Group Theory Predictions for $B \to M_1 M_2 M_3$

by

Tongyan Lin

Submitted to the Department of Physics
in partial fulfillment of the requirements for the degree of

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Abstract

The study of B meson decays to 3 pseudoscalar mesons MMM provides a promising arena for constraining CP violation from the Standard Model and searching for "new physics". In this thesis we derive decay amplitudes, rates, and CP asymmetries for B mesons decaying to MMM, in the limit of $SU(2)$ isospin and in the limit of $SU(3)$ quark flavor symmetry. Our results are classified according to the relative angular momentum of mesons in the final states. When all the mesons have relative even angular momentum, there are 56 decay channels expressed as linear combinations of 7 reduced matrix elements. There are also 7 reduced matrix elements for the 36 decay channels where all the mesons have relative odd angular momentum. These results imply relations between the decay amplitudes, including several isospin triangles for $B \to MMM$, analogous to the $B \to \pi\pi$ isospin triangle. We also derive sum rules for $B \to MMM$, which give approximate $SU(2)$ relations among branching ratios and CP asymmetries.

Thesis Supervisor: Iain W. Stewart
Title: Associate Professor

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Contents

Chapter 1

Introduction

Symmetries provide an attractive way to describe nature. Examples of important symmetry transforms are parity P, charge conjugation C, and the combined transform CP. The charge conjugation transformation, C, reverses the electric charge and all the internal quantum numbers of a particle, such as baryon number. C thus converts a particle to an antiparticle. Meanwhile, the parity operation, P, is the inversion of all spatial coordinates $\vec{x}$ to $-\vec{x}$. Charge conjugation and parity symmetries are each preserved in classical physics and in strong interactions, but are badly broken by weak interactions. The combined transform CP was historically expected to be a symmetry of weak interactions, but it too is broken by weak interactions. Careful study of the pattern by which these symmetries are broken was important to the construction of the Standard Model. [1]

In the Standard Model, CP symmetry breaking, or CP violation, occurs in electroweak quark flavor changing processes due to the complex phase in the Cabibbo-Kobayashi-Maskawa (CKM) matrix. The CKM matrix is typically written as follows:

$$\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{1.1}$$

where each matrix element V_{xy} gives the relative magnitude and phase of the quark flavor changing process $x \rightarrow y$. The quark names, masses, and charges are in given Table 1.

The CP violation in the Standard Model, however, is already known to be insufficient to explain the matter-antimatter asymmetry in the universe. Furthermore, it is still not clear whether the CP violation in the CKM matrix is enough to account for all the CP violation we can observe in collider experiments. A careful study of experimental constraints on the CKM matrix may reveal other sources of CP violation at higher energies than physicists have been able to probe in the past, which is generically called "new physics". It is possible that the CP violation due to new physics could account for the matter-antimatter asymmetry. Thus an accurate determination of sources of CP violation would have many implications in particle physics and cosmology.

Table 1.1: Quark masses and charges from [2].

Flavor		Mass	Charge (e)
Up	u	1.5-3 MeV	2/3
Down	d	3-7 MeV	-1/3
Strange	s	95 ± 35 MeV	-1/3
Charm	c	$1.25 \pm .09$ GeV	2/3
Bottom	b	$4.20 \pm .07$ GeV	-1/3
Top	t	174.2 ± 3.3 GeV	2/3

One promising area of research is the study of CP violation in B meson decays. The phenomenological study of CP violation in B-decays centers on the unitarity triangle. In order for the CKM matrix to be unitary, there are 6 relations that must hold between elements of the matrix. The unitarity triangle is one particular relation:

$$\begin{aligned} V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0, \qquad \text{or} \\ \frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} + 1 + \frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*} = 0. \end{aligned} \tag{1.2}$$

The angles of the unitarity triangle are defined as:

$$\alpha = \arg\left(\frac{-V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right), \quad \beta = \arg\left(\frac{-V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right), \quad \gamma = \arg\left(\frac{-V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right). \tag{1.3}$$

These angles are called weak phases because the quark flavor changing processes occur via the weak interaction. There are analogous weak phases defined for the other unitarity relations. These phases lead to CP violation; a larger weak phase results in more CP violation. Measurements of CP violation which cannot satisfy unitarity consistently would imply sources of new physics.

Because B decays to mesons typically involve quark processes such as $b \to (\bar{u}d)u$ (with weak phase $V_{ud}V_{ub}^*$) to $b \to (\bar{c}d)c$ (with weak phase $V_{cd}V_{cb}^*$), they are useful for studying the angles of the unitarity triangle. Decays of B to two mesons, such as $B \to \pi\pi$ or $B \to \pi K$, have been well-studied theoretically, and are being analyzed thoroughly by experimental collaborations such as BaBar and Belle [3, 4].

On the other hand, the decay of B to three mesons M, such as $B \to \pi\pi\pi$ and $B \to K\pi\pi$ and related decays, is still a fresh area to explore. With the copious amounts of B decays being observed by experimentalists, predictions for $B \to MMM$ offers additional tests and constraints on the CKM matrix and sources of new physics [5, 6, 7, 8, 9, 10]. The goal of this thesis is to provide predictions between different $B \to MMM$ channels from group theory.

This thesis begins in Chapter 2 with a brief guide to the group theory necessary for understanding the analysis done here. We discuss the observables and the Hamiltonian for the relevant decay processes in Chapter 3. Chapters 4 and 5 are devoted to a detailed analysis of the group theory of calculating matrix elements and symmetrization of the wavefunction. Finally, the predictions for $B \to MMM$ in both $SU(3)$ and $SU(2)$ limits are presented. We give a limited list of decay and amplitude relations, classified according to relative angular momentum of particles in MMM, in Chapter 6. In Chapter 7, we discuss useful relations that can be tested by the data, including isospin triangles analogous to the $B \to \pi\pi$ isospin triangle, as well as four sum rules and their corresponding CP asymmetry sum rules. We conclude in Chapter 8. The full tables of 3-body decay amplitudes and additional amplitude relations are presented in the appendices.

Chapter 2

Symmetry and Group Theory

The usefulness of group theory in studying particles composed of quarks stems from the approximate flavour symmetry of quarks in strong interactions. The strong interaction is blind to quark flavor, though not to quark mass. If all the quarks had the same mass, then there would be an $SU(6)$ symmetry of strong interactions among the quarks.

Though the quarks do not all have the same mass, as shown in Table 1, the difference in masses between several quarks is small compared to the typical scale of strong interactions, $\Lambda \approx 200-500$ MeV. In particular, there is an approximate $SU(2)$ symmetry known as isospin, between the up and down quarks, because $(m_u - m_d)/\Lambda$ is small. The other quarks are treated as 'singlets' or 'invariants' in the $SU(2)$ limit, and therefore regarded as if they had infinite mass. Isospin has only about 2% theoretical error when predictions are compared to observations.

Since $(m_s - m_d)/\Lambda \approx .3$, it is reasonable to try to assume an $SU(3)$ symmetry as well among the up, down, and strange quarks. $SU(3)$ introduces a theoretical error of about 30% in calculations, but can still provide valuable information. In this thesis we assume $SU(3)$ symmetry, but note which predictions are true assuming only isospin, as these predictions are more precise.

2.1 Representations and Tensor Methods

It is now necessary to delineate some basic definitions, methods, and theorems relevant to group theory. Further details can be found in [11]. Our focus will be on representations of $SU(2)$ and $SU(3)$. A representation is a linear mapping T of group elements g to matrices which preserves group multiplication: $T(g_1)T(g_2) = T(g_1 g_2)$. An irreducible representation is a representation that has no invariant subspace; other than the entire vector space, there is no set of vectors that only transforms into itself under multiplication by matrices of the representation. The dimension of the irreducible representation is the dimension of the vector space acted on by the matrices.

Irreducible representations determine how particles transform under symmetry operations. $SU(2)$ is the familiar group from quantum mechanics which describes the rotational symmetry of physical systems. Each irreducible representation of $SU(2)$

is typically labeled by its spin, and we follow the convention in this thesis; the 2 dimensional representation (the Pauli matrices) is labeled by $\frac{1}{2}$, and the spin S representation has dimension $2S+1$. Spin-S particles transform under the irreducible spin S representation.

In angular momentum the two basis states for spin-$\frac{1}{2}$ are typically labeled by projection of spin along the z axis: $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$. In $SU(2)$ isospin, we pick the two eigenstates to be $|u\rangle$ and $|d\rangle$, for the u and d quarks. The antiquarks $\overline{u}$ and $\overline{d}$ also have an $SU(2)$ isospin symmetry, and here the eigenstates are picked by convention to be $-|\overline{d}\rangle$ and $|\overline{u}\rangle$. Taking the direct product of representations (adding angular momenta in $SU(2)$) is typically accomplished using Clebsch-Gordan coefficients, which can be found in tables.

We can use tensor methods to arrive at the same results, since the calculations are far easier. In the tensor analysis, we place the particle states in matrices corresponding to their representations. The quark doublet in $SU(2)$ is:

$$Q^i \equiv \begin{pmatrix} u \\ d \end{pmatrix}, \quad \overline{Q}_i \equiv \begin{pmatrix} \overline{u} & \overline{d} \end{pmatrix} \tag{2.1}$$

where we use an upper index for the quarks and a lower index for the antiquarks. The fundamental representation of $SU(3)$ is the 3. In this thesis $SU(3)$ representations will labeled by their dimension D, and $SU(2)$ representations will be labeled by spin S where the dimension is $2S+1$. The quarks in $SU(3)$ are in a triplet and the antiquarks in an anti-triplet:

$$3 \ : q^i \equiv \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \quad \overline{3} \ : \overline{q}_i \equiv \begin{pmatrix} \overline{u} & \overline{d} & \overline{s} \end{pmatrix} \tag{2.2}$$

Note that while representations of $SU(2)$ are equivalent to their complex conjugate, this is not true in $SU(3)$. In $SU(3)$, the complex conjugate representation of D is denoted by $\overline{D}$.

From $SU(2)$ we know higher dimensional irreducible representations can be obtained by taking tensor products of the $\frac{1}{2}$, as long as some symmetry restrictions are imposed. Decomposition of tensor products into a direct sum of irreducible representations in $SU(2)$ is: $m \otimes n = |m-n| \oplus |m-n|+1 \oplus ... \oplus |m+n|$. In $SU(3)$ higher dimensional representations can be obtained by taking tensor products of the 3 and $\overline{3}$, again with symmetry restrictions. The decomposition of tensor products in $SU(3)$ is done with Young tableaux; this is discussed further in Chapter 5.

Thus an irreducible representation can be written as a tensor with a number of upper and lower indices, each of which refers to the lowest dimensional representation ($\frac{1}{2}$ in $SU(2)$ or 3 and $\overline{3}$ in $SU(3)$), and with symmetry constraints on the indices: $B^{i_1 i_2 \ldots i_n}_{j_1 j_2 \ldots j_m}$. In $SU(3)$, if a representation D has n upper indices and m lower indices, then $\overline{D}$ has m upper indices and n lower indices. Without proof, we now state some results regarding tensors irreducible representations. [11] The first two are necessary for some of the symmetrization done in Chapter 5.

First, an irreducible representation $B^{i_1 i_2 \ldots i_n}_{j_1 j_2 \ldots j_m}$ must be symmetric in its upper indices, symmetric in its lower indices, and must satisfy the traceless requirement: $\delta^{j_k}_{i_\ell} B^{i_1 i_2 \ldots i_n}_{j_1 j_2 \ldots j_m} = 0$, where $k \in 1, 2, \ldots m$ and $\ell \in 1, 2, \ldots n$.

Second, the Levi Civita symbols ϵ_{ijk} and ϵ^{ijk} and the Kronecker delta δ^i_k are invariant tensors in $SU(3)$. In $SU(2)$, δ^i_k, ϵ_{ij} and ϵ^{ij} are invariant. These are invariant because singlets (0-dimensional representations) can be formed by contracting upper and lower indices using ϵ_{ijk} or δ^i_k. For example $A_i B^k \delta^i_k$ and $\epsilon_{ijk} A^i B^j C^k$ are singlets. A singlet does not change under rotations in the flavor symmetry space and is therefore invariant. Note also that these invariant tensors can be used to raise or lower indices.

Finally, the Wigner Eckart Theorem is essential. This theorem is proven in [11] and [12]; here we give the result. Suppose we are given two states that transform under irreducible representations labeled by ℓ and n, respectively, and are labeled by i_ℓ and i_n within the representations. For example, in $SU(2)$, we could have particles with spin angular momentum ℓ labeled by the z component of angular momentum, m_ℓ. The two states may also have other labels which we denote by α and α'.

Next, consider a tensor operator O^k_m. A tensor operator O^k_m is defined to be a set of operators that transforms under commutation with a specific set of operators (the generators of the Lie algebra) like an irreducible representation of dimension k. Specifically, suppose the ket $|\ell, i_\ell\rangle$ transforms as the i_ℓth state in the irreducible representation ℓ. Then $O^k_m |\ell, i_\ell\rangle$ transforms like the direct product of two irreducibles: $|k, m\rangle \otimes |\ell, i_\ell\rangle$.

The Wigner-Eckart Theorem states:

$$\left\langle \ell, i_\ell, \alpha | O^k_m | n, i_n, \alpha' \right\rangle = \left\langle \ell, \alpha || O^k || n, \alpha' \right\rangle \langle \ell, i_\ell | k, m; n, i_n \rangle \tag{2.3}$$

where $\left\langle \ell, \alpha || O^k || n, \alpha' \right\rangle$ is a number, called the reduced matrix element, that is independent of m, i_ℓ and i_n. The number $\langle \ell, i_\ell | k, m; n, i_n \rangle$ is a Clebsch-Gordan coefficient that depends only on the transformation properties of states and operators, and is nonzero *only* when the irreducible $\bar{\ell}$ appears in the tensor product $k \otimes n$. Note that it is necessary for $\bar{\ell}$ to appear and not ℓ because ℓ is the label of the bra state. $\left\langle \ell, i_\ell, \alpha | O^k_m | n, i_n, \alpha' \right\rangle$ must be an $SU(3)$ invariant, or a singlet, and only a representation and its complex conjugate representation can form a singlet. In the tensor language, this means the upper and lower indices must match up in order to be contracted.

The particle states and operators in this thesis transform as irreducibles. The Wigner-Eckart theorem implies the matrix elements depend only a few reduced matrix elements which we will enumerate in Chapter 4. The group theory calculation then gives the relative dependence of a decay channel on each reduced matrix element.

Chapter 3

Observables and the Electroweak Hamiltonian

For the decays $B \to M_1 M_2 M_3$ we are concerned with the B, π, η, and K mesons, which are spin-0 (pseudoscalar) bosons composed of one quark and one anti-quark. The particle convention adopted in this thesis is:

$$\begin{array}{ccc} B^- = b\overline{u} & \overline{B}^0 = -b\overline{d} & \overline{B}^0_s = b\overline{s} \\ B^+ = \overline{b}u & B^0 = \overline{b}d & B^0_s = \overline{b}s \\ \pi^- = \overline{u}d & \pi^0 = \frac{(u\overline{u}-d\overline{d})}{\sqrt{2}} & \pi^+ = -\overline{d}u \\ K^- = s\overline{u} & \overline{K}^0 = -s\overline{d} & \\ K^+ = \overline{s}u & K^0 = \overline{s}d & \\ & \eta_8 = (\overline{u}u + \overline{d}d - 2\overline{s}s)/\sqrt{6} & \end{array} \tag{3.1}$$

The negative signs in Eq. 3.1 are included such that particle states correspond directly to isospin eigenstates in the standard sign convention of Clebsch-Gordan tables. In addition, the η_8 above is part of of the meson octet (see Eq. 3.2 below); the physical η is a mixture of η_8 and the singlet $\eta_1 = (\overline{u}u + \overline{d}d + \overline{s}s)/\sqrt{3}$. The physical η can be written as $\cos\phi\ \eta_8 + \sin\phi\ \eta_1$, with the mixing angle $\phi \approx 10 - 20°$. In this thesis, we limit our analysis to η_8. In $SU(2)$, both η and η_8 are singlets, and thus the non-negligible mixing angle does not alter the group theory predictions. However, any $SU(3)$ results derived involving η_8 have an uncertainty due to $\eta_1 - \eta_8$ mixing associated with them.

Particles which are combinations of quarks and anti-quarks can form a tensor with one upper index for the u, d, s quarks and one lower index for the $\overline{u}, \overline{d}, \overline{s}$ anti-quarks in $SU(3)$. The b quark is a singlet and has no index. The $SU(3)$ meson octet is the 8-dimensional irreducible representation formed by taking the tensor product $3 \otimes \overline{3}$. We write the states in the matrix M^i_j as:

$$M \equiv \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & -\pi^+ & K^+ \\ \pi^- & \frac{-\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & -\overline{K}^0 & -\sqrt{\frac{2}{3}}\eta \end{pmatrix} \tag{3.2}$$

Again note the negative signs in front of π^+ and $\overline{K}^0$ in keeping with the particle convention of this thesis. Each element of the matrix M^i_k is a sum of mesons M weighted by $\langle q^i\overline{q}_k|M\rangle$. Thus Eq. 3.2 is a convenient way of keeping track of the ket states [11]:

$$\sum_{i,k} M^i_k \left|q^i\overline{q}_k\right\rangle = \pi^0\left|\pi^0\right\rangle + \eta\left|\eta\right\rangle + K^+\left|K^+\right\rangle + \ K^0\left|K^0\right\rangle ... \tag{3.3}$$

We can think of the mesons in Eq. 3.2 as creation operators (on ket states) or annihilation operators (on bra states). Taking the conjugate of the above equation gives:

$$\sum_{i,k} \left\langle q^i\overline{q}_k\right| (M^*)^i_k = \left\langle\pi^0\right|\pi^0 + \left\langle\eta\right|\eta + \left\langle K^+\right|K^- + \left\langle K^0\right|\overline{K}^0 ... \tag{3.4}$$

Hence M can also be used to keep track of the bra states, by regarding the mesons in M as creation operators acting on the bra state of the *conjugate* particle. We define a tensor $\overline{M}$ to keep track of the bra states, defined by $\overline{M}^k_i \equiv (M^*)^i_k$. Note that in this case $\overline{M}^k_i = M^k_i$.

In $SU(3)$ the B mesons form an anti-triplet, or a triplet if one looks at the conjugate particles:

$$\overline{B} \equiv \begin{pmatrix} B^- & -\overline{B}^0 & \overline{B}^0_s \end{pmatrix} \qquad B \equiv \begin{pmatrix} B^+ \\ B^0 \\ B^0_s \end{pmatrix} \tag{3.5}$$

B has an upper index and $\overline{B}$ has a lower index. In practice, we will label both B and $\overline{B}$ by B and just use the index to indicate whether we mean a triplet or an anti-triplet.

In $SU(2)$ there are a number of triplets and doublets:

$$\pi \equiv \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} & -\pi^+ \\ \pi^- & \frac{-\pi^0}{\sqrt{2}} \end{pmatrix} \quad \mathbf{K} \equiv \begin{pmatrix} K^+ \\ K^0 \end{pmatrix} \quad \overline{\mathbf{K}} \equiv \begin{pmatrix} K^- & -\overline{K}^0 \end{pmatrix}$$
$$\overline{\mathbf{B}} \equiv \begin{pmatrix} B^- & -\overline{B}^0 \end{pmatrix} \quad \mathbf{B} \equiv \begin{pmatrix} B^+ \\ B^0 \end{pmatrix} \tag{3.6}$$

The kaons form doublets here because the s quark is a singlet in $SU(2)$.

While the results derived in this paper are strictly all $\overline{B} \to MMM$, which is the decay of a b quark, all results can be directly applied to the CP conjugate process $B \to MMM$ for the decay of the $\overline{b}$ quark. The only difference is that weak phases in decay amplitudes must be conjugated. We work through an explicit example in Chapter 7.

3.1 Observables

Using the Wigner-Eckart Theorem, we will be calculating the decay amplitudes $\mathcal{A} = \langle MMM|H|B\rangle$ which is dimensionless. $|\mathcal{A}|^2$ is proportional to the (unnormalized) decay rates $\Gamma(B \to MMM)$. The CP conjugate process of $\mathcal{A}$ will be denoted by $\overline{\mathcal{A}}$.

The branching ratio is the time-independent observable quoted most frequently by experimentalists and is defined as

$$\mathrm{Br} \equiv \frac{1}{\Gamma_B} \int dm_{12}^2 dm_{23}^2 \frac{1}{(2\pi)^3} \frac{1}{32 m_B^3} \left(\frac{|\mathcal{A}|^2 + |\overline{\mathcal{A}}|^2}{2} \right) \tag{3.7}$$

for three-body decays to pseudoscalars [2]. Here $m_{ij}^2 = (p_i + p_j)^2$, and p_i is the 4-momentum of the ith particle. The integral over m_{12}^2 and m_{23}^2 is equivalent to the integral is over all possible momenta of the 3 final mesons, given that the B is initially at rest and the initial energy of the system is m_B (in units where $c = 1$). The numerical constants under the integral, including the m_B, are phase space factors. Meanwhile, Γ_B is the total decay rate of the specific B particle which is decaying, for example B^-. Note that Γ_B is equal to $1/\tau_B$ in the equation above, where τ_B is the lifetime of the particle, and is not the specific decay rate $\Gamma(B \to MMM) \propto |\mathcal{A}(B \to MMM)|^2$. Also $\Gamma_{B^-} = \Gamma_{B^+}$ and $\Gamma_{B^0} \approx \Gamma_{\overline{B}^0}$.

The other parameter we will be concerned with is the CP asymmetry, defined as

$$A_{\mathrm{CP}} \equiv \frac{|\mathcal{A}|^2 - |\overline{\mathcal{A}}|^2}{|\mathcal{A}|^2 + |\overline{\mathcal{A}}|^2}. \tag{3.8}$$

Clearly if there is no CP violation A_{CP} will be 0.

3.2 CKM Matrix

Matrix elements of decays are weighted by elements of the CKM matrix, so the sizes of these numbers can suppress or enhance various reduced matrix elements, depending on the decay. This information is useful in making approximations for the values of decay amplitudes.

The Wolfenstein Parameterization of the CKM matrix is [13]:

$$\begin{pmatrix} 1 - \frac{1}{2}\lambda^2 - \frac{1}{8}\lambda^4 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda + \frac{1}{2}A^2\lambda^5[1 - 2(\rho + i\eta)] & 1 - \frac{1}{2}\lambda^2 - \frac{1}{8}\lambda^4(1 + 4A^2) & A\lambda^2 \\ A\lambda^3[1 - (1 - \frac{1}{2}\lambda^2)(\rho + i\eta)] & -A\lambda^2 + \frac{1}{2}A\lambda^4[1 - 2(\rho + i\eta)] & 1 - \frac{1}{2}A^2\lambda^4 \end{pmatrix} \tag{3.9}$$

The parameters A, ρ, η are of order unity. $|\lambda| \approx .2$ and terms of $O(\lambda^6)$ were ignored. The following representation of the CKM matrix gives only the order of magnitude of the elements and will be used below to justify the smallness or largeness of various terms:

$$\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \approx \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}. \tag{3.10}$$

3.3 Electroweak Hamiltonian

The electroweak Hamiltonian describing $\Delta b = 1$ transitions $b \to f$, where f is d or s, is given by

$$H_W = \frac{G_F}{\sqrt{2}} \sum_{p=u,c} V_{pb} V_{pf}^* \Big(C_1 O_1^p + C_2 O_2^p + \sum_{i=3}^{10,7\gamma,8g} C_i O_i \Big), \tag{3.11}$$

with G_F, C_i constants and O are operators [14]. The standard basis of operators are:

$$\begin{aligned}
O_1^u &= (\overline{u}b)_{V-A}(\overline{f}u)_{V-A}, & O_2^u &= (\overline{u}_\beta b_\alpha)_{V-A}(\overline{f}_\alpha u_\beta)_{V-A},\\
O_1^c &= (\overline{c}b)_{V-A}(\overline{f}c)_{V-A}, & O_2^c &= (\overline{c}_\beta b_\alpha)_{V-A}(\overline{f}_\alpha c_\beta)_{V-A},\\
O_3 &= (\overline{f}b)_{V-A}(\overline{q}q)_{V-A}, & O_4 &= (\overline{f}_\beta b_\alpha)_{V-A}(\overline{q}_\alpha q_\beta)_{V-A},\\
O_5 &= (\overline{f}b)_{V-A}(\overline{q}q)_{V+A}, & O_6 &= (\overline{f}_\beta b_\alpha)_{V-A}(\overline{q}_\alpha q_\beta)_{V+A},\\
O_7 &= \frac{3e_q}{2}(\overline{f}b)_{V-A}(\overline{q}q)_{V+A}, & O_8 &= \frac{3e_q}{2}(\overline{f}_\beta b_\alpha)_{V-A}(\overline{q}_\alpha q_\beta)_{V+A},\\
O_9 &= \frac{3e_q}{2}(\overline{f}b)_{V-A}(\overline{q}q)_{V-A}, & O_{10} &= \frac{3e_q}{2}(\overline{f}_\beta b_\alpha)_{V-A}(\overline{q}_\alpha q_\beta)_{V-A},
\end{aligned}$$
$$O_{7\gamma,8g} = -\frac{m_b}{8\pi^2}\, \overline{f}\, \sigma^{\mu\nu}\{eF_{\mu\nu}, gG^a_{\mu\nu}T^a\}(1+\gamma_5)b\,. \tag{3.12}$$

The sum over $q = u, d, s, c, b$ is implicit in $(\overline{q}q)$. The labels α, β are color indices and the e_q label is electric charge. The ${}_{V-A}$ and ${}_{V+A}$ subscripts on the operators denote spin structure. The $\Delta s = 0$ (strangeness preserving) and $\Delta s = 1$ (strangeness changing) Hamiltonians are obtained by respectively setting $f = d$ and $f = s$ in Eqs. 3.11 and 3.12.

