Finite-Particle Representations and States of the Canonical Commutation Relations

by

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B.S., Carnegie Institute of Technology

(1960)

Submitted in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy

at the

Massachusetts Institute of Technology

September, 1966

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Chairman, Departmental
Committee on Graduate Students

ABSTRACT

Title of Thesis: Finite-Particle Representations and States of the Canonical Commutation Relations

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Submitted to the Department of Mathematics in June, 1966, in partial fulfillment of the requirement for the degree of Doctor of Philosophy.

A mathematical analysis is made of the existence of the formal number-of-particles operator $N = \Sigma a_k{}^* a_k$ for a representation of the canonical commutation relations, where a_k is the k^{th} annihilation operator. Using a natural rigorous definition of N as a limit of

$$\sum_{k=1}^{n} a_k{}^* a_k \text{ as } n \to \infty,$$

it is shown that N exists in uncountably many inequivalent irreducible representations; they are all described here. With an alternative definition of N it is proved that N exists only in the zero-interaction (Fock) representation.

A related result shows that every regular state of the Weyl algebra which has a finite number of particles with probability one is given by a density matrix in the zero-interaction representation.

Thesis Supervisor: Irving E. Segal

Title: Professor of Mathematics

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INTRODUCTION

In 1952, Friedrichs was the first to investigate the existence of the total number-of-particles operator for a representation of the canonical commutation relations. (See [8] for a reprint.) Among the representations which he studied, he found only the standard zero-interaction (Fock) representation to have a number operator. Then, in 1954, Gårding and Wightman [9] published the statement that there is only one irreducible representation of the commutation relations for which a number-of-particles operator exists, and in the following year Wightman and Schweber [28] published a proof of this statement. Their criterion for the existence of a number operator is $\sum_{k=1}^{\infty} a_k^* a_k$ should exist, where a_k is the k^{th} annihilation operator. But since this criterion involves the convergence of a sequence of unbounded operators $N_n = \sum_{k=1}^{n} a_k^* a_k$, it is possible to give several different mathematical meanings to the statement that the limit exists. Once a rigorous meaning has been given to the existence of the limit, it is then not entirely trivial to show that the limit exists in only one representation. Recently this problem has been formulated in a satisfactory way and solved by Dell'Antonio and Doplicher*, and we also give several other formulations and solutions in Section 6.

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*G.F. Dell'Antonio, private communication.

A complication is introduced by the fact that the accepted idea of what constitutes a representation of the canonical commutation relations has changed since the result of Wightman and Schweber was published. (We describe the change in Section 1.) If we use their definition of representation, then $\Sigma\, a_k^* \, a_k$ exists in only one representation; whereas if we use the present definition, this is no longer true. In fact, as we show in Section 5, any criterion for convergence of $\Sigma\, a_k^* \, a_k$ which can be proved to hold in the Fock representation is true in uncountably many inequivalent irreducible representations. We exhibit all of them more or less explicitly.

The existence of these strange representations suggests that either the present definition of a representation of the canonical commutation relations should be restricted in some way, or else the number operator should be defined differently. Only two ways of restricting the definition of representation seem reasonable. The first is to return to the definition used by Wightman and Schweber in their proof; but this is unacceptable since the present definition was made to allow the introduction of relativistic invariance, the description of a quantum field as an operator-valued distribution (see, for example [27]), and the development of an algebraic formulation of quantum field theory (see, for example, [25]). The second alternative is to insert into the definition precisely those continuity requirements needed to eliminate the strange representations for which $\Sigma a_k^* a_k$ exists. But this does not seem to be justifiable physically; indeed reasonable models exist for which the needed continuity requirements fail to hold.*

We shall show that in fact it is the definition of the number operator which is unsatisfactory. For a careful examination of what it means to say a state has a finite number of particles shows that none of the vectors in any of the strange representations for which $\Sigma a_k^* a_k$ exists corresponds to a state with a finite number of particles. We then discuss two possible criteria for the existence of a number operator N. Neither of them is any more complicated than a rigorous definition of $\Sigma a_k^* a_k$, and moreover they both have natural physical interpretations. One of them, which uses the bounded form of the commutation relation $Na = a(N-I)$, where a is an annihilation operator, can be found in the work of Segal [24]. This criterion is, however, unrelated to the convergence of the usual number operators for a finite number of degrees of freedom, so we also give a net-convergence criterion. It generalizes the usual sequential-convergence criterion and is proved to be equivalent to the criterion of Segal. Taking either of these criteria as a definition of N, we show that N exists only in the standard zero-interaction (Fock) representation.

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*J. Glimm, private communication.

In the process of proving this result we discuss methods of finding the number of particles in a state other than by using a total number operator. The most important of these involves the notion of the "probability of finding a finite number of particles" in a state. A state can have a finite number of particles with probability one even if the vector which represents the state is not in the domain of a number operator. But in Section 7 we prove that there are no unexpected states which have a finite number of particles with probability one -- they are all given by density matrices in the standard zero-interaction representation.

Section 1 contains the definitions needed for later sections. In Section 2 we describe in detail the number-of-particle operators in the Fock representation. Many of these results are well known but are proved here for the first time in a mathematically rigorous way. Section 3 contains an extension of certain number operators to arbitrary representations. In Section 4 we discuss the probabilistic interpretation of the number of particles in a state. Section 5 contains the examples of the strange representations for which $\Sigma\, a_k{}^* a_k$ exists. In Sections 6 and 7 we prove our main results which characterize the finite-particle representations and states.

1. WEYL SYSTEMS

The most satisfactory method of considering the problem of finding self-adjoint operators Q and P on some Hilbert space K satisfying the commutation relation

$$QP - PQ \subset iI \tag{1.1}$$

is to reformulate this relation in terms of bounded operators. The method first suggested by Weyl [26] is to let

$$U(s) = e^{isQ} \quad \text{and} \quad V(t) = e^{itP},$$

and to require that U and V satisfy the relation

$$U(s)\, V(t) = e^{-ist}\, V(t)\, U(s). \tag{1.2}$$

Soon afterward, von Neumann [14] found it convenient to consider the operators

$$e^{\frac{1}{2}ist}\, U(s)\, V(t)$$

which depend on two parameters s and t. We may think of them as depending on the single complex variable $z = s + it$, and write

$$W(s + it) = e^{\frac{1}{2}ist}\, U(s)\, V(t). \tag{1.3}$$

These are called the <u>Weyl operators</u>, and they satisfy

$$W(s+it)\, W(s'+it') = \exp[\tfrac{1}{2}\, i(ts'-st')]\, W((s+s') + i(t+t'))$$

or

$$W(z)\, W(z') = \exp[\tfrac{1}{2}\, i \operatorname{Im} z\bar{z}']\, W(z + z'), \tag{1.4}$$

which are called the <u>Weyl relations</u>. (The bar indicates complex conjugation.)

We shall need the following generalization from one degree of freedom to an arbitrary number of degrees of freedom, which is obtained by replacing $\bar{z}z'$ in (1.4) by an inner product (z,z').

DEFINITION 1.1. Let H be a complex inner product space. A Weyl system over H is a map $z \to W(z)$ which assigns to each $z \in H$ a unitary operator $W(z)$ on some complex Hilbert space K satisfying

(a) for every z and z' in H

$$W(z)\, W(z') = \exp[\tfrac{1}{2}\, i\, \mathrm{Im}(z,z')]\, W(z + z') \tag{1.5}$$

(the Weyl relations), and

(b) for every $z \in H$, if we consider $W(tz)$ as a function of the real variable t, then $t \to W(tz)$ is weakly continuous at zero.

We hasten to observe that from (a) it follows that $t \to W(tz)$ is a one-parameter group of unitaries, so the continuity assumption (b) is equivalent to strong continuity in t, which is precisely what one needs, according to Stone's Theorem (See, for instance [18]), to have a self-adjoint generator of the group. In other words, (b) is the minimal assumption required to be able to get P's and Q's from the W's.

Using the Weyl relations, one easily sees that (b) is equivalent to assuming that the function W is continuous from each finite-dimensional subspace of H into the strong operator topology. But it is not equivalent to assuming that W is continuous from all of H into the strong operator topology, if H is infinite-dimensional. That is, it is not necessarily true that, given $x \in K$, we can make $W(z)x$ close to

$W(z')x$ by choosing z' close enough to z in H. [This will be proved in Section 5.] It is important to keep this in mind for what follows. Different people will have different ideas as to what the space H should be - for some it will be a space of test functions with an L^2 inner product, for others a space of solutions to a differential equation - but it is not generally possible to give some physical reason why a Weyl system should be continuous on all of H.

The connection between what we have called a Weyl system and what is usually called a representation of the commutation relations is, as in the one-dimensional case, a simple matter of algebra. If we select a real-linear subspace H_R of H such that $H = H_R + iH_R$, and define

$$U(f) = W(f) \quad \text{if} \quad f\epsilon\ H_R$$
$$V(g) = W(ig) \quad \text{if} \quad g\epsilon\ H_R$$

then $$U(f)V(g) = \exp[-i\ \mathrm{Re}(f,g)]\ V(g)\ U(f)$$

so the pair U,V is what is called a representation of the canonical commutation relations. (See, for instance, [1]). In particular if we can select an orthonormal basis $\{e_1, e_2, \cdots\}$ (finite or countably infinite) of H, and then we take for H_R the real-linear span of these vectors, we will get

$$U(se_j)\ V(te_k) = \exp(ist\ \delta_{jk})\ V(te_k)\ U(se_j)$$
$$U(se_j)\ U(te_k) = U(te_k)\ U(se_j)$$
$$V(se_j)\ V(te_k) = V(te_k)\ V(se_j).$$

So we recognize the self-adjoint generator of $s \to U(se_j)$ as Q_j and the self-adjoint generator of $t \to V(te_k)$ as P_k.

However, the original idea of a representation of the commutation relations, referred to in the introduction, differed from the definition just given in that one allowed, as elements of H, only those vectors which were finite linear combinations of the vectors $e_1, e_2, \cdots$. The vector space of all such vectors is the algebraic span of the set $\{e_1, e_2, \cdots\}$. The selection of this space for H means that one is considering only operators of the form

$$\exp\left(i \sum_{j=1}^{n} s_j Q_j\right) \quad \text{and} \quad \exp\left(i \sum_{k=1}^{m} t_k P_k\right),$$

where n and m are some integers (Cf. [28]). In practice one observes that there is no natural way to select an orthonormal basis, so it is not meaningful to select H to be the algebraic span of a basis.

On several occasions we will need to use the Stone-von Neumann Theorem [14], which states that for a finite number of degrees of freedom there is only one Weyl system up to multiplicity, so we shall state it carefully here. We say that two Weyl systems over H, say W (acting on K) and W' (acting on K'), are unitarily equivalent if there is a unitary transformation U from K to K' such that for every $z \in H$

$$W'(z) = UW(z)\, U^{-1}.$$

THEOREM (STONE - VON NEUMANN):

If M *is an n-dimensional complex inner product space,* and W is a Weyl system over M, then W is unitarily equivalent to a direct sum (possibly not countable) of copies of a Schrödinger Weyl system W_s over M, which is defined as follows: The representation space K_s is $L^2(\mathbb{R}^n)$. Select a basis of M so that we may think of M as $\mathbb{R}^n + i\,\mathbb{R}^n$. Then for every $f \in L^2(\mathbb{R}^n)$

$$W_s(x+iy)f(u) = e^{\frac{1}{2} ix\cdot y}\, e^{iu\cdot x}\, f(u+y). \tag{1.6}$$

The Schrödinger Weyl system is irreducible, which means that no unitary operator on $L^2(\mathbb{R}^n)$ commutes with all the $W_s(z)$ unless it is a multiple of the identity. The P's and Q's which come from it are the familiar ones:

$$P_k = \frac{1}{i}\frac{\partial}{\partial x_k} \quad \text{and} \quad Q_j = \text{multiplication by } x_j.$$

It has been possible to generalize the Schrödinger representation to the case where H is infinite-dimensional [20]. But because there is no adequate generalization of Lebesgue measure to an infinite number of dimensions, the measure which is used in the infinite-dimensional case (the normal distribution) is a generalization of the measure ν on $\mathbb{R}^n$ given by $d\nu(x) = \pi^{-n/2}\, e^{-|x|^2}\, d^n x$. If we transform the Schrödinger Weyl system defined by (1.6) into the equivalent system W_o on $L^2(\mathbb{R}^n,\nu)$ (using the unitary transformation $f(x) \to \pi^{\frac{1}{4}n}\, e^{\frac{1}{2}|x|^2}\, f(x)$), we get operators which have the same appearance as their infinite-dimensional generalization:

$$W_o(x+iy)\ f(u) = e^{\frac{1}{2}i(x,y)}\, e^{i(u,x)}\, e^{-(u,y)-\frac{1}{2}|y|^2}\, f(u+y). \tag{1.7}$$

2. THE STANDARD ZERO-INTERACTION SYSTEM

The generalization of the Schrödinger Weyl system W_o to the case where H is infinite-dimensional acts on $L^2(H_R,\nu)$ in a manner which is roughly indicated by (1.7). (Here H_R is a real-linear subspace of the completion H' of H such that $H' = H_R + iH_R$, and ν is the normal distribution.) W_o is unitarily equivalent to several other systems which appear quite different. One of these is given by the holomorphic functional representation [24], another by the Fock-Cook representation [7; 3]. When we do not mean to specify a particular one of these unitarily equivalent systems, we shall refer to the <u>standard zero-interaction Weyl system</u>. It is known to be irreducible whether H is complete or not [3; 21], and it is continuous on all of H. (See [20, Th. 4 and Cor. 3.3] or [2].)

Let us briefly review how it is defined. We follow Cook [3] who gave a basis-free description. We suppose here that H is a Hilbert space, i.e. is complete. (But we do not exclude the possibility that H is finite-dimensional.)

We denote by H^n the n-fold tensor product of H with itself, with H^o = the complex numbers. By S_n we denote the projection of H^n onto its symmetric subspace:

$$S_n(u_1 \otimes \cdots \otimes u_n) = \frac{1}{n!} \sum_{\sigma} u_{\sigma 1} \otimes \cdots \otimes u_{\sigma n}$$

where the sum is over all permutations σ of $\{1,\cdots,n\}$.

The representation space of the Fock-Cook system is

$$H_F = H^o \oplus H \oplus S_2 H^2 \oplus S_3 H^3 \oplus \cdots .$$

If $z \in H$, we define a bounded linear operator
$a_n(z) : H^n \to H^{n-1}$ by

$$a_n(z)(u_1 \otimes \cdots \otimes u_n) = (u_1, z)\ u_2 \otimes \cdots \otimes u_n$$

(extending to H^n by linearity and continuity). Then we define the annihilation operator for a particle with wavefunction z on H_F by

$$a(z) = 0 \oplus a_1(z) \oplus 2^{1/2}\ a_2(z) \oplus 3^{1/2}\ a_3(z) \oplus \cdots \quad (2.1)$$

$a(z)$ is a closed, unbounded linear operator on H_F. Its adjoint $a^*(z)$ is the creation operator for a particle of wavefunction z. It has the form

$$a^*(z) = a_1^*(z) \oplus 2^{\frac{1}{2}}\ S_2 a_2^*(z) \oplus 3^{\frac{1}{2}}\ S_3 a_3^*(z) \oplus \cdots \quad (2.2)$$

where $a_n^*(z) : H^{n-1} \to H^n$ is defined by

$$a_n^*(z)\ (u_1 \otimes \cdots \otimes u_n) = z \otimes u_1 \otimes \cdots \otimes u_n.$$

If we define $R(z)$ to be the closure of the operator $2^{-\frac{1}{2}}\ [a^*(z) + a(z)]$, then $R(z)$ is self-adjoint [3, p. 231], and furthermore if we define

$$W_o(z) = e^{iR(z)} \quad (2.3)$$

then W_o is a Weyl system over H [20], which is the standard zero-interaction Weyl system.

PARTICLE INTERPRETATION

The particle interpretation of the vectors in H_F is in keeping with the terminology we have been using. The vector $1 \oplus 0 \oplus 0 \oplus \cdots$ is interpreted to represent the vacuum, and the vector which is produced by applying $a^*(z)$ to the vacuum is interpreted to represent a single particle with wavefunction z. If we then apply $a^*(z')$ to this one-particle state, we get a vector representing a two-particle state, but unless z' is perpendicular to z this state is not interpreted to have exactly one particle of type z and one of type z'. Let us state carefully how one makes a particle interpretation of an arbitrary vector in H_F.

For each closed subspace M of H we will specify which elements of H_F have exactly k particles with wavefunctions in M. First let us look at vectors in H^n. Let $P_M = P_M(1)$ be the projection of H onto M, and let $P_M(0) = I - P_M(1)$ be the projection onto the orthocomplement of M. Then, since $I = P_M(0) + P_M(1)$, the identity operator on H^n is given by

$$I^n = [P_M(0) + P_M(1)] \otimes \cdots \otimes [P_M(0) + P_M(1)]$$

$$= \sum_{\alpha\epsilon\{0,1\}^n} P_M(\alpha_1) \otimes \cdots \otimes P_M(\alpha_n)$$

where $\{0,1\}^n$ is the set of all n-tuples of zeros and ones. If $\alpha\epsilon\{0,1\}^n$, let $|\alpha| = \sum_{j=1}^{n} \alpha_j$. Then we have

$$I^n = \sum_{k=0}^{n} \sum_{|\alpha|=k} P_M(\alpha_1) \otimes \cdots \otimes P_M(\alpha_n). \qquad (2.4)$$

Now the operator

$$A_k = \sum_{|\alpha|=k} P_M(\alpha_1) \otimes \cdots \otimes P_M(\alpha_n) \qquad (2.5)$$

contains all the terms in (2.4) in which exactly k P_M's show up. Furthermore A_k is a projection, since it is clearly self-adjoint, and from

$$P_M(i)\; P_M(j) = \delta_{ij}\; P_M(i)$$

we have

$$A_k{}^2 = \left[\sum_{|\alpha|=k} P_M(\alpha_1) \otimes \cdots \otimes P_M(\alpha_n)\right]\left[\sum_{|\beta|=k} P_M(\beta_1) \otimes\cdots\otimes P_M(\beta_n)\right]$$

$$= \sum_{|\alpha|=|\beta|=k} P_M(\alpha_1)P_M(\beta_1) \otimes \cdots \otimes P_M(\alpha_n)P_M(\beta_n)$$

$$= \sum_{|\alpha|=k} P_M(\alpha_1) \otimes \cdots \otimes P_M(\alpha_n)$$

$$= A_k .$$

Now the projection A_k leaves $S_n\ H^n$ invariant, since any permutation of a term in the sum (2.5) is another term in the sum. The image of $S_n\ H^n$ under A_k is precisely what is meant by the subspace of $S_n\ H^n$ consisting of vectors with exactly k particles in M. So we adopt the following notation.

DEFINITION 2.1. For each closed subspace M of H, define $P_k{}^n(M)$ to be the projection

$$A_k = \sum_{|\alpha|=k} P_M(\alpha_1) \otimes \cdots \otimes P_M(\alpha_n)$$

restricted to $S_n\ H^n$. (And for convenience define $P_o{}^o(M) = 1$, $P_k{}^n(M) = 0$ if $k > n$.)

PROPOSITION 2.1. $P_k{}^n(M)\ P_\ell{}^n(M) = \delta_{k\ell}\ P_k{}^n(M)$.

<u>Also</u> $\sum_{k=0}^{n} P_k{}^n(M) =$ <u>identity on</u> $S_n\ H^n$.

Proof: If $\ell = k$, the first statement is the same as saying $P_k{}^n(M)$ is a projection. So suppose $\ell > k$. Then any term in the sum defining $P_\ell{}^n(M)$ has at least one more $P_M(1)$

than any term in the sum for $P_k^n(M)$. When the two terms are multiplied the extra $P_M(1)$ will multiply $P_M(0)$ giving zero, so the product of the two terms is zero. The other part of the proposition is a restatement of (2.4).

