CODING FOR A DISCRETE INFORMATION SOURCE WITH A DISTORTION MEASURE

by

THOMAS JOHN GOBLICK, JR.

B. S., Bucknell University (1956)

S. M., Massachusetts Institute of Technology (1958)

E. E., Massachusetts Institute of Technology (1960)

SUBMITTED IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

October 1962

Signature of Author . . . .
Department of Electrical Engineering, October 1962

Certified by . . . .
Thesis Supervisor

Accepted by . . . .
Chairman, Departmental Committee on Graduate Students

CODING FOR A DISCRETE INFORMATION SOURCE WITH A DISTORTION MEASURE

by

THOMAS JOHN GOBLICK, JR.

Submitted to the Department of Electrical Engineering in October 1962 in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

The encoding of a discrete, independent letter information source with a distortion measure is studied. An encoding device maps blocks of source letters into a subset of blocks of output letters, reducing the number of different blocks that must be transmitted to a receiving point. The distortion measure, defined between letters of the source and output alphabets, is used to compare the actual source output with the encoded version which is then transmitted to the receiver. This problem was previously studied by Shannon who showed that the constraint that the average distortion per letter between the source output and its facsimile at the receiving point not exceed d^* implies a minimum necessary information capacity (dependent on d^*) between source and receiver.

In this work, the average distortion per letter for block codes of fixed rate and length n is upper and lower bounded, and for optimum block codes these bounds are shown to converge to the same limit, with the convergence being as a negative power of n as $n \to \infty$. The asymptotic agreement of these bounds for optimum codes leads to an alternate description of Shannon's rate-distortion function $R(d^*)$. Moreover, this analysis of optimum block codes gives an explicit computational method for calculating the rate-distortion function. The final results may be interpreted in terms of the same test channel described by Shannon, though no such test channel is actually used in the bounding arguments.

In studying the instrumentation of codes for sources, as a tractable example the binary symmetric, independent letter source with Hamming distance as the distortion measure is treated. The existence of group codes which satisfy the upper bound on average distortion for optimum block codes is proved. The average distortion and the average number of computations per encoded digit are upper bounded for sequential encoding schemes for both group codes and tree codes.

The dual nature of channel coding problems and source coding with a distortion measure is pointed out in the study of topics closely related to the zero error capacity of channels, channels with side information, and a partial ordering of channels.

Thesis Supervisor: Robert G. Gallager
Title: Assistant Professor of Electrical Engineering

ACKNOWLEDGEMENT

I wish to express special thanks to Professor Robert Gallager for his patient supervision and many very valuable suggestions during the course of this research, and also to Professor Claude Shannon for motivating and encouraging this work. I also wish to thank Professor Irwin Jacobs, Professor Peter Elias, and Dr. Barney Reiffen for helpful suggestions during the course of many discussions.

I am grateful to my fellow graduate students in the Information Processing and Transmission group of the Research Laboratory of Electronics, especially Robert Kennedy, James Massey, and Jacob Ziv, for the benefit of many helpful discussions which contributed to this research.

I wish to express my thanks to Dr. Paul Green and Dr. Robert Price of Lincoln Laboratory who encouraged me to undertake my graduate study as a Lincoln Laboratory Staff Associate. The difficult job of typing this thesis was cheerfully done by Miss Linda M. Giles, to whom I am grateful. I thank also Lincoln Laboratory for their financial support during the three years of my graduate study.

Finally, it would have been impossible to undertake and complete my graduate study if it had not been for the many sacrifices of my wife Carolyn.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

In many communication systems it is required that messages be transmitted to a receiver with extremely high reliability. For this reason there has been a great effort to put into practice Shannon's theorems on coding information for error-free communication through noisy channels.

There are also many communication systems in which not exact but merely approximate transmission of messages is required. For example, it is certainly not necessary to transmit television pictures to viewers without any errors. Let us consider a communication system to transmit television pictures across the country. It is impossible to transmit pictures over a distance of several thousand miles in the same form of an amplitude modulated carrier for local transmission to viewers because the cumulative effect of the noise along the entire transmission system would produce objectionable picture quality. Therefore, for cross-country transmission, a picture is divided into a large number of discrete picture cells or elements, and the light intensities of these picture elements are coarsely quantized into discrete levels. This discrete representation of a picture (the encoded picture) is then transmitted without error from one relay station to the next across the country. In order to reduce the transmission capacity requirements to transmit the encoded version of a picture without error, the number of quantum levels for encoding the picture element intensities may be reduced.

However, it is clear that there is a trade-off between the required transmission capacity and the distortion introduced into the picture by the quantization. If the quantization is made too coarse, the resulting distortion will render the encoded version of the picture objectionable even when the actual transmission of the encoded picture is done without error. We may conclude that such a system requires a certain minimum amount of information to be transmitted in order to maintain acceptable picture quality.

This work is concerned with a much simpler, abstract problem than the television example. We shall confine ourselves to the consideration of a discrete information source which chooses letters x from a finite alphabet X independently with probability P(x). The output of the source, a sequence of letters, is to be transmitted over a channel and reproduced, perhaps only approximately, at a receiving point. We are given a distortion measure $d(xy) \geq 0$, which defines the distortion (or cost) when source letter x is reproduced at the receiver as letter y of the output alphabet Y. The Y alphabet may be identical to the X alphabet, or it may be an enlarged alphabet which includes special symbols for unknown or partly known letters.

Consider another example in which we have a source which chooses integers from 0 to 10 inclusive, independently and with equal probability. Suppose we are given the distortion measure $d(xy) = |x-y|$, where the output alphabet is identical to the source alphabet. If we are required to reproduce each letter with no more than one unit of distortion, we find that we need to use only four output letters to represent the source output well enough to satisfy this requirement on distortion. We therefore need a transmission channel capable of sending any one of four integers without error to the decoder. (See Figure 1.1) The decoder is a device which merely looks up the output

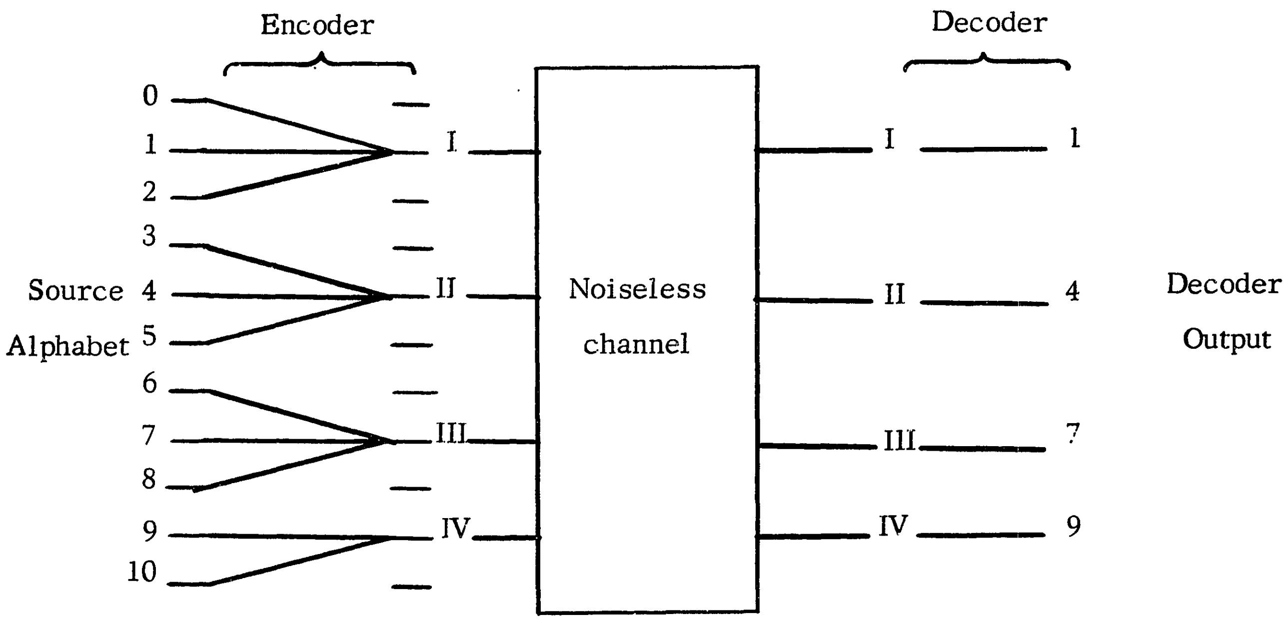


Figure 1.1 An example of a simple source encoder and decoder.

letter that a received integer corresponds to, and this output letter is the facsimile of the source output. If we were required to reproduce each source letter with zero distortion, we would require a channel capable of sending any one of eleven integers to the decoder without error. It is clear from this example that a specification of the tolerable distortion implies a certain minimum required transmission capacity for this type of source encoding. A different type of specification on the tolerable distortion, such as average distortion per letter of one unit or less, would lead to a different minimum required transmission capacity between source and receiver.

We wish to consider a more general type of source encoder which maps blocks of n source letters into a set of M blocks of n output letters called code words. When the source produces a block of n letters, the encoder maps this block into one of the M code words, say the j^{th} one. The output of the encoder is then the integer j, and this is transmitted without error over a channel to the decoder. The decoder output is the j^{th} code word, which is a sequence of output letters. The combination of the source and encoder resembles a new source which selects one of M integers to be transmitted to the decoder. A channel which is capable of sending any one of M integers to the decoder without error is needed, and in view of this we define the transmission rate per source letter for an encoder as

$$R = \frac{1}{n} \log M.$$

We wish to find encoders which minimize M for a given block length n while satisfying a given specification on the tolerable level of distortion.

Throughout this work we will assume that the transmission channel introduces no errors in sending the encoder output to the decoder. Error free transmission from encoder to decoder may actually involve a noisy channel with its own coding and decoding equipment to give the required reliability. We make the assumption of an error free transmission channel in order to keep the source encoding problem separate from the problem of combating channel noise.

There are obviously many ways in which the tolerable level of distortion could be specified. In the example of Fig. 1.1, we required that each source letter be reproduced at the receiver with no more than D units of distortion. Another widely applicable fidelity criterion is the average distortion per letter. Furthermore, this fidelity criterion is mathematically more tractable than that used in the example of Fig. 1.1, and a much more interesting theoretical development can be achieved. The majority of this research, therefore, deals with the fidelity criterion of average distortion per letter.

When a block of source letters $u = x_1 x_2 \ldots x_n$ is encoded, transmitted, and reproduced at the receiver as the block of output letters $v = y_1 y_2 \ldots y_n$, the average distortion per letter is

$$d(uv) = \frac{1}{n} \sum_{i=1}^{n} d(x_i y_i).$$

The noiseless channel assumption allows the transmission channel to be represented as a fixed transformation, and since an encoder and decoder are fixed transformations, the combination of an encoder, transmission channel, and decoder may be represented simply as a transformation T(u) defined on all possible blocks of n source letters. The T(u) are actually blocks of n output letters, and we may write the average distortion for our communication system as

$$\bar{d} = \sum_{u} P(u)\, d(u, T(u)) \; ,$$

where P(u) is the probability that the source produces the block of letters u. To minimize the average distortion of the system for a particular set of M code words, the encoder should map each block of source letters into the code word which gives the smallest average distortion per letter with the source block. The operation of the source encoder is very similar to the operation of a noisy channel decoder, which must map a channel output sequence into the code word which gives the lowest probability of error. The source decoder is also seen to be analogous to the channel encoder. From our experience with channel coding and decoding, we expect that the source encoder will be a far more complex device than the source decoder. A block diagram of the communication system that we study in this work is shown in Figure 1.2.

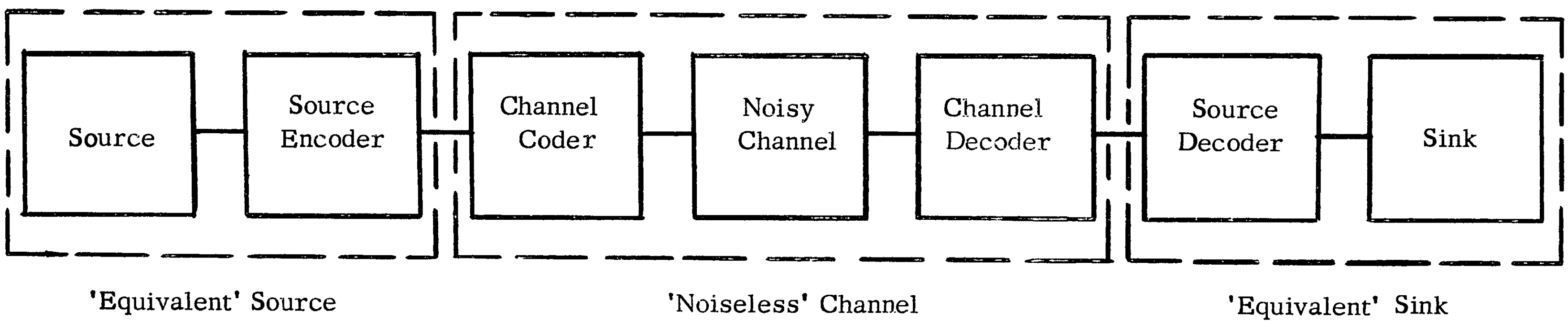


Figure 1.2 A block diagram of the communication system studied in this work.

The concept of a fidelity criterion is basic to the information theory. The information rate of an amplitude continuous source is undefined unless a tolerable level of distortion according to some distortion measure is specified, since exact transmission of the output of a continuous source to a receiver would require an infinite information capacity. This is somewhat analogous to the problem of finding the channel capacity of an additive gaussian-channel which is undefined until one puts constraints on the signals that the transmitter may use.

The fundamental work on the encoding of a discrete information sourch with a distortion measure was done by Shannon.[15] He showed that the constraint that the average distortion per letter be no more than a certain amount, say d^*, led to a unique definition of the equivalent information rate $R(d^*)$ of the source. The rate-distortion function, $R(d^*)$, was defined by Shannon as follows. Given the set of source probabilities $P(x)$, and a distortion measure $d(xy)$, we can take an arbitrary assignment of transition probabilities $q(y|x)$, $\left(q(y|x) \geq 0,\ \sum_Y q(y|x) = 1\right)$, and calculate the quantities

$$d\,(q(y|x)) = \sum_{X,Y} P(x)\; q\,(y|x)\, d(xy)$$

$$R(q(y|x)) = \sum_{X,\,Y} P(x)\, q\,(y|x)\, \log \frac{q\,(y\,|\,x)}{\sum_X P(x')\, q(y|x')} \;.$$

The rate-distortion function $R(d^*)$ is defined as the minimum value of $R(q\,(y|x))$ under the variation of the $q(y|x)$ subject to their probability constraints and subject to the constraint that $d(q(y|x)) \leq d^*$. The use of a test channel $q(y|x)$ with the source to define $R(d^*)$ is similar to the use of a test source with a channel to define channel capacity.

The test channel is adjusted to minimize the average mutual information of the source-test channel combination while the average distortion is kept equal to or less than d^*, when transmitting the source output directly through the test channel.

The significance of the function $R(d^*)$ is explained by the following powerful results. Shannon showed that there are no encoding schemes with rate less than $R(d^*)$ which give average distortion per letter d^* or less, but there are encoding schemes which give average distortion per letter d^* with rates arbitrarily close to but greater than $R(d^*)$. These results justify the interpretation of $R(d^*)$ as the <u>equivalent information rate</u> of the source.

This research is largely an elaboration of Shannon's fundamental work. Of special interest were the problems involved in putting the theory of source coding into practice.

The first results derived are upper and lower bounds on average distortion for block codes of fixed rate and block length n. The asymptotic form of the upper bound on average distortion as $n \to \infty$ leads to the parametric functions $R_U(t)$ and $d_U(t)$, $t \leq 0$, which have the following significance. For a given $t \leq 0$, there exist block codes with rates $R_U(t) + \epsilon$, $\epsilon > 0$, which give average distortion $d^*(t)$ or less. Convergence of the upper bound on average distortion to its limiting value is as a negative power of n, as $n \to \infty$.

The asymptotic form of the lower bound on average distortion as $n \to \infty$ leads to the parametric functions $R_L(t)$ and $d_L(t)$, $t \leq 0$, which are interpreted as follows. For a given $t \leq 0$, there exist no block codes with rate less than $R_L(t)$ for which the average distortion is less than $d_L(t)$. Convergence of the lower bound to its limiting form is found from an asymptotic series and the limiting value of the bound is also approached as a negative power of n, as $n \to \infty$.

The asymptotic form of the upper and lower bounds may be optimized to yield asymptotic bounds on the average distortion for optimum block codes. We show that this optimization yields $R_U(t) = R_L(t) = R^*(t)$ and $d_U(t) = d_L(t) = d^*(t)$, for all $t \leq 0$. We have therefore shown that, for a given $t \leq 0$, there are no block codes with rate less than $R^*(t)$ for which the average distortion is less than $d^*(t)$, and there are block codes with rate $R^*(t) + \epsilon$, $\epsilon > 0$, for which the average distortion is $d^*(t)$ or less. The parametric functions $R^*(t)$ and $d^*(t)$, $t \leq 0$, thus have exactly the same significance as Shannon's rate-distortion function $R(d^*)$. We find that $R^*(t)$ and $d^*(t)$, $t \leq 0$, may be calculated explicitly by solving two sets of linear equations. Although we did not use a test channel in bounding the average distortion for block codes, the expression for $R^*(t)$ is interpreted as the average mutual information of a channel $Q(y|x)$, where $Q(y|x)$ depends on $P(x)$, $d(xy)$, and t. The expression for $d^*(t)$ is also interpreted as the average distortion when the source output is transmitted directly through this test channel $Q(y|x)$. Thus we have also found an explicit description of Shannon's test channel. These results are presented in Chapters 2 and 3.

In Chapter 4, the general problem of analyzing block codes with algebraic structure for sources is discussed briefly. The remainder of the chapter treats the binary symmetric, independent letter source with Hamming distance as the distortion measure. We show the existence of group codes which satisfy the upper bound on average distortion for optimum block codes. We also study the use of group codes and tree codes together with sequential encoding schemes as a means of reducing encoder complexity. The sequential encoding of group codes is simple to instrument, but yields a weak upper bound on average distortion. The sequential encoding of binary tree codes appears to yield the optimum average distortion, but the complexity required to do so is very great.

Chapter 5 presents three separate topics, the first of which deals with the fidelity criterion mentioned above on maximum allowable distortion per letter. The analysis of source coding problems with this fidelity criterion is quite similar to the treatment of the zero error capacity of channels by Shannon[13]. The second topic treats sources with side information available at the decoder, and this problem is seen to be similar to the problem of a channel with side information available at the transmitter[14].

Finally, a partial ordering of sources is defined but only for a fidelity criterion of geometric mean fidelity. Given the measure of fidelity $\rho(xy)$ between letters of the source and output alphabets, the geometric mean fidelity (g.m.f.) that is produced when the source sequence $u = x_1 x_2 \ldots x_n$ is reproduced as the output sequence $v = y_1 y_2 \ldots y_n$ is defined as

$$\text{g.m.f. } (uv) = \left(\prod_{i=1}^{n} \rho(x_i y_i)\right)^{\frac{1}{n}}.$$

The partial ordering of sources has roughly the same significance as does Shannon's partial ordering of channels[16], with the important exception that the geometric mean distortion seems to be much less practical as a fidelity criterion. A simple partial ordering for arithmetic average distortion as fidelity criterion could not be found. All of the topics in Chapter 5 serve to emphasize the dual nature of the problems of channel coding and source coding with a distortion measure.

We present some general remarks on this research in Chapter 6 and also several interesting directions in which to extend the theory.

CHAPTER II

BLOCK CODES FOR INDEPENDENT LETTER SOURCES

2.1 Introduction

The main concern of this chapter will be the theoretical performance of block codes for reducing the equivalent information rate of an independent letter source at the expense of introducing distortion.

We restate the source encoding problem in order to introduce some notation. The information source selects letters independently from a finite alphabet X according to the probability distribution $P(x)$, $x \in X$. There is another finite alphabet Y, called the output alphabet, which is used to encode or represent the source output. We call a block or sequence of n source letters a source word and a block of n output letters an output word. An encoder is defined as a mapping of the space U of all possible source words into a subset V^* of the space V of all possible output words. The subset V^*, called a block code of length n or just a block code, consists of M output words called code words. When the source produces a particular sequence $u \in U$, the encoder output is the code word which is the image of u in the mapping.

The encoder may be specified by a block code and a partitioning of the space U into M disjoint subsets $w_1, w_2, \ldots, w_M$. Each subset w_i consists of all those source words u that are mapped into the code word $v_i \in V^*$. Every u sequence is in some subset w_i.

The distortion measure $d(xy) \geq 0$ defines the amount of distortion introduced when some letter x is mapped by the encoder into the output letter y. When the sequence $u = \xi_1 \ \xi_2 \ . \ . \ . \ \xi_n$, $\xi_i \in X_2$ is mapped by the encoder into the sequence $v = \eta_1 \ \eta_2 \ . \ . \ . \ \eta_n, \eta_i \in Y$, the distortion is defined by

$$d(uv) = \frac{1}{n} \sum_{i=1}^{n} d(\xi_i \eta_i) \ . \tag{2.1}$$

For any block code and a partitioning of the source space U, there is a definite average distortion (per letter) which is given by

$$\bar{d} = \sum_{i=1}^{M} \sum_{w_i} P(u) \, d(uv_i) \ . \tag{2.2}$$

P(u) is the probability that the source produces the sequence u, and for an independent letter source, this is given by

$$P(u) = \prod_{i=1}^{n} P(\xi_i) \ , \ \xi_i \in X \ .$$

The output of the encoder must be transmitted to the information user or <u>sink</u>. The output sequences themselves need not be transmitted if the block code is known in advance. For instance, the binary representation of the integers from 1 to M could be sent over a channel. At the output of the channel the binary numbers could be converted back to code words, giving the sink an approximation to the actual source output. It would take $\log_2 M$ binary digits to represent n source letters in this scheme. In view of this, we define the <u>information rate</u> for a block code as $R = \frac{1}{n} \log M$ nats per letter. (All logarithms are to the base e unless otherwise specified.)

Throughout this work we will make the important assumption that there are <u>no errors introduced by the transmission channel</u>. We thereby restrict ourselves to the problem of mapping a large number of possible source words into a smaller set V^* of code words, assuming that the code words are presented directly to the sink. The sink is presented with an approximate representation of the source output but the channel capacity requirements to transmit the data are reduced by encoding.

2.2 The Average Distortion for Randomly Constructed Block Codes

We will study an ensemble of randomly constructed block codes in order to prove the existence of block codes with rate R that guarantee a certain average distortion $\bar{d}$.

The random code construction is as follows. We choose at random M code words of length n, each letter of each word being chosen independently according to a probability distribution $P_c(y)$, $y \in Y$. Each output word v has probability

$$P_c(y) = \prod_{i=1}^{n} P_c(\eta_i)$$

of being chosen as a particular code word of a random block code. According to this system, the same code word may appear more than once in a block code. Each block code of the set of all possible block codes of length n with M code words has a certain probability of being selected. An ensemble of block codes is then completely specified by M, n, and $P_c(y)$.

Given a particular set of code words, we define a partitioning of the space U which minimizes the average distortion. We put $u \in w_i$ if and only if

$$d(uv_i) \leq d(uv_j), \quad j = 1, \ldots, M. \tag{2.3}$$

If for a particular u there are several values of i which satisfy Eq. 2.3, then we put u in the subset denoted by the lowest integer. Each block code now has a definite probability of being chosen and a definite average distortion when used with the source P(x) and distortion measure d(xy). We now derive an upper bound to the average over the ensemble of codes of all the average distortions of the individual codes, the weighting being the probability of choosing the individual codes. We can conclude that there exists a block code with at least as low an average distortion as that for the whole ensemble, and hence there exists a code satisfying our upper bound on average distortion over the ensemble.

Denote the number of letters in the X and Y alphabets by a and b respectively.

<u>Theorem 2.1.</u> Consider the ensemble of block codes consisting of M code words of length n with letters chosen independently according to $P_c(y)$. The average distortion over this ensemble of codes, when used with a source P(x) and a distortion measure $d(xy) \geq 0$, satisfies

$$\bar{d} \leq \gamma'(t) + n^{-1/4} + \gamma'(0) \left\{ \frac{\sigma_1 \exp(-n^{1/2}/2\sigma_1^2)}{(2\pi)^{1/2}\, n^{1/4}} + \frac{2\sigma_2 \exp(-n^{1/2}/2\sigma_2^2)}{(2\pi)^{1/2}\, n^{1/4}} \right.$$

$$+ \frac{3 \log n}{n^{1/2}} \left(\frac{\beta_{13}}{\sigma_1^{3/2}} + \frac{2\beta_{23}}{\sigma_2^{3/2}} \right)$$

$$\left. + e^{-(M-1)\, K(n) \exp - n [t\gamma'(t) - \gamma(t) + n^{-1/4} + |t|\, n^{-1/4}]} \right\} \tag{2.4}$$

for any $t \leq 0$.

$\gamma_x(t) = \log \sum_Y P_c(y)\, e^{td(xy)}$, $\gamma'_x(t) = \partial \gamma_x(t)/\partial t$ considered as random variables with probability P(x), have mean values $\gamma(t)$ and $\gamma'(t)$, and variances σ_1^2 and σ_2^2, respectively. β_{13} and β_{23} are third absolute moments of $\gamma_x(t)$ and $\gamma'_x(t)$, respectively. $\sum_Y$ indicates summation only over letters $y \in Y$ for which $P_c(y) > 0$.

$$K(n) = (2\pi n)^{-\frac{a(b-1)}{2}} \, e^{-\frac{ab}{2} - |t|\Delta - \sum_{XY} Q^{-1}(y|x)}$$

$$\text{where } Q(y|x) = P_c(y)\, e^{td(xy) - \gamma_x(t)}, \quad \Delta = \max_{XY} d(xy)$$

The proof of this theorem is rather involved and has been relegated to Appendix A. It should be pointed out that an upper bound to $\bar{d}$ could actually be computed for finite n from Eq. 2.4. However, the main use of this theorem will be to study the upper bound on $\bar{d}$ for an ensemble of block codes as the block length n gets very large. The only term of Eq. 2.4 that does not clearly vanish in the limit as $n \to \infty$ is the very last term in the brackets, which depends upon M. In this term, K(n) is an unimportant function of n whereas the exponential in the first exponent is all important since $t\gamma'(t) - \gamma(t) \geq 0$. Substitute $M = e^{nR}$ and notice that as $n \to \infty$ we must have the first exponent

$$K(n)\ (e^{nR}-1)\exp(-n\ [t\gamma'(t) - \gamma(t) + n^{-1/4} + |t|\ n^{-1/4}]) \to \infty$$

to drive this whole term to zero. This can be accomplished if we set $R > t\gamma'(t) - \gamma(t) \geq 0$ because as $n \to \infty$

$$e^{nR - n(t\gamma'(t) - \gamma(t))}$$

will then be increasing exponentially with n, overcoming the algebraic functions of n in K(n). The bound on $\bar{d}$ then becomes, as $n \to \infty$

$$\bar{d} \leq \gamma'(t) \quad .$$

The above discussion proves the following theorem.

Theorem 2.2. There exist block codes with rate $R > t\gamma'(t) - \gamma(t)$, $t \leq 0$, that give average distortion $\bar{d} \leq \gamma'(t)$. • •

We will now put the constraints on R and $\bar{d}$ of theorem 2.2 in a more useful form. From the definitions of $\gamma_x(t)$, $\gamma(t)$, $\gamma'_x(t)$, $\gamma'(t)$ in theorem 2.1 we can write

$$\gamma(t) = \sum_X P(x)\, \gamma_x(t) = \sum_X P(x) \log \sum_Y P_c(y)\; e^{td(xy)} \tag{2.5a}$$

$$\gamma'(t) = \sum_X P(x)\, \gamma'_x(t) = \sum_{XY} \frac{d(xy)\, P(x)\, P_c(y)\, e^{td(xy)}}{\sum_Y P_c(y)\, e^{td(xy)}} \quad . \tag{2.5b}$$

It is convenient to define the tilted conditional probability distribution

$$Q(y\,|\,x) = \frac{P_c(y)\;\; e^{td(xy)}}{\sum_Y P_c(y)\;\; e^{td(xy)}} \tag{2.6}$$

with which Eq. 2.5b (together with Eq. A.6) becomes

$$d_u = \gamma'(t) = \sum_{XY} Q(y\,|\,x)\, P(x)\, d(xy) \quad . \tag{2.7}$$

Directly from Eq. 2.6 we get

$$\log \frac{Q(y|\,x)}{P_c(y)} = td(xy) - \ln \sum_Y P_c(y)\, e^{td(xy)}$$

$$= td(xy) - \gamma_x(t) \quad . \tag{2.8}$$

Combining Eqs. 2.5a, 2.5b, 2.6, and 2.8 we can define R_u as

$$R_u = t\gamma'(t) - \gamma(t) = \sum_{XY} Q(y|x)\, P(x)\, (td(xy) - \gamma_x(t))$$

$$= \sum_{XY} Q(y|x)\, P(x) \log \frac{Q(y|x)}{P_c(y)} \quad . \tag{2.9}$$

Fano[4] has shown (on pages 46-47) that $\sum_Y Q(y|x) \log (Q(y|x) \;/\; P_c(y)) \geq 0$ so that we have $R_u \geq 0$.

The expression in Eq. 2.9 for R_u resembles the expression for the average mutual information $I(X;Y)$ of a channel $Q(y|x)$ driven by the source $P(x)$. However, Eq. 2.9 is not exactly an average mutual information because the channel output probabilities are $\sum_X Q(y|x)\, P(x)$ which do not in general match $P_c(y)$. The expression for d_u in Eq. 2.7 resembles the average distortion for a source-channel combination. The interpretation of $Q(y|x)$ as a channel will be used again later.

R_u and d_u are related parametrically through the variable t. We may think of d_u as the independent variable and t as the intermediate variable when we write

$$R_u = td_u - \gamma(t) \quad . \tag{2.10}$$

The derivative of the curve of R_u vs d_u is then (see Hildebrand[8], pages 348-351)

$$\frac{dR_u}{dd_u} = t - \frac{\partial R_u}{\partial t} \; \frac{dt}{dd_u}$$

but from Eq. 2.10,

$$\frac{\partial R_u}{\partial t} = d_u - \gamma'(t)$$

which is zero from Eq. 2.7 because of the way t is related to d_u. The R_u vs d_u curve has slope $t \leq 0$.

We can show the convexity of the R_u vs d_u curve for fixed $P_c(y)$ as follows.

$$\frac{d^2 R_u}{d d_u^2} = \frac{\partial}{\partial t}\left(\frac{d R_u}{d d_u}\right) \cdot \frac{dt}{d d_u} = \frac{1}{\frac{d d_u}{dt}} = \frac{1}{\gamma''(t)}$$

From Eq. 2.7

$$\gamma''(t) = \sum_{XY} P(x)\, d(xy)\, \frac{\partial}{\partial t}\, Q(y|x)$$

$$= \sum_{X} P(x) \left[\sum_{Y} Q(y|x)\, d^2\,(xy) - \left(\sum_{Y} Q(y|x)\, d(xy)\right)^2\right] .$$

We may interpret the quantity inside the square brackets as the variance of a random variable, and $\gamma''(t)$ is an average of variances, therefore $\gamma''(0) \geq 0$. We have shown that for fixed $P_c(y)$,

$$\frac{d^2 R_u}{d d_u^2} = \frac{1}{\gamma''(t)} \geq 0 . \qquad (2.11)$$

Since $\frac{d d_u}{dt} = \gamma''(t) \geq 0$, d_u is a monotone function as t decreases and this fact together with Eq. 2.11 show that R_u vs d_u is convex downward.

2.3 Optimization of the Upper Bound on Average Distortion

For each probability distribution $P_c(y)$ we have an ensemble of block codes and an R_u vs d_u curve. From Theorem 2.2 it is clear that we want to find the ensemble of codes which gives the lowest value of R_u for a fixed d_u. From another viewpoint, we want to find the lower envelope of all R_u vs d_u curves.

From Eq. 2.10 we see that if there was no parametric relation between t and d_u such as Eq. 2.7, fixing t would give R_u as a linear function of d_u for any particular $P_c(y)$.

This straight line in the R_u - d_u plane is the tangent to the R_u vs d_u curve corresponding to the $P_c(y)$ at the point at which Eq. 2.7 is satisfied for the fixed t. The slope of this straight line is t (from Eq. 2.10) and its d-axis intercept is $\gamma(t) / t$, $t < 0$. Because of the convexity of the R_u vs d_u curves, we can find a point on the lower envelope of all R_u vs d_u curves by finding the $P_c(y)$ which gives the minimum d-axis intercept. (See Figure 2.1.)

