})^t"] \arrow[u, "V^\iota"] & \mathbb{Z}/p^{n-1} \arrow[u, "V^\iota"]
\end{tikzcd}$$

By induction, the composite

$$(V^\iota(\sigma_{n-1})^t)_{m+1,n-1}(1) \doteq p^{n-g'_{\tau(m),n-2}-r'_{n-1}}.$$

Hence

$$((\sigma_n)^t V^\iota)_{m+1,n-1}(1) \doteq p^{n-g'_{\tau(m),n-2}-r'_{n-1}}.$$

Recall that $r'_{n-1} = \min(n-1, \frac{q+|\alpha|}{2}+1)$ and $r_n = \min(n, \frac{q+|\alpha|}{2}+1)$. But r'_{n-1} cannot equal $n-1$ since $k'_{\tau(m),n-2} \geq r'_{n-1}$ and the hypothesis $g'_{\tau(m),n-2} - g'_{\tau(w'+1),n-2} \geq 1$ implies that $k'_{\tau(m),n-2} < l'_{\tau(m),n-2}$. So, $r'_{n-1} = r_n$.

Also note by Proposition 5.2.1 that $l'_{\tau(m),n-2} = l'_{m+1,n-1}$ since $k'_{\tau(m),n-2} \geq r'_{n-1}$. Finally, note by Proposition 6.1.7 that $l'_{\tau(m),n-2} = l_{\tau(m),n-1}$ or $l'_{\tau(m),n-2} = l_{\tau(m),n-1} - 1$ and $k'_{\tau(m),n-2} = k_{\tau(m),n-1}$ since by hypothesis $k_{\tau(m),n-1} < l_{\tau(m),n-1}$. This implies that $l'_{m+1,n-1} = l_{m+1,n}$ or $l'_{m+1,n-1} = l_{m+1,n} - 1$. So we can rewrite the above composite as

$$((\sigma_n)^t V^\iota)_{m+1,n-1}(1) \doteq p^{n-l_{m+1,n}+k_{\tau(m),n-1}-r_n}$$

if $l'_{m+1,n-1} = l_{m+1,n}$, and

$$((\sigma_n)^t V^\iota)_{m+1,n-1}(1) \doteq p^{n-l_{m+1,n}+k_{\tau(m),n-1}-r_n+1}$$

if $l'_{m+1,n-1} = l_{m+1,n} - 1$.

We now study the map

$$V^\iota_{m+1,n-1} : \mathbb{Z}/p^{l'_{m+1,n-1}} \longrightarrow \mathbb{Z}/p^{l_{1,n}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n,n}}.$$

Assume first that $l'_{m+1,n-1} = l_{m+1,n}$. Suppose

$$V^\iota_{i,m+1,n-1}(1) \doteq p^{a_i} \in \mathbb{Z}/p^{l_{i,n}}.$$

We analyze for which i the composites $\sigma_{n-1}(V^\iota_{i,m+1,n-1})$ could contribute to the term

$p^{n-l_{m+1,n}+k_{\tau(m),n-1}-r_n}$. In order for the composite

$$\sigma_{n-1}(V^{\iota}_{i,m+1,n-1})(1) \doteq p^{n-l_{m+1,n}+k_{\tau(m),n-1}-r_n},$$

we must have

$$a_i + n - l_{i,n} \leq n - l_{m+1,n} + k_{\tau(m),n-1} - r_n.$$

So a_i must satisfy the inequality

$$a_i \leq l_{i,n} - r_n - g_{\tau(m),n-1}$$

First consider $i > w+1$, and hence $k_{\tau(i),n-1} < r_n$. By Proposition 5.2.1 either $l_{i,n} = l_{\tau(i),n-1}$ or $l_{i,n} = g_{\tau(i),n-1} + k_{\tau(z),n-1}$. In either case, since

$$g_{\tau(m),n-1} > g_{\tau(w+1),n-1} \geq g_{\tau(i),n-1},$$

the inequality above says that $a_i < 0$. This is a contradiction, and hence for these values of i, the composite $\sigma_{n-1}(V^{\iota}_{i,m+1,n-1})$ cannot contribute to $p^{n-l_{m+1,n}+k_{\tau(m),n-1}-r_n}$. We now look at $2 \leq i \leq w+1$ such that $g_{\tau(i-1),n-1} \leq g_{\tau(w+1),n-1}$. By the argument above, the composite $\sigma_{n-1}(V^{\iota}_{i,m+1,n-1})$ cannot contribute to $p^{n-l_{m+1,n}+k_{\tau(m),n-1}-r_n}$. We are left with $2 \leq i \leq w+1$ such that $g_{\tau(i-1),n-1} > g_{\tau(w+1),n-1}$. These are exactly the i satisfying the hypotheses of the lemma. We now look at $2 \leq i \leq w+1$, satisfying the hypotheses of the lemma. By the induction hypothesis, the lemma already holds for those i where

$$g_{\tau(m),n-1} > g_{\tau(i-1),n-1}.$$

Hence

$$(\sigma_n)^t_{i,n}(1) = p^{n-l_{i,n}+k_{\tau(i-1),n-1}-r_n}.$$

It follows that

$$a_i \leq (l_{i,n} - k_{\tau(i-1),n-1}) - (l_{m+1,n} - k_{\tau(m),n-1}) < 0.$$

This is a contradiction, so no such value of i can contribute to $p^{n-l_{m+1,n}+k_{\tau(m),n-1}-r_n}$. So, the only values of i for which the composite $\sigma_{n-1}(V^{\iota}_{i,m+1,n-1})$ can contribute to $p^{n-l_{m+1,n}+k_{\tau(m),n-1}-r_n}$ are those $2 \leq i \leq w+1$ such that $g_{\tau(i-1),n-1} \geq g_{\tau(m),n-1} > g_{\tau(w+1),n-1}$.

We now go through the values of j from 2 to $m+1$, which satisfy the hypotheses of the lemma, and show that $k_{j,n} = k_{\tau(j-1),n-1} - r_n$. Suppose that for these values of j the summands are ordered such that

$$g_{\tau(1),n-1} \geq g_{\tau(2),n-1} \geq \cdots g_{\tau(j),n-1} \cdots \geq g_{\tau(w),n-1}$$

where if $g_{\tau(i),n-1} = g_{\tau(i+1),n-1}$, then $l_{\tau(i),n-1} < l_{\tau(i+1),n-1}$. If the summands are not ordered in this way, we temporarily permute them to meet these conditions. Let $j = 2$. Suppose that

$$V^{\iota}_{i,2,n-1} \doteq p^{a_i}$$

for $2 \leq i \leq w+1$ such that $g_{\tau(i-1),n-1} > g_{\tau(w+1),n-1}$. Recall the commutative diagram relating the maps V and R

$$\begin{array}{ccc} \mathbb{Z}/p^{l_{1,n}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n,n}} & \xrightarrow{R^\iota} & \mathbb{Z}/p^{l_{1,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n-1,n-1}} \\ \uparrow{\scriptstyle V^\iota} & & \uparrow{\scriptstyle V^\iota} \\ \mathbb{Z}/p^{l'_{1,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l'_{n-1,n-1}} & \xrightarrow{R^\iota} & \mathbb{Z}/p^{l'_{1,n-2}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n-2,n-2}} \end{array}$$

Since the restriction maps $R^\iota_{\tau(i-1),i,n-1}$ and $R^\iota_{\tau(i-1),i,n}$ are identity maps, this implies that the Verschiebung map on the left has

$$V^\iota_{\tau(i-1),\tau(1),n-2} \doteq p^{a_i}.$$

We look at this map in the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}/p^{l_{1,n-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{n-1,n-1}} & \xrightarrow{(\sigma_{n-1})^t} & \mathbb{Z}/p^{n-1} \\ \uparrow{\scriptstyle V^\iota} & & \uparrow{\scriptstyle V^\iota} \\ \mathbb{Z}/p^{l'_{1,n-2}} \oplus \ldots \oplus \mathbb{Z}/p^{l'_{n-2,n-2}} & \xrightarrow{(\sigma_{n-2})^t} & \mathbb{Z}/p^{n-2} \end{array}$$

The composite

$$(\sigma_{n-1})^t (V^\iota_{\tau(1),n-2})(1) = \sum_i p^{n-1-g_{\tau(i-1),n-1}+a_i}.$$

By commutativity, this also equals

$$V^\iota (\sigma_{n-2})^t_{\tau(1),n-2}(1) = p^{n-1-g_{\tau(1),n-1}}.$$

For all $i > 2$, $g_{\tau(i-1),n-1} \leq g_{\tau(1),n-1}$, and in the cases where they are equal $l_{\tau(i-1),n-1} > l_{\tau(1),n-1}$ and hence $a_i \geq 1$. Thus it follows that in order for the diagram to commute, we must have $a_1 = 0$.