We are only concerned with analyzing the flavor structure; the other parts of the Hamiltonian will all be absorbed into the reduced matrix elements. We also make the further simplification of ignoring the $(\overline{q}q)$ operators, since these are singlets in the group theory. The charm quark c is also a singlet. Thus for our purposes the operator basis that contains only the non-singlet flavor operators can be written as:

$$\begin{aligned}
O_{1,2}^u &\to (\overline{u}b)(\overline{f}u) & &\text{for } b \to uf\overline{u}\\
O_{1,2}^c &\to (\overline{f}b) & &\text{for } b \to f\\
O_{3,4,5,6} &\to (\overline{f}b) & &\text{for } b \to f\\
O_{7,8,9,10} &\to \frac{3e_q}{2}(\overline{f}b)(\overline{q}q) & &\text{for } b \to f(\frac{3e_q}{2}\overline{q}q).
\end{aligned} \tag{3.13}$$

The operators in the left column of Eq. 3.13 correspond to the processes listed in the right column. The final state quarks appearing in each process are the conjugate of the quarks appearing in the operator because these operators are annihilation and creation operators on the quarks. Hence b needs to act on the initial state and the other quarks need to act on the final state (they act on the ket if b acts on the bra). The $(\frac{3e_q}{2}\overline{q}q)$ operator can be written as $\frac{3}{2}\overline{u}u + \frac{3}{2}\overline{c}c - \frac{1}{2}(\overline{q}q)$ and is thus composed of a piece that transforms like the $O_{3,4,5,6}$ operators and a piece that transforms like $O^u_{1,2}$.

In the group theory, the operators in Eq. 3.13 are the only ones contributing to $B \to MMM$. The operator $O^c_{1,2}$ contributes even though there are no charm quarks in the octet mesons because the $c\bar{c}$ pair can annihilate to a gluon, producing a $q\bar{q}$. All of these operators can be combined with an arbitrary number of singlets $\bar{q}q$, produced by flavor singlet gluons. For the decay $B \to MMM$, it is necessary to include two extra singlets for the $(\bar{f}b)$ operator and one extra singlet for every other operator in Eq. 3.13. For a group theory analysis these singlets only affect the value of the reduced matrix elements.

The operators $O^u_{1,2}$ are called the tree operators and the operators $O^c_{1,2}$ and $O_{3,4,5,6}$ are called penguin operators. The $O_{7,8,9,10}$ operators are typically called the electroweak penguin operators. Fig. 3-1 shows some possible quark diagrams corresponding to these operators. There are many possible quark diagrams contributing to a decay $B \to MMM$, such as as the annihilation diagrams also shown in Fig. 3-1. However, the operators in Eq. 3.13 and the Hamiltonian, Eq. 3.11, give all the information about the decays as far as the flavor structure is concerned, since the group theory analysis cannot distinguish between two operators which transform in the same way. The annihilation diagrams in Fig. 3-1 (c) and (d), for example, correspond to operators that transform in the same way as $O_{3,4,5,6}$.

The Hamiltonian, Eq. 3.11, gives the relevant CKM factor for each operator. To see how this comes about, consider the penguin operators $O_{3,4,5,6}$ and $O^c_{1,2}$ which correspond to $b \to f$. These operators transform as 3s in $SU(3)$ because b is a singlet. The isospin of the operators is $\Delta I = 0$ in $\Delta s = 1$ processes and is $\Delta I = 1/2$ in $\Delta s = 0$ processes. However, $O_{3,4,5,6}$ has a CKM factor of $V_{ub}V^*_{uf} + V_{cb}V^*_{cf}$ while $O^c_{1,2}$ has a CKM factor of $V_{cb}V^*_{cf}$ according to Eq. 3.11. Now using unitarity, we can write:

$$V_{ub}V^*_{uf} + V_{cb}V^*_{cf} + V_{tb}V^*_{tf} = 0. \qquad (3.14)$$

Though $O_{3,4,5,6}$ in reality has a CKM factor of $V_{tb}V^*_{tf}$, it can be written with a factor of $V_{ub}V^*_{uf} + V_{cb}V^*_{cf}$ instead. Both $V_{cb}V^*_{cf}$ and $V_{tb}V^*_{tf}$ are $\sim \lambda^3$ for $f = d$ and $\sim \lambda^2$ for $f = s$ from Eq. 3.10. However, $V_{ub}V^*_{uf}$ is $\sim \lambda^3$ for $f = d$ and $\sim \lambda^4$ for $f = s$.

The tree operator $O^u_{1,2}$ corresponds to $b \to \bar{u}f\bar{u}$ where f is d or s. An example is shown in Fig. 3-1(b). These operators are the tensor product of three non-singlet quark operators and transform as $3 \oplus \bar{6} \oplus 15$ in $SU(3)$. The electroweak penguin operators $O_{7,8,9,10}$ also transforms as $3 \oplus \bar{6} \oplus 15$ in $SU(3)$ because $(\frac{3e_q}{2}\bar{q}q) = \frac{3}{2}\bar{u}u + \frac{3}{2}\bar{c}c - \frac{1}{2}(\bar{q}q)$. While $O_{7,8,9,10}$ has a piece that transforms like the penguin operator, this piece can be combined with the 3 in $3 \oplus \bar{6} \oplus 15$. The isospin of the operators is $\Delta I = 1$ in $\Delta s = 1$ processes and $\Delta I = \frac{3}{2}$ in $\Delta s = 0$ decays. However the CKM factor for $O_{7,8,9,10}$ is $V_{tb}V^*_{tf}$ while the CKM factor for $O^u_{1,2}$ is $V_{ub}V^*_{uf}$.

To summarize, the $\Delta s = 0$ decays have contributions from processes with $V_{ub}V^*_{ud} \sim \lambda^3$ and $V_{cb}V^*_{cd} \sim \lambda^3$. (We always will remove CKM factors involving t using Eq. 3.14.) The relative weak phase of these two CKM factors is γ in Eq. 1.3. Meanwhile the $\Delta s = 1$ decays have contributions from processes with $V_{ub}V^*_{us} \sim \lambda^4$ and $V_{cb}V^*_{cs} \sim \lambda^2$. As a result, for the $\Delta s = 1$ Hamiltonian, contributions with CKM factor $V_{ub}V^*_{us}$ are suppressed by $O(\lambda^2)$ relative to contributions with CKM factor of $V_{cb}V^*_{cs}$. Even though

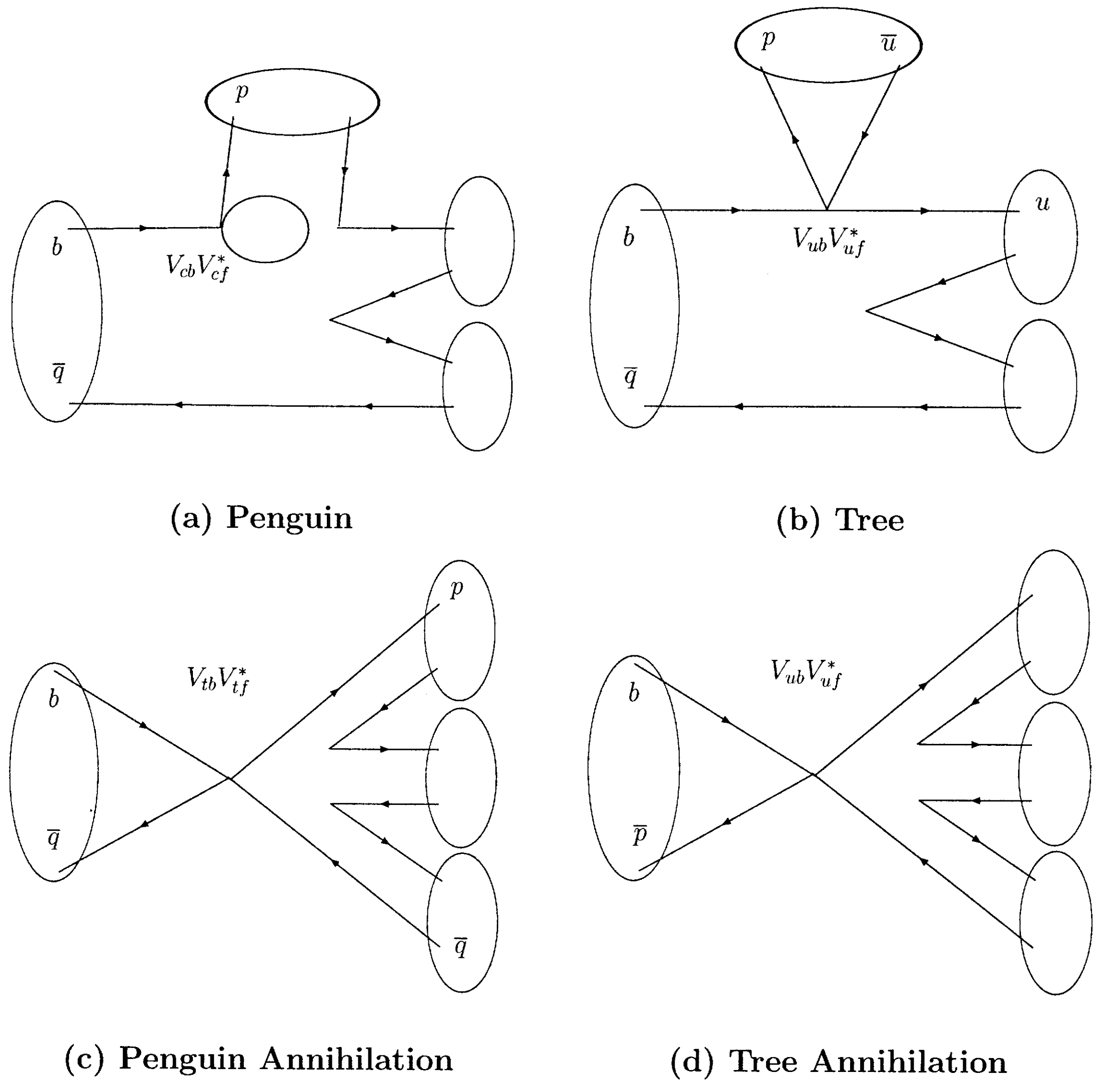


Figure 3-1: Some of the quark diagrams corresponding to operators in Eq. 3.13. There are a number of possible quark diagrams for a given operator, and only a few are shown here. However they do not affect the group theory. Possible pairings of the quarks into the final state mesons are represented by the ellipses. Each diagram is associated with a CKM matrix factor, labeled in the figure. Antiquarks are arrows pointing away from mesons. The unlabeled quark-antiquark pairs are all singlets ($\overline{q}q$), or penguins. The p means either d or s, depending on the Hamiltonian.

these coefficients are suppressed we cannot drop these terms since they are important for interference in CP violation. Finally, the CP conjugate of a process conjugates the CKM factors but not the coefficients multiplying the CKM factors. If a decay has contributions from processes with a relative weak phase, the CP conjugate decay will have a different amplitude, resulting in CP violation. The size of the CP violation is quantified by Eq. 3.8.

3.4 Components of the Hamiltonian

There are two types of operators as far as group theory is concerned. The tree operators and electroweak operators corresponding to $b \to \overline{u} f u$ are $3 \otimes (\overline{3} \otimes 3) = 3 \otimes (1 \oplus 8) = 3 \oplus 3 \oplus \overline{6} \oplus 15$, as we will show below. Next, the penguin operators corresponding to $b \to f\overline{q}q$ are simply 3s.

Therefore the Hamiltonian has three parts, transforming as a 3, $\overline{6}$, and 15, respectively. We write the parts of the Hamiltonian as $H(3)^i, H(\overline{6})^{ij}_k$ and $H(15)^{ij}_k$ [11]. The number in the parentheses indicates how that part of the Hamiltonian transforms. Note that it is necessary to give the tensor $H(\overline{6})^{ij}_k$ two upper indices and one lower index for the index contraction to work in Eq. 4.1 below. A tensor which transforms as a $\overline{6}$ typically has 2 lower indices. However, a $\overline{3}$ can be formed from an antisymmetric combination of two 3s using the ϵ_{ijk} tensor. As a result $H(\overline{6})^{ij}_k$ will transform as a $\overline{6}$ if we impose antisymmetry on the upper two indices.

In general, the components of the tensors $H(3)^i, H(\overline{6})^{ij}_k$ and $H(15)^{ij}_k$ are linear combinations of the physical constants of the electroweak Hamiltonian in Eq. 3.11, weighted by CKM matrix elements. Hence the H components can be written as $V_{ub}V^*_{uf}X + V_{cb}V^*_{cf}Y$, where $f = d$ or s and X and Y are constants. The components of $H(3)$ will have contributions from tree and penguin operators, and include CKM factors of $V_{ub}V^*_{uf}$ and $V_{cb}V^*_{cf}$. Meanwhile the components of $H(\overline{6})$ and $H(15)$ will have contributions only from tree and electroweak penguin operators, with CKM factors of $V_{ub}V^*_{uf}$ and $V_{tb}V^*_{tf}$. Since the contribution from electroweak penguin operators is expected to be small, the effect of adding $V_{tb}V^*_{tf}$ terms to $H(\overline{6})$ and $H(15)$ should also be small. Nevertheless, we keep them in order to make our analysis complete.

In the group theory analysis, however, only the symmetry properties of the irreducible representations matter. Thus the amplitude relations we derive are independent of the components used. We will only impose the restriction that an irreducible representation $B^{i_1 i_2 \ldots i_n}_{j_1 j_2 \ldots j_m}$ must be symmetric in its upper indices, symmetric in its lower indices, and be traceless. Hence $H(15)^{ij}_k$ is symmetric in its upper two indices. We will set the components of the Hamiltonian to be numbers, and separate our calculations according to $\Delta s = 0$, for $f = d$, or $\Delta s = 1$, for $f = s$. Depending on Δs, only certain components of the Hamiltonian will be nonzero. This choice of components simplifies calculation. The weak phases and other constants from Eq. 3.11 will all be absorbed into the reduced matrix elements of the Wigner-Eckart Theorem.

We use the same components as Savage and Wise, who calculated $B \to MM$ group theory relations in [15]. The components can be derived by writing out the tensor product of $3 \otimes 3 \otimes \overline{3}$ with indices. We begin with $3 \otimes \overline{3}$:

$$3 \otimes \overline{3} = 8 \oplus 1 : q^j \overline{q}_k = (q^j \overline{q}_k - \frac{1}{3}\delta^j_k q^l \overline{q}_l) + \frac{1}{3}\delta^j_k q^l \overline{q}_l \tag{3.15}$$

with an implied sum over repeated indices. The term in parentheses is an 8 and the other term is a 1. Call the 8 v^j_k. Then $3 \otimes 8$ is:

$$\begin{aligned} w^i v^j_k &= \frac{1}{2}(w^i v^j_k + w^j v^i_k - \frac{1}{4}\delta^i_k w^l v^j_l - \frac{1}{4}\delta^j_k w^l v^i_l) \\ &+ \frac{1}{4}\epsilon^{ijl}(\epsilon_{lmn} w^m v^n_k + \epsilon_{kmn} w^m v^n_l) + (\frac{3}{8}\delta^i_k w^l v^j_l - \frac{1}{8}\delta^j_i w^l v^i_l) \end{aligned} \tag{3.16}$$

where each term in parentheses transforms as a 15, $\overline{6}$ and 3 respectively. Given an operator such as $ud\overline{d}$ we know which components of $q^j, \overline{q}_k$, and w^i are nonzero: for example q^1 for u, $\overline{q}_2$ for $\overline{d}$ and w^2 for d. A convenient convention is to then set those components of $q^j, \overline{q}_k$, and w^i to be 1 because of our freedom in defining the reduced matrix elements. From this one obtains the components of the various parts of the Hamiltonian. The nonzero components used in the computation for the $\Delta s = 0$ Hamiltonian are:

$$\begin{aligned} &H(3)^2 = 1 \\ &H(\overline{6})^{12}_1 = H(\overline{6})^{23}_2 = -H(\overline{6})^{32}_2 = -H(\overline{6})^{21}_1 = 1 \\ &H(15)^{12}_1 = H(15)^{21}_1 = 3 \\ &H(15)^{22}_2 = -2 \\ &H(15)^{32}_3 = H(15)^{23}_3 = -1. \end{aligned} \tag{3.17}$$

The nonzero components used in the computation for the $\Delta s = 1$ Hamiltonian are:

$$\begin{aligned} &H(3)^3 = 1 \\ &H(\overline{6})^{13}_1 = H(\overline{6})^{32}_3 = -H(\overline{6})^{23}_3 = -H(\overline{6})^{31}_1 = 1 \\ &H(15)^{13}_1 = H(15)^{31}_1 = 3 \\ &H(15)^{33}_3 = -2 \\ &H(15)^{23}_2 = H(15)^{32}_2 = -1. \end{aligned} \tag{3.18}$$

Note that the weak phase factors for the $\Delta s = 1$ reduced matrix elements are different from that of the $\Delta s = 0$ reduced matrix elements.

Chapter 4

Effective Hamiltonian and Matrix Elements in $SU(3)$

The electroweak Hamiltonian H describes decays of B to three mesons, MMM. Therefore $\langle M_1 M_2 M_3 | H | B \rangle$ gives the decay amplitude for $B \to M_1 M_2 M_3$. To fully understand all the group theory involved in computing $\langle M_1 M_2 M_3 | H | B \rangle$, it is necessary to delve into details of tensor product decomposition and symmetrization. In this chapter, we focus on the physics; the details of the group theory are presented in the next chapter.

The procedure for computing $\langle M_1 M_2 M_3 | H | B \rangle$ is to construct $SU(3)$ invariants by contracting indices of the meson octets with the index of the B meson and the indices of the various parts of the Hamiltonian: $H(3)^i$, $H(\overline{6})^{ij}_k$ and $H(15)^{ij}_k$. The contraction of indices can be pictured as diagrams shown in Fig. 3-1. A contraction between indices of two mesons such as $M^i_k M^k_j$ indicates that a quark-antiquark singlet contributes a quark to one meson and an antiquark to the other. A contraction such as $B^i M^j_i$ indicates that the anti-quark in the original B meson contributes to the formation of the meson M^j_i. Because the indices of the mesons B and M are fixed, the Hamiltonian must have an extra upper index compared to the number of lower indices. Hence we chose the $\overline{6}$ part of the Hamiltonian as $H(\overline{6})^{ij}_k$, where $H(\overline{6})$ is antisymmetric in its two upper indices.

From the tensor analysis, we derive an effective Hamiltonian for $B \to M_1 M_2 M_3$ in $SU(3)$ as follows. To find an $SU(3)$ invariant, we find all the unique ways of contracting indices:

$$\begin{aligned} H_{eff} = {} & a_3 B_i H(3)^i (M^k_\ell M^\ell_j M^j_k) + b_3 B_i M^i_k H(3)^k (M^\ell_m M^m_\ell) + c_3 B_i M^i_j M^j_\ell M^\ell_k H(3)^k \\ & + d_6 B_i H(\overline{6})^{ij}_k M^k_j (M^\ell_m M^m_\ell) + e_6 B_i M^i_m M^m_\ell H(\overline{6})^{\ell k}_j M^j_k \\ & + f_6 B_i M^i_\ell H(\overline{6})^{\ell k}_j M^j_m M^m_k + g_6 B_i H(\overline{6})^{ij}_k M^\ell_j M^m_\ell M^k_m \\ & + h_{15} B_i H(15)^{ij}_k M^k_j (M^\ell_m M^m_\ell) + i_{15} B_i M^i_m M^m_\ell H(15)^{\ell k}_j M^j_k \\ & + j_{15} B_i M^i_\ell H(15)^{\ell k}_j M^j_m M^m_k + k_{15} B_i H(15)^{ij}_k M^\ell_j M^m_\ell M^k_m \end{aligned} \tag{4.1}$$

where there is an implied sum over all repeated indices. The constants $a_3, b_3, ..., k_{15}$ of the effective Hamiltonian are the reduced matrix elements of the Wigner-Eckart

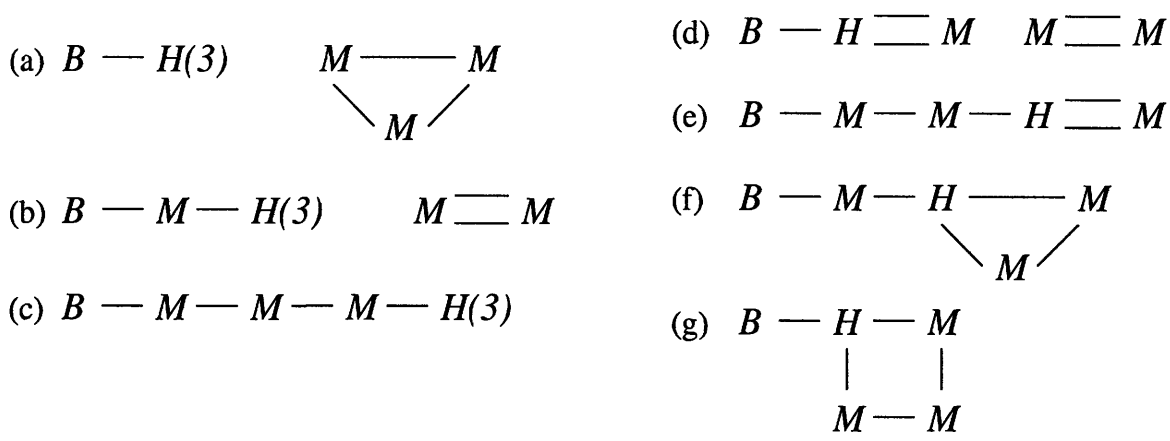


Figure 4-1: The effective Hamiltonian of Eq. 4.1. Every line represents an index contraction. Hence there is one line extending from every B and $H(3)$ and there are two lines from every M. There are three lines from every H, which could be $H(\overline{6})$ or $H(15)$. Diagrams (a), (b), and (c) correspond to the terms multiplying a_3, b_3 and c_3 respectively. Diagrams (d), (e), (f), and (g) correspond to the terms multiplying d_6, e_6, f_6 and g_6 respectively when H is replaced with $H(\overline{6})$. When H is replaced with $H(15)$, diagrams (d)-(g) correspond to the terms multiplying h_{15}, i_{15}, j_{15} and k_{15}, respectively.

Theorem. Their subscripts denote the representation of the Hamiltonian in each term. The 11 terms in the effective Hamiltonian implies that there are 11 singlets in the tensor decomposition of

$$\begin{aligned}&(8 \otimes 8 \otimes 8 \otimes 3 \otimes \overline{3})\\ &\oplus (8 \otimes 8 \otimes 8 \otimes \overline{6} \otimes \overline{3})\\ &\oplus (8 \otimes 8 \otimes 8 \otimes 15 \otimes \overline{3}).\end{aligned} \tag{4.2}$$

The terms in the Eq. 4.1 can also be seen diagrammatically in Figure 4-1. In the figure each index contraction is represented by a line connecting two tensors, and the number of lines from a tensor is the number of indices it has. Each part of the figure is a term in the Hamiltonian. The first three match the terms with $H(3)$. The last four match both the terms with $H(\overline{6})$ and those with $H(15)$ since $H(\overline{6})$ and $H(15)$ have the same number of upper and lower indices. Because a tensor is traceless no line extending from a tensor loops back to itself.

In computing $\langle M_1 M_2 M_3 | H | B \rangle$ the effective Hamiltonian should in general have $\overline{M}$ instead of M, where $\overline{M}$ is defined using Eq. 3.4. Since $\overline{M} = M$ we can just use M here. This gives the effective Hamiltonian as a sum of creation or annihilation operators. However, because the creation operator for $\langle M|$ is the annihilation operator for $|M\rangle$, all mesons M appearing in the effective Hamiltonian must be conjugated. (Practically, this can be carried out by transposing all the M matrices in Eq. 4.1.) Thus we obtain the coefficient to every $BMMM$ term in the effective Hamiltonian as the relative contribution to the decay $B \to MMM$.

4.1 Including Angular Momentum and Symmetrizing the Wavefunction

The expression for the effective Hamiltonian in Eq. 4.1 has not yet been subject to the constraints of angular momentum and symmetrization. In this section we discuss the symmetrization of the wavefunction. Further discussion of symmetrization using Young tableaux may be found in Chapter 5.

The mesons in the $SU(3)$ flavor symmetry limit are identical particles labeled by spin and flavor. The pseudoscalar mesons in the meson octet are spin 0 and must obey Bose statistics. The total wavefunction of a meson has the form:

$$\Psi = \psi_{\text{space}}\psi_{\text{flavor}} \tag{4.3}$$

where we have dropped the $\psi_{\text{color}}\psi_{\text{spin}}$ part of the wavefunction because the color and spin wavefunctions of a pseudoscalar meson are flavor singlets and identical for all mesons. The color and spin part of the wavefunction is already symmetric under interchange of two mesons. The wavefunction in Eq. 4.3 must be symmetrized with respect to every two mesons in the final state. Furthermore, the reduced matrix elements depend on the angular momentum states. In this section we discuss five different cases of relative angular momentum between the mesons, and evaluate the number of reduced matrix elements for each case.

If the relative angular momentum of two mesons is L, the symmetry under interchange of spatial wavefunctions is $(-1)^L$. An even L implies the flavor part of the wavefunction must be symmetric under exchange of flavor labels of those two mesons. Meanwhile L odd implies the flavor part must be antisymmetric. It is also possible for two mesons to have no symmetry under exchange in the spatial part but only be symmetric under the combined exchange of both spatial and flavor parts. The different possibilities for angular momentum in the final states are given in Table 4.1. Note it is not possible for two of the particles to have relative odd angular momentum, and another two to have relative even angular momentum. Suppose $\psi_1\psi_2 = -\psi_2\psi_1$ and $\psi_1\psi_3 = \psi_3\psi_1$. Then

$$\psi_1\psi_2\psi_3 = -\psi_2\psi_1\psi_3 = -\psi_2\psi_3\psi_1 = \psi_1\psi_3\psi_2 = \psi_3\psi_1\psi_2 = -\psi_3\psi_2\psi_1, \tag{4.4}$$

but we assumed $\psi_1\psi_2\psi_3 = \psi_3\psi_2\psi_1$. Hence this symmetry is not possible. Nor is it possible for two pairs of the particles to have relative odd (or even) angular momentum, but for the last pair to have no symmetry. Suppose $\psi_1\psi_2 = \pm\psi_2\psi_1$ and $\psi_1\psi_3 = \pm\psi_3\psi_1$. Then

$$\psi_1\psi_2\psi_3 = \pm\psi_2\psi_1\psi_3 = \psi_2\psi_3\psi_1 = \pm\psi_1\psi_3\psi_2 \tag{4.5}$$

which implies $\psi_2\psi_3 = \pm\psi_3\psi_2$. Then all the particles must have relative odd (or even) angular momentum. This explains why Table 4.1 has only 5 rows.

Now we impose these restrictions on the effective Hamiltonian. We must symmetrize $|M_1\psi_1\rangle\,|M_2\psi_2\rangle\,|M_3\psi_3\rangle$ where M_i and ψ_i refer to specific flavor and spatial

Table 4.1: Different cases for relative angular momentum in MMM. The $\psi_i \leftrightarrow \psi_j$ columns give symmetry under exchange of spatial wavefunctions of M_i and M_j. The / means the two mesons are not in relative angular momentum eigenstate, so that there is no symmetry property. In rows 2 and 4, which have a + or − in one column and a / in the other two columns, exactly two of the mesons are in a relative angular momentum eigenstate. These two mesons are labeled by M_1 and M_2 in this table; however they could also be labeled M_2 and M_3 or M_1 and M_3. The column $|\Psi\rangle$ indicates where the explicit wavefunction may be found. The last column of the table gives the number of reduced matrix elements.