Let us define the lower envelope of all R_u vs d_u curves as the curve R^*_u vs d^*_u. We attempt now to find the ensemble $P_c(y)$ which for fixed t gives the minimum d-axis intercept. First, we show that for fixed $t < 0$, the intercept $I(P_c(y)') = \frac{\gamma(t)}{t} = -\frac{1}{|t|}\gamma(t)$ is a convex downward function of the $P_c(y)$. Consider two different probability vectors $P_{c1}(y)$ and $P_{c2}(y)$ and denote

$$\gamma_x^{(1)}(t) = \log \sum_Y P_{c1}(y) \; e^{td(xy)}$$

$$\gamma_x^{(2)}(t) = \log \sum_Y P_{c2}(y) \; e^{td(xy)}$$

$$P_{c3}(y) = \lambda P_{c1}(y) + (1-\lambda) P_{c2}(y), \; y \in Y, \; 0 \le \lambda \le 1$$

$$\gamma_x^{(3)}(t) = \log \sum_Y P_{c3}(y) \; e^{td(xy)} \quad .$$

Since $\log \chi$ is a concave downward function of χ, we use the concave inequality from Hardy[7] (theorem 98, page 80). For any $x \in X$,

$$\gamma_x^{(3)}(t) \ge \lambda \; \gamma_x^{(1)}(t) + (1-\lambda) \; \gamma_x^{(2)}(t) \quad .$$

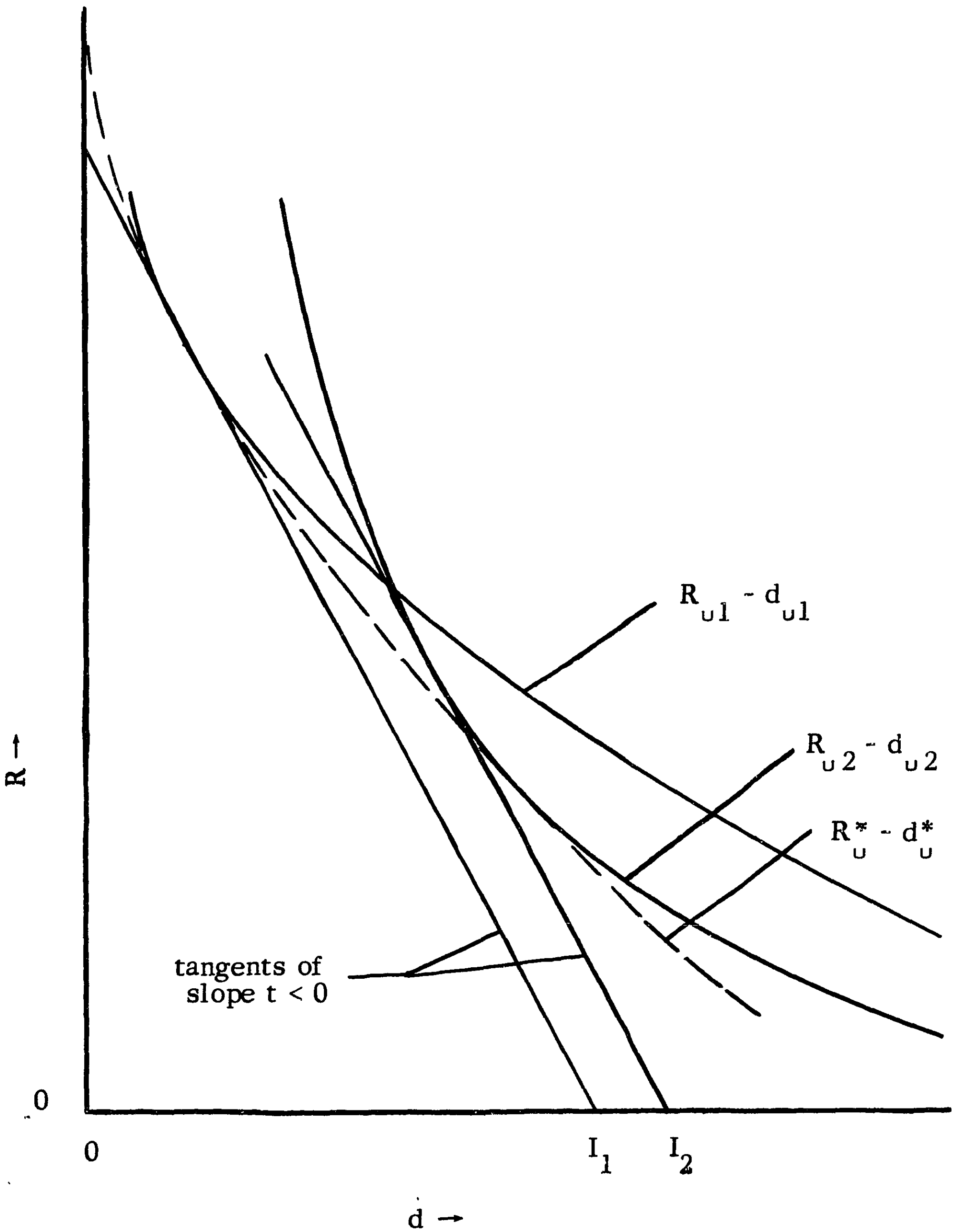


Figure 2.1. The lower envelope of all R_u - d_u curves may be determined by finding the smallest intercept I of all tangents of the same slope of the R_u - d_u curves.

For $t < 0$,

$$- \frac{\gamma^{(3)}(t)}{|t|} \leq - \lambda \frac{\gamma^{(1)}(t)}{|t|} - (1-\lambda) \frac{\gamma^{(2)}(t)}{|t|}$$

therefore, for $0 \leq \lambda \leq 1$,

$$I(\lambda P_{c1}(y) + (1-\lambda) P_{c2}(y)) \leq \lambda I(P_{c1}(y)) + (1-\lambda) I(P_{c2}(y)) \tag{2.12}$$

and the intercept I as a function of the $P_c(y)$ is a convex downward function.

We now seek to minimize $I = \gamma(t)/t$ for fixed t by varying the $P_c(y)$ under the constraints $P_c(y) \geq 0$, $\sum_Y P_c(y) = 1$. First we find a stationary point of I with respect to the $P_c(y)$ while constraining the sum of the $P_c(y)$.

$$\frac{\partial}{\partial P_c(y_k)} \left(\frac{\gamma(t)}{t} + \mu \sum_Y P_c(y) \right) = 0, \ t < 0.$$

Using Eq. 2.5a this becomes

$$\frac{1}{t} \sum_X P(x) \frac{e^{td(xy_k)}}{\sum_Y P_c(y) \ e^{td(xy)}} + \mu = 0. \tag{2.13}$$

Multiplying this last equation by $t \ P_c(y_k)$ gives

$$\sum_X P(x) \ Q(y_k/x) = - \mu t \ P_c(y_k) \tag{2.14}$$

where we have used Eq. 2.6. We have a stationary point of I if we can find $P_c(y)$, all $y \in Y$, which satisfy Eq. 2.14. It is convenient to define the probability distribution

$$Q(y) = \sum_X Q(y|x) \ P(x). \tag{2.15}$$

If we now choose $\mu = -1/t$ we see that Eq. 2.14 becomes

$$Q(y) = P_c(y), \text{ all } y \in Y \tag{2.16}$$

and this value of μ then implies that the $P_c(y)$ satisfy the constraint on their sum. However, we can not guarantee that the $P_c(y)$ which satisfy Eqs. 2.16 will be non-negative.

It is convenient to denote

$$g_x(t) = e^{\gamma_x(t)} = \sum_Y P_c(y)\, e^{td(xy)} \quad . \tag{2.17}$$

The Eqs. 2.16 imply that in order to calculate the optimum $P_c(y)$, we may first solve for $g_x^{-1}(t)$ the following set of equations which are linear in $g_x^{-1}(t)$.

$$\sum_X P(x)\, e^{td(xy)}\, g_x^{-1}(t) = 1, \; y \in Y. \tag{2.18}$$

We may then solve for $P_c(y)$ the set of equations

$$\sum_X P_c(y)\, e^{td(xy)} = g_x(t), \quad x \in X \tag{2.19}$$

which are linear in $P_c(y)$. These operations are easy to perform with the aid of modern computers. However, we may notice that P(x), d(xy) and t may be such that one or more of the $g_x^{-1}(t)$ satisfying the Eqs. 2.18 are negative or zero. A $g_x^{-1}(t)$ that is zero implies that the Eqs. 2.19 are meaningless and we cannot get a solution for $P_c(y)$. A negative $g_x(t)$ implies a negative $P_c(y)$, but more important, since $\gamma(t) = \sum_X P(x) \log g_x(t)$, we find that the solution for $P_c(y)$ leads to imaginary values of R_u and d_u,† again a

† There are such P(x), d(xy), $t \le 0$ such that $g_x^{-1}(t) < 0$ for some x. For example, consider the ternary source with letter probabilities all 1/3, set t = -1, and take

$$d_{xy} = \begin{bmatrix} 0 & \log 1.5 & \log 6 \\ \log 2 & 0 & \log 2 \\ \log 6 & \log 1.5 & 0 \end{bmatrix} .$$

meaningless situation. We can conclude that there are situations in which one does not have any meaningful solution to Eqs. 2.16 for a range of $t < 0$. This can be interpreted as an R_U^* vs d_U^* curve with discontinuities in its derivative dR_U^*/dd_U^*, since the slope of R_U^* vs d_U^* is given by t (by construction).

Finally, it should be obvious that there may be situations in which all the $g_X^{-1}(t) > 0$ and we still get from the Eqs. 2.19 a negative $P_C(y)$. We have shown that I is a convex downward function of the b arguments $P_C(y)$ (b letters $y \in Y$). We constrain the $P_C(y)$, considered as points in b-dimensional Euclidian space, to vary within a region of the (b-1)-dimensional hyperplane $\sum_Y P_C(y) = 1$. The boundary of the acceptable region of points $P_C(y)$ are the hyperplanes $P_C(y) = 0$, all $y \in Y$. If the absolute minimum of I lies outside this region, the solution to Eqs. 2.16 may have one or more negative probabilities $P_C(y)$. We can still find a minimum of I along a hyperplane boundary of the acceptable region by setting some $P_C(y) = 0$ and minimizing I again by solving the set of Eqs. 2.16. The fact that such a further constraint on $P_C(y)$ still leads to a minimum of I is guaranteed by the convexity of I.

The special case of $t = 0$ must be treated separately. We wish to find a minimum of

$$\lim_{t \to 0} \frac{\gamma(t)}{t} .$$

Since $\gamma(0) = 0$ (from Eq. 2.5a), we may write

$$\lim_{t \to 0} \frac{\gamma(t)}{t} = \lim_{t \to 0} \frac{\gamma(t) - \gamma(0)}{t - 0} \equiv \gamma'(0)$$

and so we wish to minimize $\gamma'(0)$ with respect to the $P_c(y)$. From Eq. 2.5b

$$\min_{P_c(y)} \gamma'(0) = \min_{P_c(y)} \sum_{XY} P(x)\ P_c(y)\ d(xy) \tag{2.20}$$

and the solution to Eq. 2.20 for which $\sum_Y P_c(y) = 1$ is the choice of

$$P_c(y) = \begin{cases} 1 \text{ for } y_0 \\ \\ 0 \text{ otherwise} \end{cases}$$

where y_0 is such that

$$d_{max} = \sum_X P(x)\, d(xy_0) = \min_y \left(\sum_X P(x)\, d(xy) \right) . \tag{2.21}$$

From Eq. 2.6,

$$\lim_{t \to 0} Q(y\,|\,x) = P_c(y)$$

so that $Q(y\,|\,x) \,/\, P_c(y) = 1$ and from Eq. 2.9 we see that this implies $R_u^* = 0$ for this case. (2.22)

It will be helpful to put our results on the optimum upper bound on average distortion in a form which will allow comparison with later results on a lower bound. If we define the function $f_0(x)$ as

$$f_0^t(x) = g_x^{-1}(t)$$

we may re-write Eqs. 2.20 as a linear set in $f_0^t(x)$, i.e.,

$$\sum_X P(x)\ e^{td(xy)}\ f_0^t(x) = 1, \quad y \in Y. \tag{2.22}$$

The optimum $P_c(y)$ can then be found as the solution to the set of linear equations

$$\sum_Y P_c(y)\ e^{td(xy)} = f_0^{-t}(x), \quad x \in X. \tag{2.23}$$

For the optimum ensemble of codes, we may interpret the $Q(y|x)$ as a channel driven by the source $P(x)$, with output probabilities $\sum_X Q(y|x)\,P(x) = Q(y) = P_c(y)$. We can find the dual set of transition probabilities $Q(x|y)$ which specify this channel as

$$Q(x|y) = \frac{Q(y|x)\,P(x)}{P_c(y)} = \frac{P(x)\,e^{td(xy)}\,f_o^t(x)}{\sum\limits_X P(x)\,e^{td(xy)}\,f_o^t(x)}$$

$$= P(x)\,e^{td(xy)}\,f_o^t(x), \tag{2.24}$$

where we have used Eq. 2.22. From Eq. 2.24 we find the useful relation

$$\frac{Q(x|y)}{P(x)} = \frac{Q(y|x)}{P_c(y)}, \tag{2.25}$$

so that we may re-write Eqs. 2.7 and 2.9 for the optimum $P_c(y)$ as

$$R_u^*(t) = \sum_{XY} Q(x|y)\,P_c(y)\ln\frac{Q(x|y)}{P(x)}, \tag{2.26}$$

$$d_u^*(t) = \sum_{XY} Q(x|y)\,P_c(y)\,d(xy),\ t \le 0. \tag{2.27}$$

We see that the $R_u^*(t)$ - $d_u^*(t)$ function is defined by Eqs. 2.26 and 2.27 and by the sets of Equations 2.22 and 2.23.

Theorem 2.3 For any $t \le 0$ and any $\epsilon > 0$, there exist block codes with rate $R = R_u^*(t) + \epsilon$ and average distortion at least as low as $d_u^*(t)$. • •

The $Q(y|x)$ may be interpreted as a channel whose transition probabilities depend on $P(x)$, $d(xy)$, and t. R_u^* is the average mutual information of the channel $Q(y|x)$ when driven by the source $P(x)$, and d_u^* is the average distortion from this source-channel combination.

Shannon(15) showed that any channel with input and output alphabets X and Y, respectively, could be used as a test channel to prove a coding theorem for a source with a distortion measure. His coding theorem states that there exist block codes with rate arbitrarily close to but greater than the average mutual information of the source-test channel combination with average distortion equal to that calculated for the source-channel combination. J. L. Kelly (9) used this approach to prove a coding theorem for amplitude-continuous sources by using a continuous channel. Our work here obtains a coding theorem for sources without using such a test channel, but the resulting expressions involve a fictitious channel $Q(y|x)$. Moreover, we only have a strict channel interpretation after optimization of the upper bound on average distortion with respect to the code ensemble $P_c(y)$.

As a special example, suppose we have a distortion measure with the property that for every $x \in X$ there is one and only one $y = y_x$ such that $d(xy_x) = 0$. There is only one way to represent the source output exactly (zero distortion) in this case. Since, from Eq. 2.6

$$\lim_{t \to -\infty} Q(y|x) = \delta_{y,y_x} ,$$

we have for this case

$$\lim_{t \to \infty} d_u^*(t) = \sum_{XY} P(x)\, \delta_{y,y_x}\, d(xy) \equiv 0.$$

Also

$$\sum_{Y} \delta_{y,y_x} \log \delta_{y,y_x} = 0 \qquad \text{so}$$

$$\lim_{t \to \infty} R_u^*(t) = \sum_{X} P(x) \log \frac{1}{P(x)} = H(X)$$

where H(X) is the well-known source entropy. From Eq. 2.21 we see that for $t = 0$, $d^*_u(0) = d_{max}$, and from Eq. 2.22,

$$R^*_u(0) = 0.$$

In general, $R^*_u(-\infty) = 0$ if and only if each source letter $x \epsilon X$ has some output letter y such that $d(xy) = 0$. A typical $R^*_u(t) - d^*_u(t)$ function is shown in Figure 2.2.

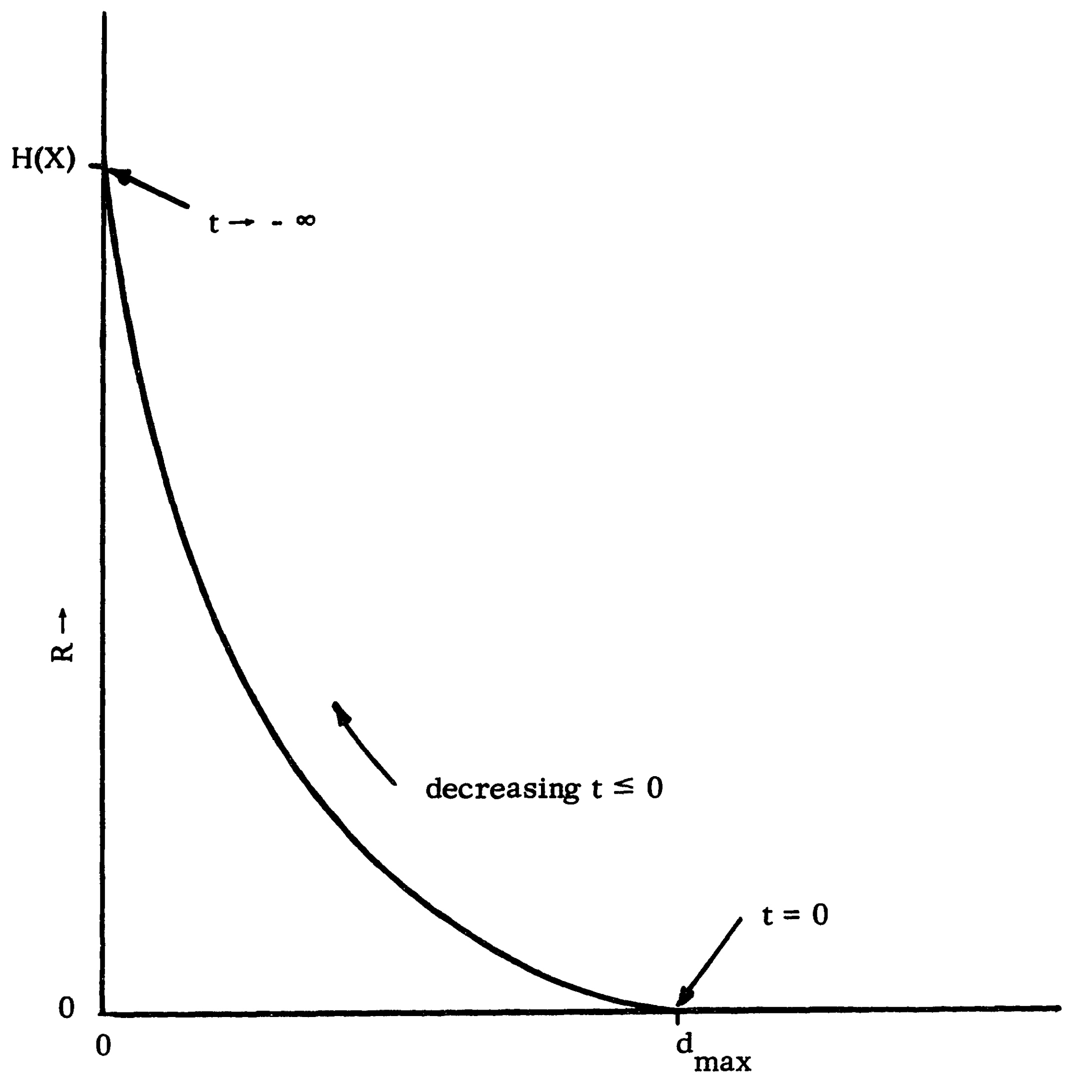


Figure 2.2 A typical function R^*_u vs d^*_u.

2.4 Lower Bound to Average Distortion for Fixed Composition Block Codes

We have demonstrated the existence of block codes that guarantee a certain average distortion. Now we seek a lower bound to average distortion applicable to all block codes, so that we may compare the performance of our randomly constructed codes to the best possible block codes.

First we define a distance function

$$D(xy) = d(xy) + \log f(x) \tag{2.28}$$

where d(xy) is the distortion measure and f(x) may be any strictly positive function defined on the source alphabet X. The distance between two sequences u and v is defined as

$$D(uv) = \frac{1}{n} \sum_{i=1}^{n} D(\xi_i \eta_i) = \frac{1}{n} \sum_{i=1}^{n} (d(\xi_i \eta_i) + \log f(\xi_i))$$

$$= d(uv) + \frac{1}{n} f(u) \tag{2.29}$$

where we have denoted

$$f(u) = \prod_{i=1}^{n} f(\xi_i) \quad .$$

For any output word v of length n we may count the number of times each letter of the Y alphabet appears. We denote by $n(y_k)$ the number of times letter y_k appears in the v sequence and we call the set of integers n(y), $y \in Y$, the composition of v. The composition of a source word u is denoted n(x). The product composition of a pair of sequences u - v is denoted n(xy) and is the number of times the letters x_k and y_j appear in corresponding positions of the u and v sequences. The product composition of a u-v pair is such that

$$\sum_{XY} n(xy) = n, \sum_{X} n(xy) = n(y), \sum_{Y} n(xy) = n(x).$$

For a u-v pair with product composition n(xy) we can write the probability of the source word u as

$$P(u) = \prod_{XY} P(x)^{n(xy)} = \prod_{X} P(x)^{\sum_{Y} n(xy)} = \prod_{X} P(x)^{n(x)} \tag{2.30}$$

and the distance between the u-v pair is

$$D(uv) = \frac{1}{n} \sum_{XY} n(xy)\, d(xy) . \tag{2.31}$$

We see that for an independent letter source the probability of a source word depends only on the composition of the word. Also, the distance function and distortion measure between sequences depend only on the product composition of a u-v pair.

The distance function D(xy) can be thought of as another distortion measure so that for any block code consisting of the code words v_i, i=1, ..., M and a given partitioning of the source space U into encoding subsets w_i, i=1, ..., M, we may write the average distance for the block code and encoder as

$$\overline{D} = \sum_{i=1}^{M} \sum_{w_i} P(u)\, d(uv_i) . \tag{2.32}$$

Theorem 2.4 Consider a source P(x), distortion measure $d(xy) \geq 0$, and a positive function f(x). Suppose we have a set of M code words of length n and all have the same composition $n_c(y)$. Let U_o represent the subset of source sequences u for which $D(uv_o) \leq D_o$ for any particular sequence v_o with composition $n_c(y)$, and D_o such that U_o is not empty. Then if M is such that

$$M \leq \frac{1}{\sum_{U_o} P(u)} \tag{2.33}$$

the average distance for the block code satisfies

$$\bar{D} \geq \frac{\sum_{U_o} P(u)\, D(uv_o)}{\sum_{U_o} P(u)} \quad . \tag{2.34}$$

••

Proof The proof of this theorem is analogous to R. G. Gallager's (unpublished) proof of a theorem on the lower bound to the probability of error for a memoryless channel.

A block code in which all code words have the same composition will be referred to as a fixed composition block code. We proceed to derive a lower bound to the average distance that any block code of fixed composition $n_c(y)$ could give for any partitioning of the source space. For each code word v_i, $i=1, \ldots, M$, we define the increasing staircase function $F_i(z)$ as follows. List all source sequences u of length n in order of increasing $D(uv_i)$ and number the sequences in the i - th ordering $u_{1i}, u_{2i}, u_{3i}, \ldots$ Now define

$$F_i(z) = 0 \quad , \quad z < 0$$

$$F_i(z) = D(u_{1i}, v_i), \quad 0 \leq z \leq P(u_{1i})$$

$$F_i(z) = D(u_{2i} v_i), \quad P(u_{1i}) < z \leq P(u_{1i}) + P(u_{2i})$$

$$\vdots$$

$$F_i(z) = D(u_{ki} v_i), \quad \sum_{j=1}^{k-1} P(u_{ji}) < z \leq \sum_{j=1}^{k} P(u_{ji})$$

$$\vdots$$

We may visualize every source word u being represented in $F_i(z)$ by a rectangle with height $D(uv_i)$ and width $P(u)$. (See Figure 2.3.)

For a given partitioning w_i, $i=1, \ldots, M$, we can write the average distance as

$$\overline{D} = \sum_{i=1}^{M} \overline{D}_i$$

where

$$\overline{D}_i = \sum_{w_i} P(u)\, D(uv_i) \quad .$$

If we shade all rectangles of $F_i(z)$ corresponding to all u sequences in w_i, we can interpret $\overline{D}_i$ as the area of these shaded rectangles. Each $\overline{D}_i$ is lower bounded by

$$\overline{D}_i \geq \int_0^{z_i} F_i(z)\, dz \quad , \quad z_i = \sum_{w_i} P(u) \quad . \tag{2.35}$$

We have underbounded $\overline{D}_i$ by the area under $F_i(z)$ on the interval $0 \leq z \leq z_i$. This bound may be interpreted as a sequential process of replacing the area of shaded rectangles for the largest $D(uv_i)$ by smaller areas in unshaded portions of $F_i(z)$, preserving the width measure of the shaded rectangles, until the entire area under $F_i(z)$ is shaded out to some z_i.

Define

$$\phi_i(z) = \int_0^{z} F_i(z')\, dz'$$

where the $\phi_i(z)$ are convex downward, monotone increasing, continuous functions of z. The main point of the proof hinges on the fact that P(u) and D(uv) depend only on the product composition of the pair u-v and so for code words u_i with identical composition

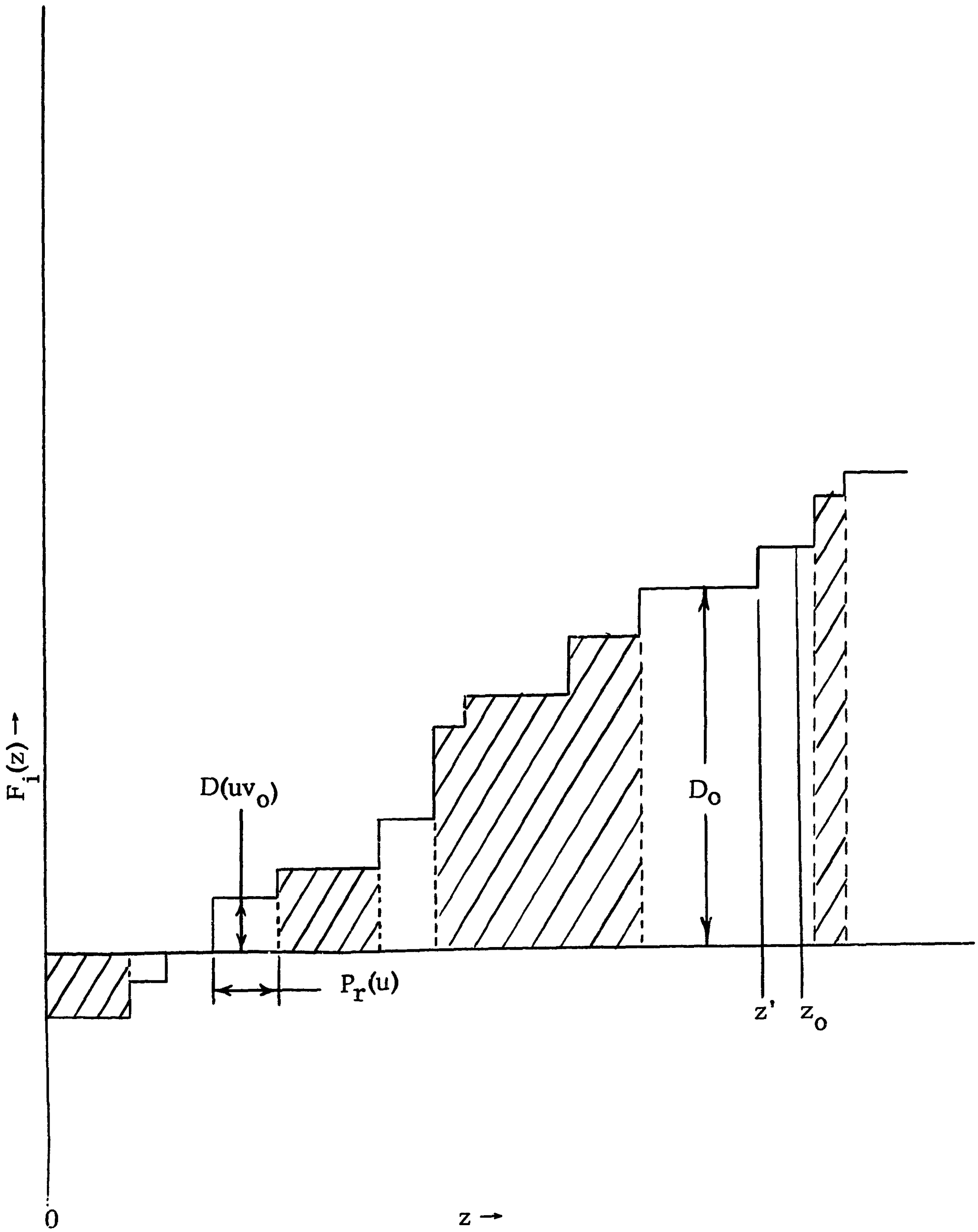


Figure 2.3 The function $F_i(z)$ for a code word v_i of composition $n_c(y)$, showing some of the $u \in w_i$, z_o, z', and D_o.

$n_c(y)$, the $\phi_i(z)$ are all identical and we may drop the subscript i. Note from Eq. 2.35 that

$$\sum_{i=1}^{M} z_i = 1$$

since every source word is in some subset w_i. We may now apply the convex inequality (see Hardy[9], theorem 86, page 72) to give us

$$\overline{D} \geq M \sum_{i=1}^{M} \frac{1}{M} \phi(z_i) \geq M \phi \left(\sum_{i=1}^{M} \frac{1}{M} z_i \right) = M \phi \left(\frac{1}{M} \right) . \tag{2.36}$$

A lower bound to $\overline{D}$ for any block code of fixed composition $n_c(y)$ is achieved if we assume that we can make all $z_i = z_o = 1/M$ so that

$$\overline{D}_i = \phi \left(\frac{1}{M} \right) .$$

We can write Eq. 2.36 as

$$\overline{D} = \frac{1}{z_o} \int_0^{z_o} F(z) \, dz \tag{2.37}$$

which may be interpreted as the average area per unit length under $F(z)$ for $0 \leq z \leq z_o$. Clearly, with this interpretation the monotonicity of $F(z)$ allows us to write

$$\frac{1}{z_o} \int_0^{z_o} F(z) \, dz \geq \frac{1}{z'} \int_0^{z'} F(z) \, dz , \quad z' \leq z_o . \tag{2.38}$$

Let us then define z' by

$$z' = \sum_{U_o} P(u) \leq z_o = \frac{1}{M} \tag{2.39}$$

where U_o is the subset† of source words for which $D(uv_o) \leq D_o$. Any D_o for which

† Hereafter we will use the standard shorthand notation in the definition of sets, e.g., $U_o = \{ u \mid D(uv_o) \leq D_o \}$.

Eq. 2.39 is true can be used to define U_o. Hence, if D_o is a constant such that

$$M \leq \frac{1}{\sum_{U_o} P(u)}$$

then from Eqs. 2.37 and 2.38 and the definition of F(z),

$$\overline{D}_l \geq \frac{\sum_{U_o} P(u) D(uv_o)}{\sum_{U_o} P(u)} .$$

Q.E.D.

This theorem on average distance leads to a lower bound on average distortion for block codes of fixed composition $n_c(y)$. From Eqs. 2.29, 2.32, and 2.2 we see that

$$\overline{D} = \sum_{i=1}^{M} \sum_{w_i} P(u)\, d(uv_i) + \frac{1}{n} \sum_{i=1}^{M} \sum_{w_i} P(u) \log f(u) = \overline{d} + \sum_{X} P(x) \log f(x) \tag{2.40}$$

since

$$\log f(u) = \sum_{i=1}^{n} \log f(\xi_i) \quad , \quad u = (\xi_i),$$

and this term is entirely independent of the block code. We may now restate Theorem 2.4 in terms of its implications to the average distortion of fixed composition block codes.

Theorem 2.5 Suppose we have a source P(x), distortion measure d(xy), and any positive function f(x). Any block code with M code words of length n, all having fixed composition $n_c(y)$, which satisfies

$$M \leq \frac{1}{\sum_{U_o} P(u)} \tag{2.41}$$

must have average distortion that satisfies

$$\bar{d} \geq \frac{\sum_{U_0} P(u)\, D(uv_0)}{\sum_{U_0} P(u)} - \sum_{X} P(x) \log f(x) \tag{2.42}$$

where

$U_0 = \{ u \mid D(uv_0) \leq D_0 \}$,

v_0 is any output sequence with composition $n_c(y)$,

D_0 is such that U_0 is not empty. ••

It is difficult to get bounds on the expressions in Eq. 2.42 for finite n which will give the correct asymptotic bound on $\bar{d}$ as $n \to \infty$. These difficulties and methods of surmounting them are the main concern of Chapter 3. Our present interest is to obtain the correct limiting forms for the constraints on M and $\bar{d}$ as $n \to \infty$.

