Second-Order Fluid Dynamics Models for Travel Times in Dynamic Transportation Networks

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Abstract—In recent years, traffic congestion in transportation networks has grown rapidly and has become an acute problem. The impetus for studying this problem has been further strengthened due to the fast growing field of Intelligent Vehicle Highway Systems (IVHS). Therefore, it is critical to investigate and understand its nature and address questions of the type: *how are traffic patterns formed?* and *how can traffic congestion be alleviated?* Understanding drivers' travel times is key behind this problem. In this paper, we present macroscopic models for determining analytical forms for travel times. We take a fluid dynamics approach by noticing that traffic macroscopically behaves like a fluid. Our contributions in this work are the following: (i) We propose two second-order non-separable macroscopic models for analytically estimating travel time functions : the Polynomial Travel Time (PTT) Model and the Exponential Travel Time (ETT) Model. These models generalize the models proposed by Kachani and Perakis [4] as they incorporate second-order effects such as reaction of drivers to upstream and downstream congestion as well as second-order link interaction effects. (ii) Based on piecewise linear and piecewise quadratic approximations of the departure flow rates, we propose different classes of travel time functions for the first-order separable PTT and ETT models, and present the relationship between these functions. (iii) We show how the analysis of the first-order separable PTT Model extends to the second-order model with non-separable velocity functions for acyclic networks. (iv) Finally, we analyze the second-order separable ETT model where the queue propagation term -corresponding to the reaction of drivers to upstream congestion or decongestion- is not neglected. We are able to reduce the analysis to a Burgers equation and then to the more tractable heat equation.

Keywords— Dynamic Traffic Flow, Dynamic Travel Time, Fluid models

I. INTRODUCTION

The way flows circulate in traffic networks, the way queues form and disappear, the spillback and the shock wave phenomena, are striking evidence that traffic flows are similar to gas and water flows. It is therefore normal to use physical laws of fluid dynamics for compressible flow to model traffic flow patterns.

In 1955, Lighthill and Whitham [5], and in 1956, Richards [17] introduced the first continuum approximations of traffic flows using kinematic wave theory. A variety of dynamic traffic flow models have been proposed in the literature that can be classified in two major categories: microscopic models and macroscopic models.

Microscopic models, or car-following models, have the ability to describe, at a level of detail, the network geometry, the traffic flow and its kinematics and the traffic control logic. Such models enable simulated tests of traffic flow control strategies, and help design safety procedures by better understanding the driver's behavior. In 1950, Reuschell [16] proposed the first car-following model. Pipes [14] and Herman *et al.* [15] extended this model. Gerlough and Huber [3], Bekey *et al.* [2], and Papageorgiou ([10], [11], [9]) and references therein provide an extensive analysis of these models.

On the other hand, analytical models usually possess mathematical properties that are useful in understanding the properties of a model and in designing solution algorithms to solve instances of the problem. In an attempt to improve modeling accuracy, the first-order model of Lighthill and Whitham [5] was extended by Payne [12] and Whitham [18].

The purpose of this paper is to address the question of *what is the travel time of a driver in traversing a network's link*. Practitioners in the transportation area have been using several families of travel time functions. Akcelik [1] proposed a polynomial-type travel time function for links at signalized intersections. The BPR function [8], that is used to estimate travel times at priority intersections, is also a polynomial function. Finally, Meneguzzer *et al.* [7] proposed an exponential travel time function for all-way-stop intersections. Our goal is to lay the theoretical foundations for using these polynomial and exponential families of travel time functions in practice. While most analytical models in traffic modeling assume an a priori knowledge of a driver's travel time functions, in this paper, travel time is part of the model and comes as an output.

To determine travel times, we examine and further extend the analytical model proposed by Perakis [13]. This model provides a macroscopic fluid dynamics approach to the dynamic network equilibrium problem. We also extend the analysis performed by Kachani and Perakis [4] to account for second-order effects such as reaction of drivers to upstream and downstream congestion, drivers' reaction time, as well as to account for second-order link interaction effects.

II. NOTATIONS AND INTRODUCTION TO THE HYDRODYNAMIC THEORY OF TRAFFIC FLOW

In Subsection A, we summarize the notation that we use throughout the paper. In Subsection B, we consider a single link network and introduce the hydrodynamic theory of traffic flow developed by Lighthill and Whitham [5].

A. Notation

The physical traffic network is represented by a directed network $G = (N, I)$, where N is the set of nodes and I is the set of directed links. Index w denotes an Origin-Destination (O-D) in the set W of origin destination pairs. Index P denotes the set of paths and index P_w denotes the set of paths between O-D w.