$\psi_1 \leftrightarrow \psi_2$	$\psi_1 \leftrightarrow \psi_3$	$\psi_2 \leftrightarrow \psi_3$	$\|\Psi\rangle$	Reduced Matrix Elements
/	/	/	Eq. 4.6	≤ 53
+	/	/	Eq. 4.10	≤ 28
+	+	+	Eq. 4.12	7
−	/	/	Eq. 4.10	≤ 25
−	−	−	Eq. 4.12	7

wavefunctions. In general the symmetrized wavefunction is:

$$\begin{aligned}|\Psi\rangle =& \frac{1}{\sqrt{6}}\big(|M_1\psi_1\rangle\, |M_2\psi_2\rangle\, |M_3\psi_3\rangle + |M_2\psi_2\rangle\, |M_1\psi_1\rangle\, |M_3\psi_3\rangle + |M_3\psi_3\rangle\, |M_2\psi_2\rangle\, |M_1\psi_1\rangle \\ &+ |M_1\psi_1\rangle\, |M_3\psi_3\rangle\, |M_2\psi_2\rangle + |M_2\psi_2\rangle\, |M_3\psi_3\rangle\, |M_1\psi_1\rangle + |M_3\psi_3\rangle\, |M_1\psi_1\rangle\, |M_2\psi_2\rangle \big)\end{aligned} \tag{4.6}$$

where the color and spin labels have been dropped since they are identical for all mesons. It is more useful for our purposes to separate the flavor and spatial tensor product spaces, in the following form:

$$\begin{aligned}|\Psi\rangle =& \frac{1}{\sqrt{6}}\big(|M_1M_2M_3\rangle\, |\psi_1\psi_2\psi_3\rangle + |M_2M_1M_3\rangle\, |\psi_2\psi_1\psi_3\rangle + |M_3M_2M_1\rangle\, |\psi_3\psi_2\psi_1\rangle \\ &+ |M_1M_3M_2\rangle\, |\psi_1\psi_3\psi_2\rangle + |M_2M_3M_1\rangle\, |\psi_2\psi_3\psi_1\rangle + |M_3M_1M_2\rangle\, |\psi_3\psi_1\psi_2\rangle \big).\end{aligned} \tag{4.7}$$

In order to keep track of the spatial wavefunction, we keep track of the ordering of the Ms in the Hamiltonian of Eq. 4.1. For example:

$$\begin{aligned}H_{(123)} =\ & a_{(123)} B_i H(3)^i ({M_1}^k_\ell {M_2}^\ell_j {M_3}^j_k) \\ &+ b_{(123)} B_i {M_1}^i_k H(3)^k ({M_2}^\ell_m {M_3}^m_\ell) + c_{(123)} B_i {M_1}^i_j {M_2}^j_\ell {M_3}^\ell_k H(3)^k \\ &+ d_{(123)} B_i H(\overline{6})^{ij}_k {M_1}^k_j ({M_2}^\ell_m {M_3}^m_\ell) + e_{(123)} B_i {M_1}^i_m {M_2}^m_\ell H(\overline{6})^{\ell k}_j {M_3}^j_k \\ &+ f_{(123)} B_i {M_1}^i_\ell H(\overline{6})^{\ell k}_j {M_2}^j_m {M_3}^m_k + g_{(123)} B_i H(\overline{6})^{ij}_k {M_1}^\ell_j {M_2}^m_\ell {M_3}^k_m \\ &+ h_{(123)} B_i H(15)^{ij}_k {M_1}^k_j ({M_2}^\ell_m {M_3}^m_\ell) + i_{(123)} B_i H(15)^{ij}_k {M_1}^\ell_j {M_2}^m_\ell {M_3}^k_m \\ &+ j_{(123)} B_i {M_1}^i_\ell H(15)^{\ell k}_j {M_2}^j_m {M_3}^m_k + k_{(123)} B_i {M_1}^i_m {M_2}^m_\ell H(15)^{\ell k}_j {M_3}^j_k\end{aligned} \tag{4.8}$$

where there is again an implied sum over all repeated indices. The constants $a_{(123)}$, $b_{(123)}$, ..., $k_{(123)}$ are reduced matrix elements. The (123) subscripts of $H_{(123)}$ and these reduced matrix elements refer to how $M_1 M_2 M_3$ is contracted in each term. This also labels the spatial wavefunctions $\psi_1\psi_2\psi_3$. For example, in $H_{(213)}$ the first term would be $a_{(213)} B_i H(3)^i (M_2{}^k_\ell M_1{}^\ell_j M_3{}^j_k)$ which differs from the $a_{(123)}$ term in $H_{(123)}$ of Eq. 4.8 by the interchange of $1 \leftrightarrow 2$. $H_{(213)}$ gives contributions to the final state $|M_2\psi_2 M_1\psi_1 M_3\psi_3\rangle$.

There are a total of 6 different effective Hamiltonians $H_{(\mu\nu\gamma)}$ we can construct, where $(\mu\nu\gamma)$ is a permutation of (123). Using Eq. 3.3, we then directly read off the amplitude $\langle M_\mu\psi_\mu M_\nu\psi_\nu M_\gamma\psi_\gamma|H|B\rangle$ from the coefficient for $BM_\mu M_\nu M_\gamma$ appearing in $H_{(\mu\nu\gamma)}$. Note that because the reduced matrix elements depend on the angular momentum, they depend on $(\mu\nu\gamma)$. In general, the total effective Hamiltonian is a sum of all the $H_{(\mu\nu\gamma)}$:

$$H_{eff} = H_{(123)} + H_{(213)} + H_{(132)} + H_{(321)} + H_{(312)} + H_{(231)} \tag{4.9}$$

so that H_{eff} gives the contribution to $\langle\Psi|H|B\rangle$.

If none of the Ms have a relative angular momentum symmetry, then there are no conditions to impose on H_{eff}. There are 11 reduced matrix elements in each $H_{(\mu\nu\gamma)}$, and 6 different $H_{(\mu\nu\gamma)}$, resulting in up to 66 independent reduced matrix elements in H_{eff}. However, there are several terms that have a manifest symmetry in the Ms: the a terms are cyclically symmetric in all three mesons, and b, d, and h terms are symmetric in two of the mesons. This can easily be seen in Fig. 4-1. Hence there are effectively only two a terms and 3 of each of the b, d, and h terms. This reduces the number of independent reduced matrix elements in H_{eff} to at most $(66 - 4 - 3 \times 3) = 53$. This number is a bound and does not consider the symmetries that may relate various terms of the effective Hamiltonian; we will see several examples of these symmetries below.

In the mixed symmetry case two of the spatial wavefunctions, say ψ_1 and ψ_2, have relative angular momentum L. We use the symmetry in the wavefunction to rewrite Eq. 4.6 as:

$$\begin{aligned} |\Psi\rangle = \frac{1}{\sqrt{6}}\big(& |M_1\psi_1\rangle |M_2\psi_2\rangle |M_3\psi_3\rangle + (-1)^L |M_2\psi_1\rangle |M_1\psi_2\rangle |M_3\psi_3\rangle \\ & + |M_3\psi_3\rangle |M_1\psi_1\rangle |M_2\psi_2\rangle + (-1)^L |M_3\psi_3\rangle |M_2\psi_1\rangle |M_1\psi_2\rangle \\ & + |M_2\psi_2\rangle |M_3\psi_3\rangle |M_1\psi_1\rangle + (-1)^L |M_1\psi_2\rangle |M_3\psi_3\rangle |M_2\psi_1\rangle \big). \end{aligned} \tag{4.10}$$

This wavefunction can also be written as:

$$\begin{aligned} |\Psi\rangle = \frac{1}{\sqrt{6}}\big(& (|M_1\rangle |M_2\rangle |M_3\rangle + (-1)^L |M_2\rangle |M_1\rangle |M_3\rangle) \otimes |\psi_1\rangle |\psi_2\rangle |\psi_3\rangle \\ & + (|M_3\rangle |M_1\rangle |M_2\rangle + (-1)^L |M_3\rangle |M_2\rangle |M_1\rangle) \otimes |\psi_3\rangle |\psi_1\rangle |\psi_2\rangle \\ & + (|M_2\rangle |M_3\rangle |M_1\rangle + (-1)^L |M_1\rangle |M_3\rangle |M_2\rangle) \otimes |\psi_2\rangle |\psi_3\rangle |\psi_1\rangle \big). \end{aligned} \tag{4.11}$$

To impose this symmetry on the effective Hamiltonian, we symmetrize or antisymmetrize M_1 and M_2 in $H_{(123)}$, $H_{(312)}$, and $H_{(231)}$ and sum these three terms. This gives us contributions to the first, second, and third lines of Eq. 4.11 respectively. Note that we could also have symmetrized M_2 and M_3 or M_3 and M_1. However, this simply corresponds to a relabeling of the mesons from $M_1M_2M_3$ to $M_2M_3M_1$ or $M_3M_1M_2$.

Thus when there is a mixed symmetry in two mesons, there are up to 33 independent reduced matrix elements, 11 for each of the symmetrized or antisymmetrized $H_{(123)}$, $H_{(312)}$, and $H_{(231)}$. However, there can only be one independent a contribution because symmetrizing (or antisymmetrizing) any two mesons in $(M^k_\ell M^\ell_j M^j_k)$ gives a totally symmetric (or totally antisymmetric) combination of all three mesons. Similarly, there can only be two independent matrix elements from the b, d and h terms $M_{1}{}^i_k(M_{2}{}^\ell_m M_{3}{}^m_\ell)$: symmetrizing the first and third mesons or the first and second mesons give equivalent contributions. Therefore there can only be up to $(33 - 2 - 1 \times 3) = 28$ independent reduced matrix elements for even mixed symmetry. The b, d, and h terms cannot be antisymmetrized in one pair of mesons so for odd mixed symmetry there are at most $(28 - 3) = 25$ independent reduced matrix elements.

The final cases are when every pair of particles has relative even angular momentum, or every pair of particles has relative odd angular momentum. Then because of the symmetry on the spatial wavefunction, the total wavefunction can be written as:

$$|\Psi_\pm\rangle = \frac{1}{\sqrt{6}}\left(|\psi_1\rangle |\psi_2\rangle |\psi_3\rangle\right) \otimes \Big(|M_1\rangle |M_2\rangle |M_3\rangle + |M_2\rangle |M_3\rangle |M_1\rangle + |M_3\rangle |M_1\rangle |M_2\rangle \\ \pm |M_1\rangle |M_3\rangle |M_2\rangle \pm |M_2\rangle |M_1\rangle |M_3\rangle \pm |M_3\rangle |M_2\rangle |M_1\rangle \Big) \quad (4.12)$$

where $|\Psi_+\rangle$ is the wavefunction for even L and $|\Psi_-\rangle$ is the wavefunction for odd L. We completely symmetrize the meson flavor states in Eq. 4.8 for even L and completely antisymmetrize for odd L. The result is that each of the $H_{(\mu\nu\gamma)}$ are equivalent up to an overall sign so only one is needed.

When the meson flavor state is totally symmetric, the labels on the Ms in Eq. 4.8 are unnecessary and we instead just use Eq. 4.1. There are at most 11 independent reduced matrix elements. We can use four additional identities to eliminate four of these. There is one identity involving all the terms with $H(3)$:

$$\frac{2}{3}B_i H(3)^i (M^k_\ell M^\ell_j M^j_k) + B_i M^i_k H(3)^k (M^\ell_m M^m_\ell) - 2 B_i M^i_j M^j_\ell M^\ell_k H(3)^k = 0. \quad (4.13)$$

As a result, only two of the terms in $H(3)$ are necessary to describe the decays. We will use a_3 and b_3 There are two identities involving $H(\overline{6})$:

$$B_i M^i_\ell H(\overline{6})^{\ell k}_j M^j_m M^m_k = -B_i M^i_m M^m_\ell H(\overline{6})^{\ell k}_j M^j_k \quad (4.14)$$

and

$$B_i H(\overline{6})^{ij}_k M^\ell_j M^m_\ell M^k_m = \frac{1}{2} B_i H(\overline{6})^{ij}_k M^k_j (M^\ell_m M^m_\ell). \quad (4.15)$$

These relations are independent of the components of $H(\overline{6})$, as long as $H(\overline{6})$ satisfies the tracelessness and antisymmetry requirements. The relations imply that only the linear combinations $e_6 - f_6$ and $d_6 + g_6/2$ will appear in the effective Hamiltonian Eq. 4.1. We can drop the f_6 and g_6 terms since they provide no new information. Hence there are only two independent $H(\overline{6})$ terms that can be formed. Finally there is one relation for $H(15)$:

$$B_i H(15)^{ij}_k M^\ell_j M^m_\ell M^k_m = \frac{1}{2} B_i H(15)^{ij}_k M^k_j (M^\ell_m M^m_\ell), \tag{4.16}$$

which again only depends on the symmetry properties of $H(15)$. This allows us to eliminate the k_{15} term. These four equations decrease the number of independent reduced matrix elements to 7 when all the mesons have even relative angular momentum. The totally symmetric effective Hamiltonian is:

$$\begin{aligned} H^S_{eff} = {} & a^S_3 B_i H(3)^i (M^k_\ell M^\ell_j M^j_k) + b^S_3 B_i M^i_k H(3)^k (M^\ell_m M^m_\ell) \\ & + d^S_6 B_i H(\overline{6})^{ij}_k M^k_j (M^\ell_m M^m_\ell) + e^S_6 B_i M^i_m M^m_\ell H(\overline{6})^{\ell k}_j M^j_k \\ & + h^S_{15} B_i H(15)^{ij}_k M^k_j (M^\ell_m M^m_\ell) + i^S_{15} B_i M^i_m M^m_\ell H(15)^{\ell k}_j M^j_k \\ & + j^S_{15} B_i M^i_\ell H(15)^{\ell k}_j M^j_m M^m_k \end{aligned} \tag{4.17}$$

where the S superscript means totally symmetric flavor wavefunction.

Finally, when all the mesons have relative odd angular momentum, we can again drop the labels in Eq. 4.8 if we require that the mesons anticommute. It is clear the b, d, and h terms cannot be totally antisymmetrized so there are at most $(11-3) = 8$ independent reduced matrix elements. When the mesons anticommute, there is an identity for $H(\overline{6})$:

$$-B_i M^i_m M^m_\ell H(\overline{6})^{\ell k}_j M^j_k - B_i M^i_\ell H(\overline{6})^{\ell k}_j M^j_m M^m_k + 2 B_i H(\overline{6})^{ij}_k M^\ell_j M^m_\ell M^k_m = 0 \tag{4.18}$$

which we will use to remove the g_6 term from the Hamiltonian. Therefore the number of independent reduced matrix elements for totally antisymmetric wavefunctions is 7. The totally antisymmetric effective Hamiltonian is:

$$\begin{aligned} H^A_{eff} = {} & a^A_3 B_i H(3)^i (M^k_\ell M^\ell_j M^j_k) + c^A_3 B_i M^i_j M^j_\ell M^\ell_k H(3)^k \\ & + e^A_6 B_i M^i_m M^m_\ell H(\overline{6})^{\ell k}_j M^j_k + f^A_6 B_i M^i_\ell H(\overline{6})^{\ell k}_j M^j_m M^m_k \\ & + i^A_{15} B_i M^i_m M^m_\ell H(15)^{\ell k}_j M^j_k + j^A_{15} B_i M^i_\ell H(15)^{\ell k}_j M^j_m M^m_k \\ & + k^A_{15} B_i H(15)^{ij}_k M^\ell_j M^m_\ell M^k_m . \end{aligned} \tag{4.19}$$

where the S superscript means totally symmetric flavor wavefunction.

In this section we addressed the different possibilities for spatial parts of the wavefunctions, and the resulting consequences for the flavor wavefunction. The Wigner Eckart Theorem is most practical for the totally antisymmetric or totally symmetric cases. Imposing the symmetry on the effective Hamiltonian is easier here, since it only involves anticommuting or commuting flavor states. Furthermore there are only 7 reduced matrix elements in both cases. In the $B \to MM$ decays, there are 5 reduced matrix elements for both the antisymmetric and symmetric cases. (However, for $B \to MM$ only the symmetric case is physically allowed because the initial state has total angular momentum equal to 0.) The cost of adding another M in the final state is a slight increase in the number of reduced matrix elements for the completely symmetric or antisymmetric cases, and a much larger number of reduced matrix elements when we consider mixed symmetries in MMM.

In the remainder of this thesis we will focus on the the totally antisymmetric and totally symmetric cases, and leave a complete analysis of the mixed symmetry case for a future publication. The totally antisymmetric or symmetric cases should contain the greatest number of simple relations between the number of reduced matrix elements is much lower. However, they also require extraction of the symmetry state from the data which may be more difficult. Furthermore, the mixed symmetry cases could be used to study decays such as resonant decays such as $B \to \rho M \to (\pi\pi)M$, where the ρ particle is an antisymmetric combination of π mesons. In a non-resonant decay, B would go directly to MMM rather than a resonant decay where $B \to MM' \to M(MM)$. An analysis of mixed symmetry decays could provide more useful information for experimentalists, who distinguish resonant decays from non-resonant decays.

4.2 Reduced Matrix Elements

We began this chapter by deriving an effective Hamiltonian from a practical viewpoint: we determined all the ways the indices of B, H, and MMM could be contracted in order to create $SU(3)$ singlets. In this section, we connect this approach to the group theory by examining representations and Clebsch-Gordan decomposition of tensor products into irreducible representations. This provides an important cross-check on the counting above. We use some results from Chapter 5 below.

The Clebsch-Gordan decomposition HB is:

$$\begin{aligned} HB &\to \overline{3} \otimes (3 \oplus \overline{6} \oplus 15) \\ \overline{3} \otimes 15 &= 8 \oplus 10 \oplus 27 \\ \overline{3} \otimes \ \overline{6} &= 8 \oplus \overline{10} \\ \overline{3} \otimes \ 3 &= 1 \oplus 8 \\ HB &\to 1 \oplus 8 \oplus 8 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27 \end{aligned} \tag{4.20}$$

This equation can be derived using the Young tableaux of Chapter 5. The Wigner-Eckart Theorem states there will be nonzero matrix elements when MMM transforms as 1, 8, 10, $\overline{10}$ or 27. Note that the representations 1, 8, and 27 are all equal to their conjugate representations, but $\overline{10}$ is the conjugate of 10.

The Clebsch-Gordan decomposition of MMM is more complicated because it is now necessary to consider symmetrization of the final particle wavefunction and thus consider the symmetries of the various representations that appear in the decomposition of MMM. The prodigious ingredients that go into the symmetrization are delayed until Chapter 5. Here, we just state the results.

$$\begin{aligned}
MMM &: 8 \otimes (8 \otimes 8) \\
&= 8 \otimes (1_S \oplus 8_A \oplus 8_S \oplus 10_A \oplus \overline{10}_A \oplus 27_S) \\
(MMM)_S &\rightarrow 1 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27 \\
(MMM)_A &\rightarrow 1 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27
\end{aligned} \tag{4.21}$$

where the subscript A or S indicates totally antisymmetric or symmetric with respect to the two eights it arose from. The final states are decomposed into irreducibles that transform like the conjugate of the representations in HB of Eq. 4.20.

From Eqs. 4.20-4.21, we obtain the reduced matrix elements. The reduced matrix elements for $(MMM)_S$ labeled in the manner of Eq. 2.3 are: $\langle 1||3||\overline{3}\rangle$, $\langle 8||3||\overline{3}\rangle$, $\langle 8||\overline{6}||\overline{3}\rangle$, $\langle 10||\overline{6}||\overline{3}\rangle$, $\langle 8||15||\overline{3}\rangle$, $\langle \overline{10}||15||\overline{3}\rangle$, and $\langle 27||15||\overline{3}\rangle$ where we dropped the extra labels α and α'. These 7 numbers are linear combinations of the 7 numbers a_3^S, b_3^S, d_6^S, e_6^S, h_{15}^S, i_{15}^S, and j_{15}^S from Eq. 4.1. For the completely antisymmetric case, or relative L odd between all the mesons, the reduced matrix elements are again labeled: $\langle 1||3||\overline{3}\rangle$, $\langle 8||3||\overline{3}\rangle$, $\langle 8||\overline{6}||\overline{3}\rangle$, $\langle 10||\overline{6}||\overline{3}\rangle$, $\langle 8||15||\overline{3}\rangle$, $\langle \overline{10}||15||\overline{3}\rangle$, and $\langle 27||15||\overline{3}\rangle$. These are linear combinations of the 7 numbers a_3^A, c_3^A, e_6^A, f_6^A, i_{15}^A, j_{15}^A, and k_{15}^A. Note that these numbers all depend on the relative angular momentum.

4.3 Reduced Matrix Elements as 'Graphical' Amplitudes

In this thesis, we are computing decay amplitudes as a linear combination of the reduced matrix elements, and we explained earlier in this chapter that this has a correspondence with contraction of quark flavor indices in diagrams. Another common practice is to decompose decay rates in terms of 'graphical amplitudes' which directly correspond to quark diagrams. [16] This practice can be useful for writing decays as sums from contributions due to specific physical processes, though it is less fundamental than doing the group theory and it is harder to count the number of independent unknowns. Graphical amplitudes are also useful because one can associate different sizes to them.

Suppose we were to try to swap between bases of reduced matrix elements and 'graphical' amplitudes. The graphical amplitudes for $\Delta s = 0$ include terms like P (penguin), PC (color-suppressed penguin), T (tree), TC (color-suppressed tree), PA (penguin annihilation), and EW (electro-weak). The origin of these terms is beyond the scope of this thesis but they can be found in [16]; suffice it to say that they correspond to different configurations of the quark diagrams like in Fig. 3-1.

We know the number of independent decay amplitudes in $\Delta s = 0$ and $\Delta s = 1$ for a totally symmetric final state is 7. If the number of graphical amplitudes is also 7, we can express each linear combination of matrix elements as a linear combination of the decay rates. (The physics tells us which processes contribute to which decays, and we are free to pick the phase definition.) Then it is just a matter of solving $A\mathbf{x} = b$, where

b is a vector of the decay rates and $\mathbf{x}$ is the vector of graphical amplitudes. This gives each graphical amplitude as a linear combination of matrix elements. Conversely, one could also define each graphical amplitude as a linear combination of matrix elements that contribute to it, and solve.

However, the number of graphical amplitudes is unfortunately not equal to the number of independent decay amplitudes, so one must pick linear combinations of certain graphical amplitudes to be a basis element, for example as in [16] where it was done for $B \to MM$ decays. For 2 body decays, the number of independent decay amplitudes was close to the number of graphical amplitudes. In $B \to MMM$, the number of graphical amplitudes is far larger than the number of independent reduced matrix elements for the completely symmetric or antisymmetric cases. This makes it more difficult to think of reduced matrix elements as sums of graphical amplitudes. Perhaps a scheme could be devised, but it is beyond the goal of this work.

4.4 Effective Hamiltonian in $SU(2)$

The group theory analysis of $B \to MMM$ is simplified when we only consider the $SU(2)$ isospin limit. In $SU(2)$, the $\pi, K, \overline{K}$ and η mesons are not identical particles. Only particles within a doublet such as $\mathbf{K}$ or within a triplet such as π are identical, and must be symmetrized. The η is now a singlet, and we are free to use the physical η because both η_8 and η_1 are isosinglets. All $SU(2)$ results can be found in the $SU(3)$ results by looking for relations that only involve final states of the same combination of $SU(2)$ representations, such as $\pi\pi\mathbf{K}$. We will discuss this in more detail in Chapter 6. However, to be complete, here we outline the method for the $B \to MMM$ $SU(2)$ analysis. We limit the discussion below to decays with relative even angular momentum between identical particles. The different angular momentum cases can also be done for $SU(2)$ by keeping track of order of the mesons, much as we did in earlier parts of this chapter for $SU(3)$.

Using the spin label for representations of $SU(2)$, the $\Delta s = 0$ Hamiltonian has isospin $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}$. Hence the Hamiltonian has $\Delta I = 1/2$ and $\Delta I = 3/2$ parts. The $\Delta I = 1/2$ part of Hamiltonian, $H(\frac{1}{2})^i$, has one index and has the following components:

$$H(\tfrac{1}{2})^i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.22}$$

Here we neglected the constants and CKM factors, for the reasons described in Chapter 3. Meanwhile the $\Delta I = 3/2$ part of the Hamiltonian has three indices, $H(\frac{3}{2})^{ij}_k$, and has the following nonzero components, again without CKM factors:

$$H(\tfrac{3}{2})^{12}_1 = H(\tfrac{3}{2})^{21}_1 = 1 = -H(\tfrac{3}{2})^{22}_2 \tag{4.23}$$

Because the π, $\mathbf{K}$, and $\overline{\mathbf{K}}$ are all in different representations, it is necessary to construct many effective Hamiltonians. For example, in $\overline{B}^0_s \to \pi\pi K$ with the π

mesons in a symmetric state, the effective Hamiltonian is:

$$\begin{aligned} H_{eff} &= a\ \overline{B}^0_s H(\tfrac{1}{2})^\ell \overline{\mathbf{K}}_\ell \pi^i_j \pi^j_i + b\ \overline{B}^0_s H(\tfrac{1}{2})^\ell \pi^i_\ell \pi^j_i \overline{\mathbf{K}}_j \\ &+ c\ \overline{B}^0_s H(\tfrac{3}{2})^{ij}_k \overline{\mathbf{K}}_i \pi^k_\ell \pi^\ell_j + d\ \overline{B}^0_s H(\tfrac{3}{2})^{ij}_k \pi^k_j \pi^\ell_i \overline{\mathbf{K}}_\ell \\ &= a'\ \overline{B}^0_s H(\tfrac{1}{2})^\ell \overline{\mathbf{K}}_\ell \pi^i_j \pi^j_i + c'\ \overline{B}^0_s H(\tfrac{3}{2})^{ij}_k \overline{\mathbf{K}}_i \pi^k_\ell \pi^\ell_j \end{aligned} \tag{4.24}$$

where $a' = a + b/2$ and $d' = d$. We have used that $\overline{\pi} = \pi$. The last line follows because there are only two reduced matrix elements, $\langle 0||\frac{1}{2}||\frac{1}{2}\rangle$ and $\langle 0||\frac{3}{2}||\frac{3}{2}\rangle$. Similar effective Hamiltonians can be constructed for $B \to \pi\pi\pi$, $\overline{B}^0_s \to KK\overline{K}$ and so on.