Let us define the sets

$$\Delta = \{ u \mid D_0 - \delta \leq D(uv_0) \leq D_0 \} , \quad \delta > 0,$$

$$U_0 - \Delta = \{ u \mid D(uv_0) < D_0 - \delta \}$$

and denote the right hand side of Eq. 2.34 as D_L. We re-write Eq. 2.34 as

$$\bar{D} \geq D_L = \frac{\sum_{U_0 - \Delta} P(u)\, D(uv_0) + \sum_{\Delta} P(u)\, D(uv_0)}{\sum_{U_0} P(u)}$$

$$\geq (D_0 - \delta) \frac{\sum_{\Delta} P(u)}{\sum_{U_0} P(u)} \; . \tag{2.43}$$

For a given v_o, $nD(uv_o)$ is a sum of independent, non-identical random variables and we may write the distribution function for the random variable $D(uv_o)$ as

$$\mathcal{P}_n(\chi) = P_r [D(uv_o) \le \chi]$$

so that Eq. 2.43 becomes

$$D_L = (D_o - \delta) \frac{\mathcal{P}_n(D_o) - \mathcal{P}_n(D_o - \delta)}{\mathcal{P}_n(D_o)}$$

$$= (D_o - \delta) \left(1 - \frac{\mathcal{P}_n(D_o - \delta)}{\mathcal{P}_n(D_o)}\right) . \tag{2.44}$$

We may apply Fano's[4] bounds (pages 265 and 275) on $\mathcal{P}_n(\chi)$ for a given v_o, which are as follows.

$$K_L(n)\ e^{-nE(\chi)} \le \mathcal{P}_n(\chi) \le e^{-nE(\chi)} , \quad t \le 0, \tag{2.45}$$

where $\quad E(\chi) = t\mu'(t) - \mu(t) \ge 0$

$$\mu(t) = \sum_Y P_c(y) \log \sum_X P(x)\ e^{tD(xy)} \tag{2.45a}$$

$$P_c(y) = \frac{n_c(y)}{n}$$

and t is chosen so that

$$\mu'(t) = \frac{\partial \mu(t)}{\partial t} = \chi \le \text{mean value of } \chi . \tag{2.46}$$

$K_L(n)$ is only algebraic in n and is similar to Eq. 8.125 of Appendix A. $\mu''(t)$ can be interpreted as an average of variances, so $\mu''(t) \ge 0$, implying $\mu'(t)$ is a continuous monotone increasing function of t which then guarantees that we can always satisfy

Eq. 2.46 for some value of $t \leq 0$. If we have

$$\mu'(t_1) = D_o$$

$$\mu'(t_2) = D_o - \delta \ , \ \delta > 0$$

then $\quad t_2 < t_1 \leq 0.$

Since $\frac{d}{dt} E(\chi) = t\mu''(t) \leq 0$, for $t \leq 0$, $E(\chi)$ is a continuous, monotone decreasing function of $t \leq 0$ and $0 \leq E(D_o - \delta) < E(D_o)$. This difference in exponents in the bounds of Eq. 2.45, when applied to Eq. 2.44, overcomes the function $K_L(n)$ as $n \to \infty$ and we have

$$\frac{\mathcal{P}_n(D_o - \delta)}{\mathcal{P}_n(D_o)} \to 0.$$

We conclude that for arbitrary $\delta > 0$,

$$\lim_{n \to \infty} D_L = D_o - \delta \quad .$$

Therefore the limiting form of the bound on average distance is, for $n \to \infty$

$$\overline{D} \geq D_o.$$

Applying Eq. 2.40, the bound on average distortion is, for $n \to \infty$

$$\overline{d} \geq \lim_{n \to \infty} D_L - \sum_X P(x) \log f(x)$$

$$= D_o - \sum_X P(x) \log f(x) \ . \tag{2.47}$$

We can write the constraint on M of Eq. 2.41 more conservatively, using Eq. 2.45, with t chosen so that $\mu''(t) = D_o$,

$$M \leq e^{nE(D_o)} \leq \frac{1}{\sum_{U_o} P(u)} \quad . \tag{2.48}$$

From the definition of R we have the constraint on the code rate

$$R = \frac{1}{n} \log M \leq E(D_o) \quad . \tag{2.49}$$

We summarize the above discussion with the statement of a theorem.

Theorem 2.6 There exist no block codes of fixed composition $n_c(y)$ with rate $R \leq t\mu'(t) - \mu(t)$, $t \leq 0$, that give average distortion

$$\bar{d} < \mu'(t) - \sum_X P(x) \log f(x) \ . \qquad \bullet\bullet$$

We now put the constraints on R and $\bar{d}$ of Theorem 2.6 in a more useful form. From the definition of $\mu(t)$ in Eq. 2.45, we proceed (as in Eqs. 2.6, 2.7, and 2.9) to define

$$R_L = t\mu'(t) - \mu(t) = \sum_{XY} Q(x \mid y) P_c(y) \log \frac{Q(x \mid y)}{P(x)} \tag{2.50}$$

$$d_L = \mu'(t) - \sum_X P(x) \log f(x)$$

$$= \sum_{XY} Q(x \mid y) P_c(y) d(xy) + \sum_X (Q(x) - P(x)) \log f(x) \tag{2.51}$$

where

$$Q(x \mid y) = \frac{P(x) e^{td(xy)} f^t(x)}{\sum_X P(x) e^{td(xy)} f^t(x)} \tag{2.52}$$

and

$$Q(x) = \sum_{Y} Q(x \mid y)\, P_C(y) \quad . \tag{2.53}$$

R_L resembles the expression for the average mutual information of a channel $Q(x \mid y)$ driven by the source $P_C(y)$, but $Q(x)$ and $P(x)$ do not match in general so there is only a resemblance.

For each function $f(x)$ and composition $P_C(y)$ we have a curve R_L vs d_L with R_L and d_L related parametrically through t. We may think of d as the independent variable and t as the intermediate variable when we write, from Eqs. 2.50 and 2.51,

$$R_L = t\,(d_L + \sum_{X} P(x) \log f(x)\,) - \mu(t) \quad . \tag{2.54}$$

The derivative of the R_L vs d_L curve for fixed $f(x)$ and $P_C(y)$ is

$$\frac{dR_L}{dd_L} = t + \frac{\partial R_L}{\partial t} \frac{dt}{dd_L}$$

but from Eqs. 2.50 and 2.51 we see that t is chosen so that $\partial R_L / \partial t = 0$, hence

$$\frac{dR_L}{dd_L} = t \le 0 \quad . \tag{2.55}$$

We can show the convexity of R_L vs d_L as follows.

$$\frac{d^2 R_L}{dd_L^2} = \frac{\partial}{\partial t}\left(\frac{dR_L}{dd_L}\right) \cdot \frac{dt}{dd_L} = \frac{1}{\frac{dd_L}{dt}} = \frac{1}{\mu''(t)} \quad .$$

$\mu''(t)$ can be interpreted as an average of variances so

$$\frac{d^2 R_L}{dd_L^2} \ge 0$$

and $\frac{dd_L}{dt} = \mu''(t) \ge 0$ is a monotone function of t. We conclude that R_L vs d_L is a continuous, convex downward function of t with continuous slope for fixed f(x) and $P_c(y)$.

2.5 A Lower Bound on Average Distortion for Block Codes

The arbitrary function f(x) may be thought of as a parameter which may be adjusted to optimize the lower bound on average distortion for block codes of fixed composition $P_c(y)$. Fano(4) used such a function in his derivation of both upper and lower bounds on probability of error for discrete, memoryless channels. To get the strongest lower bound to average distortion for a fixed $P_c(y)$ we should maximize R_L with respect to the f(x) while holding d_L fixed. Using the expression in Eq. 2.54 for R_L we obtain†

$$\left(\frac{\partial R_L}{\partial f(x_k)}\right)_{f(x_j),\, j\neq k} = \frac{\partial R_L}{\partial t}\,\frac{\partial t}{\partial f(x_k)} + \frac{\partial R_L}{\partial f(x_k)} = 0 \ .$$

Again, $\partial R_L/\partial t = 0$ by choice of t. Using Eqs. 2.54 and 2.45a,

$$\frac{\partial R_L}{\partial f(x_k)} = \frac{t}{f(x_k)}\left(P(x_k) - \sum_Y P_c(y)\, Q(x_k\,|\,y) \right) = 0.$$

Using Eq. 2.53, we find that we have a stationary point of R_L for fixed d_L if we choose f(x) so that

$$P(x) = Q(x) \quad , \quad \text{all } x \in X \ . \tag{2.56}$$

We cannot show explicitly that this stationary point of R_L is a maximum with respect to the f(x), but the Theorem 2.6 is true for <u>any</u> positive f(x). Let us then assume for the present that for fixed $P_c(y)$ and each value of $t \leq 0$ we can find a positive f(x) such that Eq. 2.56 is satisfied and let us then use this f(x) in the following work.

† We use Hildebrand's(8) notation of page 350, Eq. 4b.

We now find a lower bound to average distortion for any fixed composition block code by minimizing R_L with respect to $P_c(y)$ for fixed d_L, under the constraints that $P_c(y) \geq 0$, $\sum_Y P_c(y) = 1$. We solve

$$\frac{\partial}{\partial P_c(y_k)} \left(R_L + \lambda \sum_Y P_c(y) \right) = 0$$

for fixed d_L. Again using Hildebrand's[8] notation,

$$\left(\frac{\partial R_L}{\partial P_c(y_k)} \right)_{P_c(y_j),\, j \neq k} = \sum_X \frac{\partial R_L}{d f(x)} \frac{\partial f(x)}{\partial P_c(y_k)} + \frac{\partial R_L}{\partial P_c(y_k)}$$

since $\partial R_L / \partial t = 0$. Also $\partial R_L / \partial f(x) = 0$ for all x because the f(x) are chosen to make R_L stationary. We obtain

$$\frac{\partial R_L}{\partial P_c(y_k)} + \lambda = 0$$

and from Eqs. 2.54 and 2.45a, $P_c(y)$ should be chosen so that

$$\log \sum_X P(x)\; e^{td(xy)} f^t(x) = K \tag{2.57}$$

where K is a constant independent of y. We may re-write Eq. 2.57 as

$$\sum_X P(x)\; e^{td(xy)}\; f^t(x) = K' \tag{2.58}$$

but since $\sum_X Q(x \mid y) = 1$ for any y, we see that Eq. 2.58 together with Eq. 2.52 implies

$$\sum_X P(x)\; e^{td(xy)} f^t(x) = 1, \text{ all } y \in Y . \tag{2.59}$$

It remains for us to show that the choice of $P_C(y)$ which makes Eq. 2.59 true corresponds to a minimum of R_L for fixed d_L. Notice first that with Eq. 2.59, we can re-write Eq. 2.56 as

$$\sum_Y P_C(y)\ e^{td(xy)} = f^{-t}(x)\ , \quad \text{all } x \in X\ . \tag{2.60}$$

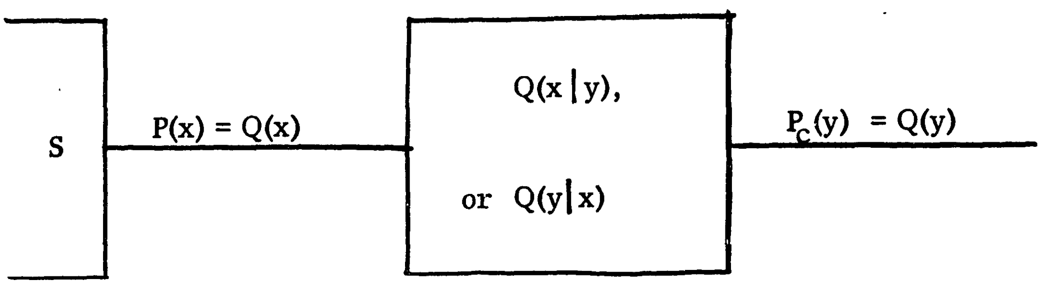
The functions $R_U^*(t)$ and $d_U^*(t)$ corresponded to a lower envelope of all R_U vs d_U for different $P_C(y)$ and this implies that we found a minimum of $R_U^*(t)$ with respect to $P_C(y)$ for fixed $d_U^*(t)$. The Eqs. 2.22, 2.23, 2.24, 2.26, and 2.27 define $R_U^*(t)$ and $d_U^*(t)$. Comparing these equations to Eqs. 2.59, 2.60, 2.52, 2.50, and 2.51, we see that the two sets of equations match exactly and we have therefore found a minimum of R_L with respect to $P_C(y)$ for fixed d_L.

Instead of attempting the solution of Eqs. 2.56 for f(x) for any given $P_C(y)$, we just solve Eqs. 2.59 for $f^t(x)$ and then solve Eqs. 2.60 for the optimum composition $P_C(y)$. We may then drop our assumption concerning the existence of solutions to Eqs. 2.56 and the statements about the existence of meaningful solutions to the Eqs. 2.18 and 2.19 defining the upper bound will apply to the solution of Eqs. 2.59 and 2.60 defining the lower bound.

We now have a lower bound on average distortion for any fixed composition block code, and we may define the functions

$$\begin{aligned} R^*(t) &= R_L^*(t) = R_U^*(t) \\ d^*(t) &= d_L^*(t) = d_U^*(t)\ . \end{aligned} \tag{2.61}$$

We have proved the following theorem.



$$Q(y|x) = \frac{P_c(y)\, e^{td(xy)}}{\sum\limits_{Y} P_c(y)\, e^{td(xy)}}$$

$$R^*(t) = \sum_{XY} Q(y|x)\, P(x) \log \frac{Q(y|x)}{P_c(y)}\ , \quad d^*(t) = \sum_{XY} Q(y|x)\, P(x)\, d(xy)$$

Figure 2.4 The test channel.

Theorem 2.7 For any fixed composition block code with $R \leq R^*(t)$, $t \leq 0$, the average distortion must satisfy $\bar{d} \geq d^*(t)$. ••

Note that our choice of f(x) satisfying Eq. 2.56 implies that the output probabilities Q(x) of the channel $Q(x|y)$ driven by the source $P_c(y)$ match P(x) and R_L corresponded to the average mutual information of this source-channel combination. We also see from Eq. 2.51 that d_L corresponds to the average distortion for the source-channel combination. We could actually use the channel $Q(x|y)$ for any $P_c(y)$, if we can satisfy Eq. 2.56, as a test channel and prove a coding theorem as Shannon does. We could show that there exist block codes with rate arbitrarily close to but greater than the average mutual information of the $Q(x|y) \cdot P_c(y)$ combination which give average distortion $d \leq \sum_{XY} Q(x|y)\, P_c(y)\, d(xy)$.

Our asymptotic upper bound on average distortion only agrees with our lower bound for fixed composition codes only for the optimum choice of $P_c(y)$. This can be seen as follows. In the upper bound derivations, we do not have a test-channel interpretation until we have optim ized with respect to $P_c(y)$. In the lower bound derivation we have a test-channel as soon as we select f(x) to satisfy Eq. 2.56 for any $P_c(y)$ for which such a solution is possible. For other than the optimum choice of $P_c(y)$ the best asymptotic lower bound leads to a test channel and the upper bound does not.

Theorem 2.8 Any block code with rate $R \leq R^*(t)$, $t \leq 0$, must have average distortion $\bar{d} \geq d^*(t)$. ••

<u>Proof</u> Any block code of length n can be broken up into sub-codes of fixed composition. There are $B \leq n^b$ composition classes of length n. Each sub-code has a probability of occurrence which is the sum of the probabilities of all source sequences included in the encoding subsets w_i of code words of the sub-code.

Let c_j denote the j-th composition class and p_j the probability of the sub-code with the j-th composition. Suppose there are M_j code words in this sub-code. Our lower bound on average distortion for fixed composition codes applies equally well to sub-codes. As in the proof of Theorem 2.4, we assume that we have disjoint subsets w_i of equal probability $\frac{p_j}{M_j}$. Then each encoding subset of the fixed composition sub-code gives the same distortion, which is a function only of $\frac{p_j}{M_j}$ or $\log \frac{p_j}{M_j}$. The average distortion of the j-th sub-code is then bounded by

$$\bar{d}_j \geq d_o\left(\frac{1}{n} \log \frac{M_j}{p_j}\right)$$

where $d_o(\cdot)$ may be thought of as the expression for $d^*(t)$ explicitly as a function of $R^*(t)$. The lower bound on average distortion for any block code of length n then becomes

$$\bar{d} = \sum_{j=1}^{B} p_j \bar{d}_j \geq \sum_{j=1}^{B} p_j d_o\left(\frac{1}{n}\log \frac{M_j}{p_j}\right) . \tag{2.62}$$

We see from the definition of $d_o(\cdot)$ that in order to lower bound $d_o(\cdot)$ we must overbound its argument.

At this point we make use of a combinatorial theorem on the distribution of a set of weights totaling one pound into B boxes. If we have $0 < q < 1$, at least q pounds of the weights are contained in a set of boxes each of which contains at least $(1-q)/B$

pounds per box. The proof of this theorem is simple. Consider the set of all boxes each of which contains less than q/B pounds of weights. This set of boxes must contain a total weight of less than q/B times the total number of boxes or q pounds. The complementary set of boxes then must contain at least 1-q pounds.

Associating boxes with composition classes and weights with probabilities of sub-codes, we define the subset C* of composition classes as

$$C^* = \{ C_j \mid p_j \geq \frac{q}{B} \text{ , } 0 < q < 1 \} ,$$

so that we know from the combinatorial theorem

$$P_r [C^*] \geq 1-q \ .$$

For $C_j \epsilon C^*$,

$$\frac{M_j}{p_j} \leq \frac{M}{p_j} \leq \frac{MB}{q}$$

and
$$\bar{d}_j \geq d_o \left(\frac{1}{n} \log \frac{MB}{q} \right) .$$

For C_j not in the subset C* we underbound d_j by zero and Eq. 2.62 becomes

$$\bar{d} \geq \sum_{C^*} p_j d_o \left(\frac{1}{n} \log \frac{MB}{q} \right) = P_r[C^*] \, d_o \left(\frac{1}{n} \log \frac{MB}{q} \right)$$

$$\geq (1-q) \, d_o \left(\frac{1}{n} \log \frac{MB}{q} \right) \ . \tag{2.63}$$

We may overbound $B \leq n^b$ and choose $q = \frac{1}{n}$ so that our lower bound on average distortion for any block code of rate $R = \frac{1}{n} \log M$ becomes

$$\bar{d} \geq \left(1-\frac{1}{n}\right) d_o \left(\frac{1}{n}\log M n^{b+1}\right) . \tag{2.64}$$

We see that as $n \to \infty$ the lower bound on average distortion for any block code approaches (from below) the bound given in Theorem 2.7 for any fixed composition block code. This lower bound is weak for finite n but is asymptotically correct for $n \to \infty$.

Q.E.D.

Example 1 - Consider the binary independent letter source with probabilities $P_o = 0.8$, $P_1 = 0.2$, and the distortion measure

$$d_{ij} = 1 - \delta_{ij} \quad ; \quad i,j = 0, 1 .$$

The distortion between binary sequences is just the Hamming distance divided by the sequence length. Computer programs for the IBM 7090 were written to calculate the R_L vs d_L curves so that f(x) and then $P_c(y)$ could be optimized. The R_U vs d_U curves were also computed and $P_c(y)$ was optimized. We show the results of these calculations in Figure 2.5. Even the simple case of the asymmetric binary source requires the use of a non-trivial function f(x).

Example 2 - Consider Shannon's [15] example of the symmetric binary source with output alphabet consisting of the three symbols 0, 1, and ?. Suppose we have the distortion measure

$$d_{ij} = \begin{array}{c|ccc} & 0 & 1 & ? \\ \hline 0 & 0 & 1 & 0.25 \\ 1 & 1 & 0 & 0.25 \end{array} .$$

If we did not have the ?, the rate-distortion function would be given by[†]

[†]This result is derived in Chapter 4.

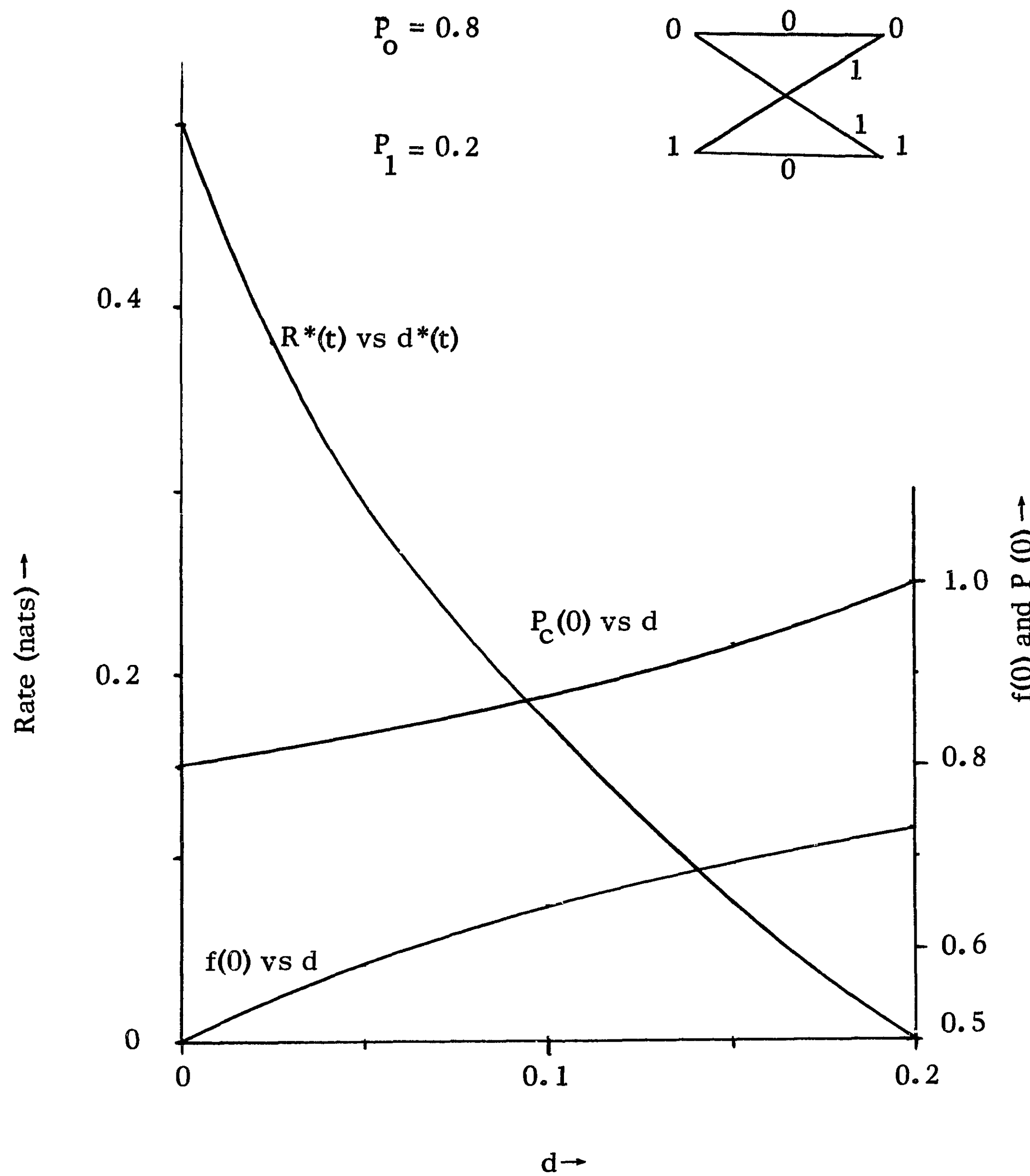


Figure 2.5. The rate-distortion function for an asymmetric binary source showing the optimum f(x) and $P_c(y)$.

$$R(d) = 1 - H(d) \ , \quad 0 \le d \le \frac{1}{2} \ ,$$

where $$H(d) = -d \log d - (1-d) \log (1-d) \ .$$

With the ?, we find $g_x(t)$, given by Eq. 2.17 is independent of x and the Eqs. 2.19 become

$$g(t) = q_0 \, (1+e^t) \ + \ q_? \, e^{t/4} \tag{2.65}$$

where the $P_c(y) = q_i$ are

$q_0 = q_1$ = probability of 0 = probability of 1

$q_?$ = probability of ? .

We may write the Eqs. 2.18 as

$$g(t) = (1+e^t) \,/\, 2 \tag{2.66a}$$

$$g(t) = e^{t/4} \tag{2.66b}$$

It is easy to determine that $t = 0$ and only one negative value of t satisfy both Eqs. 2.66a and b. Eq. 2.65 becomes

$$2q_0 + q_? = 1$$

which is satisfied for any ensemble with our restriction that $q_0 = q_1$. This implies that we have $R^*(t)$ vs $d^*(t)$ for any $0 \le q_? \le 1$ with constant slope t^* which satisfies

$$(1+e^{t^*}) \,/\, 2 = e^{t^*/4} \ .$$

For $q_? = 1$ we have $d^* = 0.25$ and $R^* = 0$.

For $q_? = 0$, we have the ordinary binary symmetric source and Hamming distance distortion measure, so for $t \le t^*$ we have

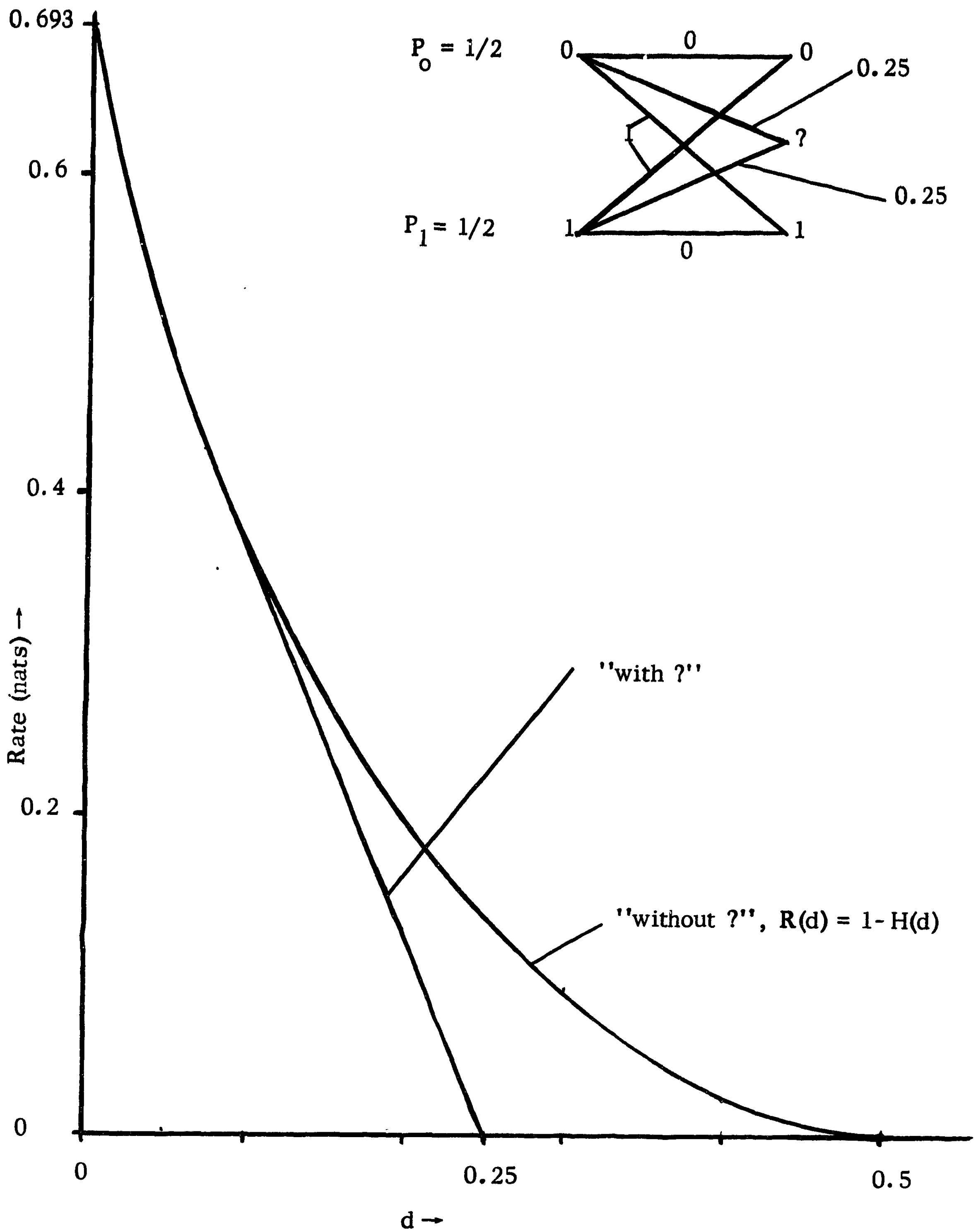


Figure 2.6 A rate-distortion function with a discontinuity in slope and a straight line segment.

$$R^*(t) = 1 - H(d^*(t)) \quad .$$

We show these results in Figure 2.6.

It is seen that the straight line portion of $R^*(t)$ vs $d^*(t)$ arises when we have one more independent constraint in the set of Eqs. 2.18 than we have in the set of Eqs. 2.19. The $P_c(y)$ are then not uniquely determined and there are many ensembles $P_c(y)$ which satisfy Eqs. 2.18 and 2.19 for only one fixed value of t. We have illustrated an $R^*(t)$ vs $d^*(t)$ with a discontinuity in slope since we have solutions for Eqs. 2.18 and 2.19 only for

$$t = 0, \; t \leq t^* < 0.$$

Another interesting point occurs in studying this example. The curve $R^*(t)$ vs $d^*(t)$ is the lower envelope of all R_u vs d_u curves which in turn are all continuous, convex downward, with continuous slope given by $t \leq 0$. It is then impossible to have a discontinuity in slope in $R^*(t)$ vs $d^*(t)$ for some range $t_2 \leq t \leq t_1$ where $t_1 < 0$. We may, however, have straight line segments in $R^*(t)$ vs $d^*(t)$ for any $t \leq 0$.

2.6 Summary

We have discussed the performance of block codes used in encoding the output of a discrete, independent letter information source with a distortion measure. First, an upper bound to average distortion was derived for block codes of finite length n in which M code words were selected at random, each letter of each code word being selected independently according to a probability distribution $P_c(y)$. The asymptotic form of this upper bound for $n \to \infty$ was studied in detail. For each different probability distribution $P_c(y)$, the asymptotic upper bound took the form of a continuous, convex curve R_u vs d_u with continuous derivative. We found the strongest upper bound on average distortion by finding the lower envelope of all R_u vs d_u curves, denoted $R^*(t)$ vs $d^*(t)$ and given parametrically as a function of $t \leq 0$.

$R^*(t)$ vs $d^*(t)$ was found to be a continuous, convex downward function with $R^*(0) = 0$ and

$$d^*(0) = d_{max} = \min_Y \sum_X P(x)\ d(xy)\ ,$$

which agrees entirely with Shannon's results[15]. Also, $R^*(t)$ is given by an expression which could be interpreted as the average mutual information of a test channel $Q(y|x)$ driven by the source $P(x)$. $d^*(t)$ is given by the calculation of average distortion when the source output is transmitted through the test channel $Q(y|x)$. Our formulation of a coding theorem had no channel in it, yet the results appear to involve a test channel $Q(y|x)$. We also found the slope of $R^*(t)$ vs $d^*(t)$ to be given simply by $t \leq 0$.

We mentioned that $d^*(t) \to 0$ for $t \to -\infty$ if and only if each source letter x had some output letter y for which $d(xy) = 0$. The case of $d^*(t)$ not approaching zero is analogous to the problem of the zero error capacity of a discrete channel and is taken up in Chapter 5 of this thesis.

Next, a lower bound to average distortion for block codes of fixed composition $P_c(y)$ was derived. This bound involved an arbitrary positive function $f(x)$, similar to that used by Fano[4] in bounding the probability of error in discrete, memoryless channels. The asymptotic form of the lower bound to average distortion for $n \to \infty$ for block codes of a fixed composition $P_c(y)$ was found and the bound took the form of a curve R_L vs d_L for each $f(x)$ and $P_c(y)$. We optimized the lower bound with respect to $f(x)$ and $P_c(y)$, obtaining a lower bound on average distortion for any fixed composition code.

We obtained exactly the same parametric functions $R^*(t)$ and $d^*(t)$ over the same range of $t \leq 0$ as we obtained in the upper bound. Optimizing first with respect to $f(x)$ lead to the interpretation of a test channel $Q(x|y)$ for any $P_c(y)$. The test channel for the optimum composition $P_c(y)$ was shown to be identical with the test channel $Q(y|x)$ found in the upper bound.

