Two Interactions Between Combinatorics and Representation Theory: Monomial Immanants and Hochschild Cohomology

by

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Abstract

This thesis consists of two independent parts. The first part concerns Stanley's symmetric function generalization of the chromatic polynomial, the series of immanant conjectures made by Stembridge, and Zaslavsky's theory of signed graphs. The conjectures made by Stembridge that the so-called "monomial" immanants are nonnegative on totally positive matrices and monomial positive on Jacobi-Trudi matrices are shown to hold for several infinite families of these monomial immanants. We make use of results of Stembridge and Goulden-Jackson which reduce both of these conjectures to a statement about some elements in the group algebra of the symmetric group. We also show that a more general conjecture by Stembridge concerning acyclic digraphs with certain path-intersection properties can be reduced to the conjectures considered above. A particular consequence is that the result of Greene that (ordinary) immanants of Jacobi-Trudi matrices are monomial positive can be extended to a wider class of combinatorially-defined matrices satisfying some simple conditions. Some new relationships between these conjectures and graph coloring are developed. Analogues of Stanley's chromatic symmetric function are given for signed graphs, and their basic properties are studied. There are some interesting connections with hyperplane arrangements. These also imply some new results about ordinary graphs.

The second part concerns the so-called "Hodge-type" decompositions of Hochschild (co)homology. Let A be a commutative algebra over a field of characteristic zero, and M be a symmetric A-bimodule. Gerstenhaber and Schack have shown that there are decompositions $H_n(A,M) = \oplus_k H_{k,n-k}(A,M)$, $H^n(A,M) = \oplus_k H^{k,n-k}(A,M)$ of the Hochschild (co)homology. The first summands, $H_{1,n-1}(A,M)$ and $H^{1,n-1}(A,M)$, are known to be the Harrison (co)homology defined in terms of shuffles. We discuss interpretations of the decompositions in terms of k-shuffles and how these relate to versions of the Poincaré-Birkoff-Witt theorem. We then turn to a detailed study of how the decomposition behaves with respect to the Gerstenhaber operations (cup and Lie products) in cohomology. We show by example that neither product is generally graded, but that $\mathcal{F}_q = \bigoplus_{r\geq q} H^{*,r}(A,A)$ are ideals for both products with $\mathcal{F}_p \cup \mathcal{F}_q \subseteq \mathcal{F}_{p+q}$ and $[\mathcal{F}_p, \mathcal{F}_q] \subseteq \mathcal{F}_{p+q}$. The statements for the cup product were conjectured in this form by Gerstenhaber and Schack.

The results in the second part were obtained in collaboration with Nantel Bergeron.

Thesis Supervisor: Richard P. Stanley
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Acknowledgments

I should begin by thanking my advisor, Richard Stanley, particularly for introducing me to the topics covered in the first part of this thesis, and also for his classes on posets and generating functions, where I learned many of the classical results and techniques of combinatorics which are used (or even taken for granted) here.

I owe a great deal of gratitude to Nantel Bergeron, who served as my undergraduate advisor at Harvard and collaborated with me on Hochschild cohomology during my first year at MIT. The results of our collaboration are contained in the second part of this thesis. It was Nantel's enthusiasm for the subject that first attracted me to combinatorics, and he has been an unfailing source of encouragement throughout the years that I have known him.

All of my teachers have contributed, indirectly or otherwise, to this document. Most of what I know about symmetric functions comes from lectures given by Sergey Fomin, and my notes from those lectures are still my principal reference on the subject. The influence of Gian-Carlo Rota's class on the role of Hopf algebras in combinatorics can also be felt here.

I would like to thank the many people who have taken the time to discuss the topics covered here with me and have provided useful comments, including Stavros Garoufalidis, Murray Gerstenhaber, Michael Hutchings, Gadi Perets, Bruce Sagan, and John Stembridge. I am particularly indebted to Dror Bar-Natan for giving me the opportunity to present several lectures on Hopf algebras and Hochschild cohomology in his knot theory class.

Finally, I would like to thank the members of my thesis committee, Sergey Fomin, Curtis Greene, and Richard Stanley, for their time and valuable comments.

Contents

List of Figures

Introduction

This thesis consists of two independent parts. The first part concerns Stanley's symmetric function generalization of the chromatic polynomial, the series of conjectures made by Stembridge about immanants, and Zaslavsky's theory of signed graphs. The second part concerns the so-called "Hodge-type" decomposition of the Hochschild (co)homology of a commutative algebra. The results in the second part were obtained in collaboration with Nantel Bergeron.

In this introduction, we discuss some of the background material related to these topics and give an overview of the results which will be presented in the chapters to follow.

Part I

Stanley's Chromatic Symmetric Function

Let $\Gamma = (V, E)$ be a graph with d vertices. A proper coloring of Γ is a map $\kappa : V \to \mathbb{P}$ such that adjacent vertices are colored differently. The chromatic polynomial $\chi_\Gamma(n)$ counts the number of proper colorings with images in $[n] = \{1, 2, \ldots, n\}$. The chromatic polynomial appears in Birkhoff [3] in terms of colorings of "maps" on surfaces and is one of the fundamental tools of graph theory. It is easily seen that $\chi_\Gamma(n)$ is a polynomial in n.

In [41], Stanley defines a symmetric function generalization of the chromatic polynomial which contains information not only about the number of proper colorings, but also about how many times each color is used:

$$X_\Gamma = \sum_{\kappa \text{ proper}} x^\kappa$$

where $x^\kappa = \prod_{v \in V} x_{\kappa(v)}$. The property of having adjacent vertices colored differently is not affected by permutations of $\mathbb{P}$ (the names of the colors), so X_Γ is a symmetric function in the variables $\{x_1, x_2, \ldots\}$. Specializing the variables to $x_1 = x_2 = \cdots = x_n = 1$, $x_{n+1} = x_{n+2} = \cdots = 0$ yields $\chi_\Gamma(n)$. Clearly X_Γ is homogeneous of degree d (the number of vertices).

Let Sym denote the ring of symmetric functions with rational coefficients in the variables $\{x_1, x_2, \ldots\}$. The elementary properties of symmetric functions referred to in this introduction can be found in [32]. The homogeneous component of Sym of degree d has several natural bases, indexed by partitions, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, of d. So, a reasonable question to ask is: What properties of Γ are reflected in the coefficients obtained when X_Γ is expanded in these various bases?

A partition π of the vertices of Γ is called *stable* if the induced subgraph on each block of π is totally disconnected, i.e. if there are no edges connecting two vertices in the same block. Similarly, π is said to be *connected* if the induced subgraph on each block is connected.

The coefficients in the expansion of X_Γ in the augmented monomial basis,

$$\widetilde{m}_\lambda = \sum_{\substack{(i_1,i_2,\ldots,i_k)\\ i_1,i_2,\ldots,i_k \text{ distinct}}} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_k}^{\lambda_k},$$

are given by the following result, which follows easily from the definition of X_Γ.

Proposition 1 (Stanley [41, Prop. 2.4]) *If a_λ denotes the number of stable partitions of Γ of type λ, then*

$$X_\Gamma = \sum_{\lambda \vdash d} a_\lambda \widetilde{m}_\lambda.$$

Stanley proves several results regarding the expansion of X_Γ in the power sum basis, $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$, where $p_j = \sum_i x_i^j$. These expansions can be viewed as generalizations of properties of the chromatic polynomial. The most important of these from our point of view will be the expansion in terms of the *lattice of contractions* of Γ, L_Γ (also known as the *bond lattice* of Γ). L_Γ is the set of connected partitions of Γ, partially ordered by refinement.