We return to the composite

$$(\sigma_n)^t (V^\iota_{2,n-1})(1) = \sum_i p^{n-g_{i,n}+a_i}.$$

By commutativity this equals

$$V^\iota (\sigma_{n-1})^t_{2,n-1}(1) = p^{n-1-l_{2,n}+k_{\tau(1),n}}.$$

For the summands above to contribute to the term $p^{n-1-l_{2,n}+k_{\tau(1),n}}$, we must have

$$a_i \leq g_{i,n} - (l_{2,n} - k_{\tau(1),n-1}) - r_n.$$

However, it follows from the diagram relating σ and R that $k_{i,n} \geq k_{\tau(i-1),n-1} - r_n$.

Thus this inequality implies that

$$a_i \leq g_{\tau(i-1),n} - g_{\tau(1),n}.$$

The right side of this inequality is less than or equal to zero if $i \neq 2$. Further, when it is zero, $l_{\tau(i-1),n-1} > l_{\tau(1),n-1}$ and hence $a_i \geq 1$. This is a contraction, so no such a_i exist. Thus we conclude that $k_{2,n} = k_{\tau(1),n-1} - r$. We then suppose that we have shown that $k_{j,n} = k_{\tau(j-1),n-1} - r$, and prove that $k_{v,n} = k_{\tau(v-1),n-1} - r_n$ for the next largest $j < v \leq w+1$ satisfying the hypotheses of the lemma. Again assume that

$$V^{\iota}_{i,v,n-1} \doteq p^{b_i}$$

for some values b_i. The commutative diagram relating V and R shows that $b_v = 0$. For $i > v$ the same argument as above shows that these terms cannot contribute to the composite

$$V^{\iota}(\sigma_{n-1})^t_{v,n-1}(1) = p^{n-1-l_{v,n}+k_{\tau(v-1),n}}.$$

We now restrict our attention to the terms where $i \leq v$. The diagram relating V and σ gives us the relation

$$\sum_{i=2}^{i=v} p^{n-g_{i,n}+b_i} \doteq p^{n-l_{v,n}+k_{\tau(v-1),n-1}-r_n}$$

where i only takes the values such that $g_{\tau(i-1),n-1} > g_{\tau(w+1),n-1}$. We have assumed that for such $i < v$, $k_{i,n} = k_{\tau(i-1),n-1} - r_n$, so we can rewrite the above equality as

$$\sum_{i=2}^{i=v-1} p^{n-l_{i,n}+k_{\tau(i-1),n-1}-r_n+b_i} + p^{n-g_{v,n}} \doteq p^{n-l_{v,n}+k_{\tau(v-1),n-1}-r_n}.$$

The diagram relating R and V in conjunction with the diagram relating V and σ gives the relation

$$\sum_{i=2}^{i=v} p^{n-l_{i,n}+k_{\tau(i-1),n-1}+a_i} \doteq p^{n-l_{v,n}+k_{\tau(v-1),n-1}}.$$

Together these two equalities imply that $k_{v,n} = k_{\tau(v-1),n-1} - r_n$. Thus by induction $k_{m+1,n} = k_{\tau(m),n-1} - r_n$. In the case where $l_{m+1,n} = l'_{m+1,n-1} + 1$, we find that $V^{\iota}_{m+1,m+1,n-1}(1) \doteq p$ and $F^{\iota}_{m+1,m+1,n-1}(1) \doteq 1$. Then we proceed with the same argument as above. $\square$

Chapter 7

Computation of $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p)$

7.1 Proofs of main theorems

We now give a proof of the main theorem.

Proof. The computation of $\mathrm{TR}^1_{q+\alpha^{(n-1)}}(\mathbb{F}_p;p)$ is discussed in section 5. Suppose

$$\mathrm{TR}^{j-1}_{q+\alpha^{(n-j+1)}}(\mathbb{F}_p;p) \cong \mathbb{Z}/p^{l_{1,j-1}} \oplus \mathbb{Z}/p^{l_{2,j-1}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{j-1,j-1}}$$

and $k_{i,j-1} = \mathrm{length}_{\mathbb{Z}_p}(\ker\hat{\Gamma}_{i,j-1})$. The lengths $l_{i,j}$ follow directly from Proposition 5.2.1 and we have

$$\mathrm{TR}^{j}_{q+\alpha^{(n-j)}}(\mathbb{F}_p;p) \cong \mathbb{Z}/p^{l_{1,j}} \oplus \mathbb{Z}/p^{l_{2,j}} \oplus \ldots \oplus \mathbb{Z}/p^{l_{j,j}}.$$

In order to proceed inductively, we need to compute $k_{i,j}$ for $1 \leq i \leq j$.

Lemma 6.1.1 determines the value $k_{1,j} = 0$. We now determine values $k_{m,j}$ for $1 < m \leq j$. We divide our analysis into three cases:

$$k_{\tau(m),j-1} < r_j,$$

$$k_{\tau(m),j-1} \geq r_j \text{ and } g_{\tau(m),j-1} \leq g_{\tau(w+1),j-1},$$

$$k_{\tau(m),j-1} \geq r_j \text{ and } g_{\tau(m),j-1} > g_{\tau(w+1),j-1}.$$

Lemma 6.1.4 determines that in the first case, $k_{m,j} \geq l_{m,j} - l_{1,j}$. Since we determined above that $k_{1,j} = 0$, this implies that $g_{1,j} \geq g_{m,j}$.Similarly, Lemma 6.1.5 determines that in the second case $k_{m+1,j} \geq l_{m+1,j} - l_{1,j}$ and consequently $g_{1,j} \geq g_{m+1,j}$.

In the first case, in the computation of $\mathrm{TR}^{j+1}_{q+\alpha^{(n-j-1)}}(\mathbb{F}_p;p)$, $g_{1,j} \geq g_{m,j}$. Hence we do not need to know the exact value of $k_{m,j}$ to compute the group $\mathrm{TR}^{j+1}_{q+\alpha^{(n-j-1)}}(\mathbb{F}_p;p)$. In particular, although Proposition 5.2.1 assumes by hypothesis that the summands are ordered such that the values of $k_{i,j}$ are non-decreasing from left to right, if we know that $k_{1,j} \leq k_{m,j}$ and $g_{1,j} \geq g_{m,j}$, we don't need to know the relation of $k_{m,j}$ to the other $k_{i,j}$, $2 \leq i \leq j$.

Further, the inequality $g_{1,j} \geq g_{m,j}$ implies that we are again in case 1 or 2 when we compute the value $k_{\tau^{-1}(m),j+1}$. Thus, knowing that $k_{m,j} \geq l_{m,j} - l_{1,j}$ is all the information we need to proceed inductively. Although we don't necessarily need to specify a value for $k_{m,j}$, for simplicity we set $k_{m,j} = l_{m,j}$. In the second case we apply the same argument and set $k_{m+1,j} = l_{m+1,j}$.

In case three above, Lemma 6.1.6 determines that $k_{m+1,j} = k_{\tau(m),j-1} - r_j$. This completes the proof of the theorem.

□

We now compute the orders of the groups $\mathrm{TR}^n_{q-1+\alpha}(\mathbb{F}_p;p)$ when q is even. These are the groups indexed by the odd-dimensional representations in $RO(S^1)$.

We denote by $|\mathrm{TR}^n_{q-1+\alpha}(\mathbb{F}_p;p)|$ the order of the group $\mathrm{TR}^n_{q-1+\alpha}(\mathbb{F}_p;p)$ With the notation introduced in Theorem 1.0.1, we have the following proposition

Proposition 7.1.1. *The group* $\mathrm{TR}^1_{q-1+\alpha^{(n-1)}}(\mathbb{F}_p;p) = 0$. *The order of the group* $\mathrm{TR}^n_{q-1+\alpha}(\mathbb{F}_p;p)$ *is given by*

$$|\mathrm{TR}^n_{q-1+\alpha}(\mathbb{F}_p;p)| = p^S|\mathrm{TR}^{n-1}_{q-1+\alpha'}(\mathbb{F}_p;p)|$$

where S is

$$S = \begin{cases} \max(0, r_n - 1) & \text{if } r_n + g_{\tau(w+1),n-1} \leq n \\ r_n + g_{\tau(w+1),n-1} - n & \text{else} \end{cases}$$

for $r_n = \min(n, \frac{q+|\alpha|}{2} + 1)$.