PROPOSITION 2.2. _If_ P_M _commutes with_ $P_{M'}$, _then_ $P_k^n(M)$ _commutes with_ $P_\ell^n(M')$.

Proof: This is obvious from the fact that $P_M(i)$ commutes with $P_{M'}(j)$ for $i,j = 1$ or 2.

Now observe that the operator B_n on H_F defined by

$$B_n = P_k^0(M) \oplus P_k^1(M) \oplus \cdots \oplus P_k^n(M) \oplus 0 \oplus 0 \oplus \cdots$$

is a projection, and if $n' > n$, then $B_{n'} \geq B_n$. Hence $\text{st-lim}_{n\to\infty} B_n$ exists and is a projection which we shall call $P_k(M)$.

DEFINITION 2.2. The projection $P_k(M)$ on H_F is defined by

$$P_k(M) = \bigoplus_{n=0}^{\infty} P_k^n(M).$$

Its range is called the _subspace of vectors which have exactly_ k _particles with wavefunctions in_ M, or the k-_particle subspace over_ M.

From Proposition 2.1 we know that

$$\sum_{k=0}^{\infty} P_k(M) = I, \qquad (2.6)$$

and from Proposition 2.2 we know that if P_M commutes with $P_{M'}$, then $P_k(M)$ commutes with $P_\ell(M')$.

By selecting appropriate orthonormal bases, we can exhibit the projections we have just defined in familiar form. First we choose an orthonormal basis $\{e_\gamma : \gamma\varepsilon\Gamma_o\}$ of M, and then we extend it to an orthonormal basis $\{e_\gamma : \gamma\varepsilon\Gamma\}$ of H. From this we can construct an orthonormal basis of H_F whose typical element we shall denote by $\Pi_{\gamma\varepsilon\Gamma}\ e_\gamma{}^{n(\gamma)}$. Here n is any nonnegative-integer-valued function on Γ such that only finitely many values, say $n(\gamma_1),\cdots, n(\gamma_k)$, are not zero; and the symbol $\Pi_\gamma\ e_\gamma{}^{n(\gamma)}$ stands for the vector which results from symmetrizing and normalizing

$$\left[\otimes^{n(\gamma_1)}\ e_{\gamma_1}\right]\otimes\left[\otimes^{n(\gamma_2)}\ e_{\gamma_2}\right]\otimes\cdots\otimes\left[\otimes^{n(\gamma_k)}\ e_{\gamma_k}\right]$$

Every element of this basis is either in the range of $P_k(M)$ or in its nullspace; indeed $\Pi_\gamma\ e_\gamma{}^{n(\gamma)}$ is in the range of $P_k(M)$ if and only if the number of factors which lie in M is k, i.e. if and only if $\Sigma_{\gamma\varepsilon\Gamma_o}\ n(\gamma) = k$.

NUMBER OPERATORS

Now for any unit vector $z\varepsilon H$, let us see the connection between the familiar "number operator" $N(z) = a^*(z)a(z)$ and the projections $P_k([z])$. ($[z]$ is the one-dimensional subspace of H spanned by z.) Choose an orthonormal basis $\{e_\gamma : \gamma\varepsilon\Gamma\}$ of H such that $0\varepsilon\Gamma$ and $e_o = z$. Then each basis vector $\Pi e_\gamma{}^{n(\gamma)}$ of H_F is an eigenvector of $N(z)$ with eigenvalue $n(0)$. For if $n(0) = 0$, then

$$a(z)\ \Pi e_\gamma{}^{n(\gamma)} = 0$$

by the definition (2.1). Now supposing $n(0) \neq 0$, $\Pi e_\gamma^{n(\gamma)}$ has the form

$$c\, S_m\, (u_1 \otimes \cdots \otimes u_m),$$

where c is a normalization constant, the first n(0) u_i's equal z, and all the other u_i's are perpendicular to z. Thus

$$a(z) \prod_\gamma e_\gamma^{n(\gamma)} = c\, m^{\frac{1}{2}} \frac{1}{m!} \sum_\sigma{}' u_{\sigma 2} \otimes \cdots \otimes u_{\sigma m}$$

where the prime on the summation sign means we sum over only those permutations σ for which $u_{\sigma 1} = z$. (There are $n(0)(m-1)!$ such permutations.)

So we have

$$\begin{aligned} N(z) \Pi e_\gamma^{n(\gamma)} &= a^*(z) a(z) \Pi e_\gamma^{n(\gamma)} \\ &= c\, S_m\, m \frac{1}{m!} \sum_\sigma{}' z \otimes u_{\sigma 2} \otimes \cdots \otimes u_{\sigma m} \\ &= c\, S_m \frac{1}{(m-1)!} \sum_\sigma{}' u_{\sigma 1} \otimes u_{\sigma 2} \otimes \cdots \otimes u_{\sigma m} \\ &= c \frac{1}{(m-1)!}\, n(0)\, (m-1)!\, S_m(u_1 \otimes \cdots \otimes u_m) \\ &= n(0)\, c\, S_m\, (u_1 \otimes \cdots \otimes u_m) \\ &= n(0)\, \Pi e_\gamma^{n(\gamma)}. \end{aligned}$$

What we have shown is that every basis vector in $P_k([z]) H_F$ is an eigenvector of N(z) with eigenvalue k. Since N(z) is self-adjoint (in particular closed), it follows that the range of $P_k([z])$ consists entirely of eigenvectors of N(z) with eigenvalue k. Now consider the self-adjoint operator N' whose spectral resolution is $\sum_{k=0}^{\infty} k\, P_k([z])$. For any u in the domain of N', let $u_n = \sum_{k=0}^{n} P_k([z])\, u$. Then $u_n \to u$ and $N'u_n \to N'u$. But $N'u_n = N(z)u_n$, so $N(z)u_n$ also converges to $N'u$. Hence u is in the domain N(z) and $N(z)u = N'u$. Thus $N(z) \supset N'$, so since both N(z) and N' are self-adjoint we have

$$N(z) = N' = \sum_{k=0}^{\infty} k\, P_k([z]).$$

We make this result into a definition.

DEFINITION 2.3. For every closed subspace M of H, define the <u>number operator over</u> M <u>for the zero-interaction representation</u> to be the non-negative self-adjoint operator given by the spectral resolution

$$N_o(M) = \sum_{k=0}^{\infty} k\, P_k(M).$$

$N_o(H)$ is also called the <u>total number operator</u>. One sees from the definition that it has the form

$$N_o(H) = 0 \oplus 1 \oplus 2 \oplus 3 \oplus \cdots$$

on H_F.

According to this definition, what we have proved above is that $N([z]) = a^*(z)a(z)$.

Since the spectral projections of $N_o(M)$ commute with those of $N_o(M')$ if P_M commutes with $P_{M'}$ (See the remarks after Definition 2.2), we conclude that $N_o(M)$ commutes with $N_o(M')$ if P_M commutes with $P_{M'}$.

Now we shall give a characterization of $N_o(M)$ in terms of the relation of the unitary group it generates to the standard zero-interaction Weyl system W_o. This is the criterion we shall use to characterize number operators for systems other than the standard zero-interaction system. It states, in bounded form, the pair of commutation relations $Na^*(z) = a^*(z)(N + I)$ and $Na(z) = a(z)(N - I)$.

PROPOSITION 2.3. Let M be a closed subspace of H, P_M the projection of H onto M. Then for all $t \in \mathbb{R}$

$$e^{itN_o(M)} W_o(z) e^{-itN_o(M)} = W_o(e^{itP_M} z). \tag{2.7}$$

Proof:

First suppose $z \in M$. We will show

$$e^{itN_o(M)} a^*(z) e^{-itN_o(M)} = a^*(e^{it} z).$$

If $z = 0$, this is obvious. So suppose $z \neq 0$. Select an orthonormal basis $\{e_\gamma : \gamma \in \Gamma_o\}$ of M such that $0 \in \Gamma_o$ and $e_o = \frac{z}{|z|}$ and extend to an orthonormal basis $\{e_\gamma : \gamma \in \Gamma\}$ of H. Then if

$$x = \sum_n c(n) \prod_\gamma e_\gamma^{n(\gamma)}$$

is in H_F, we have

$$e^{-itN_o(M)} x = \sum_n \exp(-it \sum_{\gamma \in \Gamma_o} n(\gamma))\, c(n) \prod_\gamma e_\gamma^{n(\gamma)}.$$

Now since the domain of $a^*(z)$ is the set of x for which $\Sigma\, n(0)\, |c(n)|^2 < \infty$, which is also the domain of $a^*(e^{it} z)$ [3, p. 228], and since multiplying $c(n)$ by e^{itk} has no effect on this sum, we see that $e^{-itN_o(M)} x$ is in the domain of $a^*(z)$ if and only if x is in the domain of $a^*(e^{it} z)$. Further, for such an x we have

$$e^{itN_o(M)} a^*(z) e^{-itN_o(M)} x$$

$$= e^{itN_o(M)} \sum_n [n(0)]^{\frac{1}{2}} \exp(-it \sum_{\gamma\epsilon\Gamma_o} n(\gamma))c(n)|z| e_o^{n(0)+1} \prod_{\gamma\neq 0} e_\gamma^{n(\gamma)}$$

$$= e^{it} \sum_n [n(0)]^{\frac{1}{2}} c(n)|z| \; e_o^{n(0)+1} \prod_{\gamma\neq 0} e_\gamma^{n(\gamma)}$$

$$= a^*(e^{it} z) \; x.$$

Hence

$$e^{itN_o(M)} a^*(z) e^{-itN_o(M)} = a^*(e^{it} z) \text{ if } z\epsilon M.$$

From this we get

$$e^{itN_o(M)} a(z) e^{-itN_o(M)} = a(e^{\,it} z) \text{ if } z\epsilon M.$$

Now suppose z is orthogonal to M, $z \neq 0$. This time select an orthonormal basis $\{e_\gamma : \gamma\epsilon\Gamma_1\}$ of M such that $0 \notin \Gamma_1$, and extend to an orthonormal basis $\{e_\gamma : \gamma\epsilon\Gamma\}$ such that $0\epsilon\Gamma$ and $e_o = |z|^{-1}z$. As before the domain of $a^*(z)e^{-itN_o(M)}$ is the domain of $a^*(z)$; and if $x = \Sigma\, c(n)\Pi e_\gamma^{n(\gamma)}$ is in that domain

$$e^{itN_o(M)} a^*(z) e^{-itN_o(M)} x$$

$$= e^{itN_o(M)} \sum_n [n(0)]^{\frac{1}{2}} \exp\left(-it \sum_{\gamma\epsilon\Gamma_1} n(\gamma)\right) c(n)|z| e_o^{n(0)+1} \prod_{\gamma\neq 0} e_\gamma^{n(\gamma)}$$

$$= \sum_n [n(0)]^{\frac{1}{2}} c(n)|z| \; e_o^{n(0)+1} \prod_{\gamma\neq 0} e_\gamma^{n(\gamma)}$$

$$= a^*(z) \; x.$$

So $e^{itN_o(M)} a^*(z) e^{-itN_o(M)} = a^*(z)$ if $z \perp M$, and consequently $e^{itN_o(M)} a(z) e^{-itN_o(M)} = a(z)$ if $z \perp M$.

Finally, for arbitrary z, using the fact that $a^*(z)$ is the closure of $a^*(P_M z) + a^*((I - P_M)z)$ [3, p. 225], we have

$$e^{itN_o(M)} a^*(z) e^{itN_o(M)} = a^*(e^{it} P_M z + (I - P_M)z)$$
$$= a^*(e^{itP_M} z)$$

and similarly for $a(z)$. Then using the fact that $R(z)$ is the closure of $2^{-1/2} [a(z) + a^*(z)]$, we find

$$e^{itN_o(M)} R(z) e^{itN_o(M)} = R(e^{itP_M} z).$$

Since $W_o(z)$ is defined to be $e^{iR(z)}$, the Proposition is proved.

PROPOSITION 2.4: <u>If</u> $N'(M)$ <u>is any self-adjoint operator on the standard zero-interaction space which satisfies</u>

$$e^{itN'(M)} W_o(z) e^{-itN'(M)} = W_o(e^{itP_M} z),$$

<u>then</u> $N'(M) = N_o(M) + aI$ <u>for some real number</u> a. <u>If also</u> $N'(M)$ <u>annihilates the vacuum then</u> $N'(M) = N_o(M)$.

Proof;

We see from the hypothesis and (2.7) that

$$\left[e^{-itN_o(M)} e^{itN'(M)} \right] W_o(z) \left[e^{-itN_o(M)} e^{itN'(M)} \right]^{-1}$$
$$= W_o(z) \text{ for every } z$$

so $e^{-itN_o(M)} e^{itN'(M)}$ commutes with every $W_o(z)$. By irreducibility,

$$e^{-itN_o(M)} e^{itN'(M)} = c(t)I$$

where $c(t)$ is some complex number such that $|c(t)| = 1$.

One easily sees that $c(t_1 + t_2) = c(t_1)\ c(t_2)$ and c is continuous, so $c(t) = e^{iat}$ for some real a (See [17], p. 140). From this the proposition follows easily.

Proposition 2.4 is a uniqueness result which allows us to identify number operators in various guises. As an example of its use we prove a result which could also be proved using the unbounded operators.

COROLLARY 1. <u>If</u> M <u>is finite-dimensional, and</u> $\{e_1, e_2, \cdots, e_n\}$ <u>is an orthonormal basis of</u> M, <u>then</u>

$$N_o(M) = \sum_{k=1}^{n} a^*(e_k)\ a(e_k) = \sum_{k=1}^{n} N_o([e_k]).$$

(This shows the result doesn't depend on what basis is chosen.)

Proof: Using the fact that $N_o([e_k])$ commutes with $N_o([e_j])$ we have

$$\exp\left(it \sum_{k=1}^{n} N_o([e_k])\right) W_o(z) \exp\left(-it \sum_{k=0}^{n} N_o([e_k])\right)$$

$$= \prod_{k=1}^{n} \exp\left(itN_o\ ([e_k])\right) \quad W_o(z) \prod_{k=1}^{n} \exp\left(-it\ N_o([e_k])\right)$$

$$= W_o \left(\prod_{k=1}^{n} \exp(it\ P_{[e_k]})\ z\right)$$

$$= W_o\ (\exp(it\ P_M)\ z)$$

Now use Proposition 2.4.

COROLLARY 2. <u>If</u> $\{M_k\}$ <u>is any sequence of subspaces of</u> H <u>such that</u> $\{P_{M_k}\}$ <u>converges strongly to</u> I, <u>then</u> $\exp(it\ N_o(M_k))$

converges strongly to $\exp(it\ N_o(H))$ in the standard zero-interaction space.

Proof:

Let v be the vacuum vector. By the irreducibility of the standard zero-interaction system we know that the set $\{W_o(z)\ v : z \epsilon H\}$ generates (algebraically) a dense subset of H_F.

But from (2.7) and the fact that $N_o(M)$ annihilates the vacuum, we have

$$\exp(itN_o(M_k))\ W_o(z)\ v = W_o(\exp(it\ P_{M_k})\ z)\ v. \qquad (2.8)$$

Since

$$\exp(it\ P_{M_k})z = e^{it}\ P_{M_k} z + (I - P_{M_k})z$$

which converges to $e^{it}\ z$, and since the standard zero-interaction system is continuous on H [20; 2],

$$W_o(\exp(itP_{M_k})\ z) \to W(e^{it}\ z).$$

It follows from (2.8) that the sequence $\exp(it\ N_o(M_k))$ converges strongly on a dense subset of H_F, hence on all of H_F.

Let $U(t) = \text{st-lim}_{k\to\infty} \exp(it\ N_o(M_k))$. Since

$$U(t)\ W_o(z)\ v = W_o(e^{it}\ z)\ v$$

we see that U is a strongly continuous one-parameter unitary group on H_F. If N is its self-adjoint generator, then evidently N annihilates the vacuum, since

$$e^{itN}\ v = v \text{ for all } t.$$

Furthermore

$$e^{itN} W_o(z)\, e^{-itN} = \text{st-lim}_{k\to\infty} \exp(itN_o(M_k))\, W_o(z)\, \exp(-itN_o(M_k))$$

$$= W_o(e^{it} z).$$

Hence, from Proposition 2.4, $N = N_o(H)$.

Corollaries 1 and 2 together give us a rigorous way of stating that if H is separable and $\{e_1, e_2, \cdots\}$ is any orthonormal basis of H, then $N_o(H) = \sum_{k=1}^{\infty} a^*(e_k)\, a(e_k)$. Namely, for every t

$$\text{st-lim}_{n\to\infty} \exp\left(it \sum_{k=1}^{n} a^*(e_k)\, a(e_k)\right) = e^{itN_o(H)}.$$

FOR INCOMPLETE INNER PRODUCT SPACES.

Throughout this section we have been assuming H is a Hilbert space. If, however, we allow H to be an incomplete inner product space, then all the results of this section apply to the completion H' of H. If W_o' is the standard zero-interaction Weyl system over H', acting on H_F', we define the <u>standard zero-interaction Weyl system</u> W_o <u>over</u> H by

$$W_o(z) = W_o'(z) \quad \text{for all } z \in H.$$

Then the standard zero-interaction system over H acts on H_F'.

Similarly, if M is any closed subspace of H' such that $M \subset H$, we define the number operator $N_o(M)$ just as before. In particular $N_o(M)$ is defined for every finite-dimensional subspace M of H.

REMARK: It is often stated that a number operator does not exist in a representation other than the standard zero-interaction representation because the representation space has a basis $\{x_\gamma\}$ consisting of vectors with an infinite number of particles. We have purposely not made this statement very precise, for we will discuss it in detail in Section 4, but it has something to do with the fact that

$$\sum_{k=1}^{\infty} \langle a^*(e_k)\ a(e_k)\ x_\gamma,\ x_\gamma \rangle = +\infty.$$

We wish to point out that if we make this interpretation, then the argument is not very convincing, since <u>even in the standard zero-interaction representation</u> one can find a basis having this property. One has to expect that an unbounded operator will have a great many vectors not in its domain, so that finding a basis consisting of them is not surprising and doesn't prove that the operator does not exist.

3. GENERAL NUMBER OPERATORS

Let W be a Weyl system over H on K. Although there may not be a total number operator for W, we will now show that for each finite-dimensional subspace M of H, there is a number operator analogous to the operator $N_o(M)$ for the zero-interaction representation.

PROPOSITION 3.1. <u>For every finite-dimensional subspace</u> M <u>of</u> H <u>there exists a unique non-negative self-adjoint operator</u> N(M) <u>on</u> K <u>such that</u>

(a) $0 \in$ <u>spectrum</u> N(M).

(b) $e^{itN(M)}$ <u>is in the weakly closed algebra generated by</u> $\{W(z) : z \in M\}$ <u>for every</u> $t \in \mathbb{R}$.

(c) $e^{itN(M)} W(z) e^{-itN(M)} = W(e^{it} z)$ <u>for every</u> $z \in M$, $t \in \mathbb{R}$.

<u>In addition,</u> N(M) <u>has the following properties</u>:

(d) <u>There exists a Hilbert space</u> X <u>and a unitary operator</u> V <u>from the tensor product</u> $M_F \otimes X$ <u>of the standard zero-interaction space</u> M_F <u>over</u> M <u>with</u> X <u>onto</u> K <u>such that</u>

$$W(z) = V[W_o(z) \otimes I] V^{-1} \quad \text{for every } z \in M$$

<u>and</u>

$$e^{itN(M)} = V[e^{itN_o(M)} \otimes I] V^{-1}$$

$$= \sum_{k=0}^{\infty} e^{itk} V[P_k(M) \otimes I] V^{-1}$$

<u>where</u> I <u>is the identity on</u> X. (See Definitions 2.2 and 2.3.)