Theorem 1.2.1[1] ([41, Thm. 2.6])

$$X_\Gamma = \sum_{\pi \in L_\Gamma} \mu(\hat{0}, \pi) p_{\text{type}(\pi)},$$

where μ is the Möbius function of L_Γ.

Theorem 1.2.1 generalizes Whitney's computation of the coefficients of the chromatic polynomial (which can be seen to be equivalent to a determinant formula given by Birkhoff).

Theorem 2 (Whitney [49], see also Birkhoff [3])

$$\chi_\Gamma(n) = \sum_{\pi \in L_\Gamma} \mu(\hat{0}, \pi) n^{|\pi|},$$

where $|\pi|$ denotes the number of blocks of π.

Theorem 2 is important for a number of reasons. It expresses a connection between the chromatic theory of graphs and lattice theory. Generalizations of this idea lead to the theory of *matroids* (see [48], for example), and in particular, to the theory of hyperplane arrangements and Zaslavsky's *signed graphs* which are discussed below.

[1] Results which are discussed in detail in the body of the text are numbered according to their appearance there.

We should note that while the chromatic polynomial is completely determined by the poset structure of L_Γ, X_Γ is not — the expansion in Theorem 1.2.1 also contains information about the sizes of the blocks. In particular, while all trees on d vertices have isomorphic bond lattices (i.e. boolean algebras on the set of $d-1$ edges), there are no known examples of nonisomorphic trees having the same X_Γ (see [41]).

It can be shown that L_Γ is a geometric lattice, and it follows, via a result of Rota [36], that the Möbius function of L_Γ strictly alternates in sign:

$$(-1)^{d-|\pi|}\mu(\hat{0},\pi) > 0.$$

The usual involution, ω, on Sym can be defined by $\omega p_\lambda = (-1)^{d-\ell(\lambda)}p_\lambda$ when λ is a partition of d with $\ell(\lambda)$ parts. The results considered above imply that ωX_Γ has nonnegative coefficients when expanded in the power sum basis. In general, if $\{b_\lambda\}$ is a basis of Sym, we will say that $F \in$ Sym is b-positive when the expansion of F in $\{b_\lambda\}$ has nonnegative coefficients. We will also say that Γ is b-positive when X_Γ is b-positive.

A number of interesting results and conjectures are related to the expansion of X_Γ in the elementary and Schur bases of Sym. The elementary basis is defined by

$$e_n = \sum_{i_1<i_2<\cdots<i_n} x_{i_1}x_{i_2}\cdots x_{i_n},$$

and $e_\lambda = e_{\lambda_1}\cdots e_{\lambda_k}$. The Schur basis can be defined by letting s_λ denote the sum of monomials corresponding to semi-standard tableaux of shape λ. The Littlewood-Richardson rule can be used to show that each e_λ is s-positive. So the e-positivity is a stronger condition than s-positivity.

In general, X_Γ need not be s-positive. For example, the "claw" K_{13} fails to be s-positive, and hence also fails to be e-positive. However, certain families of graphs are known or conjectured to be e-positive or s-positive. Of particular interest are the *incomparability graphs* of posets: if P is a poset, let $\mathrm{inc}(P)$ denote the graph whose vertices are the elements of P and whose edges connect pairs of incomparable elements.

We will call a poset $(\mathbf{a}+\mathbf{b})$*-free* when it contains no induced subposet isomorphic to the disjoint union of chains of cardinalities a and b. Gasharov has shown the following theorem.

Theorem 1.2.7 ([10]) *If P is a $(\mathbf{3}+\mathbf{1})$-free poset, then $X_{\mathrm{inc}(P)}$ is s-positive.*

There are two natural ways to strengthen the statement in Theorem 1.2.7. Since a poset is $(\mathbf{3}+\mathbf{1})$-free if and only if its incomparability graph has no induced subgraph isomorphic to $K_{13} = \mathrm{inc}(\mathbf{3}+\mathbf{1})$, it is natural to conjecture that s-positivity might hold for all *clawfree* graphs. It is also possible that the graphs in Theorem 1.2.7 might actually be e-positive.

Conjecture 1.2.8 (Gasharov, see [42, Conj. 1.4]) *If Γ is clawfree (i.e. has no induced subgraph isomorphic to K_{13}), then Γ is s-positive.*

Conjecture 1.2.9 ([43, Conj. 5.5], [41, Conj. 5.1]) *If P is a $(\mathbf{3}+\mathbf{1})$-free poset, then $X_{\mathrm{inc}(P)}$ is e-positive.*

We should note that examples of Stanley [41] show that clawfree graphs need not be e-positive and that e-positive graphs need not be clawfree. Stanley and Stembridge [43] report that Conjecture 1.2.9 has been verified for all posets with fewer than eight elements, and Stembridge has verified (see [41]) that, among the 16999 posets with eight elements, the 2469 which are $(\mathbf{3}+\mathbf{1})$-free all satisfy Conjecture 1.2.9. So there is significant empirical evidence, at least, to support Conjecture 1.2.9.

Posets which are both $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free are known as *semi-orders*, and their incomparability graphs are called *indifference graphs* or *unit interval orders*. It can be shown that a poset P is $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free if and only if it possesses a linear extension $\alpha : P \to [n]$ such that

$$\left.\begin{array}{r}(x <_P y \text{ and } \alpha(y) <_{\mathbb{P}} \alpha(z)) \\ \text{or } (\alpha(x) <_{\mathbb{P}} \alpha(y) \text{ and } y <_P z)\end{array}\right\} \Longrightarrow x <_P z,$$

or equivalently, there is a labeling of the vertices of $\Gamma = \mathrm{inc}(P)$ by elements of $[n]$ and a set of intervals $[i,j] \subseteq [n]$ such that Γ is the edge-union of cliques on these intervals (the same α can be used for the labeling). General information about indifference graphs can be found in [7].

Stanley and Stembridge [43] have shown that the (conjectured) e-positivity of indifference graphs can be viewed as a special case of the monomial immanant conjectures discussed in the next section. In particular, the results we will present in Chapter 2 imply that for certain λ's, the coefficient of e_λ in the expansion of X_Γ is nonnegative for every indifference graph Γ.

One of the new results we will present in Chapter 1 is the following proposition, which gives a necessary condition for a graph to be e-positive.

Proposition 1.3.3 *If X_Γ is e-positive, and Γ has a connected partition of type λ, then Γ has a connected partition of type μ for every μ which is a refinement of λ. In particular, a connected, e-positive graph must have a connected partition of every type.*

We show by example that the converse is false. However, Proposition 1.3.3 still raises a lot of interesting questions. For example, we know of no non-e-positive graph for which every induced subgraph satisfies the conclusions of the proposition.

We will also show that if Γ is the incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free poset, then every connected subgraph of Γ has a Hamiltonian path. This shows that Conjecture 1.2.9 is consistent with Proposition 1.3.3, and is a somewhat interesting result in its own right. (For an indifference graph, the existence of these Hamiltonian paths follows trivially from the characterization discussed above.) This helps to explain the difference between incomparability graphs of $(\mathbf{3}+\mathbf{1})$-free posets and arbitrary clawfree graphs, since we will show by example that a connected clawfree graph need not have a connected partition of each type.

Although X_Γ need not be e-positive, the following result shows that certain sums of coefficients in the elementary expansion are nonnegative.