Path variables:

x_p	:	position on path p;
L_p	:	length of path p;
$F_p(x_p, t)$	:	flow rate at time t on path p at position x_p;
$F(0, t)$	:	vector of departure path flow rates;
$T_p(L_p, t)$	:	travel time on path p departing at time t;

Link variables:

x_i	:	position on link i;
L_i	:	length of link i;
$f_i(x_i, t)$	:	flow rate at time t on link i at position x_i;
$f(0, t)$	:	vector of departure link flow rates;
$T_i(L_i, t)$	:	travel time on link i departing at time t;
$u_i(x_i, t)$	:	speed on link i at position x_i at time t;
$\overline{u}_i(k, \nabla k)$	:	function of the vectors of link densities and link densities' gradients;
$\widehat{u}_i(k)$	:	function of the vector of link densities;
$k_i(x_i, t)$	:	density on link i at position x_i at time t;
u_i^{max}	:	maximum traffic speed on link i;
k_i^{max}	:	maximum traffic density on link i;

Link-path flow variables:

ip	:	link-path pair;
δ_{ip}	=	1 if link i belongs to path p, 0 otherwise;
L_{ip}	:	length from the origin of path p until the beginning of link i;
$T_{ip}(L_{ip}, t)$	:	partial path travel time from the origin of path p until the beginning of link i departing at time t;

B. Hydrodynamic Theory of Traffic Flow on a Single Stretch of Road

In this subsection, we describe the laws of fluid dynamics for compressible flow in a single stretch of road. Lighthill and Whitham [5] introduced these laws.

Let us consider a link of length L. We denote by $\tau = \tau(x, t)$ the travel time to reach position x when departing at time t. The three fundamental traffic variables of fluid dynamics are the flow rate function $f(x, t+\tau)$, the density function $k(x, t+\tau)$, and the velocity function $u(x, t+\tau)$ at point x at time $t + \tau$.

Two relationships connect these three variables: $f(x, t+\tau) = k(x, t+\tau).u(x, t+\tau)$, and $\frac{\partial f(x,t+\tau)}{\partial x} + \frac{\partial k(x,t+\tau)}{\partial \tau} = 0$, $\forall x, \tau$. The latter expresses that there is conservation of cars in a road with no exits.

If we knew the velocity function $u(.)$, then the above two laws would allow us to obtain the flow rate $f(.)$ and as a result the density $k(.)$. In the mid-1950's Lighthill and Whitham [5] and independently Richards [17], proposed an additional assumption, that is, that the velocity at any point depends only on the density. In mathematical terms: $u = \widehat{u}(k)$.

The function $\widehat{u}$ is empirically measured and is an input to the model. Several models have been proposed in the literature for the velocity function $\widehat{u}(.)$. Mahmassani and Hernan [6] proposed a linear model: $\widehat{u}(k) = u^{max}(1 - \frac{k}{k^{max}})$. Therefore, the free flow speed is the maximum speed: $\widehat{u}(0) = u^{max}$, and at maximum density, the speed is zero: $\widehat{u}(k^{max}) = 0$.

III. GENERAL MODELS FOR TRAVEL TIME FUNCTIONS

In Subsection A, we propose a second-order non-separable model (Model 1) for travel time functions that incorporates the drivers' reactions to upstream congestion or decongestion as well as link interaction. This model generalizes the first-order model proposed by Perakis [13]. In Subsection B, we propose two simplified versions of the general model: the Polynomial Travel Time (PTT) Model and the Exponential Travel Time (ETT) Model. The analysis of these two models is the focus of the following sections.

A. A Second-Order Model

The purpose of this subsection is to model the following two traffic phenomena:
1- Drivers' reaction to upstream congestion or decongestion. In particular, when a driver realizes the formation of a queue upstream, he/she starts slowing down. Similarly, drivers start accelerating when the queue starts dissipating.
2- Effects on a link of densities as well as variations in densities of neighboring links.