We were then able to show that the $R^*(t)$ vs $d^*(t)$ curve applied also as an asymptotic lower bound on average distortion for any block code. This allowed us to identify our parametric functions $R^*(t)$ and $d^*(t)$, $t \leq 0$, with Shannon's rate-distortion function $R(d)$. Our test channel $Q(x|y)$ (or $Q(y|x)$) may be identified with Shannon's test channel in his definition of the $R(d)$ function. However, we provide an explicit solution for the transition probabilities of the test channel and, hence, also for $R^*(t)$ vs $d^*(t)$ in the Eqs. 2.6, 2.7, 2.9, 2.18, and 2.19. (See Fig. 2.4.)

An example showed that we may have straight line segments in $R^*(t)$ vs $d^*(t)$ but the only discontinuity in slope must occur on the $R^*(0) = 0$ axis. Each straight line segment of $R^*(t)$ vs $d^*(t)$ could be attributed to one more independent constraint in the set of Eqs. 2.18 than in the set of Eqs. 2.19. We then have a non-unique solution to the Eqs. 2.19 for a certain value of t (the slope of $R^*(t)$ vs $d^*(t)$), which implies that we have many compositions $P_c(y)$, and hence many values of $R^*(t)$ and $d^*(t)$ for which $R^*(t)$ vs $d^*(t)$ has the same slope.

The lower bound on average distortion for finite block length codes is a very difficult problem which is treated separately in Chapter 3. A different fidelity criterion from average distortion per letter is also treated in Chapter 5.

CHAPTER III

ASYMPTOTIC CONVERGENCE OF THE LOWER BOUND ON AVERAGE DISTORTION

3.1 A Lower Bound to Average Distortion

Our treatment of the lower bound on average distortion for any block code resulted in the limiting expressions as the block length $n \to \infty$. Shannon[15] has shown that the rate-distortion function $R^*(t)$ vs $d^*(t)$ is a firm lower bound on average distortion for any block length. We have no stronger lower bound on average distortion for finite n, and hence no estimate of the convergence of the lower bound to the limiting form as $n \to \infty$. This is a weaker result than that given in Theorem 2.1 for the upper bound on average distortion. We will show the inherent difficulties in obtaining such strong results in the lower bound case, and we will instead find asymptotic expansions for the lower bound expressions showing the convergence with large n to the limiting form.

We will study the expressions in Theorem 2.4 (given again below) as functions of n for constant D_o and a given output sequence v_o with composition $n_c(y)$.

$$R_L = \frac{1}{n} \log M_L = -\frac{1}{n} \log P_r[U_o] \tag{3.1a}$$

$$D_L = \frac{\sum_{U_o} P(u)\, D(uv_o)}{P_r[U_o]} \tag{3.1b}$$

where

$$U_o = \{ u \mid D(uv_o) \leq D_o \}.$$

Since the above expression for D_L differs from the lower bound on $\bar{d}$ given in Theorem 2.5 only by the term $\sum_X P(x) \log f(x)$, which is independent of n, it is sufficient to study R_L and D_L for large n to find the rate of convergence of the bound on $\bar{d}$ to its limiting form.

We are concerned with the random variable

$$n D(uv_o) = \sum_{i=1}^{n} D(\xi_i \eta_i)$$

which, for a given $v_o = (\eta_i)$ is a sum of n independent, but non-identical random variables. Define the distribution function

$$\mathfrak{p}_n(\chi) = P_r[n D(uv_o) \leq \chi]$$

so that Eq. 3.1b may be re-written

$$n D_L = \frac{\int_{-\infty}^{n D_o} \chi \, d\mathfrak{p}_n(\chi)}{\int_{-\infty}^{n D_o} d\mathfrak{p}_n(\chi)} = \frac{I_n}{I_D} \quad . \tag{3.2}$$

Consider a new random variable whose distribution function $\mathcal{H}_n(\chi)$ is defined by

$$\mathcal{H}_n(\chi) = \frac{\int_{-\infty}^{\chi} e^{t\chi'} \, d\mathfrak{p}_n(\chi')}{\int_{-\infty}^{\infty} e^{t\chi''} \, d\mathfrak{p}_n(\chi'')} \quad , \text{ (t real)}, \tag{3.3}$$

so that

$$d\mathcal{H}_n(\chi) = \frac{e^{t\chi} \, d\mathfrak{p}_n(\chi)}{\int_{-\infty}^{\infty} d^{t\chi'} \, d\mathfrak{p}_n(\chi')} \quad . \tag{3.4}$$

Define the moment generating function of $\mathfrak{p}_n(\chi)$ as

$$g_n(t) = e^{n\mu_n(t)} = \int_{-\infty}^{\infty} e^{t\chi'} \, dp_n(\chi') \quad . \tag{3.5}$$

Since we are dealing with the sum of independent random variables we can write $\mu_n(t)$ as

$$\mu_n(t) = \frac{1}{n} \sum_Y n_c(y) \log \sum_X e^{tD(xy)} P(x) = \sum_Y P_c(y) \, \mu_y(t) \tag{3.6}$$

where $P_c(y) = \dfrac{n_c(y)}{n}$.

The mean value of $D(uv_o)$ is

$$D_m = \sum_Y P_c(y) \sum_X P(x) \, D(xy) = \mu_n'(0) \quad . \tag{3.7}$$

By an argument analogous to that used to show $\gamma''(t) \geq 0$ in Chapter 2, we can show $n\mu_n''(t)$ is the variance of the distribution $H_n(\chi)$, so $\mu_n''(t) \geq 0$ and $\mu_n'(t)$ is a monotone increasing function of t. Let us fix the value of t so that

$$D_o = \mu_n'(t) \leq D_m \quad . \tag{3.8}$$

We can always find such a $t \leq 0$ for $D_o \leq D_m$ because $\mu_n'(t)$ is continuous and monotone in t. Now let us re-write Eq. (3.2), using Eqs. 3.4, 3.5, and 3.8.

$$nD_L = \frac{e^{n\mu_n(t)} \displaystyle\int_{-\infty}^{n\mu_n'(t)} \chi \, e^{t\chi} \, dH_n(\chi)}{e^{n\mu_n(t)} \displaystyle\int_{-\infty}^{n\mu_n'(t)} e^{t\chi} \, dH_n(\chi)} \quad , \quad t \leq 0. \tag{3.9}$$

The distribution $H_n(\chi)$ has mean value $n\mu_n'(t)$ and variance $n\mu_n''(t)$. If we make the substitution

$$z = \frac{\chi - n\mu_n'(t)}{\sqrt{n\mu_n''(t)}}$$

and write $\mathfrak{F}_n(z)$ for $H_n\left(\sqrt{n\mu_n''(t)}\; z + n\mu_n'(t)\right)$, we obtain a distribution $\mathfrak{F}_n$ with mean zero and variance one, suitable for application of ordinary central limit results. The Eq. (3.9) becomes

$$nD_L = \frac{e^{n\mu(t) - nt\mu'(t)} \int_{-\infty}^{0} \left(\sqrt{n\mu''(t)}\; z + n\mu'(t)\right) e^{-t\sqrt{n\mu''(t)}\, z}\, d\mathfrak{F}_n(z)}{e^{n\mu(t) - nt\mu'(t)} \int_{-\infty}^{0} e^{-t\sqrt{n\mu''(t)}\, z}\, d\mathfrak{F}_n(z)} ; \quad (3.10)$$

We have dropped the n subscripts on μ, μ', μ'', remembering that these quantities have an n-dependence because v_o must be selected for each n.

We could use the central limit theorem by Cramer[(1)] (page 77-78) to write

$$\left| \mathfrak{F}_n(z) - \Phi(z) \right| < C \frac{\beta_{3n}}{\mu_n''(t)^{3/2}} \frac{\log n}{\sqrt{n}} \quad (3.11)$$

where β_{3n} is the third absolute moment of H_n and C is an absolute constant. A substitution for $d\mathfrak{F}_n(z)$ from Eq. 3.11 would enable us to use integration by parts to obtain bounds on the integrals in Eq. 3.10. Shannon[(17)] has derived upper and lower bounds on the integral in the denomination of Eq. 3.10 in exactly this manner. Although his bounds were derived for identically distributed random variables, it is clear from Cramer's work on asymptotic expansions of $\mathfrak{F}_n(z) - \Phi(z)$ ((1), Part II, Chapter VII) that our case of non-identically distributed random variables only introduces the n-dependence in μ, μ', μ'', ... as we have defined it in Eq. 3.6.

Shannon's upper bound to the denominator of Eq. 3.10 is

$$\frac{e^{n\mu(t) - nt\mu'(t)}}{|t| \sqrt{2\pi n\mu''(t)}} K(n)$$

where K(n) is a function of t, μ'', μ'''', and powers of n. The limiting form of K(n) satisfies

$$\lim_{n \to \infty} K(n) > 1 \, .$$

The numerator integrals involving $C \beta_{3n} \log n / \sqrt{n} \, (\mu''_n (t))^{3/2}$ can all be bounded uniformly in n so the factor $1/\sqrt{n}$ will cause these terms to approach zero as $n \to \infty$. The numerator integrals involving $\Phi(z)$ are lower bounded by the expression

$$\frac{e^{n\mu(t) - nt\mu'(t)}}{|t| \sqrt{2\pi n \mu''(t)}} \; n\mu'(t) \left(1 - \frac{2t\mu''}{\mu'} + \frac{1}{nt\mu'} - \frac{1}{nt^2\mu''}\right), \; t < 0.$$

We can see that the limiting form of the bound on D_L is

$$D_L \sim \lim_{n \to \infty} \frac{\mu'(t) - 2t\mu''(t)}{K(n)} \neq \mu'(t)$$

whereas we know from Chapter 2 the correct limit is $\mu'(t)$.

We conclude that a much stronger central limit theorem than Eq. 3.11 is needed to get the desired lower bound to D_L to converge to $\mu'(t)$ as $n \to \infty$. The factor

$$\frac{e^{n\mu(t) - nt\mu'(t)}}{|t| \sqrt{2\pi n \mu''(t)}}$$

is common to the tight bounds on the numerator and denominator of Eq. 3.2 and cancels. Our bounds must therefore be asymptotically correct in the terms of lower order in n than the cancelling factor. In the next section, we will then study an asymptotic expansion of D_L.

3.2 Asymptotic Expansion of D_L

In order to get an asymptotic expansion for nD_L, we must have asymptotic expansions for the integrals in Eq. 3.10. Since $\mathfrak{F}_n$ is the distribution function for a normalized sum of independent random variables, we can expand it in an asymptotic series and proceed to derive an asymptotic series for the integrals of Eq. 3.10. Let us suppose for the moment that we have the expansions

$$\int_{-\infty}^{0} e^{\alpha z}\, d\mathfrak{F}_n(z) = \frac{d_1}{\sqrt{n}} + \frac{d_2}{n} + \frac{d_3}{n^{3/2}} + \dots .$$

$$\int_{-\infty}^{0} z e^{\alpha z}\, d\mathfrak{F}_n(z) = \frac{c_1}{n} + \frac{c_2}{n^{3/2}} + \frac{c_3}{n^2} + \dots \tag{3.12}$$

where $\alpha = |t| \sqrt{n\mu''(t)}$. We can derive the expansion for the numerator of Eq. 3.10 in a straightforward manner.

$$I_N = \sqrt{n\mu''} \left(\frac{c_1}{n} + \frac{c_2}{n^{3/2}} + \cdots \right) + n\mu' \left(\frac{d_1}{\sqrt{n}} + \frac{d_2}{n} + \cdots \right)$$

$$= \frac{n\mu'}{|t| \sqrt{2\pi n \mu''}} \left(\frac{\hat{c}_1}{n} + \frac{\hat{c}_2}{n^{3/2}} + \cdots + \hat{d}_1 + \frac{\hat{d}_2}{\sqrt{n}} + \frac{\hat{d}_3}{n} + \dots \right)$$

$$= \frac{n\mu'}{|t| \sqrt{2\pi n \mu''}} \left(\hat{d}_1 + \frac{\hat{d}_2}{\sqrt{n}} + \frac{\hat{c}_1 + \hat{d}_3}{n} + \dots \right) \tag{3.13}$$

where $\hat{c}_1 = \dfrac{|t| \sqrt{2\pi\mu''}}{\mu'}\, c_1$, etc.

$\hat{d}_1 = |t| \sqrt{2\pi \mu''}\, d_1$, etc.

$$I_D = \frac{1}{|t|\sqrt{2\pi n\mu''}} \left(\hat{d}_1 + \frac{\hat{d}_2}{\sqrt{n}} + \frac{\hat{d}_3}{n} + \dots \right) \tag{3.14}$$

If we expand nD_L in the asymptotic series†

$$nD_L \sim n\mu' \left(e_o + \frac{e_1}{\sqrt{n}} + \frac{e_2}{n} + o\left(\frac{1}{n}\right) \right)$$

we can find the coefficients e_i in terms of the $\hat{c}_i$ and $\hat{d}_i$.

$$I_N \sim n\mu' \left(\hat{d}_1 + \frac{\hat{d}_2}{\sqrt{n}} + \frac{\hat{c}_1 + \hat{d}_3}{n} + o\left(\frac{1}{n}\right) \right)$$

$$\sim n\mu' \left(e_o + \frac{e_1}{\sqrt{n}} + \frac{e_2}{n} + o\left(\frac{1}{n}\right) \right) \left(\hat{d}_1 + \frac{\hat{d}_2}{\sqrt{n}} + \frac{\hat{d}_3}{n} + o\left(\frac{1}{n}\right) \right) \sim nD_L \cdot I_D$$

$$\sim n\mu' \left(e_o\hat{d}_1 + \frac{e_o\hat{d}_2 + e_1\hat{d}_1}{\sqrt{n}} + \frac{e_o\hat{d}_3 + e_1\hat{d}_2 + e_2\hat{d}_1}{n} + o\left(\frac{1}{n}\right) \right) \tag{3.15}$$

Equating coefficients(2) of like powers of n in the expansions for I_N and $nD_L \cdot I_D$ gives the result

$$e_o = 1, \quad e_1 = 0, \quad e_2 = \frac{\hat{c}_1}{\hat{d}_1}$$

so that

$$D_L \sim \mu'(t) \left(1 + \frac{\hat{c}_1}{\hat{d}_1} \frac{1}{n} + o\left(\frac{1}{n}\right) \right) . \tag{3.16}$$

In order to see the asymptotic approach of D_L to $\mu'(t)$ we need only the coefficients $\hat{c}_1$ and $\hat{d}_1$ of the asymptotic expansions for the integrals in Eq. 3.10.

---

† The notation $o(\frac{1}{n})$ is used for terms which, for arbitrary $\epsilon > 0$, can be made smaller in magnitude than $\frac{\epsilon}{n}$ for large enough n.

In Appendix B we derive the coefficients $\hat{c}_1$ and $\hat{d}_1$ resulting in

$$D_L \sim \mu'(t) - \frac{1}{n|t|} + o\left(\frac{1}{n}\right) \tag{3.17}$$

whether the distance D(xy) is a lattice or non-lattice random variable. This result is intuitively appealing since the limiting value of D_L is $\mu'(t)$, as we know from Chapter 2, and also the limiting value is approached from below. We may interpret Eq. 3.2 as the calculation of the location of the centroid along the χ-axis of the tail of the distribution function $\mathcal{P}_n$ out to $n\mu'(t)$. We know the centroid must be located at a point $\chi_o < n\mu'(t)$.

We may also find an asymptotic expansion for R_L from Eqs. 3.1a and 3.14.

$$R_L \sim -\frac{1}{n}\log\left(\frac{e^{-n[t\mu'(t)-\mu(t)]}}{|t|\sqrt{2\pi n\mu''(t)}}\left(1+o\left(\frac{1}{\sqrt{n}}\right)\right)\right)$$

$$\sim t\mu'(t) - \mu(t) + \frac{1}{n}\log\sqrt{n} + \frac{1}{n}\log K - \frac{1}{n}\log\left(1+o\left(\frac{1}{n}\right)\right)$$

$$\sim t\mu'(t) - \mu(t) + \frac{1}{n}\log\sqrt{n} + o\left(\frac{1}{n}\log\sqrt{n}\right) \tag{3.18}$$

(We have used $\log(1-\epsilon) \sim -\epsilon$ for $\epsilon \to 0$.) The asymptotic expression for R_L is approached from above and convergence is like $\frac{1}{n}\log\sqrt{n}$. Convergence in D_L (like $\frac{1}{n}$) is seen to be faster than convergence in R_L.

If we consider $R^*_{Ln} - d^*_{Ln}$ to be the equivalent of the $R^*_L - d^*_L$ curve for finite n, we may interpret our asymptotic results graphically as the convergence of the $R^*_{Ln} - d^*_{Ln}$ curve to the $R^*_L - d^*_L$ curve as $n \to \infty$. Since $R^*_L - d^*_L$ is a firm lower bound on average distortion for any n, $R^*_{Ln} - d^*_{Ln}$ must approach $R^*_L - d^*_L$ from above. Figure 3.1 is a sketch of the locus of points on $R^*_{Ln} - d^*_{Ln}$ for fixed D_o as n increases. This locus of points shows D_L converging more rapidly than R_L, thus insuring that $R^*_{Ln} - d^*_{Ln}$ converges to $R^*_L - d^*_L$ from above.

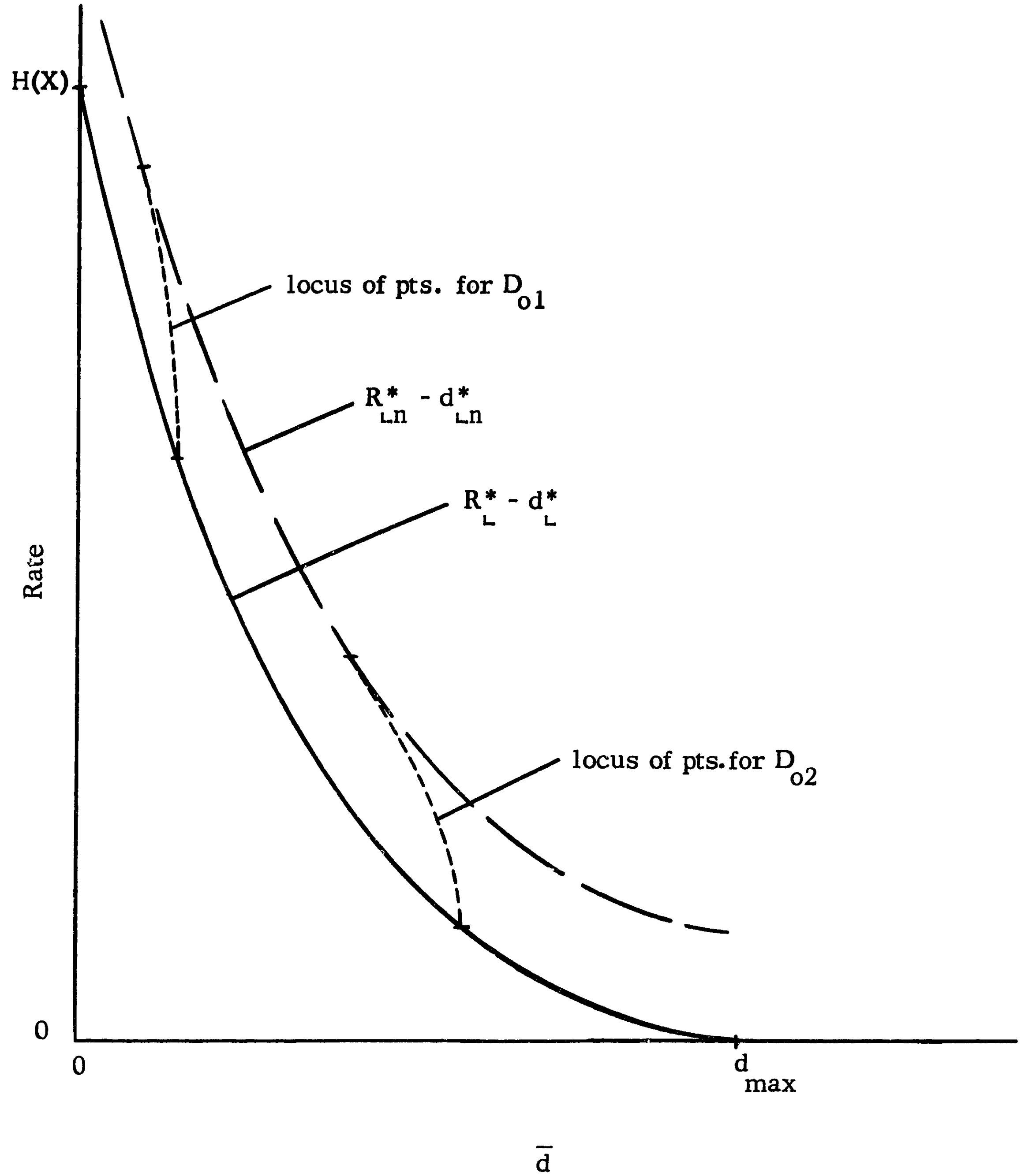


Figure 3. 1 -- Convergence of points on $R^*_{Ln} - d^*_{Ln}$ to $R^*_L - d^*_L$.

CHAPTER IV

BINARY SOURCE ENCODING

4.1 Introduction

In order to use a block code to encode a source, the encoder must be capable of storing $M = e^{nR}$ code words of length n. To encode each source sequence of length n, the encoder must also compute the distortion between the source sequence and each code word of the block code. The general theory of the previous chapters indicates that if we wish to achieve an average distortion close to the minimum attainable average distortion for a particular code rate R, the code length, n, must be very large. In implementing a block code, therefore, both the amount of storage space for the code and the number of computations of distortion per source letter increase exponentially with n. Our purpose in this chapter is to explore coding systems which give average distortion approaching the ideal performance indicated by the rate-distortion function $R^*(t) - d^*(t)$ with much less equipment complexity than block codes require.

Following the approaches to the complexity problem which were taken in coding for noisy channels,[(3),(12),(18)] we attempt to build some algebraic structure into codes to enable us to generate the code by using an algorithm with a reduced amount of stored data. For instance, if we were dealing with a binary source, we should first study binary linear or group codes in which the block code consists of all possible linear combinations (addition modulo 2) of nR generator or basis sequences. Group codes, therefore, only require the storage of nR generator sequences of length n. The algebraic structure of group codes also allows a simpler encoding procedure than the comparison of a source sequence with every code word.[12] It would then be of great interest to demonstrate that the ensemble of randomly chosen group codes gives as good an upper bound to average distortion as the ensemble of random block codes.

We derived an upper bound to average distortion for an ensemble of random block codes in Theorem 2.1 by using a non-optimum encoding procedure which led to the correct asymptotic bound on average distortion. In attempting to encode a source sequence, the encoder searched the list of code words to find one which gave less than a certain amount, say d_o, of distortion with the source sequence. If no code word in the list satisfied the d_o threshold, we could bound the distortion by $\Delta = \max_{XY} d(xy)$. As before, let us define P_o as the probability that the source chooses a word u and we choose a code at random such that we find no code word v for which $d(uv) \le d_o$. The upper bound on average distortion over the ensemble of random codes can be written as

$$\bar{d} \le d_o + \Delta \cdot P_o . \tag{4.1}$$

Suppose all source words are equiprobable, to simplify things for the present. If we denote the code words of a randomly chosen code as $v_1, v_2, \ldots, v_M$, then

$$P_o = \text{Prob. } (v_1 \text{ N.A. } \underline{\text{and}}\ v_2 \text{ N.A. } \underline{\text{and}} \ldots \underline{\text{and}}\ v_M \text{N.A.}),\ \text{N.A.} = \text{not acceptable},$$

which may in turn be written as

$$P_o = \text{Prob. } (v_1 \text{ N.A.})^M \tag{4.2}$$

for block codes in which each code word is chosen independently at random. However, if we have some algebraic structure to the code we cannot write P_o in a factored form as in Eq. 4.2. For instance, a group code is completely determined by nR generator sequences. A random group code is then selected by choosing only the nR generator sequences independently at random and so all $M = e^{nR}$ code words are not independent. In this case, as in any code with algebraic structure, we see that the probability P_o involves an intersection of events with subtle dependencies between them. The algebraic dependencies introduced between code words by a group structure are difficult to characterize and so we cannot derive an upper bound on average distortion for the ensemble of block codes with this relatively simple algebraic structure. In bounding the probability of error in channels for an ensemble of randomly chosen group codes, an upper bound to the union of dependent events is needed. This is conveniently gotten since the probability of a union of events, whether dependent or not, is always upper bounded by the sum of the probabilities of the individual events. In the source encoding problem, the treatment of an ensemble of random codes with algebraic structure involves a fundamental difficulty, namely, the upper bound on an intersection of dependent events. We have not been able to overcome this difficulty in a general way in this research.

In view of the above discussion, we will consider the simplest of source encoding problems to gain some insight into the methods of analyzing the performance of coding systems as well as the complexity involved in their use. We therefore discuss in the remainder of this chapter our results concerning the binary source with equiprobable ones and zeroes and the distortion measure

$$d_{ij} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

The distortion between two binary sequences, according to this distortion measure, is just the Hamming distance between the sequences.

Suppose we wish to encode this binary source with no more than r errors in a sequence of length n. If we are given a particular source word of length n, the probability of choosing a code word of length n with independent, equiprobable binary digits which gives r or fewer errors with the source word is just $2^{-n} \sum_{i=0}^{r} \binom{n}{i}$, where $\binom{n}{i}$ is the binomial coefficient. Since all source words are equiprobable, we may write P_o for a block code with M independently chosen code words as

$$P_o = [\, 1 - 2^{-n} N_r \,]^M < 2^{-M\, 2^{-n} N_r} \tag{4.3}$$

where

$$N_r = \sum_{i=0}^{r} \binom{n}{i} .$$

The upper bound on average distortion of Eq. 4.1 becomes, in this case,

$$\bar{d} \leq \frac{r}{n} + P_o . \tag{4.4}$$

In this chapter only, we will use as the definition of the code rate, $R = \frac{1}{n} \log_2 M$, so that $M = 2^{nR}$. From Fano[4] (page 216) we get the bounds on $\binom{n}{r}$,

$$e^{-\frac{1}{12nd(1-d)}} < \binom{n}{r} \frac{2^{-nH(d)}}{\sqrt{2\pi\, nd(1-d)}} < 1, \quad 0 < d = \frac{r}{n} < 1, \tag{4.5}$$

where

$$H(d) = - d \log_2 d - (1-d) \log_2 (1-d).$$

We may lower bound N_r by $\binom{n}{r}$ and using the above lower bound on $\binom{n}{r}$ in Eq. 4.3, we may get as an upper bound on P_o

$$P_o < 2^{-K(n)\, 2^{n[R-1+H(d)]}}$$

where

$$K(n) = \frac{e^{-\frac{1}{12nd(1-d)}}}{\sqrt{2\pi\, nd(1-d)}} \quad .$$

If we choose r so that $\frac{r}{n} = d$ remains fixed as we increase n, it is clear that we must have $R - 1 + H(d) > 0$ in order to have P_o tend to zero with increasing n. From Eq. 4.4 we see that if $R > 1 - H(d)$, then as $n \to \infty$ the upper bound on average distortion becomes $\bar{d} \le d$.

Let us now apply our lower bound on average distortion of Chapter 2, to this binary case. The symmetry of the binary source and distortion measure allow us to disperse with the distance function D(uv) in our derivation of Chapter 2, and we may take $D(uv_o)$ in Theorem 2.4 to be the distortion between u and v_o. From Theorem 2.4, if we have a block code for which

$$M \le \frac{1}{2^{-n} \sum_{i=0}^{r} \binom{n}{i}} \quad , \tag{4.6}$$

then

$$\bar{d} \ge \frac{\frac{1}{n} \sum_{i=0} i \cdot \binom{n}{i}}{\sum_{i=0}^{r} \binom{n}{i}} \quad , \quad 0 \le r \le n. \tag{4.7}$$

It should be clear that in the binary case these inequalities are true whether the block code is a fixed composition code or not.† Using Eq. 4.5 we may upper bound the sum of binomial coefficients by

$$\sum_{i=0}^{r} \binom{n}{r} < \frac{(r+1)\, 2^{nH(d)}}{\sqrt{2\pi nd(1-d)}} = \frac{(nd+1)\, 2^{nH(d)}}{\sqrt{2\pi nd(1-d)}} . \tag{4.8}$$

We now write the constraint on M in Eq. 4.6 more conservatively using the bound in Eq. 4.8,

$$M \leq 2^{n(1-H(d))} \left(\frac{\sqrt{2\pi\, nd(1-d)}}{nd+1} \right)$$

and from this we see that as $n \to \infty$, we get the constraint on $R = \frac{1}{n} \log_2 M$ as

$$R \leq 1 - H(d) . \tag{4.9}$$

Directly from our development in Chapter 2, we know that the limit of the right hand side of Eq. 4.7 as $n \to \infty$ is $d = \frac{r}{n}$. We conclude that as $n \to \infty$, any binary block code for which $R \leq 1 - H(d)$ gives an average distortion $\bar{d} \geq d$. The rate-distortion function of the binary symmetric source with Hamming distance as distortion measure is then

$$R(d) = 1 - H(d). \tag{4.10}$$

4.2 Binary Group Codes

We first treat the simplest binary block codes with structure, namely group codes. (See Peterson's book[12], Chapters 2 and 3). A group code is determined by k generator sequences of length n, the code consisting of all 2^k possible linear combinations

---

† This is true because the functions $F_i(z)$ in the proof of Theorem 2.4 are identical for all output sequences.

(addition modulo 2) of the generators. Since $M = 2^k$, the code rate is

$$R = \frac{1}{n} \log_2 M = \frac{k}{n} .$$

Suppose we number the k code generators as the first k code words. We may specify any linear combination of the generators by a sequence of k binary digits. If the j^{th} digit is a one, the j^{th} generator is added into the linear combination. Any code word is then specified uniquely by a sequence of k binary digits.

The source encoder may operate exactly like any group code decoder used with a binary symmetric channel.[(12)] Once a source sequence has been encoded into a code word, k binary digits which specify the code word must be transmitted over a channel to the information sink. At the output of the channel the decoder forms the mod 2 sum of the generator sequences specified by the k digits and presents the sink with a code word approximating the actual source output.

We now give a theorem which shows that group codes can be constructed to give performance close to the ideal rate-distortion performance.

> Theorem 4.1 There exist binary group codes with rate $R \leq 1\text{-}H(\frac{d}{2})$ that give average distortion d or less. ••

Proof We give a construction proof. Since we can consider the zero sequence (denoted $\underline{0}$) as a linear combination of a set of code generators, $\underline{0}$ is a code word in every group code. Choose v_1, the first code generator, as any sequence with Hamming distance from $\underline{0}$ greater than r. Then choose as the second generator, v_2, any sequence with Hamming distance, from both $\underline{0}$ and v_1 greater than r. Then the code word which is the mod 2 sum of v_1 and v_2 (denoted $v_1 \oplus v_2$) has distance greater than r from $\underline{0}$ because

v_2 was chosen so that $v_1 \oplus v_2$ has weight greater than r. (Computing the Hamming distance between two sequences is the same as computing the weight of the mod 2 sum of the sequences.) We see that $v_1 \oplus v_2$ has distance greater than r from $\underline{0}$, v_1, and v_2.

We choose as v_3 any sequence not within Hamming distance r of all sequences already in the code ($\underline{0}$, v_1, v_2, $v_1 \oplus v_2$). In other words, we have chosen v_3 so that the sequences $v_3 \oplus \underline{0} = v_3$, $v_3 \oplus v_1$, $v_3 \oplus v_2$, $v_3 \oplus v_2 \oplus v_1$ all have weight greater than r. This implies that the group code with v_1, v_2, and v_3 as generators has no code words within distance r of each other.

We proceed to construct a group code in this manner until we add no more generators, i.e., there are no more sequences greater than distance r from all code words. This implies that we guarantee the number of errors (the distortion) in encoding any source word to be r or less. Since no two code words are within distance r of each other, the sets of sequences within distance $\frac{r}{2}$ of each code word are disjoint sets. Therefore, in order to reach the point where we can add no more generators to the code, we need no more than $2^n / \sum_{i=0}^{r/2} \binom{n}{i}$ code words or more conservatively

$$2^k \binom{n}{\frac{r}{2}} \leq 2^n .$$

Taking $\log_2$ of this equation we get

$$\frac{k}{n} + \frac{1}{n} \log_2 \binom{n}{\frac{r}{2}} \leq 1 .$$

If $d = \frac{r}{n}$ is held constant as n increases, we get as $n \to \infty$

$$\bar{d} \leq d \text{ and } R \leq 1 - H(\tfrac{d}{2}) .$$

Q.E.D.

This bound and the R(d) function of Eq. 4.2 are plotted in Figure 4.1. We see that our construction bound is quite weak for average distortion near 0.5. Suppose we have two block codes with rates R_1 and R_2 and giving average distortion d_1 and d_2, respectively. These may be plotted in the R-d plane as two points (R_1, d_1) and (R_2, d_2). If the code lengths are n_1 and n_2, we can construct a block code of length $n_1 + n_2$ by alternating the use of the two codes. The rate for the new code is $\frac{n_1 R_1 + n_2 R_2}{n_1 + n_2}$ and the average distortion is $\frac{n_1 d_1 + n_2 d_2}{n_1 + n_2}$. We may then plot this code in the R-d plane. We can easily see that mixing two block codes in any proportion gives codes with points on the straight line connecting (R_1, d_1) and (R_2, d_2). We can therefore tighten our construction bound by a code mixing argument which enables us to draw a tangent to the construction bound passing through the point (0, 0.5). This is also shown in Figure 4.1.