Theorem 1.2.3 (Stanley [41, Thm. 3.3]) *Let* $X_\Gamma = \sum_{\lambda \vdash d} c_\lambda e_\lambda$ *and let* $\mathrm{sink}(\Gamma, j)$ *be the number of acyclic orientations of* Γ *with* j *sinks. Then*

$$\mathrm{sink}(\Gamma, j) = \sum_{\substack{\lambda \vdash d \\ \ell(\lambda)=j}} c_\lambda.$$

Another relationship between X_Γ and the acyclic orientations of Γ is given by the following reciprocity result.

Theorem 1.2.4 (Stanley [41, Thm. 4.2])

$$\omega X_\Gamma = \sum_{(\mathfrak{o},\kappa)} x^\kappa$$

where the sum runs over pairs $(\mathfrak{o}, \kappa)$ *of acyclic orientations and colorings satisfying* $\kappa(u) \leq \kappa(v)$ *if* (v, u) *is an edge of* $\mathfrak{o}$.

In the last section of Chapter 1, we will consider the images of X_Γ under some algebra maps related to the Hopf algebra structure of Sym. These considerations will lead to the proof of Proposition 1.3.3 discussed above. We will also see that Theorems 1.2.3 and 1.2.4 have a common generalization.

Monomial Immanants

If A is an $n \times n$ matrix with coefficients in any monoid, we will employ the notation:

$$[A] = \sum_{\sigma \in \mathfrak{S}_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \cdot \sigma^{-1}. \tag{0.1}$$

The familiar formula for the determinant of A can be obtained by applying the sign character to $[A]$, i.e. replacing each permutation σ^{-1} with its sign $(-1)^\sigma$. If one, instead, replaces each σ^{-1} by 1 (i.e. applies the trivial character), the result is called the *permanent* of A ($\mathrm{per}(A)$). More generally, if one applies any irreducible character, χ^λ, of the symmetric group, $\mathfrak{S}_n$, the corresponding matrix function,

$$\chi^\lambda[A] = \sum_{\sigma \in \mathfrak{S}_n} \chi^\lambda(\sigma^{-1}) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)},$$

is called an *immanant*. It is common to see $\chi^\lambda(\sigma)$ in the definition above, but any class function on $\mathfrak{S}_n$ satisfies $\chi(\sigma^{-1}) = \chi(\sigma)$.

Historically, part of the interest in these matrix functions stems from results and conjectures about their values on positive semi-definite Hermitian matrices. For example, there is Schur's Dominance Theorem and the Permanental Dominance Conjecture of Lieb.

Theorem 3 ([37]) *If* A *is a positive semi-definite Hermitian matrix, then*

$$\frac{1}{\chi^\lambda(1)} \chi^\lambda[A] \geq \det(A) \geq 0.$$

Conjecture 4 ([29]) *If A is a positive semi-definite Hermitian matrix, then*

$$\mathrm{per}(A) \geq \frac{1}{\chi^\lambda(1)}\chi^\lambda[A].$$

In the present thesis, we will be concerned mostly with the *monomial immanants* introduced by Stembridge [46]. These are the matrix functions corresponding to *monomial characters*, ϕ^λ, which can be defined as follows.

A fundamental result in the representation theory of the symmetric group (one form of the Frobenius character formula) says that the character table of $\mathfrak{S}_n$ is given by

$$\chi^\lambda(\sigma) = \langle s_\lambda, p_{\mathrm{type}(\sigma)}\rangle$$

where the inner product on Sym used on the right hand side is defined by $\langle m_\lambda, h_\mu\rangle = \delta_{\lambda\mu}$, or equivalently, by saying that the Schur functions s_λ form an orthonormal basis. The partition type(σ) has parts equal to the lengths of the cycles of σ. Here, h_μ is an element of the complete basis of Sym, defined by $h_\mu = h_{\mu_1} \cdots h_{\mu_k}$ and

$$h_r = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

With these definitions in mind, the monomial characters can be defined by

$$\phi^\lambda(\sigma) = \langle m_\lambda, p_{\mathrm{type}(\sigma)}\rangle,$$

or equivalently, via the expansion $p_{\mathrm{type}(\sigma)} = \sum_\lambda \phi^\lambda(\sigma) h_\lambda$. Following Stembridge, we will call the corresponding matrix function $A \mapsto \phi^\lambda[A]$ a *monomial immanant*. We will usually refer to the immanants defined above (for the irreducible characters χ^λ) as *ordinary* immanants, in order to avoid any confusion.

Note that the monomial characters are actually virtual characters (differences between characters). Since every Schur function s_λ is a sum of monomial symmetric functions, it follows that each irreducible character is a sum of monomial characters. In particular, the sign character is $\phi^{(1^n)}$ and the trivial character is the sum of all the monomial characters ($\sum_{\lambda \vdash n} \phi^\lambda$).

In [46], Stembridge presents a series of conjectures involving ordinary and monomial immanants. His primary motivation for considering monomial characters stems from the fact (which he shows) that any class function on $\mathfrak{S}_n$ satisfying these conjectures must be a nonnegative linear combination of ϕ^λ's.

There are several results and conjectures related to *Jacobi-Trudi* matrices. Jacobi-Trudi matrices are defined for any skew shape μ/ν, by

$$H_{\mu/\nu} = [h_{(\mu_i - i)-(\nu_j - j)}]_{1 \leq i,j \leq n}$$

(where, by convention, $h_0 = 1$ and $h_{-k} = 0$ if $k > 0$). The Jacobi-Trudi identity says that

$$s_{\mu/\nu} = \det H_{\mu/\nu},$$

where $s_{\mu/\nu}$ is a skew Schur function enumerating semi-standard tableaux of shape μ/ν. For any class function χ on $\mathfrak{S}_n$, $\chi[H_{\mu/\nu}]$ is a homogeneous symmetric function of degree $|\mu|-|\nu|$.

It was Goulden and Jackson [18] who first conjectured and Greene who first proved that ordinary immanants of Jacobi-Trudi matrices are monomial-positive:

Theorem 2.0.2 ([20]) $\chi^\lambda[H_{\mu/\nu}]$ *is m-positive.*

Haiman has shown, using Kazhdan-Lusztig theory, that ordinary immanants of Jacobi-Trudi matrices are actually Schur-positive:

Theorem 5 ([22]) $\chi^\lambda[H_{\mu/\nu}]$ *is s-positive.*

The analogous conjectures for monomial immanants remain open:

Conjecture 2.0.3 ([46, Conj. 4.2(b)]) $\phi^\lambda[H_{\mu/\nu}]$ *is m-positive.*

Conjecture 6 ([46, Conj. 4.1]) $\phi^\lambda[H_{\mu/\nu}]$ *is s-positive.*

Let $s_{[i,j]} \in \mathbb{Z}\mathfrak{S}_n$ denote the sum of all permutations of $\{i, i+1, \dots, j\} \subseteq [n]$, and let Π denote the set of all finite products of the $s_{[i,j]}$. The first step of Greene's proof of Theorem 2.0.2 (which is actually due to Goulden and Jackson [18]) shows that

$$[H_{\mu/\nu}] = \sum_{\pi\in\Pi} f_\pi \pi, \tag{0.2}$$

where f_π is monomial-positive, and the $[\,\cdot\,]$ notation is as defined in (0.1). The second step shows that $\chi^\lambda(\pi) \geq 0$ for every $\pi \in \Pi$.

Greene's proof that $\chi^\lambda(\pi) \geq 0$ involves an explicit computation of the matrices $\rho_\lambda(s_{[i,j]})$, where ρ_λ is the Young seminormal representation with character χ^λ. In particular, Greene shows that all of the entries of these matrices are nonnegative. (So taking a product of these matrices and then taking the trace yields a nonnegative value for $\chi^\lambda(\pi)$).