To account for the two phenomena, we replace the speed-density relationship $u_i = \widehat{u}_i(k)$ by $u_i = \overline{u}_i(k, \nabla k)$. The variables k and ∇k contain the term $\frac{\partial k_i}{\partial x_i}$ that allows us to model the reaction of drivers to changes in the link density. They also contain the terms k_j and $\frac{\partial k_j}{\partial x_j}$ for the set of links j in the neighborhood of link i, that allow us to effectively model link interaction. We propose the following general form of the velocity of link i, at position x_i and at time t:

$$\overline{u}_i(k, \nabla k) = u_i^{max} - b_i(u_i^{max})^2 k_i(x_i, t) -$$
$$\frac{\lambda_i(x_i)}{k_i(x_i, t)} \frac{\partial k_i(x_i, t)}{\partial x_i} + \sum_{j \in I} \alpha_{ij}(x_i) k_j(\overline{x}_j, t - \Delta_{ij}) +$$
$$\sum_{j \in I} \frac{\beta_{ij}(x_i)}{k_j(\overline{x}_j, t - \Delta_{ij})} \frac{\partial k_j(\overline{x}_j, t - \Delta_{ij})}{\partial x_j}, \quad (1)$$

where $\alpha_{ij}(x_i)$ and $\beta_{ij}(x_i)$ are density correlation functions between link i and link j and depend on the position x_i on link i; $\overline{x}_j$ is a fixed position of a detector of density on link j and Δ_{ij} is a propagation time between link i and link j.

The term $-\frac{\lambda_i(x_i)}{k_i(x_i, t)} \frac{\partial k_i}{\partial x_i}$ is borrowed from heat transfer and accounts for the drivers' awareness of upstream and downstream conditions. The heat transfer term $\lambda_i(x_i)$ is a positive term expressed in squared miles per unit of time. The propagation term $\frac{\lambda_i(x_i)}{k_i(x_i, t)} \frac{\partial k_i}{\partial x_i}$ expresses the variation in the speed induced by a variation in the density. For instance, when a queue is expanding on link i, the term

$-\frac{\lambda_i(x_i)}{k_i(x_i,t)}\frac{\partial k_i}{\partial x_i}$ is negative and hence the velocity function $u_i(x_i,t)$ decreases.

Model 1 can be formulated as follows:

Model 1

For all $t \in [0,T]$, $p \in P$, and $i \in I$, we have:

$$T_p(L_p,t) = \sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip},t))\delta_{ip}, \quad (2)$$

$$f_i(x_i,t) = \sum_{p \in P} F_p(x_i,t)\delta_{ip}, \quad (3)$$

$$u_i(x_i,t) = \overline{u}_i(k, \nabla k), \quad (4)$$

$$f_i(x_i,t) = k_i(x_i,t)u_i(x_i,t), \quad (5)$$

$$\frac{\partial f_i(x_i,T_i)}{\partial x_i} + \frac{\partial k_i(x_i,T_i)}{\partial T_i} = 0, \quad (6)$$

$$\frac{dT_i(x_i,t)}{dx_i} = \frac{1}{u_i}, \quad (7)$$

$$T_i(0,t) = 0. \quad (8)$$

When $\lambda_i(.)$, $\alpha_{ij}(.)$ and $\beta_{ij}(.)$ go to 0, the above speed-density relationship becomes $u_i = u_i^{max} - b_i(u_i^{max})^2 k_i(x_i,t)$. The latter corresponds to the first-order model proposed by Perakis [13].

Model 1 is very hard to analyze in its current form. For this reason, in the following subsection, we consider two simplified models of Model 1.

B. Two Simplified Second-Order Separable Models for Travel Time Functions

Our goal in this subsection is to solve Model 1 and propose specific travel time functions. To achieve this, the first step is to eliminate some of the variables involved in the model. We eliminate the density variables by expressing them as functions of the flow rates. This leads to proposing two simplified versions of Model 1. We impose the following assumptions:

A1 $\overline{u}_i(k, \nabla k)$ is a separable function of the density k_i. Further, $\overline{u}_i(k, \nabla k) = u_i^{max} - b_i(u_i^{max})^2 k_i(x_i,t) - \frac{\lambda_i(x_i)}{k_i(x_i,t)}\frac{\partial k_i(x_i,t)}{\partial x_i}$, where b_i is a constant.

A2 The term $\frac{1}{(u_i^{max})^2} \ll 1$.

A3 The term $\lambda_i(x_i)\frac{\partial k_i}{\partial x_i} \ll 1$.

A4 The link flow rate $f_i(0, t + \tau_i)$ can be approximated through a continuously differentiable function $h_i^t(\tau_i)$ of τ_i.

Lemma 1: Under Assumption (A1), the link density as a function of the link flow rate function and the queue propagation term can be expressed as:

$$k_i = \frac{1}{2b_i u_i^{max}}(1 - (1 - 4b_i(f_i + \lambda_i(x_i)\frac{\partial k_i}{\partial x_i}))^{\frac{1}{2}}). \quad (9)$$

Proof: Since $\widehat{u}(k_i) = u_i^{max} - b_i(u_i^{max})^2 k_i$, combining the speed-density and the flow-speed-density relationships, we derive $f_i = u_i^{max}k_i - b_i(u_i^{max})^2 k_i^2 - \lambda_i(x_i)\frac{\partial k_i}{\partial x_i}$. By solving in terms of k_i for stable flows, we obtain the result of the lemma. Q.E.D.