We can also consider the $\Delta s = 1$ Hamiltonian, which has isospin $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$. The $\Delta I = 0$ part is simply a singlet. The $\Delta I = 1$ part, $H(1)^i_j$, has components:

$$H(1)^i_j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.25}$$

There are again a number of a effective Hamiltonians, such as that for $B \to K\overline{K}\overline{K}$:

$$H_{eff} = a\mathbf{B}_i \mathbf{K}^i \mathbf{K}^j \overline{\mathbf{K}}_j + b\mathbf{B}_i H(1)^i_j \mathbf{K}^j \overline{\mathbf{K}}_k \overline{\mathbf{K}}^k + c\mathbf{B}_i \mathbf{K}^i H(1)^j_k \overline{\mathbf{K}}_j \mathbf{K}^k \tag{4.26}$$

This matrix multiplication is equivalent to looking up numbers in Clebsch-Gordan coefficient tables. The tensor analysis is useful here because in general we have 5 objects in the effective Hamiltonian and would need to use the Clebsch-Gordan tables three times.

Chapter 5

Tensor Products and Symmetrizing the Wavefunction

In this chapter we work out the details of Clebsch-Gordan decomposition and symmetrization with Young tableaux. We find the tensor decomposition of MMM or $8 \otimes 8 \otimes 8$, with symmetries. We also determine the symmetry of the Ms in each irreducible representation in the tensor product of $8 \otimes 8 \otimes 8$. This enforces the symmetries in the wavefunctions. Results in this chapter justify Equation 4.21 and the counting of the reduced matrix elements in Chapter 4. However, the discussion here is limited to the totally symmetric and totally antisymmetric cases.

Young tableaux are a compact way of describing representations and their symmetries. Details about the origin of the tableaux and their use in tensor product decomposition can be found in [11]. In a Young tableau, each box represents an upper index, a 3 (or an N in $SU(N)$). The rule is to symmetrize indices separately in each row, and then antisymmetrize the indices in each column. Since an $\overline{3}$ is an antisymmetric combination of two 3s, it has a lower index represented by a pair of boxes in a column:

$$3 : \begin{array}{|c|}\hline \; \\ \hline\end{array} \qquad\qquad \overline{3} : \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline\end{array}$$

To see the $\overline{3}$ in index notation, note that $A_i = \epsilon_{ijk} B^j C^k$. The ϵ_{ijk} is an invariant tensor in $SU(3)$. Contracting of an upper and lower index in Young tableau is accomplished by putting 3 boxes in a column:

$$D^i A_i = \epsilon_{ijk} D^i B^j C^k \rightarrow \begin{array}{|c|}\hline i \\ \hline j \\ \hline k \\ \hline\end{array}$$

If we now start with an arbitrary reducible tensor A with indices A^{ijkl}, then we can use a Young tableau to pick out a part of the tensor that transforms as a $\overline{6}$ by putting the indices in a tableau for a $\overline{6}$:

$$\begin{array}{|c|c|}\hline i & j \\ \hline k & l \\ \hline\end{array} \tag{5.1}$$

We first symmetrize separately in the upper indices and lower indices:

$$A^{ijkl} + A^{jikl} + A^{ijlk} + A^{jilk}$$

Next antisymmetrize in pairs of upper and lower indices in the same column:

$$\begin{aligned}
&A^{ijkl} + A^{jikl} + A^{ijlk} + A^{jilk} \\
&-A^{kjil} - A^{jkil} - A^{kjli} - A^{jkli} \\
&-A^{ilkj} - A^{likj} - A^{iljk} - A^{lijk} \\
&+A^{klij} + A^{lkij} + A^{klji} + A^{lkji}
\end{aligned} \tag{5.2}$$

The resulting tensor transforms as $\overline{6}$. In general the indices could be placed in different orders in the boxes to obtain a different representation. In this case, this are two $\overline{6}$s possible. This tensor is symmetric in its first two and second two indices, and antisymmetric in its first and third indices and second and fourth indices. Hence the Young tableaux gives the symmetry properties of the tensor. We can think of each box as an index of the irreducible tensor, where boxes in rows are symmetrized and boxes in columns are antisymmetrized.

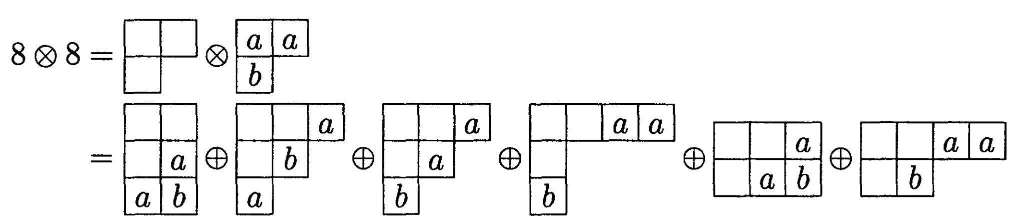
There is an algorithm for using Young tableaux to find the Clebsch-Gordan decomposition of a tensor product, which may be found in [11]. This gives:

$$\begin{aligned}
MMM &\to 8 \otimes (8 \otimes 8) \\
&= 8 \otimes (1_S \oplus 8_A \oplus 8_S \oplus 10_A \oplus \overline{10}_A \oplus 27_S) \\
&= 8 \oplus (8 \otimes 8_A) \oplus (8 \otimes 8_S) \oplus (8 \oplus 10 \oplus 27 \oplus 35) \oplus (8 \oplus \overline{10} \\
&\quad \oplus 27 \oplus \overline{35}) \oplus (8 \oplus 10 \oplus \overline{10} \oplus 27 \oplus 27 \oplus 35 \oplus \overline{35} \oplus 64)
\end{aligned} \tag{5.3}$$

We examine this in greater detail, and with symmetries, in the following section.

5.1 A Totally Symmetric Wavefunction

Using the rules for Clebsch-Gordan decomposition from [11], we have for $8 \otimes 8$:



$$= 1 \oplus 8_1 \oplus 8_2 \oplus 10 \oplus \overline{10} \oplus 27 \tag{5.4}$$

The as and bs are just placeholders to keep track of which tableaux are valid and to avoid double counting. Note that the as indicate the first two indices are symmetrized, so a tableaux in the decomposition cannot have as in the same column.

Next we use the rules of the Young tableaux to symmetrize in rows and then antisymmetrize in columns, and from this deduce the symmetrization properties of the two 8s. For example, in the first term of Eq. 5.4, the singlet, we use symmetry under exchange of columns to rewrite the tableau:

$$\begin{array}{|c|c|}\hline c & c \\ \hline d & a \\ \hline a & b \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|}\hline c & c \\ \hline a & d \\ \hline b & a \\ \hline \end{array} \tag{5.5}$$

where we temporarily keep track of the tableau of the first 8 with c and d. Then we can exchange boxes in the columns; for every exchange there is a factor of -1 for antisymmetry. There are 4 exchanges:

$$\begin{array}{|c|c|}\hline c & c \\ \hline a & d \\ \hline b & a \\ \hline \end{array} \longrightarrow - \begin{array}{|c|c|}\hline c & a \\ \hline a & d \\ \hline b & c \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|}\hline a & a \\ \hline c & d \\ \hline b & c \\ \hline \end{array} \longrightarrow - \begin{array}{|c|c|}\hline a & a \\ \hline b & d \\ \hline c & c \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|}\hline a & a \\ \hline b & c \\ \hline c & d \\ \hline \end{array} \tag{5.6}$$

The result is a tableau the same as the starting tableau, with the exchange of (a,b) and (c,d). We found that given a set of indices specifying M_1 and M_2, there is no change of sign when the indices are interchanged. Hence this singlet is a symmetric combination of two 8s.

The same process follows for the rest of the products in Eq. 5.4. For example, in the 8_1 we begin by exchanging labels in rows and then in columns:

$$\begin{array}{|c|c|c|}\hline c & c & a \\ \hline d & b & \multicolumn{1}{|c}{} \\ \cline{1-2} a & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array} \longrightarrow \begin{array}{|c|c|c|}\hline c & a & c \\ \hline b & d & \multicolumn{1}{|c}{} \\ \cline{1-2} a & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array} \longrightarrow - \begin{array}{|c|c|c|}\hline a & a & c \\ \hline b & d & \multicolumn{1}{|c}{} \\ \cline{1-2} c & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array} \tag{5.7}$$

which is antisymmetric. The $8_1, 10$ and $\overline{10}$ in Eq. 5.4 are antisymmetric and the 1, 8_2, and 27 are symmetric by similar arguments.

Thus to find the totally symmetric combination of 8s the relevant tensor products in $8\otimes(8\otimes 8)$ are $8\otimes 1_S$, $8\otimes 8_S$, and $8\otimes 27_S$, where the S subscript indicates symmetric. Now it is necessary to determine which representations in $8\otimes(8\otimes 8)$ are symmetric or can be symmetrized. We begin by keeping track of the indices of all three particles with the pairs $(a,b),(c,d)$ and (e,f), where the first index in each pair refers to the top row of the 8 tableau and the second index refers to the bottom box of the 8 tableau. The results for $8\otimes(8\otimes 8)$ are:

$$1_S \otimes 8 = \begin{array}{|c|c|}\hline a & a \\ \hline b & c \\ \hline c & d \\ \hline \end{array} \otimes \begin{array}{|c|c|}\hline e & e \\ \hline f & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|c|}\hline a & a & e & e \\ \hline b & c & f & \multicolumn{1}{|c}{} \\ \cline{1-3} c & d & \multicolumn{2}{|c}{} \\ \cline{1-2} \end{array} = 8 \tag{5.8}$$

$$\begin{aligned} 8_S \otimes 8 &= \begin{array}{|c|c|c|}\hline a & a & c \\ \hline b & c & \multicolumn{1}{|c}{} \\ \cline{1-2} d & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array} \otimes \begin{array}{|c|c|}\hline e & e \\ \hline f & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \\ &= \begin{array}{|c|c|c|}\hline a & a & c \\ \hline b & c & e \\ \hline d & e & f \\ \hline \end{array} \oplus \begin{array}{|c|c|c|c|}\hline a & a & c & e \\ \hline b & c & f & \multicolumn{1}{|c}{} \\ \cline{1-3} d & e & \multicolumn{2}{|c}{} \\ \cline{1-2} \end{array} \oplus \begin{array}{|c|c|c|c|}\hline a & a & c & e \\ \hline b & c & e & \multicolumn{1}{|c}{} \\ \cline{1-3} d & f & \multicolumn{2}{|c}{} \\ \cline{1-2} \end{array} \oplus \begin{array}{|c|c|c|c|c|}\hline a & a & c & e & e \\ \hline b & c & \multicolumn{3}{|c}{} \\ \cline{1-2} d & f & \multicolumn{3}{|c}{} \\ \cline{1-2} \end{array} \oplus \begin{array}{|c|c|c|c|}\hline a & a & c & e \\ \hline b & c & e & f \\ \hline d & \multicolumn{3}{|c}{} \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|c|c|c|c|}\hline a & a & c & e & e \\ \hline b & c & f & \multicolumn{2}{|c}{} \\ \cline{1-3} d & \multicolumn{4}{|c}{} \\ \cline{1-1} \end{array} \\ &= 1 \oplus 8_1 \oplus 8_2 \oplus 10 \oplus \overline{10} \oplus 27 \end{aligned} \tag{5.9}$$

The 8_1 is antisymmetric in (a, b) and (e, f) so it cannot be completely symmetrized. Finally, the decomposition of $8 \otimes 27_S$ is:

$$
\begin{aligned}
27_S \otimes 8 = & \begin{array}{|c|c|c|c|}\cline{1-4} a&a&c&c\\\cline{1-4} b&d\\\cline{1-2}\end{array} \otimes \begin{array}{|c|c|}\cline{1-2} e&e\\\cline{1-2} f\\\cline{1-1}\end{array} \\
= & \begin{array}{|c|c|c|c|}\cline{1-4} a&a&c&c\\\cline{1-4} b&d&e\\\cline{1-3} e&f\\\cline{1-2}\end{array} \oplus \begin{array}{|c|c|c|c|c|}\cline{1-5} a&a&c&c&e\\\cline{1-5} b&d\\\cline{1-2} e&f\\\cline{1-2}\end{array} \oplus \begin{array}{|c|c|c|c|}\cline{1-4} a&a&c&c\\\cline{1-4} b&d&e&e\\\cline{1-4} f\\\cline{1-1}\end{array} \oplus \begin{array}{|c|c|c|c|c|}\cline{1-5} a&a&c&c&e\\\cline{1-5} b&d&f\\\cline{1-3} e\\\cline{1-1}\end{array} \oplus \begin{array}{|c|c|c|c|c|}\cline{1-5} a&a&c&c&e\\\cline{1-5} b&d&e\\\cline{1-3} f\\\cline{1-1}\end{array} \\
& \oplus \begin{array}{|c|c|c|c|c|c|}\cline{1-6} a&a&c&c&e&e\\\cline{1-6} b&d\\\cline{1-2} f\\\cline{1-1}\end{array} \oplus \begin{array}{|c|c|c|c|c|}\cline{1-5} a&a&c&c&e\\\cline{1-5} b&d&e&f\\\cline{1-4}\end{array} \oplus \begin{array}{|c|c|c|c|c|c|}\cline{1-6} a&a&c&c&e&e\\\cline{1-6} b&d&f\\\cline{1-3}\end{array} \\
= & \ 8 \oplus 10 \oplus \overline{10} \oplus 27 \oplus 27 \oplus 35 \oplus \overline{35} \oplus 64
\end{aligned} \tag{5.10}
$$

The first 27 is antisymmetric in (a, b) and (e, f) and cannot contribute.

To create a totally symmetric wavefunction, we symmetrize in the labels of each irreducible representation. For example, the 8 in $1_S \otimes 8$, in Eq. 5.8:

$$
\begin{array}{|c|c|c|c|}\cline{1-4} a&a&e&e\\\cline{1-4} b&c&f\\\cline{1-3} c&d\\\cline{1-2}\end{array} + \begin{array}{|c|c|c|c|}\cline{1-4} a&a&c&c\\\cline{1-4} b&e&d\\\cline{1-3} e&f\\\cline{1-2}\end{array} + \begin{array}{|c|c|c|c|}\cline{1-4} c&c&a&a\\\cline{1-4} d&e&b\\\cline{1-3} e&f\\\cline{1-2}\end{array} + (\text{cyclic permutations of } (a, b) \to (c, d) \to (e, f)\ ) \tag{5.11}
$$

The next step is to examine all the different representations of the same dimension after symmetrization. There are three possible 8s:

$$
\begin{aligned}
1_S \otimes 8 : & \begin{array}{|c|c|c|c|}\cline{1-4} a&a&e&e\\\cline{1-4} b&c&f\\\cline{1-3} c&d\\\cline{1-2}\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\cline{1-4} a&a&e&e\\\cline{1-4} b&f&c\\\cline{1-3} d&c\\\cline{1-2}\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\cline{1-4} a&a&c&e\\\cline{1-4} b&c&e\\\cline{1-3} d&f\\\cline{1-2}\end{array} \\
27_S \otimes 8 : & \begin{array}{|c|c|c|c|}\cline{1-4} a&a&c&c\\\cline{1-4} b&d&e\\\cline{1-3} e&f\\\cline{1-2}\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\cline{1-4} a&a&c&c\\\cline{1-4} b&d&e\\\cline{1-3} f&e\\\cline{1-2}\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\cline{1-4} a&a&e&c\\\cline{1-4} b&e&c\\\cline{1-3} f&d\\\cline{1-2}\end{array} \\
8_S \otimes 8 : & \begin{array}{|c|c|c|c|}\cline{1-4} a&a&c&e\\\cline{1-4} b&c&e\\\cline{1-3} d&f\\\cline{1-2}\end{array}
\end{aligned} \tag{5.12}
$$

The 8 of $27_S \otimes 8$ has a term proportional to the symmetrized 8 in $8_S \otimes 8$, and so does the 8 in $1_S \otimes 8$. Hence there is only one independent 8 when we totally symmetrize.

The other representations we consider are 1, 10, $\overline{10}$ and 27. In the above decomposition there was only one 1 that appeared. The 27s are:

$$
\begin{aligned}
8_S \otimes 8 : & \begin{array}{|c|c|c|c|c|}\cline{1-5} a&a&c&e&e\\\cline{1-5} b&c&f\\\cline{1-3} d\\\cline{1-1}\end{array} \\
27_S \otimes 8 : & \begin{array}{|c|c|c|c|c|}\cline{1-5} a&a&c&c&e\\\cline{1-5} b&d&e\\\cline{1-3} f\\\cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|c|c|c|}\cline{1-5} a&a&e&c&c\\\cline{1-5} b&e&d\\\cline{1-3} f\\\cline{1-1}\end{array}
\end{aligned} \tag{5.13}
$$

The 27 in $8_S \otimes 8$ of Eq. 5.9 and the second 27 in $27_S \otimes 8$ of Eq. 5.10 are proportional upon symmetrization, so there is one 27. Finally, using arguments similar to those presented in Eqs. 5.12-5.13, one can show that the two 10s are proportional, and the two $\overline{10}$s are proportional as well. This shows that the relevant representations appearing in $(MMM)_S$ are 1, 8, 10, $\overline{10}$ and 27.

5.2 A Totally Antisymmetric Wavefunction

For relative odd angular momentum in all of the mesons, the relevant terms in $8 \otimes (8 \otimes 8)$ are instead $8 \otimes 8_A$, $8 \otimes 10_A$, and $8 \otimes \overline{10}_A$ where the subscript A means antisymmetric. Keeping track of all the indices with pairs $(a, b), (c, d)$ and (e, f) again, we have

$$
\begin{aligned}
8_A \otimes 8 &= \begin{array}{|c|c|c|} a & a & c \\ b & d & \\ c & & \end{array} \otimes \begin{array}{|c|c|} e & e \\ f & \end{array} \\
&= \begin{array}{|c|c|c|} a & a & c \\ b & d & e \\ c & e & f \end{array} \oplus \begin{array}{|c|c|c|c|} a & a & c & e \\ b & d & f & \\ c & e & & \end{array} \oplus \begin{array}{|c|c|c|c|} a & a & c & e \\ b & d & e & \\ c & f & & \end{array} \oplus \begin{array}{|c|c|c|c|c|} a & a & c & e & e \\ b & d & & & \\ c & f & & & \end{array} \oplus \begin{array}{|c|c|c|c|} a & a & c & e \\ b & d & e & f \\ c & & & \end{array} \oplus \begin{array}{|c|c|c|c|c|} a & a & c & e & e \\ b & d & f & & \\ c & & & & \end{array} \\
&= 1 \oplus 8_1 \oplus 8_2 \oplus 10 \oplus \overline{10} \oplus 27
\end{aligned} \tag{5.14}
$$

The 8_1 cannot contribute because it is symmetric under exchange of (c, d) and (e, f), and the 27 also cannot because it is symmetric under exchange of (a, b) and (e, f). Even though the 1 is a symmetric combination of an 8 and an antisymmetrized pair of two 8s, it can still be completely antisymmetrized.

For $8 \otimes 10_A$ the decomposition is:

$$
\begin{aligned}
8 \otimes 10_A &= \begin{array}{|c|c|c|c|} a & a & c & c \\ b & & & \\ d & & & \end{array} \otimes \begin{array}{|c|c|} e & e \\ f & \end{array} \\
&= \begin{array}{|c|c|c|c|} a & a & c & c \\ b & e & e & \\ d & f & & \end{array} \oplus \begin{array}{|c|c|c|c|c|} a & a & c & c & e \\ b & e & & & \\ d & f & & & \end{array} \oplus \begin{array}{|c|c|c|c|c|} a & a & c & c & e \\ b & e & f & & \\ d & & & & \end{array} \oplus \begin{array}{|c|c|c|c|c|c|} a & a & c & c & e & e \\ b & f & & & & \\ d & & & & & \end{array} \\
&= 8 \oplus 10 \oplus 27 \oplus 35
\end{aligned} \tag{5.15}
$$

And $8 \otimes \overline{10}_A$:

$$
\begin{aligned}
\overline{10}_A \otimes 8 &= \begin{array}{|c|c|c|} a & a & c \\ b & c & d \end{array} \otimes \begin{array}{|c|c|} e & e \\ f & \end{array} \\
&= \begin{array}{|c|c|c|c|} a & a & c & e \\ b & c & d & \\ e & f & & \end{array} \oplus \begin{array}{|c|c|c|c|} a & a & c & e \\ b & c & d & f \\ e & & & \end{array} \oplus \begin{array}{|c|c|c|c|c|} a & a & c & e & e \\ b & c & d & & \\ f & & & & \end{array} \oplus \begin{array}{|c|c|c|c|c|} a & a & c & e & e \\ b & c & d & f & \end{array} \\
&= 8 \oplus \overline{10} \oplus 27 \oplus \overline{35}
\end{aligned} \tag{5.16}
$$

Now, we must antisymmetrize. The 1 from $8_A \otimes 8$ in Eq. 5.14 can be antisymmetrized. Let us now compare the three possible 8 terms:

$$
\begin{aligned}
8_A \otimes 8 &: \young(aace,bde,cf) \rightarrow \young(aace,bde,fc) \rightarrow -\young(aace,bce,fd) \rightarrow \young(aaee,bcc,fd) \\
10_A \otimes 8 &: \young(aacc,bee,df) \\
\overline{10}_A \otimes 8 &: \young(aace,bcd,ef) \rightarrow \young(aace,bcd,fe) \rightarrow \young(eaca,bcd,fe) \rightarrow \young(eeca,fcd,ba)
\end{aligned}
\tag{5.17}
$$

The first and second 8 are proportional after antisymmetrization. (They do not cancel out because there may be different constants multiplying the two.) The last 8 is symmetric in (a, b) and (e, f) because the order of the e and f doesn't matter in the last row of the tableau; it is just to avoid double counting. This leaves one 8 when we totally antisymmetrize.

In the 27s:

$$
\begin{aligned}
10_A \otimes 8 &: \young(aacce,bef,d) \\
\overline{10}_A \otimes 8 &: \young(aacee,bcd,f) \rightarrow \young(aaeec,bcd,f)
\end{aligned}
\tag{5.18}
$$

The last 27 in the equation above is proportional to the first after antisymmetrization, so there will be one independent 27 representation. The two $\overline{10}$s and the two 10s are also proportional. This shows that the relevant symmetric representations appearing in MMM are 1, 8, 10, $\overline{10}$, and 27.

5.3 $SU(2)$ Subgroups

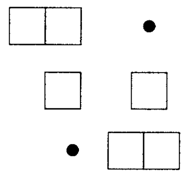
It is also useful to know how the $SU(3)$ Hamiltonian looks like in $SU(2)$, since this corresponds to isospin. This can also be accomplished using Young tableaux, which we present without proof following [11]. The idea is again around symmetrization. Take an $SU(2)$ tableaux and take the product of it with a $U(1)$ tableaux. The number of times the $SU(3)$ representation appears in the product is the number of times the $SU(2)$ representation appears. Here is the isospin decomposition of 3, $\overline{6}$, and 15:

The 3 decomposition is:

$$
\begin{aligned}
\square \quad \bullet \quad &\rightarrow I = \frac{1}{2} \\
\bullet \quad \square \quad &\rightarrow I = 0
\end{aligned}
\tag{5.19}
$$

where the $\bullet$ denotes a singlet. The first column gives the $SU(2)$ tableaux, and the second column gives the $U(1)$ tableaux. One index in $SU(2)$ is $I = \frac{1}{2}$ and a singlet in $SU(2)$ is $I = 0$.

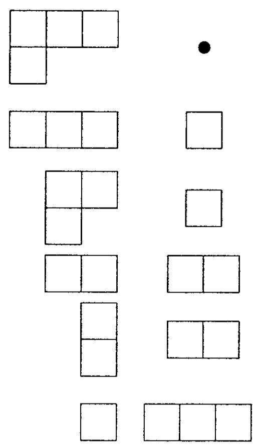
The $\overline{6}$ decomposition is:



$$\begin{aligned} &\rightarrow I = 1 \\ &\rightarrow I = \frac{1}{2} \\ &\rightarrow I = 0 \end{aligned} \tag{5.20}$$

The first row is $I = 1$ because the $SU(2)$ tableaux is a symmetric combination of two spins.

Finally, the 15 decomposition is:



$$\begin{aligned} &\rightarrow I = 1 \\ &\rightarrow I = \frac{3}{2} \\ &\rightarrow I = \frac{1}{2} \\ &\rightarrow I = 1 \\ &\rightarrow I = 0 \\ &\rightarrow I = \frac{1}{2} \end{aligned} \tag{5.21}$$

The first row is $I = 1$ because it looks like a singlet with a symmetric combination of 2 spins. The second row is $I = \frac{3}{2}$ because it's completely symmetric addition of 3 spins. The third row has to be $I = \frac{1}{2}$ because it has 3 spins but isn't completely symmetric. In the fifth row, the antisymmetric combination of two spins is $I = 0$.

In $\Delta s = 0$ decays, where the process is $b \rightarrow (d\overline{u})u$ for the tree and electroweak operator, the Hamiltonian has only $\Delta I = \frac{1}{2}$ and $\Delta I = \frac{3}{2}$ parts. The penguin operators are only $\Delta I = \frac{1}{2}$. From the above decomposition, we know only the 15 contributes to $\Delta I = \frac{3}{2}$, so the 15 is only tree and electroweak, as expected. In $\Delta s = 1$ decays, where the process is $b \rightarrow (s\overline{u})u$ for the tree and electroweak operator, the Hamiltonian has $\Delta I = 0$ and $\Delta I = 1$ parts. The penguin operators are only $\Delta I = 0$.

Chapter 6

Relations Among Decay Amplitudes

The effective Hamiltonian of Eqs. 4.1 and 4.8 gives the decay amplitudes $\mathcal{A}(B \to MMM)$ as linear combinations of reduced matrix elements. The full tables of results are presented in Appendix A, with columns labeled by reduced matrix element and rows labeled by decay. The results are separated according to totally symmetric meson wavefunction or totally antisymmetric meson wavefunction. There are 56 decay channels expressed in terms of 7 reduced matrix elements for totally symmetric and 36 decay channels expressed in terms of 7 reduced matrix elements for totally antisymmetric.

The purpose of this chapter is to summarize the relations and discuss the implications when two amplitudes are related. When 3 or 4 amplitudes are related the implications are more involved, and this is left to Chapter 7. In addition, because of η mixing and because there are a large number of relations involving η_8, the $SU(3)$ relations involving η_8 are summarized in Appendix B. Finally, all the CP conjugated decay amplitudes satisfy the same relations, with the weak phase conjugated.