Unfortunately, this representation-theoretical approach can't be applied to the **virtual** characters ϕ^λ (at least not directly). However, it does follow from these considerations that Conjecture 2.0.3 is implied by the following:

Conjecture 2.0.1 (Stembridge [46, Conj. 5.2]) $\phi^\lambda(\pi) \geq 0$ *for all* $\pi \in \Pi$.

Another family of matrices considered by Stembridge are the $n \times n$ matrices with real entries for which every minor is nonnegative. Such matrices are called *totally positive.* Let TP denote the set of totally positive matrices.

Theorem 7 (Stembridge [45]) *If* $A \in \mathrm{TP}$, *then* $\chi^\lambda[A] \geq 0$.

Conjecture 8 ([46, Conj. 2.1]) *If* $A \in \mathrm{TP}$, *then* $\phi^\lambda[A] \geq 0$.

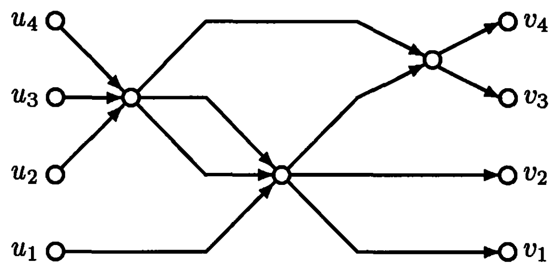

Figure 0.1: The skeleton corresponding to $s_{[3,4]}s_{[1,3]}s_{[2,4]}$.

Stembridge's proof of Theorem 7 also relies on Greene's result that $\chi^\lambda(\pi) \geq 0$ for any $\pi \in \Pi$. He shows that for any $A \in \mathrm{TP}$, $[A]$ is a nonnegative linear combination of elements of Π. It follows that Conjecture 2.0.1 implies Conjecture 8.

Most of Chapter 2 will be devoted to investigating Conjecture 2.0.1. In particular, we will show that the conjecture holds for certain families of partitions λ, including the partitions with two parts, the "hooks" $(r, 1, 1, \ldots, 1)$, and the partitions with no part larger than two. We will also show that if the conjecture holds for λ, then it holds for $(m\lambda_1, m\lambda_2, \ldots, m\lambda_k)$. It follows from results in [46] that the conjecture holds when $\lambda = (r^j), (n-1, 1)$, or $(2, 1^{n-2})$. Stembridge has recently indicated (personal communication) that he has found a proof for the case when λ has no part larger than three.

Rather than working directly with the $s_{[i,j]} \in \mathbb{Z}\mathfrak{S}_n$, we will take a "geometrical" approach. Goulden and Jackson's derivation of (0.2) is based on a lattice path interpretation of $[H_{\mu/\nu}]$ which was used by Gessel and Viennot [17] to give a combinatorial proof of the Jacobi-Trudi identity. We will discuss this interpretation in detail in Section 2.1.1, but for the present it will suffice to note that products of the $s_{[i,j]}$'s can be associated with certain planar digraphs (directed graphs), as is demonstrated in Figure 0.1.

We will use the term "skeleton" to denote the digraphs obtained in this way (together with the labelings of the sources and sinks), and we will always assume that we have a fixed planar embedding with the sources and sinks lying on vertical lines and the edges moving left to right, as in the example in Figure 0.1. Our choice of language is motivated by (but slightly different from) Greene [20], where a "skeleton" is a multiset of edges appearing in a family of lattice paths. He attributes the terminology to John Stembridge.

If S is a skeleton, we will write $\langle S \rangle$ for the corresponding product of $s_{[i,j]}$'s and $Z[S] \in \mathrm{Sym}$ for the image of $\langle S \rangle$ under the map $\sigma \mapsto p_{\mathrm{type}(\sigma)}$. Then Conjecture 2.0.1 says that $\phi^\lambda\langle S \rangle \geq 0$ for every S and λ, or equivalently, that $Z[S]$ is always h-positive.

We mentioned earlier that Stanley and Stembridge [43] have shown that the conjectured e-positivity of indifference graphs is equivalent to a special case of the monomial immanant conjectures. What they show, more specifically, is that if A is the matrix obtained by replacing the nonzero entries of $H_{\mu/\nu}$ by 1's, then the conjectured e-positivity of indifference graphs is equivalent to the nonnegativity of $\phi^\lambda[A]$. For example, if $\mu = (5, 5, 5, 3, 2)$ and

$\nu = (2,2,2,1,0)$, then A would be

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

The $(0,1)$-matrices obtained in this way are characterized by the fact that the positions of the zeroes form a Young diagram (or shape), justified in the lower left corner, and lying entirely below the main diagonal. The permutations whose coefficient in $[A]$ is nonzero correspond to "rook placements" in the 1's of A.

It can be shown that the *cycle indicators* of these $(0,1)$-matrices (i.e. the image of $[A]$ under $\sigma \mapsto p_{\text{type}(\sigma)}$) are, up to multiplication by a positive constant, the same as the $Z[S]$'s for skeletons corresponding to products $s_{[i_1,j_1]}s_{[i_2,j_2]}\cdots s_{[i_k,j_k]}$ with $j_1 \le j_2 \le \cdots \le j_k$.

One of the new results we will present in Chapter 2 is that for any skeleton, $\omega Z[S]$ can be expressed in a natural way as a sum of chromatic symmetric functions of incomparability graphs. And so, Conjecture 2.0.1 is equivalent to conjecturing that certain sums of chromatic symmetric functions are e-positive. Unfortunately, the individual incomparability graphs obtained by this construction have no special properties and need not be e-positive. However, any and all "general" results concerning the elementary expansions of the X_Γ's can be applied to $Z[S]$. For example, Theorem 1.2.3 can be applied to show that

$$\sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k}} \phi^\lambda\langle S\rangle \ge 0.$$

In [46], Stembridge also considers more general matrices defined in terms of paths in acyclic digraphs. If $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ are ordered n-tuples of vertices, he says that $\mathbf{u}$ and $\mathbf{v}$ are *D-compatible* if

$$i<j,\ k>l \implies \begin{array}{l}\text{every path in } D \text{ from } u_i \text{ to } v_k \text{ intersects every path} \\ \text{from } u_j \text{ to } v_l.\end{array}$$

If the edges of D are weighted by independent (commuting) indeterminates, Stembridge considers the matrix $A(\mathbf{u},\mathbf{v})$, whose (i,j) entry is the generating function for paths from u_i to v_j (in terms of these weights), and he makes the following conjecture:

Conjecture 2.6.1 ([46, Conj. 6.3]) *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are D-compatible, then* $\phi^\lambda[A(\mathbf{u},\mathbf{v})]$ *is monomial positive.*

He also shows that this conjecture holds in the cases when $\lambda = (r^\ell)$ (including (n) and (1^n)), when $\lambda = (21^{n-2})$, and when $\lambda = (n-1,1)$.

We will show that, as in the case of totally positive matrices and Jacobi-Trudi matrices (which can be viewed as a special case of this construction), this conjecture can be reduced to Conjecture 2.0.1 — at least in the case when D is finite or when $A(\mathbf{u}, \mathbf{v})$ can be viewed as a limit of finite cases. In particular, it will follow from our results that ordinary immanants of these matrices are monomial positive.