B.1 The Polynomial Travel Time (PTT) Model

In this subsection, we consider an approximation of equation (9). This approximation enables us to describe the conservation law of cars (6) only in terms of the link flow rate functions.

Lemma 2: Under Assumptions (A1)-(A2), the link density as a function of the link flow rate function and the queue propagation term can be expressed as:

$$k_i = \frac{f_i + \lambda_i(x_i)\frac{\partial k_i}{\partial x_i}}{u_i^{max}} + \frac{b_i(f_i + \lambda_i(x_i)\frac{\partial k_i}{\partial x_i})^2}{u_i^{max}}. \quad (10)$$

Proof: From equation (9), $k_i = \frac{1}{2b_i u_i^{max}}(1 - (1 - 4b_i(f_i + \lambda_i(x_i)\frac{\partial k_i}{\partial x_i}))^{\frac{1}{2}})$. Assumption (A2) and the definition of b_i in Assumption (A1) imply that all the terms of order higher than or equal to 3 in the Taylor expansion of the above equation are negligible. That is, $1 - (1 - \epsilon)^{\frac{1}{2}} = \frac{\epsilon}{2} + \frac{\epsilon^2}{8} + O(\epsilon^3)$. The result of the lemma follows. Q.E.D.

Using the above result, the following theorem provides a partial differential equation that provides a new version of the conservation law (6) described only by the link flow rate functions .

Theorem 1: Under Assumptions (A1)-(A3) and equation (10), the link flow rate functions f_i are solutions of the second-order partial differential equation:

$$\frac{\partial f_i}{\partial t} + \frac{u_i^{max}}{1 + 2b_i f_i}\frac{\partial f_i}{\partial x_i} = \lambda_i(x_i)\frac{\partial^2 f_i}{\partial x_i^2}. \quad (11)$$

Assumption (A4) provides a boundary condition and, when $\lambda_i(x_i)$ is non-zero, $f_i(x_i,0)$, for $x_i \in [0,L_i]$ and $i \in I$, provides an initial condition.

Proof: Replacing the value of k_i from 10 in the flow conservation equation gives rise to:

$$\frac{\partial f_i}{\partial x_i} + \frac{1 + 2b_i f_i}{u_i^{max}}\frac{\partial f_i}{\partial t} + \frac{1}{u_i^{max}}\frac{\partial^2 k_i}{\partial t \partial x_i}(\lambda_i(x_i)$$

$$+ 2b_i \lambda_i(x_i)^2\frac{\partial k_i}{\partial x_i} + 2b_i \lambda_i(x_i)f_i) + \frac{2b_i \lambda_i(x_i)}{u_i^{max}}\frac{\partial k_i}{\partial x_i}\frac{\partial f_i}{\partial t} = 0.$$

Differentiating the flow conservation equation with respect to x_i leads to $\frac{\partial^2 k_i}{\partial t \partial x_i} = -\frac{\partial^2 f_i}{\partial x_i^2}$. Replacing the above value of $\frac{\partial^2 k_i}{\partial t \partial x_i}$ in the above second-order equation and using Assumption (A3) leads to $\frac{\partial f_i}{\partial x_i} + \frac{1 + 2b_i f_i}{u_i^{max}}\frac{\partial f_i}{\partial t} - \frac{\lambda_i(x_i) + 2b_i \lambda_i(x_i)f_i}{u_i^{max}}\frac{\partial^2 f_i}{\partial x_i^2} = 0$. Dividing each term by $\frac{1 + 2b_i f_i}{u_i^{max}}$ gives rise to the result of the theorem. Q.E.D.

Conservation law (11) is the basis of our analysis of the PTT Model in the following sections.

B.2 The Exponential Travel Time (ETT) Model

In this subsection, we use a different approach. We first eliminate the density variables through equation (9), and use this to derive a conservation law. We then approximate this equation to obtain a conservation law in the link flow rate.