We now demonstrate the existence of group codes which satisfy the same upper bound to average distortion as the ensemble of random block codes (Eqs. 4.3 and 4.4), implying that there exist group codes giving performance as near the ideal rate-distortion performance as we wish. First we present a useful lemma.

<u>Lemma</u> Let v and s denote points in an n dimensional binary space (binary n-tuples), and denote the operation of addition of n-tuples mod 2 by $\oplus$. Suppose we have a set S_o of n_o points s_o and a set S_1 of n_1 points s_1. For any particular point v_o we define the set $S(v_o) = \{v \mid v = v_o \oplus s_o \text{ for some } s_o \in S_o\}$. Then there exists a point v_o such that the union of $S(v_o)$ and S_1 includes at least $n_o + n_1 - n_o n_1 2^{-n}$ points.

••

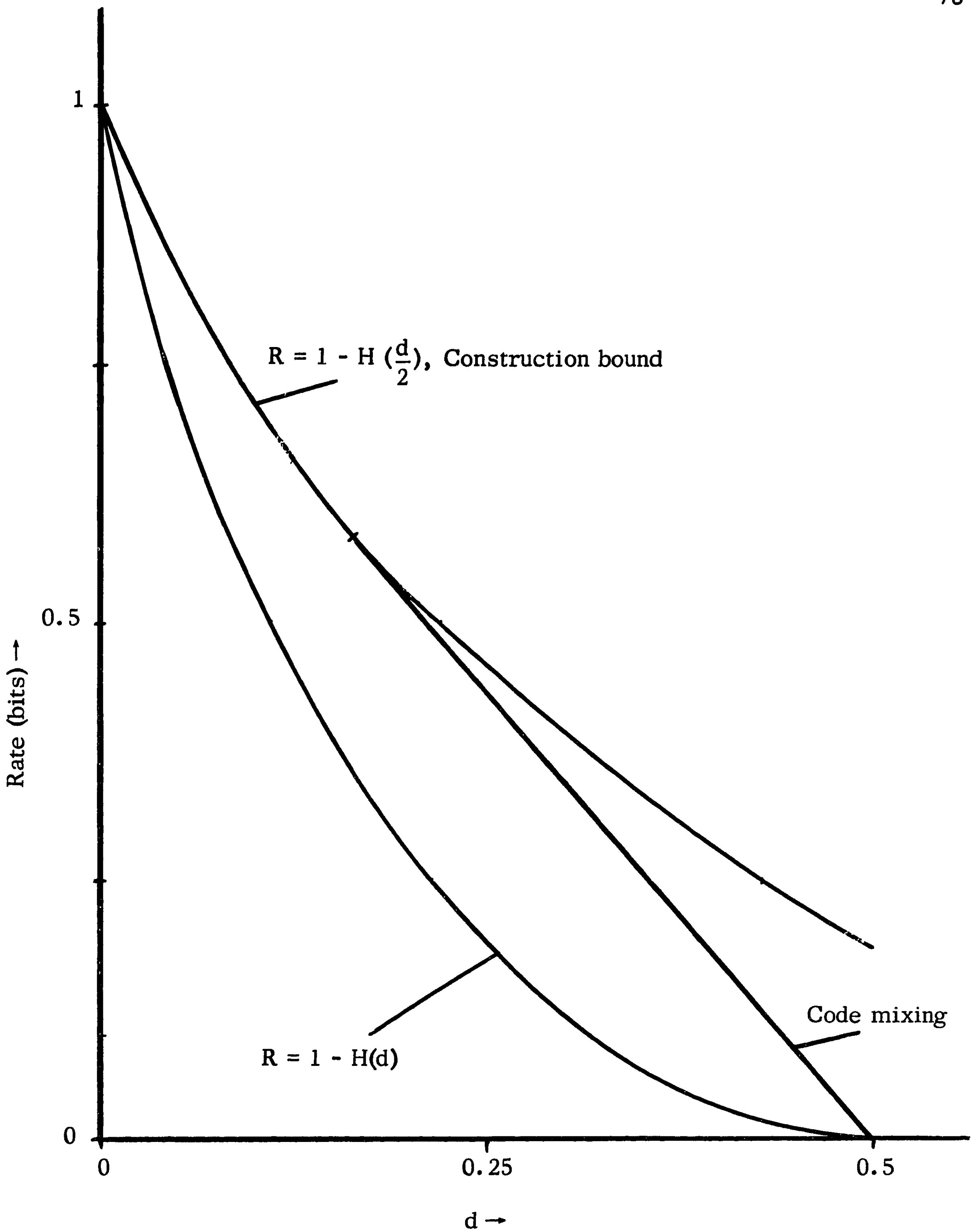


Figure 4.1 The rate-distortion function for the binary symmetric source with Hamming distance as the distortion measure, and a construction bound on average distortion for binary group codes.

Proof Consider the set of points $v_o \oplus s_o$ for a particular $s_o \epsilon S_o$ and all 2^n possible choices of v_o. It is clear that $v_o \oplus s_o$ takes on each of the 2^n points in the space once and only once as v_o takes on all 2^n values. Therefore $v_o \oplus s_o$ coincides with a particular $s_1 \epsilon S_1$ once and only once for the 2^n choices of v_o. We then conclude that each point in $S(v_o)$ corresponding to a particular $s_o \epsilon S_o$ coincides only once with each $s_1 \epsilon S_1$ for all possible choices of v_o. For the 2^n different sets $S(v_o)$ there are then a total of $n_o n_1$ coincidences of points of $S(v_o)$ and S_1 and, hence, the average number of coincidences per choice of v_o is $n_o n_1 2^{-n}$.

If $S(v_o)$ and S_1 are disjoint, then $S(v_o) \cup S_1$ contains $n_o + n_1$ points. If for a particular v_o we have I points of $S(v_o)$ and S_1 coinciding, then $S(v_o) \cup S_1$ contains $n_o + n_1 - I$ points. Since the average intersection of $S(v_o)$ and S_1 is $n_o n_1 2^{-n}$ for the set of all possible choices of v_o, there exists a particular v_o which gives an intersection at least as small as the average. We conclude then, that there exists a particular v_o such that $S(v_o) \cup S_1$ contains at least $n_o + n_1 - n_o n_1 2^{-n}$ points.

Q. E. D.

We are now in a position to prove the following theorem on group codes.

Theorem 4.2 For any r, $0 \leq r \leq n$, there exists a binary group code of length n with rate $\frac{k}{n}$ and with average distortion satisfying

$$\bar{d} \leq \frac{r}{n} + \left[1 - 2^{-n} \sum_{i=0}^{r} \binom{n}{i}\right]^{2^k}. \tag{4.11}$$

• •

Proof The proof is by induction. Consider sequences of length n as points in an n-dimensional binary space. Define the set S_o as the set of all points with weight r or less. There are then $\sum_{i=0}^{r} \binom{n}{i}$ points in S_o. For any particular point v' we define the set

$$S(v') = \{ v \mid v = v' \oplus s \text{ for some } s \in S_o \} .$$

It is clear that if we interpret v' as a code word, every point in S(v') can be encoded as v' with r or fewer errors.

The point $\underline{0}$ is in every group code. For the code consisting only of $\underline{0}$ we have $N_o = \sum_{i=0}^{r} \binom{n}{i}$ points that can be encoded with r or fewer errors. The probability of the source producing a point which cannot be encoded with r or fewer errors with this code is

$$Q_o = \frac{2^n - N_o}{2^n} = 1 - \frac{N_o}{2^n} \tag{4.12}$$

and so an upper bound to average distortion is

$$\bar{d}_o = \frac{r}{n} + Q_o. \tag{4.13}$$

Now suppose we have a group code with j generators v_i^*, $i = 1, \ldots, j$, ($v_1^* = \underline{0}$). We then have 2^j code points and the probability of the source producing a point which cannot be encoded with r or fewer errors is

$$Q_j = 1 - \frac{N_j}{2^n} \tag{4.14}$$

where N_j is the number of points in $S_j = \bigcup_{i=1}^{2^j} S(v_i)$, the v_i are the code points. We can write an upper bound on average distortion for this code as

$$\bar{d}_j \leq \frac{r}{n} + Q_j.$$

Suppose we wish to add another generator v** to the code. We actually double the number of code points because $v^{**} \oplus v_i$, $i = 1, \ldots, 2^j$ are all new code words. The set defined by $\bigcup_{i=1}^{2^j} S(v^{**} \oplus v_i)$ is topologically the same as S_j, i.e. they have the same number of points N_j, and if $v^{**} = \underline{0}$, the sets are identical. We have 2^n possible choices for v^{**} and by the previous Lemma there is a choice for v^{**} such that the number of points N_{j+1} in

$S_{j+1} = \bigcup_{i=1}^{2^{j+1}} S(v_i)$, ($v_i$ include now the new code points due to v^{**}),

is guaranteed to satisfy

$$N_{j+1} \geq 2N_j - N_j^2\, 2^{-n}. \tag{4.15}$$

We may then write Q_{j+1}, the probability of the source producing a point which cannot be encoded with r or fewer errors with the new code of $j+1$ generators,

$$Q_{j+1} \leq 1 - \frac{N_{j+1}}{2^n} = 1 - 2\frac{N_j}{2^n} + \left(\frac{N_j}{2^n}\right)^2 = \left(1 - \frac{N_j}{2^n}\right)^2$$

$$= Q_j^2$$

where we have used Eqs. 4.14 and 4.15.

Since we have defined in Eq. 4.12

$$Q_0 = 1 - \sum_{i=0}^{r} \binom{n}{i} 2^{-n}$$

and now the recursion relation $Q_{j+1} = Q_j^2$, our inductive proof is complete and we may write Q_k for a group code with k generators as

$$Q_k = \left[\, 1 - 2^{-n} \sum_{i=0}^{r} \binom{n}{i} \right]^{2^k},$$

where Q_k is the probability that the source produces a point which cannot be encoded with r or fewer errors.

Notice that we have demonstrated the existence of at least one group code with rate $\frac{k}{n}$ with average distortion satisfying

$$\bar{d}_k \leq \frac{r}{n} + [\,1 - 2^{-n} \sum_{i=0}^{r} \binom{n}{i}\,]^{2^k}$$

which is identical with Eqs. 4.3 and 4.4 with $M = 2^k$. We have shown the existence of group codes which satisfy the same upper bound on average distortion as the ensemble of random block codes.

Q.E.D.

Although Theorem 4.2 is much stronger than Theorem 4.1, the latter presents a construction method which may actually be used on a digital computer to obtain a group code, while Theorem 4.2 would be more difficult to implement in this way.

The algebraic structure of group codes allows a simple encoding procedure. Suppose we have a group code of length n with k generators and we wish to encode a source word u. We would first compute the syndrome or parity check pattern corresponding to u, look up the coset leader s(a binary sequence of length n) corresponding to the computed syndrome, and then form $v = u \oplus s$. The algebraic structure is such that v is a code word and the distortion produced in encoding u as v is given by the weight of s. The details of such a procedure have been described many times[(3), (4), (12), (18)] and we will not discuss this system any further here. We wish only to point out that the number of possible syndromes is 2^{n-k} and so the required storage space for such a system grows exponentially with n.

Every code word of a group code may be expressed as a linear combination of the k generators of the code. Suppose we consider the list of k generators of a code to form a $k \times n$ binary matrix with the generator sequences as rows. We may assume that no generators are zero and we can then put this generator matrix in a standard form by diagonalizing the first k columns. The group code is then actually determined by only $(n-k) \cdot k$ binary digits in the remaining n-k columns of the generator matrix. We wish now to describe a simple scheme to search for an acceptable linear combination of these diagonalized generators to encode a source word. Given a source word u of length n, we form the code word v_o which agrees exactly in its first k digits with u. This is easily done by considering the first k digits of u to specify a linear combination of the k diagonalized generators. A one in the j^{th} position of u indicates that the j^{th} generator is added into the linear combination to form v_o. Now the sequence $u \oplus v_o$ has zeroes in its first k places and all errors between u and v_o occur in the last n-k places. We now try to improve the number of errors between u and v_o by comparing the weights of the sequences $u \oplus v_o$ and $u \oplus v_o \oplus v_1$, where v_1 is the first code generator. If the weight of $u \oplus v_o \oplus v_1$ is less than the weight of $u \oplus v_o$, we define the sequence $S_1 = v_o \oplus v_1$, and otherwise we define $S_1 = v_o$. Next we compare the weights of the sequences $u \oplus S_1$ and $u \oplus S_1 \oplus v_2$. If the weight of $u \oplus S_1 \oplus v_2$ is less than the weight of $u \oplus S_1$, we define $S_2 = S_1 \oplus v_2$, and otherwise we define $S_2 = S_1$. We thus proceed to test all generators in this manner. In general, having tested j generators, we have a linear combination of v_o and the first j generators, S_j, and we compare the weights of the sequences $u \oplus S_j \oplus v_{j+1}$ and $u \oplus S_j$. We then define $S_{j+1} = S_j \oplus v_{j+1}$ if $u \oplus S_j \oplus v_{j+1}$ has smaller weight than $u \oplus S_j$, and otherwise $S_{j+1} = S_j$. After testing all k generators of the code we have the linear combination of generators S_k which we then use as the code word to encode u.

If we consider the encoder to make n-k computations in testing one generator sequence (one for each of the last n-k digits), the encoder then does only $k(n-k) = n^2R(1-R)$

computations to find S_k and a code word for the source sequence. This scheme uses a simple rule to construct a linear combination of generators. Since we can always encode u as v_o, we are sure of having a code word with distortion no more than n-k with u. We then try to improve things by testing each generator to see if together with the tentative code word it will give another code word with even less distortion with u. In adding a generator to S_j, one error is introduced in the first k places, but more than one error in the last n-k places may be removed. We add the new generator to S_j only if it improves the distortion. This is similar to the threshold decoding scheme for channels presented by J. L. Massey[11] and the step-by-step channel decoding scheme discussed by Peterson[12].

It is clear that our source encoding scheme is not the optimum one since all 2^k possible linear combinations of generators are not tested, but this is exactly what we are trying to avoid. If only the order of the generators is changed before diagonalization, the results in general would be different. In fact, having computed S_k, we could start the whole process of testing generators over again using $u \oplus S_k$ instead of $u \oplus v_o$ and the result, in general, would not be S_k again.

In view of the greatly reduced computation for long block codes, let us study in more detail the scheme of testing each generator once to compute S_k. We write the probability distribution for the weight of a binary sequence of length n-k chosen at random with equiprobable zeroes and ones as

$$p_o(w) = 2^{-n+k} \binom{n-k}{w}, \quad w = 0,\ 1,\ \ldots,\ n\text{-}k.$$

The probability distribution for the weight of the sequence $u \oplus v_o$ is given by $p_o(w)$. If the last n-k digits of v_1 are chosen at random with independent and equiprobable letters, the weight of the last n-k digits of $u \oplus v_o \oplus v_1$, given $u \oplus v_o$, also has the probability distribution $p_o(w)$.

Suppose we choose a group code at random by choosing the last n-k digits of all k diagonalized generators equiprobably and independently at random. Suppose also that for a particular u selected at random by the source, we have constructed v_o and have formed S_j by testing the first j generators. Assume that we know completely $P_j(c,w)$, the probability that there are c ones in the first k places of S_j and w ones in the last n-k places. The c ones in the first k places would be due to c generators already added into S_j. Then since the last n-k digits of v_{j+1} are chosen at random, and the j^{th} column has never been changed before by any of the first j generators, we may write for $P_{j+1}(c,w)$

$$P_{j+1}(c,w) = \text{Prob. } [S_j \text{ has c ones in the first k digits and w ones in the last n-k digits } \underline{\text{and}}\ v_{j+1} \text{ has w-1 ones or more in the last n-k places.}]$$

$$+ \text{ Prob } [S_j \text{ has c-1 ones in the first k digits and w + 2 ones or more in the last n-k digits } \underline{\text{and}}\ v_{j+1} \text{ has w ones in the last n-k digits})$$

$$= P_j(w,c) \sum_{i=w-1}^{n-k} p_o(i) + p_o(w) \sum_{i=w+2}^{n-k} P_j(i,c-1). \qquad (4.16)$$

The first term of $P_{j+1}(c,w)$ corresponds to the event that the randomly selected v_{j+1} does not improve the distortion between the tentative code word at step j and the source word. The distortion at this step is c + w and even if v_{j+1} resulted in w-1 ones in the last n-k digits of S_{j+1}, the change in the $(j+1)^{th}$ digit results in $w-1+c+1 = w+c$ errors again. The second term of $P_{j+1}(c,w)$ corresponds to the event that v_{j+1} does improve the distortion between the tentative code word and the source word. Since each generator is selected independently, we can write the joint probabilities of $P_{j+1}(w,c)$ as factors in Eq. 4.16. But if we were to start testing generators again from the top of the list with S_k, the tests would not be independent any longer and we could not write Eq. 4.16 so simply. This is the reason for treating only this scheme which tests each randomly selected generator once and only once.

The average distortion in encoding u with a randomly selected group code and the above step-by-step encoding procedure is given by

$$\bar{d} = \sum_{c=0}^{k} \sum_{w=0}^{n-k} (c+w)\, P_k(c,w) \tag{4.17}$$

and so this is the average distortion over the ensemble of randomly selected group codes together with the step-by-step encoding procedure. The recursion relation of Eq. 4.16 has not been solved explicitly for $P_k(c,w)$, but it is easily programmed on a digital computer. A computer program was written for the IBM 7090 digital computer to calculate $P_k(c,w)$ and $\bar{d}$ for code lengths up to 100 and many rates between zero and one. The results are reasonably good in that this encoding method gives rate-distortion performance comparable to but not as good as the rate-distortion function $R(d) = 1-H(d)$. For a code length of 20, we have plotted code rate vs average distortion for step-by-step encoding in Fig. 4.2. Longer code lengths up to about 60 gave essentially this same curve and even longer codes gave poorer performance. At rate $R = \frac{1}{2}$, for example, we see the lowest possible average distortion is about 0.11 and step-by-step encoding gives an average distortion of 0.185. The straight line in Figure 4.2 represents the performance we could expect if we encoded u as the v_o calculated from u, since

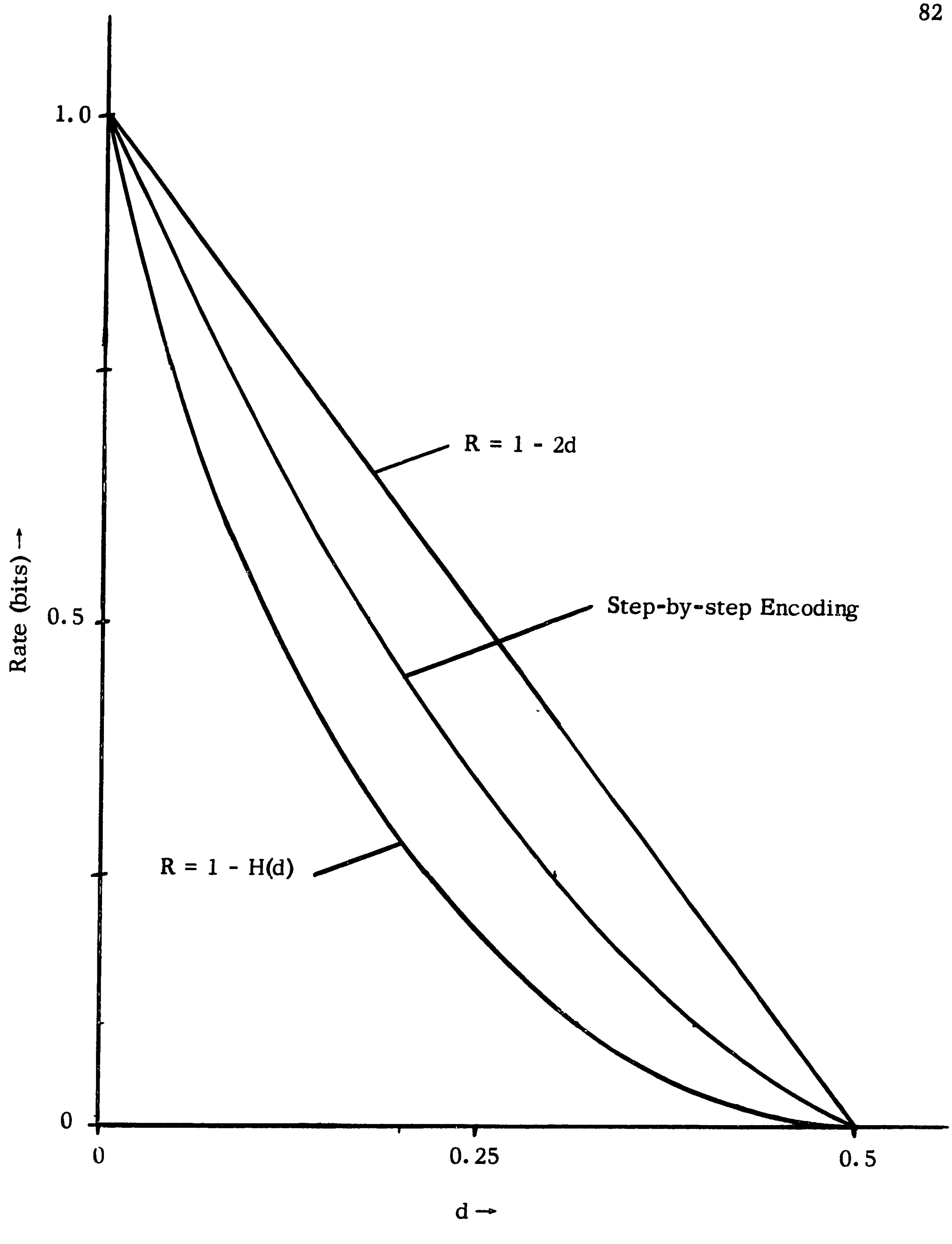


Figure 4.2 Comparison of R(d) and the upper bound on average distortion for step-by-step encoding with codes of length 20.

there could only be errors in the last n-k digits of $u \oplus v_o$ and each of these n-k digits has probability $\frac{1}{2}$ of being in error. Hence, encoding u as v_o results in $\bar{d} = \frac{1}{n}\frac{n-k}{2} = \frac{1}{2}(1-R)$. For rate $R = \frac{1}{2}$ and $n = 20$, we also see that we do only $n^2R(1-R) = 100$ computations to encode u or only 5 computations per encoded digit.

In the following section we adopt another viewpoint that has been applied successfully to channel decoding with limited equipment complexity.

4.3 Sequential Encoding with Random Tree Codes

We now discuss a sequential encoding system for the binary symmetric source and a Hamming distance distortion measure using randomly chosen tree codes. Consider an infinite length binary tree code with two binary digits per branch and two branches emerging from each node (See Figure 4.3a). We wish to represent the source output by the path in the tree code which gives the least distortion with the source output. The distortion between a source sequence of length n and a path of length n in a tree code is just the Hamming distance between the source sequence and the path considered as a sequence of length n.

It takes half the number of binary digits in a source sequence to specify a path in the tree code of the same length as the source sequence since only the binary choice of branches at each node must be specified. Therefore, for every two source letters, the encoder will only put out one binary digit. In a tree of length n there are only $M=2^{n/2}$ paths and the rate of the tree code is then $\frac{1}{n}\log_2 2^{n/2} = \frac{1}{2}$. The binary digits which specify a path in the tree code can be transmitted through a channel and a decoder then produces for the information sink the corresponding path as an approximation to the actual source output.

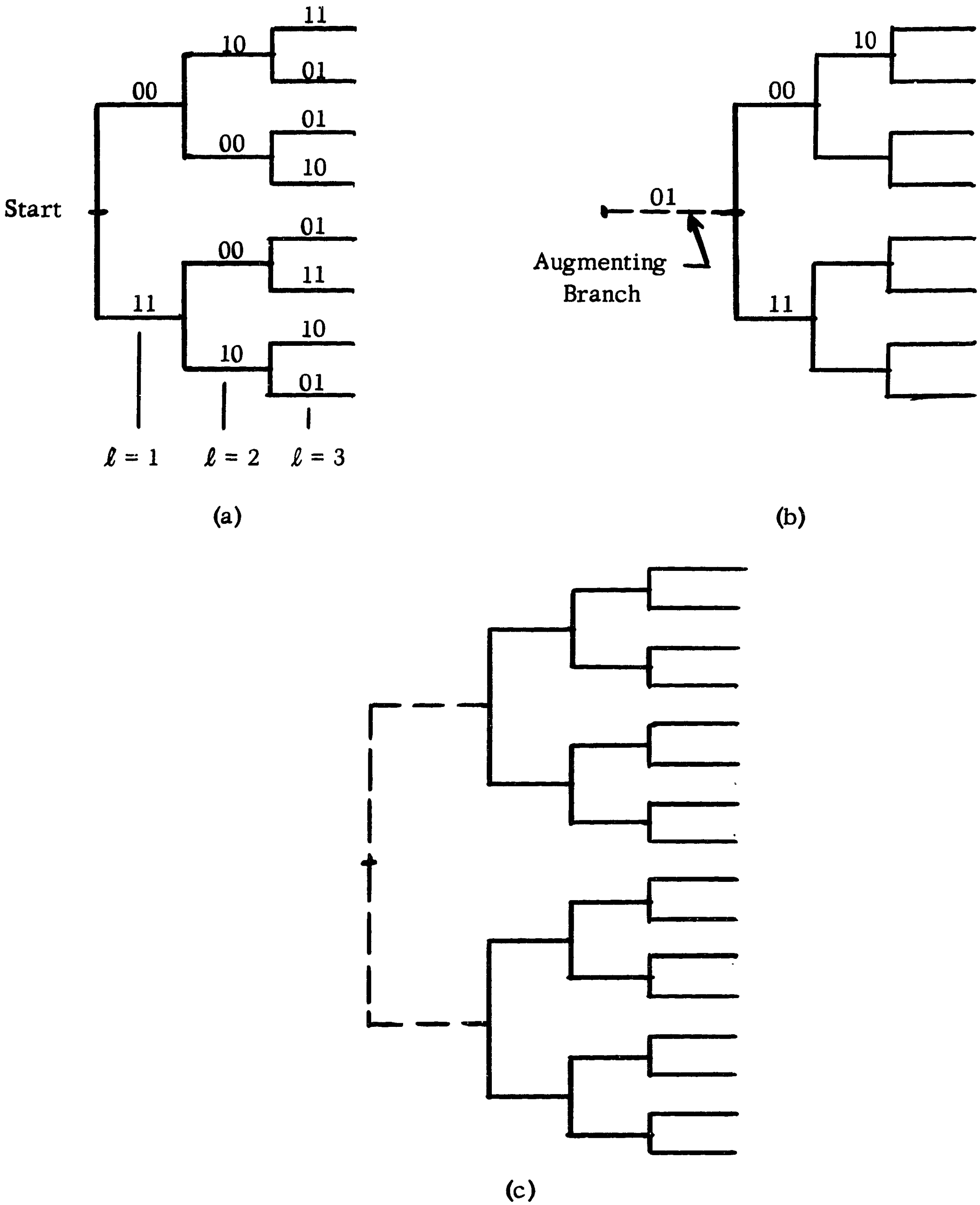


Figure 4.3 (a) Binary tree code of length 3.
(b) Augmented binary tree of length 4.
(c) Tree code of length 4 formed from two independent augmented trees of length 4.

We wish to have the encoder find a path in the tree code which has no more than a certain amount, say d^*, average distortion per letter with the source sequence. Finding a path in the tree code which gives d^* or less average distortion per letter is somewhat analogous to finding the most likely transmitted path of a binary tree code at the output of a binary symmetric channel[18]. The source encoder begins by calculating the distortion between the first two digits of the source sequence and the two branches emerging from the first node of the tree code. If neither of these branches gives distortion $2d^*$ or less (average distortion per letter d^* or less), the encoder extends its computation of distortion to all paths of length four emerging from the first node of the tree code, and checks to see if any of these paths gives distortion $4d^*$ or less with the first four digits of the source sequence. The encoder proceeds in this manner until it finds some path with ℓ links (path of length ℓ) which gives $2\ell d^*$ or less distortion with the first 2ℓ digits of the source sequence. The encoder then accepts this path and specifies this path by putting out ℓ binary digits. It then attempts to encode the source sequence, beginning at the $(2\ell+1)$th binary digit by using the tree emerging from the node at the end of the accepted path. Once a path of some length is accepted, the encoder is faced with an entirely new encoding problem and begins its searching anew.

The encoding system also uses a set of reject thresholds B_ℓ, $\ell = 1, 2$, etc. Whenever any path of length ℓ gives a distortion of B_ℓ or more the encoder no longer considers that path as a possible representative for the source sequence, and no further distortion calculations are done on the paths emerging from the last node of a rejected path. The encoder also gives up its search for an acceptable path when it has progressed down the tree code far enough to check paths of length ℓ_t and if no acceptable path is found at this length, a standard path of length ℓ_t is chosen to represent the source sequence. (For instance, the standard path may be the path corresponding to the choice of the upper branch at each node of the tree.) The encoder begins searching again in the tree emerging from the last node of the standard path. It may happen that the encoder rejects all paths at some length $\ell < \ell_t$, whereupon the portion of the standard path of length ℓ is used as a representation of the source output and the encoding operation begins again at the node at the end of the accepted part of the standard path.

If no path of length ℓ is accepted, the encoder extends the distortion calculation of all paths not rejected to paths of length $\ell + 1$, and again checks the accept threshold $2(\ell + 1)d^*$ and reject threshold $B_{\ell+1}$. We will define a single computation of the encoder as the calculation of the distortion between one of the braches of the code emerging from a certain path of length ℓ and the corresponding two digits of the source sequence, and the addition of this distortion to the distortion of the path from which the branch emerges.

We will now consider this encoding system operation with the ensemble of random tree codes in which all digits in a tree are chosen independently with zeroes and ones equiprobable. We will upper bound N, the average number of computations to find an acceptable path and also upper bound P_F, the probability of failing to find an acceptable path.

Failure occurs in two ways; all paths in the tree may be rejected at some length $\ell < \ell_t$, and an acceptable path may not be found at any length $\ell \leq \ell_t$. Let C_ℓ denote the event of all paths being rejected at length ℓ, and E_ℓ the event of no acceptable path at length ℓ. For the ensemble of random tree codes we have

$$P_F = \text{Prob}\,[C_1 \text{ or } C_2 \text{ or } C_3 \text{ or } \ldots \text{ or } C_{\ell_t - 1} \; \underline{\text{or}} \text{ no acceptable paths at any length } \ell \leq \ell_t]$$

$$\leq P_r\,[C_1] + P_r[C_2] + \ldots + P_r[C_{\ell_t - 1}] + P_r[E_1 \text{ and } E_2 \text{ and } \ldots \text{ and } E_{\ell_t}]$$

$$\leq P_r\,[C_1] + P_r[C_2] + \ldots + P_r[C_{\ell_t - 1}] + P_r[E_{\ell_t}]. \tag{4.18}$$

The average number of computations at length ℓ is upper bounded by two times the average number of paths not rejected at length $\ell - 1$, which in turn is upper bounded by

$$2 \cdot 2^{\ell - 1} (1 - P_r[C_{\ell - 1}])$$

which assumes no rejection prior to length $\ell - 1$. We have

$$N \leq 2 + 2 \sum_{\ell = 2}^{\ell_t} 2^{\ell - 1} \; (1 - P_r[C_{\ell - 1}])$$

where the first two computations at the start of the encoding process are expressed separately and not included in the summation on ℓ.

We now note from symmetry that the probability of finding a path in a randomly chosen tree code "close" (in the distortion sense) to the zero sequence is the same as the probability of finding a path close to any particular sequence. Since all source sequences are equiprobable we have

$$P_r[E_{\ell_t}] = \text{probability that the minimum weight path in the tree has weight} > 2\ell_t d^*.$$

Notice also that $P_r[C_\ell]$ is upper bounded by the probability that all paths at length ℓ are rejected assuming no previous rejection. This is the same as the probability that the minimum weight path of length ℓ of a randomly chosen tree code has weight B_ℓ or greater.

Our analysis of this source encoding system depends now only on the distribution function which we shall call $P_\ell(w)$, which is the probability that the minimum weight path of a randomly chosen tree code of length ℓ has weight w or greater.

If we picked 2^ℓ binary sequences of independent random digits, each of length 2ℓ, we could write the probability $\hat{P}_\ell(w)$ that the minimum weight sequence has weight w or greater as

$$\hat{P}_\ell(w) = [1 - P_{o\ell}(w)]^{2^\ell}$$

where $P_{o\ell}(w)$ is the probability of choosing a single sequence of length 2ℓ which has weight less than w. $P_{o\ell}(w)$ is a multinomial distribution function. $\hat{P}_\ell(w)$ is the distribution function for the minimum weight code word of a randomly chosen block code of length 2ℓ and 2^ℓ code words.