Signed and Voltage Graphs

Signed graphs were introduced by Harary [24], but most of the theory we will use was developed in a series of papers by Zaslavsky [52–57].

A signed graph $\Sigma = (\Gamma, \varphi)$ is an ordinary[2] graph, Γ, together with an assignment of "signs" to the edges, $\varphi : E(\Gamma) \to \{+1, -1\}$.

Let V denote the set of vertices of Γ. We will consider colorings $\kappa : V \to \mathbb{Z}$. A coloring is called *proper* when for each edge e from v to w, $\kappa(w) \neq \kappa(v)\varphi(e)$. In our considerations here, we will assume that Σ has no positive loops (otherwise, there are no proper colorings). Zaslavsky's chromatic polynomials for signed graphs come in "unbalanced" and "balanced" (zero-free) versions:

$$\begin{aligned} \chi^u_\Sigma(2n+1) &= \text{number of proper colorings using } [-n, n] \\ \chi^b_\Sigma(2n) &= \text{number of proper colorings using } ([-n, n] \setminus \{0\}) \end{aligned}$$

In [52], Zaslavsky interprets signed graph colorings in terms of certain hyperplane arrangements. Let Σ be a signed graph on d vertices, labelled by $[d]$. Σ can be associated with an arrangement in $\mathbb{R}^d$ by including the hyperplane $\mathsf{x}_i = \epsilon \mathsf{x}_j$ if there is an edge (i, j) with sign ϵ. Denote the set of these hyperplanes by $H[\Sigma]$. It is immediate from the definition that proper colorings of Σ exactly correspond to those points in $\mathbb{R}^d$ which have integer coordinates and lie in the complement of $H[\Sigma]$.

Let $\mathcal{B}^*_d$ denote the set of hyperplanes dual to the elements of the root system, $\mathcal{B}_d$, of the hyperoctahedral group, $\mathfrak{B}_d$. Then $H[\Sigma]$ is a subarrangement of $\mathcal{B}^*_d$, and every subarrangement can obtained in this way.

Likewise, let $\mathcal{A}^*_{d-1}$ denote the set of hyperplanes of the form $\mathsf{x}_i = \mathsf{x}_j, \quad (i \neq j)$. These correspond to positive edges which are not loops, and the duals of these in $\mathcal{B}_d$ can be identified with the root system of $\mathfrak{S}_d$. Ordinary (unsigned) graphs, Γ, on $[d]$ correspond to subarrangements, $H[\Gamma]$, of $\mathcal{A}^*_{d-1}$ (i.e. the edges can be considered to have a positive sign).

For an ordinary graph Γ, there is a direct correspondence (see Greene [19] and Zaslavsky [51]) between acyclic orientations of Γ and regions of the arrangement $H[\Gamma]$ (the regions into which $\mathbb{R}^d$ is divided by the hyperplanes). Choosing a direction for each edge corresponds to choosing one of the half-spaces defined by the corresponding hyperplane, and the acyclic orientations correspond to the non-empty intersections of half-spaces.

[2]Zaslavsky also considers a more general situation, where Σ is allowed to have "free loops" and "half-arcs." However, these ideas are introduced to allow for deletion-contraction methods. These methods will not be applicable to our chromatic functions, so we won't need to consider these other Σ's.

In [57], Zaslavsky develops an orientation theory for signed graphs such that acyclic orientations correspond to regions of $H[\Sigma]$. This theory involves oriented matroids and the details are significantly more complicated than in the case of ordinary graphs. However, from our point of view, it will suffice to just work directly with the regions of the arrangement.

The direct analogues of Stanley's chromatic symmetric function for signed graphs would be:

$$X_\Sigma^u = \sum x^\kappa, \qquad X_\Sigma^b = \sum x^\kappa \tag{0.3}$$

where the sums run over all proper colorings and zero-free proper colorings, respectively. These are formal series in variables x_j, with $j \in \mathbb{Z}$.

It will also be useful to consider the images of these obtained by "forgetting" the signs of the colors. We will write $|x_j| = x_{|j|}$, and similarly for formal power series in these variables. It is fairly obvious from the definitions that $|X_\Sigma^u| \in \mathbb{Q}[x_0] \otimes \mathrm{Sym}$ and $|X_\Sigma^b| \in \mathrm{Sym}$. These functions have some nice properties which are not shared by X_Σ^u, X_Σ^b, and are generally speaking less cumbersome to work with.

Any of these functions can be considered analogues of Stanley's X_Γ. From the point of view of acyclic orientations and hyperplane arrangements, X_Σ^u is the natural thing to consider, but "algebraic" results about X_Γ generalize more directly to $|X_\Sigma^b|$.

One of the interesting results we will present is an expansion of X_Σ^u in terms of an analogue of the fundamental basis of the ring of quasi-symmetric functions, which is directly related to the regions of the corresponding hyperplane arrangement. We will also show a "reciprocity" result which relates X_Σ^u and the enumerator of pairs of acyclic orientations and compatible colorings in a way which is analogous to Theorem 1.2.4, or "geometrically speaking," it takes (the lattice points in) each region to (the lattice points in) the closure of the region.

We will also derive some interesting new results for ordinary graphs which relate X_Γ to some other hyperplane arrangements (apart from the usual one).

For example, if R is a region of $H[+\Gamma^\circ]$ (the usual arrangement plus the coordinate hyperplanes), let $\mathrm{neg}(R)$ be the number of coordinates of an interior point of R which are negative and let $\mathrm{sink}(R)$ denote the number of faces of R which lie in coordinate hyperplanes. Then we have the following result:

Proposition 3.4.6 *The algebra map* $\mathrm{Sym} \to \mathbb{Q}[s,t]$ *induced by*

$$e_k \mapsto (1+s^k)t + (s+s^2+\cdots+s^{k-1})t^2$$

sends X_Γ *to* $\sum s^{\mathrm{neg}(R)} t^{\mathrm{sink}(R)}$, *where the sum runs over the regions for* $H[+\Gamma^\circ]$.

This proposition implies that certain nonnegative combinations of the c_λ are positive (where $X_\Gamma = \sum c_\lambda e_\lambda$). If we set $s = 0$, we obtain Theorem 1.2.3.

We will also consider generalizations of these chromatic functions for *voltage graphs* (a generalization of signed graphs in which the edges are labeled by elements of an arbitrary group). At this level of generality, there are analogues of the monomial and power sum expansions of X_Γ, but no corresponding theory of acyclic orientations.

Part II

The results described in Chapters 4 and 5 were obtained in collaboration with Nantel Bergeron and have been published in [2].

Hochschild Homology, Shuffles, and Free Lie Algebras

Let A be an associative, unital algebra over a field K and M be an A-bimodule (a two-sided A-module). In [27], Hochschild introduces a cohomology theory for the pairs (A, M) as an analogue of the cohomology theory for groups developed by Eilenberg and MacLane.

The cochains for this theory are $C^n(A, M) = \mathrm{Hom}_K(A^{\otimes n}, M)$ and the Hochschild coboundary map $\delta = \delta_n : C^n(A, M) \to C^{n+1}(A, M)$ is given by

$$\begin{aligned}\delta f(a_1, \ldots, a_{n+1}) &= a_1 \cdot f(a_2, \ldots, a_n) + \\ &\sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) + \\ &(-1)^{n+1} f(a_1, \ldots, a_n) \cdot a_{n+1},\end{aligned}$$

where we have written $f(a_1, \ldots, a_n)$ for $f(a_1 \otimes a_2 \otimes \cdots \otimes a_n)$. The maps δ_n make $C^*(A, M)$ a complex, and the homology $H^n(A, M) = (\ker \delta_n)/(\mathrm{Im}\, \delta_{n-1})$ is called the Hochschild cohomology of A with coefficients in M.