Proof. The first claim follows from the discussion of TR^1-groups in Section 5. In Proposition 5.2.1 we classified the possible extensions that arise in the long exact sequence of TR-groups for q even. We consider what happens with the adjacent odd groups in each case. Recall the long exact sequence is

$$\xrightarrow{0} \mathbb{H}_q \xrightarrow{N} \mathrm{TR}^n_{q-1+\alpha} \xrightarrow{R} \mathrm{TR}^{n-1}_{q+\alpha'} \xrightarrow{\partial} \mathbb{H}_{q-1} \xrightarrow{N} \mathrm{TR}^n_{q-1+\alpha} \xrightarrow{R} \mathrm{TR}^{n-1}_{q-1+\alpha'} \xrightarrow{0}$$

where we have omitted some arguments in the interest of readability. Recall that for q even,

$$\mathbb{H}_q(C_{p^{n-1}}, T(\mathbb{F}_p) \wedge S^{-\alpha}) \cong \mathbb{Z}/p^{\max(0,r_n)},$$

and

$$\mathbb{H}_{q-1}(C_{p^{n-1}}, T(\mathbb{F}_p) \wedge S^{-\alpha}) \cong \mathbb{Z}/p^{\max(0,r_n-1)}.$$

So if $r_n \leq 1$, we have an exact sequence

$$0 \xrightarrow{N} \mathrm{TR}^n_{q-1+\alpha} \xrightarrow{R} \mathrm{TR}^{n-1}_{q-1+\alpha'} \xrightarrow{0} \cdots$$

and hence $|\mathrm{TR}^n_{q-1+\alpha}(\mathbb{F}_p;p)| = |\mathrm{TR}^{n-1}_{q-1+\alpha'}(\mathbb{F}_p;p)|$, and $S = 0$. In the cases of our classification where $r + g_{\tau(w+1),n-1} \leq n$ the map ∂ is zero. Hence $|\mathrm{TR}^n_{q-1+\alpha}(\mathbb{F}_p;p)| = p^{r_n-1}|\mathrm{TR}^{n-1}_{q-1+\alpha'}(\mathbb{F}_p;p)|$.

In the case where $r + g_{\tau(w+1),n-1} > n$,

$$\partial(1) \doteq p^{n-1-g_{\tau(w+1),n-1}}.$$

It then follows that $p^{r_n+g_{\tau(w+1),n-1}-n}|\mathrm{TR}^n_{q-1+\alpha}(\mathbb{F}_p;p)| = |\mathrm{TR}^{n-1}_{q-1+\alpha'}(\mathbb{F}_p;p)|$.

$\square$

From this proposition we get the following recursive method for computing the orders of the groups indexed by odd-dimensional representations.

Theorem 7.1.2. *For q even, the order of the group $\mathrm{TR}^n_{q-1+\alpha}(\mathbb{F}_p;p)$ is p^{d_n} where d_n is given as follows. Let $d_1 = 0$. For $2 \le j \le n$ define $r_j = \min(j, \frac{q+|\alpha^{(n-j)}|}{2}+1)$. Then:*

$$d_j = d_{j-1} + \begin{cases} \max(0, r_j - 1) & \text{if } r_j + g_{\tau(w+1),j-1} \le j \\ r_j + g_{\tau(w+1),j-1} - j & \text{else} \end{cases}$$

Proof. The theorem follows from iterated applications of Proposition 7.1.1. $\square$

While this theorem computes the order of the abelian group $\mathrm{TR}^n_{q-1+\alpha}(\mathbb{F}_p;p)$, it does not compute which abelian group it is.

Remark 1. Recall from the computations in section 4 that for q even,

$$\hat{\mathbb{H}}^{q-1}(C_{p^n}, T(\mathbb{F}_p) \wedge S^{-\alpha}) \cong \mathbb{H}^{q-1}(C_{p^n}, T(\mathbb{F}_p) \wedge S^{-\alpha}) = 0.$$

So the norm-restriction diagram of long exact sequences is of the form

$$\begin{tikzcd} {} \arrow[r, "\partial"] & \mathbb{H}_{q-1} \arrow[r, "N"] \arrow[d, equal] & \mathrm{TR}^n_{q-1+\alpha}(\mathbb{F}_p;p) \arrow[r, "R"] \arrow[d, "\Gamma_{n-1}"] & \mathrm{TR}^{n-1}_{q-1+\alpha'}(\mathbb{F}_p;p) \arrow[r] \arrow[d, "\hat{\Gamma}_{n-1}"] & \mathbb{H}_{q-2} \arrow[r] \arrow[d, equal] & {} \\ {} \arrow[r] & \mathbb{H}_{q-1} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{H}_{q-2} \arrow[r] & {} \end{tikzcd}$$

Thus, while we can use our computation of the map ∂ to determine the order of the group $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p)$, we cannot use the map $\hat{\Gamma}_{n-1}$ to determine the extension as we did in the case of q even.

7.2 Special cases

We now interpret the main theorem in a few special cases.

Corollary 7.2.1. *Let α be a virtual representation of the form $\alpha = \beta - 0$. Then for all q even we have*

$$\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p;p) \cong \mathbb{Z}/p^L$$

where L is given as follows: Let m be such that $-|\alpha^{(n-m)}| < q \le -|\alpha^{(n-m+1)}|$. Let $r_j = \min(j, \frac{q+|\alpha^{(n-j)}|}{2} + 1)$. Set $l_{1,m} = r_m$. Letting j range from $m+1$ to n,

$$l_{1,j} = \min(j, l_{1,j-1} + r_j).$$

Then $L = l_{1,n}$.

Proof. For a representation α of this form,

$$-|\alpha| \leq -|\alpha'| \leq \ldots \leq -|\alpha^{(n-1)}|$$

and given the hypotheses of the corollary,

$$r_1 \leq r_2 \leq \ldots r_{m-1} \leq 0 < r_m \leq \ldots r_n$$

From the recursive definition in Theorem 1.0.1 we know that for $j < m$, all $l_{i,j} = 0$, $1 \leq i \leq j$. It follows that $l_{1,m} = r_m$, $k_{1,m} = 0$, and $l_{i,m} = k_{i,m} = 0$ for $2 \leq i \leq m$. We show by induction that for all j from m to n, $l_{1,j} \neq 0$, $k_{1,j} = 0$, and $l_{i,j} = k_{i,j} = 0$ for $2 \leq i \leq j$. We have already noted that this holds for $j = m$. If it holds for j it holds for $j + 1$ by the recursive definition. Thus $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p; p)$ is a cyclic group. The order of this group as stated in the corollary follows directly from the recursive definition. □

Corollary 7.2.2. *Let α be a representation of the form $\alpha = 0 - \gamma$. Then for all q even*

$$\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p; p) \cong \mathbb{Z}/p^m$$

where m is the number such that

$$-|\alpha^{(n-m)}| \leq q < -|\alpha^{(n-m-1)}|.$$

Proof. For a representation α of this form,

$$-|\alpha^{(n-1)}| \leq -|\alpha^{(n-2)}| \leq \ldots \leq -|\alpha|$$

and

$$r_n \leq r_{n-1} \leq \ldots \leq r_{m+1} \leq 0 < r_m \leq \ldots \leq r_1.$$

Following the recursive definition in Theorem 1.0.1, $l_{1,1} = 1$ and $k_{1,1} = 0$. We show inductively that $l_{1,j} = j$ and $k_{1,j} = 0$ for $1 \leq j \leq m$. We have just observed that this holds for $j = 1$. Now suppose it holds for $j - 1$. Then by the recursive definition

$$l_{1,j} = \min(j, r_j + g_{\tau(w+1),j-1}) = \min(j, r_j + l_{1,j-1}) = j.$$

It is also clear that $l_{i,j} = 0$ for all $2 \leq i \leq j$. So $l_{1,m} = m$ and $l_{i,m} = 0$ for $2 \leq i \leq m$. Continuing the recursion, we see that the group is unchanged for $j > m$, since in those cases $w = j-1$, and $r_j < 0$ so the summand $l_{1,j} = 0$. Thus $\mathrm{TR}^n_{q+\alpha}(\mathbb{F}_p; p) \cong \mathbb{Z}/p^m$. □

The second corollary agrees with the result given by Hesselholt and Madsen in this case ([12], Proposition 8.1). We have now seen that if α is an actual real representation of S^1, or zero minus an actual representation, the TR-groups we are calculating are all cyclic.

Chapter 8

TR as a Mackey Functor

We can also consider TR-theory as a $\mathbb{Z}$-graded or $RO(S^1)$-graded Mackey functor. Then we are able to exploit the homological algebra tools developed by Lewis and Mandell [18] for Mackey functors to further study TR-groups.

For a G-spectrum Y, the homotopy groups of Y form a $\mathbb{Z}$-graded Mackey functor. To a coset G/H we associate the abelian group

$$(\underline{\pi}_q Y)(G/H) = \pi_q^H Y = [S^q \wedge G/H_+, Y]_G.$$

Let $K \subset H$ be a subgroup. Inclusion of fixed points gives a map $\pi_q^H Y \to \pi_q^K Y$ and an associated transfer $\pi_q^K Y \to \pi_q^H Y$. Similarly, for such a G-spectrum Y, there is an associated $RO(G)$-graded Mackey functor with abelian group

$$(\underline{\pi}_\alpha Y)(G/H) = \pi_\alpha^H Y = [S^\beta \wedge G/H_+, Y \wedge S^\gamma]_G.$$

for $\alpha \in RO(G)$, $\alpha = [\beta] - [\gamma]$.

These homotopy Mackey functors allow us to consider the TR-theory of $\mathbb{F}_p$ as an S^1-Mackey functor. We write $\underline{\mathrm{TR}}_*$ for the Mackey functor:

$$\underline{\mathrm{TR}}_*(S^1/C_{p^n}) = \underline{\mathrm{TR}}_*^{n+1}(\mathbb{F}_p; p).$$

We can consider this as either a $\mathbb{Z}$-graded Mackey functor or as an $RO(S^1)$-graded Mackey functor, depending on which version of the TR-groups we are interested in considering. To simplify matters we also want to look at the C_p-Mackey functor $\underline{\mathrm{TR}}_*^{\leq 2}$ which is our Mackey functor $\underline{\mathrm{TR}}_*$ restricted to the subgroup C_p of S^1.