A number of decays have no contributions from $H(3)$. Because penguin processes transform as a 3 and tree and electroweak processes transform as $3 \oplus \overline{6} \oplus 15$ in $SU(3)$, these decays must result from tree and electroweak processes. For the $\Delta s = 0$ decays, discussed in Section 3.3, both tree and electroweak operators have a CKM factor of $O(\lambda^3)$. Since here electroweak contributions are small, these decays are tree dominated:

$$\begin{aligned}
&\overline{B}_s^0 \to (\eta_8 \pi^- K^+)_S \\
&\overline{B}_s^0 \to (K^0 \pi^0 \eta_8)_S \\
&\overline{B}_s^0 \to (\pi^0 \pi^- K^+)_S \\
&B^- \to (K^0 \pi^0 K^-)_S \\
&B^- \to (K^0 \eta_8 K^-)_S \\
&B^- \to (\pi^0 \eta_8 \pi^-)_S \\
&B^- \to (K^0 \pi^0 K^-)_A
\end{aligned} \tag{6.1}$$

where the S subscript is totally symmetrized final state and A is totally antisymmetrized. However, for $\Delta s = 1$ decays, tree contributions are suppressed by a factor of $O(\lambda^2)$ relative to electroweak contributions so both can be important.

A final comment on the decay amplitudes in Appendix A: there is one decay amplitude $\mathcal{A}(\overline{B}_s^0 \to (\pi^0\pi^- K^+)_S)$ which has only one contribution from the reduced matrix elements, from the 15 part of the Hamiltonian. This is interesting because it means it is possible to measure the magnitude of one of the reduced matrix elements, i_{15}, thus simplifying the analysis of other decay amplitudes which also involve i_{15}.

In the rest of this chapter we discuss relations between decay amplitudes. These relations were obtained by finding nullspaces of the matrices formed by the tables of decay amplitudes in Appendix A. Note that relations only hold for a given spatial wavefunction of MMM, as the reduced matrix elements depend on the spatial wavefunction. In this thesis only relations involving up to four decay amplitudes are discussed. We also give a number of decay rate relations. Some relations can be obtained by adding together several of the lines we list, so these extra dependent relations will not be listed.

A number of the $SU(3)$ results below have analogous $SU(2)$ relations which are slightly different and listed separately. In $SU(2)$ we only symmetrize within each triplet and doublet, since each multiplet is regarded as distinguishable from the rest. In addition, because the η_8 and η are both singlets in $SU(2)$, we can replace η_8 with η when we extract $SU(2)$ relations from their $SU(3)$ counterparts. Examples are given below.

6.1 Even Angular Momentum Relations

6.1.1 $\Delta s = 0$, $SU(2)$ Relations

We begin with $\Delta s = 0$ relations for even L that hold in $SU(2)$. Two simple ones are:

$$\begin{aligned} \mathcal{A}(B^- \to (\pi^-\pi^-\pi^+)_S) &= -\sqrt{2}\, \mathcal{A}(B^- \to (\pi^-\pi^0\pi^0)_S) \\ \sqrt{2}\, \mathcal{A}(\overline{B}^0 \to (\pi^0\pi^0\pi^0)_S) &= -\sqrt{3}\, \mathcal{A}(\overline{B}^0 \to (\pi^-\pi^+\pi^0)_S) \end{aligned} \tag{6.2}$$

leading to the decay rate relations:

$$\begin{aligned} \Gamma(B^- \to (\pi^-\pi^-\pi^+)_S) &= 4\, \Gamma(B^- \to (\pi^-\pi^0\pi^0)_S) \\ \Gamma(\overline{B}^0 \to (\pi^0\pi^0\pi^0)_S) &= \frac{3}{2}\, \Gamma(\overline{B}^0 \to (\pi^-\pi^+\pi^0)_S). \end{aligned} \tag{6.3}$$

These can be directly tested once an angular momentum decomposition of the data becomes available.

There are also two triangle relations. In $SU(3)$ they are:

$$\begin{aligned}\sqrt{2}\,\mathcal{A}(\overline{B}^0_s \to (\pi^0\pi^-K^+)_S) + \sqrt{2}\,\mathcal{A}(\overline{B}^0_s \to (K^0\pi^0\pi^0)_S) &= -\mathcal{A}(\overline{B}^0_s \to (K^0\pi^-\pi^+)_S)\\ \mathcal{A}(\overline{B}^0 \to (\pi^+\pi^-\eta_8)_S) + \sqrt{2}\,\mathcal{A}(\overline{B}^0 \to (\pi^0\pi^0\eta_8)_S) &= \sqrt{2}\,\mathcal{A}(B^- \to (\pi^-\pi^0\eta_8)_S).\end{aligned} \tag{6.4}$$

but these relations also hold in $SU(2)$. In $SU(2)$, only identical particles are symmetrized and the two relations become:

$$\begin{aligned}\sqrt{2}\,\mathcal{A}(\overline{B}^0_s \to K^+(\pi^0\pi^-)_S) + \sqrt{2}\,\mathcal{A}(\overline{B}^0_s \to K^0(\pi^0\pi^0)_S) &= -\mathcal{A}(\overline{B}^0_s \to K^0(\pi^-\pi^+)_S)\\ \mathcal{A}(\overline{B}^0 \to \eta(\pi^+\pi^-)_S) + \sqrt{2}\,\mathcal{A}(\overline{B}^0 \to \eta(\pi^0\pi^0)_S) &= \sqrt{2}\,\mathcal{A}(B^- \to \eta(\pi^-\pi^0)_S).\end{aligned} \tag{6.5}$$

These two triangle relations are discussed further in Chapter 7 on triangle relations.

6.1.2 $\Delta s = 0$, $SU(3)$ Relations

The simplest $\Delta s = 0, SU(3)$ relations are:

$$\sqrt{2}\,\mathcal{A}(B^- \to (\pi^-K^-K^+)_S) = 2\,\mathcal{A}(B^- \to (\pi^-\pi^0\pi^0)_S) = -\mathcal{A}(B^- \to (\pi^-\pi^-\pi^+)_S) \tag{6.6}$$

leading to the relation between decay rates:

$$2\,\Gamma(B^- \to (\pi^-K^-K^+)_S) = 4\,\Gamma(B^- \to (\pi^-\pi^0\pi^0)_S) = \Gamma(B^- \to (\pi^-\pi^-\pi^+)_S) \tag{6.7}$$

HFAG [17] gives the following information for branching ratios of CP conjugates of the decays in the equation above:

$$\begin{aligned} B^+ \to \pi^+\pi^+\pi^- :&\quad (16.2 \pm 1.5) \quad \times 10^{-6}\\ B^+ \to \pi^+\pi^+\pi^-(NR) :&\quad < 4.6 \quad \times 10^{-6}\\ B^+ \to K^+\pi^+\pi^- :&\quad (54.9 \pm 2.9) \quad \times 10^{-6}\\ B^+ \to K^+\pi^+\pi^-(NR) :&\quad (2.9^{+1.0}_{-0.8}) \quad \times 10^{-6}\end{aligned} \tag{6.8}$$

This information for $B^+ \to \pi^+\pi^+\pi^-$ and $B^+ \to K^+\pi^+\pi^-$ from HFAG seems to contradict Eq. 6.7. However, the relation given in Eq. 6.7 was for 3-body decays where all the final mesons are in relative even angular momentum states. In order to test the relations, the experimental results in Eq. 6.8 must be decomposed into angular momentum symmetry states.

There is one triangle relation in $SU(3)$ not involving η_8:

$$\sqrt{2}\,\mathcal{A}(B^- \to (K^-K^0\pi^0)_S) + \sqrt{2}\,\mathcal{A}(\overline{B}^0_s \to (K^0\pi^0\pi^0)_S) = \mathcal{A}(\overline{B}^0_s \to (K^0K^-K^+)_S) \tag{6.9}$$

and two quadrangle relations:

$$\begin{aligned}
&\mathcal{A}(\overline{B}^0_s \to (K^0\pi^-\pi^+)_S) \qquad (6.10)\\
&= \sqrt{2}\,(\mathcal{A}(B^- \to (K^0\pi^0K^-)_S) - \mathcal{A}(\overline{B}^0 \to (\pi^0K^-K^+)_S) - \mathcal{A}(\overline{B}^0 \to (\pi^-\pi^+\pi^0)_S))\\
&= \mathcal{A}(\overline{B}^0 \to (K^0K^-\pi^+)_S) - \mathcal{A}(\overline{B}^0_s \to (K^0K^-K^+)_S) - \mathcal{A}(\overline{B}^0 \to (\overline{K}^0K^+\pi^-)_S).
\end{aligned}$$

In addition the relation $\sqrt{3}\,\mathcal{A}(\overline{B}^0 \to (\pi^-\pi^+\pi^0)_S) = -\sqrt{2}\,\mathcal{A}(\overline{B}^0 \to (\pi^0\pi^0\pi^0)_S)$ can be used in the equation above to obtain one more quadrangle relation.

6.1.3 $\Delta s = 1$, $SU(2)$ Relations

Next we look at $\Delta s = 1$ decays for L even. There are 3 simple $SU(3)$ relations:

$$\begin{aligned}
\mathcal{A}(\overline{B}^0 \to (K^-\pi^0\pi^+)_S) &= -\,\mathcal{A}(B^- \to (\overline{K}^0\pi^0\pi^-)_S)\\
\sqrt{2}\,\mathcal{A}(\overline{B}^0_s \to (\pi^0\pi^0\pi^0)_S) &= -\,\sqrt{3}\,\mathcal{A}(\overline{B}^0_s \to (\pi^-\pi^+\pi^0)_S)\\
-\sqrt{2}\,\mathcal{A}(\overline{B}^0_s \to (\eta_8\pi^0\pi^0)_S) &= \mathcal{A}(\overline{B}^0_s \to (\eta_8\pi^-\pi^+)_S). \qquad (6.11)
\end{aligned}$$

From this we obtain the following $SU(2)$ relations between decay rates:

$$\begin{aligned}
\Gamma(\overline{B}^0 \to K^-(\pi^0\pi^+)_S) &= \Gamma(B^- \to \overline{K}^0(\pi^0\pi^-)_S)\\
2\,\Gamma(\overline{B}^0_s \to (\pi^0\pi^0\pi^0)_S) &= 3\,\Gamma(\overline{B}^0_s \to (\pi^-\pi^+\pi^0)_S)\\
2\,\Gamma(\overline{B}^0_s \to \eta(\pi^0\pi^0)_S) &= \Gamma(\overline{B}^0_s \to \eta(\pi^-\pi^+)_S). \qquad (6.12)
\end{aligned}$$

There is one $SU(2)$ triangle relation:

$$\sqrt{2}\,\mathcal{A}(\overline{B}^0 \to K^-(\pi^0\pi^+)_S) + \sqrt{2}\,\mathcal{A}(\overline{B}^0 \to \overline{K}^0(\pi^0\pi^0)_S) + \mathcal{A}(\overline{B}^0 \to \overline{K}^0(\pi^-\pi^+)_S) = 0. \qquad (6.13)$$

In addition, there are several $SU(2)$ quadrangle relations:

$$\begin{aligned}
&\sqrt{2}\,\mathcal{A}(\overline{B}^0 \to \overline{K}^0(\pi^0\pi^0)_S) + \mathcal{A}(B^- \to K^-(\pi^-\pi^+)_S)\\
&\quad = -\sqrt{2}\,\mathcal{A}(B^- \to K^-(\pi^0\pi^0)_S) - \mathcal{A}(\overline{B}^0 \to \overline{K}^0(\pi^-\pi^+)_S)\\
&\sqrt{2}\,\mathcal{A}(\overline{B}^0 \to K^+(\overline{K}^0K^-)_S) + \mathcal{A}(\overline{B}^0 \to K^0(\overline{K}^0\overline{K}^0)_S)\\
&\quad = \sqrt{2}\,\mathcal{A}(B^- \to K^0(K^-\overline{K}^0)_S) + \mathcal{A}(B^- \to K^+(K^-K^-)_S)\\
&\sqrt{2}\,\mathcal{A}(B^- \to \pi^0K^-\eta) + \mathcal{A}(B^- \to \pi^-\overline{K}^0\eta)\\
&\quad = \sqrt{2}\,\mathcal{A}(\overline{B}^0 \to \pi^0\overline{K}^0\eta) + \mathcal{A}(\overline{B}^0 \to \pi^+K^-\eta)\\
&\sqrt{2}\,\mathcal{A}(\overline{B}^0_s \to K^-K^+\pi^0) + \mathcal{A}(\overline{B}^0_s \to K^-K^0\pi^+)\\
&\quad = -\sqrt{2}\,\mathcal{A}(\overline{B}^0_s \to \overline{K}^0K^0\pi^0) - \mathcal{A}(\overline{B}^0_s \to \overline{K}^0K^+\pi^-). \qquad (6.14)
\end{aligned}$$

The last two relations in the equation above involve three distinguishable particles in the final state in $SU(2)$. Hence these relations hold independent of the relative angular momenta, provided that the spatial wavefunction for MMM is the same in all four decays. All of these relations are discussed further in Chapter 7 on sum rules.

6.1.4 $\Delta s = 1$, $SU(3)$ Relations

There is one $\Delta s = 1$, $SU(3)$ relation between decay rates:

$$\begin{aligned} \mathcal{A}(B^- \to (K^-K^-K^+)_S) &= -\sqrt{2}\,\mathcal{A}(B^- \to (K^-\pi^-\pi^+)_S) \\ \Gamma(B^-(\to K^-K^-K^+)_S) &= 2\,\Gamma(B^- \to (K^-\pi^-\pi^+)_S). \end{aligned} \tag{6.15}$$

HFAG provides a limited set of data to compare with the above relation [17]:

$$\begin{aligned} B^+ \to K^+K^+K^- : &\quad (33.7 \pm 1.5) \quad \times 10^{-6} \\ B^+ \to K^+\pi^+\pi^- : &\quad (54.9 \pm 2.9) \quad \times 10^{-6} \\ B^+ \to K^+\pi^+\pi^-(NR) : &\quad (2.9^{+1.0}_{-0.8}) \quad \times 10^{-6} \end{aligned} \tag{6.16}$$

Again the data is inconclusive as the above relation only holds for totally symmetric meson wavefunction. Finally, we have a quadrangle relation:

$$\begin{aligned} &\mathcal{A}(B^- \to (K^-K^-K^+)_S) - 2\,\mathcal{A}(B^- \to (K^-\pi^0\pi^0)_S) \\ &\quad = 2\,\mathcal{A}(\overline{B}^0 \to (\overline{K}^0\pi^0\pi^0)_S) + \sqrt{2}\,\mathcal{A}(\overline{B}^0 \to (\overline{K}^0\pi^-\pi^+)_S). \end{aligned} \tag{6.17}$$

6.1.5 $SU(3)$ Relations with both $\Delta s = 0$ and $\Delta s = 1$ Decays

There are also $SU(3)$ relations involving both $\Delta s = 0$ and $\Delta s = 1$ decays. In the $SU(3)$ tensor analysis, the difference between $\Delta s = 1$ and $\Delta s = 0$ decay amplitudes is the different CKM factors. Any $\Delta s = 0$ decay amplitude can be written as $V_{ub}V_{ud}^*W + V_{cb}V_{cd}^*X$ and any $\Delta s = 1$ decay amplitude can be written as $V_{ub}V_{us}^*Y + V_{cb}V_{cs}^*Z$. Let $\mathcal{A}(B \to MMM)_1 = W$, $\mathcal{A}(B \to MMM)_2 = X$ for a $\Delta s = 0$ decay, and $\mathcal{A}(B \to MMM)_1 = Y$, $\mathcal{A}(B \to MMM)_2 = Z$ for a $\Delta s = 1$ decay. The following $SU(3)$ relations are relations between the coefficients W, X, Y, Z, which are combinations of the reduced matrix elements without the CKM factor absorbed in them. In the equations below the $\Delta s = 0$ decays are listed to the left of the equality and the $\Delta s = 1$ decays listed to the right. There is an abundance of simple relations:

$$\begin{aligned} \mathcal{A}(\overline{B}_s^0 \to (K^0\pi^0\pi^0)_S)_i &= \mathcal{A}(\overline{B}^0 \to (\overline{K}^0\pi^0\pi^0)_S)_i \\ \mathcal{A}(\overline{B}_s^0 \to (K^0K^0\overline{K}^0)_S)_i &= \mathcal{A}(\overline{B}^0 \to (K^0\overline{K}^0\overline{K}^0)_S)_i \\ \mathcal{A}(\overline{B}_s^0 \to (K^0K^-K^+)_S)_i &= -\,\mathcal{A}(\overline{B}^0 \to (\overline{K}^0\pi^-\pi^+)_S)_i \\ \mathcal{A}(\overline{B}_s^0 \to (K^0\pi^-\pi^+)_S)_i &= -\,\mathcal{A}(\overline{B}^0 \to (\overline{K}^0K^-K^+)_S)_i \\ \sqrt{2}\,\mathcal{A}(B^- \to (K^0\overline{K}^0\pi^-)_S)_i &= \mathcal{A}(B^- \to (K^0\overline{K}^0K^-)_S)_i \\ \sqrt{2}\,\mathcal{A}(B^- \to (\pi^-\pi^-\pi^+)_S)_i &= \mathcal{A}(B^- \to (K^-\pi^-\pi^+)_S)_i \\ \mathcal{A}(B^- \to (K^0\pi^0K^-)_S)_i &= \mathcal{A}(\overline{B}^0 \to (\pi^0K^-\pi^+)_S)_i \\ \mathcal{A}(\overline{B}^0 \to (\overline{K}^0\pi^-K^+)_S)_i &= -\,\mathcal{A}(\overline{B}_s^0 \to (K^0K^-\pi^+)_S)_i \\ \mathcal{A}(\overline{B}^0 \to (\overline{K}^0K^-\pi^+)_S)_i &= -\,\mathcal{A}(\overline{B}_s^0 \to (\overline{K}^0\pi^-K^+)_S)_i. \end{aligned} \tag{6.18}$$

Using these relations to obtain information about decay rates and CP asymmetries requires making 3 measurements, if one assumes the CKM factors are known. The simple relations in the sections above relating only $\Delta s = 0$ decays or only $\Delta s = 1$ decays require making 2 measurements, for example $|\mathcal{A}|^2$ and $|\overline{\mathcal{A}}|^2$.

There are a large number of relations between $\Delta s = 0$ and $\Delta s = 1$ decays that can be formed using the above amplitude relations. Hence they will not be listed. The relations below are less obvious. Two triangle relations are:

$$\begin{aligned}
&\mathcal{A}(B^- \to (K^-\pi^-K^+)_S)_i = \sqrt{2}\,\mathcal{A}(B^- \to (\pi^0\pi^0K^-)_S)_i - \sqrt{2}\,\mathcal{A}(\overline{B}^0 \to (\pi^0K^-\pi^+)_S)_i \\
&-\sqrt{2}\,\mathcal{A}(\overline{B}^0_s \to (K^0\pi^0\pi^0)_S)_i + \mathcal{A}(\overline{B}^0_s \to (K^0K^-K^+)_S)_i = \sqrt{2}\,\mathcal{A}(\overline{B}^0 \to (\pi^0K^-\pi^+)_S)_i
\end{aligned} \tag{6.19}$$

and two quadrangle relations are:

$$\begin{aligned}
\mathcal{A}(\overline{B}^0_s \to (K^0K^-K^+)_S)_i + \mathcal{A}(\overline{B}^0_s \to (K^0\pi^-\pi^+)_S)_i + \mathcal{A}(\overline{B}^0 \to (\overline{K}^0\pi^-K^+)_S)_i& \\
= -\mathcal{A}(\overline{B}^0_s \to (\overline{K}^0\pi^-K^+)_S)_i& \\
\mathcal{A}(\overline{B}^0 \to (K^0K^-\pi^+)_S)_i + \sqrt{2}\mathcal{A}(\overline{B}^0 \to (\pi^0K^-K^+)_S)_i + \sqrt{2}\mathcal{A}(\overline{B}^0 \to (\pi^0\pi^-\pi^+)_S)_i& \\
= \sqrt{2}\,\mathcal{A}(\overline{B}^0 \to (\pi^0K^-\pi^+)_S)_i& \\
\sqrt{2}\,\mathcal{A}(\overline{B}^0_s \to (K^0\pi^0\pi^0)_S)_i - \sqrt{2}\,\mathcal{A}(B^- \to (\pi^-\pi^0\pi^0)_S)_i - \mathcal{A}(\overline{B}^0_s \to (K^0K^-K^+)_S)_i& \\
= -\sqrt{2}\,\mathcal{A}(B^- \to (K^-\pi^0\pi^0)_S)_i.&
\end{aligned} \tag{6.20}$$

6.2 Odd Angular Momentum Relations

Next we discuss relations for when MMM is completely antisymmetric. Relations involving η_8 are given in Appendix B. We will use a subscript A here to denote totally antisymmetric.

For $\Delta s = 0$ decays there is just one $SU(3)$ relation not involving η_8:

$$\begin{aligned}
&\mathcal{A}(\overline{B}^0 \to (K^0K^-\pi^+)_A) + \mathcal{A}(\overline{B}^0 \to (\overline{K}^0K^+\pi^-)_A) \\
&\quad = \mathcal{A}(\overline{B}^0_s \to (K^0K^-K^+)_A) + \mathcal{A}(\overline{B}^0_s \to (K^0\pi^-\pi^+)_A).
\end{aligned} \tag{6.21}$$

This relation also holds in $SU(3)$ in the totally symmetrized case.

For $\Delta s = 1$ decays there are two $SU(2)$ quadrangle relations, which were listed above in the even angular momentum section. These relations do not depend on the relative symmetry of the mesons:

$$\begin{aligned}
&\sqrt{2}\,\mathcal{A}(B^- \to \pi^0K^-\eta) + \mathcal{A}(B^- \to \pi^-\overline{K}^0\eta) \\
&\quad = \sqrt{2}\,\mathcal{A}(\overline{B}^0 \to \pi^0\overline{K}^0\eta) + \mathcal{A}(\overline{B}^0 \to \pi^+K^-\eta) \\
&\sqrt{2}\,\mathcal{A}(\overline{B}^0_s \to K^-K^+\pi^0) + \mathcal{A}(\overline{B}^0_s \to K^-K^0\pi^+) \\
&\quad = -\sqrt{2}\,\mathcal{A}(\overline{B}^0_s \to \overline{K}^0K^0\pi^0) - \mathcal{A}(\overline{B}^0_s \to \overline{K}^0K^+\pi^-).
\end{aligned} \tag{6.22}$$

6.2.1 $SU(3)$ Relations with both $\Delta s = 0$ and $\Delta s = 1$ Decays

As in Sec.6.1.5, there are $SU(3)$ relations between amplitudes where the CKM factors have been factored out of the reduced matrix elements. Again we write equalities with $\Delta s = 0$ decays to the left of the equality.

$$\begin{aligned}
\mathcal{A}(\overline{B}^0_s \to (K^0 K^- K^+)_A)_i &= \mathcal{A}(\overline{B}^0 \to (\overline{K}^0 \pi^- \pi^+)_A)_i \\
\mathcal{A}(\overline{B}^0_s \to (K^0 \pi^- \pi^+)_A)_i &= -\mathcal{A}(\overline{B}^0 \to (\overline{K}^0 K^- K^+)_A)_i \\
\mathcal{A}(B^- \to (\pi^- \overline{K}^0 K^0)_A)_i &= -\mathcal{A}(B^- \to (K^0 \overline{K}^0 K^-)_A)_i \\
\mathcal{A}(B^- \to (\pi^- K^- K^+)_A)_i &= -\mathcal{A}(B^- \to (K^- \pi^- \pi^)_A)_i \\
\mathcal{A}(\overline{B}^0 \to (K^0 K^- \pi^+)_A)_i &= \mathcal{A}(\overline{B}^0_s \to (\pi^- \overline{K}^0 K^+)_A)_i \\
\mathcal{A}(\overline{B}^0 \to (\overline{K}^0 K^+ \pi^-)_A)_i &= \mathcal{A}(\overline{B}^0_s \to (K^0 K^- \pi^+)_A)_i.
\end{aligned} \tag{6.23}$$

There are a number of relations that can be formed by simple substitution using the above relations, which will not be listed. There is also a quadrangle relation:

$$\begin{aligned}
&-\sqrt{2}\,\mathcal{A}(\overline{B}^0_s \to (\pi^0 \pi^- K^+)_A)_i \\
&\quad = \mathcal{A}(\overline{B}^0_s \to (\pi^- \overline{K}^0 K^+)_A)_i + \mathcal{A}(\overline{B}^0_s \to (K^0 K^- \pi^+)_A)_i + \sqrt{2}\,\mathcal{A}(\overline{B}^0 \to (K^- \pi^0 \pi^+)_A)_i.
\end{aligned} \tag{6.24}$$

Chapter 7

Triangle Relations and Sum Rules

In Chapter 6 we discussed relations between decay amplitudes. Relations between two decay amplitudes are easier to test because they simply relate different decay rates (once the appropriate angular momentum projection is done). In this chapter we connect some of the more complicated relations in Chapter 6, involving more than 2 decay amplitudes, with physical observables. We will use our knowledge of the CKM matrix and the operators in the Hamiltonian of Chapter 4. We begin by discussing triangle relations between three decay amplitudes and then continue on to quadrangle relations involving four decay amplitudes. Each section is preceded with a review of the analyses done for $B \to MM$ decays, because the strategy is similar.

7.1 Isospin Triangles

The Gronau London triangle relation for strangeness-preserving $B \to \pi\pi$ decays is a well known result from an isospin analysis of $B \to \pi\pi$ [18]. This is a triangle relation between three $B \to \pi\pi$ amplitudes:

$$\mathcal{A}(\overline{B}^0 \to (\pi^+\pi^-)_S) + \sqrt{2}\,\mathcal{A}(\overline{B}^0 \to (\pi^0\pi^0)_S) = \sqrt{2}\,\mathcal{A}(B^- \to (\pi^-\pi^0)_S) \tag{7.1}$$

along with the CP conjugate relation:

$$\overline{\mathcal{A}}(B^0 \to (\pi^+\pi^-)_S) + \sqrt{2}\,\overline{\mathcal{A}}(B^0 \to (\pi^0\pi^0)_S) = \sqrt{2}\,\overline{\mathcal{A}}(B^+ \to (\pi^+\pi^0)_S). \tag{7.2}$$

Using group theory, one can show that the two amplitudes ($\mathcal{A}(B^- \to (\pi^-\pi^0)_S)$ and its CP conjugate $\overline{\mathcal{A}}(B^+ \to (\pi^+\pi^0)_S)$) have no penguin contribution because they cannot result from the $\Delta I = 0$ part of the Hamiltonian. If we neglect electroweak penguin amplitudes, which are typically small, then these amplitudes are pure tree and thus differ only by an overall phase of 2α, where α was defined in Eq. 1.3. In addition the angle between $\mathcal{A}(\overline{B}^0 \to (\pi^+\pi^-)_S)$ and its CP conjugate can be related directly to an observable, the time-dependent asymmetry $S(\pi^+\pi^-)$. This angle is defined to be $2\alpha_{eff}$, and in the absence of CP violation $\alpha_{eff} = \alpha$.