(e) $e^{itN(M)} W(z) e^{-itN(M)} = W(e^{itP_M} z)$

for all $z \in H$, where P_M is the projection of H onto M.

DEFINITION 3.1. The operator $N(M) = N(M;W)$ on K specified by Proposition 3.1 is called the number operator over M for W. Its k^{th} spectral projection is denoted $P_k(M;W)$:

$$N(M;W) = \sum_{k=0}^{\infty} k \, P_k(M;W).$$

The range of $P_k(M;W)$ is called the k-particle subspace over M for W. Even if M is not finite-dimensional there may exist an operator $N(M)$ satisfying (c):

$$e^{itN(M)} W(z) e^{-itN(M)} = W(e^{it} z)$$

for every $z \in M$. In this case we say there exists a number operator over M for W.

REMARKS:

(1) We can rephrase the Proposition as follows. Denoting by $\mathcal{A}_M(W)$ the weakly closed algebra generated by $\{W(z): z \in M\}$, the Proposition says that the map $W(z) \to W(e^{it} z)$ induces an inner automorphism of $\mathcal{A}_M(W)$, and there is a unique one-parameter unitary group in $\mathcal{A}_M(W)$ which has non-negative self-adjoint generator, induces the automorphism, and leaves some vector invariant.

(2) Using the Proposition one can show that the number operators $N(M;W)$ are related in the correct way to the relevant creation and annihilation operators found from W, but we shall not need this in what follows.

To prove the Proposition we will need

LEMMA. Let U_o be a continuous one-parameter unitary group on a Hilbert space K_o whose self-adjoint generator has spectral resolution $\sum_{k=0}^{\infty} k\, P_k$. Then for any Hilbert space X, the operators $U(t) = U_o(t) \otimes I$ on $K_o \otimes X$ form a continuous one-parameter unitary group whose self-adjoint generator is

$$\sum_{k=0}^{\infty} k\,(P_k \otimes I).$$

Proof:

Let $B(K_o)$ be the algebra of all bounded operators on K_o. The map $\varphi: B(K_o) \to B(K_o \otimes X)$ given by $\varphi(A) = A \otimes I$ is continuous (and has continuous inverse) with respect to the strongest (ultraforte) operator topology [15; or 4, p. 57]. Moreover, the strongest topology coincides with the strong topology on the unit ball of the operators [4, p. 36]. So we see that $U = U_o \otimes I$ is a strongly continuous one-parameter unitary group; and also, since $\sum_{k=0}^{n} e^{itk} P_k$ converges to $U_o(t)$ in the strongest topology as $n \to \infty$ (by the spectral theorem), we see that

$$U(t) = \text{st-lim}_{n\to\infty} \sum_{k=0}^{n} e^{itk}\,(P_k \otimes I).$$

Hence the self-adjoint generator of U is

$$\sum_{k=0}^{\infty} k\,(P_k \otimes I).$$

Proof of Proposition 3.1:

By the Stone-von Neumann Theorem (Sec. 1), the restriction of W to M is unitarily equivalent to a direct sum of Schrödinger Weyl systems over M. By the same theorem again, and the fact that the standard zero-interaction Weyl system W_o over M is irreducible, we conclude that each Schrödinger system is unitarily equivalent to W_o acting on M_F. Hence we can find a Hilbert space X and a unitary operator V from $M_F \otimes X$ onto K such that

$$W(z) = V[W_o(z) \otimes I]\ V^{-1}$$

for every $z \in M$. (This is simply an alternative description of a direct sum. See [4, pp. 23-24]).

Let $$U(t) = V[e^{itN_o(M)} \otimes I]\ V^{-1}.$$

By the Lemma, U is a continuous one-parameter unitary group, and if we call N(M) its self-adjoint generator then

$$N(M) = \sum_{k=0}^{\infty} V[P_k(M) \otimes I]\ V^{-1}.$$

So, for this choice of N(M), (a) and (d) are true.

To see that (b) is true observe that, because W_o is irreducible, the weakly closed algebra generated by $\{W_o(z) : z \in M\}$ is $B(M_F)$ —— all bounded operators on M_F. Then since the weak closure of a *-algebra equals its strongest closure [4, p. 43], we see that the weakly closed algebra generated by $\{W(z) : z \in M\}$ is $V[B(M_F) \otimes I]\ V^{-1}$. Evidently $e^{itN(M)}$ is in this algebra.

To prove (c): If $z \in M$

$$U(t)\ W(z)\ U(-t) = V\left[e^{itN_o(M)} \otimes I\right] V^{-1}\ V[W_o(z) \otimes I]\ V^{-1}$$

$$\times\ V\left[e^{-itN_o(M)} \otimes I\right] V^{-1}$$

$$= V[W_o(e^{it}\ z) \otimes I]\ V^{-1}$$

$$= W(e^{it}\ z).$$

From (b) and (c) we get (e). For if y is perpendicular to M, $W(y)$ commutes with $\{W(z) : z \in M\}$, so by (b) $W(y)$ commutes with $e^{itN(M)}$. Hence for any $z \in H$

$$e^{itN(M)}\ W(z)\ e^{-itN(M)} = e^{itN(M)}\ W(P_M z)\ W((I - P_M)z)\ e^{-itN(M)}$$

$$= W(e^{it}\ P_M z)\ W((I - P_M)z)$$

$$= W((e^{it}\ P_M + I - P_M)z)$$

$$= W(e^{itP_M}\ z)$$

Now to prove that $N(M)$ is the unique non-negative operator satisfying (a), (b), and (c), suppose N' is another. Then by (b)

$$e^{itN'} \in V[B(M_F) \otimes I]\ V^{-1}$$

So there is an operator $S(t)$ on M_F such that

$$S(t) \otimes I = V^{-1}\ e^{itN'}\ V$$

It follows by the strongest continuity of the map $A \otimes I \to A$ that S is a strongly continuous one-parameter group.

Using (c) and Proposition 2.4, we conclude that

$$S(t) = e^{it[N_o(M)+aI]}$$

for some real a. But then since the spectrum of N' is non-negative and contains 0, a must be zero.

REMARK: If W_o is the standard zero-interaction Weyl system over H and M is a finite-dimensional subspace of H, we now have two different definitions of a number operator over M. One is the operator $N_o(M)$ of Definition 2.3, and the other is the operator $N(M;W_o)$ given by Proposition 3.1. But they are, of course, the same operator. For Proposition 3.1 tells us that $N(M;W_o)$ is non-negative with zero in its spectrum, and satisfies

$$e^{itN(M;W_o)} W_o(z) e^{itN(M;W_o)} = W_o(e^{itP_M} z).$$

Hence from Proposition 2.4 we conclude that $N(M; W_o) = N_o(M) + aI$ for some real number a. But then the spectrum condition on $N(M;W_o)$ implies a = 0. So we conclude $N(M;W_o) = N_o(M)$.

DEFINITION 3.2. For W a Weyl system over H acting on K, and M a finite-dimensional subspace of H, define

$Q_n(M;W) = \sum_{k=0}^{n} P_k(M;W)$ (see Definition 3.1). Then $Q_n(M;W)$ is the projection of K onto its <u>subspace consisting of vectors with</u> n <u>or fewer particles with wavefunctions in</u> M.

Suppose now that M' ⊂ M. Then it is intuitively clear that if a state has more than n particles with wavefunctions in M', then it certainly has more than n particles with wavefunctions in M. That is the content of the next proposition.

PROPOSITION 3.2. <u>If</u> M <u>and</u> M' <u>are finite-dimensional subspaces of</u> H <u>with</u> $M' \subset M$, <u>then</u>

$$Q_n(M';W) \geq Q_n(M;W) \quad \text{for every } n.$$

Proof:

First let us see that the result is true in the standard zero-interaction representation over M. Then we will reduce the general case to this one.

So let W_o be the standard zero-interaction representation over M, acting on M_F. Then $N(M;W_o)$ is, by the Remark above, the total number operator for W_o, so its k-particle subspace is precisely $S_k\, M^k$. Thus

$$Q_n(M;W_o) = I \oplus I \oplus \cdots \oplus I \oplus 0 \oplus 0 \oplus \cdots$$

where the identity appears n times. (See Definition 2.3.) But for the number operator $N(M';W_o)$, the projection onto the k-particle subspace is the operator

$$P_k(M') = \sum_{m=0}^{\infty} P_k^{\,m}\,(M')$$

given by Definition 2.2.

So

$$Q_n(M';W_o) = \sum_{k=0}^{n} \sum_{m=0}^{\infty} P_k^{\,m}(M') \geq \sum_{k=0}^{n} \sum_{m=0}^{n} P_k^{\,m}(M').$$

From Proposition 2.1 we see that

$$\sum_{k=0}^{n} P_k^{\,m}(M') = \sum_{k=0}^{m} P_k^{\,m}(M') = \text{identity on } S_m\, H^m$$

if $m \leq n$, so

$$Q_n(M';W_o) \geq I \oplus I \oplus \cdots \oplus I \oplus 0 \oplus 0 \oplus \cdots$$
$$\geq Q_n(M;W_o).$$

Hence the proposition is proved in the case W is the standard zero-interaction system over M.

Now let us look at the general W over M. From Proposition 3.1 we get a unitary operator V from $M_F \otimes X$ onto K having the property that

$$W(z) = V[W_o(z) \otimes I]\, V^{-1} \quad \text{for all } z \in M$$

and

$$e^{itN(M;W)} = V\left[e^{itN_o(M)} \otimes I\right]V^{-1}.$$

Define an operator N'(M') on K by

$$e^{itN'(M')} = V\left[e^{itN_o(M')} \otimes I\right]V^{-1}.$$

We wish to show that N'(M') = N(M';W). But since $N_o(M') = N(M';W_o)$ (See Remark above), we know from Proposition 3.1 that $N_o(M')$ has spectrum {0, 1, 2, ···}, $e^{itN_o(M')}$ is in the weakly closed algebra generated by $\{W_o(z) : z \in M'\}$, and

$$e^{itN_o(M')}\, W_o(z)\, e^{-itN_o(M')} = W_o(e^{itP_{M'}} z).$$

Using these properties of $N_o(M')$ and the defining equation

$$e^{itN'(M')} = V\left[e^{itN_o(M')} \otimes I\right] V^{-1},$$

we draw the following conclusions: From the Lemma following Proposition 3.1 we have

(a) N'(M') is non-negative and has zero in its spectrum.

From the equivalence of weak closure and strongest closure we have

(b) $e^{itN'(M')}$ is in the weakly closed algebra generated by $\{W(z) : z \in M'\}$.

And a simple calculation shows

(c) $e^{itN'(M')} W(z) e^{-itN'(M')} = W(e^{it} z)$ for all $z \in M$.
From Proposition 3.1 we conclude that $N'(M') = N(M';W)$.
As a consequence of this we have

$$N(M';W) = \sum_{k=0}^{\infty} k \ V[P_k(M';\ W_o) \otimes I]\ V^{-1}.$$

But we also know that

$$N(M;W) = \sum_{k=0}^{\infty} k \ V[P_k(M;W_o) \otimes I]\ V^{-1}.$$

Hence we just have to use the fact that the desired result is true for the system W_o to get

$$\begin{aligned} Q_n(M';W) &= \sum_{k=0}^{n} V[P_k(M';W_o) \otimes I]\ V^{-1} \\ &= V[Q_n(M';W_o) \otimes I]\ V^{-1} \\ &\geq V[Q_n(M;W_o) \otimes I]\ V^{-1} \\ &= Q_n(M;W). \end{aligned}$$

CONVERGENCE RESULTS

This result leads us to introduce a type of convergence which will be useful to us throughout the rest of the paper.

DEFINITION 3.3. Let $\mathfrak{F}(H)$ be the set of all finite-dimensional subspaces of the inner-product space H, directed by inclusion. If for each $M \in \mathfrak{F}(H)$, a_M is some element of a topological space $\mathcal{S}$, the notation

$$\lim_{M \to H} a_M = a \quad \text{or} \quad a_M \to a \ \text{ as } \ M \to H$$

means that the <u>net</u> (or <u>generalized sequence</u>) $\{a_M\}$ converges to a. To be precise, given any neighborhood V of a in $\mathcal{S}$, there exists $M_o \in \mathfrak{F}(H)$ such that $a_M \in V$ for every $M \supset M_o$. (We shall use the word <u>net</u> only to refer to the case where $\mathfrak{F}(H)$ is the directed set.)

The reader should be aware of some of the pecularities of net convergence. First, the convergence of the net $\{a_M\}$ does <u>not</u> imply the convergence of $a_{M^1}, a_{M^2}, \cdots$ where $M^1, M^2, \cdots$ is an increasing sequence in $\mathfrak{F}(H)$ converging to H. Second, the convergence of $a_{M^1}, a_{M^2}, \cdots$ for <u>every</u> increasing sequence does not imply that the net converges. Third, the pointwise limit of a convergent net of measurable functions need not be measurable; in particular the usual convergence theorems of integration theory do not extend to net convergence.

However, many convergence theorems which are familiar for sequences are also true for nets. One of these is that a Cauchy net in a Hilbert space converges [6, p. 28], which we need for the next result.

PROPOSITION 3.3. <u>Let</u> W <u>be any Weyl system over</u> H, <u>acting on</u> K. <u>If</u> $Q_n(M;W)$ <u>is the projection onto the subspace of</u> K <u>having</u> n <u>or fewer particles with wavefunctions in</u> M, (Definition 3.2) <u>then the net</u> $M \to Q_n(M;W)$ <u>converges strongly to a projection</u> Q_n.

Proof: In view of Proposition 3.2, it suffices to prove that if $\{Q(M)\}$ is a decreasing net of projections on K, then it converges strongly to a projection. The proof we give is similar to the usual proof for sequences.

The assumption that $Q(M) \leq Q(M')$ if $M \supset M'$ implies that $Q(M') - Q(M)$ is a projection if $M \supset M'$. Now let $x \in K$. If $M \supset M'$, we have

$$\langle Q(M)\, x,x \rangle \leq \langle Q(M')\, x,x \rangle .$$

Let $a = \inf \{ \langle Q(M)\, x,x \rangle : M \in \mathfrak{F}(H)\}$. Then we have $\lim_{M \to H} \langle Q(M)\, x,x \rangle = a$. (Proof: we can choose M_n such that $\langle Q(M_n)\, x,x \rangle - a < n^{-1}$. Given $\epsilon > 0$, choose any n such that $n^{-1} < \epsilon$. Then if $M \supset M_n$, we have $a \leq \langle Q(M)\, x,x \rangle \leq \langle Q(M_n)x,x \rangle \leq a + \epsilon$, which proves the convergence.)

Thus if $M \supset M'$ we have

$$\begin{aligned} ||Q(M')x - Q(M)x||^2 &= ||[Q(M') - Q(M)]\, x\, ||^2 \\ &= \langle [Q(M') - Q(M)]\, x,x \rangle \\ &= \langle Q(M')\, x,x \rangle - \langle Q(M)\, x,x \rangle \\ &\to a - a \quad \text{as} \quad M,\ M' \to H. \end{aligned}$$

Thus $\{Q(M)\, x\}$ is a Cauchy net which therefore converges to some Qx.

Q is easily seen to be linear, and to show it is a projection, we just observe that for each $M \in \mathfrak{F}(H)$ and each $x,\ y \in K$,

$$\langle Q(M)\, x,\ Q(M)\, y \rangle = \langle Q(M)x,y \rangle = \langle x,\ Q(M)\, y \rangle .$$

So, taking the limit,

$$\langle Qx,\ Qy \rangle = \langle Qx,\ y \rangle = \langle x,\ Qy \rangle,$$

or $$Q^2 = Q = Q^*.$$

COROLLARY 1. <u>For each</u> k <u>the net</u> $M \to P_k(M;W)$ (see Definition 3.1) <u>converges to a projection</u> $P_k = P_k(W)$.

Proof:

$$P_k(M;W) = Q_k(M;W) - Q_{k-1}(M;W) \to Q_k - Q_{k-1}.$$

DEFINITION 3.4. The range of the projection $P_k(W)$ defined by Corollary 1 is called the k-<u>particle subspace for</u> W.

COROLLARY 2. <u>The sequence</u> $\{Q_n\}$ <u>converges strongly to a projection</u> $Q = Q(W)$.

Proof: If $n' > n$, we have for each $M \in \mathfrak{F}(H)$

$$Q_n(M)\ Q_{n'}(M) = Q_{n'}(M)\ Q_n(M) = Q_n(M).$$

Taking the limit we have

$$Q_n Q_{n'} = Q_{n'} Q_n = Q_n.$$

Therefore $Q_{n'} \geq Q_n$ if $n' > n$.

Since an increasing sequence of projections converges to a projection, the proof is complete.

DEFINITION 3.5. The range of the projection $Q(W)$ defined by Corollary 2 is called the <u>finite-particle subspace for</u> W.

4. THE NUMBER OF PARTICLES IN A REGULAR STATE

STATES OF THE WEYL ALGEBRA

DEFINITION 4.1. Given an inner product space H and a Weyl system W over H acting on K, the <u>Weyl algebra</u> $\underset{\sim}{A}(W)$ <u>for</u> W is a C*-algebra of operators on K which is constructed as follows: For each finite-dimensional subspace M of H, let $\underset{\sim}{A}_M(W)$ be the weakly closed algebra generated by $\{W(z) : z \in M\}$. Letting $\underset{\sim}{B} = \cup \{\underset{\sim}{A}_M(W) : M \in \mathfrak{F}(H)\}$, the algebra $\underset{\sim}{A}(W)$ is defined to be the uniform closure of B.

When H is infinite-dimensional, there are many Weyl systems over H which are not unitarily equivalent to the standard zero-interaction Weyl system W_o over H. Nonetheless, from a result of Segal [22] we know that for any Weyl system W over H there is a unique C*-isomorphism $\varphi : \underset{\sim}{A}(W_o) \to \underset{\sim}{A}(W)$ such that

$$\varphi(W_o(z)) = W(z)$$

for every $z \in H$. We will call φ the <u>canonical</u> isomorphism of $\underset{\sim}{A}(W_o)$ with $\underset{\sim}{A}(W)$. It has the property that for every finite-dimensional subspace M of H, φ maps $\underset{\sim}{A}_M(W_o)$ onto $\underset{\sim}{A}_M(W)$.

Segal's result shows that, as C*-algebras, all the $\underset{\sim}{A}(W)$'s are isomorphic, so we may refer to any one of them, say $\underset{\sim}{A}(W_o)$, as <u>the</u> Weyl algebra A over H. When we do use the expression "<u>the</u> Weyl algebra," we mean that only the C*-algebra structure and the labelling of certain operators as W(z)'s is to be considered. If, on the other hand, we wish to refer specifically to a Weyl algebra of operators on a Hilbert space, we shall use the expression "concrete C*-algebra."

The next notion we shall need is that of a state of a C*-algebra.

DEFINITION 4.2. Let $\underset{\sim}{A}$ be a C*-algebra with unit I. A linear functional E: $\underset{\sim}{A} \to \mathbb{C}$ is a state of $\underset{\sim}{A}$ if and only if

(a) E is positive, namely

$$E(A^*A) \geq 0 \quad \text{for every} \quad A \in \underset{\sim}{A}.$$

(b) E is normalized, namely $E(I) = 1$.

If $\underset{\sim}{A}$ is a concrete C*-algebra of operators on a Hilbert space K, and v is a unit vector in K, the state E of $\underset{\sim}{A}$ defined by

$$E(A) = \langle Av, v \rangle$$

will be called the state determined by v. Not every state of $\underset{\sim}{A}$ need come from a vector $v \in K$ in this way; those that are determined by vectors in K will be called normalizable states of $\underset{\sim}{A}$ in K. Even though the state E may not be normalizable in K, it is possible to find a representation of $\underset{\sim}{A}$ by operators on another Hilbert space K' such that E is normalizable in K'. More precisely, given the state E one can construct, by the Gelfand-Segal construction [10;19; see also 5] a (unique up to unitary equivalence) cyclic representation π of $\underset{\sim}{A}$ by operators on a Hilbert space K' such that there is a unit cyclic vector $v' \in K'$, satisfying

$$E(A) = \langle \pi(A)v', v' \rangle$$

for all $A \in \underset{\sim}{A}$. [Cyclic means that $\{\pi(A)v' : A \in \underset{\sim}{A}\}$ is dense in K.]