Lewis and Mandell have developed a theory of homological algebra for $\mathbb{Z}$ and $RO(S^1)$-graded Mackey functors. In particular they have developed universal coefficient and Kunneth spectral sequences for graded Mackey functors.

The categories of $\mathbb{Z}$-graded Mackey functors and $RO(S^1)$-graded Mackey functors each have a symmetric monoidal product, the box product, denoted $\Box$. They also have an adjoint function object functor, denoted $\langle -, - \rangle$. Then for $\underline{R}_*$ a graded Mackey functor ring, and $\underline{L}_*$, $\underline{M}_*$ Mackey functor modules over $\underline{R}_*$, one can construct derived functors of $\Box$ and $\langle -, - \rangle$. Let $\underline{\mathrm{Tor}}_s^{\underline{R}_*}(\underline{N}_*, \underline{M}_*)$ be the sth left derived functor of $\underline{N}_* \Box_{\underline{R}_*} \underline{M}_*$. Let $\underline{\mathrm{Ext}}_{\underline{R}_*}^s(\underline{L}_*, \underline{M}_*)$ be the sth right derived functor of $\langle \underline{L}_*, \underline{M}_* \rangle^{\underline{R}_*}$.

8.1 Tor computation

To demonstrate the computational tools that this viewpoint provides, we want to consider the following spectral sequence of Lewis and Mandell ([18], Theorem 1.3).

Theorem 8.1.1. *Let X and Y be G-spectra indexed on the same universe. There is a naturally strongly convergent homology spectral sequence of $\underline{R}_*$ - modules*

$$E^2_{s,\tau} = \underline{\mathrm{Tor}}^{\underline{R}_*}_{s,\tau}(\underline{R}_*X, \underline{R}_*Y) \Rightarrow \underline{R}_{s+\tau}(X \wedge Y).$$

Here we have the convention that $\underline{\mathrm{Tor}}^{\underline{R}_*}_{s,\tau}(\underline{R}_*X, \underline{R}_*Y) = (\underline{\mathrm{Tor}}^{\underline{R}_*}_{s}(\underline{R}_*X, \underline{R}_*Y))_\tau$. The homological grading s is a nonnegative integer, and the grading τ can be in $\mathbb{Z}$ or $RO(S^1)$. We consider this spectral sequence when $X = S^{-\alpha}, Y = S^\alpha$, $\underline{R}_* = \underline{\mathrm{TR}}^{\leq 2}_*$, and the grading $\tau \in \mathbb{Z}$. Note that we are considering this spectral sequence as a spectral sequence of $\mathbb{Z}$-graded Mackey functors.

We introduce some notation, and then do an example of how this spectral sequence can be used to study TR-theory. Recall that elements of $RO(S^1)$ are formal differences of isomorphism classes of S^1-representation, or virtual representations. Let $\lambda_{k_1,\ldots k_n}$ denote the real $2n$-dimensional representation of S^1 given by $\mathbb{C}(k_1)\oplus\mathbb{C}(k_2)\oplus\ldots\oplus\mathbb{C}(k_n)$ where $\mathbb{C}(k)$ denotes the representation of S^1 on $\mathbb{C}$ by the $k-$fold power map. Then a general element α of $RO(S^1)$ can be written in the form

$$\alpha = \lambda_{k_1,\ldots k_n} - \lambda_{l_1,\ldots l_m} + q,$$

for some $q \in \mathbb{Z}$.

Example 8.1.2. As an example, we study the spectral sequence

$$\underline{\mathrm{Tor}}^{\underline{\mathrm{TR}}^{\leq 2}_*}(\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}}, \underline{\mathrm{TR}}^{\leq 2}_{*-\lambda_{1,1}}) \Rightarrow \underline{\mathrm{TR}}^{\leq 2}_*.$$

In order to do this we need to form a projective resolution of $\underline{\mathrm{TR}}^{\leq 2}_{*-\lambda_{1,1}}$ as a $\underline{\mathrm{TR}}^{\leq 2}_*$ - module. We first recall the Mackey functor $\underline{\mathrm{TR}}^{\leq 2}_*$

$$\mathbb{Z}/p[\sigma] \underset{F}{\overset{V}{\rightleftarrows}} \mathbb{Z}/p^2[\sigma]$$

where σ is a polynomial generator in degree 2. The Frobenius map is determined by $F(1) = 1$, and $F(\sigma) = \sigma$. The Verschiebung is determined by $V(1) = p$, and $V(\sigma) = p\sigma$. We next need to determine the structure of $\underline{\mathrm{TR}}^{\leq 2}_{*-\lambda_{1,1}}$ as a Mackey functor module over $\underline{\mathrm{TR}}^{\leq 2}_*$. Applying the recursive method before Theorem 1.0.1 to the representation $\alpha = -\lambda_{1,1}$ computes the groups $\mathrm{TR}^n_{q-\lambda_{1,1}}(\mathbb{F}_p; p)$ and the action of σ on $\mathrm{TR}^n_{q-\lambda_{1,1}}(\mathbb{F}_p; p)$. In particular, for TR^1 we get:

$$\mathrm{TR}^1_{2+\alpha}(\mathbb{F}_p; p) \xrightarrow{\sigma_1} \mathrm{TR}^1_{4+\alpha}(\mathbb{F}_p; p) \xrightarrow{\sigma_1} \mathrm{TR}^1_{6+\alpha}(\mathbb{F}_p; p) \xrightarrow{\sigma_1} \cdots$$

is given by

$$0 \longrightarrow \mathbb{Z}/p \xrightarrow{\text{id}} \mathbb{Z}/p \xrightarrow{\text{id}} \mathbb{Z}/p \xrightarrow{\text{id}} \cdots$$

For TR^2 we compute that

$$\mathrm{TR}^2_{0+\alpha}(\mathbb{F}_p;p) \xrightarrow{\sigma_2} \mathrm{TR}^2_{2+\alpha}(\mathbb{F}_p;p) \xrightarrow{\sigma_2} \mathrm{TR}^2_{4+\alpha}(\mathbb{F}_p;p) \xrightarrow{\sigma_2} \mathrm{TR}^2_{6+\alpha}(\mathbb{F}_p;p) \xrightarrow{\sigma_2} \cdots$$

is given by

$$\mathbb{Z}/p \xrightarrow{0} \mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{\text{id}} \mathbb{Z}/p^2 \xrightarrow{\text{id}} \cdots$$

Thus, if we rewrite our above Mackey functor $\underline{\mathrm{TR}}^{\leq 2}_*$ as

$$\mathrm{TR}^1_* \underset{F}{\overset{V}{\rightleftarrows}} \mathrm{TR}^2_*$$

then we can write $\underline{\mathrm{TR}}^{\leq 2}_{*-\lambda_{1,1}}$ as

$$\mathrm{TR}^1_*[u_4] \underset{F}{\overset{V}{\rightleftarrows}} \mathrm{TR}^2_*[e_0,e_2,e_4]/(pe_0,\sigma e_0, pe_2, \sigma e_2 - pe_4)$$

where $|u_4| = |e_4| = 4$, $|e_2| = 2$, and $|e_0| = 0$. The maps F, V are given by $F(e_4) = u_4$, and $F(e_0) = F(e_2) = 0$, $V(u_4) = pe_4$. We would like to resolve this Mackey functor module via projective $\underline{\mathrm{TR}}^{\leq 2}_*$ Mackey functor modules. We first need to ask what such a projective module looks like. Lewis and Mandell determine that if $\underline{R}_*$ is a Mackey functor ring, then an $\underline{R}_*$-module is projective if and only if it is a direct summand of a direct sum of $\underline{R}_*$-modules of the form $\underline{R}_*\Box_*\sum^\tau \underline{B}^{G/H}_*$. Here, $\underline{B}^X_*$ is the graded Mackey functor concentrated in degree zero, such that in degree zero it is given by $\mathcal{B}_G(-,X)$, maps in the Burnside category. Lewis and Mandell prove that

$$(\textstyle\sum^\tau \underline{B}^X_*\Box_*\underline{M}_*)_\alpha \cong \underline{M}_{-\tau+\alpha}(X\times Y)$$

so we are able to undertand these projective modules fairly easily. Let us consider the two Mackey functor modules $\sum^\tau \underline{B}^{C_p/C_p}_*\Box_*\underline{\mathrm{TR}}^{\leq 2}_*$ and $\sum^\tau \underline{B}^{C_p/*}_*\Box_*\underline{\mathrm{TR}}^{\leq 2}_*$. We can see that

$$\begin{aligned}
(\textstyle\sum^\tau \underline{B}^{C_p/C_p}_*\Box_*\underline{\mathrm{TR}}^{\leq 2}_*)_q(C_p/*) &= \underline{\mathrm{TR}}^{\leq 2}_{-\tau+q}(C_p/*) = \mathrm{TR}^1_{\tau+q}\\
(\textstyle\sum^\tau \underline{B}^{C_p/C_p}_*\Box_*\underline{\mathrm{TR}}^{\leq 2}_*)_q(C_p/C_p) &= \underline{\mathrm{TR}}^{\leq 2}_{-\tau+q}(C_p/C_p) = \mathrm{TR}^2_{\tau+q}
\end{aligned}$$