It can be shown that Hochschild cohomology is a generalization of simplicial cohomology, although the proof of this fact is quite complicated. In particular, Gerstenhaber and Schack [12] have shown that if Σ is a locally finite simplicial complex, then the simplicial cohomology $H^*(\Sigma, K)$ is isomorphic to the Hochschild cohomology $H^*(A, A)$, where A is the *incidence algebra* of the poset consisting of the simplices in Σ, ordered by inclusion.

Hochschild homology is defined analogously. The chains are given by $C_n(A, M) = M \otimes A^{\otimes n}$. The boundary map $\partial = \partial_n : C_n(A, M) \to C_{n-1}(A, M)$ is given by

$$\begin{aligned}\partial_n(m[a_1, \ldots, a_n]) &= m a_1[a_2, \ldots, a_n] + \\ &\sum_{i=1}^{n-1} (-1)^i m[a_1, \ldots, a_i a_{i+1}, \ldots, a_n] + \\ &(-1)^n a_n m[a_1, \ldots, a_{n-1}],\end{aligned}$$

where we use the "shorthand" $m[a_1, \ldots, a_n]$ for $m \otimes a_1 \otimes \cdots \otimes a_n$.

Hochschild homology and cohomology can also be defined in terms of the derived functors Tor and Ext. This approach allows for a uniform development of the homology theories for groups, Lie algebras, and associative algebras (see, for example, Cartan and Eilenberg [5]). However, since we will be concerned with the action of $\mathfrak{S}_n$ on $A^{\otimes n}$, the "classical" definitions given above are better suited to our purposes.

Now we consider the case when A is a commutative, unital algebra over a field K of characteristic zero and M is a symmetric A-bimodule (i.e. just an ordinary A-module).

Harrison [26] introduced a different cohomology theory for commutative algebras which is defined in terms of *shuffles.*

There is a (signed) shuffle product on the tensor algebra of A, $\mathrm{TA} = \bigoplus_{n\geq 0} A^{\otimes n}$, which can be defined as follows. Let $\mathfrak{S}_n$ act on the left of $A^{\otimes n}$ via

$$\sigma[a_1, a_2, \ldots, a_n] = [a_{\sigma_1^{-1}}, a_{\sigma_2^{-1}}, \ldots, a_{\sigma_n^{-1}}],$$

where we write $[a_1, \cdots, a_n]$ for $a_1 \otimes \cdots \otimes a_n$. Then the shuffle product can be defined as

$$[a_1, a_2, \ldots, a_i] ⧢ [b_1, b_2, \ldots, b_j] = \sum_{\substack{\sigma\in\mathfrak{S}_{i+j} \\ \sigma(1)<\cdots<\sigma(i) \\ \sigma(i+1)<\cdots<\sigma(i+j)}} (-1)^{\sigma}\sigma[a_1, \ldots, a_i, b_1, \ldots, b_j].$$

For shuffle products involving elements of $A^{\otimes 0} = K$, we can just take the usual scalar product. Let us write $\mathrm{Sh}^2_n(\mathrm{TA})$ for the subspace of $A^{\otimes n}$ spanned by non-trivial ($i, j \geq 1$ in the above definition) shuffle products of two elements of TA, and let $\mathrm{Sh}^2_*(\mathrm{TA})$ denote the sum over all n of these. Similarly, define $\mathrm{Sh}^k_*(\mathrm{TA})$ to be the span of non-trivial (shuffle) products of k elements of TA. We will call these k-shuffles.

It can be shown that the set of cochains $f : A^{\otimes n} \to M$ which vanish on $\mathrm{Sh}^2_n(\mathrm{TA})$ form a subcomplex under δ. The homology of this complex is called Harrison cohomology.

We should note that this definition of Harrison cohomology is actually due to MacLane, who was the referee of Harrison's paper [26]. Harrison's original definition was quite different and is omitted from the published version of his paper. Although we will not need to consider this original definition, we mention in the way of background that Harrison's unpublished manuscript [25] is fairly widely circulated and occasionally referenced in the literature. Both definitions and a proof of their equivalence are discussed in [13].

The situation for homology is similar. $M \otimes \mathrm{Sh}^2_*(\mathrm{TA})$ is a subcomplex of $C_*(A, M)$ and the Harrison homology of A with coefficients in M is defined to be the homology of the quotient of $C_*(A, M)$ by $M \otimes \mathrm{Sh}^2_*(\mathrm{TA})$.

We note that all of these definitions make sense without the assumption that K has characteristic zero. Barr [1] showed that, with the characteristic zero assumption, Harrison (co)homology is actually a summand of Hochschild (co)homology (i.e. that the appropriate subcomplexes are complemented). He also showed that this fails to be the case if the characteristic zero assumption is dropped.

Gerstenhaber and Schack [13] and (independently) Loday [30] have shown that there are decompositions of the Hochschild (co)homology in the commutative case:

$$H_n(A, M) = H_{1,n-1}(A, M) \oplus H_{2,n-2}(A, M) \oplus \cdots \oplus H_{n,0}(A, M)$$

$$H^n(A, M) = H^{1,n-1}(A, M) \oplus H^{2,n-2}(A, M) \oplus \cdots \oplus H^{n,0}(A, M)$$

These decompositions are obtained by a family of orthogonal idempotents $e_n^{(k)}$ in the group algebra $\mathbb{Q}[\mathfrak{S}_n]$ of the symmetric group, which acts on the (classical) Hochschild complexes by trivially extending the action on $A^{\otimes n}$ described above. The arguments in [13] and [30] are generalizations of Barr's construction such that the first summands $H_{1,n-1}(A,M)$ and $H^{1,n-1}(A,M)$ are the Harrison (co)homology.

The summands $H_{n,0}(A,M)$ and $H^{n,0}(A,M)$ can be described in terms of differential forms and skew multiderivations, respectively. The intermediate components have not been well understood.

The idempotents $\rho_n^{(k)} = \theta(e_n^{(k)})$, where θ is induced by $\sigma \mapsto (-1)^\sigma \sigma$, have been used by Garsia and Reutenauer [8, 9, 33, 34] to study the combinatorics of free Lie algebras. In particular, they show that the $\rho_n^{(k)}$ give projections into the direct summands of the symmetric algebra of the free Lie algebra, making explicit the Poincaré-Birkhoff-Witt theorem.

We will use free Lie algebras to study the decompositions of Hochschild (co)homology.

Harrison (co)homology is defined in terms of shuffles. In Section 4.3, we will see that all of the components of the decomposition can be described by generalizing to k-shuffles:

$$\begin{aligned} H_{k,n-k}(A,M) &\cong H_n(M \otimes \mathrm{Sh}_*^k(\mathrm{TA})/M \otimes \mathrm{Sh}_*^{k+1}(\mathrm{TA})) \\ H^{k,n-k}(A,M) &\cong H_n(\mathrm{Hom}_K(\mathrm{Sh}_*^k(\mathrm{TA}),M)/\mathrm{Hom}_K(\mathrm{Sh}_*^{k+1}(\mathrm{TA}),M)). \end{aligned} \tag{0.4}$$

In particular, we have a nice characterization of $\oplus_{j=1}^k H^{j,n-j}(A,M)$ as the cohomology of the Hochschild n-cochains which vanish on $(k+1)$-shuffles. Section 4.3 is (essentially) taken from Wolfgang [50]. (0.4) has been shown independently by Ronco [35] and by Sletsjøe [38].