Theorem 2: Under Assumption (A1), the link flow rate functions f_i are solutions of the partial differential equation:

$$\frac{\partial f_i}{\partial t} + u_i^{max}(1 - 4b_i(f_i + \lambda_i(x_i)\frac{\partial k_i}{\partial x_i}))^{\frac{1}{2}}\frac{\partial f_i}{\partial x_i} = \lambda_i(x_i)\frac{\partial^2 f_i}{\partial x_i^2}.$$

Furthermore, under Assumptions (A2) and (A3), the link flow rate functions f_i are solution of the second-order partial differential equation:

$$\frac{\partial f_i}{\partial t} + u_i^{max}(1 - 2b_i f_i)\frac{\partial f_i}{\partial x_i} = \lambda_i(x_i)\frac{\partial^2 f_i}{\partial x_i^2}. \qquad (12)$$

Assumption (A4) provides a boundary condition and, when $\lambda_i(x_i)$ is non-zero, $f_i(x_i, 0)$, $x_i \in [0, L_i]$ and $i \in I$, provides an initial condition.

Proof: Under Assumption (A1), equation (9) holds. Differentiating this equation with respect to t gives rise to $\frac{\partial k_i}{\partial t} = \frac{\frac{\partial f_i}{\partial t} + \lambda_i(x_i)\frac{\partial^2 k_i}{\partial x_i \partial t}}{u_i^{max}(1 - 4b_i(f_i + \lambda_i(x_i)\frac{\partial k_i}{\partial x_i}))^{\frac{1}{2}}}$. Moreover, differentiating the flow conservation equation with respect to x_i leads to $\frac{\partial^2 k_i}{\partial t \partial x_i} = -\frac{\partial^2 f_i}{\partial x_i^2}$. Therefore, it follows that $\frac{\partial k_i}{\partial t} = \frac{\frac{\partial f_i}{\partial t} - \lambda_i(x_i)\frac{\partial^2 f_i}{\partial x_i^2}}{u_i^{max}(1 - 4b_i(f_i + \lambda_i(x_i)\frac{\partial k_i}{\partial x_i}))^{\frac{1}{2}}}$. Substituting the above value of $\frac{\partial k_i}{\partial t}$ in the flow conservation equation leads to $\frac{\partial f_i}{\partial t} + u_i^{max}(1 - 4b_i(f_i + \lambda_i(x_i)\frac{\partial k_i}{\partial x_i}))^{\frac{1}{2}}\frac{\partial f_i}{\partial x_i} = \lambda_i(x_i)\frac{\partial^2 f_i}{\partial x_i^2}$.

Assumption (A2) implies that all the terms of order higher than or equal to 2 in the Taylor expansion of the above equation are negligible. That is, $(1 - \epsilon)^{\frac{1}{2}} = 1 - \frac{\epsilon}{2} + O(\epsilon^2)$. Assumption (A3) gives then rise to equation (12). Q.E.D.

Conservation law (12) is the basis of our analysis of the ETT Model in the following sections.

Our purpose is to reduce the analysis of the Second-Order ETT Model to the analysis of a known problem in fluid dynamics. This reduction will be achieved in two steps. The first reduction consists of transforming the bottleneck operation of the model to a Burgers equation. In fluid dynamics, Burgers equations are considered to be the simplest equations combining both nonlinear propagation effects and diffusive effects. The second reduction consists of a standard reduction of a Burgers' equation to a heat equation.

Equation (20) is a second-order partial differential equation in the link flow rate f_i. Solving this PDE is the bottleneck operation in the solution of this model. The following result achieves the two-stage reduction outlined above.

Theorem 3: (i) If $Y_i = u_i^{max}(1 - 2b_i f_i)$, then, Y_i satisfies $\frac{\partial Y_i}{\partial t} + Y_i\frac{\partial Y_i}{\partial x_i} = \lambda_i\frac{\partial^2 Y_i}{\partial x_i^2}$.

(ii) Let Z_i be defined by $Y_i = (1 - 2\lambda_i\frac{\frac{\partial Z_i}{\partial x_i}}{Z_i})$. Equation (20) reduces to a heat equation of the type

$$\frac{\partial Z_i}{\partial t} = \lambda_i\frac{\partial^2 Z_i}{\partial x_i^2}, \qquad (13)$$

Note that $f_i = -\frac{1}{2b_i}(1 - 2\frac{\lambda_i}{u_i^{max}}\frac{\frac{\partial Z_i}{\partial x_i}}{Z_i})$. Equation (13) is a heat equation. The heat equation has been extensively studied in the literature. The application of literature results to our specific problem is the subject of ongoing research.

IV. ANALYSIS OF FIRST-ORDER SEPARABLE VELOCITY FUNCTIONS

In this section, we derive and analyze the first-order Separable PTT and the ETT Models. This corresponds to the case where the queue propagation term $\lambda_i(.)$ is neglected in Assumption (A1). In this case, the PTT and ETT models can be viewed as two simplified versions of the model proposed by Perakis [13]. We summarize the results of Kachani and Perakis [4] and refer the reader to [4] for proofs of these results.