Similarly,

$$\begin{aligned}
(\textstyle\sum^\tau \underline{B}^{C_p/*}_*\Box_*\underline{\mathrm{TR}}^{\leq 2}_*)_q(C_p/*) &= \underline{\mathrm{TR}}^{\leq 2}_{-\tau+q}(C_p/*\times C_p/*) = \oplus_p \mathrm{TR}^1_{\tau+q}\\
(\textstyle\sum^\tau \underline{B}^{C_p/*}_*\Box_*\underline{\mathrm{TR}}^{\leq 2}_*)_q(C_p/C_p) &= \underline{\mathrm{TR}}^{\leq 2}_{-\tau+q}(C_p/*\times C_p/C_p) = \mathrm{TR}^1_{\tau+q}
\end{aligned}$$

We use projective modules of these two forms. The first form gives us shifts of the

Mackey functor ring that we are resolving over. We can write this as

$$\mathrm{TR}^1_*[i_m] \underset{F}{\overset{V}{\rightleftarrows}} \mathrm{TR}^2_*[j_m]$$

where $|i_m| = |j_m| = m$, and $F(j_m) = i_m$ and $V(i_m) = pj_m$. The second type of projective module can be written

$$\bigoplus_p \mathrm{TR}^1_*[v_m] \underset{F}{\overset{V}{\rightleftarrows}} \mathrm{TR}^1_*[w_m] \qquad .$$

Here the Frobenius map is the diagonal map and the Verschiebung is the fold map. Now we would like to resolve the Mackey functor $\underline{\mathrm{TR}}^{\leq 2}_{*-\lambda 1,1}$ as a $\underline{\mathrm{TR}}^{\leq 2}_*$-module. Figure 8-1 is such a projective resolution.

The maps ϕ_i are given as follows. The map ϕ_0 is determined by:

$$\begin{array}{ccc} i_4 \mapsto u_4 & i_2 \mapsto 0 & i_0 \mapsto 0 \\ j_4 \mapsto e_4 & j_2 \mapsto e_2 & j_0 \mapsto e_0 \end{array}$$

The map ϕ_1 is given by:

$$\begin{array}{cccc} i_4 \mapsto -\sigma i_2 & (v_2, 0, \ldots, 0) \mapsto i_2 & (v_0, 0, \ldots, 0) \mapsto i_0 & l_2 \mapsto \sigma i_0 \\ j_4 \mapsto (pj_4, -\sigma j_2) & w_2 \mapsto pj_2 & w_0 \mapsto pj_0 & k_2 \mapsto \sigma j_0 \end{array}$$

The map ϕ_2 is generated by:

$$\begin{array}{cc} (v_4, 0, \ldots, 0) \mapsto ((\sigma v_2, 0, \ldots, 0), (-v_4, v_4, 0, \ldots 0)) & i_2 \mapsto (v_2, v_2, v_2, \ldots, v_2) \\ (x_2, 0, \ldots, 0) \mapsto ((\sigma v_0, 0, \ldots, 0), (-x_2, x_2, 0 \ldots, 0)) & i_0 \mapsto (v_0, v_0, v_0, \ldots, v_0) \\ w_4 \mapsto (pj_4, \sigma w_2) & w_2 \mapsto 0 \\ w_0 \mapsto 0 & y_2 \mapsto (\sigma w_0, -pk_2) \end{array}$$

The map ϕ_3 is generated by:

$$\begin{array}{cc} (v_4, 0, \ldots, 0) \mapsto ((\sigma v_2, 0, \ldots, 0), i_4) & (v_2, 0, \ldots, 0) \mapsto (v_2, -v_2, 0, \ldots, 0) \\ (v_0, 0, \ldots, 0) \mapsto (v_0, -v_0, 0, \ldots, 0) & (x_2, 0, \ldots, 0) \mapsto ((\sigma v_0, 0, \ldots, 0), i_2) \\ w_4 \mapsto \sigma w_2 & j_2 \mapsto w_2 \\ j_0 \mapsto w_0 & y_2 \mapsto \sigma w_0 \end{array}$$

Finally, the map ϕ_4 is determined by:

$$\begin{array}{cccc} i_4 \mapsto (\sigma i_2, -(v_4, v_4, \ldots, v_4)) & i_2 \mapsto 0 & i_0 \mapsto 0 & j_4 \mapsto (\sigma j_2, -w_4) \\ l_2 \mapsto (\sigma i_0, -(x_2, x_2, \ldots x_2)) & j_2 \mapsto pj_2 & j_0 \mapsto pj_0 & y_2 \mapsto (\sigma j_0, -y_2) \end{array}$$

On any coordinates we have omitted in the images, the mappings above go to zero. Observe also that for maps from terms of the form $\bigoplus_p \mathrm{TR}^1_*[v_m]$ we have only spec-

$$
\begin{tikzcd}
0 & 0 \\
\mathrm{TR}^1_*[u_4] \arrow[u] \arrow[r, shift left] \arrow[r, leftarrow, shift right] & \mathrm{TR}^2_*[e_0, e_2, e_4]/(pe_0, \sigma e_0, pe_2, \sigma e_2 - pe_4) \arrow[u] \\
\mathrm{TR}^1_*[i_4] \oplus \mathrm{TR}^1_*[i_2] \oplus \mathrm{TR}^1_*[i_0] \arrow[u, "\phi_0"] \arrow[r, shift left] \arrow[r, leftarrow, shift right] & \mathrm{TR}^2_*[j_4] \oplus \mathrm{TR}^2_*[j_2] \oplus \mathrm{TR}^2_*[j_0] \arrow[u, "\phi_0"'] \\
\begin{array}{c} \mathrm{TR}^1_*[i_4] \oplus \bigoplus_p \mathrm{TR}^1_*[v_2] \oplus \\ \bigoplus_p \mathrm{TR}^1_*[v_0] \oplus \mathrm{TR}^1_*[l_2] \end{array} \arrow[u, "\phi_1"] \arrow[r, shift left] \arrow[r, leftarrow, shift right] & \mathrm{TR}^2_*[j_4] \oplus \mathrm{TR}^1_*[w_2] \oplus \mathrm{TR}^1_*[w_0] \oplus \mathrm{TR}^2_*[k_2] \arrow[u, "\phi_1"'] \\
\begin{array}{c} \bigoplus_p \mathrm{TR}^1_*[v_4] \oplus \bigoplus_p \mathrm{TR}^1_*[v_2] \oplus \\ \bigoplus_p \mathrm{TR}^1_*[v_0] \oplus \bigoplus_p \mathrm{TR}^1_*[x_2] \end{array} \arrow[u, "\phi_2"] \arrow[r, shift left] \arrow[r, leftarrow, shift right] & \mathrm{TR}^1_*[w_4] \oplus \mathrm{TR}^1_*[w_2] \oplus \mathrm{TR}^1_*[w_0] \oplus \mathrm{TR}^1_*[y_2] \arrow[u, "\phi_2"'] \\
\begin{array}{c} \bigoplus_p \mathrm{TR}^1_*[v_4] \oplus \mathrm{TR}^1_*[i_2] \oplus \\ \mathrm{TR}^1_*[i_0] \oplus \bigoplus_p \mathrm{TR}^1_*[x_2] \end{array} \arrow[u, "\phi_3"] \arrow[r, shift left] \arrow[r, leftarrow, shift right] & \mathrm{TR}^1_*[w_4] \oplus \mathrm{TR}^2_*[j_2] \oplus \mathrm{TR}^2_*[j_0] \oplus \mathrm{TR}^1_*[y_2] \arrow[u, "\phi_3"'] \\
\begin{array}{c} \mathrm{TR}^1_*[i_4] \oplus \mathrm{TR}^1_*[i_2] \oplus \\ \mathrm{TR}^1_*[i_0] \oplus \mathrm{TR}^1_*[l_2] \end{array} \arrow[u, "\phi_4"] \arrow[r, shift left] \arrow[r, leftarrow, shift right] & \mathrm{TR}^2_*[j_4] \oplus \mathrm{TR}^2_*[j_2] \oplus \mathrm{TR}^2_*[j_0] \oplus \mathrm{TR}^2_*[k_2] \arrow[u, "\phi_4"'] \\
\begin{array}{c} \mathrm{TR}^1_*[i_4] \oplus \bigoplus_p \mathrm{TR}^1_*[v_2] \oplus \\ \bigoplus_p \mathrm{TR}^1_*[v_0] \oplus \mathrm{TR}^1_*[l_2] \end{array} \arrow[u, "\phi_1"] \arrow[r, shift left] \arrow[r, leftarrow, shift right] & \mathrm{TR}^2_*[j_4] \oplus \mathrm{TR}^1_*[w_2] \oplus \mathrm{TR}^1_*[w_0] \oplus \mathrm{TR}^2_*[k_2] \arrow[u, "\phi_1"'] \\
\vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd}
$$

Figure 8-1: Projective resolution of $\mathrm{TR}^{\leq 2}_{*-\lambda_{1,1}}$

ified the maps on the first summand of this direct sum. All maps in the projective resolution are required to be C_p-equivariant. On most terms C_p acts as the identity, but on terms of the form $\bigoplus_p \mathrm{TR}^1_*[v_m]$ it permutes factors. Thus we need only specify the above maps on one of the summands.