In Section 4.4, we show that the complex $C_{(k)}$ defining $H_{k,*-k}(A,A)$ is the k-th shuffle power of $C_{(1)}$. The same methods are used to show that the $e_n^{(k)}$ are the projection maps for a dual, graded version of the Poincaré-Birkhoff-Witt theorem due to Hain [23]. This generalizes what Hain has shown for $e_n^{(1)}$ and gives an intuitive explanation for the relationship between the $e_n^{(k)}$ and the $\rho_n^{(k)}$.

Gerstenhaber Operations

The Hochschild cohomology $H^*(A,A)$ is endowed with two *Gerstenhaber operations*: a graded commutative cup product and a graded Lie bracket. For the sake of clarity, we should mention that the word "super" is also commonly used to describe the graded analogues of various structures (as in "super Lie algebra"). In the present thesis we will use the more descriptive word, "graded."

Although the cup product is used in Hochschild [28], most of the interesting properties are due to Gerstenhaber [11] — this is our reason for calling these Gerstenhaber operations. In [13], Gerstenhaber and Schack ask if these operations are graded with respect to the decomposition, i.e. does one have $H^{(j)} \cup H^{(k)} \subseteq H^{(j+k)}$ and $[H^{(j)}, H^{(k)}] \subseteq H^{(j+k-1)}$, where $H^{(k)} = \oplus_n H^{k,n-k}(A,A)$?

In general, the answer to this question is "no". In Section 5.3 we will give counterexamples. The products are, however, filtered with respect to $H^{(\leq k)} = \oplus_{j\leq k} H^{(j)}$, as we will show in Section 5.1. That is, $H^{(j)} \cup H^{(k)} \subseteq H^{(\leq j+k)}$ and $[H^{(j)}, H^{(k)}] \subseteq H^{(\leq j+k-1)}$. The intuition here is that the projections of $f \cup g$ and $[f,g]$ into the smaller components of the decomposition should be viewed as "error terms". We will relate these operations on cohomology to certain operations on words and show that the error terms above are related to the "error terms" obtained from writing an arbitrary product of Lie elements in terms of symmetrized products of Lie elements.

Equivalently, one can say that $\mathcal{F}_q = \bigoplus_{r\geq q} H^{*,r}(A,A)$ are ideals for the cup product and the Lie bracket in $H^*(A,A)$, with

$$\mathcal{F}_p \cup \mathcal{F}_q \subseteq \mathcal{F}_{p+q}$$

$$[\mathcal{F}_p, \mathcal{F}_q] \subseteq \mathcal{F}_{p+q}.$$

The results for the cup product were conjectured in this form in [14, 15]. These results generalize results in [13] for the Harrison component and $\mathcal{F}_1$.

Somewhat stronger statements can be made for the Harrison component. For example, we show that the $H^{(k)}$ are $H^{(1)}$-modules under the bracket action.

In Section 5.2, we examine the behavior of the Gerstenhaber operations for a (not necessarily commutative) algebra with an involution. We obtain a decomposition into two parts for which the cup and Lie products are $\mathbb{Z}/2\mathbb{Z}$-graded.

A paper by Sletsjøe [38] incorrectly asserts that the Gerstenhaber operations are graded with respect to the decomposition. This is discussed at the end of Section 5.1.

Part I

Chromatic Symmetric Functions, Monomial Immanants, and Related Topics

Chapter 1

Stanley's Chromatic Symmetric Function

1.1 Symmetric Functions

In the first part of this section, we will review some of the properties of symmetric functions and representations of the symmetric group which we will need in the sequel. Proofs of these facts and a more complete development can be found in Macdonald [32].

Let $\mathrm{Sym} = \mathrm{Sym}(x_1, x_2, \ldots)$ denote the ring of symmetric functions in the variables $\{x_1, x_2, \ldots\}$ with rational coefficients. More generally, we will write Sym_R for the symmetric functions with coefficients in (some ring) R. The homogeneous component of Sym consisting of functions of degree n has several natural bases indexed by *partitions* of n. A partition of n with k *parts* is a k-tuple of positive integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \geq \cdots \geq \lambda_k$ and $\lambda_1 + \cdots + \lambda_k = n$. In this case, we will write $\lambda \vdash n$, $|\lambda| = n$, and $\ell(\lambda) = k$. We will let $\{e_\lambda\}$, $\{h_\lambda\}$, $\{p_\lambda\}$, $\{m_\lambda\}$, $\{\widetilde{m}_\lambda\}$, and $\{s_\lambda\}$ denote, respectively, the elementary, complete, power sum, monomial, augmented monomial, and Schur function bases of Sym, which can be defined as follows.

The most "obvious" basis of Sym is the monomial basis, defined by letting m_λ be the sum of all monomials whose exponents are given by λ. It is sometimes more convenient to work with the *augmented* monomial basis, given by

$$\widetilde{m}_\lambda = \sum_{\substack{(i_1, i_2, \ldots, i_k) \\ i_1, i_2, \ldots, i_k \text{ distinct}}} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_k}^{\lambda_k} = \left(\prod_i r_i!\right) m_\lambda,$$

where r_i denote the multiplicity of i as a part in λ.

If we let (for any $n \in \mathbb{P}$)

$$\begin{aligned} e_n &= \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}, \\ h_n &= \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n}, \\ p_n &= x_1^n + x_2^n + x_3^n + \cdots, \end{aligned}$$

then it can be shown that Sym is a polynomial algebra, and any of the sets, $\{e_i\}$, $\{h_i\}$, $\{p_i\}$, can be taken to be a set of generators. The elementary, complete, and power sum bases are defined to be the corresponding sets of monomials in these: $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}$, etc. We will employ the usual convention and let $e_0 = h_0 = 1$.

The reason for calling $\{h_\lambda\}$ the *complete* basis is that

$$h_n = \sum_{\lambda \vdash n} m_\lambda.$$

The generating functions for the e_n's and h_n's are given by

$$E(t) = \sum_{n=0}^{\infty} e_n t^n = \prod_{i=1}^{\infty} (1 + x_i t) \tag{1.1}$$

and $$H(t) = \sum_{n=0}^{\infty} h_n t^n = \prod_{i=1}^{\infty} (1 - x_i t)^{-1}. \tag{1.2}$$

These are related to the power sum symmetric functions via

$$\log E(t) = \sum_{i \geq 1} p_i \frac{(-1)^{i-1} t^i}{i}, \quad \log H(t) = \sum_{i \geq 1} p_i \frac{t^i}{i}. \tag{1.3}$$

There are many ways to define the Schur functions. For our purposes, the most convenient definition will be the "combinatorial" one in terms of ***semi-standard tableaux***. A semi-standard tableau for the partition λ is an arrangement of positive integers in left-justified rows of lengths given by the parts of λ, such that the integers are weakly increasing in each row and strictly increasing in each column. λ is called the ***shape*** of the tableau. For example, a semi-standard tableau of shape $(4, 4, 2)$ is given by:

strictly increasing
↑

4	6		
2	3	4	4
1	2	2	3

→ weakly increasing

(Another common convention is to draw these so that the columns read downwards.) The Schur function s_λ is the sum over all semi-standard tableaux of shape λ of the corresponding monomials (the product of the variables corresponding to the entries in the tableau). The set of squares of shape λ into which these entries go is called a *Young diagram* or *Ferrers board*.