The difficulty which arises in the case where $\underset{\sim}{A}$ is the Weyl algebra is that for an arbitrary state E, the Gelfand-Segal construction may yield a representation π such that the operators $\pi(W_o(z))$ do not form a Weyl system. The Weyl relations (1.5) will be satisfied by the $\pi(W_o(z))$'s, but the continuity condition (Definition 1.1(b)) may not be. The states for which we do get a Weyl system are called regular and these are the only ones which will interest us here. (See the discussion by Segal [25]).

PROPOSITION 4.1 The following conditions on a state E of the Weyl algebra $\underset{\sim}{A}(W_o)$ over H are equivalent and mean E is regular:

(a) If π is the Gelfand-Segal representation of $\underset{\sim}{A}(W_o)$ determined by E, then $z \to \pi(W_o(z))$ is a Weyl system over H.

(b) $E(W_o(z))$ is a continuous function of z on every finite-dimensional subspace of H.

(c) For every Weyl system W over H and every finite-dimensional subspace M of H, E is weakly continuous on the unit ball of $\underset{\sim}{A}_M(W)$.

(d) For every Weyl system W over H acting on, say, K and for every finite-dimensional subspace M of H, there exists a non-negative trace class operator D_M on K such that

$$E(A) = \text{Trace } (AD_M)$$

for every $A \in \underset{\sim}{A}_M(W)$. ($D_M$ is called the density matrix for E on $\underset{\sim}{A}_M(W)$.)

The equivalences stated here are well-known [23; 4, p. 54].

THE NUMBER OF PARTICLES IN A REGULAR STATE

Now let us return our attention to the number operators. Proposition 3.1(b) shows that the operator $e^{itN(M;W)}$ is in the Weyl algebra for W over H. Moreover the mapping

$$A \to V[A \otimes I]\, V^{-1}$$

where V is the operator which appears in Proposition 3.1(d) is an explicit isomorphism between $\underset{\sim}{A}_M(W_o)$ and $\underset{\sim}{A}_M(W)$ which takes $W_o(z)$ into $W(z)$ for every $z \in M$. Since the canonical isomorphism φ between $\underset{\sim}{A}(W_o)$ and $\underset{\sim}{A}(W)$ does the same thing and, when restricted to $\underset{\sim}{A}_M(W_o)$, is necessarily continuous in the strongest topology [4, p. 57], we conclude that

$$\varphi(A) = V[A \otimes I]\, V^{-1}\ , \text{ for every } A \in \underset{\sim}{A}_M(W_o).$$

Hence, in particular,

$$\varphi\left(e^{itN_o(M)}\right) = e^{itN(M;W)} \tag{4.1}$$

by Proposition 3.1(d). The significance of this equation is roughly the following: the canonical correspondence between two concrete Weyl algebras puts the number operator over M for the first into correspondence with the number operator over M for the second.

Similarly one sees that

$$\varphi(P_k(M)) = P_k(M;W) \tag{4.2}$$

where $P_k(M;W)$ is the projection onto the k-particle subspace over M for W.

Now suppose Ξ is a state of the Weyl algebra $\underset{\sim}{A}(W_o)$. Since $e^{itN_o(M)} \in \underset{\sim}{A}_M(W_o)$ we may define a function $\psi_{E,M} : \mathbb{R} \to \mathbb{C}$ by

$$\psi_{E,M}(t) = \Xi(e^{itN_o(M)}). \tag{4.3}$$

PROPOSITION 4.2. <u>Let</u> E <u>be a regular state of the Weyl algebra</u> $\underset{\sim}{A}(W_o)$ <u>over</u> H, <u>let</u> W <u>be the Weyl system over</u> K <u>which is given by the Gelfand-Segal representation for</u> E, <u>and let</u> $v \in K$ <u>be the vector which determines</u> E. <u>Then</u>

(a) <u>For each finite-dimensional subspace</u> M <u>of</u> H, <u>the function</u> $\psi_{E,M}$ of Eq. (4.3) <u>is continuous and periodic with period</u> 2π.

(b) <u>There is a unique probability measure</u> $\mu_{E,M}$ <u>on the non-negative integers such that</u>

$$\psi_{E,M}(t) = \sum_{k=0}^{\infty} e^{itk} \mu_{E,M}(k)$$

<u>for every</u> $t \in \mathbb{R}$; $\mu_{E,M}$ <u>is given by</u>

$$\mu_{E,M}(k) = \langle P_k(M;W) v, v \rangle$$

<u>where</u> $P_k(M;W)$ <u>is the projection onto the subspace of</u> K <u>having</u> k <u>particles with wavefunctions in</u> M (Cf. Definition 3.1).

(c) <u>For each non-negative integer</u> n, $\mu_{E,M}(\{0,1, \cdots, n\})$ <u>is a decreasing function of</u> M; <u>and</u> $\mu_E(n) = \lim_{M \to H} \mu_{E,M}(n)$ <u>exists for every</u> n.

(d) $\mu_E(k) = \langle P_k(W) v, v \rangle$ <u>and</u> $\mu_E(\{0,1,2, \cdots\}) = \langle Q(W)v,v \rangle$, <u>where</u> Q(W) <u>is the projection onto the finite-particle subspace for</u> W (Definition 3.5).

Proof:

Since E is regular, it is continuous in the weak topology on the unit ball of $\underset{\sim}{A}_M(W_o)$. But $e^{itN_o(M)} \in \underset{\sim}{A}_M(W_o)$ so $E(e^{itN_o(M)})$ is a continuous function of t, which proves $\psi_{E,M}$ is continuous.

Now let φ be the Gelfand-Segal representation of $\underset{\sim}{A}(W_o)$ determined by E. Then if we define

$$W(z) = \varphi(W_o(z))$$

for every $z \in H$, we see that W is a Weyl system over H and φ is the canonical isomorphism of $\underset{\sim}{A}(W_o)$ onto $\underset{\sim}{A}(W)$. Since

$$E(A) = \langle \varphi(A)\ v,v \rangle \tag{4.4}$$

for every $A \in \underset{\sim}{A}(W_o)$, we have

$$\begin{aligned}\psi_{E,M}(t) &= E\left(e^{itN_o(M)}\right) \\ &= \langle \varphi\left(e^{itN_o(M)}\right) v,v \rangle \\ &= \langle e^{itN(M)}\ v,v \rangle \end{aligned} \tag{4.5}$$

by Equation (4.1).

Now we use the spectral resolution of N(M;W) given by Proposition 3.1 and Definition 3.1 to conclude that

$$\psi_{E,M}(t) = \sum_{k=0}^{\infty} e^{itk} \langle P_k(M;W)\ v,v \rangle.$$

This proves the periodicity of $\psi_{E,M}$, so that (a) is proved, and it also proves (b).

To prove (c):

$$\begin{aligned}\mu_{E,M}(\{0,1,\cdots,n\}) &= \sum_{k=0}^{n} \mu_{E,M}(k) = \langle \sum_{k=0}^{n} P_k(M;W)\ v,v \rangle \\ &= \langle Q_n(M;W)\ v,v \rangle\end{aligned}$$

by Definition 3.2. Then by Proposition 3.2 we have that

$$\mu_{E,M}(\{0,1,\cdots, n\}) \leq \mu_{E,M'} (\{0,1,\cdots, n\})$$

if $M \supset M'$. This is the first part of (c). Since decreasing nets of real numbers converge (cf. proof of Proposition 3.3), we know

$$\lim_{M\to H} \mu_{E,M} (\{0,1,\cdots, n\}) = \mu_E(\{0,1,\cdots, n\})$$

exists. But then

$$\lim_{M\to H} \mu_{E,M}(n) = \lim_{M\to H} [\mu_{E,M}(\{0,1,\cdots, n\}) - \mu_{E,M}(\{0,1,\cdots,n-$$

exists.

To prove (d), we simply have to observe that by definition

$$P_k(W) = \text{st-lim}_{M\to H} P_k(M;W) \text{ and } Q(W) = \text{st-lim}_{M\to H} Q(M;W).$$

THE PROBABILISTIC INTERPRETATION

The functions $\psi_{E,M}$ may be conveniently utilized to give a probabilistic interpretation of the number of particles in the state E. From Proposition 4.2(b) we see that $\psi_{E,M}$ is the Fourier transform of the probability measure $\mu_{E,M}$. In the terminology of probability theory, $\psi_{E,M}$ is the "characteristic function" of a certain random variable $n_{E,M}$ which takes on values which are non-negative integers. For concreteness we may take n to be the unique non-decreasing function on the unit interval [0, 1] for which the set

$$\{x : n_{E,M}(x) = k\}$$

is a (left-closed, right-open) interval of length $\mu_{E,M}(k)$. Namely, $n_{E,M}(x) = k$ if and only if

$$\sum_{j=0}^{k-1} \mu_{E,M}(j) \leq x < \sum_{j=0}^{k} \mu_{E,M}(j). \tag{4.6}$$

Each random variable $n_{E,M}$ is finite everywhere, but its expected value $\int_0^1 n_{E,M}$ may be finite or infinite. Indeed, for non-negative random variables the expected value is finite if and only if the characteristic function is differentiable at zero, and in this case the expected value equals (-i) times the derivative at zero (See [29]). So, $n_{E,M}$ has finite expected value if and only if $\psi_{E,M}'(0)$ exists, and then

$$\int_0^1 n_{E,M} = \sum_{k=0}^{\infty} k\, \mu_{E,M}(k)$$

$$= \frac{1}{i}\frac{d}{dt}\, \psi_{E,M}(t)\Big|_{t=0}$$

$$= \frac{1}{i}\frac{d}{dt} \langle e^{itN(M;W)}\, v, v \rangle \Big|_{t=0}.$$

Hence we see that $n_{E,M}$ has finite expected value if and only if v is in the domain of $N(M;W)^{1/2}$, and in this case

$$\int_0^1 n_{E,M} = \| N(M;W)^{\frac{1}{2}} v \|.$$

If v is actually in the domain of N(M;W) we have

$$\int_0^1 n_{E,M} = \langle N(M;W)\, v,\, v \rangle.$$

In other words, the expected value (finite or infinite) of the random variable $n_{E,M}$ is precisely the expected value of the operator N(M;W) in the state determined by v in the usual quantum-mechanical sense.

The random variable $n_{E,M}$ actually contains much more information about the relation of N(M) to E than just this expected value. For the probability that $n_{E,M}$ equals k is precisely

$$\mu_{E,M}(k) = \langle P_k(M;W)\ v, v \rangle = E(P_k(M)) \qquad (4.7)$$

by Equations (4.2) and (4.4).

The physical interpretation of this equation is that $\mu_{E,M}(k)$ equals the probability of finding, in the state E, k particles with wavefunctions in M. For suppose we write $v = \sum_{k=0}^{\infty} a_k v_k$, where v_k is a unit vector in the k-particle subspace for W over M (i.e. $P_k(M;W)v_k = v_k$). Then the quantum-mechanical interpretation would be that v is a superposition of the states given by $v_1, v_2, \cdots$, and the probability of finding v_k is $|a_k|^2$. But from (4.7) we see that

$$\mu_{E,M}(k) = |a_k|^2$$

so $\mu_{E,M}(k)$ is precisely the probability of finding k particles from M. In particular, the only way that $\mu_{E,M}(k)$ can equal one is if v lies in the k-particle subspace for W over M.

Another way to look at (4.7) is in terms of particles in the zero-interaction representation. Let D_M be the density matrix for E on $\underset{\sim}{A}_M(W_o)$. (See Proposition 4.1(d)). D_M is a non-negative trace class operator on H_F of trace one, so we may select an orthonormal basis $\{x_\gamma : \gamma \in \Gamma\}$ of H_F such that $D_M x_\gamma = \lambda_\gamma x_\gamma$ and $\Sigma\ \lambda_\gamma = 1$.

Then

$$\mu_{E,M}(k) = E(P_k(M))$$
$$= \text{Trace } (P_k(M)\ D_M)$$
$$= \Sigma\ \lambda_\gamma \langle P_k(M)\ x_\gamma,\ x_\gamma \rangle .$$

So we see that the only way $\mu_{E,M}(k)$ can equal <u>one</u> is if every non-zero λ_γ corresponds to an x_γ which lies in the k-particle subspace over M; or in other words, if D_M is supported by the k-particle subspace over M.

Even though the state E may not be represented by a density matrix on all of $\underset{\sim}{A}(W_o)$, it is clear from Proposition 4.2(d) that the probabilistic interpretation of the values of the limit measure μ_E is analogous to that for $\mu_{E,M}$.

DEFINITION 4.3. $\mu_E(k)$ is the <u>probability of finding</u> k <u>particles in the state</u> E. $\mu_E(\{0,1,2, \cdots \})$ is the <u>probability of finding a finite number of particles in the state</u> E.

To see what the significance of $\mu_E(k)$ is with respect to the random variables $n_{E,M}$, we prove the next result.

PROPOSITION 4.3. <u>Let</u> E <u>be a regular state of the Weyl algebra over</u> H.

(a) <u>If</u> $M \supset M'$, <u>then</u> $n_{E,M} \geq n_{E,M'}$.

(b) <u>The pointwise limit</u> n_E <u>of</u> $n_{E,M}$ <u>as</u> $M \to H$ <u>exists (perhaps not finite-valued)</u>.

(c) <u>The probability that the limit random variable</u> n_E <u>has the value</u> k <u>is</u> $\mu_E(k)$, <u>and the probability that</u> n_E <u>is finite is</u> $\mu_E(\{0,1,2, \cdots \})$.

Proof: Again, this is a standard result for sequences, but we shall give the proof since nets are involved.

Let W be the Weyl system for the Gelfand-Segal representation determined by E, and v the vector which determines E.

If M and M' are finite-dimensional subspaces of H such that $M \supset M'$, and $n_{E,M'} = k$, then by (4.6)

$$\langle Q_{k-1}(M';W)\, v,\, v \rangle \leq x < \langle Q_k(M';W)\, v,\, v \rangle .$$

Then by Proposition 3.2

$$\langle Q_{k-1}(M;W)\, v,\, v \rangle \leq \langle Q_{k-1}(M';W)\, v,\, v \rangle \leq x$$

so that $n_{E,M}(x) \geq n_{E,M'}(x)$. This proves (a).

To prove (b), we just have to observe that for each x, $\sup \{n_{E,M}(x) : M \in \mathfrak{F}(H)\}$ exists (finite or infinite). If this supremum is finite, say k, then because $n_{E,M}$ is integer-valued, there exists M' such that $n_{E,M'}(x) = k$. It follows from (a) that for all $M \supset M'$, $n_{E,M}(x) = k$, so $\lim_{M\to H} n_{E,M}(x) = k$. If the supremum is infinite, given any k we can find an M' such that $n_{E,M'}(x) > k$. It follows that $n_{E,M}(x) > k$ for every $M \supset M'$. Thus $\lim_{M\to H} n_{E,M}(x) = +\infty$.

Now we prove (c). For each k we must find the measure of the set $\{x : n_E(x) = k\}$. But in the previous paragraph we showed that $n_E(x) = k$ if and only if there exists an M' such that for all $M \supset M'$, $n_{E,M}(x) = k$. This is the same as the condition

$$\langle Q_{k-1}(M;W)\, v,v \rangle \leq x < \langle Q_k(M;W)\, v,v \rangle$$

for every $M \supset M'$, which is equivalent to

$$\langle Q_{k-1}(W)\ v,v \rangle \leq x < \langle Q_k(W)\ v,v \rangle$$

where Q_k is the limit defined in Proposition 3.3. Thus the measure of the set $\{x : n_E(x) = k\}$ is

$$\langle Q_k\ v,v \rangle - \langle Q_{k-1}\ v,v \rangle = \langle P_k v,v \rangle = \mu_E(k).$$

This proves (c).

WHAT CAN GO WRONG

Even though we now know that we can write

$$\psi_{E,M}(t) = E\left(e^{itN_o(M)}\right) = \sum_{n=0}^{\infty} e^{itn}\ \mu_{E,M}(n)$$

and the measures $\mu_{E,M}$ will converge to a measure μ_E, nonetheless we can not conclude that the functions $\psi_{E,M}$ converge to the function

$$\psi_E(t) = \sum_{n=0}^{\infty} e^{itn}\ \mu_E(n).$$

EXAMPLE: We shall give an example of a state E which has an infinite number of particles with probability one and which has the property that the functions $\psi_{E,M}$ do not converge as $M \to H$. (In the next section we shall give a similar example in which the $\psi_{E,M}$ do converge.)

Let H be the <u>algebraic</u> span of a countably infinite orthonormal system $\{e_1, e_2, e_3, \cdots\}$. Let W_s be the usual Schrödinger Weyl system over $\mathbb{C}$ acting on $L^2(\mathbb{R})$ (see Section 1), and let $N_s = N(\mathbb{C}\ ; W_s)$ be the total number operator for this Weyl system. Let v_1 be a unit vector in the one-particle subspace for W_s (i.e. v_1 is some multiple of the first Hermite function).

Let K be the incomplete tensor product of $L^2(\mathbb{R})$ with itself countably many times with distinguished vector $v = v_1 \otimes v_1 \otimes \cdots$. (See [16] for the definition and properties of an incomplete tensor product.)

We define a Weyl system W over H acting on K as follows: If $z \in H$, then it can be written in the form $z = \sum_{i=1}^{n} a_i e_i$ for some finite n. Define

$$W(z) = W_s(a_1) \otimes \cdots \otimes W_s(a_n) \otimes I \otimes I \otimes \cdots .$$

(It is easy to verify that this is indeed a Weyl system over H.)

If M_n is the finite-dimensional subspace of H spanned by $\{e_1, e_2, \cdots, e_n\}$, then the number operator $N(M_n;W)$ is given by

$$e^{itN(M_n;W)} = \left[\otimes^n e^{itN_s}\right] \otimes I \otimes I \otimes \cdots .$$

(It is easy to prove this. One simply has to check that the conditions (a), (b), and (c) of Proposition 3.1 are satisfied by the given operator.)

Now if E is the state determined by the vector $v = v_1 \otimes v_1 \otimes \cdots$, we have

$$\psi_{E,M_n}(t) = \langle \otimes^n (e^{itN_s} v_1) \otimes v_1 \otimes v_1 \otimes \cdots, v_1 \otimes v_1 \otimes \cdots \rangle$$

$$= e^{itn} .$$

Hence the <u>sequence</u> $n \to \psi_{E,M_n}$ does not converge. But if the <u>net</u> $M \to \psi_{E,M}$ converged, the above sequence would converge, for every finite-dimensional subspace of H is contained in one of the M_n's.

To show that E has an infinite number of particles with probability one, observe that

$$\mu_{E,M_n}(k) = \begin{cases} 0 & k \neq n \\ 1 & k = n \end{cases}$$

so $\mu_E(k) = \lim_{M \to H} \mu_{E,M}(k) = 0$ for every k. By Definition 4.3, the probability of finding a finite number of particles in the state E is zero.

One could in fact show that the state determined by any unit vector $x \in K$ has an infinite number of particles with probability one, but we will not do this here since later we will give a general theorem which proves the same thing. At any rate, we wish to point out that this in no way proves that there is no total number operator for W, i.e. that there is no self-adjoint operator N on K such that

$$e^{itN}\, W(z)\, e^{-itN} = W(e^{it}\, z)$$

for every $z \in H$. We shall discuss this in detail in Section 6.