In figure 8-2 we rewrite this projective resolution using the notation

$$\Sigma^{\tau}_{C_p} \underline{\mathrm{TR}}^{\leq 2}_* = \Sigma^{\tau} \underline{B}^{C_p/C_p}_* \Box_* \underline{\mathrm{TR}}^{\leq 2}_*$$

and

$$\Sigma^{\tau}_{\{e\}}\underline{\mathrm{TR}}^{\leq 2}_* = \Sigma^{\tau}\underline{B}^{C_p/\{e\}}_* \Box_* \underline{\mathrm{TR}}^{\leq 2}_*$$

$$\begin{array}{c}
0 \\
\uparrow \\
\underline{\mathrm{TR}}^{\leq 2}_{*-\lambda_{1,1}} \\
\uparrow \\
\Sigma^4_{C_p}\underline{\mathrm{TR}}^{\leq 2}_* \oplus \Sigma^2_{C_p}\underline{\mathrm{TR}}^{\leq 2}_* \oplus \Sigma^0_{C_p}\underline{\mathrm{TR}}^{\leq 2}_* \\
\uparrow \\
\Sigma^4_{C_p}\underline{\mathrm{TR}}^{\leq 2}_* \oplus \Sigma^2_{\{e\}}\underline{\mathrm{TR}}^{\leq 2}_* \oplus \Sigma^0_{\{e\}}\underline{\mathrm{TR}}^{\leq 2}_* \oplus \Sigma^2_{C_p}\underline{\mathrm{TR}}^{\leq 2}_* \\
\uparrow \\
\Sigma^4_{\{e\}}\underline{\mathrm{TR}}^{\leq 2}_* \oplus \Sigma^2_{\{e\}}\underline{\mathrm{TR}}^{\leq 2}_* \oplus \Sigma^0_{\{e\}}\underline{\mathrm{TR}}^{\leq 2}_* \oplus \Sigma^2_{\{e\}}\underline{\mathrm{TR}}^{\leq 2}_* \\
\uparrow \\
\Sigma^4_{\{e\}}\underline{\mathrm{TR}}^{\leq 2}_* \oplus \Sigma^2_{C_p}\underline{\mathrm{TR}}^{\leq 2}_* \oplus \Sigma^0_{C_p}\underline{\mathrm{TR}}^{\leq 2}_* \oplus \Sigma^2_{\{e\}}\underline{\mathrm{TR}}^{\leq 2}_* \\
\uparrow \\
\Sigma^4_{C_p}\underline{\mathrm{TR}}^{\leq 2}_* \oplus \Sigma^2_{C_p}\underline{\mathrm{TR}}^{\leq 2}_* \oplus \Sigma^0_{C_p}\underline{\mathrm{TR}}^{\leq 2}_* \oplus \Sigma^2_{C_p}\underline{\mathrm{TR}}^{\leq 2}_* \\
\uparrow \\
\Sigma^4_{C_p}\underline{\mathrm{TR}}^{\leq 2}_* \oplus \Sigma^2_{\{e\}}\underline{\mathrm{TR}}^{\leq 2}_* \oplus \Sigma^0_{\{e\}}\underline{\mathrm{TR}}^{\leq 2}_* \oplus \Sigma^2_{C_p}\underline{\mathrm{TR}}^{\leq 2}_* \\
\uparrow \\
\vdots
\end{array}$$

Figure 8-2: Projective resolution of $\mathrm{TR}^{\leq 2}_{*-\lambda_{1,1}}$ (revisited)

To form the Mackey functor $\underline{\mathrm{Tor}}^{\underline{\mathrm{TR}}^{\leq 2}_*}(\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}}, \underline{\mathrm{TR}}^{\leq 2}_{*-\lambda_{1,1}})$ that we are interested in, we need to take the box product of the above resolution with the Mackey functor module $\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}}$ over the Mackey functor ring $\underline{\mathrm{TR}}^{\leq 2}_*$. We can write this Mackey functor module as

$$\mathrm{TR}^1_*[w_{-4}] \underset{F}{\overset{V}{\rightleftarrows}} \mathrm{TR}^2_*[f_{-4}, g_{-2}, m_{-3}]/(pf_{-4}, pm_{-3}, \sigma m_{-3}, \sigma f_{-4} - pg_{-2})$$

where the subscripts indicate the degrees of the generators. The maps F, V are given by $F(f_{-4}) = F(m_{-3}) = 0, F(g_{-2}) = \sigma w_{-4}$ and $V(w_{-4}) = f_{-4}$. We take the box product of the above projective resolution with $\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}}$ over $\underline{\mathrm{TR}}^{\leq 2}_*$. Using the property that

$$\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \Box_{\underline{\mathrm{TR}}^{\leq 2}_*} \underline{\mathrm{TR}}^{\leq 2}_* \cong \underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}}$$

we get the complex in figure 8-3

$$
\begin{array}{c}
0 \\
\uparrow \\
\Sigma^4_{C_p}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \oplus \Sigma^2_{C_p}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \oplus \Sigma^0_{C_p}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \\
\uparrow \\
\Sigma^4_{C_p}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \oplus \Sigma^2_{\{e\}}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \oplus \Sigma^0_{\{e\}}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \oplus \Sigma^2_{C_p}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \\
\uparrow \\
\Sigma^4_{\{e\}}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \oplus \Sigma^2_{\{e\}}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \oplus \Sigma^0_{\{e\}}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \oplus \Sigma^2_{\{e\}}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \\
\uparrow \\
\Sigma^4_{\{e\}}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \oplus \Sigma^2_{C_p}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda 1,1} \oplus \Sigma^0_{C_p}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \oplus \Sigma^2_{\{e\}}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \\
\uparrow \\
\Sigma^4_{C_p}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \oplus \Sigma^2_{C_p}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \oplus \Sigma^0_{C_p}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \oplus \Sigma^2_{C_p}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \\
\uparrow \\
\Sigma^4_{C_p}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \oplus \Sigma^2_{\{e\}}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \oplus \Sigma^0_{\{e\}}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \oplus \Sigma^2_{C_p}\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}} \\
\uparrow \\
\vdots
\end{array}
$$

Figure 8-3: Complex computing $\underline{\mathrm{Tor}}$

The maps are induced from the maps described above.

We then compute the groups $\underline{\mathrm{Tor}}^{\underline{\mathrm{TR}}^{\leq 2}_*}_{s,t}(\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}}, \underline{\mathrm{TR}}^{\leq 2}_{*-\lambda_{1,1}})$ by computing the homology of this complex. It is easy to verify that if we evaluate the complex above at $C_p/\{e\}$, it has homology only in degree 0, and there we find that

$$\underline{\mathrm{Tor}}^{\underline{\mathrm{TR}}^{\leq 2}_*}_{0,*}(\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}}, \underline{\mathrm{TR}}^{\leq 2}_{*-\lambda_{1,1}})(C_p/\{e\}) \cong \mathrm{TR}^1_* \cong \mathbb{Z}/p[\sigma]$$

where $|\sigma| = 2$. Thus, the spectral sequence we are interested in collapses at the E^2-term in this case. We now evaluate the complex at C_p/C_p. For ease of notation we write $\underline{\mathrm{Tor}}^{\underline{\mathrm{TR}}^{\leq 2}_*}_{s,t}(C_p/C_p)$ for $\underline{\mathrm{Tor}}^{\underline{\mathrm{TR}}^{\leq 2}_*}_{s,t}(\underline{\mathrm{TR}}^{\leq 2}_{*+\lambda_{1,1}}, \underline{\mathrm{TR}}^{\leq 2}_{*-\lambda_{1,1}})(C_p/C_p)$. We find that

$$\underline{\mathrm{Tor}}^{\underline{\mathrm{TR}}^{\leq 2}_*}_{0,*}(C_p/C_p) \cong \mathrm{TR}^2_*[f_0, g_2, m_1, g_0, m_{-1}, g_{-2}, m_{-3}]/\sim$$

where the relations $\sim$ are given by

$$(pg_{-2}, pf_0, pm_1, pm_{-3}, pm_{-1}, pg_0, \sigma g_0, \sigma g_{-2}, \sigma m_{-3}, \sigma m_1, \sigma m_{-1}, \sigma f_0 - pg_2).$$