This approach cannot be used for $P_\ell(w)$ because the paths of a tree are not all independent of one another. An alternate approach suggested by C. E. Shannon is to develop a set of recursion relations for $P_\ell(w)$. Suppose we know $P_{\ell-1}(w)$ completely. Let us form an augmented tree of length ℓ by adding a randomly chosen branch to the beginning or first node of the tree of length $\ell-1$. (See Figure 4.3b) We can derive from $P_{\ell-1}(w)$ the distribution function $Q_\ell(w)$, the probability that the minimum weight path in the augmented tree of length ℓ has weight w or more. In fact,

$$Q_\ell(w) = p_o P_{\ell-1}(w) + p_1 P_{\ell-1}(w-1) + p_2 P_{\ell-1}(w-2)$$

where p_i is the probability of the extra branch having weight i. In our case of equiprobable binary digits, $p_o = 1/4$, $p_1 = 1/2$, $p_2 = 1/4$. The term $p_o P_{\ell-1}(w)$ is the probability of a tree of length $\ell-1$ having its minimum weight path with weight w or more, and adding an augmenting branch of weight zero. The explanation of the other terms follows similarly. Knowing $Q_\ell(w)$, we choose independently at random another augmented tree of length ℓ, and join the ends of the augmenting branches to form a standard tree of length ℓ. (See Figure 4.3c) The probability that the new tree thus formed has its minimum weight path of weight w or more is simply

$$P_\ell(w) = [Q_\ell(w)]^2$$

because of the independence of the augmented trees.

These recursion relations have not been solved explicitly for $P_\ell(w)$, but it is reasonable to expect that there is some limiting form for $P_\ell(w)$ as $\ell \to \infty$. Since these recursion relations are easily programmed on a digital computer, the actual distributions $P_\ell(w)$, $\ell \leq 200$, were thus calculated and studied. It turns out that the limiting form for $P_\ell(w)$, which emerges distinctly at about $\ell = 25$, is such that the distribution function does not change shape with increasing ℓ but merely changes its position along the w-axis. The position of the $P_\ell(w) = \frac{1}{2}$ point is located approximately at $.11 \times 2\ell$. This is in contrast to the multinomial distribution function $P_{o\ell}(w)$ in which the position of the $P_{o\ell}(w) = \frac{1}{2}$ point is proportional to ℓ but the shape of the distribution spreads out as $\ell^{1/2}$.

The distribution function $\hat{P}_\ell(w)$ of the minimum weight binary sequence of a length 2ℓ block code behaves very much like $P_\ell(w)$ with respect to its limiting behavior $\ell \to \infty$. The limiting form of $\hat{P}_\ell(w)$ also appeared to keep its shape constant and change its position in proportion to ℓ. The limiting form of $\hat{P}_\ell(w)$ was slightly less spread than the limiting form of $P_\ell(w)$. An upper bound on $\hat{P}_\ell(w)$ shows the $\hat{P}_\ell(w) = \frac{1}{2}$ points approximately take on the positions $2\ell d$, where d is defined by the solution of

$$R(d) = \frac{1}{2} \tag{4.19}$$

and R(d) is defined in Eq. 4.10. The approximate valueof d is 0.11. In Figure 4.4 we plot the envelope of the probability distributions $p_\ell(w)$ and $\hat{p}_\ell(w)$ which correspond to the distribution functions $P_\ell(w)$ and $\hat{P}_\ell(w)$ respectively.

A computer program for the IBM 7090 was written to compute $P_\ell(w)$ and also to compute the bounds on P_F and N for $\ell_t \leq 100$ for the sequential source encoding system described here. A single number, d*, specifies completely the set of accept thresholds. The set of reject thresholds B_ℓ were programmed to take the form of a linear function of ℓ with one break point. (See Figure 4.5.)

After some experimenting, a good reject threshold could be found which could be set high enough to give a low probability of failure due to rejection of all paths at some length (dependent upon $P_r[C_\ell]$), while still keeping the average computation down (dependent upon $1 - P_r[C_\ell]$). Some performance figures for the encoding system are given in Table 4.1 for $\ell_t = 100$. In this table, N* is the average number of computations to find an acceptable path when no reject threshold is employed by the encoder.

TABLE 4.1 ($\ell_t = 100$)

d*	P_F	N	N*
0.14	0.359	1.07×10^5	4×10^{28}
0.15	0.28×10^{-2}	1.44×10^5	2×10^{25}
0.16	0.52×10^{-4}	1.37×10^5	2×10^{19}
0.17	0.61×10^{-6}	1.05×10^5	3×10^{14}

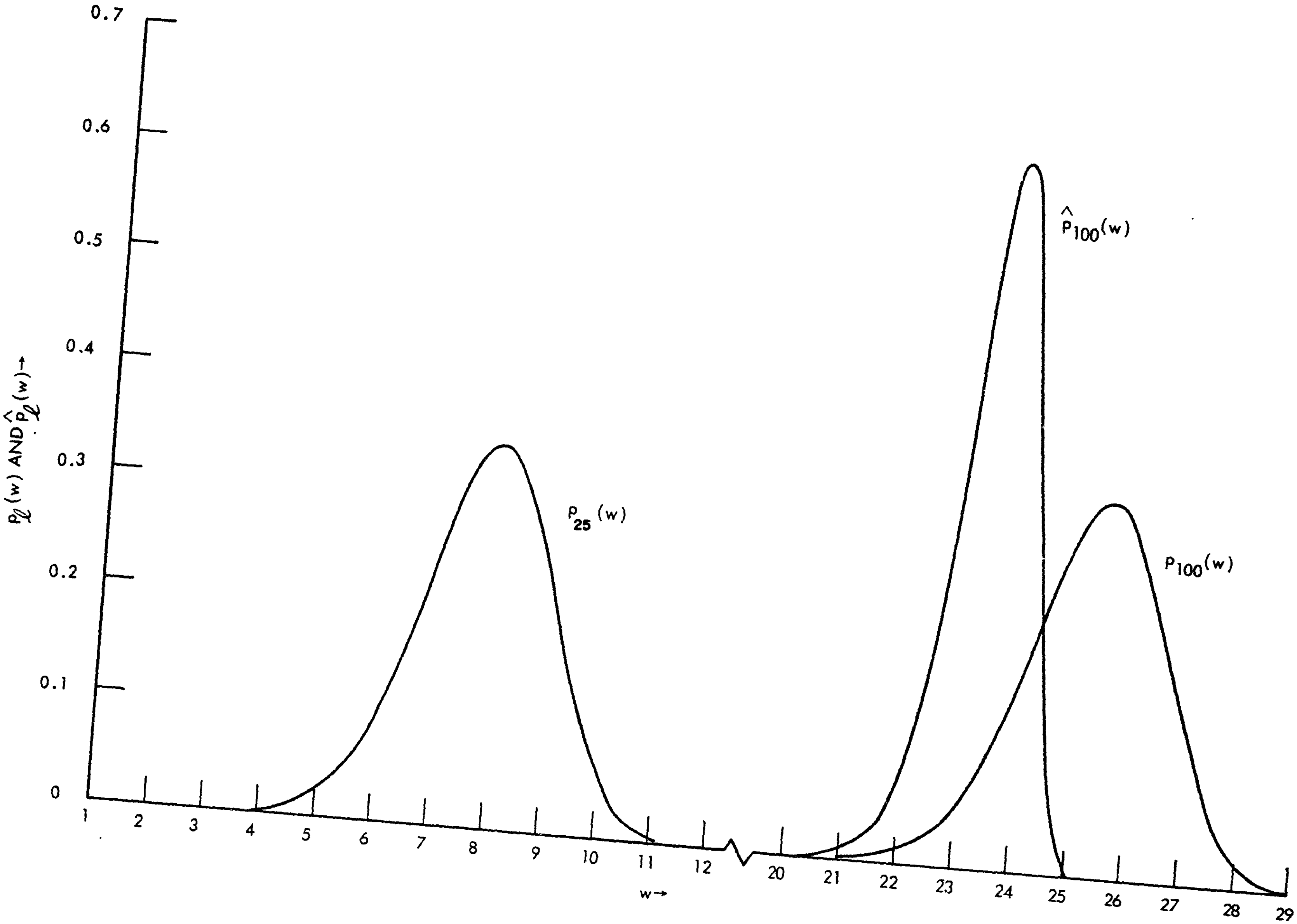


Figure 4.4 The function $p_\ell(w)$ for a tree code, $\ell = 25, 100$. Also included for comparison is the function $\hat{p}_\ell(w)$ for a block code, $\ell = 100$.

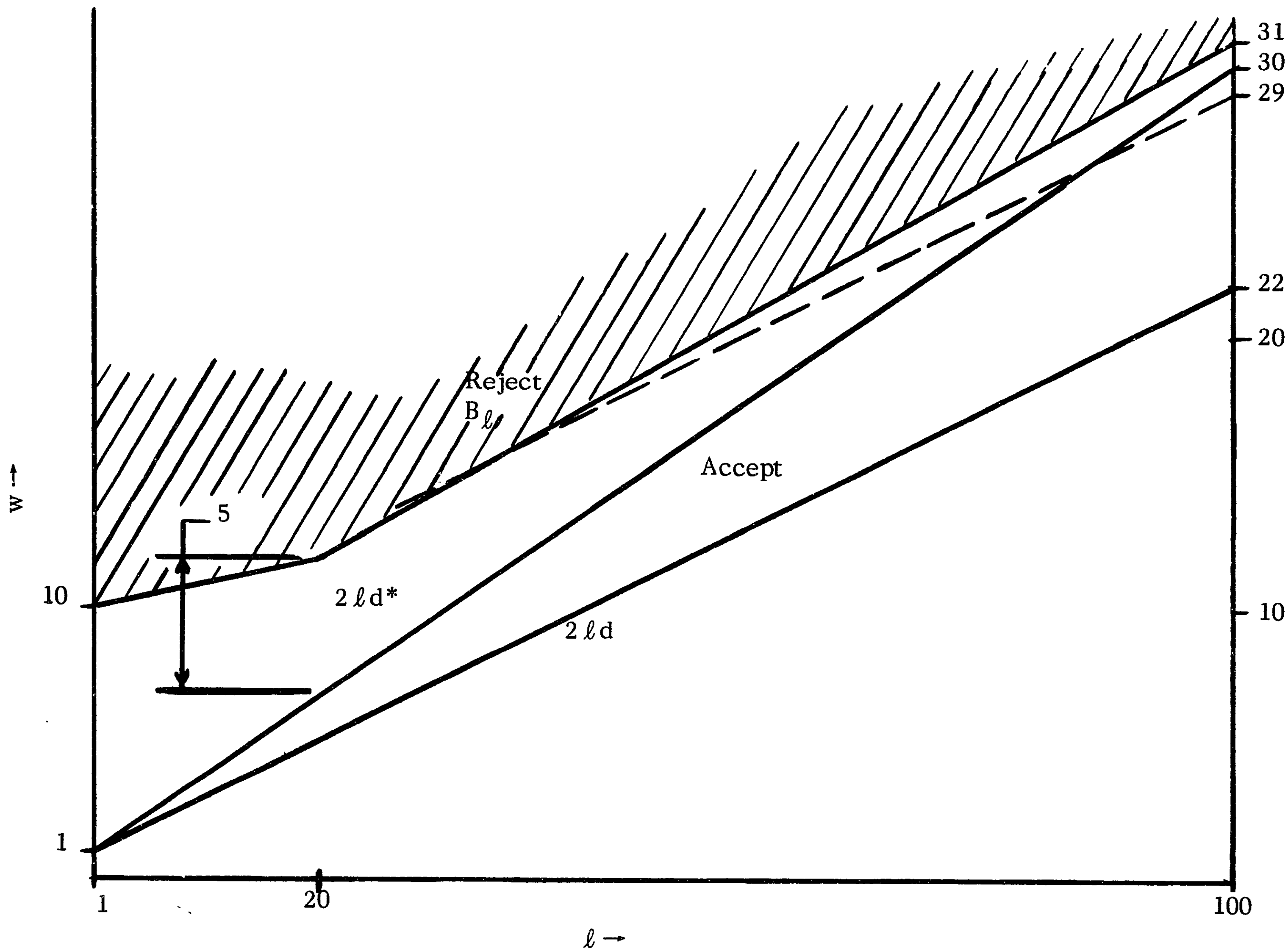


Figure 4.5 Accept thresholds $2\ell d^*$ and reject thresholds B_ℓ that give good results. The broken line is the locus of $P_\ell(w) \leq 10^{-4}$.

The bounds on both N and N* actually converge to a constant number independent of ℓ_t for large enough ℓ_t, which indicates that there is a fixed amount of work on the average to find an acceptable path with either scheme. Table 1 indicates that for $d^* = 0.14$, $\ell_t = 100$ is not long enough to bring down the probability of failure P_F. The reject thresholds were chosen so that almost all of the contribution to P_F in Eq. 4.16 came from $P_r[E_{\ell_t}]$.

According to the rate-distortion function, the best possible encoding scheme with rate $R = \frac{1}{2}$ could not give average distortion per letter less than 0.11, as we saw from Eq. 4.19. While P_F and N converged well for $d^* = 0.15$ and $\ell_t = 100$, the probability of failure converges much more slowly for $d^* = 0.14$, which is significantly closer to the absolute limit for average distortion of 0.11.

It is interesting to note that since $P_\ell(w)$ only moves along the w-axis with increasing n without changing shape, we can extrapolate safely the dashed line of Fig. 4.5 to obtain the locus of the 10^{-4} points of $P_\ell(w)$ for large ℓ. Since we know the locus of the $P_\ell(w) = \frac{1}{2}$ points as approximately $0.11 \times 2\ell$, we can also write the locus of the $P_\ell(w) = 10^{-4}$ points as $K_o + 0.11 \times 2\ell$, where K_o is some fixed constant. We can then estimate the code length ℓ_t at which P_F converges for any accept threshold $2\ell d^*$ by noting the value of ℓ_t where $2\ell_t d^* = K_o + 0.11 \times 2\ell_t$. As long as we choose the reject thresholds so that P_F given in Eq. 4.18 is due mostly to $P_r[E_{\ell_t}]$, the suggested extrapolation of the $P_\ell(w) = 10^{-4}$ line gives a fair estimate of the code length required to have P_F converge to an acceptable level. Based on such an extrapolation of Figure 4.5, for $d^* = 0.14$ we find that P_F converges to a satisfactory level for $\ell_t \approx 180$.

In this connection we also notice that for any $\epsilon > 0$ we can always find an ℓ_t large enough so that P_F will converge to a low value for an accept threshold $2\ell(0.11 + \epsilon)$. It would be of great interest then to actually calculate the exact rate at which the limiting form of $P_\ell(w)$ moves along the w-axis with increasing ℓ. Knowing the exact rate of propagation of $P_\ell(w)$ with ℓ, we could determine the limiting distortion d* for which accept the threshold $2\ell d^*$ would allow P_F to converge to zero. From the experimental work done on the digital computer, it seems that P_F could be made to converge for large enough ℓ_t for any $d^* = d + \epsilon$, where d satisfies Eq. 4.19. This carries the implication that random tree codes may give the ideal rate-distortion performance with a complexity which is strictly bounded independently of ℓ_t. This, however, remains to be strictly proved, although this special case provides a reasonable basis for speculation.

The techniques used in the above analysis could obviously be applied in the analysis of non-binary systems, more general randomly chosen tree codes, and different code rates. The sequential encoding procedure could also be modified to use a set of rejection thresholds $B_{1\ell}$, $B_{2\ell}$, etc. such as J. M. Wozencraft and B. Reiffen describe in channel decoding[18]. The source encoder would first attempt to encode using the reject thresholds $B_{1\ell}$, and if it failed to find an acceptable path it would then go through the whole procedure again using the set $B_{2\ell}$, etc. This system gives a slight reduction in the bound on N.

It is obvious that it is as difficult to instrument a random tree code as it is a block code, and in the final results the average number of computations is very high even with this elaborate sequential scheme. Our purpose in this chapter was not to produce an immediately workable scheme, but to explore the possible methods of analyzing schemes to evaluate both average distortion and average number of computations per encoded letter. We have presented several useful viewpoints in approaching such problems and we also have produced some interesting results.

As P. Elias comments[3] (page 40) with regard to channel codes, it would be completely consistent with the results of Chapter 2 if no code with any simplicity or algebraic symmetry properties were a good source code. This first investigation shows that this is fortunately not the case. We may speculate a bit more by adding that perhaps the basic difficulties pointed out in this chapter are entirely mathematical difficulties and we may find in future work that there are quite economical schemes which perform as well as the optimum code.

CHAPTER V

MISCELLANY

5.1 Maximum Allowable Letter Distortion as a Fidelity Criterion

The essence of the source encoding problem as we have discussed it in the previous chapters has been the coding of the source output to minimize the information capacity required to transmit a facsimile of the source output to the information user or sink. We were given a distortion measure with which to evaluate the facsimile of the source output that is presented to the sink and we arbitrarily stated our fidelity criterion, or the tolerable performance level, in terms of average distortion per letter. We then found the rate-distortion function as the fundamental restriction on code rates under the constraint of the fidelity criterion, namely, that the average distortion per letter must be kept at or below some specified level. Another type of fidelity criterion or constraint on tolerable distortion would naturally lead to a completely different rate-distortion function.

In this section we will study a fidelity criterion other than average distortion per letter. We will require that each individual source letter be transmitted to the sink with not more than a certain amount of distortion, say D. This is a fidelity criterion encountered quite commonly in practice. For instance, the specifications on an analog-to-digital converter (a quantizer) for an analog signal source often state that the quantization error should not exceed a certain amount, say 0.1 volts.

All the information essential to the coding problem for the letter distortion as fidelity criterion is contained in a line diagram such as Figure 5.1, in which a source letter and an output letter are connected by a line if the distortion between these letters is D or less. Letters connected by such lines will be called equivalent , and if two or more source letters are equivalent to a single output letter, we refer to these source letters as being adjacent. In the example of Figure 5.1, we see that letters a and B are equivalent and a, b, and c are adjacent since they are all equivalent to letter B. A source word is equivalent to an output word only if each concurrent letter pair of the words is equivalent.

A block code will not be acceptable according to the maximum letter distortion criterion unless the probability that any letter is transmitted to the sink with more than D distortion is precisely zero. The line diagram of the mapping which describes an encoder must be such that it can be superimposed upon the line diagram of equivalent source and output words without adding any new lines to the latter diagram. We again define the rate of a block code as $R = \frac{1}{n} \log M$, but it is interesting to note that this quantity does not now have the significance of the usual measure of information rate. If we suppose that the source produces one letter per second, the rate of the block code is R units per second. We cannot now use any transmission channel with ordinary information capacity C nats per second, where $C > R$, because such a channel is only guaranteed to transmit any one of M integers to the decoder with probability of error approaching zero, in general. The fidelity criterion demands that each of the integers representing a code word must be transmitted with probability of error precisely zero. From this, we conclude that we must have a transmission channel with zero error

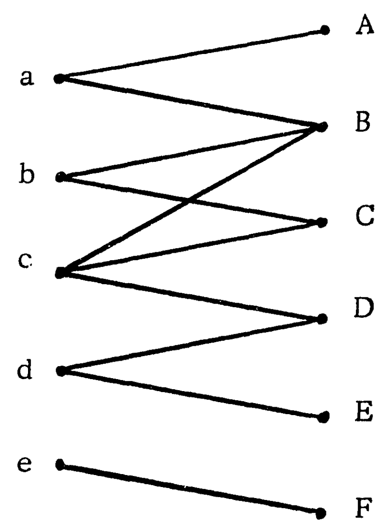


Figure 5. 1 Line diagram, showing equivalent source and output letters.

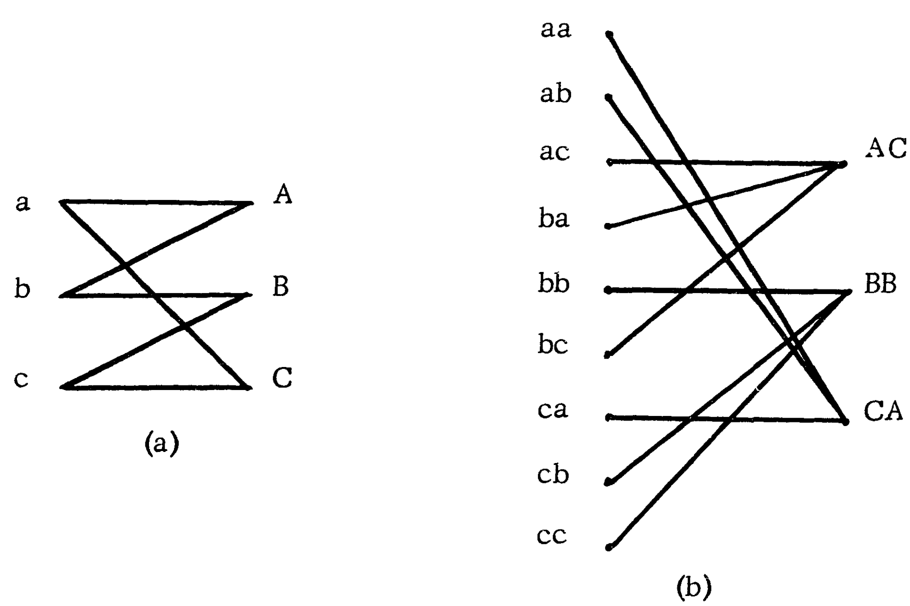


Figure 5.2 (a) Line diagram of a 3-letter source.
(b) Acceptable encoder of length 2 with 3 code words.

capacity C_o units per second[13], where $C_o > R$, in order to satisfy the fidelity criterion. In Figure 5.1 we see that an acceptable block code of length 1 must consist of at least three output letters, such as B, D, and F, and the rate of this code is $\log_3$.

From the example of Figure 5.1 we see that the source probabilities do not enter into the coding problem for this fidelity criterion on letter distortion. In fact, all sources with distortion measures which lead to the same line diagram can be encoded in exactly the same way. We define D_{min} as the smallest value of D for which each source letter has at least one equivalent output letter. For $D < D_{min}$ we cannot satisfy the fidelity criterion with any code because there exists at least one source letter which is encoded with more than D distortion. The probability, P_F, of the occurrence of at least one such letter in a sequence of length n which cannot be encoded properly by any block code of length n, for $D < D_{min}$, is then bounded away from zero by $P_F \geq 1-(1-p_{min})^n$, where $p_{min} > 0$ is the minimum source probability.

For $D \geq D_{min}$, it is possible to encode the source output with letter distortion D or less using codes with bounded rate. In view of the interpretation of the code rate, we define the greatest lower bound of all rates that can be achieved with letter distortion D or less to be the zero error rate of the source and will be denoted by $\hat{R}(D)$. If we let $M_D(n)$ be the smallest number of code words in a code of length n giving letter distortion D or less, then

$$\hat{R}(D) = \text{g. l. b.} \ \frac{1}{n} \log M_D(n) \tag{5.1}$$

when n varies over all positive integers.

A simple example shows that, in general, we do not have $\hat{R}(D) = \log M_D(1)$, where $M_D(1)$ is the number of output letters in the smallest set of output letters for which each source letter has an equivalent letter. In Figure 5.2a we have a 3-letter source and a line diagram for a certain D. $M_D(1) = 2$ and so $\log M_D(1) = \log 2$. We show in Figure 5.2b that three output letter pairs form an acceptable code, so $M_D(2) = 3$ and the rate of this code is $\frac{1}{2}\log 3 = \log\sqrt{3} < \log 2$. Therefore $\hat{R}(D)$ for this line diagram is at most $\log\sqrt{3}$.

If we have a line diagram for a particular value of D for a certain distortion measure, we may be able to increase D to $D+\epsilon$, $\epsilon > 0$ and not change the line diagram, implying $\hat{R}(D) = \hat{R}(D+\epsilon)$ in this case. We may raise D to some value D_1 when the line diagram suddenly changes, i.e. new lines appear with the original line diagram, implying a relaxation of coding requirements and so $\hat{R}(D) > \hat{R}(D_1)$. From this we can see that $\hat{R}(D)$ for $D \geq D_{min}$ is a positive, decreasing staircase function of D. If we define D_{max} as the smallest value of D for which a single output letter is equivalent to the entire source alphabet, we see that $\hat{R}(D) = 0$ for $D \geq D_{max}$.

Since the important information in encoding for a fixed letter distortion is contained in a line diagram, let us put this information in the form of an "equivalence" matrix where

$$E_{ij}(D) = \begin{cases} 1, & \text{if source letter } i \text{ is "equivalent" to output letter } j \\ 0, & \text{otherwise} \end{cases}$$

when the tolerable letter distortion is D. Every line of a line diagram will have a corresponding 1 in the equivalence matrix.

Theorem 5.1 $\hat{R}(D)$ for a discrete source with a distortion measure is bounded from below by

$$\hat{R}(D) \geq \max_{P_i} R(0), \; P_i \geq 0, \; \sum_i P_i = 1,$$

where R(d) is the rate-distortion function of the discrete source with letter probabilities P_i and distortion measure d_{ij} in which

$$d_{ij} = \begin{cases} 1, & \text{if } E_{ij}(D) = 0 \\ 0, & \text{if } E_{ij}(D) = 1 \end{cases}$$ ••

Proof From Chapter 2 we know that there exist no block codes with $\frac{1}{n} \log M$ less than R(0) for which the probability of non-zero distortion is zero. The probability of non-zero distortion with the source P_i and distortion measure d_{ij} can be interpreted as the probability of a source letter being encoded as a non-equivalent output letter, in the terminology of the letter distortion criterion.

Q.E.D.

We have shown the existence of block codes for which the probability P_F of encoding a source letter with more than D distortion is approaching zero, a relaxation of our requirement that P_F be precisely zero. Therefore we require block codes with rate at least as great as R(0) in order to guarantee P_F precisely zero.

Theorem 5.2 $\hat{R}(D)$ for a discrete source is bounded from above by

$$\hat{R}(D) \leq -\log \max_{P_i, Q_j} \sum_{i,j} P_i E_{ij}(D) Q_j$$

where $P_i \geq 0, \sum_i P_i = 1, Q_j \geq 0, \sum_j Q_j = 1.$ ••

Proof A random coding argument will be used in this proof. Consider the ensemble of block codes with M code words of length n, each letter of each code word chosen independently with letter probabilities Q_j. If source words are now chosen at random by selecting letters of each word independently according to the probabilities P_i, we can find the probability over the ensemble of codes of finding a source word which cannot be encoded with all letter distortions D or less.

Suppose we pick a source word and a block code at random. The probability that the first letter of the first code word is equivalent to the first letter of the source word is

$$\sum_{i,j} P_i E_{ij}(D) Q_j$$

since this sum includes probabilities of pairs of letters for which $E_{ij}(D) = 1$, i.e. letters which are equivalent. The probability that the first code word is equivalent to the source word is

$$\left(\sum_{i,j} P_i E_{ij}(D) Q_j\right)^n$$

and the probability that the first code word is not equivalent to the source word is therefore

$$1 - \left(\sum_{i,j} P_i E_{ij}(D) Q_j\right)^n.$$

Since code words are selected independently, we may write the probability P_F that no code word is equivalent to the source word as

$$P_F = \left[1 - \left(\sum_{i,j} P_i E_{ij}(D) Q_j\right)^n\right]^M . \tag{5.2}$$

Denote by Λ the quantity $\sum_{i,j} P_i E_{ij}(D) Q_j$. Since for $D \geq D_{min}$, $0 < \Lambda < 1$, and we may bound P_F using an inequality from Appendix A,

$$P_F = (1 - \Lambda^n)^M < e^{-M\Lambda^n} . \tag{5.3}$$

From the above discussion we see that over the ensemble of random block codes the average probability of choosing a source word which cannot be properly encoded with a random block code is just P_F. Therefore, there must exist at least one block code for which the actual probability of choosing a source word which cannot be properly encoded is as small as P_F. If we define

$$p_{min} = \min_i P_i > 0$$

then there are at most P_F / p^n_{min} source words of length n which cannot be properly encoded. We need only add P_F/p^n_{min} code words to our block code in order to encode every source word properly. Our augmented block code has M' code words, where

$$M' = M + P_F / p^n_{min} = M + p^{-n}_{min} e^{-M\Lambda^n} . \tag{5.4}$$

Suppose we choose M so that

$$M = \frac{n^2}{\Lambda^n} \text{ or } M\Lambda^n = n^2. \tag{5.5}$$

Then Eq. 5.4 becomes

$$M' = \frac{1}{\Lambda^n} \left(n^2 + \Lambda^n \; e^{-M\Lambda^n - n \log p_{min}}\right)$$

$$= \frac{1}{\Lambda^n} \left(n^2 + e^{-n^2 + n \log \Lambda - n \log p_{min}}\right) . \tag{5.6}$$

We conclude that the actual code rate necessary to have all source words properly encoded is

$$R = \frac{1}{n} \log M' = - \log \Lambda + \frac{1}{n} \log \quad n^2 + e^{-n^2 + n \log \Lambda / p_{min}} . \tag{5.7}$$

Λ is a constant independent of n. We see that as $n \to \infty$, the exponent in the second term of Eq. 5.7 is essentially $-n^2$ which causes the second term to approach $\frac{1}{n} \log n^2$. The result is

$$\lim_{n \to \infty} R = -\log \Lambda \,. \tag{5.8}$$

There exist block codes with rate $-\log \Lambda$ which properly encode the source. Therefore $\hat{R}(D) \leq -\log \Lambda$ and we may now tighten this bound by maximizing Λ with respect to the arbitrary probability distributions P_i and Q_j.

Q.E.D.

The bounds on $\hat{R}(D)$ are dependent only upon the matrix $E_{ij}(D)$ which is a function of D. We see then that the upper and lower bounds on $\hat{R}(D)$ are decreasing staircase functions with location of the steps along the D-axis coinciding with the steps of $\hat{R}(D)$. However, for sources with small alphabets, it will probably be easier to obtain $\hat{R}(D)$ by construction of codes rather than by using the above bounds.

If we have a complicated source which can be decomposed into the product of two sources, we may relate the $\hat{R}(D)$ function for the product source to the individual sources. Let us first define a product source as one which produces a pair of letters (x_1, x_2) at a time. Supppse we have a sum distortion measure such that the distortion between (x_1, x_2) and (y_1, y_2) is given by $d_1(x_1 y_1) + d_2(x_2 y_2)$. The letters x_1, x_2, y_1, and y_2 may all be from different alphabets. In fact, we can treat the product source as though it consisted of two single letter sources operating simultaneously.

Let us consider the source S_1 with its distortion measure $d_1(x_1 y_1)$ and zero rate distortion function $\hat{R}_1(D_1)$ and the source S_2 with $d_2(x_2 y_2)$ and $\hat{R}_2(D_2)$. Suppose we have a block code of length n, for S_1 with $M_{D_1}^{(1)}$ code words which guarantees D_1 or less letter distortion and a block code of length n for S_2 with $M_{D_2}^{(2)}$ code words which guarantees D_2 or less letter distortion. Then we can clearly construct a code for the product source with $M_{D_1}^{(1)} \cdot M_{D_2}^{(2)}$ code words of length n which guarantees letter distortion $D_1 + D_2$ or less. We can in fact define $\hat{R}_{12}(D)$, the zero error rate of the product source, as

$$\hat{R}_{12}(D) = \min_{0 \leq D_1 \leq D} [\hat{R}_1(D_1) + \hat{R}_2(D-D_1)] \tag{5.9}$$

since we can never get codes of lower rate that give D or less letter distortion, and by actually combining existing codes for the two single letter sources we can realize a code for the product source with this rate.

Even though the fixed letter distortion as a fidelity criterion seems simpler to work with than the average distortion per letter criterion, this is not actually true. For instance, it is more difficult to calculate $\hat{R}(D)$ than the rate distortion function for the average distortion per letter criterion. Moreover, there are probably not very many more interesting results that can be derived for the letter distortion criterion, whereas the average distortion criterion could be pursued to obtain many and detailed results.

The theorems presented in this section of the chapter bear a close resemblance to Shannon's theorems on the zero error capacity of a discrete, memoryless channel.[13] We have already mentioned the zero error capacity in connection with the interpretation of the code rate for the letter distortion criterion. If we demand that each source letter be transmitted to the sink with D or less distortion, then we must have $C_o > \hat{R}(D)$, where

C_o is the zero error capacity (per unit time) of the channel and $\hat{R}(D)$ is the zero error rate (per unit time) of the source. The problem of finding codes which give letter distortion D or less is a sort of dual to the problem of finding codes which give zero probability of error with a discrete channel. A typical function $\hat{R}(D)$ is shown in Figure 5.3.