In particular, in Subsection A, we examine the PTT Model for piecewise linear and piecewise quadratic functions $h_i^t(T_i)$ (see Assumption (A4)). In Subsection B, we examine the ETT Model by approximating the initial flow rate with piecewise linear functions $h_i^t(T_i)$. In Subsection C, we show how the families of travel time functions we propose in Subsections A and B relate.

A. First-Order Separable PTT Model

In the case where the queue propagation term is neglected, the analysis of the PTT Model model in the previous section gives rise to the following formulation:

PTT Model
For all $t \in [0, T]$, $p \in P$ and $i \in I$:

$$\frac{\partial f_i}{\partial T_i} + \frac{u_i^{max}}{1 + 2b_i f_i}\frac{\partial f_i}{\partial x_i} = 0, \qquad (14)$$

$$f_i(0, t + T_i) = h_i^t(T_i), \qquad (15)$$

$$k_i = \frac{f_i}{u_i^{max}} + \frac{b_i f_i^2}{u_i^{max}}, \qquad (16)$$

$$u_i = \frac{f_i}{k_i}, \qquad (17)$$

$$\frac{dT_i(x_i, t)}{dx_i} = \frac{1}{u_i}, \qquad (18)$$

$$T_i(0, t) = 0, \qquad (19)$$

$$T_p(L_p, t) = \sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip}, t))\delta_{ip}. \qquad (20)$$

The following theorem provides an existence result for a continuously differentiable solution of the PTT Model as formulated above.

Theorem 4: [13] The PTT Model as formulated above possesses a solution if and only if the first derivative of the link flow rate function $h_i^t(T_i)$ satisfies the following boundedness condition:

$$\frac{dh_i^t(T_i)}{dT_i} > -\frac{u_i^{max}}{2b_i L_i}.$$

Special Cases:
1- Linear PTT Model: We assume that during a time period $[t, t + \Delta]$, drivers make the approximation that the departure link flow rate for subsequent times $t + T_i$ is linear in terms of the travel time T_i. That is, $f_i(0, t + T_i) = h_i^t(T_i) = A_i(t) + B_i(t)T_i$. Over the time period $[0, T]$, this results into a piecewise linear approximation of link departure flow rates.

Theorem 5: If equation (21) holds, then:
(i) The Linear PTT Model possesses a solution.

(ii) The link flow rate functions $f_i(x_i, t + T_i)$ are continuously differentiable,

$$f_i(x_i, t + T_i) = \frac{B_i(t)u_i^{max}T_i - B_i(t)x_i + A_i(t)u_i^{max}}{u_i^{max} + 2b_iB_i(t)x_i}. \quad (21)$$

(iii) The link travel time functions $T_i(x_i, t)$ are given by:

$$T_i(x_i, t) = \frac{x_i}{u_i^{max}} + \frac{A_i(t)}{B_i(t)}((1 + \frac{2b_iB_i(t)x_i}{u_i^{max}})^{\frac{1}{2}} - 1). \quad (22)$$

2- Quadratic PTT Model: We now assume that during a time period $[t, t + \Delta]$, drivers make the approximation that the departure link flow rate for subsequent times $t + T_i$ is quadratic in terms of the travel time T_i. That is, $f_i(0, t + T_i) = h_i^t(T_i) = A_i(t) + B_i(t)T_i + C_i(t)(T_i)^2$. Over the time period $[0, T]$, this results into a piecewise quadratic approximation of link departure flow rates.

Let $\alpha_1 = \frac{b_iB_i(t)}{u_i^{max}}$, $\alpha_2 = \frac{b_iA_i(t)}{u_i^{max}}$ and $\alpha_3 = \frac{4b_iC_i(t)}{u_i^{max}}$.

Theorem 6: Assume that

$$|B_i(t) + 2C_i(t)(t + \Delta)| << \frac{u_i^{max}}{2b_iL_i}. \quad (23)$$

Then, the following holds
(i) The Quadratic PTT Model possesses a solution.
(ii) The link travel time functions $T_i(x_i, t)$ become

$$T_i(x_i, t) = \frac{1}{u_i^{max}}[(1 + A_i(t)b_i)x_i - \frac{A_i(t)B_i(t)(b_i)^2}{2u_i^{max}}x_i^2 + (\frac{2A_i(t)^2C_i(t)(b_i)^3}{3(u_i^{max})^2} - \frac{7A_i(t)B_i(t)^2(b_i)^3}{6(u_i^{max})^2} + \frac{3B_i(t)^2C_i(t)(b_i)^3}{(u_i^{max})^2})x_i^3]. \quad (24)$$