To see how this works, in Fig. 7-1 we present the triangle relations and the various angles connected to observables. (See [3] for details.) In order to acquire the weak

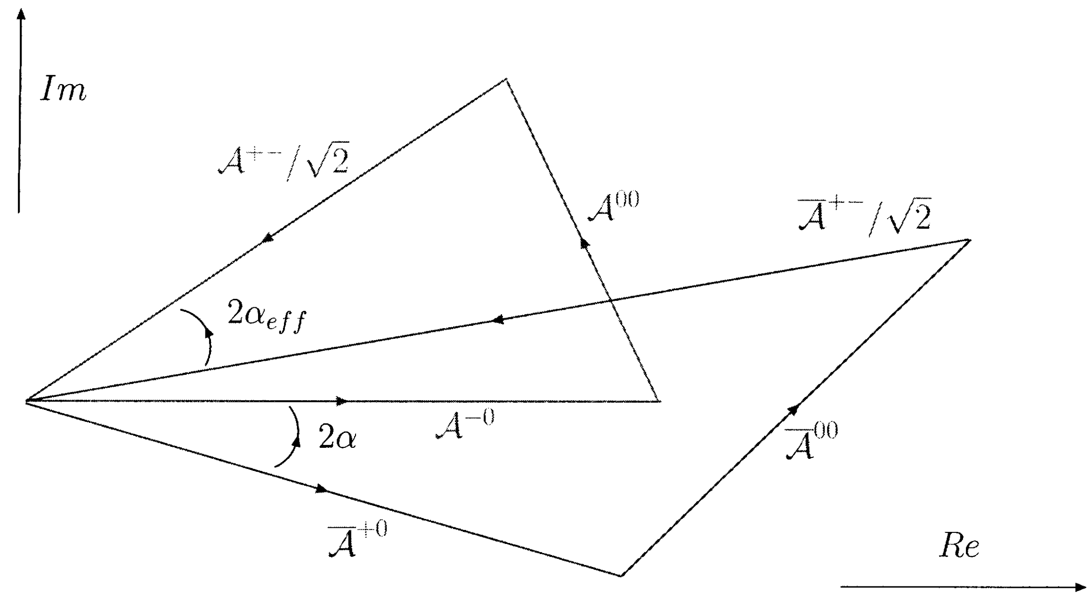


Figure 7-1: The Gronau London isospin triangles; the blue triangle formed by decay amplitudes $\mathcal{A}$ is from Eq. 7.1 and the red triangle formed by the CP conjugate decay amplitudes $\overline{\mathcal{A}}$ is from Eq. 7.2. The overall phase of the amplitude $\mathcal{A}(B^- \to (\pi^-\pi^0)_S)$ has been factored out of all other amplitudes. The superscripts on $\mathcal{A}$ and $\overline{\mathcal{A}}$ indicate the charge of the pions in each decay.

phase α, there are 6 observables that must be measured. One of these is α_{eff}, obtained from $S(\pi^+\pi^-)$. The other 5 observables are the magnitudes of the sides of the two triangles, where we have used $|\mathcal{A}(B^- \to (\pi^-\pi^0)_S)| = |\overline{\mathcal{A}}(B^+ \to (\pi^+\pi^0)_S)|$. In terms of the observables of Chapter 3, we need to measure:

$$\begin{aligned}
\mathrm{Br}(B \to (\pi^+\pi^-)_S) &\propto |\mathcal{A}(\overline{B}^0 \to (\pi^+\pi^-)_S)|^2 + |\overline{\mathcal{A}}(B^0 \to (\pi^+\pi^-)_S)|^2 \\
A_{\mathrm{CP}}(B \to (\pi^+\pi^-)_S) &= \frac{|\mathcal{A}(\overline{B}^0 \to (\pi^+\pi^-)_S)|^2 - |\overline{\mathcal{A}}(B^0 \to (\pi^+\pi^-)_S)|^2}{|\mathcal{A}(\overline{B}^0 \to (\pi^+\pi^-)_S)|^2 + |\overline{\mathcal{A}}(B^0 \to (\pi^+\pi^-)_S)|^2} \\
\mathrm{Br}(B \to (\pi^0\pi^0)_S) &\propto |\mathcal{A}(\overline{B}^0 \to (\pi^0\pi^0)_S)|^2 + |\overline{\mathcal{A}}(B^0 \to (\pi^0\pi^0)_S)|^2 \\
A_{\mathrm{CP}}(B \to (\pi^0\pi^0)_S) &= \frac{|\mathcal{A}(\overline{B}^0 \to (\pi^0\pi^0)_S)|^2 - |\overline{\mathcal{A}}(B^0 \to (\pi^0\pi^0)_S)|^2}{|\mathcal{A}(\overline{B}^0 \to (\pi^0\pi^0)_S)|^2 + |\overline{\mathcal{A}}(B^0 \to (\pi^0\pi^0)_S)|^2} \\
\mathrm{Br}(B \to (\pi^0\pi^-)_S) &\propto \frac{\tau_{\overline{B}^0}}{\tau_{B^-}} \left(|\mathcal{A}(B^- \to (\pi^-\pi^0)_S)|^2 + |\overline{\mathcal{A}}(B^+ \to (\pi^+\pi^0)_S)|^2\right). \qquad (7.3)
\end{aligned}$$

Note that $A_{\mathrm{CP}}(B \to (\pi^0\pi^-)_S) = 0$ if we neglect electroweak penguins. These 5 numbers give the magnitudes of the sides of the isospin triangles. With the addition

$S(\pi^+\pi^-)$ we can determine Fig. 7-1 completely.

The Gronau London isospin triangles are a very tidy example of using group theory to directly measure weak phases. There are a few limitations of the analysis above that merit discussion. First, we neglected the effect of electroweak penguins, which would result in small differences between the magnitudes of $\mathcal{A}(B^- \to (\pi^-\pi^0)_S)$ and $\overline{\mathcal{A}}(B^+ \to (\pi^+\pi^0)_S)$ as well as the angle between them. However, this is actually not a limitation as proven in [19]. In addition, decays to neutral pions π^0 are difficult to measure. In fact the uncertainty in measuring $A_{\rm CP}(B \to \pi_0\pi_0)$ currently does not make this analysis very accurate.

In $B \to MMM$ decays, there are also several isospin triangle relations for which we can apply the same analysis as above. For each triangle relation below, there is one amplitude that has no penguin contribution. Hence we can draw diagrams analogous to Fig. 7-1, and measure the six observables by which we can obtain weak phases. However, the limitations of the $B \to \pi\pi$ isospin triangles also apply to the decays below.

There are two $SU(2)$ triangle relations for $B \to MMM$ and $\Delta s = 0$, given in the previous chapter. The first relation is an obvious extension of the $B \to MM$ triangle relation:

$$\mathcal{A}(\overline{B}^0 \to \eta(\pi^+\pi^-)_S) + \sqrt{2}\,\mathcal{A}(\overline{B}^0 \to \eta(\pi^0\pi^0)_S) = \sqrt{2}\,\mathcal{A}(B^- \to \eta(\pi^-\pi^0)_S). \quad (7.4)$$

Because η is an isosinglet, its presence in the decay does not affect the group theory in $SU(2)$. Therefore, like the decay $B^- \to (\pi^-\pi^0)_S$ and its CP conjugate, the decay $B^- \to \eta(\pi^-\pi^0)_S$ and its CP conjugate also cannot have a penguin term which is $\Delta I = 0$.

The second triangle relation for $B \to MMM$ and $\Delta s = 0$ in $SU(2)$ is:

$$\mathcal{A}(\overline{B}^0_s \to K^+(\pi^0\pi^-)_S) + \mathcal{A}(\overline{B}^0_s \to K^0(\pi^0\pi^0)_S) + \frac{1}{\sqrt{2}}\,\mathcal{A}(\overline{B}^0_s \to K^0(\pi^-\pi^+)_S) = 0. \quad (7.5)$$

Here it is the decay $\overline{B}^0_s \to K^+(\pi^0\pi^-)_S$ that has no penguin contribution. This can be seen from the $SU(3)$ results given in Appendix A, where the decay $\overline{B}^0_s \to (K^+\pi^0\pi^-)_S$ has no terms from the $H(3)$ part of the Hamiltonian while the other two decays do. Since the penguin operator transforms as a 3, it cannot contribute to $\overline{B}^0_s \to (K^+\pi^0\pi^-)_S$.

Also, for $\Delta s = 1$ decays there is the following triangle relation in $SU(2)$:

$$\mathcal{A}(\overline{B}^0 \to K^-(\pi^0\pi^+)_S) + \mathcal{A}(\overline{B}^0 \to \overline{K}^0(\pi^0\pi^0)_S) + \frac{1}{\sqrt{2}}\,\mathcal{A}(\overline{B}^0 \to \overline{K}^0(\pi^-\pi^+)_S) = 0. \quad (7.6)$$

In this decay $\mathcal{A}(\overline{B}^0 \to K^-(\pi^0\pi^+)_S)$ has no penguin contribution while the other two decays do. However, because this is a $\Delta s = 1$ decay the tree part is suppressed by $\sim \lambda^2$. Based on data from $B \to K\pi$ we expect ${\rm Br}(\overline{B}^0 \to K^-(\pi^0\pi^+)_S)$ to be smaller than the other two branching ratios, which might make analyzing the triangle here more difficult.

7.2 Sum Rules for $B \to MMM$

Sum rules give approximate relations among the branching ratios and CP asymmetries, and hence can provide strong constraints on observed decays. First we review the Lipkin and CP sum rules for $\Delta s = 1, B \to \pi K$ decays. There are four $B \to \pi K$ decays which have amplitudes that are related in $SU(2)$ by a quadrangle relation:

$$\begin{aligned}\sqrt{2}\,\mathcal{A}(B^- \to \pi^0 K^-) &- \mathcal{A}(\overline{B}^0 \to \pi^+ K^-) \\ &- \sqrt{2}\,\mathcal{A}(\overline{B}^0 \to \pi^0 \overline{K}^0) + \mathcal{A}(B^- \to \pi^- \overline{K}^0) = 0.\end{aligned} \tag{7.7}$$

A decomposition of the amplitudes that satisfies this relation is:

$$\begin{aligned}\mathcal{A}(B^- \to \pi^- \overline{K}^0) &= V_{ub}V_{us}^* A + V_{cb}V_{cs}^* P_{\pi K} \\ \sqrt{2}\,\mathcal{A}(B^- \to \pi^0 K^-) &= -\,V_{ub}V_{us}^*(C + T + A) - V_{cb}V_{cs}^*(P_{\pi K} + EW^T) \\ \mathcal{A}(\overline{B}^0 \to \pi^+ K^-) &= -\,V_{ub}V_{us}^* T - V_{cb}V_{cs}^*(P_{\pi K} + EW^C) \\ \sqrt{2}\mathcal{A}(\overline{B}^0 \to \pi^0 \overline{K}^0) &= -\,V_{ub}V_{us}^* C + V_{cb}V_{cs}^*(P_{\pi K} - EW^T + EW^C).\end{aligned} \tag{7.8}$$

This parameterization and the derivation below follows the notation and convention of [20]. We have written the amplitudes as linear combinations of graphical amplitudes with appropriate CKM factors. Next recall that $V_{ub}V_{us}^* \sim \lambda^4$ and $V_{cb}V_{cs}^* \sim \lambda^2$, where $\lambda \approx .2$. This suppresses terms that have a CKM factor of $V_{ub}V_{us}^*$, such as the tree graphical amplitude T. In addition, the electroweak contributions to the decay amplitudes (EW^T and EW^C) are small relative to the penguin amplitude $P_{\pi K}$. Hence, these amplitudes can be rewritten as:

$$\begin{aligned}\mathcal{A}(B^- \to \pi^- \overline{K}^0) &= P(1 + e^{i\phi}\epsilon_A) \\ \sqrt{2}\,\mathcal{A}(B^- \to \pi^0 K^-) &= -P\big(1 + \epsilon_T^{ew} + e^{i\phi}(\epsilon_A + \epsilon_T + \epsilon_C)\big) \\ \mathcal{A}(\overline{B}^0 \to \pi^+ K^-) &= -P(1 + \epsilon_C^{ew} + e^{i\phi}\epsilon_T) \\ \sqrt{2}\,\mathcal{A}(\overline{B}^0 \to \pi^0 \overline{K}^0) &= P(1 + \epsilon_C^{ew} - \epsilon_T^{ew} - e^{i\phi}\epsilon_C)\end{aligned} \tag{7.9}$$

where $P = V_{cb}V_{cs}^* P_{\pi K}$ and $\phi \equiv \arg\left(\frac{V_{ub}V_{us}^*}{V_{cb}V_{cs}^*}\right)$. The other terms in the equation above are small and are defined by:

$$\begin{aligned}r &= \left|\frac{V_{ub}V_{us}^*}{V_{cb}V_{cs}^*}\right| \qquad & \epsilon_T^{ew} &= \frac{EW^T}{P_{\pi K}} \qquad & \epsilon_C^{ew} &= \frac{EW^C}{P_{\pi K}} \\ \epsilon_A &= r\left(\frac{A}{P_{\pi K}}\right) & \epsilon_T &= r\left(\frac{T}{P_{\pi K}}\right) & \epsilon_C &= r\left(\frac{C}{P_{\pi K}}\right).\end{aligned} \tag{7.10}$$

From Eq. 7.9 we obtain the following decay rates:

$$\begin{aligned}\Gamma(B^- \to \pi^- \overline{K}^0) &= |P|^2\big(1 + e^{i\phi}\epsilon_A + e^{-i\phi}\epsilon_A^* + O(\epsilon^2)\big) \\ 2\,\Gamma(B^- \to \pi^0 K^-) &= |P|^2\big(1 + 2\,\mathrm{Re}(\epsilon_T^{ew}) + e^{i\phi}(\epsilon_A + \epsilon_T + \epsilon_C) + e^{-i\phi}(\epsilon_A^* + \epsilon_T^* + \epsilon_C^*) + O(\epsilon^2)\big) \\ 2\,\Gamma(\overline{B}^0 \to \pi^0 \overline{K}^0) &= |P|^2\big(1 + 2\,\mathrm{Re}(\epsilon_C^{ew}) - 2\,\mathrm{Re}(\epsilon_T^{ew}) - e^{i\phi}\epsilon_C - e^{-i\phi}\epsilon_C^* + O(\epsilon^2)\big) \\ \Gamma(\overline{B}^0 \to \pi^+ K^-) &= |P|^2\big(1 + 2\,\mathrm{Re}(\epsilon_C^{ew}) + e^{i\phi}\epsilon_T + e^{-i\phi}\epsilon_T^* + O(\epsilon^2)\big).\end{aligned} \tag{7.11}$$

To obtain the decay rates of the CP conjugate of the above decays, only the weak phase ϕ is conjugated.

Recall from Eq. 3.7 that the branching ratio is proportional to the average of a decay and its CP conjugate, the lifetime of the particle. The proportionality constant is independent of the specific decay. Therefore the relevant branching ratios are:

$$\begin{aligned}
\mathrm{Br}(B^- \to \pi^- \overline{K}^0) &\propto \tau_{B^-}|P|^2\big(1 + \cos(\phi)\mathrm{Re}(\epsilon_A) + O(\epsilon^2)\big) \\
2\ \mathrm{Br}(B^- \to \pi^0 K^-) &\propto \tau_{B^-}|P|^2\big(1 + 2\ \mathrm{Re}(\epsilon_T^{ew}) + 2\cos(\phi)\mathrm{Re}(\epsilon_A + \epsilon_T + \epsilon_C) + O(\epsilon^2)\big) \\
2\ \mathrm{Br}(\overline{B}^0 \to \pi^0 \overline{K}^0) &\propto \tau_{\overline{B}^0}|P|^2\big(1 + 2\ \mathrm{Re}(\epsilon_C^{ew}) - 2\ \mathrm{Re}(\epsilon_T^{ew}) - 2\cos(\phi)\mathrm{Re}(\epsilon_C) + O(\epsilon^2)\big) \\
\mathrm{Br}(\overline{B}^0 \to \pi^+ K^-) &\propto \tau_{\overline{B}^0}|P|^2\big(1 + 2\ \mathrm{Re}(\epsilon_C^{ew}) + 2\cos(\phi)\mathrm{Re}(\epsilon_T) + O(\epsilon^2)\big).
\end{aligned} \tag{7.12}$$

The Lipkin sum rule is a relation between these four branching ratios:

$$\frac{2\ \mathrm{Br}(B^- \to \pi^0 K^-) - \mathrm{Br}(B^- \to \pi^- \overline{K}^0)}{\mathrm{Br}(B^- \to \pi^- \overline{K}^0)} + \frac{\tau_{B^-}}{\tau_{\overline{B}^0}} \frac{2\ \mathrm{Br}(\overline{B}^0 \to \pi^0 \overline{K}^0) - \mathrm{Br}(\overline{B}^0 \to \pi^+ K^-)}{\mathrm{Br}(B^- \to \pi^- \overline{K}^0)} = O(\epsilon^2). \tag{7.13}$$

The branching ratio we chose to place in the denominator is arbitrary.

The CP-sum rule relates the set of rescaled asymmetries [21, 22, 23, 24]:

$$\begin{aligned}
\Delta_1 &= \frac{2\mathrm{Br}(B^- \to \pi^0 K^-)}{\mathrm{Br}(B^- \to \pi^- \overline{K}^0)} \times A_{\mathrm{CP}}(\pi^0 K^-) \\
\Delta_2 &= \frac{\mathrm{Br}(\overline{B}^0 \to \pi^+ K^-)}{\mathrm{Br}(B^- \to \pi^- \overline{K}^0)} \times A_{\mathrm{CP}}(\pi^+ K^-) \\
\Delta_3 &= \frac{2\mathrm{Br}(\overline{B}^0 \to \pi^0 \overline{K}^0)}{\mathrm{Br}(B^- \to \pi^- \overline{K}^0)} \times A_{\mathrm{CP}}(\pi^0 \overline{K}^0) \\
\Delta_4 &= A_{\mathrm{CP}}(\pi^- \overline{K}^0).
\end{aligned} \tag{7.14}$$

The statement of the CP-sum rule is

$$\Delta_1 - \Delta_2 + \Delta_3 - \Delta_4 = O(\epsilon^2). \tag{7.15}$$

The usefulness of these decays hinges on the assumption made above that the ϵ terms in Eq.7.10 are small.

The sum rules for $B \to \pi K$ in Eq. 7.13 and 7.15 can be extended to the $\Delta s = 1, B \to MMM$ decays in $SU(2)$. We will discuss four cases. Each case arises from a quadrangle relation that was given in Sec. 6.1.3. The parameterization of amplitudes in Eq. 7.9 can be applied to all four decays in each quadrangle relation below. In a given quadrangle relation, each decay must have a penguin part P, which is enhanced over tree contributions T by a factor of approximately $1/\lambda^2$. The penguin operator is a 3 in $SU(3)$, so it is necessary that each decay has a nonzero coefficients in the a_3 or b_3 columns in the amplitude tables in Appendix A. We also must determine that each decay is proportional to the same linear combination of a_3 and b_3 in order to use a parameterization like the one in Eq. 7.8. Finally, we can check that each of the decays below has a large penguin part P by ascertaining that we can draw the corresponding quark diagram of Fig. 3-1(a) for each decay.

The four amplitudes in each quadrangle relation are needed to cancel out both the order unity terms and the order ϵ terms. What we mean by ϵ below is terms with the size of about $(\lambda^2 T/P)$ or terms with the size of the ratio of electroweak to penguin amplitudes (EW/P). An advantage of one of these sum rules is that it does not involve decays to neutral pions.

The first amplitude relation relates decays of type $B \to K\pi\pi$ in $SU(2)$:

$$\sqrt{2}\,\mathcal{A}(\overline{B}^0 \to \overline{K}^0(\pi^0\pi^0)_S) + \mathcal{A}(B^- \to K^-(\pi^-\pi^+)_S) \\ + \sqrt{2}\,\mathcal{A}(B^- \to K^-(\pi^0\pi^0)_S) + \mathcal{A}(\overline{B}^0 \to \overline{K}^0(\pi^-\pi^+)_S) = 0. \qquad (7.16)$$

The sum rule relating the branching ratios is:

$$\frac{2\,\mathrm{Br}(\overline{B}^0 \to \overline{K}^0(\pi^0\pi^0)_S) - \mathrm{Br}(\overline{B}^0 \to \overline{K}^0(\pi^-\pi^+)_S)}{\mathrm{Br}(\overline{B}^0 \to \overline{K}^0(\pi^-\pi^+)_S)} \\ + \frac{\tau_{\overline{B}^0}}{\tau_{B^-}} \frac{2\,\mathrm{Br}(B^- \to K^-(\pi^0\pi^0)_S) - \mathrm{Br}(B^- \to K^-(\pi^-\pi^+)_S)}{\mathrm{Br}(\overline{B}^0 \to \overline{K}^0(\pi^-\pi^+)_S)} = O(\epsilon^2). \qquad (7.17)$$

And the CP sum rule is:

$$\frac{2\,\mathrm{Br}(\overline{B}^0 \to \overline{K}^0(\pi^0\pi^0)_S) A_{\mathrm{CP}}(\overline{B}^0 \to \overline{K}^0(\pi^0\pi^0)_S)}{\mathrm{Br}(\overline{B}^0 \to \overline{K}^0(\pi^-\pi^+)_S)} - A_{\mathrm{CP}}(\overline{B}^0 \to \overline{K}^0(\pi^-\pi^+)_S) \\ - \frac{\tau_{\overline{B}^0}}{\tau_{B^-}} \frac{\mathrm{Br}(B^- \to K^-(\pi^-\pi^+)_S) A_{\mathrm{CP}}(B^- \to K^-(\pi^-\pi^+)_S)}{\mathrm{Br}(\overline{B}^0 \to \overline{K}^0(\pi^-\pi^+)_S)} \\ + \frac{\tau_{\overline{B}^0}}{\tau_{B^-}} \frac{2\,\mathrm{Br}(B^- \to K^-(\pi^0\pi^0)_S) A_{\mathrm{CP}}(B^- \to K^-(\pi^0\pi^0)_S)}{\mathrm{Br}(\overline{B}^0 \to \overline{K}^0(\pi^-\pi^+)_S)} = O(\epsilon^2). \qquad (7.18)$$

This sum rule has two drawbacks: not only is it necessary to make accurate measurements of neutral pion decays, but it is also necessary to determine the relative symmetries of the pions.

The second set of $SU(2)$ relations has $B \to K\overline{KK}$ decays:

$$\sqrt{2}\,\mathcal{A}(\overline{B}^0 \to K^+(\overline{K}^0 K^-)_S) - \mathcal{A}(B^- \to K^+(K^-K^-)_S) \\ - \sqrt{2}\,\mathcal{A}(B^- \to K^0(K^-\overline{K}^0)_S) + \mathcal{A}(\overline{B}^0 \to K^0(\overline{K}^0\overline{K}^0)_S) = 0 \qquad (7.19)$$

giving the branching ratio sum rule:

$$\frac{2\,\mathrm{Br}(\overline{B}^0 \to K^+(\overline{K}^0 K^-)_S) - \mathrm{Br}(\overline{B}^0 \to K^0(\overline{K}^0\overline{K}^0)_S)}{\mathrm{Br}(\overline{B}^0 \to K^0(\overline{K}^0\overline{K}^0)_S)} \\ + \frac{\tau_{\overline{B}^0}}{\tau_{B^-}} \frac{2\,\mathrm{Br}(B^- \to K^0(K^-\overline{K}^0)_S) - \mathrm{Br}(B^- \to K^+(K^-K^-)_S)}{\mathrm{Br}(\overline{B}^0 \to K^0(\overline{K}^0\overline{K}^0)_S)} = O(\epsilon^2) \qquad (7.20)$$

and the CP sum rule:

$$\frac{2\,\mathrm{Br}(\overline{B}^0 \to K^+(\overline{K}^0 K^-)_S) A_{\mathrm{CP}}(\overline{B}^0 \to K^+(\overline{K}^0 K^-)_S)}{\mathrm{Br}(\overline{B}^0 \to K^0(\overline{K}^0\overline{K}^0)_S)} - A_{\mathrm{CP}}(\overline{B}^0 \to K^0(\overline{K}^0\overline{K}^0)_S)$$
$$- \frac{\tau_{\overline{B}^0}}{\tau_{B^-}} \frac{\mathrm{Br}(B^- \to K^+(K^-K^-)_S) A_{\mathrm{CP}}(B^- \to K^+(K^-K^-)_S)}{\mathrm{Br}(\overline{B}^0 \to K^0(\overline{K}^0\overline{K}^0)_S)}$$
$$+ \frac{\tau_{\overline{B}^0}}{\tau_{B^-}} \frac{2\,\mathrm{Br}(B^- \to K^0(K^-\overline{K}^0)_S) A_{\mathrm{CP}}(B^- \to K^0(K^-\overline{K}^0)_S)}{\mathrm{Br}(\overline{B}^0 \to K^0(\overline{K}^0\overline{K}^0)_S)} = O(\epsilon^2). \tag{7.21}$$

This sum rule is nice because it does not involve any pions, making it easier to test. However, it is still necessary to sort out the symmetries of the Ks. It is also perhaps pertinent to point out that the amplitudes in Eq. 7.16 and 7.19 have not only a penguin contribution, but also a color-suppressed penguin contribution. This simply means that there is more than one kind of penguin quark diagram that can be drawn for these decays; this does not change the sum rules as all the penguin terms can be grouped into the P amplitude. The other amplitudes in the rest of this section do not have any color-suppressed penguin contribution.