5.2 Sources with Side Information

Consider a discrete source which has a finite number of states, $s_1, s_2, \ldots, s_h$. Before each source letter is chosen, a new state is chosen independent of previous states and previous source letters, with probability p_i for state s_i. When in state s_i, the source selects a letter $x \in X$ according to the probability distribution $P_i(x)$. A distortion measure $d(xy)$ is given and we again use the standard fidelity criterion of average distortion per letter. We shall consider the situations in which the encoder, or the decoder, or both have the state of the source available as side information.

If we first suppose that both the encoder and the decoder are given the state of the source in advance of each source letter, a different block code may be used for each source state. Since both the encoder and information user know the state of the source, they also know which block code is being used at any instant. Each block code is governed by a rate distortion function $R_i(d)$ for a discrete, independent letter source $P_i(x)$ and the distortion measure $d(xy)$. If we have block codes with rate $R_i(d_i)$ which give average distortion d_i, and if $\sum_i p_i d_i = \bar{d}$, then we can encode the source with side information with rate $\sum_i p_i R_i(d_i)$ and average distortion $\bar{d}$. The rate-distortion function for the source with such side information is given by

$$R(d) = \min_{d_i} \sum_{i=1}^{h} p_i R_i(d_i)$$

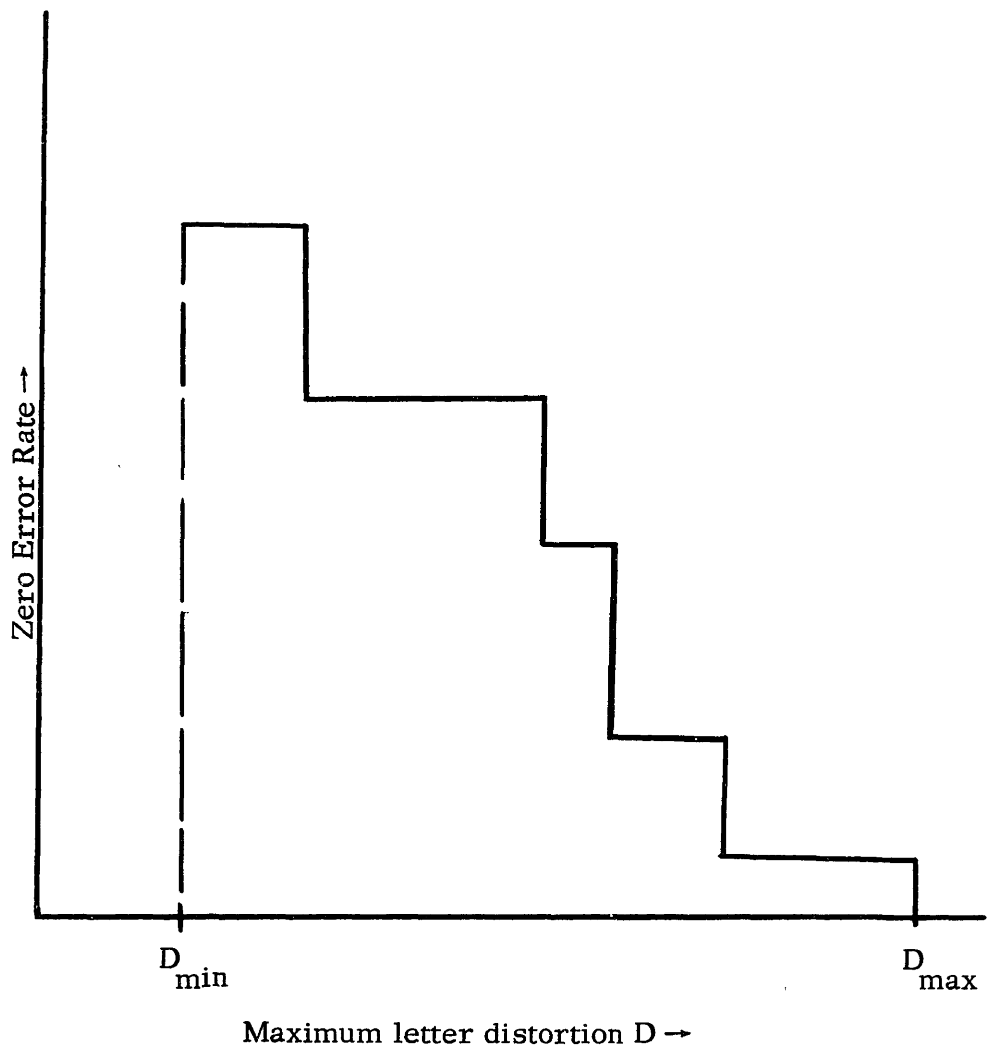
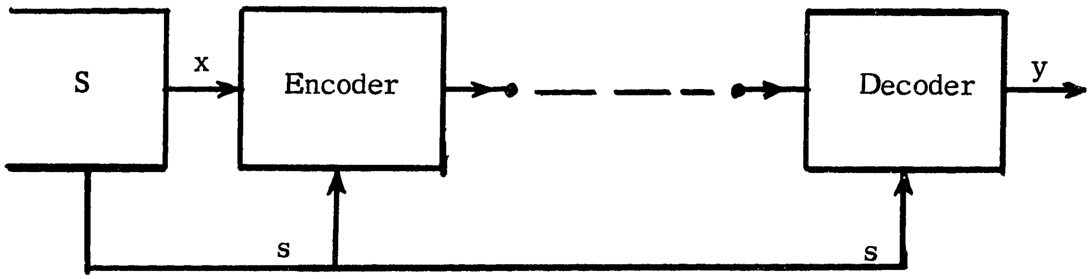


Figure 5.3 A typical $\hat{R}(D)$ function.

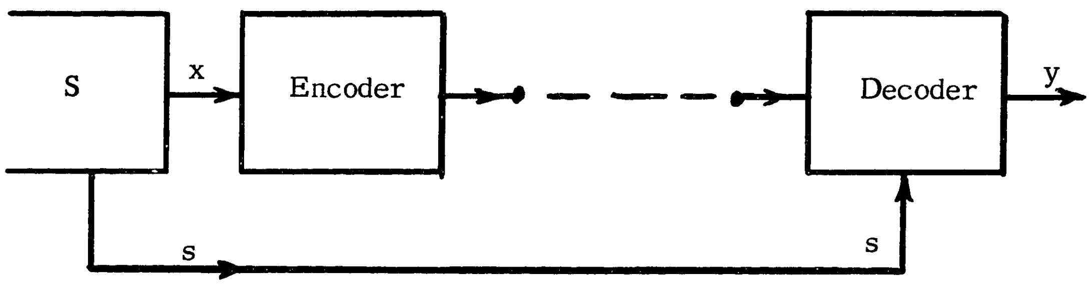
subject to the constraint that $\sum_{i=1}^{h} p_i d_i \leq \bar{d}$. This situation is schematized in Figure 5.4a.

A more interesting case arises when only the decoder has access to the side information. (See Figure 5.4b) The encoder maps the source output into one of M code words to be transmitted to the decoder. If each code word is merely a sequence of output letters, the decoder cannot take advantage of the state information since he has no freedom to operate on the code words. The code words should be selected so that there is some remaining freedom for the decoder to take advantage of the state information in interpreting the facsimile of the source output. If the length of the block code is n, in general, the decoder will use n functions, $f_1(m;\bar{s})$, $f_2(m;\bar{s})$, ..., $f_n(m;\bar{s})$, where $\bar{s} = r_1, r_2, \ldots, r_n$ is the sequence of source states corresponding to the encoded source sequence. In these functions m ranges over the integers from 1 to M and the r_i range over all possible source states. The functions themselves take on values in the output alphabet. The decoder operates as follows. The encoder operates only on the source sequence and encodes it as some code word denoted by an integer from 1 to M. This integer m is transmitted without error to the decoder which presents the sink with the letters $y_i = f_i(m;\bar{s})$. We should notice that the decoder may base its operation on the entire sequence of source states because the encoding operation cannot be completed until the source sequence and hence the state sequence is completed.

Theorem 5.3 Suppose we have a discrete, independent letter source S with a distortion measure d(xy), and suppose also the decoder only has available side information, defined by p_i and $P_i(x)$, i=1, 2,..., h. The rate-distortion function for S is identical to the rate distortion function for a discrete, independent letter source S'



(a)



(b)

Figure 5.4 A source with side information available (a) to both encoder and decoder, and (b) only at the decoder.

with letter probabilities $P'(x)$, distortion measure $d'(x\bar{y})$, and without side information. The source alphabet of S' is the same as that of S and

$$P'(x) = \sum_{i=1}^{h} p_i P_i(x).$$

The output alphabet of S' has b^h letters $\bar{y} = (y_1, y_2, \ldots, y_h)$, where the y_i are in the output alphabet of $d(xy)$. The distortion measure for S' is

$$d'(x\bar{y}) = \sum_{i=1}^{h} p_i \, d(xy_i). \qquad \bullet\bullet$$

Proof We reduce the anaylsis of the source with side information to a source with a different distortion measure and more output letters but without side information. Codes derived for source S' can be used with S and the statistical properties of these codes are identical with either source.

Let us discuss how a code for S' could be used with S. The encoder for S is identical to the encoder for S'. The encoder maps the source word into one of M code words with letters $\bar{y} \in \bar{Y}$, say the m^{th} one. The decoder for S is given the integer m and from this and the sequence of source states, which it has stored, it must produce a sequence of output letters $y \in Y$ to present to the sink. A particular letter $\bar{y}$ of the $\bar{Y}$ alphabet may be thought of as a function from the state alphabet to the output alphabet Y. The whole alphabet $\bar{Y}$ consists of all b^h such possible functions. The decoder merely treats each of the letters $\bar{y}$ of a code word as independent functions from the state alphabet to the output alphabet Y. If the state is s_1, the decoder presents y_1 of the letter $\bar{y}$ to the sink. For state k, the decoder decodes as y_k. The translation is letter by letter since there is no memory involved in the generation of source states and letters.

The codes for S' are actually a specialized set of the decoding functions defined above, where $f_i(m;\bar{s})$ is really a function only of m and of s_i, the i^{th} state. In fact, the encoder actually uses decoding functions as output letters in encoding the source output, thus giving the decoder the freedom to decode using the side information. The average distortion of such a code with S is exactly the same as the average distortion of the code with S'.

Q.E.D.

This result is an obvious adaptation of Shannon's analysis of a discrete, memoryless channel with side information at the transmitter only.(14) As in Section I of this chapter, we have a sort of dual to a channel coding problem.

The case of the source with side information available only to the encoder is trivial. The uncertainty about source letters is the same as for a source with letter probabilities $\sum_i p_i P_i(x) = P(x)$, but merely broken down into two uncertainties, one about the state and the other about the actual source letter. The job of the encoder may be aided with the state information but the actual block codes and their statistical properties will be exactly the same as for the source P(x).

5.3 A Partial Ordering of Information Sources

Consider a discrete, independent letter source with letter probabilities P(x), and a measure of fidelity $\rho(xy)$ which gives the amount of fidelity (or the reward) involved in reproducing source letter x as output letter y at the decoder. We restrict the value of $\rho(xy)$ to be non-negative. We define the average fidelity per letter between sequences $u = x_1, x_2 \dots x_n$ and $v = y_1\, y_2 \dots y_n$ as

$$\rho(uv) = \left(\prod_{i=1}^{n} \rho(x_i y_i)\right)^{\frac{1}{n}} ,$$

which is the geometric mean fidelity per letter. For a block code of length n with M code words $v_1, v_2, \ldots, v_M$, and a partitioning of the source space into M disjoint encoding subsets w_i, we may write the geometric mean fidelity as

$$\text{g.m.f.} = \sum_{i=1}^{M} \sum_{w_i} P(u)\, \rho(uv_i).$$

An example† in which a geometric mean fidelity criterion may be preferrred over the usual arithmetic average distortion would be in comparing an encoder to a noisy transmission channel. Suppose

$$\rho(xy) = \exp - (x-y)^2/2$$

for the case of source and output alphabets consisting of the integers from 0 to 10. The measure of fidelity between a source and an output word then resembles the probability that the source word results in the output word when transmitted directly through a particular noisy channel. (The transition probabilities of the channel may not be properly normalized in this interpretation.) A high fidelity corresponds to accurate reproduction of the source word at the receiver. Notice that if any one letter of the source word is reproduced giving zero fidelity, the fidelity measure in reproducing the entire source word is zero. Low fidelity corresponds to poor reproduction of the source sequence, and zero fidelity letter transitions are very costly, ruining the entire sequence fidelity, and these transitions should be avoided.

---

† Suggested by B. Reiffen in a private communication.

We can carry out our upper and lower bounding techniques of Chapter 2 for the geometric mean fidelity criterion and derive a rate-fidelity function for the source. The only point to be mentioned in this connection is that we encounter probabilities such as $P_r[\rho(uv) \leq \rho_0]$, where u or both u and v are chosen at random. We merely restate this as $P_r[\log \rho(uv) \leq \log \rho_0]$, and since $\log \rho(uv) = \frac{1}{n}\sum_{i=1}^{n} \log \rho(x_i y_i)$, we have only to bound a sum of the independent random variables $\log \rho(x_i y_i)$. In fact, if we define $\rho(xy)$ in terms of a given distortion measure d(xy) as

$$\rho(xy) = e^{-d(xy)},$$

the fidelity-distortion curve is merely the rate-distortion curve with a scale change and reversal of the distortion axis.

Let us define the matrix $\theta(xy)$ for a source with a fidelity measure as

$$\theta(xy) = P(x)\,\rho(xy). \tag{5.10}$$

A block code of length 1 is merely a transformation T(x) defined on source letters into input letters. The g.m.f. for a given code of length 1 is then

$$\text{g.m.f.} = \sum_{X} P(x)\,\rho(x,T(x)) = \sum_{X} \theta(x, T(x)). \tag{5.11}$$

We can visualize the code of length 1 with the aid of the $\theta(xy)$ matrix. If we circle all the $\theta(xy)$ elements with subscripts (x, T(x)), the g.m.f. is then the sum of the circled elements of $\theta(xy)$. Every row of $\theta(xy)$ must have one circled element in this representation of a block code. We can use this same representation for block codes of any length n by merely using the n^{th} order direct or Kronecker product of $\theta(xy)$ with itself.

In Eq. 5.12 we have the Θ matrices of two different sources.

$$\Theta_1 = \begin{bmatrix} 6 & 1 & 3 \\ 3 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix} \qquad \Theta_2 = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix} \tag{5.12}$$

Notice that comparing the columns of Θ_2 with the columns of Θ_1 with the last row deleted, we see the first columns of Θ_2 and Θ_1 are the same and the second and third columns of Θ_1 are both larger† than the second column of Θ_2. This implies that we can use any block code of length 1 for the two letter source to obtain a code for the three letter source with at least as large a g.m.f. The circled elements of the Θ_2 matrix lead to circled elements in the first two rows of Θ_1 which sum to at least the g.m.f. associated with Θ_2, and any elements circled in the third row of Θ_1 can only add to the g.m.f. associated with Θ_1. Therefore, a code of length 1 for source No. 2 leads to a code of the same rate for source No. 1 with at least the same g.m.f. as the code for source No. 1 gives. This argument can be generalized to block codes of any length.

In a sense, we may think of source No. 2 in the above example as being included in source No. 1. We are especially interested in generalizing the notion of adapting a block code for one source for use with another source, giving at least as good a g.m.f. We now present a definition of source inclusion which will lead to a useful coding theorem.

Definition Consider the discrete memoryless source S_1 with letter probabilities $P_1(x_1)$ and fidelity measure $\rho_1(x_1y_1)$, and source S_2 with $P_2(x_2)$ and $\rho_2(x_2y_2)$. We shall say that S_1 includes S_2, ($S_1 \supseteq S_2$ or $S_2 \subseteq S_1$),

---

† A column is considered larger than another column if every element of the first column is larger than the corresponding element of the second column.

if and only if there exist two sets of transition probabilities, $p_\alpha(x_2|x_1)$ and $q_\alpha(y_1|y_2)$, with

$$p_\alpha(x_2|x_1) \geq 0, \quad \sum_{X_2} p\ (x_2|x_1) = 1,$$

and

$$q_\alpha(y_1|y_2) \geq 0, \quad \sum_{Y_1} q_\alpha(y_1|y_2) = 1,$$

and there exists

$$g_\alpha \geq 0, \quad \sum_\alpha g_\alpha = 1$$

with

$$\sum_{\alpha, X_1, Y_1} g_\alpha\, p_\alpha(x_2|x_1)\ \theta_1(x_1\, y_1)\ q_\alpha(y_1|y_2) = \theta_2(x_2 y_2),$$

where

$$\theta_1(x_1 y_1) = P_1(x_1)\ \rho_1(x_1 y_1),\ \theta_2(x_2 y_2) = P_2(x_2)\ \rho_2(x_2 y_2). \tag{5.13}$$

We may think of the sets of transition probabilities $p_\alpha(x_2|x_1)$ and $q_2(y_1|y_2)$ as channels used in pairs, g_α being the probability for the pair with subscript α . Any discrete channel may be interpreted as being composed of a weighted sum of pure channels in which all transition probabilities are either 0 or 1. A pure channel carries each input letter with certainty to some output letter. A pure channel may also be thought of as a mapping of the input alphabet into a subset of the output alphabet. For each α, this decomposition of the channels $p_\alpha(x_2|x_1)$ and $q_\alpha(y_1|y_2)$ may be carried out. A more complete description of the decomposition of a channel into a weighted sum of mappings is given by Shannon.[16] In particular we wish to make the point that the randomly chosen channel pairs can be decomposed into randomly chosen pairs of pure channels or transformations. In other words, source inclusion can be defined equivalently to the above definition but with the added condition that $p_\alpha(x_2|x_1)$ and $q_\alpha(y_1|y_2)$ correspond to pure channels.

The relation of source inclusion is transitive. If $S_1 \supseteq S_2$ and $S_2 \supseteq S_3$, then $S_1 \supseteq S_3$. In fact, if g_α, P_α, Q_α are the probabilities † for the first inclusion relation, and g'_β, P'_β, Q'_β those for the second, then

$$\sum_{\alpha,\beta} g_\alpha g_\beta \, (P'_\beta \cdot P_\alpha) \cdot \Theta_1 \cdot (Q_\alpha \cdot Q'_\beta) = \Theta_3 ,$$

where $\cdot$ denotes the ordinary matrix product. If $S_1 \supseteq S_2$ and $S_2 \supseteq S_1$, we will say that these are equivalent sources and write $S_1 \equiv S_2$. We see that $S_1 \equiv S_1$ always. Grouping sources with fidelity measures into these equivalence classes, we have a partial ordering of sources. A universal lower bound of all sources is the source with one letter and Θ matrix (1×1) with entry zero.

The ordering relation is preserved under the source operation of multiplication. That is, if $S_1 \supseteq S'_1$ and $S_2 \supseteq S'_2$, then

$$S_1 \times S_2 \supseteq S'_1 \times S'_2 ,$$

where $\times$ denotes the direct or Kronecker matrix product. A product of sources corresponds to a source which produces letter pairs (x_1, x_2) with probability $P_1(x_1)\,P_2(x_2)$ and fidelity measure between (x_1, x_2) and (y_1, y_2) given by $\rho_1(x_1 y_1)\,\rho_2(x_2 y_2)$. Suppose again g_α, P_α, Q_α are the probabilities of the inclusion relation $S_1 \supseteq S'_1$, and g'_β, P'_β, Q'_β are those of the relation $S_2 \supseteq S'_2$. Then it is easy to show

$$\sum_{\alpha,\beta} g_\alpha g'_\beta \, (P_\alpha \times P'_\beta) \cdot (\Theta_1 \times \Theta_2) \cdot (Q_\alpha \times Q'_\beta) = \Theta'_1 \times \Theta'_2 .$$

† P_α and Q_α denote the stochastic matrices of the pure channels $p_\alpha(x_2|x_1)$ and $q_\alpha(y_1|y_2)$, respectively.

Our chief reason for defining source inclusion as above is the following theorem which relates the ordering to the coding problem for sources.

Theorem 5.4 Suppose $S_1 \supseteq S_2$ and there is a block code of length n for S_2 with M code words which gives g.m.f. f_2^*. Then there exists for source S_1 a block code of length n and M code words which gives g.m.f. $f_1^* \geq f_2^*$. ••

Proof We have a block code for S_2 which may be represented by a transformation $T(u_2) = v_2^*$ defined on all possible source words of length n of S_2. We may write the g.m.f. for the encoder as

$$f_2^* = \sum_{U_2} P_2(u_2)\, \rho_2(u_2, T(u_2)) = \sum_{U_2} \theta_2(u_2 v_2^*). \tag{5.14}$$

By the preservation of the ordering under source multiplication, we know that we have g_α, $p_\alpha(u_2|u_1)$, $q_\alpha(v_1|v_2)$ such that

$$\sum_{\alpha, U_1, V_1} g_\alpha\, p_\alpha(u_2|u_1)\, P_1(u_1)\, \rho_1(u_1 v_1)\, q_\alpha(v_2|v_1) = P_2(u_2)\, \rho_2(u_2 v_2), \tag{5.15}$$

where $p_\alpha(u_2|u_1)$ and $q_\alpha(v_1|v_2)$ are pure, n^{th} power channels. Let us drive the channel $p_\alpha(u_2|u_1)$ with the source S_1^n and let us also connect the output of this channel to the input to the encoder for S_2. We also connect the channel $q_\alpha(v_1|v_2)$ to the output of the encoder for S_2. For a particular α, $p_\alpha(u_2|u_1)$ maps the output of source S_1 into the input to the encoder for S_2, while $q_\alpha(v_1|v_2)$ maps the set of M code words for S_2 into a set of M v_1 words, which then may be considered a block code for S_1. We then have defined a mapping of the source words of length n of S_1 into M output words v_1 for each α, and we then have an encoder for S_1 for a block code of length n and M code words.

Each of the encoders for S_1 for a particular α then gives a certain g.m.f., say $f^*_{1\alpha}$. We may think of using the ensemble of encoders with S_1, choosing each encoder with probability g_α. We may then write the g.m.f. for the ensemble of randomly chosen encoders as

$$f^*_1 = \sum_{\alpha} g_\alpha \, f^*_{1\alpha} \, . \tag{5.16}$$

We can calculate f^*_1 as follows. The probability that the source word u_1 will be mapped into output word v_1 by a randomly chosen set of transformations P_α and Q_α is just

$$P_1(u_1) \sum_{U_2} \sum_{\alpha} g_\alpha \, p_\alpha(u_2|u_1) \, q_\alpha(v_1|v_2^*),$$

where v_2^* is defined by the encoder for S_2 as $T(u_2) = v_2^*$. When the source word u_1 is mapped in this manner into v_1, the fidelity is $\rho_1(u_1v_1)$, hence, the g.m.f. over the ensemble of randomly chosen encoders is given by

$$f^*_1 = \sum_{U_1} \sum_{V_1} \sum_{U_2} \sum_{\alpha} g_\alpha \, p_\alpha(u_2|u_1) \Big(P_1(u_1)\, \rho_1(u_1v_1) \Big) \; q_\alpha(v_1|v_2^*)$$

$$= \sum_{U_2} \Big(\sum_{\alpha} \sum_{U_1} \sum_{V_1} g_\alpha \, p_\alpha(u_2|u_1) \;\; \theta_1(u_1v_1)\, q_\alpha(v_1|v_2^*) \Big)$$

$$= \sum_{U_2} \theta_2(u_2v_2^*) = f^*_2$$

where we have used Eqs. 5. 10, 5. 15, and 5. 14. In view of Eq. 5. 16, there exists a particular pair of transformations P_α, Q_α such that $f^*_{1\alpha} \geq f^*_1 = f^*_2$. We have shown the existence of a block code of length n and M code words for S_1 which giv g. m. f. f^*_2 or more.

Q. E. D.

In particular, we can conclude that if $S_1 \supseteq S_2$, the best possible code of length n and M code words for S_2 could not give greater g. m. f. than the best possible code of length n and M code words for S_1. It is possible to define the inclusion relation with an inequality sign in Eq. 5. 13, thus making it more general, but this is not a basic change from what we present here. Shannon[16] presented a partial ordering for discrete, memoryless communication channels which looks very much like our source ordering from a mathematical viewpoint. However, we find such an ordering only for a geometric mean fidelity criterion and not for the more interesting arithmetic average distortion criterion. We may conclude that there is a dual source problem to the partial ordering of channels, but the duality involves the less practical geometric mean fidelity criterion, and is thus not as useful as the channel ordering.

CHAPTER VI

CONCLUDING REMARKS AND SUGGESTIONS FOR FURTHER RESEARCH

6.1 Summary and Conclusions

It was intended that this research consist of a general study of the problems involved in attempting to encode an information source with a distortion measure. We have presented several aspects of this general problem area from an information theoretic viewpoint. Our results on the rate-distortion function elaborate on the previous work of Shannon[15], whereas the other topics of i) binary source encoding, ii) sources with side information, and iii) the fidelity criterion on maximum letter distortion are treated for the first time.

The upper and lower bounds on average distortion for block codes of fixed rate were very useful in arriving at the rate-distortion function of a source. We optimized the asymptotic form as $n \rightarrow \infty$ of the bounds on average distortion until the two bounds were identical. This gave us a unique relation between rate and average distortion which had the significance of Shannon's rate-distortion function. Moreover, this approach gave an explicit method of computing the rate-distortion function by simply solving two sets of linear equations. Our results agreed completely with previous results even to the extent of interpreting our parametric expressions for rate and average distortion in terms of a test channel. The upper and lower bounds on average distortion were shown to converge to their limiting values only as negative powers of n.

The discussion of binary source coding gave some insight into the basic mathematical difficulties involved in applying codes with algebraic structure to sources. We showed the existence of group codes which gave rate-distortion performance as good as the ensemble of random block codes, which is asymptotically ideal. The encoding complexity in applying codes to the binary source was investigated for two sequential encoding schemes. We presented a simple scheme for binary group codes of length n and rate R which required only $nR(1-R)$ computations per source letter to encode the source output. The ensemble of random group codes together with this sequential encoding scheme was shown to give an upper bound on average distortion which is useful but weaker than the rate distortion function.

The analysis of a sequential encoding scheme for randomly chosen binary tree codes of rate $R = \frac{1}{2}$ gave an upper bound to average distortion which seemed to approach the rate-distortion function. An upper bound on the average number of computations per source letter to encode the source output was found to be independent of the length of the code. While the upper bound on average computation was independent of the code length, it indicated a large number of computations per source letter. It is not clear whether the bound is weak or whether the average number of computations is, in fact, high. It would be very desirable to analyze this sequential encoding scheme with convolutional codes[18], which are extremely simple to store and generate.

A logical and interesting extension of this study of source encoding would be the study of group codes for use with non-binary sources and a more general class of distortion measures. In analyzing codes with algebraic structure, we are usually faced with the problem of finding the probability distribution of the smallest value obtained in several selections of some random variable. The statistical theory of extreme values is well developed and may be of use in this aspect of the problem. In view of what we present here, there is a reasonable chance that a good solution can be found to the complexity problem in a fairly general class of source encoding problems.

The fidelity criterion on maximum allowable distortion per letter is interesting in that it is used often in practice, but also because it is directly connected with a problem in channel coding, i.e., the zero error capacity of a channel. The problem of encoding a source according to this fidelity criterion is a sort of mathematical dual to the problem of coding for a channel for zero probability of error.

Our work on sources with side information and a partial ordering of sources again bring out the dual nature of problems in channel coding and source coding. Prior to Shannon's work on coding sources according to a fidelity criterion, interest in sources generally centered on finding expressions for source entropy and noiseless coding schemes to give low probability of misrepresenting the exact source output. This research shows clearly that the fidelity criterion generalizes the notions of source coding and produces as many interesting facets of this newer problem as have been found in the general area of channel coding. Even in the work on the rate-distortion function, we see that the techniques used in getting upper and lower bounds on average distortion are quite similar to those used in deriving bounds on the optimum probability of error for channel codes.

It must appear to the reader as it certainly has to the author that the problems in source coding are generally more awkward than their dual counterparts in channel coding. We point out as examples the difficulty of analyzing codes with algebraic structure and our difficulties in obtaining a lower bound to average distortion for finite code length. This awkwardness appears to be due in some degree to the sum distortion measure which measures the distortion between sequences of letters as the sum of the individual letter distortions. The simplicity of the results on the partial ordering for sources for a geometric mean fidelity criterion leads one to believe that a product distortion measure may be closer to the mathematical dual to channel coding problems. However, the geometric mean distortion criterion seems much less useful as a measure of the performance of a communication system.

6.2 Extension of the Theory to Markov Sources and Markov Distortion Measures

Throughout this work we have assumed the simplest of sources, one which selects letters independently at random. It would certainly be of value to have a method of calculating the rate-distortion function for a source with memory, i.e. random letter selections dependent on the past history of the source. Perhaps the simplest source with memory is the finite state Markov source, for which the entropy is already well known[4]. It is clear that our general approach of analyzing the performance of block codes for such Markov sources would be a logical first attempt to obtain the rate-distortion function. We have shown that we needed probability bounds on a sum of independent random variables which were exponentially correct to obtain the rate-distortion function for independent letter sources. However, the analogous bounds on sums of random variables generated by Markov processes do not yet exist, and this is the first and main problem encountered in the extension of the theory to Markov sources.

The distortion measures that we have used were single letter distortion measures in which the distortion was defined by concurrent source and output letters, i.e. d(xy) was the distortion when the source letter x was reproduced as output letter y. Shannon discussed a local distortion measure[15] in which the distortion was defined by concurrent blocks of source and output letters. (The length of the blocks was finite.) Let us call a pair of successive source letters in a sequence a transition. For example, if x_{t-1} and x_t are the source letter produced at time $t-1$ and the succeeding letter, respectively, we call x_{t-1}, x_t the transition occurring at time $t-1$. A local distortion measure with finite memory can be represented as a distortion measure which depends only on concurrent source and output letter transitions, if we are willing to deal with expanded alphabets consisting of sequences of source and output letters. A distortion measure which depends only on source and output letter transitions, such as $d(x_{t-1}, x_t, y_{t-1}, y_t)$ is called a Markov distortion measure.

We now discuss a simple example of a Markov distortion measure. Consider a source which selects independently, with equal probability, one of A discrete wheel positions numbered from 0 to A-1. Let us define the operations of addition and multiplication of these source letters as modulo A. Suppose we are now faced with the situation in which the information user or sink does not 'see' the actual decoder output directly, but has access to the data only after it has passed through an input device which cannot be by-passed. (See Figure 6.1). The input device of the sink may be characterized as a data processing system, possibly with memory. The distortion measure may be defined in terms of the difference between what the sink receives when the actual source output and the decoder output are passed through the sink input device. The situation in which the sink has access to data only through a fixed input device is not entirely academic, for a human being has access to optical and auditory stimuli only through his eyes and ears.

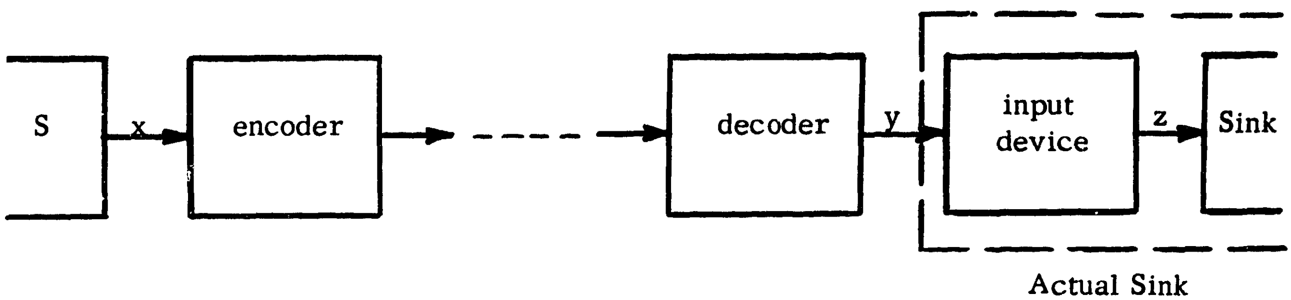


Figure 6. 1 Block diagram showing a sink with a fixed input device.

Let us suppose that we are given the distortion measure $d(x' \ominus y')$ which depends only on the difference $x' \ominus y'$, where x' is the response of the input device to the actual source output x and y' is the response of the input device to the decoder output y. If we now suppose that the input device is a linear system whose response depends only on input letter transitions, the linearity allows us to write

$$\begin{aligned} d(x'_t \ominus y'_t) &= d((a_1 x_{t-1} \oplus a_o x_t) \ominus (a_1 y_{t-1} \oplus a_o y_t)) \\ &= d(a_1(x_{t-1} \ominus y_{t-1}) \oplus a_o(x_t \ominus y_t)) \\ &= d'(x_{t-1} \ominus y_{t-1}, x_t \ominus y_t) \end{aligned}$$

where $d'(x_{t-1}, x_t, y_{t-1}, y_t)$ is a Markov distortion measure which depends only on the transitions of the difference (mod A) between the source output and decoder output. We can see that an input device with memory and a distortion measure $d(x,z)$ (using the notation of Figure 6.1) will lead to a local distortion measure d' between the X and Y alphabets.