We compute

$$\underline{\mathrm{Tor}}_{1,*}^{\underline{\mathrm{TR}}_*^{\leq 2}}(C_p/C_p) \cong \mathrm{TR}_*^2[m_{-1}, m_1]/\sigma, p,$$

$$\underline{\mathrm{Tor}}_{2,*}^{\underline{\mathrm{TR}}_*^{\leq 2}}(C_p/C_p) \cong \mathrm{TR}_*^1[t_{-4}, t_{-2}]/\sigma,$$

and

$$\underline{\mathrm{Tor}}_{3,*}^{\underline{\mathrm{TR}}_*^{\leq 2}}(C_p/C_p) \cong \mathrm{TR}_*^2[m_{-1}, m_{-3}, f_{-4}, f_{-2}, (g_{-2} - t_{-2}), (g_0 - t_0)]/\sim$$

where the relations $\sim$ are given by

$$pm_{-1}, \sigma m_{-1}, pm_{-3}, \sigma m_{-3}, \sigma f_{-4}, \sigma f_{-2}, p(g_{-2} - t_{-2}), \sigma(g_{-2} - t_{-2}), p(g_0 - t_0), \sigma(g_0 - t_0)$$

Finally, we compute

$$\underline{\mathrm{Tor}}_{4,*}^{\underline{\mathrm{TR}}_*^{\leq 2}}(C_p/C_p) \cong \mathrm{TR}_*^2[m_1, m_{-1}, m_{-3}, h_{-1}]/\sigma, p.$$

Since the projective resolution repeats, the remaining $\underline{\mathrm{Tor}}_{s,*}^{\underline{\mathrm{TR}}_*^{\leq 2}}(C_p/C_p)$ groups are given by

$$\underline{\mathrm{Tor}}_{s,*}^{\underline{\mathrm{TR}}_*^{\leq 2}}(C_p/C_p) \cong \underline{\mathrm{Tor}}_{s+4,*}^{\underline{\mathrm{TR}}_*^{\leq 2}}(C_p/C_p)$$

for $s \geq 1$.

These computations give the E^2-term of the spectral sequence we are interested in. This E^2-term is shown in Figure 8-4.

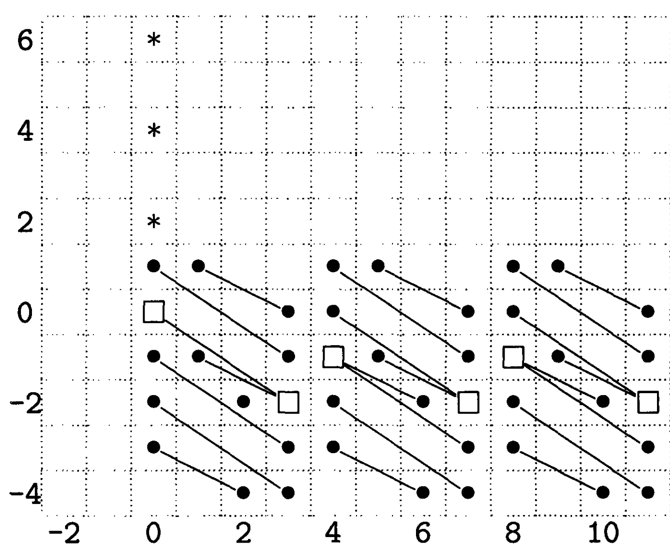

Figure 8-4: E^2-term of the $\underline{\mathrm{Tor}}$- Kunneth spectral sequence

$\bullet = \mathbb{Z}/p$ $\qquad * = \mathbb{Z}/p^2$ $\qquad \square = \mathbb{Z}/p \oplus \mathbb{Z}/p$

The pattern continues to repeat to the right. Knowing that the spectral sequence converges to $\underline{\mathrm{TR}}_*^{\leq 2} \cong \mathbb{Z}/p^2[\sigma]$ forces the differentials as shown in the spectral sequence above.

8.2 Ext computation

Similarly, we can study the TR-Mackey functors via the $\underline{\mathrm{Ext}}$ Mackey functor. Lewis and Mandell give a spectral sequence ([18], Theorem 1.3)

Theorem 8.2.1. *Let X and Y be G-spectra indexed on the same universe. There is a natural conditionally convergent spectral sequence*

$$E_2^{s,\tau} = \underline{\mathrm{Ext}}_{\underline{R}^*}^{s,\tau}(\underline{R}_{-*}X, \underline{R}^*Y) \Rightarrow \underline{R}^{s+\tau}(X \wedge Y).$$

By convention $\underline{\mathrm{Ext}}_{\underline{R}^*}^{s,\tau}(\underline{R}_{-*}X, \underline{R}^*Y) = (\underline{\mathrm{Ext}}_{\underline{R}^*}^{s}(\underline{R}_{-*}X, \underline{R}^*Y))_{-\tau}$ where s is a nonnegative integer and the grading τ is either in $\mathbb{Z}$ or $RO(S^1)$, depending on which version of the spectral sequence we are using.

In our case we consider the spectral sequence of integer-graded Mackey functors where $\underline{R}_* = \underline{\mathrm{TR}}_*^{\leq 2}$, $X = S^{-\lambda_{1,1}}$, and $Y = S^0$. Then we have a spectral sequence

$$E_2^{s,\tau} = \underline{\mathrm{Ext}}_{\underline{\mathrm{TR}}_{\leq 2}^*}^{s,\tau}(\underline{\mathrm{TR}}_{-*}^{\leq 2}S^{\lambda_{1,1}}, \underline{\mathrm{TR}}_{\leq 2}^*S^0) \Rightarrow \underline{\mathrm{TR}}_{\leq 2}^{s+\tau}(S^{-\lambda_{1,1}}).$$

Written differently, we have

$$E_2^{s,\tau} = \underline{\mathrm{Ext}}_{\underline{\mathrm{TR}}_{-*}^{\leq 2}}^{s,\tau}(\underline{\mathrm{TR}}_{-*-\lambda_{1,1}}^{\leq 2}, \underline{\mathrm{TR}}_{-*}^{\leq 2}) \Rightarrow \underline{\mathrm{TR}}_{-s-\tau+\lambda_{1,1}}^{\leq 2}.$$

More generally, given the input of a Mackey functor $\underline{\mathrm{TR}}_{*-\alpha}^{\leq 2}$, this spectral sequence converges to the Mackey functor $\underline{\mathrm{TR}}_{*+\alpha}^{\leq 2}$. In our example, we use the projective resolution of $\underline{\mathrm{TR}}_{*-\lambda_{1,1}}^{\leq 2}$ above. We then apply $\underline{\mathrm{Hom}}(-, \underline{\mathrm{TR}}_*^{\leq 2})$ to this resolution to get the complex in Figure 8-5.

Here the gradings are negative in the sense that a class indexed by the number m is in degree $-m$, and $|\sigma| = -2$.

The maps ϕ_i^* are given as follows. The map ϕ_1^* is given by:

$$\begin{array}{llll} i_4^* \mapsto 0 & i_2^* \mapsto -\sigma i_4^* + (v_2^*, v_2^*, \dots v_2^*) & l_2^* \mapsto 0 & j_4^* \mapsto pj_4^* \\ j_2^* \mapsto w_2^* - \sigma j_4^* & i_0^* \mapsto -\sigma l_2^* + (v_0^*, v_0^*, \dots v_0^*) & n_0^* \mapsto w_0^* - \sigma k_2^* & k_2^* \mapsto pk_2^* \end{array}$$

The map ϕ_2^* is generated by:

$$\begin{array}{cc} (v_2^*, 0, \dots, 0) \mapsto (\sigma v_4^*, 0 \dots 0) + (v_2^*, -v_2^*, 0 \dots 0) & i_4^* \mapsto (v_4^*, v_4^*, \dots v_4^*) \\ (v_0^*, 0, \dots, 0) \mapsto (\sigma x_2^*, 0 \dots 0) + (v_0^*, -v_0^*, 0 \dots 0) & l_2^* \mapsto (x_2^*, x_2^*, \dots x_2^*) \\ j_4^* \mapsto w_4^* & w_2^* \mapsto \sigma w_4^* \\ w_0^* \mapsto \sigma w_2^* & k_2^* \mapsto w_2^* \end{array}$$