REMARKS: When one has a pointwise convergent sequence of probability measures on the integers, the Lévy Continuity Theorem [13, p. 191] prescribes exactly what must happen to the characteristic functions ψ_n: Either the limit measure μ is a probability measure, in which case the characteristic functions ψ_n converge to the characteristic function of μ, or the limit measure μ is not a probability measure, in which case the characteristic functions ψ_n diverge or else converge to a discontinuous function. However, the same situation is not true for a pointwise-convergent net of measures.

It is possible that the net μ_M converges pointwise to, say, zero, and yet the characteristic functions ψ_M converge to a continuous function ψ; indeed even if all the μ_M are non-zero only on non-negative integers the Fourier series of the limit ψ may have non-zero coefficients for the terms e^{-it}, $e^{-2it}, \cdots$! (We are indebted to L. Gross and C. Herz for producing an example of this phenomenon.)

In view of the vastly more complicated technical problems related to nets, it would seem sensible to try to prove the desired results using a sequence of subspaces of H. At least in the case that H is separable one can select a sequence $M^1, M^2, \ldots$ of finite-dimensional subspaces of H converging to H. One would like to prove that if the measures $\mu_{M^1}, \mu_{M^2}, \cdots$ converge to a probability measure, then E must be normalizable in some direct sum of standard zero-interaction systems. This conjecture is false, as the next section shows.

5. SOME DISCONTINUOUS WEYL SYSTEMS

Select an orthonormal basis $\mathcal{E} = \{e_1, e_2, e_3, \cdots\}$ of the separable inner product space H and let H_o be the <u>algebraic</u> span of $\mathcal{E}$. Suppose that $H \neq H_o$. We will show that a Weyl system W over H such that $W(z) = W_o(z)$ for every $z \in H_o$ need not equal W_o everywhere on H, nor need it be unitarily equivalent to W_o. This shows that the continuity assumption in Definition 1.1(b) is strictly weaker than assuming continuity on all of H.

To see the relevance of this result to the question of when a number operator exists, suppose we have found such a W which agrees with W_o on H_o but does not equal W_o. Let M_n be the finite-dimensional subspace of H spanned by $\{e_1, \cdots, e_n\}$. Since $M_n \subset H_o$, the number operator $N(M_n;W)$ will equal $N(M_n;W_o)$ so we can prove

$$\lim_{n\to\infty} e^{itN(M_n;W)} = e^{itN_o(H)} .$$

In other words, $N(M_n;W) = \sum_{k=1}^{n} a^*(e_k)\, a(e_k)$ does <u>converge</u> as $n \to \infty$ to the number operator $N_o(H)$. So we have a Weyl system W not unitarily equivalent to the standard zero-interaction system for which $\sum_{k=1}^{\infty} a^*(e_k)\, a(e_k)$ converges. It is precisely this difficulty which forces us to use the net of number operators rather than a sequence of number operators in what follows.

We shall see that, despite the convergence of $\Sigma\, a^*(e_k)\, a(e_k)$, the <u>net</u> $e^{itN(M;W)}$ can not converge. Moreover there does not exist a number operator over H for W in the sense of Definition 3.1.

To review the notation to be used in this section:

H is a separable complex inner product space.

W_o is the standard zero-interaction Weyl system over H acting on K.

$\mathcal{E} = \{e_1, e_2, \cdots\}$ is a fixed orthonormal basis of H, and H_o is the algebraic span of $\mathcal{E}$.

First we will show the extent to which a Weyl system over H can differ from W_o if it agrees with W_o on H_o.

PROPOSITION 5.1. _Suppose_ W _is a Weyl system over_ H _on_ K _such that_ $W(z) = W_o(z)$ _for every_ $z \in H_o$. _Then there exists a real-linear transformation_ $T : H \to \mathbb{R}$ _such that_

$$W(z) = e^{iT(z)}\, W_o(z)$$

and

$$T(H_o) = \{0\}$$

Proof:

For any $z \in H$ and $z_o \in H_o$ we have

$$[W(z)\, W_o(z)^{-1}]W_o(z_o) = W(z)\, W_o(-z + z_o) \exp\left[\tfrac{1}{2}\, i\mathrm{Im}(-z,\ z_o)\right]$$

$$= W(z)\, W_o(z_o - z) \exp\left[\tfrac{1}{2}\, i\ \mathrm{Im}(z_o, z)\right] \quad (5.1)$$

using the Weyl relations (1.5).

On the other hand, since $W_o(z_o) = W(z_o)$, we have

$$W_o(z_o)[W(z)\, W_o(z)^{-1}] = W(z_o)\, W(z)\, W_o(z)^{-1}$$

$$= W(z)\, W(z_o) \exp\left[i\ \mathrm{Im}(z_o, z)\right] W_o(z)^{-1}$$

$$= W(z)\, W_o(z_o - z) \exp\left[\tfrac{1}{2}\, i\ \mathrm{Im}(z_o, -z)\right] \exp\left[i\ \mathrm{Im}(z_o, z)\right]$$

$$= W(z)\, W_o(z_o - z) \exp\left[\tfrac{1}{2}\, i\ \mathrm{Im}(z_o, z)\right]. \quad (5.2)$$

Comparing (5.1) with (5.2) we see that $W(z)\ W_o(z)^{-1}$ commutes with $W_o(z_o)$ for every $z_o \in H_o$. Using the irreducibility [3] of $\{W_o(z_o) : z_o \in H_o\}$, we conclude that $W(z)\ W_o(z)^{-1}$ is a multiple of the identity. Hence for each $z \in H$ there exists a complex number $\chi(z)$ such that $|\chi(z)| = 1$ and $W(z) = \chi(z)\ W_o(z)$.

Now for every $z, z' \in H$,

$$\begin{aligned}\chi(z + z')\ W_o(z + z') &= W(z + z') \\ &= W(z)\ W(z')\ \exp\left[\, -\tfrac{1}{2}\, i\ \mathrm{Im}(z,z')\right] \\ &= \chi(z)\chi(z')\ W_o(z)\ W_o(z') \\ &\quad \times \exp\left[-\tfrac{1}{2}\, i\ \mathrm{Im}(z,z')\right] \\ &= \chi(z)\ \chi(z')\ W_o(z + z').\end{aligned}$$

This shows that for every $z, z' \in H$

$$\chi(z + z') = \chi(z)\ \chi(z'). \tag{5.3}$$

Now we will show $\chi(z) = e^{iT(z)}$, where T is real-linear. Define, for each $z \in H$, $\chi^z : \mathbb{R} \to \mathbb{R}$ by $\chi^z(t) = \chi(tz)$. Then by (5.3) $\chi^z(s + t) = \chi^z(s)\ \chi^z(t)$. Furthermore χ^z is continuous. For by Definition 1.1(b) for any $x, y \in K$, the function $t \to \langle W(tz)\ x, y \rangle$ is continuous from $\mathbb{R}$ to $\mathbb{C}$. But this says $t \to \chi^z(t) \langle W_o(tz)\ x, y \rangle$ is continuous, from which, by an appropriate choice of x and y, we find that χ^z is continuous.

Hence χ^z is a continuous character of $\mathbb{R}$. Hence there exists $T(z) \in \mathbb{R}$ such that $\chi^z(t) = e^{itT(z)}$ for all $t \in \mathbb{R}$ (See [17, p. 140]).

It follows from (5.3) that T is real-linear, and since if $z_o \in H_o$, $e^{iT(z_o)} = 1$, we have $T(H_o) = \{0\}$, so the Proposition is proved.

DEFINITION 5.1. A function $\chi : H \to \mathbb{C}$ of the form $\chi(z) = e^{iT(z)}$ where T is real-linear and $T(H_o) = \{0\}$ is a <u>character of</u> H <u>modulo</u> H_o; χ is called <u>non-trivial</u> if it is not identically 1.

PROPOSITION 5.2. <u>Let</u> χ <u>be any character of</u> H <u>modulo</u> H_o, <u>and define</u>

$$W(z) = \chi(z)\ W_o(z) \quad \text{for all } z \in H.$$

<u>Then</u> W <u>is a Weyl system over</u> H <u>such that</u> $W(z_o) = W_o(z_o)$ <u>for all</u> $z_o \in H_o$. <u>Furthermore</u> W <u>is irreducible, and it is not unitarily equivalent to</u> W_o <u>if</u> χ <u>is non-trivial</u>.

Proof:

Let $\chi(z) = e^{iT(z)}$. Since $T(H_o) = \{0\}$ it is obvious that $W(z_o) = W_o(z_o)$ for all $z_o \in H_o$.

To show the Weyl relations are satisfied:

$$\begin{aligned} W(z)\ W(z') &= \chi(z)\ \chi(z')\ W_o(z)\ W_o(z') \\ &= \chi(z + z')\ W_o(z + z') \exp\left[\tfrac{1}{2}\ i\ \mathrm{Im}(z, z')\right] \\ &= W(z + z') \exp\left[\tfrac{1}{2}\ i\ \mathrm{Im}\ (z, z')\right]. \end{aligned}$$

To show $t \to W(tz)$ is continuous, we just observe that

$$W(tz) = e^{itT(z)}\ W_o(tz).$$

$W_o(tz)$ is a weakly continuous function of t, and $e^{itT(z)}$ is a continuous function of t, so the weak continuity of $t \to W(tz)$ is proved.

To show irreducibility, suppose U is a unitary operator on K such that $U\ W(z)\ U^{-1} = W(z)$ for all $z \in H$.

Then in particular

$$U\, W(z_o)\, U^{-1} = W(z_o) \text{ for all } z_o \in H_o,$$

or $$U\, W_o(z_o)\, U^{-1} = W_o(z_o) \text{ for all } z_o \in H_o.$$

Then by the irreducibility of the restriction of W_o to H_o [3], U is a multiple of the identity.

Similarly one shows that if V is a unitary such that

$$V\, W(z)\, V^{-1} = W_o(z) \text{ for all } z \in H,$$

then V is a multiple of the identity, which implies $W = W_o$. Hence if $W \neq W_o$ (i.e. if χ is nontrivial), then W is not unitarily equivalent to W_o.

REMARK: We have not yet proved the existence of non-trivial characters of H modulo H_o in case $H \neq H_o$. But in fact it is obvious how to construct the most general character of H modulo H_o. First extend the orthonormal basis of H to a Hamel basis $\mathcal{E} \cup \{v_\beta\colon \beta \in B\}$ of H. (See [6, p. 36] for a definition of Hamel basis.) Then for each index β select two real numbers c_β and d_β. Define $T(v_\beta) = c_\beta$ and $T(iv_\beta) = d_\beta$. Extend T to all of H by real-linearity and the condition that $T(H_o) = \{0\}$.

Despite the fact that $\chi\, W_o$ and W_o are not unitarily equivalent if χ is non-trivial, we shall now show that for every finite-dimensional subspace M of H, the restriction of χW_o to M is unitarily equivalent to the restriction of W_o to M. This not quite a trivial consequence of the Stone-vonNeumann Theorem (Section 1), since there is a question of multiplicities. We shall exhibit the unitary operator explicitly.

PROPOSITION 5.3 *Let* $\chi = e^{iT}$ *be a character of* H *modulo* H_o, M *a finite-dimensional subspace of* H. *Let* $y(\chi, M)$ *be the unique element of* M *such that*

$$T(z) = \operatorname{Re}(y(\chi, M), z) \tag{5.4}$$

for all $z \in M$. *Then* $W_o(iy(\chi, M))$ *transforms* $W_o|_M$ *into* $\chi W_o|_M$. *That is*

$$W(z) = \chi(z)\, W_o(z) = W_o(i\, y(\chi,M))\, W_o(z)\, W_o(-iy(\chi,M)) \tag{5.5}$$

for every $z \in M$.

Proof:

To see that a $y(\chi, M)$ satisfying (5.4) exists, recall that T is a real-linear functional on M. Since M is finite-dimensional, T is necessarily continuous with respect to any (real) inner product on M, for instance the inner product $y, z \to \operatorname{Re}(y, z)$, where (,) is the complex inner product on H. By the Riesz representation theorem [6, p. 249] there exists a unique $y(\chi, M) \in M$ satisfying (5.4).

If $z \in M$ and $y = y(\chi, M)$, we have

$$\begin{aligned} &W_o(iy)\, W_o(z)\, W_o(-iy) \\ &\quad = W_o(iy)\, W_o(z - iy) \exp\left[\tfrac{1}{2} i \operatorname{Im}(z, -iy)\right] \\ &\quad = W_o(z) \exp\left[\tfrac{1}{2} i \operatorname{Im}(iy, z-iy)\right] \exp\left[\tfrac{1}{2} i \operatorname{Im}(z, -iy)\right] \\ &\quad = W_o(z) \exp\left[i \operatorname{Im}(iy, z)\right] \\ &\quad = W_o(z) \exp\left[i \operatorname{Re}(y, z)\right] \\ &\quad = W(z). \end{aligned} \tag{5.6}$$

NUMBER OPERATORS FOR χW_o.

Let χ be any character of H modulo H_o, and let $W = \chi W_o$. For each finite-dimensional subspace M of H, the number operator N(M;W) (Definition 3.1) is related to $N_o(M)$ (Definition 2.3) by

$$N(M;W) = W_o(iy(\chi, M))\ N_o(M)\ W_o(-iy(\chi, M)) \tag{5.7}$$

where $y(\chi, M)$ is the vector of Proposition 5.3. To see that this is true, one just has to prove that $W_o(iy(\chi,M))\ N_o(M) \times W_o(-iy(\chi,M))$ satisfies conditions (a), (b), and (c) of Proposition 3.1. Using (5.5) this is trivial.

PROPOSITION 5.4 <u>Let</u> χ <u>be a non-trivial character of</u> H <u>modulo</u> H_o, <u>and</u> $W = \chi W_o$.

(a) <u>There does not exist a number operator for</u> W <u>over</u> H <u>in the sense of Definition</u> 3.1.

(b) <u>The net</u> $M \to e^{itN(M;W)}$ <u>does not converge strongly for all</u> $t \in \mathbb{R}$.

(c) <u>The sequence</u> $n \to e^{itN(M_n;W)}$ <u>does converge strongly for every</u> $t \in \mathbb{R}$, <u>where</u> M_n = <u>span of</u> $\{e_1, \cdots, e_n\}$.

Proof:

(a): Suppose there exists a number operator N for W over H. Letting $U(t) = e^{itN}$, by definition we have

$$U(t)\ W(z)\ U(t)^{-1} = W(e^{it} z) \text{ for all } z \in H. \tag{5.8}$$

In particular

$$U(t) W_o(z_o) = W_o(e^{it} z_o)\ U(t)$$

for all $z_o \in H_o$. By irreducibility of $\{W_o(z_o): z_o \in H_o\}$

we conclude that $U(t)$ is some multiple $c(t)$ of $e^{itN_o(H)}$.

But then

$$U(t)\ W(z)\ U(t)^{-1}$$

$$= c(t)\ e^{itN_o(H)}\ W(z)\ c(t)^{-1}\ e^{-itN_o(H)}$$

$$= \chi(z)\ W_o(e^{it}\ z)$$

$$= \chi((1 - e^{it})z)\ W(e^{it}\ z). \tag{5.9}$$

However, if χ is non-trivial, there exist $z \in H$ and $t \in \mathbb{R}$ such that $\chi((1 - e^{it})z) \neq 1$, so (5.8) and (5.9) can't both be true.

(b): If $U(t) = \text{St-lim}_{M\to H}\ e^{itN(M;W)}$ exists for every t, then $U(-t) = U(t)^{-1}$. Since, if $z \in M$,

$$e^{itN(M;W)}\ W(z)\ e^{-itN(M;W)} = W(e^{itP_M}\ z) = W(e^{it}\ z)$$

we get, by taking the strong limit,

$$U(t)\ W(z)\ U(t)^{-1} = W(e^{it}\ z),$$

and we just showed this is impossible.

(c): If $M_n = \text{span}\ \{e_1, \cdots, e_n\}$, then $M_n \subset H_o$, so $y(\chi, M_n) = 0$ (See (5.4)) and so by (5.7)

$$N(M_n;W) = N_o(M_n).$$

By Corollary 2 to Proposition 2.4 we have

$$\lim_{n\to\infty} e^{itN(M_n;W)} = e^{itN_o(H)}$$

which proves (c).

The remarkable fact is that it is posssible for the sequence $N(M'_n;W)$ to converge as in Proposition 5.4(c) for other sequences M'_n and, indeed, to converge to a <u>different</u> operator.

EXAMPLE 1. Suppose H is a separable, infinite-dimensional complex Hilbert space (i.e. is complete). We need to know that H does not have a countable Hamel basis. (Proof: Suppose $\{v_1, v_2, \cdots\}$ is a countable Hamel basis of H. This means that every $x \in H$ can be written as a finite linear combination of the v_i's. By the Gram-Schmidt orthogonalization process (See, for instance, [11, p. 27]), one can produce from $\{v_1, v_2, \cdots\}$ an orthonormal basis $\{e_1, e_2, \cdots\}$ of H having the property that $\langle v_i, e_j \rangle = 0$ if $j > i$. Now let $\{a_1, a_2, \cdots\}$ be a sequence of real numbers such that $\Sigma(a_i)^2 < \infty$ and $a_i \neq 0$ for all i. Then $x = \Sigma\, a_i e_i \in H$. From the hypothesis that $\{v_1, v_2, \cdots\}$ is a Hamel basis we have $x = \sum_{i=1}^{n} b_i v_i$ for some n. But then $\langle x, e_j \rangle = 0$ for $j > n$, which contradicts the fact that $\langle x, e_j \rangle = a_j$.

Now we shall show that it is possible to select an uncountable collection $\{\mathcal{E}^\gamma : \gamma \in \Gamma\}$ of orthonormal bases of H such that the algebraic span H_γ of $\mathcal{E}^\gamma$ intersect $H_{\gamma'}$ in $\{0\}$ if $\gamma \neq \gamma'$. (We say H_γ is disjoint from $H_{\gamma'}$.)

First select a Hamel basis $\{v_\delta : \delta \in \Delta\}$ of H. As we just showed, Δ is uncountable. Let $\mathcal{E}^1 = \{e_1^1, e_2^1, e_3^1, \cdots\}$ be any orthonormal basis of H. Then we can write each e_j^1 uniquely as a finite linear combination of some v_δ's. Let Δ^1 be the set of all the δ's for which v_δ appears in one of these linear combinations. Then Δ^1 is countable, and the algebraic span H_1 of $\mathcal{E}^1$ is contained in the algebraic span of $\{v_\delta : \delta \in \Delta^1\}$ Now consider the span G of $\{v_\delta : \delta \in \Delta - \Delta^1\}$. This is a linear submanifold of H which is disjoint from H_1 since the set $\{v_\delta : \delta \in \Delta - \Delta^1\}$ is linearly independent from $\{v_\delta : \delta \in \Delta^1\}$. We select an orthonormal basis $\mathcal{E}^2$ of G. (The usual argument shows this is possible even though G is not complete.) The algebraic span H_2 of $\mathcal{E}^2$ is contained in G and so is disjoint from H_1.

Now it is clear how to procede. Let $\{\mathcal{E}^\gamma : \gamma \in \Gamma\}$ be a maximal collection of orthonormal bases of H such that the algebraic span H_γ of $\mathcal{E}^\gamma$ is disjoint from $H_{\gamma'}$ if $\gamma \neq \gamma'$. If Γ is countable we reach a contradiction to the maximality by considering those δ's for which v_δ appears in the sum for one of the elements of $\underset{\gamma}{\cup}\mathcal{E}^\gamma$ and constructing an extra $\mathcal{E}$ as we constructed $\mathcal{E}^2$ above. Hence Γ must be uncountable.