This particular case of the linear data processor can be analyzed, since the distortion between source and output sequences depends only on the difference between these sequences. The random coding argument was carried out for this example for the ensemble of random codes with equiprobable and independent letters. The probabilities $P_r[d(uv) \leq d_o]$ do not depend on the particular u and v sequences, and we need only bound the probability $P_r[d(w) \leq d_o]$, where $w = u \ominus v$ and $d(w) = \frac{1}{n-1} \sum_{i=2}^{n} d(w_i, w_{i-1})$. R. S. Kennedy[(10)] has derived exponentially correct bounds on $P_r[d(w) \leq d_o]$ when w is generated by a finite state Markov process. These probability bounds enabled us to find the rate-distortion function for this special case of a Markov distortion measure. The general Markov source with a Markov distortion measure requires very general exponential bounds on Markov processes, which do not yet exist.

6.3 Fidelity Criterion on Several Distortion Measures

Suppose we have an independent letter source, P(x), and two distortion measures $d_1(xy)$ and $d_2(xy)$. We may have a fidelity criterion which requires that the average distortion with respect to $d_1(xy)$ is d_1^* or less while, simultaneously, the average distortion with respect to $d_2(xy)$ is d_2^* or less. An example of such a situation would be one in which $d_1(xy)$ is an ordinary single letter distortion measure while $d_2(y)$ is merely a cost attached to the use of each output letter. We could also conceivable encounter the situation in which $d_1(xy)$ was a single letter distortion measure and $d_2(x_{t-1}, x_t, y_{t-1}, y_t)$ was a Markov distortion measure on letter transitions, which would be somewhat analogous to having a fidelity criterion on the derivative of a continuous waveform.

For the case where $d_1(xy)$ and $d_2(xy)$ are single letter distortion measures, the necessary exponential bounds would not be difficult to derive. We conjecture that the random coding bound on average distortion could be derived and that the asymptotic form of this bound as $n \to \infty$ would lead to parametric functions $R_u(t)$, $d_{1u}(t)$, $d_{2u}(t)$, $t \le 0$, which have the following significance. There exist block codes with rate $R > R_u(t)$ for which the average distortions satisfy $\bar{d}_1 \le d_{1u}(t)$ and $\bar{d}_{2u}(t)$, $t \le 0$. A lower bound on the average distortions would lead to a rate-distortion surface $R(d_1, d_2)$. A typical rate-distortion surface has been sketched in Figure 6.2. A fidelity criterion on several distortion measures seems to be an interesting extension of the theory with some practical application, and it should be only slightly more involved than the single distortion measure case.

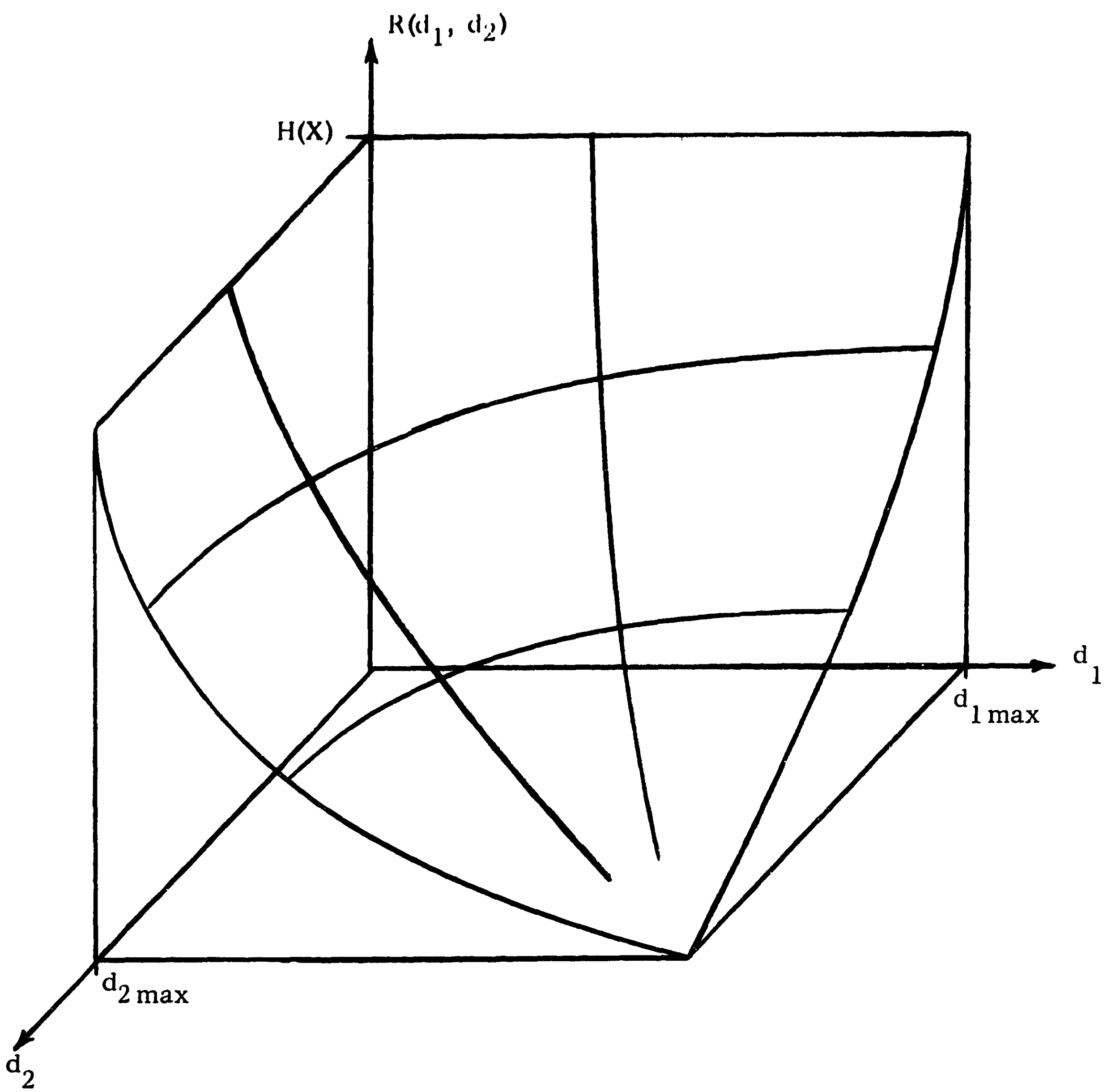


Figure 6.2 A typical rate-distortion surface $R(d_1, d_2)$.

6.4 General Remarks

The mathematical framework within which this research was done pertains only to the simplest abstraction of a physical information source. Yet the present theory does provide a useful way of looking at communication systems which attempt to reduce transmission capacity requirements by encoding the output of an information source before transmission to the receiver.

The most interesting applications of such a theory would obviously be with very complex signal sources which would ordinarily demand a very large transmission capacity without coding. Whenever subjective appreciation of the facsimile of the source output is encountered, it is clear that this may be interpreted as the application of a distortion measure to evaluate system performance. A good example of just such a physical situation is an ordinary television system. The signal source is governed by extremely complicated statistical constraints, and it is known that viewers do not require very accurate transmission of pictures. Moreover, the eye is an input device to the information user which cannot be by-passed. The visual acuity of the human eye is such that only blocks of picture elements (as contrasted to individual picture elements) can be resolved and that the average light intensity over a block of picture elements is all that is important to a viewer. It seems that a local distortion measure with a fidelity criterion on average distortion should be general enough to characterize this situation.

One of the standard engineering approaches to encoding a television source has been the experimental determination of the coarseness of quantization in picture element intensities that renders a picture objectionable to viewers. Other standard approaches attempt to encode pictures by first separating the signals that represent the light intensities of the picture into several frequency ranges and then quantizing the signals in each frequency range differently. More sophisticated methods of processing television pictures attempt to exploit the resolution of the eye by scrambling the quantization noise in adjacent picture elements so that the eye will average them out. The latter technique is similar to scrambling signals for transmission through a rapidly fading channel (a channel with memory) to remove correlation in errors in received data. The channel decoder then processes the received data as if the channel was memoryless. This technique is simple and it works to some extent, but it provides little insight in learning the fundamental limitations on the performance of the system.

The information theory indicates that the statistical constraints governing the operation of the information source must be studied and a suitable statistical model chosen to represent the source. The next key step indicated is the determination of the distortion measure and the tolerable level of distortion used by the information user to evaluate system performance. The job of modeling something as complex as a television picture source is obviously an extremely difficult task because of the wide variety of pictures that can occur. Likewise, the determination of the distortion measure used by television viewers is complicated by the fact that the eye characteristics and artistic tastes of different viewers implies that there are many distortion measures actually in use simultaneously.

If a suitable source model and distortion measure could be found, and analyzed, it would yield the limiting rate reduction that could be achieved by any encoding technique with the given source and distortion measure. This would provide a yardstick with which to evaluate any proposed encoding scheme. Since the source encoder is the complicated link in the system, the search for good codes would have to consider the complexity in instrumenting proposed codes. The decoder at the various information users is much less complicated than the source encoder, which is fortunate since there may be many users of the encoded source output, e.g. in the television system example, there are many, many information users interested in one signal source. The block diagram of our proposed communication system is given in Figure 1.2.

The actual detailed design of systems which attempt to reduce transmission capacity requirements of complex signal sources will most certainly be influenced by the information theory someday, but that day is a long way off. However, the theory does at present provide a design philosophy by indicating the basic steps involved in building signal processing equipment to conserve transmission channel capacity.

APPENDIX A

PROOF OF THEOREM 2.1

Consider the situation in which the source P(x) has just generated a word u of length n and we are about to choose a list of M code words of length n at random according to $P_c(y)$. We want to get an upper bound to the average distortion that results after choosing the code and encoding u for minimum distortion. We will study a non-optimum encoding system to get the desired bound. We have a threshold d_n and we start calculating the distortion between u and each code word in the list of code words. The source word u is mapped into the first code word v that we come to the list for which $d(uv) \le d_n$. If the first M-1 code words in the list fail this test, we map u into the last code word v_M.

If any code word satisfies the $d(uv) \le d_n$ test, the resulting distortion in encoding u is upper bounded by d_n. If none of the first M-1 code words satisfy the test, the average distortion in mapping u into v_M is just

$$\sum_{XY} P(x)\, P_c(y)\, d(xy) = d_m \quad . \tag{A.1}$$

The upper bound on the average distortion in encoding u with the described system, when averaged over all u, is an upper bound on the average distortion over the ensemble of random block codes with the best partitioning of U. We then get the bound on $\bar{d}$, the average distortion for the ensemble of block codes,

$$\bar{d} \le d_n(1-P_o) + d_m P_o \le d_n + d_m P_o \tag{A.2}$$

where P_o is the probability that the source selects a sequence u and that none of the first M-1 code words satisfy $d(uv) \leq d_n$.

For the case of M = 2 we have only one code word to test so

$$P_o = \sum_U P(u) \left(1 - P_r \left[d(uv) \leq d_n \mid u \right] \right).$$

It follows that for M code words

$$P_o = \sum_U P(u) \left(1 - P_r \left[d(uv) \leq d_n \mid u \right] \right)^{M-1} . \tag{A.3}$$

We wish to next show the conditions under which P_o can be made to tend to zero with increasing block length n so that the bound on $\bar{d}$ in Eq. A.2 tends to d_n. For any particular source word u we may count the number of times each letter of the X alphabet appears. We denote by $n(x_k)$ the number of times letter x_k appears in the u sequence and we call the set of integers $n(x)$, $x \in X$, the composition of u.

The probability $P_r\ [d(uv) \leq d_n | u]$ depends on u only through the composition of the u sequence. It is intuitively clear that the letter composition n(x) of very long u sequences tends to n P(x) with high probability. We therefore divide the space U into the subsets U^* and $\overline{U^*}$, the complement of U^*. U^* is so defined that the composition of $u \in U^*$ is approximately $n\,P(x)$, which defines $P_r\ [d(uv) \leq d_n | u \in U^*]$ within narrow limits. The part of P_o involving the subset U^* then depends on M and can be made to vanish only by restriction of M. The subset $\overline{U^*}$ has vanishing probability for very large n.

We need a lower bound on $P_r\ [d(uv) \leq d_n | u]$ in order to upper bound $\exp(-(M-1)\, P_r [d(uv) \leq d_n | u])$ in P_o. R. M. Fano's lower bound[4] (pages 275-279) is suitable for our purposes and is restated below in our notation.

$$P_r[d(uv) \le d_n \mid u] \ge K_L(n)\, e^{\Gamma_n(u,t) - t\Gamma_n'(u,t)}, \quad t \le 0 \qquad \underline{(8.125)}^\dagger$$

where $u = \xi_1 \xi_2 \cdots \xi_n$, $\xi_i \in X$ with composition $n(x)$, and

$$\gamma_x(t) = \log \sum_Y P_c(y)\, e^{td(xy)} \qquad \underline{(8.127)}$$

$$\Gamma_n(u,t) = \sum_{i=1}^{n} \gamma_{\xi_i}(t) = \sum_X n(x)\, \gamma_x(t) \qquad \underline{(8.129)}$$

$$\gamma_x'(t) = \frac{\partial \gamma_x(t)}{\partial t}$$

$$\Gamma_n'(u,t) = \sum_{i=1}^{n} \gamma_{\xi_i}'(t) = \sum_X n(x)\, \gamma_x'(t)$$

$$Q(y \mid x) = P_c(y)\, e^{td(xy) - \gamma_x(t)} \qquad \underline{(8.128)}$$

$$K_L(n) = (2\pi n)^{-\frac{a(b-1)}{2}} \exp -\left(\frac{ab}{12} + |t|\Delta + \sum_{XY} (n(x)\, Q(y \mid x))^{-1}\right) \qquad \underline{(8.125)}$$

$$\Delta = \max_{XY} d(xy) \text{ (larger than Fano's } \Delta) \qquad \underline{(8.130)}$$

The value of t is specifically chosen in (4) to satisfy

$$\Gamma_n'(u,t) = n d_n, \quad t \le 0 \qquad \underline{(8.146)}$$

but it can be seen from the derivation of this bound that we are free to choose any value of $t \le 0$ such that

$$\Gamma_n'(u,t) \le n d_n. \qquad (A.4)$$

---

† The underlined equation numbers are from (4) to aid in comparing the different notations.

We define $\gamma(t)$ and $\gamma'(t)$ as the mean values of $\gamma_x(t)$ and $\gamma_x'(t)$, respectively considered as random variables with probability P(x). We see that the mean values of $\Gamma_n(u,t)$ and $\Gamma_n'(u,t)$ are $n\gamma(t)$ and $n\gamma'(t)$, respectively. Let us take the threshold

$$d_n = d_U + n^{-1/4} \tag{A.5}$$

where d_U is a constant. Then as $n \to \infty$, $d_n \to d_U$. Now we may choose the value of t independent of any u sequence by

$$\gamma'(t) = d_U, \; t \leq 0. \tag{A.6}$$

Also, we see from Eq. A.1 and the definition of $\gamma'(t)$ that

$$\gamma'(0) = d_m \quad . \tag{A.7}$$

Define the subset U* of u sequences such that

$$n\gamma(t) - n^{3/4} \leq \Gamma_n(u,t)$$

and

$$n\gamma'(t) - n^{3/4} \leq \Gamma_n'(u,t) \leq n\gamma'(t) + n^{3/4} \quad .$$

For $u \in U^*$, we have from Eqs. A.5 and A.6

$$\Gamma_n'(u,t) \leq n\gamma'(t) + n^{3/4} = n d_n$$

so that t is chosen to give a valid lower bound to $P_r\,[d(uv) \leq d_n | u]$ for $u \in U^*$. The definition of U* is used to lower bound $\exp(\Gamma_n(u,t) - t\Gamma_n'(u,t))$ for $u \in U^*$ so that we get

$$P_r\,[d(uv) \leq d_n \mid u \in U^*] \geq K(n)\, e^{n\gamma(t) - nt\gamma'(t) - n^{3/4} - |t|\, n^{3/4}} \tag{A.8}$$

where K(n) differs from $K_L(n)$ in that each n(x) has been replaced by 1. This gives $K_L(n) \geq K(n)$ and K(n) now has no u dependence.

Eq. A.2 now becomes, using Eq. A.3,

$$\bar{d} \le d_n + d_m \left(\sum_{\overline{U^*}} P(u) + \sum_{U^*} P(u) \left(1 - P_r\left[d(uv) \le dn/u\right]\right)^{M-1} \right) \quad . \qquad (A.9)$$

Using Eqs. A.5, A.6, A.7, and the inequality for $0 < p \le 1$,

$$(1-p)^{M-1} = e^{(M-1)\log(1-p)} < e^{-(M-1)p}$$

we get

$$\bar{d} \le d_n + d_m \left(P_r[\overline{U^*}] + \sum_{U^*} P(u)\; e^{-(M-1)\, P_r[d(uv) \le d_n / u \in U^*]} \right)$$

$$\le \gamma'(t) + n^{-1/4} + \gamma'(0) \left(P_r[\overline{U^*}] + e^{-(M-1)\, K(n) \exp - n[t\gamma'(t) - \gamma(t) + n^{-1/4} + |t| n^{-1/4}]} \right)$$

(A.10)

where we have upper bounded $\sum_{U^*} P(u)$ by unity.

From the definition of U^* we see that the probability of $\overline{U^*}$ is the probability of the union of the three events

$$\Gamma_n(u,t) < n\gamma(t) - n^{3/4}$$

$$\Gamma_n'(u,t) < n\gamma'(t) - n^{3/4}$$

$$\Gamma_n'(u,t) > n\gamma'(t) + n^{3/4}$$

which can always be upper bounded by the sum of the probabilities of the individual events.

$$P_r[\overline{U^*}] \leq P_r[\Gamma_n(u,t) < n\gamma(t) - n^{3/4}]$$

$$+ P_r[\Gamma'_n(u,t) < n\gamma'(t) - n^{3/4}]$$

$$+ P_r[\Gamma'_n(u,t) > n\gamma'(t) + n^{3/4}] \tag{A.11}$$

Denote the distribution function for the random variable

$\Gamma_n(u,t) = \Gamma_n$ by $\mathcal{L}_n(\Gamma) = P_r(\Gamma_n \leq \Gamma)$. We define the random variable

$$z_n = \frac{\Gamma_n - n\gamma}{\sqrt{n}\,\sigma_1}$$

where $\gamma = \gamma(t)$ and σ_1^2 is the variance of $\gamma_x(t)$.

The distribution function for z is then

$$\mathfrak{F}_n(z) = \mathcal{L}_n(\sqrt{n}\,\sigma_1 z + n\gamma) \tag{A.12a}$$

which implies

$$\mathcal{L}_n(\Gamma) = \mathfrak{F}_n\left(\frac{\Gamma - n\gamma}{\sqrt{n}\,\sigma_1}\right) \tag{A.12b}$$

and $\mathfrak{F}_n(z)$ has mean zero and variance one so that we may apply ordinary central limit results. $\mathfrak{F}_n(z)$ can be estimated by using a theorem by Cramer[1] (pages 77-78) with the constant c in the theorem estimated by Cramer to be 3. (See comments of Gnedenko and Kolmogorov[6] page 201.) Thus

$$\mathfrak{F}_n(z) < \Phi(z) + B_{1n}(z) \tag{A.13}$$

where $B_{1n}(z) = \dfrac{3\beta_{13}\log n}{\sqrt{n}\,\sigma_1^{3/2}}$ and β_{13} is the third absolute moment of $\mathcal{L}_n$. Then combining Eqs. A.12b and A.13,

$$P_r\,[\,\Gamma_n(u,t) < n\gamma(t) - n^{3/4}\,] = \mathfrak{F}_n\left(-\frac{n^{1/4}}{\sigma_1}\right) < \Phi\left(-\frac{n^{1/4}}{\sigma_1}\right) + B_{1n}$$

$$\leq \frac{\sigma_1 \exp(-n^{1/2}/2\sigma_1^2)}{\sqrt{2\pi}\; n^{1/4}} + B_{1n}. \qquad (A.14)$$

We have used a bound for $\Phi(-\alpha)$, $\alpha > 0$ given by Feller(5)(page 166, Eq. 1.8),

$$\Phi(-\alpha) \leq \frac{\exp - \alpha^2/2}{\sqrt{2\pi}\;\alpha}, \quad \alpha > 0.$$

In an entirely analogous manner we bound the other terms of Eq. A.11. Define σ_2^2 as the variance of the random variable $\gamma_x'(t)$, and β_{23} as the third absolute moment of $\gamma_x'(t)$. We get

$$P_r\,[\,\Gamma_n'(u,t) < n\gamma'(t) - n^{3/4}\,] < \frac{\sigma_2 \exp(-n^{1/2}/2\sigma_2^2)}{\sqrt{2\pi}\,n^{1/4}} + B_{2n} \qquad (A.15)$$

$$P_r\,[\,\Gamma_n'(u,t) > n\gamma'(t) + n^{3/4}\,] \leq 1 - \Phi\left(\frac{n^{1/4}}{\sigma_2}\right) + B_{2n}$$

$$= \Phi\left(-\frac{n^{1/4}}{\sigma_2}\right) + B_{2n}$$

$$< \frac{\sigma_2 \exp(-n^{1/2}/2\sigma_2^2)}{\sqrt{2\pi}\; n^{1/4}} + B_{2n} \qquad (A.16)$$

where $B_{2n} = \dfrac{3\,\beta_{23}\log n}{\sqrt{n}\;\sigma_2^{3/2}}$.

Combining Eqs. A.14, A.15, and A.16 in Eq. A.10 gives Eq. 2.4 and proves Theorem 2.1. This bound on $\bar{d}$ is for any $t \leq 0$ and involves only n and functions of P(x) and d(xy).

Q.E.D.

APPENDIX B

ASYMPTOTIC EXPANSION OF CERTAIN INTEGRALS

We wish to obtain an asymptotic expansion of integrals of the form

$$\int_{-\infty}^{0} e^{\alpha z}\, d\mathfrak{F}_n(z) \quad \text{and} \quad \int_{-\infty}^{0} z e^{\alpha z}\, d\mathfrak{F}_n(z)$$

where $\mathfrak{F}_n$ is the normalized distribution function for a sum of discrete, independent, non-identical random variables. We shall assume that the random variables are non-lattice, i.e., there are no numbers r and h such that all values of D(xy) can be expressed as $r + hn$, $n = 0, \pm 1, \pm 2, \ldots$ From Esseen's theorem (Gnedenko and Kolmogorov[6] page 210), we may write $\mathfrak{F}_n$ as

$$\mathfrak{F}_n(z) = \Phi(z) - \frac{\beta_{3n}}{6\sqrt{n}(\mu_n'')^{3/2}} \quad \frac{(z^2-1)\exp - z^2/2}{\sqrt{2\pi}} + \frac{\epsilon}{\sqrt{n}}, \ \epsilon > 0 \tag{B.1}$$

where β_{3n} is the third absolute moment of the distribution $H_n(z)$ given in Eq. 3.3. From Cramer's work[1] it is clear that this theorem applies to sums of non-identical random variables.

Denote the function

$$\varphi^{(i)}(z) = \frac{d^i}{dz^i} \frac{\exp - z^2/2}{\sqrt{2\pi}} \tag{B.2}$$

and the integral

$$I_i(\alpha) = \int_{-\infty}^{0} e^{\alpha z}\, \varphi^{(i)}(z)\, dz. \tag{B.3}$$

Since

$$\frac{(z^2 - 1) \exp - z^2/2}{\sqrt{2\pi}} = \varphi^{(2)}(z)$$

and $d\varphi^{(1)}(z) = \varphi^{(i+1)}(z)\,dz$, we see that the integrals we are interested in are $I_o(\alpha)$ and $I_3(\alpha)$. Note that

$$\frac{d}{d\alpha} I_i(\alpha) = \int_{-\infty}^{0} z\, e^{\alpha z}\, \varphi^{(i)}(z)\, dz = J_i(\alpha) \qquad \text{(B. 4)}$$

and we are interested in the integrals $J_o(\alpha)$ and $J_3(\alpha)$ as well.

By completing the square in the exponent we find

$$I_o(\alpha) = \Phi(-\alpha) \exp \alpha^2/2 \qquad \text{(B. 5)}$$

Integration by parts yields

$$I_3(\alpha) = -1 - \alpha I_2(\alpha)$$

$$I_2(\alpha) = -\alpha I_1(\alpha)$$

$$I_1(\alpha) = 1 - \alpha I_o(\alpha)$$

so that

$$I_3(\alpha) = -1 + \alpha^2 - \alpha^3 I_o(\alpha). \qquad \text{(B. 6)}$$

By use of Eq. (B. 4) we obtain from Eqs. B. 5 and B. 6

$$J_o(\alpha) = -\frac{1}{\sqrt{2\pi}} + \alpha\, \Phi(-\alpha) \exp \alpha^2/2 \qquad \text{(B. 7)}$$

$$J_3(\alpha) = \frac{2\alpha}{\sqrt{2\pi}} - 3\alpha^2 I_o(\alpha) - \alpha^3 J_o(\alpha) \; . \qquad \text{(B. 8)}$$

From Feller[5] (page 179, Eq. 6.1) we obtain an asymptotic expansion for $\Phi(-\alpha)$, $\alpha > 0$.

$$\Phi(-\alpha) \sim \frac{\exp - \alpha^2/2}{\sqrt{2\pi}} \left(\frac{1}{\alpha} - \frac{1}{\alpha^3} + \frac{3}{\alpha^5} - \frac{15}{\alpha^7} \cdots \right) \tag{B.9}$$

If we now use $\alpha = |t| \sqrt{n\mu''}$ together with Eq. B.9 we obtain the asymptotic expansion of the desired integrals as

$$I_0(\alpha) \sim \frac{1}{\sqrt{2\pi}} \left(\frac{1}{|t| \sqrt{n\mu''}} + o\left(\frac{1}{\sqrt{n}}\right) \right)$$

$$I_3(\alpha) \sim \frac{1}{\sqrt{2\pi}} \left(- \frac{3}{|t|^2 n\mu''} + o\left(\frac{1}{n}\right) \right)$$

$$J_0(\alpha) \sim - \frac{1}{\sqrt{2\pi}} \left(\frac{1}{|t|^2 n\mu''} + o\left(\frac{1}{n}\right) \right)$$

$$J_3(\alpha) \sim \frac{1}{\sqrt{2\pi}} \left(\frac{6}{|t|^2 (n\mu'')^{3/2}} + o\left(\frac{1}{n^{3/2}}\right) \right). \tag{B.10}$$

The last term of Eq. B.1 in the expansion of $\mathfrak{F}_n(z)$ yields integrals

$$\frac{\epsilon}{\sqrt{n}} \int_{-\infty}^{0} e^{\alpha z}\, dz = \frac{\epsilon}{\sqrt{n}\,\alpha} \sim o\left(\frac{1}{n}\right)$$

$$\frac{\epsilon}{\sqrt{n}} \int_{-\infty}^{0} z e^{\alpha z}\, dz = - \frac{\epsilon}{\sqrt{n}\,\alpha^2} \sim o\left(\frac{1}{n^{3/2}}\right). \tag{B.11}$$

Now we may combine our results in Eqs. B.10 and B.11 to obtain

$$\int_{-\infty}^{0} e^{\alpha z}\, d\mathfrak{F}_n(z) \sim \frac{1}{|t|\sqrt{2\pi n \mu''}} + o\left(\frac{1}{\sqrt{n}}\right)$$

$$\int_{-\infty}^{0} z e^{\alpha z}\, d\mathfrak{F}_n(z) \sim -\frac{1}{\sqrt{2\pi}\,|t|^2 n\mu''} + o\left(\frac{1}{n}\right) \tag{B. 12}$$

and from Eqs. 3. 12 and 3. 13 we obtain

$$c_1 = \frac{-1}{\sqrt{2\pi}\,|t|^2\,\mu''}\quad,\quad d_1 = \frac{1}{|t|\sqrt{2\pi\mu''}}$$

$$\hat{c}_1 = -\frac{1}{|t|\,\mu'}\quad,\quad \hat{d}_1 = 1\ . \tag{B. 13}$$

The case of a lattice random variable is substantially the same with the complication of one extra term in the expansion of $\mathfrak{F}_n(z)$. Esseen's theorem on lattice distributions (Gnedenko and Kolmogorov[(6)], page 213) is used and the result for $\hat{c}_1/\hat{d}_1$ is exactly the same as Eq. B. 13.

BIOGRAPHICAL NOTE

Thomas John Goblick, Jr. was born in Old Forge, Pennsylvania on March 30, 1935. He attended Wilkes College, Wilkes-Barre, Penna. from 1952 to 1954, and Bucknell University, Lewisburg, Penna. from 1954 to 1956, where he was awarded the degree of Bachelor of Science in Electrical Engineering, cum laude. He enrolled at the Massachusetts Institute of Technology in September 1956 and was awarded the degree of Master of Science in Electrical Engineering in February 1958. He attended Imperial College, University of London during the fall term of 1958 as a Fulbright Scholar. In September 1959 he returned to the Massachusetts Institute of Technology as a Lincoln Laboratory Staff Associate, and has been with the Information Processing and Transmission group of the Research Laboratory of Electronics until the present time. He was awarded the degree of Electrical Engineer in June 1960 from the Massachusetts Institute of Technology.

In the summers of 1955 and 1956, he was employed at the Aircraft Products Department of General Electric Company, Johnson City, New York. During the summers of 1958 and 1959, he was employed by Lincoln Laboratory, Lexington, Massachusetts. He is a member of Tau Beta Pi, Sigma Xi, and Pi Mu Epsilon. He has been a student member of the Institute of Radio Engineers since 1954.

He is married to the former Carolyn Rita Morrison of Brookline, Massachusetts. They have two children, Pamela (1960) and Sandra (1961).

PUBLICATIONS OF THE AUTHOR

A Study of Mode Coupling in Periodic Structures, S. M. Thesis, Dept. of Electrical Engineering, Mass. Inst. of Tech., Cambridge, Mass. (January 1958).

"Radar Echoes from Venus," (with co-authors Price, Green, Kingston, Kraft, Pettengill, Silver, and Smith), Science, Vol. 129, No. 3351, 751 - 753, (March 20, 1959).

"Variational Principles and Mode Coupling in Periodic Structures," (with co-author R. M. Bevensee), I.R.E. Trans., Vol. MTT-8, No. 5, 500-509 (September 1960).

REFERENCES

1. Cramer, H., "Random Variables and Probability Distributions," Cambridge Tracts in Mathematics, Vol. 36, (1937).

2. DeBruijn, N. G., Asymptotic Methods in Analysis, (Interscience Publishers, Inc., New York, 1958).

3. Elias, P., "Coding for Noisy Channels," IRE Convention Record, Part 4, 37-46, (1955).

4. Fano, R. M., Transmission of Information, (The M.I.T. Press and John Wiley and Sons, Inc., New York, 1961).

5. Feller, W., An Introduction to Probability Theory and its Applications, Vol. 1, 2nd Edition (John Wiley and Sons, Inc., New York, 1957).

6. Gnedenko, B. V. and Kolmogorov, A. N., Limit Distributions for Sums of Independent Random Variables, (Translated by K. L. Chung), (Addison-Wesley Publishing Co., Inc., Cambridge, Mass., 1954).

7. Hardy, G. H., Littlewood, J. E. and Polya, G., Inequalities, (University Press, Cambridge, England, 1952).

8. Hildebrand, F. B., Advanced Calculus for Engineers, (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1949).

9. Kelly, J.L., Jr., " Random Coding of a Continuous Source," Bell Telephone Laboratories, New Jersey, Unpublished paper (1961).

10. Kennedy, R.S., "Finite State Binary Symmetric Channels," Sc. D. Thesis to be submitted to the Dept. of Electrical Engineering, M.I.T. (1962).

11. Massey, J.L., "Threshold Decoding," Ph. D. Thesis submitted to the Dept. of Electrical Engineering, M.I.T. (August 1962).

12. Peterson, W. W., Error-Correcting Codes, (The M.I.T. Press and John Wiley and Sons, Inc., New York, 1961).

13. Shannon, C. E., "The Zero Error Capacity of a Noisy Channel," IRE Trans. I.T., Vol. 2, No. 8, 8-19, (September 1958).

14. Shannon, C. E., "Channels with Side Information at the Transmitter," IBM Journal of Research, Vol. 2, No. 4, 289-293, (October 1958).

15. Shannon, C. E., "Coding Theorems for a Discrete Source with a Fidelity Criterion," IRE National Convention Record, Part 4, 142-163 (1959).

16. Shannon, C. E., "A Partial Ordering of Communication Channels," Information and Control, Vol. 1, No. 4, 390-397 (December 1959).

17. Shannon, C. E., "The Central Limit Theorem with Large Deviations," Unpublished paper, (1957).

18. Wozencraft, J. M. and Reiffen, B., Sequential Decoding, (The Technology Press of M.I.T. and John Wiley and Sons, Inc., New York, 1961).