The third set of sum rules relate $B \to \pi K \eta$ decay channels, with an $SU(2)$ amplitude relation:

$$\sqrt{2}\,\mathcal{A}(B^- \to \pi^0 K^- \eta) - \mathcal{A}(\overline{B}^0 \to \pi^+ K^- \eta)$$
$$- \sqrt{2}\,\mathcal{A}(\overline{B}^0 \to \pi^0 \overline{K}^0 \eta) + \mathcal{A}(B^- \to \pi^- \overline{K}^0 \eta) = 0. \tag{7.22}$$

This relation is an obvious extension of the Lipkin sum rule; the ηs are isosinglets, so the group theory is exactly the same. The sum rule is:

$$\frac{2\,\mathrm{Br}(B^- \to \pi^0 K^- \eta) - \mathrm{Br}(B^- \to \pi^- \overline{K}^0 \eta)}{\mathrm{Br}(B^- \to \pi^- \overline{K}^0 \eta)}$$
$$+ \frac{\tau_{B^-}}{\tau_{\overline{B}^0}} \frac{2\,\mathrm{Br}(\overline{B}^0 \to \pi^0 \overline{K}^0 \eta) - \mathrm{Br}(\overline{B}^0 \to \pi^+ K^- \eta)}{\mathrm{Br}(B^- \to \pi^- \overline{K}^0 \eta)} = O(\epsilon^2) \tag{7.23}$$

and the CP sum rule is:

$$\frac{2\,\mathrm{Br}(B^- \to \pi^0 K^- \eta) A_{\mathrm{CP}}(B^- \to \pi^0 K^- \eta)}{\mathrm{Br}(B^- \to \pi^- \overline{K}^0 \eta)} - A_{\mathrm{CP}}(B^- \to \pi^- \overline{K}^0 \eta)$$
$$- \frac{\tau_{B^-}}{\tau_{\overline{B}^0}} \frac{\mathrm{Br}(\overline{B}^0 \to \pi^+ K^- \eta) A_{\mathrm{CP}}(\overline{B}^0 \to \pi^+ K^- \eta)}{\mathrm{Br}(B^- \to \pi^- \overline{K}^0 \eta)}$$
$$+ \frac{\tau_{B^-}}{\tau_{\overline{B}^0}} \frac{2\,\mathrm{Br}(\overline{B}^0 \to \pi^0 \overline{K}^0 \eta) A_{\mathrm{CP}}(\overline{B}^0 \to \pi^0 \overline{K}^0 \eta)}{\mathrm{Br}(B^- \to \pi^- \overline{K}^0 \eta)} = O(\epsilon^2). \tag{7.24}$$

Note that because the π, K, and η are distinguishable in the $SU(2)$ limit there are no enforced symmetries in this relation. The $SU(3)$ amplitude relation corresponding to

Eq. 7.22 has an η_8 and all mesons symmetrized. Appendix A indicates that the $SU(3)$ decays $B \to \pi K \eta_8$ have no terms from the 3 part of the Hamiltonian, implying there is no penguin term. However, the decay $B \to \pi K \eta$ does have a penguin amplitude.

The final set of relations is for the decays $B \to K\overline{K}\pi$. Again, because we have three distinguishable particles in $SU(2)$, there is no enforced symmetry on the decays in the following $SU(2)$ amplitude relation:

$$\begin{aligned}\sqrt{2}\,\mathcal{A}(\overline{B}_s^0 \to K^-K^+\pi^0) + \mathcal{A}(\overline{B}_s^0 \to K^-K^0\pi^+)& \\ + \sqrt{2}\,\mathcal{A}(\overline{B}_s^0 \to \overline{K}^0K^0\pi^0) + \mathcal{A}(\overline{B}_s^0 \to \overline{K}^0K^+\pi^-) &= 0\end{aligned} \tag{7.25}$$

with the sum rule:

$$\begin{aligned}&\frac{2\,\mathrm{Br}(\overline{B}_s^0 \to K^-K^+\pi^0) - \mathrm{Br}(\overline{B}_s^0 \to K^-K^0\pi^+)}{\mathrm{Br}(\overline{B}_s^0 \to \overline{K}^0K^+\pi^-)} \\ &+ \frac{2\,\mathrm{Br}(\overline{B}_s^0 \to \overline{K}^0K^0\pi^0) - \mathrm{Br}(\overline{B}_s^0 \to \overline{K}^0K^+\pi^-)}{\mathrm{Br}(\overline{B}_s^0 \to \overline{K}^0K^+\pi^-)} = O(\epsilon^2).\end{aligned} \tag{7.26}$$

Finally our last CP sum rule is:

$$\begin{aligned}&\frac{2\,\mathrm{Br}(\overline{B}_s^0 \to K^-K^+\pi^0)A_{\mathrm{CP}}(\overline{B}_s^0 \to K^-K^+\pi^0)}{\mathrm{Br}(\overline{B}_s^0 \to \overline{K}^0K^+\pi^-)} - A_{\mathrm{CP}}(\overline{B}_s^0 \to \overline{K}^0K^+\pi^-) \\ &- \frac{\mathrm{Br}(\overline{B}_s^0 \to K^-K^0\pi^+)A_{\mathrm{CP}}(\overline{B}_s^0 \to K^-K^0\pi^+)}{\mathrm{Br}(\overline{B}_s^0 \to \overline{K}^0K^+\pi^-)} \\ &+ \frac{2\,\mathrm{Br}(\overline{B}_s^0 \to \overline{K}^0K^0\pi^0)A_{\mathrm{CP}}(\overline{B}_s^0 \to \overline{K}^0K^0\pi^0)}{\mathrm{Br}(\overline{B}_s^0 \to \overline{K}^0K^+\pi^-)} = O(\epsilon^2).\end{aligned} \tag{7.27}$$

All the sum rules have exactly the same form. In the sum rules relating branching ratios, the branching ratios with the prefactor of 2 have the same sign, and the other branching ratios have opposite sign. The denominators in the sum rules are arbitrary. Any of the four possible branching ratios in each sum rule can be placed in the denominator; different branching ratios will only effect a change of $O(\epsilon^3)$.

The last two sets of sum rules we gave for $B \to \pi K \eta$ and $B \to K\overline{K}\pi$ can be applied more readily than the first two, as these make no assumptions about the relative angular momentum of the three mesons. In the first two sum rules for $B \to \pi\pi K$ and $B \to K\overline{K}K$, there are two particles that must have relative even angular momentum. There is also the issue of the neutral pion, which is more troublesome for experimentalists. This difficulty with π^0 unfortunately infects three of our sum rules. Hence despite the symmetry restriction, $B \to K\overline{K}K$ remains a particularly appealing way of testing the data because of the lack of neutral pions.

Chapter 8

Conclusion

Determining the sources of CP violation is an important test of the Standard Model. By comparing experimental data with constraints on the CKM matrix, we can ascertain whether the Standard Model is enough to explain the processes observed in colliders. As a result, the study of CP violation is also a promising route to finding "new physics".

The study of B meson decays has been particularly fruitful to our knowledge of the CKM matrix. One way to make predictions for B meson decays is by using group theory in the limit of $SU(2)$ and $SU(3)$ flavor symmetry, which allows us to determine decay amplitudes as linear combinations of a few constants, the reduced matrix elements. This makes it possible to relate different decay channels. Group theoretical analyses of decays such as $B \to \pi\pi$ and $B \to \pi K$ have provided precise predictions for B decays. Here the decays depend on 5 reduced matrix elements.

In this thesis we analyzed the decay $B \to MMM$ using group theory. In three-body decays there are several possibilities for the relative symmetries of the mesons M: totally symmetric when every pair of mesons has relative even angular momentum eigenvalue, totally antisymmetric when every pair has relative odd angular momentum, and mixed symmetry when some pairs of mesons have no relative angular momentum eigenvalue. These symmetries determine the number of reduced matrix elements. In the $SU(3)$ limit, there are 7 reduced matrix elements which completely describe B decays to three mesons M in a totally symmetric state. There are also 7 reduced matrix elements when the three mesons M are in a totally antisymmetric state. However, when considering mixed symmetries there are as many as 25 or 28 reduced matrix elements for odd and even mixed symmetry, respectively.

The discussion in this thesis focused on totally symmetric and totally symmetric wavefunctions in the $SU(3)$ and $SU(2)$ limits. Because there are many three-body decays and relatively few reduced matrix elements, one can find a large number of simple $SU(3)$ relations between decay amplitudes. In the $SU(2)$ limit, we could expand our discussion to decays with mixed symmetries. In summary, we found simple relations between two decay amplitudes that led to relations between the observable decay rates. We also discussed triangle and quadrangle relations between three and four decay amplitudes respectively.

Combined with facts about CKM factors and the electroweak Hamiltonian, a number of these triangle and quadrangle relations yield tidy methods for determining weak phases from observables. The triangle relations we presented for $B \to MMM$ were analogous to the $B \to \pi\pi$ isospin triangles. Here one exploits the fact that one decay amplitude in the triangle relations has no penguin term, and is consequently dominated by a tree amplitude T. One can directly obtain the weak phase by measuring 6 different numbers: 3 branching ratios and 2 CP asymmetries listed in Eq. 7.3, and a time-dependent asymmetry.

We also derived sum rules for $\Delta s = 1$ $B \to MMM$ decays, which are analogous to the sum rules for $B \to \pi K$. Here we take advantage of the difference in size of the two relevant CKM factors. Because the CKM factor multiplying the tree amplitudes T is smaller by about λ^2 than the CKM factor multiplying penguin amplitudes P, these decays are penguin dominated. From a quadrangle amplitude relation, we can thus derive sum rules directly relating the observable branching ratios and the CP asymmetries, listed in Eqs. 7.17-7.27. These sum rules are accurate to order ϵ^2, where ϵ is a number that has the size of about $\lambda^2 T/P$ or the size of the ratio of electroweak (EW) to penguin amplitudes.

The effectiveness of the relations presented in this thesis depends in part on the ability of experimentalists to make accurate measurements about decays to neutral pions and to separate decays according to relative angular momentum. All of our isospin triangles and two of our sets of sum rules stipulate a particular symmetry between two of the mesons in the decay. However, two sets of sum rules we derived do not depend on any particular symmetry and are free of this restriction. Many of the relations we derived also require reliable observations of decays involving neutral pions, a challenge for experimental particle physics. Here we were able to derive one set of sum rules, for the $B \to K\overline{K}\overline{K}$ channel, which has no neutral pions. In conclusion, the abundance of new constraints we have provided should offer a variety of options for testing the Standard Model.

Appendix A

Full Decay Amplitudes in Terms of Reduced Matrix Elements

In Appendix A we give all the $SU(3)$ decay amplitudes as linear combinations of 7 reduced matrix elements. This appendix is separated according to $\Delta s = 0$ and $\Delta s = 1$ decays and relative angular momentum. Only decay amplitudes involving totally symmetric or antisymmetric final states are given here. The amplitude relations from Chapter 6 are all derived using these decay amplitudes. In these tables, the leftmost column gives the decay. The other columns give the relative contribution of each reduced matrix element. The labels for the reduced matrix elements corresponds to the coefficients in Eq. 4.17 and Eq. 4.19. The numerical values of the coefficients are in general different for each set of relative L and cannot be determined by group theory alone.

A.1 Δs= 0, MMM Completely Symmetric

	a_3^S	b_3^S	d_6^S	e_6^S	h_{15}^S	i_{15}^S	j_{15}^S
$\overline{B}_s^0$decays:							
$K^0\ \pi^0\ \pi^0$	0	$\sqrt{2}$	$-\sqrt{2}$	$\frac{1}{\sqrt{2}}$	$-\sqrt{2}$	$-\frac{5}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$K^0\ \eta_8\ \eta_8$	0	$\sqrt{2}$	$-\sqrt{2}$	$-\frac{1}{3\sqrt{2}}$	$-\sqrt{2}$	$-\frac{7}{3\sqrt{2}}$	$-\frac{5}{3\sqrt{2}}$
$K^0\ K^0\ \overline{K^0}$	0	$-2\sqrt{2}$	$2\sqrt{2}$	$\sqrt{2}$	$2\sqrt{2}$	$\sqrt{2}$	$3\sqrt{2}$
$K^0\ K^-\ K^+$	0	2	-2	-1	-2	-1	5
$K^0\ \pi^-\ \pi^+$	0	-2	2	-1	2	-3	-1
$\eta_8\ \pi^-\ K^+$	0	0	0	$-2\sqrt{\frac{2}{3}}$	0	0	$4\sqrt{\frac{2}{3}}$
$K^0\ \pi^0\ \eta_8$	0	0	0	$\frac{2}{\sqrt{3}}$	0	$-\frac{4}{\sqrt{3}}$	$\frac{4}{\sqrt{3}}$
$\pi^0\ \pi^-\ K^+$	0	0	0	0	0	$4\sqrt{2}$	0
B^-decays:							
$\pi^0\ \pi^0\ \pi^-$	0	$\sqrt{2}$	$\sqrt{2}$	$\frac{1}{\sqrt{2}}$	$3\sqrt{2}$	$\frac{3}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\eta_8\ \eta_8\ \pi^-$	0	$\sqrt{2}$	$\sqrt{2}$	$-\frac{5}{3\sqrt{2}}$	$3\sqrt{2}$	$\frac{3}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$K^0\ \overline{K^0}\ \pi^-$	0	-2	-2	1	-6	1	3
$K^-\ \pi^-\ K^+$	0	2	2	1	6	3	1
$\pi^-\ \pi^-\ \pi^+$	0	$-2\sqrt{2}$	$-2\sqrt{2}$	$-\sqrt{2}$	$-6\sqrt{2}$	$-3\sqrt{2}$	$-\sqrt{2}$
$K^0\ \pi^0\ K^-$	0	0	0	$-\sqrt{2}$	0	$2\sqrt{2}$	$2\sqrt{2}$
$K^0\ \eta_8\ K^-$	0	0	0	$-\sqrt{\frac{2}{3}}$	0	$2\sqrt{\frac{2}{3}}$	$2\sqrt{\frac{2}{3}}$
$\pi^0\ \eta_8\ \pi^-$	0	0	0	0	0	$\frac{8}{\sqrt{3}}$	$\frac{8}{\sqrt{3}}$
$\overline{B}^0$decays:							
$K^0\ \pi^0\ \overline{K^0}$	$-\frac{3}{\sqrt{2}}$	$-\sqrt{2}$	$-\sqrt{2}$	0	$5\sqrt{2}$	$3\sqrt{2}$	$2\sqrt{2}$
$K^0\ \eta_8\ \overline{K^0}$	$-\sqrt{\frac{3}{2}}$	$\sqrt{\frac{2}{3}}$	$-\sqrt{6}$	$-\sqrt{\frac{2}{3}}$	$\sqrt{6}$	$2\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$
$\overline{K^0}\ \pi^-\ K^+$	3	0	0	1	0	3	-1
$K^0\ K^-\ \pi^+$	3	0	0	-1	0	-1	3
$\pi^0\ \pi^0\ \pi^0$	0	$\sqrt{3}$	$\sqrt{3}$	$\frac{\sqrt{3}}{2}$	$-5\sqrt{3}$	$-\frac{5\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
$\pi^0\ \pi^0\ \eta_8$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{1}{2\sqrt{3}}$	$-\sqrt{3}$	$\frac{7}{2\sqrt{3}}$	$\frac{3\sqrt{3}}{2}$
$\eta_8\ \eta_8\ \eta_8$	1	-1	3	$\frac{1}{2}$	-3	$-\frac{1}{2}$	$\frac{1}{2}$
$\eta_8\ K^-\ K^+$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{2}{3}}$	$\sqrt{6}$	0	$-\sqrt{6}$	0	$-\sqrt{\frac{2}{3}}$
$\eta_8\ \pi^-\ \pi^+$	$\sqrt{6}$	$\sqrt{\frac{2}{3}}$	$-\sqrt{6}$	$-\frac{1}{\sqrt{6}}$	$\sqrt{6}$	$3\sqrt{\frac{3}{2}}$	$\frac{7}{\sqrt{6}}$
$\pi^0\ K^-\ K^+$	$-\frac{3}{\sqrt{2}}$	$\sqrt{2}$	$\sqrt{2}$	0	$-5\sqrt{2}$	0	$\sqrt{2}$
$\pi^0\ \eta_8\ \eta_8$	0	1	1	$-\frac{5}{6}$	-5	$\frac{1}{6}$	$-\frac{13}{6}$
$\pi^0\ \pi^-\ \pi^+$	0	$-\sqrt{2}$	$-\sqrt{2}$	$-\frac{1}{\sqrt{2}}$	$5\sqrt{2}$	$\frac{5}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$

A.2 Δs= 1, MMM Completely Symmetric

	a_3^S	b_3^S	d_6^S	e_6^S	h_{15}^S	i_{15}^S	j_{15}^S
$\overline{B}^0$decays:							
$\pi^0\ \pi^0\ \overline{K^0}$	0	$\sqrt{2}$	$-\sqrt{2}$	$\frac{1}{\sqrt{2}}$	$-\sqrt{2}$	$-\frac{5}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\eta_8\ \eta_8\ \overline{K^0}$	0	$\sqrt{2}$	$-\sqrt{2}$	$-\frac{1}{3\sqrt{2}}$	$-\sqrt{2}$	$-\frac{7}{3\sqrt{2}}$	$-\frac{5}{3\sqrt{2}}$
$K^0\ \overline{K^0}\ \overline{K^0}$	0	$-2\sqrt{2}$	$2\sqrt{2}$	$\sqrt{2}$	$2\sqrt{2}$	$\sqrt{2}$	$3\sqrt{2}$
$\overline{K^0}\ \pi^-\ \pi^+$	0	-2	2	1	2	1	-5
$\overline{K^0}\ K^-\ K^+$	0	2	-2	1	-2	3	1
$\eta_8\ K^-\ \pi^+$	0	0	0	$\sqrt{\frac{2}{3}}$	0	$2\sqrt{6}$	$-2\sqrt{\frac{2}{3}}$
$\pi^0\ \eta_8\ \overline{K^0}$	0	0	0	$\frac{2}{\sqrt{3}}$	0	$-\frac{4}{\sqrt{3}}$	$\frac{4}{\sqrt{3}}$
$\pi^0\ K^-\ \pi^+$	0	0	0	$-\sqrt{2}$	0	$2\sqrt{2}$	$2\sqrt{2}$
B^-decays:							
$\pi^0\ \pi^0\ K^-$	0	$\sqrt{2}$	$\sqrt{2}$	$-\frac{1}{\sqrt{2}}$	$3\sqrt{2}$	$\frac{7}{\sqrt{2}}$	$\frac{5}{\sqrt{2}}$
$\eta_8\ \eta_8\ K^-$	0	$\sqrt{2}$	$\sqrt{2}$	$\frac{1}{3\sqrt{2}}$	$3\sqrt{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}}$
$K^0\ \overline{K^0}\ K^-$	0	-2	-2	1	-6	1	3
$K^-\ K^-\ K^+$	0	$2\ \sqrt{2}$	$2\sqrt{2}$	$\sqrt{2}$	$6\sqrt{2}$	$3\sqrt{2}$	$\sqrt{2}$
$K^-\ \pi^-\ \pi^+$	0	-2	-2	-1	-6	-3	-1
$\pi^0\ \eta_8\ K^-$	0	0	0	$\frac{2}{\sqrt{3}}$	0	$\frac{4}{\sqrt{3}}$	$\frac{4}{\sqrt{3}}$
$\pi^0\ \overline{K^0}\ \pi^-$	0	0	0	$\sqrt{2}$	0	$-2\sqrt{2}$	$-2\sqrt{2}$
$\eta_8\ \overline{K^0}\ \pi^-$	0	0	0	$\sqrt{\frac{2}{3}}$	0	$-2\sqrt{\frac{2}{3}}$	$-2\sqrt{\frac{2}{3}}$
$\overline{B}_s^0$decays:							
$\pi^0\ \pi^0\ \eta_8$	$\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	0	0	$2\sqrt{3}$	0	$-\frac{2}{\sqrt{3}}$
$\pi^0\ K^-\ K^+$	$\frac{3}{\sqrt{2}}$	0	$-2\sqrt{2}$	$-\frac{1}{\sqrt{2}}$	$4\sqrt{2}$	$\frac{7}{\sqrt{2}}$	$\frac{3}{\sqrt{2}}$
$K^0\ \pi^0\ \overline{K^0}$	$\frac{3}{\sqrt{2}}$	0	$2\sqrt{2}$	$\frac{1}{\sqrt{2}}$	$-4\sqrt{2}$	$-\frac{5}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
$K^0\ \eta_8\ \overline{K^0}$	$\sqrt{\frac{3}{2}}$	$2\sqrt{\frac{2}{3}}$	0	$-\frac{1}{\sqrt{6}}$	$-2\sqrt{6}$	$-\frac{7}{\sqrt{6}}$	$-\frac{7}{\sqrt{6}}$
$\overline{K^0}\ \pi^-\ K^+$	-3	0	0	1	0	1	-3
$K^0\ K^-\ \pi^+$	-3	0	0	0	0	-3	1
$\eta_8\ \eta_8\ \eta_8$	-1	-2	0	0	6	4	2
$\eta_8\ K^-\ K^+$	$-\sqrt{\frac{3}{2}}$	$-2\sqrt{\frac{2}{3}}$	0	$-\frac{1}{\sqrt{6}}$	$2\sqrt{6}$	$\sqrt{\frac{3}{2}}$	$-\frac{5}{\sqrt{6}}$
$\eta_8\ \pi^-\ \pi^+$	$-\sqrt{6}$	$2\sqrt{\frac{2}{3}}$	0	0	$-2\sqrt{6}$	0	$2\sqrt{\frac{2}{3}}$
$\pi^0\ \pi^0\ \pi^0$	0	0	$-2\ \sqrt{3}$	0	$4\sqrt{3}$	0	0
$\pi^0\ \eta_8\ \eta_8$	0	0	-2	$-\frac{4}{3}$	4	$\frac{8}{3}$	$-\frac{8}{3}$
$\pi^0\ \pi^-\ \pi^+$	0	0	$2\sqrt{2}$	0	$-4\sqrt{2}$	0	0

A.3 Δs= 0, MMM Completely Antisymmetric

	a_3^A	c_3^A	e_6^A	f_6^A	i_{15}^A	j_{15}^A	k_{15}^A
$\overline{B}_s^0$decays:							
$K^0\ K^-\ K^+$	0	-1	1	-3	1	1	-1
$K^0\ \pi^-\ \pi^+$	0	1	1	1	3	-5	1
$\eta_8\ \pi^-\ K^+$	0	$\sqrt{\frac{2}{3}}$	$\sqrt{6}$	$-\sqrt{\frac{2}{3}}$	$\sqrt{6}$	$-\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$
$K^0\ \pi^0\ \eta_8$	0	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	$-3\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$
$\pi^0\ \pi^-\ K^+$	0	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$3\sqrt{2}$	$\sqrt{2}$
B^-decays:							
$K^0\ \pi^-\ \overline{K^0}$	0	-1	1	1	1	1	3
$\pi^-\ K^-\ K^+$	0	1	1	-3	3	3	-3
$K^0\ K^-\ \pi^0$	0	0	$\sqrt{2}$	$-\sqrt{2}$	$-2\sqrt{2}$	$-2\sqrt{2}$	0
$K^0\ K^-\ \eta_8$	0	$-2\sqrt{\frac{2}{3}}$	$\sqrt{6}$	$\sqrt{\frac{2}{3}}$	0	0	$2\sqrt{6}$
$\pi^0\ \eta_8\ \pi^-$	0	$-\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$-\sqrt{3}$	$\sqrt{3}$
$\overline{B}^0$decays:							
$K^0\ \pi^0\ \overline{K^0}$	$-\frac{3}{\sqrt{2}}$	$-\sqrt{2}$	0	$\sqrt{2}$	$3\sqrt{2}$	$\sqrt{2}$	$-\frac{5}{\sqrt{2}}$
$K^0\ \eta_8\ \overline{K^0}$	$3\sqrt{\frac{3}{2}}$	$2\sqrt{\frac{2}{3}}$	$-\sqrt{6}$	$-2\sqrt{\frac{2}{3}}$	0	$-2\sqrt{\frac{2}{3}}$	$\frac{13}{\sqrt{6}}$
$\overline{K^0}\ K^+\ \pi^-$	3	1	1	-1	3	-1	0
$K^0\ K^-\ \pi^+$	-3	-1	1	-1	1	-3	0
$\eta_8\ K^-\ K^+$	$-3\sqrt{\frac{3}{2}}$	0	0	$\sqrt{\frac{2}{3}}$	0	$-2\sqrt{\frac{2}{3}}$	$\frac{7}{\sqrt{6}}$
$\eta_8\ \pi^-\ \pi^+$	0	$-\frac{1}{\sqrt{6}}$	$\sqrt{\frac{3}{2}}$	$-\frac{1}{\sqrt{6}}$	$-\sqrt{\frac{3}{2}}$	$\frac{5}{\sqrt{6}}$	$-\frac{5}{\sqrt{6}}$
$\pi^0\ K^-\ K^+$	$-\frac{3}{\sqrt{2}}$	0	0	$-\sqrt{2}$	0	$2\sqrt{2}$	$\frac{5}{\sqrt{2}}$
$\pi^0\ \pi^-\ \pi^+$	$3\sqrt{2}$	$\frac{3}{\sqrt{2}}$	$\frac{3}{\sqrt{2}}$	$\frac{3}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}}$

A.4 Δs= 1, MMM Completely Antisymmetric

	a_3^A	c_3^A	e_6^A	f_6^A	i_{15}^A	j_{15}^A	k_{15}^A
$\overline{B}^0$decays:							
$\overline{K^0}\ \pi^-\ \pi^+$	0	1	-1	3	-1	-1	1
$\overline{K^0}\ K^-\ K^+$	0	-1	-1	-1	-3	5	-1
$\eta_8\ \pi^+\ K^-$	0	$-\sqrt{\frac{2}{3}}$	0	$-2\sqrt{\frac{2}{3}}$	$\sqrt{6}$	$-5\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$
$\pi^0\ \eta_8\ \overline{K^0}$	0	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$\frac{1}{\sqrt{3}}$	$3\sqrt{3}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$
$K^-\ \pi^0\ \pi^+$	0	$-\sqrt{2}$	$-2\sqrt{2}$	0	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$
B^-decays:							
$K^0\ \overline{K^0}\ K^-$	0	1	-1	-1	-1	-1	-3
$K^-\ \pi^-\ \pi^+$	0	-1	-1	3	-3	-3	3
$\pi^0\ \eta_8\ K^-$	0	$\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$
$\pi^0\ \pi^-\ \overline{K^0}$	0	$-\sqrt{2}$	$2\sqrt{2}$	0	$-\sqrt{2}$	$-\sqrt{2}$	$3\sqrt{2}$
$\eta_8\ \pi^-\ \overline{K^0}$	0	$\sqrt{\frac{2}{3}}$	0	$-2\sqrt{\frac{2}{3}}$	$-\sqrt{6}$	$-\sqrt{6}$	$-\sqrt{6}$
$\overline{B}_s^0$decays:							
$\pi^0\ K^-\ K^+$	$\frac{3}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{3}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{3}{\sqrt{2}}$	$-2\sqrt{2}$
$K^0\ \pi^0\ \overline{K^0}$	$\frac{3}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{3}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$2\sqrt{2}$
$K^0\ \eta_8\ \overline{K^0}$	$-\sqrt{6}$	$-\frac{5}{\sqrt{6}}$	$\sqrt{\frac{3}{2}}$	$\frac{5}{\sqrt{6}}$	$3\sqrt{\frac{3}{2}}$	$\frac{5}{\sqrt{6}}$	$-7\sqrt{\frac{2}{3}}$
$\pi^-\ \overline{K^0}\ K^+$	-3	-1	1	-1	1	-3	0
$K^0\ K^-\ \pi^+$	3	1	1	-1	3	-1	0
$\eta_8\ K^-\ K^+$	$3\sqrt{\frac{3}{2}}$	$\frac{5}{\sqrt{6}}$	$\sqrt{\frac{3}{2}}$	$\frac{5}{\sqrt{6}}$	$\sqrt{\frac{3}{2}}$	$-\frac{1}{\sqrt{6}}$	$-\sqrt{\frac{2}{3}}$
$\eta_8\ \pi^-\ \pi^+$	0	0	0	$-2\sqrt{\frac{2}{3}}$	0	$4\sqrt{\frac{2}{3}}$	$2\sqrt{\frac{2}{3}}$
$\pi^0\ \pi^-\ \pi^+$	$-3\sqrt{2}$	0	0	0	0	0	$3\sqrt{2}$

Appendix B

$SU(3)$ Relations With η_8

In Chapter 6 our discussion of $SU(3)$ relations was limited to relations without η_8 due to $\eta - \eta_8$ mixing. The $SU(3)$ relations including decays to η_8 are all provided in Appendix B. Again we divide the appendix according to $\Delta s = 0$ or $\Delta s = 1$ and totally symmetric wavefunction or totally antisymmetric wavefunction.