B. First-Order Separable ETT Model

The analysis of the ETT Model in the previous section gives rise to the following model in the first order case:

ETT Model
For all $t \in [0, T]$, $p \in P$ and $i \in I$:

$$\frac{\partial f_i}{\partial T_i} + u_i^{max}(1 - 2b_if_i)\frac{\partial f_i}{\partial x_i} = 0, \quad (25)$$

$$f_i(0, t + T_i) = h_i^t(T_i), \quad (26)$$

$$k_i = \frac{f_i}{u_i^{max}} + \frac{b_if_i^2}{u_i^{max}}, \quad (27)$$

$$u_i = \frac{f_i}{k_i}, \quad (28)$$

$$\frac{dT_i(x_i, t)}{dx_i} = \frac{1}{u_i}, \quad (29)$$

$$T_i(0, t) = 0, \quad (30)$$

$$T_p(L_p, t) = \sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip}, t))\delta_{ip}. \quad (31)$$

The following theorem provides an existence result for a continuously differentiable solution of the ETT Model as formulated above.

Theorem 7: The ETT Model possesses a solution if and only if the first derivative of the link flow rate function $h_i^t(T_i)$ satisfies the following boundedness condition:

$$\frac{dh_i^t(T_i)}{dT_i} > -\frac{u_i^{max}}{2b_iL_i}(1 - 2b_ih_i^t(T_i))^2. \quad (32)$$

Special Case: Linear ETT Model
As in the analysis of the PTT Model, we assume that during a time period $[t, t + \Delta]$, users make the approximation that the departure link flow rate for subsequent times $t + T_i$ is linear in terms of the travel time T_i. We introduce variables $\theta_1 = \frac{b_iB_i(t)}{1-2b_iA_i(t)}$, $\theta_2 = \frac{b_iB_i(t)}{u_i^{max}}$ and $\theta_3 = \frac{1+b_iA_i(t)}{u_i^{max}} - \frac{1}{u_i^{max}(1-2b_iA_i(t))}$.

Theorem 8: Assume that

$$|B_i(t)| << \frac{u_i^{max}}{2b_iL_i}(1 - 2b_iA_i(t) - 2b_iB_i(t)(t + \Delta))^2. \quad (33)$$

The following holds,
(i) The Linear ETT Model possesses a solution.
(ii) The link travel time functions $T_i(x_i, t)$ are given by

$$T_i(x_i, t) = \theta_3(\frac{e^{\theta_2 x_i} - 1}{\theta_2}) + \frac{\theta_1 x_i}{\theta_2(u_i^{max})^2}. \quad (34)$$

(iv) If condition (33) holds, the link travel time functions $T_i(x_i, t)$ become

$$T_i(x_i, t) = \frac{1}{u_i^{max}}[(1 + A_i(t)b_i)x_i - \frac{A_i(t)B_i(t)(b_i)^2}{2u_i^{max}}x_i^2].$$

Equation (34) gives rise to an exponential family of travel time functions. In the following subsection, we analyze the relationship between the exponential family of travel time functions from this subsection and the one we obtained through the Linear PTT Model and the Quadratic PTT Model.

C. Models Comparison

While the Linear PTT Model leads to the polynomial family of travel time functions in equation (22), the Linear ETT Model leads to the exponential family of travel time functions in equation (34).

The relationship between these two families of travel time functions will become clear after the following observation. Equations (22) and (34) coincide when $|B_i(t)| << \frac{u_i^{max}}{2b_iL_i}$ holds. This condition seems to suggest that the variation of flow with time is small. Then, the travel time function becomes:

$$T_i(x_i, t) = \frac{1}{u_i^{max}}[(1 + A_i(t)b_i)x_i - \frac{A_i(t)B_i(t)(b_i)^2}{2u_i^{max}}x_i^2]. \quad (35)$$

This relationship shows that the assumptions made for both the Linear PTT Model and the Linear ETT Model are indeed reasonable.

V. AN EXTENSION

In this section, we extend the results of the previous section for the first-order separable PTT Model to the second-order non-separable PTT Model. For the sake of simplifying notation, we introduce $J_i = 1 + \sum_{j \in B(i)} \frac{1}{u_i^{max}}(\alpha_{ij}(x_i)k_j(\overline{x}_j, t - \Delta_{ij}) + \frac{\beta_{ij}(x_i)}{k_j(\overline{x}_j, t-\Delta_{ij})}\frac{\partial k_j(\overline{x}_j, t-\Delta_{ij})}{\partial x_j})$.