Now let $\{\mathcal{E}^\gamma : \gamma \in \Gamma\}$ be such a collection of orthonormal bases of H, and suppose $0 \in \Gamma$. For each $\gamma \neq 0$ select two non-zero real sequences $\{a_n^\gamma\}$ and $b_n^\gamma\}$ such that $\Sigma(a_n^\gamma)^2$ and $\Sigma(b_n^\gamma)^2$ converge. We can select these sequences to be different for different γ, since there are as many such sequences as there are elements of H. Let $a_n^0 = b_n^0 = 0$. Denoting the elements of $\mathcal{E}^\gamma$ by $e_1^\gamma, e_2^\gamma, \cdots$ define $T(e_n^\gamma) = a_n^\gamma$, $T(ie_n^\gamma) = b_n^\gamma$, and extend T to a real-linear functional on H. Then $T(H_0) = \{0\}$, so $\chi = e^{iT}$ is a character of H modulo H_0. Let W be the Weyl system $\chi\, W_0$.

For all $z \in H_\gamma$ we have

$$\chi(z)\, W_0(z) = W_0(iy_\gamma)\, W_0(z)\, W_0(-iy_\gamma)$$

where $y_\gamma = \sum_n (a_n^\gamma + ib_n^\gamma)e_n^\gamma$. (See Equation (5.6).) Since $y_\gamma \neq 0$, $W_0(iy_\gamma)$ is not a multiple of the identity. In fact

$W_o(iy_\gamma)$ does not leave the vacuum state invariant [cf. (1.7) with $y = 0$, $x = iy_\gamma$. Here $f \equiv 1$ represents the vacuum.]

Now letting $M_n{}^\gamma$ = span of $\{e_1{}^\gamma, \cdots, e_n{}^\gamma\}$, it is clear that $y(\chi, M_n{}^\gamma) \to y_\gamma$ as $n \to \infty$, so that $W_o(iy(\chi, M_n{}^\gamma)) \to W_o(iy_\gamma)$. Hence from (5.7) we see that

$$\exp\,[itN(M_n{}^\gamma;\,W)] \to W_o(iy_\gamma)\; e^{itN_o(H)}\; W_o(-iy_\gamma)$$

as $n \to \infty$. In other words

$$\sum_{k=1}^{\infty} a^*(e_k{}^\gamma)\; a(e_k{}^\gamma) = W_o(iy_\gamma)\; N_o(H)\; W_o(-iy_\gamma). \qquad (5.10)$$

This operator does not equal $N_o(H)$ since it does not annihilate the vacuum. What we have shown is that <u>for uncountably many different orthonormal bases $\mathcal{E}^\gamma$, the operator</u> (5.10) <u>exists and is different from the usual number operator</u> $N_o(H)$.

EXAMPLE 2. This is a special case of Example 1. In this example we will show explicitly what happens to the net

$$\psi_{E,M}(t) = E\left(e^{itN_o(M)}\right)$$

where E is a certain normalizable state of one of the discontinuous Weyl systems. This state E has an infinite number of particles with probability one, but the behavior of $\psi_{E,M}$ is different from that in Section 3. In this example the functions $\psi_{E,M}$ <u>converge</u> to a discontinuous function.

Let $= \{e_1, e_2, \cdots\}$ be an orthonormal basis of the separable complex Hilbert space H, and let $\mathcal{B} \supset \mathcal{E}$ be a Hamel basis (consisting of unit vectors) for the real Hilbert space generated by $\mathcal{E}$.

Select any $z_o \in \mathcal{B}$ such that $z_o \notin \mathcal{E}$, let $\mathcal{B}' = \mathcal{B} - \{z_o\}$ and define

$$T(z) = 0 \quad \text{for all} \quad z \in \mathcal{B}'$$

$$T(z_o) = 1 \ , \ T(iz_o) = 0.$$

Extend T to a real-linear functional on H.

Let $\chi = e^{iT}$ and $W = \chi W_o$. Then for every vector z in the span of $\mathcal{B}'$, $W(z) = W_o(z)$, and for every finite-dimensional subspace M of H contained in the span of $\mathcal{B}'$, $N(M;W) = N_o(M)$. However, as proved in Propositions 5.2, 5.4 W is not unitarily equivalent to W_o, and there does not exist a number operator over H for W.

Now let $v_o \in H_F$ be the vector which determines the zero-interaction vacuum state of $\underset{\sim}{A}(W_o)$. If we define a state E of the Weyl algebra by

$$E(W_o(z)) = \langle W(z) \ v_o, \ v_o \rangle = \chi(z) \langle W_o(z) \ v_o, v_o \rangle \quad (5.11)$$

(extending in the obvious way to all of $\underset{\sim}{A}(W_o)$) then E is <u>not</u> the standard zero-interaction vacuum state of the Weyl algebra. In fact we shall show E has an infinite number of particles with probability one. By Definition 4.3, for each finite-dimensional subspace M of H, we have to find

$$\psi_{E,M}(t) = E\left(e^{itN_o(M)}\right) = \langle e^{itN(M;W)} \ v_o, v_o \rangle ,$$

then we must find the measure $\mu_{E,M}$ such that

$$\psi_{E,M}(t) = \sum_{k=0}^{\infty} e^{itk} \mu_{E,M}(k) ,$$

and then show

$$\lim_{M \to H} \mu_{E,M}(k) = 0.$$

First we calculate $\psi_{E,M}$. Let $F(M)$ be the intersection of the span of $\mathcal{B}'$ with M, and let Q_M be the projection of H onto $F(M)$. Then if $z \in M$, z can be uniquely written in the form

$$z = Q_M z + (c + id)\, z_0 \quad \text{where } c + id \in \mathbb{C}.$$

Note $T(Q_M z) = 0$, since $Q_M z \in \text{span } \mathcal{B}'$.

LEMMA 1. If $z_0 \in M$, then the vector $y(\chi, M)$ of Proposition 5.3 is given by

$$y(\chi, M) = y = \|(I - Q_M)z_0\|^{-2} \; (I - Q_M)z_0. \qquad (5.13)$$

If $z_0 \notin M$, $y(\chi, M) = 0$.

Proof:

If $z_0 \notin M$, then $W(z) = W_0(z)$ for all M, so $y(\chi, M) = 0$. If $z_0 \in M$, then $(I - Q_M)z_0 \in M$ and $(I - Q_M)z_0 \neq 0$ since $Q_M z_0 \in \text{span } \mathcal{B}'$.

Furthermore

$$\begin{aligned} \exp\,[i \operatorname{Re}\,(y,z)] &= \exp\left[i\; \|(I - Q_M)z_0\|^{-2} \operatorname{Re}\left((I-Q_M)z_0,\; Q_M z + (c + id)z_0 \right)\right. \\ &= \exp\,[i \operatorname{Re}\,(c - id)] \\ &= \exp\,[ic] \\ &= \chi(z). \end{aligned}$$

Now, using (5.7) we have

$$e^{itN(M;W)} = W_o(iy)\ e^{itN_o(M)}\ W_o(-iy)$$

$$= W_o(iy)\ e^{itN_o(M)}\ W_o(-iy)\ e^{-itN_o(M)}\ e^{itN_o(M)}$$

$$= W_o(iy)\ W_o(e^{it}(-iy))\, e^{itN_o(M)}$$

$$= W_o(iy - ie^{it}y)\ \exp\left[\frac{1}{2}\, i\ \mathrm{Im}(iy, -e^{it}y)\right] e^{itN_o(M)}$$

$$= W_o(i(1 - e^{it})y)\ \exp\left[\frac{1}{2}\, i\ \mathrm{sint}\ ||y||^2\right] e^{itN_o(M)}.$$

To calculate $E(e^{itN_o(M)})$ from this, let H_R be some real-linear subspace of H such that $i(1 - e^{it})\ y \in H_R$. Then representing v_o by the function 1 in $L^2(H_R, \nu)$, where ν is the normal distribution with variance $\frac{1}{2}$ [20], we have (cf. (1.7))

$$E(e^{itN_o(M)}) = \exp\left[\frac{1}{2}\, i\ ||y||^2\ \mathrm{sint}\right] \langle W_o(i(1-e^{it})y)\ v_o, v_o \rangle$$

$$= \exp\left[\frac{1}{2}\, i\ ||y||^2\ \mathrm{sint}\right] \int_{H_R} \exp\left[i \langle u, (1-e^{it})y \rangle\right] d\nu(u)$$

$$= \exp\left[\frac{1}{2}\, i\ ||y||^2\ \mathrm{sint}\right] \int_{H_R} \exp\left[iu\ ||(1-e^{it})y|| - u^2\right] du$$

$$= \exp\left[\frac{1}{2}\, i\ ||y||^2\ \mathrm{sint} - \frac{1}{4}\, ||(1 - e^{it})\ y||^2\right]$$

$$= \exp\left[\frac{1}{2}\, ||y||^2\ (i\ \mathrm{sint} + \mathrm{cost} - 1)\right]$$

$$= \exp\left[\frac{1}{2}\, ||y||^2\ (e^{it} - 1)\right]$$

Now, referring to (5.13) we see that if $z_o \in M$

$$||y||^2 = ||y(\chi, M)||^2 = ||(I - Q_M)z_o||^{-2}.$$

Thus

$$\psi_{E,M}(t) = E\left(e^{itN_o(M)}\right) = \begin{cases} \exp\left[\frac{1}{2} \|(I - Q_M)z_o\|^{-2}(e^{it}-1)\right], & z_o \in M \\ 1 & z_o \notin M \end{cases}$$

which gives

$$|\psi_{E,M}(t)| = \begin{cases} \exp\left[-\frac{1}{2}\|(I - Q_M)z_o\|^{-2} (1 - \cos t)\right] & z_o \in M \\ 1 & z_o \notin M \end{cases} \tag{5.14}$$

LEMMA 2. $\lim_{M\to H} |\psi_{E,M}(t)| = 0, \ t \neq 0, \pm 2\pi, \pm 4\pi, \cdots$

Proof:

First we prove

$$\lim_{M\to H} \|(I - Q_M)\, z_o\| = 0.$$

If $M \supset M'$, then $F(M) \supset F(M')$, so $I - Q_M \leq I - Q_M{}'$. Thus it suffices to prove that there exists an increasing sequence $M^1, M^2, \cdots$ of finite dimensional subspaces of H such that $\lim_{n\to\infty} \|(I - Q_{M^n})z_o\| = 0$. Select $M^n = \text{span}\,\{e_1, \cdots, e_n\}$, then Q_{M^n} = projection on M^n, so clearly st-lim $Q_{M^n} = I$. Therefore $\lim_{n\to\infty} \|(I - Q_{M^n})\, z_o\| = 0$.

Now given $1 > \epsilon > 0$, $t \neq 0, \pm 2\pi, \cdots$, choose M_o such that for all $M \supset M_o$, $\|(I - Q_M)z_o\|^2 < \frac{1 - \cos t}{-2 \log \epsilon}$. Let M_1 be the span of $M_o \cup \{z\}$. Then for all $M \supset M_1$, we have, by (5.14),

$$|\psi_{M,E}(t)| < \epsilon ,$$

which proves the Lemma.

LEMMA 3.

$$\lim_{M\to H} \mu_{E,M} = 0$$

Proof:

From the proof of Lemma 2 we see that choosing $M(k) = \text{span}\,\{e_1, \cdots, e_k, z_o\}$, we have $\lim_{k\to\infty} \psi_{E,M(k)} = 0$ almost everywhere. Also each function $\psi_{E,M(k)}$ is bounded by 1. Then since

$$\mu_{E,M(k)}(n) = \frac{1}{2\pi}\int_0^{2\pi} \psi_{E,M(k)}(t)\, e^{-int}\, dt$$

we have by the Lebesque Dominated Convergence Theorem

$$\lim_{k\to\infty} \mu_{E,M(k)}(n) = 0.$$

But it is clear from Proposition 4.2(c) that

$$\begin{aligned}\mu_E(\{0,1, \cdots, n\}) &= \lim_{M\to H} \mu_{E,M}(\{0,1,\cdots, n\}) \\ &\leq \lim_{k\to\infty} \mu_{E,M(k)}(\{0,1,\cdots, n\}) \\ &= 0.\end{aligned}$$

This completes the proof of the fact that despite the "very small" difference between W_o and W, the vector v_o which determines the no-particle state of $\underset{\sim}{A}(W_o)$ determines a state of $\underset{\sim}{A}(W)$ which has an infinite number of particles with probability one.

6. CHARACTERIZATION OF THE STANDARD ZERO-INTERACTION REPRESENTATION

It has been known for a few years [24] that there are Weyl systems over H other than the standard zero-interaction system for which total number operators exist in the sense of Definition 3.1. These Weyl systems may be constructed, using the Gelfand-Segal construction, from so-called <u>universally invariant</u> states. (These are states E which have the property that E(W(Uz)) = E(W(z)) for every unitary operator U on H.)

Segal has proved that the only universally invariant state for which the related number operator is non-negative is the standard zero-interaction vacuum [24, Theorem 2]. It is also implicit in his work [24, Theorem 3] that the only cyclic Weyl system which has a non-negative total number operator which annihilates a cyclic vector is (unitarily equivalent to) the standard zero-interaction Weyl system. However, the published statement of this result seems to require a certain additional continuity assumption. We shall indicate the structure of Segal's proof, so that we can see that the continuity assumption is not needed in this context.

On pages 515-516 of [24], Segal proves

LEMMA: <u>Let</u> W <u>be a</u> Weyl <u>system over</u> H <u>on</u> K, <u>and let</u> A <u>be a non-negative self-adjoint operator on</u> H. <u>If there exists a non-negative self-adjoint operator</u> B <u>on</u> K <u>such that</u>

$$e^{itB}\, W(z)\, e^{-itB} = W(e^{itA}\, z)$$

for all $z \in H$, and if $v \in K$ is invariant under all the operators e^{itB}, then

$$\langle W(e^{-itA} u-u) v, v \rangle = \exp\left[-\frac{1}{4} || e^{-itA} u-u||^2\right] \quad (6.1)$$

for all $u \in H$, $t \in \mathbb{R}$.

From this we have

PROPOSITION 6.1. If W is a cyclic Weyl system with a non-negative total number operator N which annihilates a cyclic vector v_o, then W is unitarily equivalent to the standard zero-interaction Weyl system.

Proof: We use the Lemma with A = the identity on H, $B = N$ on K. If $z \in H$, let $u = -\frac{1}{2} z$. Then for $t = \pi$, (6.1) gives

$$\langle W(z) v_o, v_o \rangle = e^{-\frac{1}{4}||z||^2}$$

By [23, Theorem 1 and Section 4] we know that the only regular state E of the Weyl algebra such that

$$E(W(z)) = e^{-\frac{1}{4}||z||^2}$$

for all $z \in H$ is the standard zero-interaction vacuum. Since there is, up to unitary equivalence, only one cyclic Weyl system with cyclic vector whose state is E [19] the Proposition is proved.

From Section 5 we know that there are cyclic Weyl systems W other than the zero-interaction system which have a cyclic vector v_o which is annihilated by every number operator $N([e_k];W)$, where $\{e_1, e_2, \cdots\}$ is some fixed orthonormal basis.

However the following criterion specifies the zero-interaction system.

PROPOSITION 6.2 Let W be a cyclic Weyl system with a cyclic vector v_o which is annihilated by the number operator $N([z];W)$ for every vector $z \in H$. (See Definition 3.1). Then W is unitarily equivalent to the standard zero- interaction system.

Proof: If $z \in H$, select the operator A of the Lemma to be the projection $P_{[z]}$ of H onto the subspace spanned by z. By Proposition 3.1

$$e^{itN([z];W)}\ W(y)\ e^{-itN([z];W)} = W(\exp(itP_{[z]}y))$$

for all $y \in H$, and $N([z];W)$ is non-negative. The hypothesis here is that v_o is invariant under $e^{itN([z];W)}$. So in the Lemma we can select $B = N([z];W)$, $t = \pi$, $u = -\frac{1}{2}z$, and we have

$$\langle W(z)v_o,v_o\rangle = e^{-\frac{1}{4}||z||^2}.$$

The proof is completed as in Proposition 6.1.

From this result and the results of Section 5 we have the following result in case H is separable.

PROPOSITION 6.3. Let $\{e_1,e_2, \cdots\}$ be an orthonormal basis of the separable Hilbert space H, and let W be a cyclic Weyl system over H with cyclic vector v_o which is annihilated by the number operator $N([e_k];W)$ for every k. If either H equals the algebraic span of $\{e_1, e_2, \cdots\}$, or the function

$$z \to \langle W(z)\ v_o,v_o\rangle$$

is continuous on all of H, then W is unitarily equivalent to the standard zero-interaction Weyl system.

Proof: Let H_o = the algebraic span of $\{e_1, e_2, \cdots\}$. If $z \in H_o$, then z is contained in the finite-dimensional subspace M spanned by, say, $\{e_1, \cdots, e_n\}$. By Cor. 1 to Prop. 2.4 we know N(M;W) annihilates v_o, so it follows from Prop. 3.2 with M' = [z] that N([z];W) annihilates v_o. Since z was an arbitrary element of H_o, by Prop. 6.2 we have $W = W_o$ on H_o. If $H = H_o$, we are finished. If $H \neq H_o$, the hypothesis is that W is continuous on H. But by Prop. 5.2, $W = \chi W_o$, where χ is a character of H modulo H_o. Since the only continuous character is $\chi \equiv 1$, we have $W = W_o$.

Now we want to characterize all Weyl systems for which there are normalizable states having a finite number of particles with probability one. Our first result along these lines is the following.

THEOREM 1. Let H be a complex inner product space, W a Weyl system over H acting on the Hilbert space K. The following conditions are equivalent:

(1) W is unitarily equivalent to a direct sum (of arbitrary cardinality) of standard zero-interaction Weyl systems.

(2) The representation of the Weyl algebra $\underset{\sim}{A}(W_o)$ given by the canonical isomorphism of $\underset{\sim}{A}(W_o)$ onto $\underset{\sim}{A}(W)$ (See Definition 4.1) is a direct sum of cyclic representation, each

having the property that there is a cyclic vector which determines a state which has a finite number of particles with probability one.

(3) The finite-particle subspace of K is K itself. (See Definition 3.5.)

(4) Every normalizable state of $\underset{\sim}{A}(W)$ in K has a finite number of particles with probability one. (See Definitions 4.2, 4.3.)

(5) For each $t \in \mathbb{R}$, there is an operator $V(t)$ on K such that for every $x \in K$

$$\| [e^{itN(M;W)} - V(t)]\, x \| \to 0$$

uniformly in t as $M \to H$.

(6) There exists a self-adjoint operator N on K whose spectrum is contained in the non-negative integers and which satisfies

$$e^{itN}\, W(z)\, e^{-itN} = W(e^{it} z)$$

for all $z \in H$, $t \in \mathbb{R}$.

REMARK 1. Since the equation in condition (6) is precisely our criterion for N to be a number operator over H (Definition 3.1), condition (6) can be restated as: There exists a nonnegative-integer-valued total number operator.

REMARK 2. We shall see that the operators in condition (5) actually converge to the operator e^{itN} of condition (6), so (5) is one way of stating rigorously the condition "The number operators N(M) converge to a number operator."

REMARK 3. The net convergence specified in (5) is not particularly easy to verify in practice. However, we have seen by example that one can not prove the theorem assuming very much less. If the space H is <u>separable</u>, the following condition can be substituted for (5):

(5s) If M^1, M^2, $\cdots$ is any increasing sequence of finite-dimensional subspaces of H such that $\lim_{n\to\infty} P_{M^n} = I$ where P_M is the projection of H onto M, then there exists, for each t, an operator V(t) such that the sequence $n \to \exp(itN_{M^n})$ converges strongly to V(t), and $t \to V(t)$ is weakly continuous at zero.

It is to be noted that (5s) is a condition on <u>every</u> increasing sequence $\{M^n\}$ converging to H, not just one particular sequence.

In case the representation space K is known to be <u>separable</u>, the condition (5s) can be simplified by removing the assumption that $t \to V(t)$ is continuous. For the strong limit of the sequence of one-parameter unitary groups is easily seen to be a (weakly) measurable one-parameter unitary group (See the proof of Proposition 6.4.) In case K is separable, a measurable unitary group on K is automatically continuous. (See, for example, [12].)