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\mathrm{TR}^1_{-*}[i_4^*] \oplus \mathrm{TR}^1_{-*}[i_2^*] \oplus \mathrm{TR}^1_{-*}[i_0^*] \arrow[r, shift left] \arrow[d, "\phi_1^*"] & \mathrm{TR}^2_{-*}[j_4^*] \oplus \mathrm{TR}^2_{-*}[j_2^*] \oplus \mathrm{TR}^2_{-*}[j_0^*] \arrow[l, shift left] \arrow[d, "\phi_1^*"] \\
\begin{array}{c} \mathrm{TR}^1_{-*}[i_4^*] \oplus \bigoplus_p \mathrm{TR}^1_{-*}[v_2^*] \oplus \\ \bigoplus_p \mathrm{TR}^1_{-*}[v_0^*] \oplus \mathrm{TR}^1_{-*}[l_2^*] \end{array} \arrow[r, shift left] \arrow[d, "\phi_2^*"] & \mathrm{TR}^2_{-*}[j_4^*] \oplus \mathrm{TR}^1_{-*}[w_2^*] \oplus \mathrm{TR}^1_{-*}[w_0^*] \oplus \mathrm{TR}^2_{-*}[k_2^*] \arrow[l, shift left] \arrow[d, "\phi_2^*"] \\
\begin{array}{c} \bigoplus_p \mathrm{TR}^1_{-*}[v_4^*] \oplus \bigoplus_p \mathrm{TR}^1_{-*}[v_2^*] \oplus \\ \bigoplus_p \mathrm{TR}^1_{-*}[v_0^*] \oplus \bigoplus_p \mathrm{TR}^1_{-*}[x_2^*] \end{array} \arrow[r, shift left] \arrow[d, "\phi_3^*"] & \mathrm{TR}^1_{-*}[w_4^*] \oplus \mathrm{TR}^1_{-*}[w_2^*] \oplus \mathrm{TR}^1_{-*}[w_0^*] \oplus \mathrm{TR}^1_{-*}[y_2^*] \arrow[l, shift left] \arrow[d, "\phi_3^*"] \\
\begin{array}{c} \bigoplus_p \mathrm{TR}^1_{-*}[v_4^*] \oplus \mathrm{TR}^1_{-*}[i_2^*] \oplus \\ \mathrm{TR}^1_{-*}[i_0^*] \oplus \bigoplus_p \mathrm{TR}^1_{-*}[x_2^*] \end{array} \arrow[r, shift left] \arrow[d, "\phi_4^*"] & \mathrm{TR}^1_{-*}[w_4^*] \oplus \mathrm{TR}^2_{-*}[j_2^*] \oplus \mathrm{TR}^2_{-*}[j_0^*] \oplus \mathrm{TR}^1_{-*}[y_2^*] \arrow[l, shift left] \arrow[d, "\phi_4^*"] \\
\begin{array}{c} \mathrm{TR}^1_{-*}[i_4^*] \oplus \mathrm{TR}^1_{-*}[i_2^*] \oplus \\ \mathrm{TR}^1_{-*}[i_0^*] \oplus \mathrm{TR}^1_{-*}[l_2^*] \end{array} \arrow[r, shift left] \arrow[d, "\phi_1^*"] & \mathrm{TR}^2_{-*}[j_4^*] \oplus \mathrm{TR}^2_{-*}[j_2^*] \oplus \mathrm{TR}^2_{-*}[j_0^*] \oplus \mathrm{TR}^2_{-*}[k_2^*] \arrow[l, shift left] \arrow[d, "\phi_1^*"] \\
\begin{array}{c} \mathrm{TR}^1_{-*}[i_4^*] \oplus \bigoplus_p \mathrm{TR}^1_{-*}[v_2^*] \oplus \\ \bigoplus_p \mathrm{TR}^1_{-*}[v_0^*] \oplus \mathrm{TR}^1_{-*}[l_2^*] \end{array} \arrow[r, shift left] \arrow[d] & \mathrm{TR}^2_{-*}[j_4^*] \oplus \mathrm{TR}^1_{-*}[w_2^*] \oplus \mathrm{TR}^1_{-*}[w_0^*] \oplus \mathrm{TR}^2_{-*}[k_2^*] \arrow[l, shift left] \arrow[d] \\
\vdots & \vdots
\end{tikzcd}
$$

Figure 8-5: Complex computing <u>Ext</u>

The map ϕ_3^* is generated by:

$$
\begin{array}{cc}
(v_4^*, 0, \ldots, 0) \mapsto (v_4^*, -v_4^*, 0 \ldots 0) & (v_2^*, 0, \ldots, 0) \mapsto i_2^* + (\sigma v_4^*, 0 \ldots 0) \\
(v_0^*, 0, \ldots, 0) \mapsto i_0^* + (\sigma x_2^*, 0 \ldots 0) & (x_2^*, 0, \ldots, 0) \mapsto (x_2^*, -x_2^*, 0 \ldots 0) \\
w_4^* \mapsto 0 & w_2^* \mapsto 0 \\
w_0^* \mapsto p j_0^* + \sigma y_2^* & y_2^* \mapsto p j_2^* + \sigma w_4^*
\end{array}
$$

Finally, the map ϕ_4 is determined by:

$$
\begin{array}{cccc}
(v_4^*, 0 \ldots 0) \mapsto -i_4^* & i_2^* \mapsto \sigma i_4^* & i_0^* \mapsto \sigma x_2^* & (x_2^*, 0 \ldots 0) \mapsto -x_2^* \\
w_4^* \mapsto p j_4^* & j_2^* \mapsto p j_2^* + \sigma j_4^* & j_0^* \mapsto p j_0^* + \sigma k_2^* & y_2^* \mapsto p k_*^2
\end{array}
$$

We then compute $\underline{\mathrm{Ext}}^{s,\tau}_{\underline{\mathrm{TR}}^{\leq 2}_{-*}}(\underline{\mathrm{TR}}^{\leq 2}_{-*-\lambda_{1,1}}, \underline{\mathrm{TR}}^{\leq 2}_{-*})$ by computing the homology of this complex. When the complex above is evaluated at $C_p/\{e\}$, it has homology only in degree 0. There we find that

$$\underline{\mathrm{Ext}}^{0,*}_{\underline{\mathrm{TR}}^{\leq 2}_{-*}}(\underline{\mathrm{TR}}^{\leq 2}_{-*-\lambda_{1,1}}, \underline{\mathrm{TR}}^{\leq 2}_{-*})(C_p/e) \cong \mathrm{TR}^1_{-*}[i_4^*].$$

Thus the spectral sequence we are interested in collapses at the E^2-term in this case, giving us the result

$$\mathrm{TR}^1_{q+\lambda_{1,1}}(\mathbb{F}_p;p) \cong \begin{cases} \mathbb{Z}/p & \text{if } q \geq -4, \text{ even} \\ 0 & \text{else} \end{cases}$$

We now evaluate the complex at C_p/C_p. We find that

$$\underline{\mathrm{Ext}}^{0,*}_{\underline{\mathrm{TR}}^{\leq 2}_{-*}}(\underline{\mathrm{TR}}^{\leq 2}_{-*-\lambda_{1,1}}, \underline{\mathrm{TR}}^{\leq 2}_{-*})(C_p/C_p) \cong \mathrm{TR}^2_{-*}[pj_4^*, pj_2^* - \sigma j_4^*],$$

$$\underline{\mathrm{Ext}}^{1,*}_{\underline{\mathrm{TR}}^{\leq 2}_{-*}}(\underline{\mathrm{TR}}^{\leq 2}_{-*-\lambda_{1,1}}, \underline{\mathrm{TR}}^{\leq 2}_{-*})(C_p/C_p) \cong \mathrm{TR}^2_{-*}[pk_2^*].$$

For all $s > 1$

$$\underline{\mathrm{Ext}}^{s,*}_{\underline{\mathrm{TR}}^{\leq 2}_{-*}}(\underline{\mathrm{TR}}^{\leq 2}_{-*-\lambda_{1,1}}, \underline{\mathrm{TR}}^{\leq 2}_{-*})(C_p/C_p) = 0$$

So the E^2-term of our spectral sequence is given in figure 8-6

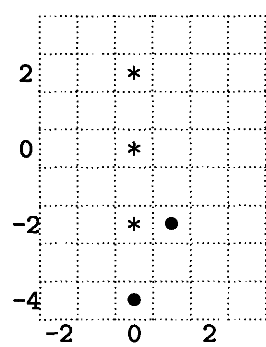

Figure 8-6: E^2-term of the $\underline{\mathrm{Ext}}$- Kunneth spectral sequence

$$\bullet = \mathbb{Z}/p \qquad\qquad * = \mathbb{Z}/p^2$$

For degree reasons there are no differentials. Hence it follows that

$$\mathrm{TR}^2_{q+\lambda_{1,1}}(\mathbb{F}_p;p) \cong \begin{cases} \mathbb{Z}/p^2 & \text{if } q \geq -2, \text{ even} \\ \mathbb{Z}/p & \text{if } q = -3 \\ \mathbb{Z}/p & \text{if } q = -4 \\ 0 & \text{else} \end{cases}$$

Further, the descriptions of these classes in the Ext computations determine the action of σ on $\mathrm{TR}^2_{q+\lambda_{1,1}}(\mathbb{F}_p;p)$. Thus, we can describe $\mathrm{TR}^2_{*+\lambda_{1,1}}(\mathbb{F}_p;p)$ as a $\mathrm{TR}^2_*(\mathbb{F}_p;p)$-module as follows:

$$\mathrm{TR}^2_{*+\lambda_{1,1}} \cong \mathrm{TR}^2_*[f_{-4}, g_{-2}, m_{-3}]/(pf_{-4}, pm_{-3}, \sigma m_{-3}\sigma f_{-4} - pg_{-2})$$

Note that this agrees with the computation in Theorem 1.0.1.

8.3 Future directions

While the above examples demonstrate the extra tools that this viewpoint provides, they only recreate results we could compute using other methods. However, these tools can hopefully be used to better understand the TR-groups indexed by odd-dimensional representations. Further, we would like to use these Mackey functor methods to study the product structure on $RO(S^1)$-graded TR, and a possible duality between the TR-groups for the representation α and the representation $-\alpha$.

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