Therefore, the Non-Separable Second-Order PTT Model becomes:

Non-Separable Polynomial Travel Time Model

For all $t \in [0, T]$, $p \in P$, and $i \in I$, we have:

$$u_i^{max} \frac{1}{(\frac{1}{J_i} + \frac{2b_i f_i}{J_i^3})} \frac{\partial f_i}{\partial x_i} + \frac{\partial f_i}{\partial T_i} = \lambda_i(x_i) \frac{\partial^2 f_i}{\partial x_i^2}, \qquad (36)$$

$$f_i(0, t + T_i) = h_i^t(T_i), \qquad (37)$$

$$k_i = \frac{f_i}{u_i^{max} J_i} + \frac{b_i f_i^2}{u_i^{max} J_i^3}, \qquad (38)$$

$$u_i = \frac{f_i}{k_i}, \qquad (39)$$

$$\frac{dT_i(x_i, t)}{dx_i} = \frac{1}{u_i}, \qquad (40)$$

$$T_i(0, t) = 0, \qquad (41)$$

$$T_p(L_p, t) = \sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip}, t)) \delta_{ip}. \qquad (42)$$

We now consider that during a time period $[t, t + \Delta]$, drivers make the approximation that the link flow rate for subsequent times $t + T_i$ is linear in terms of the travel time T_i. That is, that $f_i(0, t + T_i) = h_i^t(T_i) = A_i(t) + B_i(t)T_i$. We consider the case of linear density correlation functions. That is, for every link $j \in B(i)$, $\alpha_{ij}(x_i) = a_{ij} + b_{ij}x_i$. In addition to the acyclicity assumption we impose on the network, we further assume that the influence of neighboring links has only a first order effect. For every integer n, let $\theta_{in} = n \sum_{j \in B(i)} b_{ij}k_j$ and $\gamma_{in} = u_i^{max} + n \sum_{j \in B(i)} a_{ij}k_j$. The following result provides a linear ordinary differential equation satisfied by link travel time functions T_i for the case of linear density correlation functions.

Theorem 9: If $B_i > -\frac{u_i^{max} J_i^3}{2b_i L_i}$ holds, then:
(i) The Non-Separable Linearized PTT Model possesses a solution,
(ii) The link travel time functions $T_i(x_i, t)$ satisfy

$$\frac{dT_i}{dx_i} - \frac{b_i B_i(t)}{\gamma_{i3} + (\theta_{i3} + 2b_i B_i(t))x_i} T_i$$
$$- \frac{\frac{b_i x_i B_i(t)}{J_i} + b_i A_i(t)u_i^{max} + u_i^{max} J_i^2}{u_i^{max}(u_i^{max} J_i^3 + 2b_i x_i B_i(t))} = 0.$$

If we further consider the case of constant density correlation functions. That is, for every link $j \in B(i)$, $\alpha_{ij}(x_i) = a_{ij}$, then we have the following result:

Theorem 10: [4] If $B_i > -\frac{u_i^{max} J_i^3}{2b_i L_i}$ holds, then:
(i) The Non-Separable Linearized PTT Model possesses a solution,
(ii) The link flow rate functions $f_i(x_i, t + T_i)$ are continuously differentiable, and we have:

$$f_i(x_i, t + T_i) = \frac{B_i(t)u_i^{max}T_i - \frac{B_i(t)x_i}{J_i} + A_i(t)u_i^{max}}{u_i^{max} + \frac{2b_i B_i(t)x_i}{J_i^3}},$$

(iii) The link travel time functions $T_i(x_i, t)$ are given by:

$$T_i(x_i, t) = \frac{x_i}{u_i^{max} J_i} + \frac{A_i(t)}{B_i(t)}((1 + \frac{2b_i B_i(t)x_i}{u_i^{max} J_i^3})^{\frac{1}{2}} - 1).$$

Note that when the density correlation functions are set to zero, we have $J_i = 1$. The results of Theorem 10 then reduce to the results of Theorem 4.

VI. Conclusions

Continuing this work, we intend to investigate the extension of our results in the case of non-separable velocity functions as they apply to non-acyclic networks. We intend to examine other fluid dynamics models. For example, we can consider a different model for relating speed and density. Moreover, we plan to investigate alternate approaches including queuing models. We wish to connect these models with the dynamic user-equilibrium problem. We plan to investigate the solution to this problem and propose algorithms for computing the solution to our models. We also intend to perform a numerical study for realistic networks using the models and the analysis that we already performed in order to show how a numerical solution approach compares to an analytical one.

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