REMARK 4. The assumption of uniform convergence in (5) is probably unavoidable, as was indicated in the remarks at the end of Section 4. However, it is not an artificial assumption since we shall reduce it to a question of uniform convergence of certain characteristic functions; but if a <u>sequence</u>

of characteristic functions converges to a continuous function, the convergence is necessarily uniform on every compact interval [13, p. 191]. In the case in question here, we are only interested in the values of the functions on an interval of length 2π anyway. As we procede with the proof, it will become clear that the assumption of uniform convergence in (5) can be replaced by the assumption of uniform convergence on an arbitrarily small neighborhood of zero. However, we have avoided the extra verbiage required to prove the result in this form.

The proof of the theorem procedes through a number of lemmas, some of which are interesting in their own right. Throughout this section the symbol W stands for the given Weyl system over H, K is the space on which W acts, and a symbol such as N(M) means the number operator over M for the particular given W, i.e. N(M;W).

First observe that (3) $\Longleftrightarrow$ (4) is obvious. For a state E is normalizable in K, by Definition 4.2, if and only if there is a unit vector $x \in K$ such that $E(A) = \langle Ax, x \rangle$ for all A in the Weyl algebra. Now let Q be the projection onto the finite particle subspace (Definition 3.5), and we see that E has a finite number of particles with probability one if and only if $\langle Qx, x \rangle = 1$, and this is true if and only if $Qx = x$.

Now (4) plays no further role in the proof which follows the outline

$$(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (6) \Longrightarrow (1)$$
$$(3) \Updownarrow (5)$$

LEMMA 1. $(1) \Longrightarrow (2)$: A direct sum of standard zero-interaction representations is a direct sum of cyclic representations each having a cyclic vector which has a finite number of particles with probability one.

Proof: Let v_o be the standard zero-interaction vacuum vector. Then the state E determined by v_o has zero particles with probability one. (Proof: $N_o(M)\ v_o = 0$ for every finite-dimensional M, so it follows that $\mu_{E,M}(k) = \delta_{ko}$. Hence $\mu_E(k) = \delta_{ko}$. Now use Definition 4.3.) Furthermore v_o is cyclic for $\underset{\sim}{A}(W_o)$, since any non-zero vector is cyclic for an irreducible algebra of operators. This proves the Lemma.

Now we introduce the following

NOTATION: Define $U(t)$: $QK \to QK$ by

$$U(t) = \sum_{k=0}^{\infty} e^{itk} P_k$$

where P_k is the projection onto the k-particle subspace for W (Definition 3.4) and Q is the projection onto the finite-particle subspace for W (Definition 3.5)

Note that since we have defined $U(t)$ to be e^{itN}, where N is the self-adjoint operator on QK whose spectral resolution is $N = \Sigma\ k\ P_k$, $t \to U(t)$ is a strongly continuous one-parameter unitary group on QK.

The first main point of the proof is contained in

LEMMA 2. <u>If</u> $x \in QK$, <u>then</u> $\lim_{M \to H} ||e^{itN(M)} x - U(t) x|| = 0$ <u>uniformly in</u> t.

Proof:

Let $P_k(M)$ be the projection onto the k-particle subspace for the number operator $N(M)$ over M. Choose any $x \in K$. Then

$$\psi_M(t) = \langle e^{itN(M)}x, x \rangle = \sum_{k=0}^{\infty} e^{itk} \langle P_k(M)\ x,\ x \rangle$$

$$= \sum_{k=0}^{\infty} e^{itk}\ \mu_{x,M}(k).$$

From Corollary 1 to Proposition 3.3 we know that as $M \to H$, $\mu_{x,M}(k)$ converges to

$$\mu_x(k) = \langle P_k\ x,\ x \rangle.$$

If $x \in QK$, then, from the fact that $Q = \sum_{k=0}^{\infty} P_k$, we have

$$\sum_{k=0}^{\infty} \langle P_k\ x,\ x \rangle = \langle x,\ x \rangle,$$

so that in this case the total variation of μ_x is the same as the total variation of each $\mu_{x,M}$, namely $||x||^2$. From this we can show that the Fourier transforms ψ_M converge uniformly to

$$\psi(t) = \sum_{k=0}^{\infty} e^{itk}\ \mu_x(k).$$

For convergence of sequences, this is a familiar result. (See, for instance, Loeve [13] or the Vitali-Hahn-Saks Theorem, which can be found in many functional analysis texts.) However, most of the available proofs can not be adapted to net convergence, although the proof is quite simple in the case we have here. So we shall give a complete proof.

Let $\varepsilon > 0$ be given, and choose n_o such that

$$||x||^2 - \mu_x(\{0,1,\cdots, n_o\}) < \tfrac{1}{3}\varepsilon.$$

Then

$$\begin{aligned}
|\psi_M(t) - \psi(t)| &= \left| \sum_{k=0}^{\infty} e^{itk} \mu_{x,M}(k) - \sum_{k=0}^{\infty} e^{itk} \mu_x(k) \right| \\
&\leq \left| \sum_{k=0}^{\infty} e^{itk} \mu_{x,M}(k) - \sum_{k=0}^{n_o} e^{itk} \mu_{x,M}(k) \right| \\
&+ \left| \sum_{k=0}^{n_o} e^{itk} \mu_{x,M}(k) - \sum_{k=0}^{n_o} e^{itk} \mu_x(k) \right| \\
&+ \left| \sum_{k=0}^{n_o} e^{itk} \mu_x(k) - \sum_{k=0}^{\infty} e^{itk} \mu(k) \right| . \qquad (6.2)
\end{aligned}$$

The first term and the third term here are less than $\varepsilon/3$, independent of M and t. For

$$\left| \sum_{k=0}^{\infty} e^{itk} \mu_{x,M}(k) - \sum_{k=0}^{n_o} e^{itk} \mu_{x,M}(k) \right|$$

$$= \left| \sum_{k=n_o+1}^{\infty} e^{itk} \mu_{x,M}(k) \right|$$

$$\leq \sum_{k=n_o+1}^{\infty} \mu_{x,M}(k) = ||x||^2 - \mu_{x,M}(\{0,1,\cdots, n_o\})$$

$$\leq ||x||^2 - \mu_x(\{0,1,\cdots, n_o\})$$

$$< \frac{\varepsilon}{3},$$

and the calculation for the third term is similar.

Now select M_o such that if $M \supset M_o$ we have

$$\sum_{k=0}^{n_o} \left| \mu_{x,M}(k) - \mu_x(k) \right| < \frac{\varepsilon}{3}.$$

(This is possible since $\mu_{x,M}(k) \to \mu_x(k)$ for each k.) Then the middle term in (6.2) is also less than $\frac{\varepsilon}{3}$, independent of t, if $M \supset M_o$, for

$$\left| \sum_{k=0}^{n_o} e^{itk} [\mu_{x,M}(k) - \mu_x(k)] \right| \leq \sum_{k=0}^{n_o} |\mu_{x,M}(k) - \mu_x(k)| .$$

So we have shown that for each $x \in QK$

$$\left| \langle e^{itN(M)} x,x \rangle - \langle U(t) x,x \rangle \right| \to 0 \text{ unif. in } t$$

as $M \to H$.

Then by polarization (see [11, p. 13]) we find that for all $x,y \in QK$

$$| \langle e^{itN(M)}x,y \rangle - \langle U(t) x,y \rangle | \to 0 \text{ unif. in } t. \qquad (6.3)$$

What we want to show is that if $x \in QK$, then

$$\| [e^{itN(M)} - U(t)] x \| \to 0 \text{ unif. in } t.$$

But

$$\| [e^{itN(M)} - U(t)] x \|^2 = 2 \langle x,x \rangle - 2\mathrm{Re} \langle e^{itN(M)} x, U(t)x \rangle,$$

so it suffices to show

$$\langle e^{itN(M)} x, U(t) x \rangle \to \langle x,x \rangle \text{ unif. in } t.$$

We may assume $x \neq 0$.

Now since $t \to U(t)x$ is continuous and periodic, the set $\{U(t) x : t \in \mathbb{R}\}$ is compact in K. So, given $\varepsilon > 0$, we can select a finite set of vectors $\{y_1, \cdots, y_n\} \subset K$ such that for every t there is a y_i satisfying

$$\| U(t)x - y_i \| < \frac{\varepsilon}{4\|x\|} .$$

Fix t for the moment, and select a y_i such that this inequality is satisfied. Then

$$\begin{aligned} &| \langle e^{itN(M)} x, U(t) x \rangle - \langle x,x \rangle | \\ &\quad = | \langle e^{itN(M)} x, U(t) x \rangle - \langle U(t)x, U(t)x \rangle | \\ &\quad = | \langle [e^{itN(M)} - U(t)] x, U(t)x \rangle | \\ &\quad \leq | \langle [e^{itN(M)} - U(t)] x, U(t)x - y_i \rangle | \\ &\qquad + | \langle [e^{itN(M)} - U(t)] x, y_i \rangle | \\ &\quad \leq 2 \|x\| \, \|U(t)x - y_i\| \\ &\qquad + | \langle [e^{itN(M)} - U(t)] x, y_i \rangle | . \end{aligned}$$

From this result we conclude that for any t we have

$$\begin{aligned} &| \langle e^{itN(M)} x, U(t)x \rangle - \langle x, x \rangle | \\ &\quad < \frac{\varepsilon}{2} + \max_i | \langle [e^{itN(M)} - U(t)] x, y_i \rangle | . \end{aligned}$$

By (6.3) we can select an M_o such that if $M \supset M_o$ the indicated maximum is less than $\frac{1}{2}\varepsilon$ for all t, so the proof is complete.

Next we need the converse to Lemma 2.

LEMMA 3. <u>If</u> $x \in K$, <u>and if for each</u> t $V(t)x = \lim_{M\to H} e^{itN(M)}x$ <u>exists, and</u> $\| [e^{itN(M)}x - V(t)x] \| \to 0$ <u>uniformly in</u> t, <u>then</u> $x \in QK$.

Proof: Let $x \in K$.

$$\text{Let} \quad \psi_{x,M}(t) = \langle e^{itN(M)}x, x \rangle = \sum_{k=0}^{\infty} e^{itk} \mu_{x,M}(k),$$

$$\text{and} \quad \psi_x(t) = \langle V(t)\, x, x \rangle.$$

From the hypothesis and Schwarz' inequality we have

$$\| \psi_{x,M} - \psi_x \|_\infty \to 0.$$

But we also know that for every k $\mu_{x,M}(k) \to \mu_x(k)$, where $\mu_x(k) = \langle P_k\, x, x \rangle$ (Corollary 1 to Proposition 3.3).

For each integer n we can select a finite-dimensional subspace M^n of H such that for every $M \supset M^n$ we have

$$\| \psi_{x,M} - \psi_x \|_\infty < \frac{1}{n}$$

and

$$| \mu_{x,M}(k) - \mu_x(k) | < \frac{1}{n} \quad \text{for } k = 0, 1, \cdots, n.$$

It follows that

$$\lim_{n\to\infty} \| \psi_{x,M^n} - \psi_x \|_\infty = 0$$

<u>and</u>

$$\lim_{n\to\infty} \mu_{x,M^n}(k) = \mu_x(k) \quad \text{for } k = 0, 1, 2, \cdots.$$

Then by the Lévy continuity theorem [13, p. 191], since ψ_x is evidently continuous, the total variation of μ_x is

$\langle x, x \rangle$, which is to say $\langle Qx, x \rangle = \langle x, x \rangle$, or $x \in QK$.

LEMMA 4. (3) $\Longleftrightarrow$ (5): <u>A necessary and sufficient condition for</u> QK <u>to equal</u> K <u>is that for each</u> $t \in \mathbb{R}$ <u>there exist an operator</u> V(t) <u>such that for every</u> $x \in K$

$$|| [e^{itN(M)} - V(t)] x || \to 0 \tag{6.4}$$

<u>uniformly in</u> t <u>as</u> $M \to H$.

Proof:

Suppose $QK = K$. Then by Lemma 2, for every $x \in K$, $|| e^{itN(M)}x - U(t) x || \to 0$ uniformly in t. Conversely, if some operator V(t) exists satisfying (6.4), then every $x \in K$ satisfies the hypotheses of Lemma 3, so $K \subset QK$.

LEMMA 5. <u>If</u> $QK = K$, <u>then</u> $t \to U(t)$ <u>is a continuous one-parameter unitary group on</u> K <u>such that</u>

$$U(t) W(z) U(-t) = W(e^{it} z) \tag{6.5}$$

<u>for all</u> $z \in H$, $t \in \mathbb{R}$.

Proof: We already observed when defining U(t) that $t \to U(t)$ is a continuous unitary group on QK, which by the assumption here equals K.

By Lemma 2, if $QK = K$

$$U(t) = \text{st-lim}_{M \to H} e^{itN(M)}.$$

It follows that

$$U(t) W(z) U(-t) = \text{st-lim}_{M \to H} e^{itN(M)} W(z) e^{-itN(M)}.$$

But if M is any finite-dimensional subspace of H containing z, we have

$$e^{itN(M)}\, W(z)\, e^{-itN(M)} = W(e^{it}\, z).$$

It follows that the limit is $W(e^{it}\, z)$ also.

LEMMA 6. (3) $\Longrightarrow$ (6): <u>If</u> QK = K <u>then there exists a self-adjoint operator</u> N <u>on</u> K <u>whose spectrum is contained in the non-negative integers and which satisfies</u>

$$e^{itN}\, W(z)\, e^{-itN} = W(e^{it}\, z).$$

Proof: By Lemma 5, if QK = K, then the operator $U(t) = \sum_{k=0}^{\infty} e^{itk}\, P_k$ satisfies (6.5). But the self-adjoint generator of U is $\sum_{k=0}^{\infty} kP_k$, whose spectrum is contained in the non-negative integers. Taking $N = \sum_{k=0}^{\infty} kP_k$, the Lemma is proved.

The next lemma is now easy to prove, but is the second main step in the proof of the theorem.

LEMMA 7. QK <u>is invariant under the action of the Weyl algebra</u>.

Proof: If $x \in QK$ and $z \in H$, then

$$e^{itN(M)}\, W(z)x = W(e^{it}\, z)\, e^{itN(M)}x$$

for every finite-dimensional subspace M of H containing z. So

$$\begin{aligned} &\| e^{itN(M)}\, W(z)x - W(e^{it}\, z)\, U(t)\, x\| \\ &= \| W(e^{it}\, z)\, e^{itN(M)}x - W(e^{it}\, z)\, U(t)x\| \\ &= \| e^{itN(M)}x - U(t)x \| \end{aligned}$$

which converges to zero uniformly in t since $x \in QK$ (Lemma 2).

By Lemma 3, using $V(t)[W(z)x] = W(e^{it} z)\, U(t)x$, we find $W(z)x \in QK$. Hence we have shown

$$x \in QK \Longrightarrow W(z)x \in QK \quad \text{for all } z \in H.$$

Now suppose A is in the weakly closed algebra $\underset{\sim}{A}_M(W)$ generated by $\{W(z) : z \in M\}$, where M is a finite-dimensional subspace of H. Then using the Weyl relations (1.5), we see that A is a weak limit of operators A_n having the property that each A_n is a finite linear combination of $W(z)$'s.

Hence each $A_n\, x \in QK$ if $x \in QK$, so that for all $y \in (I-Q)K$ we have

$$\langle Ax, y \rangle = \lim_{n\to\infty} \langle A_n x, y \rangle = \lim_{n\to\infty} 0 = 0.$$

Therefore $A\, x \in QK$.

Finally, each operator B in the Weyl algebra is a uniform limit of A's of the above type, so $B\, x \in QK$ if $x \in QK$.

LEMMA 8. (2) $\Longrightarrow$ (3): *Suppose the canonical isomorphism* φ *of* $\underset{\sim}{A}(W_o)$ *onto* $\underset{\sim}{A}(W)$ *is a direct sum* $\oplus_{i\in I}\, \varphi_i$ *where each* φ_i *is a cyclic representation on, say,* K_i, *with cyclic vector* v_i *such that* $v_i \in QK$. *Then* $QK = K$.

Proof: To say v_i is cyclic means that $\{\varphi_i(A)v_i : A \in \underset{\sim}{A}(W_o)\}$ is dense in K_i. Then, since $v_i \in QK$, we know from Lemma 7 that this dense subset of K_i is contained in QK. From the fact that QK is closed, we conclude that each K_i is contained in QK; and from this we see that $K \subset QK$.

Now we have

$$(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (6)$$

$$(3) \Longleftrightarrow (4), \qquad (3) \Longleftrightarrow (5)$$

So we just have to prove (6) $\Longrightarrow$ (1). We shall do this in two steps, of which the first is the last main point of the proof of the theorem.

LEMMA 9. *If $K \neq \{0\}$ and there exists a self-adjoint operator N on K whose spectrum is bounded below by an element of the point spectrum and which satisfies*

$$e^{itN} W(z) e^{-itN} = W(e^{it} z)$$

for all $z \in H$, $t \in \mathbb{R}$, then there is a sub-Weyl system of W which is unitarily equivalent to the standard zero-interaction system.

Proof: Let b = inf spectrum (N), and let $N' = N - bI$. If $V(t) = e^{itN'}$, then the following are true:

(i) The self-adjoint generator of V is nonnegative;

(ii) $V(t) W(z) V(-t) = W(e^{it} z)$ for all $z \in H$, $t \in \mathbb{R}$;

(iii) There is a unit vector $x \in K$ such that $V(t)x = x$ for all t. (By hypothesis we can find an eigenvector x of N with eigenvalue b, and we may choose it to have unit length.)

Let K_x be the smallest subspace of K which contains x and is invariant under the Weyl algebra. Namely

$$K_x = \text{closure of } \underset{\sim}{A}(W)x$$

where $\underset{\sim}{A}(W)$ is the Weyl algebra for W. Then if we restrict each W(z) to K_x we get a cyclic Weyl system W_x with cyclic vector x. Furthermore K_x is invariant under V(t) since

$$\begin{aligned} V(t)\, W(z) &= V(t)\, W(z)\, V(-t)\, V(t)x \\ &= W(e^{it} z)x \quad \in K_x . \end{aligned}$$

It is clear that if $V_x(t)$ is the restriction of $V(t)$ to K_x, then $t \to V_x(t)$ is a continuous unitary group on K_x which also satisfies (i), (ii), and (iii) above with V replaced by V_x and W replaced by W_x. Hence by Proposition 6.1 W_x is unitarily equivalent to the standard zero-interaction Weyl system, which proves the Lemma.

Now to complete the proof of (6) $\Longrightarrow$ (1) it is only necessary to use Zorn's Lemma in the usual way.

LEMMA 10. (6) $\Longrightarrow$ (1): If there exists a self-adjoint operator N on K, whose spectrum is contained in the non-negative integers, and which satisfies

$$e^{itN} W(z) e^{-itN} = W(e^{it} z)$$

for all $z \in H$, $t \in \mathbb{R}$, then W is unitarily equivalent to a direct sum of standard zero-interaction Weyl systems.

Proof: From Zorn's Lemma and Lemma 9, we know that there exists a maximal invariant subspace $\overline{K}$ of K having the property that the restriction of W to $\overline{K}$ is unitarily equivalent to a direct sum of zero-interaction Weyl systems. We will show $\overline{K} = K$ by deriving a contradiction from the assumption that the orthocomplement $K^{\perp}$ of $\overline{K}$ is not zero.