B.1 $\Delta s = 0$, MMM Completely Symmetric

The simplest relation is:

$$\mathcal{A}(B^- \to (K^-K^0\pi^0)_S) = \sqrt{3}\,\mathcal{A}(B^- \to (K^-K^0\eta_8)_S). \tag{B.1}$$

Next we have several triangle relations:

$$\begin{aligned}\mathcal{A}(\overline{B}{}^0_s \to (K^0\pi^0\pi^0)_S) - 3\,\mathcal{A}(\overline{B}{}^0_s \to (K^0\eta_8\eta_8)_S) &= \mathcal{A}(\overline{B}{}^0_s \to (K^0K^0\overline{K}{}^0)_S)\\ \sqrt{3}\,\mathcal{A}(\overline{B}{}^0_s \to (K^0\pi^0\pi^0)_S) - \sqrt{3}\,\mathcal{A}(\overline{B}{}^0_s \to (K^0\eta_8\eta_8)_S) &= \sqrt{2}\,\mathcal{A}(\overline{B}{}^0_s \to (K^0\pi^0\eta_8)_S).\end{aligned} \tag{B.2}$$

The above two triangle relations can be combined to give two more. There are at least three more triangle relations:

$$\begin{aligned}\sqrt{3}\,\mathcal{A}(\overline{B}{}^0_s \to (\eta_8\pi^-K^+)_S) + \mathcal{A}(\overline{B}{}^0_s \to (\pi^0\pi^-K^+)_S) &= 2\sqrt{3}\,\mathcal{A}(B^- \to (K^-K^0\eta_8)_S)\\ \sqrt{6}\,\mathcal{A}(\overline{B}{}^0 \to (\pi^0\eta_8\eta_8)_S) + \sqrt{3}\,\mathcal{A}(\overline{B}{}^0 \to (\pi^0\pi^-\pi^+)_S) &= -2\sqrt{2}\,\mathcal{A}(\overline{B}{}^0_s \to (K^0\pi^0\eta_8)_S)\\ \mathcal{A}(\overline{B}{}^0 \to (K^0\pi^0\overline{K}{}^0)_S) - \sqrt{3}\,\mathcal{A}(\overline{B}{}^0 \to (K^0\eta_8\overline{K}{}^0)_S) &= \mathcal{A}(\overline{B}{}^0_s \to (K^0K^0\overline{K}{}^0)_S).\end{aligned} \tag{B.3}$$

There are a large number of quadrangle relations involving $B^- \to \pi^-\pi^-\pi^+$ (which is proportional to the $B^- \to \pi^0\pi^0\pi^-$ decay amplitude). We begin with the following:

$$\begin{aligned}&\sqrt{6}\,\mathcal{A}(\overline{B}{}^0_s \to (K^0\pi^0\eta_8)_S) + 2\,\mathcal{A}(\overline{B}{}^0_s \to (\pi^0\pi^-K^+)_S)\\ &\quad= \sqrt{6}\,\mathcal{A}(B^- \to (\eta_8\pi^-\pi^0)_S) - 2\sqrt{3}\,\mathcal{A}(B^- \to (\eta_8K^-K^0)_S)\\ &\quad= \sqrt{2}\,\mathcal{A}(B^- \to (\pi^-K^0\overline{K}{}^0)_S) - \mathcal{A}(B^- \to (\pi^-\pi^-\pi^+)_S).\end{aligned} \tag{B.4}$$

In addition are there several $B^- \to \pi^-\pi^-\pi^+$ relations which only involve B^- decays:

$$\begin{aligned}
&\sqrt{3}\,\mathcal{A}(B^- \to (\pi^-\pi^-\pi^+)_S) + 2\sqrt{2}\,\mathcal{A}(B^- \to (\pi^0\pi^-\eta_8)_S) \\
&\qquad = 8\,\mathcal{A}(B^- \to (K^-K^0\eta_8)_S) - 2\sqrt{3}\,\mathcal{A}(B^- \to (\pi^-\eta_8\eta_8)_S) \\
&\mathcal{A}(B^- \to (\pi^-\pi^-\pi^+)_S) - \sqrt{2}\,\mathcal{A}(B^- \to (\pi^-K^0\overline{K}^0)_S) \\
&\qquad = 2\sqrt{3}\,\mathcal{A}(B^- \to (K^-K^0\eta_8)_S) - \sqrt{6}\,\mathcal{A}(B^- \to (\pi^0\pi^-\eta_8)_S).
\end{aligned} \tag{B.5}$$

Combining the above two relations gives three more relations.

The following two decays can also be combined to give 3 more relations:

$$\begin{aligned}
&2\sqrt{3}\,\mathcal{A}(\overline{B}^0 \to (\eta_8K^-K^+)_S) + \mathcal{A}(\overline{B}^0_s \to (K^0\pi^0\pi^0)_S) \\
&\qquad = -3\,\mathcal{A}(\overline{B}^0_s \to (K^0\eta_8\eta_8)_S) - 2\,\mathcal{A}(B^0 \to (K^0\pi^0\overline{K}^0)_S) \\
&\mathcal{A}(\overline{B}^0_s \to (K^0\pi^0\pi^0)_S) + \sqrt{3}\,\mathcal{A}(\overline{B}^0 \to (K^0\eta_8\overline{K}^0)_S) \\
&\qquad = 3\,\mathcal{A}(\overline{B}^0_s \to (K^0\eta_8\eta_8)_S) + \mathcal{A}(\overline{B}^0 \to (K^0\pi^0\overline{K}^0)_S).
\end{aligned} \tag{B.6}$$

Next we look at quadrangle relations involving only $\overline{B}^0$ decays:

$$\begin{aligned}
&4\,\mathcal{A}(\overline{B}^0 \to (K^0\eta_8\overline{K}^0)_S) \\
&= -4\,\mathcal{A}(\overline{B}^0 \to (\eta_8K^-K^+)_S) + \sqrt{3}\,\mathcal{A}(\overline{B}^0 \to (\pi^0\pi^-\pi^+)_S) + \sqrt{6}\,\mathcal{A}(\overline{B}^0 \to (\pi^0\eta_8\eta_8)_S) \\
&= 4\,\mathcal{A}(\overline{B}^0 \to (\eta_8K^-K^+)_S) + \sqrt{2}\,\mathcal{A}(\overline{B}^0 \to (\pi^0\pi^0\eta_8)_S) - 3\sqrt{6}\,\mathcal{A}(\overline{B}^0 \to (\eta_8\eta_8\eta_8)_S)
\end{aligned} \tag{B.7}$$

and

$$\begin{aligned}
&\sqrt{3}\,\mathcal{A}(\overline{B}^0 \to (\pi^0\pi^0\pi^0)_S) - \sqrt{3}\,\mathcal{A}(\overline{B}^0 \to (\pi^0\pi^0\eta_8)_S) \\
&\qquad = \sqrt{2}\,\mathcal{A}(\overline{B}^0 \to (K^0\pi^0\pi^0)_S) - \sqrt{2}\,\mathcal{A}(\overline{B}^0 \to (K^0\pi^0\overline{K}^0)_S).
\end{aligned} \tag{B.8}$$

There are also a number of $\overline{B}^0_s$ decay relations:

$$\begin{aligned}
&2\,\mathcal{A}(\overline{B}^0_s \to (K^0K^-K^+)_S) - \sqrt{6}\,\mathcal{A}(\overline{B}^0_s \to (K^+\pi^-\eta_8)_S) \\
&\qquad = -\sqrt{2}\,\mathcal{A}(\overline{B}^0_s \to (K^+\pi^-\pi^0)_S) - 2\,\mathcal{A}(\overline{B}^0_s \to (K^0\pi^-\pi^+)_S) \\
&\qquad = \sqrt{2}\,\mathcal{A}(\overline{B}^0_s \to (K^0\pi^0\pi^0)_S) - \mathcal{A}(\overline{B}^0_s \to (K^0\pi^-\pi^+)_S)
\end{aligned} \tag{B.9}$$

and

$$\begin{aligned}
&2\sqrt{3}\,\mathcal{A}(\overline{B}^0_s \to (\pi^0\pi^-K^+)_S) + \sqrt{6}\,\mathcal{A}(\overline{B}^0_s \to (K^0\pi^-\pi^+)_S) \\
&\qquad = \sqrt{3}\,\mathcal{A}(\overline{B}^0_s \to (K^0K^0\overline{K}^0)_S) - 3\sqrt{2}\,\mathcal{A}(\overline{B}^0_s \to (K^0\pi^0\eta_8)_S) \\
&\qquad = -2\sqrt{3}\,\mathcal{A}(\overline{B}^0_s \to (K^0\eta_8\eta_8)_S) - 2\sqrt{2}\,\mathcal{A}(\overline{B}^0_s \to (K^0\pi^0\eta_8)_S).
\end{aligned} \tag{B.10}$$

Finally we give relations among decays of all three B mesons:

$$\begin{aligned}
\sqrt{6}\,&\mathcal{A}(\overline{B}^0_s \to (K^0\eta_8\eta_8)_S) - \sqrt{2}\,\mathcal{A}(\overline{B}^0 \to (K^0\eta_8\overline{K}^0)_S) \\
&= \mathcal{A}(\overline{B}^0 \to (\pi^0\pi^0\pi^0)_S) - \mathcal{A}(\overline{B}^0 \to (\pi^0\pi^0\eta_8)_S) \\
&\mathcal{A}(\overline{B}^0_s \to (\eta_8\pi^-K^+)_S) + \sqrt{3}\,\mathcal{A}(\overline{B}^0_s \to (\pi^0\pi^-K^+)_S) \\
&= \sqrt{2}\,\mathcal{A}(B^- \to (\pi^0\eta_8\pi^-)_S) - \sqrt{2}\,\mathcal{A}(\overline{B}^0_s \to (K^0\pi^0\eta_8)_S) \\
&\mathcal{A}(\overline{B}^0_s \to (K^0K^-K^+)_S) - \sqrt{6}\,\mathcal{A}(B^- \to (K^0\eta_8K^-)_S) \\
&= 3\sqrt{2}\,\mathcal{A}(\overline{B}^0_s \to (K^0\eta_8\eta_8)_S) - \sqrt{2}\,\mathcal{A}(\overline{B}^0_s \to (K^0K^0\overline{K}^0)_S).
\end{aligned} \tag{B.11}$$

B.2 $\Delta s = 1$, MMM Completely Symmetric

In this section we give $\Delta s = 1$ relations when all the mesons have relative even angular momentum. There are several simple relations:

$$\begin{aligned}
\mathcal{A}(\overline{B}^0 \to (K^-\pi^0\pi^+)_S) &= -\mathcal{A}(B^- \to (\overline{K}^0\pi^0\pi^-)_S) \\
\mathcal{A}(\overline{B}^0_s \to (\pi^0\pi^0\pi^0)_S) &= -\sqrt{\frac{3}{2}}\,\mathcal{A}(\overline{B}^0_s \to (\pi^-\pi^+\pi^0)_S) \\
-\sqrt{2}\,\mathcal{A}(\overline{B}^0_s \to (\eta_8\pi^0\pi^0)_S) &= \mathcal{A}(\overline{B}^0_s \to (\eta_8\pi^-\pi^+)_S) \\
\mathcal{A}(B^- \to (K^-K^-K^+)_S) &= -\sqrt{2}\,\mathcal{A}(B^- \to (K^-\pi^-\pi^+)_S).
\end{aligned} \tag{B.12}$$

The following triangle relations also result in a number of other triangle and quadrangle relations:

$$\begin{aligned}
&\mathcal{A}(\overline{B}^0 \to (K^0\overline{K}^0\overline{K}^0)_S) \\
&= \mathcal{A}(\overline{B}^0 \to (\overline{K}^0\pi^0\pi^0)_S) - 3\,\mathcal{A}(\overline{B}^0 \to (\overline{K}^0\eta_8\eta_8)_S) \\
&= \sqrt{\frac{2}{3}}\,\mathcal{A}(\overline{B}^0 \to (\pi^0\eta_8\overline{K}^0)_S) - 2\,\mathcal{A}(\overline{B}^0 \to (\overline{K}^0\eta_8\eta_8)_S) \\
&= \sqrt{6}\,\mathcal{A}(B^- \to (K^-\eta_8\pi^0)_S) - \sqrt{2}\,\mathcal{A}(\overline{B}^0 \to (\overline{K}^0K^-K^+)_S) \\
&= \mathcal{A}(\overline{B}^0_s \to (\overline{K}^0K^0\pi^0)_S) - \sqrt{3}\,\mathcal{A}(\overline{B}^0_s \to (\overline{K}^0K^0\eta_8)_S).
\end{aligned} \tag{B.13}$$

There are three more independent triangle relations:

$$\begin{aligned}
2\sqrt{2}\,\mathcal{A}(\overline{B}^0 \to (\pi^0\eta_8\overline{K}^0)_S) + \sqrt{6}\,\mathcal{A}(\overline{B}^0_s \to (\pi^0\eta_8\eta_8)_S) + \sqrt{3}\,\mathcal{A}(\overline{B}^0_s \to (\pi^0\pi^-\pi^+)_S) &= 0 \\
\mathcal{A}(B^- \to (\pi^-\overline{K}^0\pi^0)_S) = \sqrt{3}\,\mathcal{A}(B^- \to (\pi^-\overline{K}^0\eta_8)_S) - \mathcal{A}(\overline{B}^0 \to (K^-\pi^0\pi^+)_S) &= 0 \\
\mathcal{A}(B^- \to (K^-K^0\overline{K}^0)_S) - \mathcal{A}(B^- \to (K^-\pi^-\pi^+)_S) - \sqrt{3}\,\mathcal{A}(B^- \to (K^-\eta_8\pi^0)_S) &= 0.
\end{aligned} \tag{B.14}$$

As for quadrangle relations we begin with B^- decay relations. The first two relations can be combined to obtain two more:

$$\begin{aligned}
&2\ \mathcal{A}(B^- \to (\pi^0\eta_8K^-)_S) + \sqrt{6}\ \mathcal{A}(B^- \to (K^-\eta_8\eta_8)_S) \\
&\quad = \sqrt{2}\ \mathcal{A}(B^- \to (\eta_8\overline{K}^0\pi^-)_S) - \sqrt{3}\ \mathcal{A}(B^- \to (K^-\pi^-\pi^+)_S) \\
&\mathcal{A}(B^- \to (K^-\pi^0\pi^0)_S) + 3\ \mathcal{A}(B^- \to (K^-\eta_8\eta_8)_S) \\
&\quad = \sqrt{6}\ \mathcal{A}(B^- \to (K^-\eta_8\pi^0)_S) - 2\sqrt{2}\ \mathcal{A}(B^- \to (K^-K^0\overline{K}^0)_S) \\
&2\ \mathcal{A}(B^- \to (\overline{K}^0\pi^0\pi^-)_S) + \mathcal{A}(B^- \to (K^-\pi^0\pi^0)_S) \\
&\quad = 3\ \mathcal{A}(B^- \to (K^-\eta_8\eta_8)_S) + \sqrt{2}\ \mathcal{A}(B^- \to (K^-K^0\overline{K}^0)_S).
\end{aligned} \tag{B.15}$$

There is one relation involving only $\overline{B}^0$:

$$\begin{aligned}
&\sqrt{6}\ \mathcal{A}(\overline{B}^0 \to (\eta_8\pi^+K^-)_S) + \sqrt{2}\ \mathcal{A}(\overline{B}^0 \to (\overline{K}^0\pi^0\pi^0)_S) = \\
&\quad = 2\ \mathcal{A}(\overline{B}^0 \to (\overline{K}^0K^-K^+)_S) + \mathcal{A}(\overline{B}^0 \to (\overline{K}^0\pi^-\pi^+)_S).
\end{aligned} \tag{B.16}$$

Finally there are four quadrangle relations involving decays of two or more B mesons:

$$\begin{aligned}
&\sqrt{2}\ \mathcal{A}(\overline{B}^0 \to (K^-\pi^0\pi^+)_S) + 3\sqrt{2}\ \mathcal{A}(B^- \to (K^-\eta_8\eta_8)_S) \\
&\quad = -2\ \mathcal{A}(B^- \to (K^-K^0\overline{K}^0)_S) - \mathcal{A}(B^- \to (K^-\pi^-\pi^+)_S) \\
&\sqrt{2}\ \mathcal{A}(\overline{B}^0_s \to (\eta_8\pi^-\pi^+)_S) + \sqrt{2}\ \mathcal{A}(\overline{B}^0_s \to (\eta_8K^-K^+)_S) = \\
&\quad = \sqrt{2}\ \mathcal{A}(\overline{B}^0 \to (\eta_8\pi^+K^-)_S) + \sqrt{2}\ \mathcal{A}(\overline{B}^0_s \to (K^0K^-\pi^+)_S) \\
&\sqrt{2}\ \mathcal{A}(\overline{B}^0_s \to (K^0K^-\pi^+)_S) + \mathcal{A}(\overline{B}^0_s \to (\pi^0\pi^-\pi^+)_S) \\
&\quad = \sqrt{3}\ \mathcal{A}(\overline{B}^0_s \to (\eta_8\pi^-\pi^+)_S) - \sqrt{2}\ \mathcal{A}(\overline{B}^0 \to (\overline{K}^0K^-K^+)_S) \\
&\mathcal{A}(\overline{B}^0_s \to (\pi^0\pi^0\eta_8)_S) - 3\sqrt{3}\ \mathcal{A}(\overline{B}^0_s \to (\eta_8\eta_8\eta_8)_S) \\
&\quad = 4\sqrt{2}\ \mathcal{A}(\overline{B}^0_s \to (K^0\eta_8\overline{K}^0)_S) + 2\ \mathcal{A}(\overline{B}^0 \to (\pi^0\eta_8\overline{K}^0)_S).
\end{aligned} \tag{B.17}$$

B.3 $\Delta s = 0$, MMM Completely Antisymmetric

Beginning with $\Delta s = 0$ relations, there are several triangle relations:

$$\begin{aligned}
\mathcal{A}(\overline{B}^0 \to (K^0\eta_8\overline{K}^0)_A) + \sqrt{3}\ \mathcal{A}(\overline{B}^0 \to (K^0\pi^0\overline{K}^0)_A) + \sqrt{2}\ \mathcal{A}(\overline{B}^0_s \to (K^0\pi^0\eta_8)_A) &= 0 \\
\sqrt{3}\ \mathcal{A}(B^-0 \to (K^0K^-\eta_8)_A) -\ \mathcal{A}(B^- \to (K^0K^-\pi^0)_A) - 2\sqrt{2}\ \mathcal{A}(B^- \to (\pi^-\overline{K}^0K^0)_A) &= 0 \\
\sqrt{2}\ \mathcal{A}(B^-0 \to (K^0K^-\eta_8)_A) -\ \mathcal{A}(B^- \to (\pi^0\eta_8\pi^-)_A) - \sqrt{3}\ \mathcal{A}(B^- \to (\pi^-\overline{K}^0K^0)_A) &= 0.
\end{aligned} \tag{B.18}$$

Combining the last two listed above gives 2 more triangle relations. The quadrangle relations are:

$$\begin{aligned}
&\sqrt{3}\ \mathcal{A}(\overline{B}^0_s \to (\eta_8\pi^-K^+)_A) \qquad\qquad (B.19)\\
&\quad = 3\ \mathcal{A}(\overline{B}^0_s \to (\pi^0\pi^-K^+)_A) + 6\ \mathcal{A}(B^- \to (K^0K^-\pi^0)_A) - 2\sqrt{3}\ \mathcal{A}(\overline{B}^0_s \to (K^0\pi^0\eta_8)_A)\\
&\quad = \mathcal{A}(\overline{B}^0_s \to (\pi^0\pi^-K^+)_A) + \sqrt{2}\ \mathcal{A}(\overline{B}^0_s \to (K^0K^-K^+)_A) + \sqrt{2}\ \mathcal{A}(\overline{B}^0_s \to (K^0\pi^-\pi^+)_A)\\
&\quad = \sqrt{3}\ \mathcal{A}(\overline{B}^0 \to (\eta_8K^-K^+)_A) + \sqrt{3}\ \mathcal{A}(\overline{B}^0 \to (\eta_8\pi^-\pi^+)_A) + \frac{3}{\sqrt{2}}\ \mathcal{A}(\overline{B}^0 \to (\overline{K}^0K^+\pi^-)_A).
\end{aligned}$$

Again, more quadrangle relations can be found using the equations above.

B.4 $\Delta s = 1$, MMM Completely Antisymmetric

For $\Delta s = 1$ decays with antisymmetric MMM, we begin with the triangle relation:

$$\sqrt{3}\ \mathcal{A}(\overline{B}^0_s \to (\overline{K}^0K^0\pi^0)_A) + \ \mathcal{A}(\overline{B}^0_s \to (\overline{K}^0K^0\eta_8)_A) = \sqrt{2}\ \mathcal{A}(\overline{B}^0 \to (\pi^0\eta_8\overline{K}^0)_A). \qquad (B.20)$$

There are a number of quadrangle relations that, when combined, result in several triangle relations. The quadrangle relations are:

$$\begin{aligned}
&2\ \mathcal{A}(\overline{B}^0 \to (\eta_8\pi^+K^-)_A) - 2\sqrt{2}\ \mathcal{A}(\overline{B}^0 \to (\pi^0\eta_8\overline{K}^0)_A)\\
&\quad = \sqrt{6}\ \mathcal{A}(B^- \to (K^0K^-\overline{K}^0)_A) - 3\sqrt{2}\ \mathcal{A}(B^- \to (\pi^0K^-\eta_8)_A)\\
&\quad = \sqrt{3}\ \mathcal{A}(B^- \to (\overline{K}^0\pi^0\pi^-)_A) + 3\ \mathcal{A}(B^- \to (\eta_8\pi^-\overline{K}^0)_A)\\
&\quad = 2\sqrt{3}\ \mathcal{A}(B^- \to (\pi^0\pi^-\overline{K}^0)_A) + 2\sqrt{6}\ \mathcal{A}(B^- \to (K^0K^-\overline{K}^0)_A)\\
&\quad = 6\ \mathcal{A}(B^- \to (\eta_8\pi^-\overline{K}^0)_A) - 2\sqrt{6}\ \mathcal{A}(B^- \to (K^0K^-\overline{K}^0)_A) \qquad (B.21)
\end{aligned}$$

and combining the equations just listed gives the triangle relations:

$$\begin{aligned}
\mathcal{A}(B^- \to (\eta_8\pi^-\overline{K}^0)_A) + \sqrt{3}\ \mathcal{A}(B^- \to (\pi^0\pi^-\overline{K}^0)_A) + 2\sqrt{2}\ \mathcal{A}(B^- \to (K^-\pi^0\eta_8)_A) &= 0\\
2\sqrt{2}\ \mathcal{A}(B^- \to (K^0K^-\overline{K}^0)_A) + \mathcal{A}(B^- \to (\overline{K}^0\pi^0\pi^-)_A) - \sqrt{3}\ \mathcal{A}(B^- \to (\eta_8\pi^-\overline{K}^0)_A) &= 0\\
\sqrt{3}\ \mathcal{A}(B^- \to (K^0K^-\overline{K}^0)_A) - \mathcal{A}(B^- \to (K^-\pi^0\eta_8)_A) - \sqrt{2}\ \mathcal{A}(B^- \to (\eta_8\pi^-\overline{K}^0)_A) &= 0\\
\mathcal{A}(B^- \to (K^0K^-\overline{K}^0)_A) + \sqrt{3}\mathcal{A}(B^- \to (K^-\pi^0\eta_8)_A) + \sqrt{2}\ \mathcal{A}(B^- \to (\overline{K}^0\pi^0\pi^-)_A) &= 0.
\end{aligned} \qquad (B.22)$$

Finally we have another set of quadrangle relations:

$$\begin{aligned}
&\sqrt{2}\ \mathcal{A}(\overline{B}^0 \to (\overline{K}^0K^-K^+)_A) \qquad\qquad (B.23)\\
&\quad = \sqrt{3}\ \mathcal{A}(\overline{B}^0_s \to (\eta_8\pi^-\pi^+)_A) - \sqrt{2}\ \mathcal{A}(\overline{B}^0_s \to (K^0K^-\pi^+)_A) - \mathcal{A}(\overline{B}^0_s \to (\pi^0\pi^-\pi^+)_A)\\
&\quad = \mathcal{A}(\overline{B}^0_s \to (\pi^0K^-K^+)_A) - \sqrt{2}\ \mathcal{A}(\overline{B}^0_s \to (\pi^-\overline{K}^0K^+)_A) - \sqrt{3}\ \mathcal{A}(\overline{B}^0_s \to (\eta_8K^-K^+)_A)\\
&\quad = \mathcal{A}(\overline{B}^0 \to (K^-\pi^0\pi^+)_A) - \sqrt{2}\ \mathcal{A}(\overline{B}^0 \to (\overline{K}^0\pi^-\pi^+)_A) - \sqrt{3}\ \mathcal{A}(\overline{B}^0 \to (\eta_8\pi^+K^-)_A).
\end{aligned}$$

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