If $K^{\perp} \neq \{0\}$, then $K^{\perp}$ is invariant under the Weyl algebra (since $\overline{K}$ is), and so the restriction $W^{\perp}$ of W to $K^{\perp}$ is a Weyl system. Furthermore e^{itN} leaves $K^{\perp}$ invariant too, since e^{-itN} leaves $\overline{K}$ invariant and hence for any $x \in K^{\perp}$ and $y \in \overline{K}$, we have

$$0 = \langle x, e^{-itN} y \rangle = \langle e^{itN} x, y \rangle.$$

It follows that the self-adjoint generator N^1 of the restriction of the group $t \to e^{itN}$ to K^1 is the restriction of N to K^1. In particular N^1 has spectrum contained in the non-negative integers, because N does. Finally

$$\exp[itN^1]\, W^1(z)\, \exp[-itN^1] = e^{itN}\, W(z)\, e^{-itN}\Big|_{K^1}$$

$$= W(e^{it} z)\Big|_{K^1} = W^1(e^{it} z).$$

Thus we see that the operator N^1 satisfies the hypotheses of Lemma 9 with respect to the Weyl system W^1 on K^1. So, that Lemma tells us there is a sub-Weyl system of W^1 acting on, say ${K_x}^1 \subset K^1$, which is unitarily equivalent to the standard zero-interaction Weyl system.

But then $\overline{K} \oplus {K_x}^1$ is a subspace of K which is larger than $\overline{K}$ and has the property with respect to which $\overline{K}$ was supposed to be maximal, namely the restriction of W to $\overline{K} \oplus {K_x}^1$ is unitarily equivalent to a direct sum of standard zero-interaction systems. This is the desired contradiction.

This completes the proof of Theorem 1.

Now we will prove that the net-convergence criterion (5) is equivalent to the sequential convergence criterion (5s). (See Remark 3 after the statement of Theorem 1.) (Of course these two criteria are not equivalent in general, only in the context here.)

Supposing (5) is true, from Theorem 1 we know the representation is a direct sum of standard zero-interaction representations, in which case (5s) is true by Corollary 2 to Proposition 2.4.

On the other hand, suppose (5s) is true. Let x be a unit vector in K and E the state it determines. Select an increasing sequence $\{M^n\}$ such that $\mu_{E,M^n}(k) \to \mu_E(k)$ for every k, and $P_{M^n} \to I$. Using (5s) we conclude that the sequence $n \to \exp[itN(M^n)]x$ converges to, say, x(t), and the function ψ defined by $\psi(t) = \langle x(t), x \rangle$ is continuous at zero. Hence, letting $\psi_{M^n}(t) = \langle \exp[itN(M^n)]\, x, x \rangle$ we see that the sequence $n \to \psi_{M^n}$ converges to ψ. Thus by the Lèvy continuity theorem, ψ is the Fourier transform of μ_E, and consequently μ_E has total variation 1. Since this is true for any unit vector $x \in K$, we have shown that (5s) implies condition (3) of Theorem 1, which implies (5).

REMARKS: We have proved a number of facts which are not actually stated in the theorem, so we shall point out a few consequences here.

First, if W is any Weyl system acting on, say, K, we can write K as a direct sum

$$K = QK \oplus (I - Q)K.$$

By Lemma 7 and the fact that the Weyl algebra $\underset{\sim}{A}(W)$ is self-adjoint, we see that $\underset{\sim}{A}(W)$ leaves both summands invariant. On the first, by Theorem 1, W acts like a direct sum of standard zero-interaction Weyl systems. In $(I - Q)K$, <u>every</u> normalizable state has an infinite number of particles with probability one, because for any $v \in (I-Q)K$ such that $|v| = 1$ the probability of finding a finite number of particles in the state determined by v is

$$\sum_{k=0}^{\infty} \langle P_k v, v \rangle = \langle Qv, v \rangle = 0.$$

From this we see that if neither QK nor (I-Q)K is trivial, then given any p between 0 and 1 it is possible to find normalizable states for which the probability of finding a finite number of particles is the given number p. However if QK = {0}, that is, no subsystem of W is the standard zero-interaction system, then this phenomenon can not occur - every state has an infinite number of particles with probability one.

All the above considerations simplify to relatively transparent statements in the case that the Weyl system is irreducible. Since for a quantum field of "elementary" particles, the associated Weyl system is expected to be irreducible anyway, these are the systems of greatest interest.

THEOREM 2. *Let* H *be a complex inner product space,* W *a Weyl system over* H *acting irreducibly on the Hilbert space* K. *The following are equivalent*:

(1') W *is unitarily equivalent to the standard zero-interaction Weyl system*.

(2') One *normalizable state of the Weyl algebra of* W *has a finite number of particles with non-zero probability*.

(3') *The finite-particle subspace of* K *is* K *itself*.

(4') Every *normalizable state of the Weyl algebra of* W *has a finite number of particles with probability one*.

(5') For one non-zero vector $v \in K$, the net $M \to \exp[itN(M)]\, v$ converges to some $v(t)$ uniformly in t (as $M \to H$ through the finite-dimensional subspaces of H).

(6') There exists a self-adjoint operator N on K whose spectrum is bounded below, which satisfies

$$e^{itN}\, W(z)\, e^{-itN} = W(e^{it}\, z)$$

for all $z \in H$, $t \in \mathbb{R}$.

Proof: Evidently (1') $\Longrightarrow$ (2').

If (2') is true, then there is a vector $v \in K$ such that $Qv \neq 0$, where Q is the projection on the finite-particle subspace. This implies that QK is a non-trivial subspace of K, which by Lemma 7 is invariant. Hence $QK = K$, so (3') is true.

(3') $\Longrightarrow$ (4') as in Theorem 1.

(4') $\Longrightarrow$ (5') since by Theorem 1 (4') implies an even stronger result than (5').

(5') $\Longrightarrow$ (6') since by Lemmas 3 and 7, the set $QK = \{v \in K : V(t) = \lim_{M \to H} e^{itN(M)}\, v$ exists uniformly in $t\}$ is an invariant subspace, so since it is not zero, it is K. Then

$$U(t) = \sum_{k=0}^{\infty} e^{itk}\, P_k = \text{st-lim}_{M \to H}\, e^{itN(M)}$$

is a continuous one-parameter unitary group whose self-adjoint generator N is non-negative and satisfies the condition in (6').

To show (6') $\Longrightarrow$ (1'), we will show (6') $\Longrightarrow$ condition (6) of Theorem 1. For (6) $\Longrightarrow$ (1), and evidently (1), together with irreducibility, implies (1').

To show (6') $\Longrightarrow$ (6), we must show that the existence of a self-adjoint N whose spectrum is bounded below and which satisfies

$$e^{itN} W(z) e^{-itN} = W(e^{it} z)$$

implies the existence of an $\overline{N}$ doing the same thing, but whose spectrum is contained in $\{0,1,2,\cdots\}$. Let b be the infimum of the spectrum of N. Notice that

$$e^{2\pi iN} W(z) e^{-2\pi iN} = W(e^{2\pi i} z) = W(z)$$

so $e^{2\pi iN}$ commutes with all the W(z)'s. Hence $e^{2\pi iN} = e^{2\pi ia} I$ for some real number a such that $b \leq a < b + 1$. It follows that

$$\lambda \epsilon \text{ spectrum } N \Longrightarrow \lambda = a + \text{an integer}.$$

But the spectrum of N is bounded below by b, so $\lambda \epsilon$ spectrum N $\Longrightarrow \lambda = a$ + non-negative integer. Hence N - aI is a self-adjoint operator with spectrum contained in $\{0,1,2, \cdots\}$. Clearly

$$\exp[it(N-aI)] W(z) \exp[-it(N-aI)]$$
$$= e^{itN} W(z) e^{-itN}$$
$$= W(e^{it} z).$$

So take $\overline{N}$ = N - aI, and the proof is complete.

It may not yet be clear that Theorem 2 implies the result originally stated by Wightman and Schweber [28], so we shall prove a rigorous form of that result now.

PROPOSITION 6.4 <u>Let</u> $\mathcal{E} = \{e_1, e_2, \cdots\}$ <u>be an orthonormal basis of the separable complex inner product space</u> H, <u>and</u>

let H_o be the algebraic span of $\mathcal{E}$. Let M^n be the span of $\{e_1, \cdots, e_n\}$. Suppose W is an irreducible Weyl system over H on a separable space K such that

$$V(t) = \text{st-lim}_{n\to\infty} \exp [itN(M^n)]$$

exists for every $t \in \mathbb{R}$. Then W is unitarily equivalent to χW_o, where W_o is the standard zero-interaction representation over H and χ is a character of H mod H_o (Definition 5.1).

Proof:

Consider the Weyl system W' over H_o which is the restriction of W to H_o. Then for each finite-dimensional subspace M of H_o, $N(M;W') = N(M;W)$, since both operators satisfy (a), (b), and (c) of Proposition 3.1. So we shall use the notation N(M) to represent either one.

The hypothesis is that

$$V(t) = \text{st-lim}_{n\to\infty} \exp [itN(M^n)]$$

exists. Then $V(t + t') = V(t)\, V(t')$, so V is a one-parameter group. Moreover, V(t) is an isometry, since it is the strong limit of unitaries. This together with the fact that $V(t)\, V(-t) = V(0) = I$, implies that V(t) is unitary. Furthermore, for any $x, y, \in K$,

$$\langle V(t)\, x, y \rangle = \lim_{n\to\infty} \langle \exp[itN(M^n)]\, x, y \rangle,$$

so $t \to \langle V(t)\, x, y \rangle$ is measurable. Thus we have shown that V is a weakly measurable one-parameter unitary group, which, since K is separable, implies V is continuous [12].

By Stone's Theorem (see [18]), there exists a self-adjoint operator N such that $V(t) = e^{itN}$. To see N is non-negative, let $x \in K$, and consider

$$\psi(t) = \langle e^{itN} x,x \rangle = \lim_{n\to\infty} \langle \exp [itN(M^n)] x,x \rangle$$

$$= \lim_{n\to\infty} \sum_{k=0}^{\infty} e^{itk} \langle P_k(M^n) x,x \rangle.$$

By the Lèvy continuity theorem [13], the fact that ψ is continuous implies

$$\langle e^{itN} x,x \rangle = \sum_{k=0}^{\infty} e^{itk} \lim_{n\to\infty} \langle P_k(M^n)x, x \rangle$$

so the spectrum of N is in fact contained in $\{0,1,\cdots\}$.

Also, for every $z \in H_o$

$$e^{itN} W'(z)\, e^{-itN} = \lim_{n\to\infty} \left[\exp[itN(M^n)]\, W'(z)\, \exp[-itN(M^n)]\right]$$

$$= \lim_{n\to\infty} W'(\exp[itP_{M^n}]z)$$

$$= W'(e^{it} z),$$

since there exists an n_o such that

$$P_{M^n}\, z = z \quad \text{for every } n > n_o.$$

Thus by Theorem 2 (6'), W' is unitarily equivalent to the standard zero-interaction system over H_o. Since W' is simply the restriction of W to H_o, we know from Proposition 5.1 that W is unitarily equivalent to $\chi\, W_o$ for some character χ of H mod H_o.

7. STATES WITH A FINITE NUMBER OF PARTICLES

Given any regular state of the Weyl algebra $\underset{\sim}{A}(W_0)$ over H, one can produce, using the Gelfand-Segal construction, a concrete Weyl system W over H acting on, say, K such that the state E is normalizable in K. (See Section 4.) Using this procedure and the results of the previous section, we can derive a characterization of those regular states of the Weyl algebra which have a finite number of particles with probability one.

DEFINITION 7.1. A state E of a C*-algebra $\underset{\sim}{A}$ is pure if and only if it is not a convex linear combination of two other states.

It is known [19; or see 5] that the Gelfand-Segal representation corresponding to a pure state is irreducible, and conversely a vector in an irreducible representation determines a pure state. Using these facts it is easy to prove

THEOREM 3. For any regular pure state of the Weyl algebra over a complex inner product space, the probability of finding a finite number of particles is either zero or one. If the probability is one, then the state is normalizable in the standard zero-interaction representation (i.e. there is a unit vector v in the space K_0 of the standard zero-interaction Weyl system such that

$$E(A) = \langle A v, v \rangle$$

for all $A \epsilon \underset{\sim}{A}(W_0)$.)

Proof: Let E be a regular pure state. Then by the Gelfand-Segal construction we can produce an irreducible representation π of the Weyl algebra $\underset{\sim}{A}(W_o)$ on some Hilbert space K such that there is a unit vector $x \in K$ such that

$$E(A) = \langle \pi(A)\, x,\, x \rangle \quad \text{for all } A \in \underset{\sim}{A}(W_o).$$

Moreover, by the regularity of the state E, $z \to \pi(W(z))$ is a Weyl system on K. By Theorem 2, if the probability of finding a finite number of particles in E is not <u>zero</u>, then it is <u>one</u>. Furthermore, if the probability is one, then Theorem 2 tells us that π is unitarily equivalent to the identity. The unitary operator which effects this equivalence takes x into a vector v (in the standard zero-interaction space K_o) satisfying the condition stated in Theorem 3. So the Theorem is proved.

At this point we could procede in several ways to characterize all the regular states (not just the pure ones) which have a finite number of particles with probability one. The procedure which gets us least involved in irrelevant topological questions is to look once again at the proof of Theorem 1, and derive the next result from there rather than from the more specialized Theorem 3.

THEOREM 4. <u>Let</u> $\underset{\sim}{A}(W_o)$ <u>be the Weyl algebra over</u> H, E <u>a regular state of</u> $\underset{\sim}{A}(W_o)$. <u>The following are equivalent</u>:

(a) <u>The probability of finding a finite number of particles in the state</u> E <u>is</u> one.

(b) <u>The Weyl system which results from using the Gelfand-Segal construction with the state</u> E <u>is unitarily equivalent to a direct sum of standard zero-interaction Weyl systems</u>.

(c) <u>There exists a non-negative trace-class operator</u> D <u>on</u> H_F <u>such that</u> Trace D = 1 <u>and</u>

$$E(A) = \text{Trace }(AD) \quad \underline{\text{for all}}\ A \in \underset{\sim}{A}(W_o).$$

(d) <u>Letting</u> $\psi_M(t) = E\left(e^{itN(M)}\right)$, <u>then the net</u> $M \to \psi_M$ <u>converges uniformly</u>.

Proof:

(a) $\Longrightarrow$ (b):

Consider the Gelfand-Segal representation of $\underset{\sim}{A}(W_o)$ determined by E. This representation is cyclic with a cyclic vector whose state is E. If (a) is true, then E has a finite number of particles with probability one, so by Theorem 1, the representation is a direct sum of standard zero-interaction systems, from which we see that (b) is true.

(b) $\Longrightarrow$ (c): Suppose (b) is true. Let v be the cyclic vector whose state is E. If we write the space K on which W acts as $K = \oplus_{i\epsilon I} K_i$, where the restriction of W to K_i is unitarily equivalent to W_o, then we see that the projection of v into K_i can be non-zero for only a countable number of indices. Since v is cyclic, we conclude that the direct sum is actually countable.

So without loss of generality we may assume

$$K = K_1 \oplus K_2 \oplus \cdots \qquad \text{(finite or countably infinite)}$$

where the restriction of W to each K_n is W_o, and the projection of v into each K_n is non-zero. Write $v = a_1 v_1 \oplus a_2 v_2 \oplus \cdots$ where each $a_n > 0$ and each v_n is a unit vector. Then $1 = ||v||^2 = \Sigma\, a_n^2$. Furthermore, for every $A \in \underset{\sim}{A}(W_o)$

$$E(A) = \langle \textstyle\sum_n \oplus\, A(a_n v_n), \quad \sum_m \oplus\, a_m v_m \rangle$$

$$= \textstyle\sum_n a_n^2 \langle A\, v_n, \; v_n \rangle .$$

Let R_n be the projection of H_F onto the subspace spanned by v_n, and let $D = \Sigma\, a_n^2 R_n$. Then D is evidently non-negative, and Trace $D = \Sigma\, a_n^2 = 1$.

Now we just have to show E is related to D by

$$E(A) = \text{Trace } (AD).$$

In case the direct sum is infinite we need to observe first that the sum $\Sigma\, a_n^2 R_n$ converges in the L^1 sense. This follows from the fact that

$$\text{Trace } \left(\left| \sum_{n=1}^{\infty} a_n^2 R_n - \sum_{n=1}^{n_o} a_n^2 R_n \right| \right)$$

$$= \text{Trace } \left(\sum_{n=n_o+1}^{\infty} a_n^2 R_n \right)$$

$$= \sum_{n=n_o+1}^{\infty} a_n^2$$

which $\longrightarrow$ 0 as $n_o \to \infty$. So whether the sum is finite or infinite, we have

$$E(A) = \sum_n a_n^2 \quad \langle A\, v_n, v_n \rangle$$

$$= \sum a_n^2 \text{ Trace } (AR_n)$$

$$= \text{Trace } [A(\sum_n a_n^2 R_n)]$$

$$= \text{Trace } AD.$$

Hence (c) is proved.

(c) $\Longrightarrow$ (d):

If $E(A) = \text{Trace } (AD)$ for all $A \in \underset{\sim}{A}(W_o)$, then in particular for every finite-dimensional subspace M of H

$$E(\exp[itN_o(M)]) = \text{Trace } (\exp[itN_o(M)]D)$$

where $N_o(M)$ is the number operator over M in the standard zero-interaction representation. (Definition 2.3)

Now the function $A \to \text{Trace } (AD)$ is strongly continuous on the unit sphere of operators, so, given $\varepsilon > 0$, we can find a basic strong neighborhood $\mathcal{U}$ of $e^{itN_o(H)}$ in the unit sphere such that for all $A \in \mathcal{U}$

$$\left| \text{Trace } (AD) - \text{Trace } \left(e^{itN_o(H)} D \right) \right| < \varepsilon.$$

$\mathcal{U}$ has the form

$$\mathcal{U} = \left\{ A: \ ||A|| = 1, \ \text{ and } \ || \left[A - e^{itN_o(H)} \right] x_i || < 1, \ i=1,\cdots,n \right\}$$

But we know, from Theorem 1, that

$$\left\| \left[e^{itN_o(M)} - e^{itN_o(H)} \right] x_i \right\| \to 0$$

uniformly in t for $i = 1,\cdots, n$, so we can select M_o such that for all $M \supset M_o$

$$\exp\, [itN_o(M)] \in \mathcal{U} \text{ for every } t.$$

It follows that if $M \supset M_o$

$$|\psi_M(t) - \text{Trace}\left[e^{itN_o(H)} D\right]| < \varepsilon \text{ for all } t.$$

Hence we have the ψ_M's converging uniformly, and (d) is proved.

(d) $\Longrightarrow$ (a): This is proved as in the proof of Lemma 3 to Theorem 1. From the fact that $||\psi_M - \psi||_\infty \to 0$ and $\mu_{E,M} \to \mu_E$ we conclude that μ_E has total variation 1, which is (a).

REMARK 1. As in Theorem 1, the condition (d) in Theorem 4 can be replaced, in case H is separable, by

(ds) For every increasing sequence $\{M^n\}$ of finite-dimensional subspaces of H such that $P_{M^n} \to I$, the sequence $n \to \psi_{M^n}$ converges to a function ψ which is continuous at zero.

REMARK 2. Suppose v is a unit vector in the representation space for a Weyl system over H. From Theorem 4 we can determine the behavior of the number operators N(M) in the <u>entire</u> cyclic representation generated by v simply by checking whether (a) or (d) holds on the <u>single</u> vector v. In particular if v is in the domain of $[N(M)]^{1/2}$ for every finite-dimensional $M \subset H$, and $\lim_{M \to H} ||[N(M)]^{1/2} v||$ exists, then surely (a) is true for the state determined by v, so that (b) follows. This is slightly stronger than a recent result of Dell'Antonio and Doplicher.*

*G.F. Dell'Antonio, private communication.

ACKNOWLEDGMENTS

It is a pleasure to thank I.E. Segal for the inception of this problem and for encouragement and advice during its solution; L. Gross for many stimulating and helpful conversations; and J. Robertson, C. Herz, and H. Kesten for technical suggestions.

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