If $\mu \vdash n$ and $\nu \vdash m$ are partitions of $n \geq m$ satisfying $\mu_i \geq \nu_i$, then we can consider the *skew shape*, μ/ν, obtained by removing the Young diagram for ν from the Young diagram for μ. Fillings of these shapes which increase weakly in rows and strictly in columns are also called semi-standard tableaux, and the sum over all these of the corresponding monomials is called a *skew Schur function*, and is denoted $s_{\mu/\nu}$. An example of a semi-standard tableau of shape $(4,4,2)/(2,1)$ would be:

$$\begin{array}{|c|c|c|c|} 4 & 4 & & \\ & 2 & 2 & 4 \\ & & 1 & 3 \end{array}.$$

It will be convenient to use the notation $\epsilon_\lambda = (-1)^{|\lambda|-\ell(\lambda)}$ and $z_\lambda = \prod_i i^{r_i} r_i!$, where r_i (as above) is the number of times i appears as a part of λ.

There is an involution ω on Sym, which can be defined by any of the following conditions:

$$\omega(e_\lambda) = h_\lambda,\ \omega(s_\lambda) = s_{\lambda'},\ \omega(p_\lambda) = \epsilon_\lambda p_\lambda.$$

Here, λ' denotes the *conjugate* shape to λ, which can be defined by reflecting the Young diagram across the principal diagonal (i.e. exchanging the rows and columns).

Sym is related to the representation theory of the symmetric groups via the *characteristic map*, which gives an isomorphism between the space of class functions on $\mathfrak{S}_n$ and the homogeneous elements of Sym of degree n:

$$\mathrm{ch}(\chi) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \chi(\sigma) p_{\mathrm{type}(\sigma)},$$

where type(σ) is the cycle type of σ (the partition whose parts are the lengths of the cycles of σ). There is a one-to-one correspondence between the conjugacy classes of $\mathfrak{S}_n$ and the cycle types of the permutations in each class.

We will also need to make use of some identities involving the so-called *Cauchy product*:

$$\begin{aligned} \prod_{i,j}(1 - x_i y_j)^{-1} &= \sum_\lambda z_\lambda^{-1} p_\lambda(x) p_\lambda(y) \\ &= \sum_\lambda m_\lambda(x) h_\lambda(y) = \sum_\lambda h_\lambda(x) m_\lambda(y) \\ &= \sum_\lambda s_\lambda(x) s_\lambda(y) \end{aligned} \tag{1.4}$$

Appropriate applications of the involution ω can be used to derive the following:

$$\prod_{i,j}(1 + x_i y_j) = \sum_\lambda m_\lambda(x) e_\lambda(y) = \sum_\lambda e_\lambda(x) m_\lambda(y) \tag{1.5}$$

A consequence of (1.4) is that Sym possesses an inner product satisfying

$$\langle s_\lambda, s_\mu \rangle = \langle m_\lambda, h_\mu \rangle = \langle h_\mu, m_\lambda \rangle = \delta_{\lambda\mu} \tag{1.6}$$

$$\langle p_\lambda, p_\mu \rangle = z_\lambda^{-1} \delta_{\lambda\mu}. \tag{1.7}$$

In terms of this inner product, the character table of $\mathfrak{S}_n$ can be given by the Frobenius formula:

$$\chi^\lambda(\sigma) = \langle s_\lambda, p_{\mathrm{type}(\sigma)} \rangle \tag{1.8}$$

The Quasi-symmetric Functions

The *quasi-symmetric* functions introduced by Gessel [16] are defined to be formal series $F(x_1, x_2, \ldots)$ such that the coefficients of the monomials $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ and $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}$ are equal when $i_1 < \cdots < i_k$ and $j_1 < \cdots j_k$. We will denote the ring of quasi-symmetric functions (with rational coefficients) by QSym and the subspace consisting of the homogeneous elements of degree d by QSym_d. The *fundamental basis* $\{Q_{S,d} \mid S \subseteq [d-1]\}$ of QSym_d is given by

$$Q_{S,d} = \sum_{\substack{a_1 \le a_2 \le \cdots \le a_d \\ a_i < a_{i+1} \text{ if } i \in S}} x_{a_1} x_{a_2} \cdots x_{a_d}.$$

We will just write Q_S when d is understood by context. Note that $Q_{\emptyset,d} = h_d$ and $Q_{[d-1],d} = e_d$.

QSym is a ring, and the product can be described in terms of the fundamental basis as follows:

$$Q_{S,d} Q_{S',d'} = \sum Q_{T,d+d'},$$

where the sum runs over all rearrangements of the word $a_1 \cdots a_d b_1 \cdots b_{d'}$ which preserve the order of the a_i's and the order of the b_i's (i.e. shuffles), and the corresponding subset T is the set of i's such that the letters in the ith and $(i+1)$st positions are

$$\begin{aligned} &a_k a_{k+1} \text{ with } k \in S, \\ &b_k b_{k+1} \text{ with } k \in S', \\ \text{or } &b_i a_j \text{ (any } i, j). \end{aligned}$$

There is an involution on QSym_d, defined by $\omega(Q_{S,d}) = Q_{\overline{S},d}$ (where $\overline{S} = [d-1] - S$). This involution extends the involution on Sym.

QSym has another basis,

$$M_{S,d} = \sum_{\substack{a_i < a_{i+1} \text{ if } i \in S \\ a_i = a_{i+1} \text{ if } i \notin S}} x_{a_1} x_{a_2} \cdots x_{a_d}.$$

Note that

$$Q_S = \sum_{T \supseteq S} M_T, \text{ and hence } M_S = \sum_{T \supseteq S} (-1)^{|T|-|S|} Q_S.$$

An elementary calculation shows that

$$\omega(M_{S,d}) = (-1)^{d-1}(-1)^{|S|} \sum_{T \subseteq S} M_{T,d} \tag{1.9}$$

The Hopf Algebra Structure of Sym

In Section 1.3, we will make use of the Hopf algebra structure of Sym. This structure and its relationship to the representation theory of $\mathfrak{S}_n$ are studied in great detail in Zelevinsky [58]. Our brief presentation here is based on some lectures given by Sergey Fomin in Gian-Carlo Rota's class on Hopf algebras at MIT during the spring of 1994.

Consider the ring $\mathrm{Sym}(\mathbf{x}, \mathbf{y}) = \mathrm{Sym}(x_1, x_2, x_3, \ldots, y_1, y_2, \ldots)$, i.e. replace the variables by the union of two countable sets of variables. Any element of this ring must be invariant under permutations of the x_i's or the y_j's (separately), so in particular, $\mathrm{Sym}(\mathbf{x}, \mathbf{y})$ can be viewed as a subspace of $\mathrm{Sym} \otimes \mathrm{Sym}$. On the other hand, any countable set of variables defines the same ring of symmetric functions, i.e. $\mathrm{Sym}(\mathbf{x}, \mathbf{y})$ is isomorphic to Sym. So, given an element of $F \in \mathrm{Sym}$, we can consider $F(x_1, x_2, \ldots, y_1, y_2, \cdots)$ and this can be written as a sum

$$\sum_i F_{1,i}(x_1, x_2, \ldots) F_{2,i}(y_1, y_2, \ldots),$$

where each $F_{1,i}$ and $F_{2,i}$ are symmetric functions.

The map $F \mapsto F_{1,i} \otimes F_{2,i}$ defines a coassociative coproduct

$$\Delta : \mathrm{Sym} \to \mathrm{Sym} \otimes \mathrm{Sym}\,.$$

It is easy to see that Δ is an algebra map from this definition, and with this coproduct and the usual product, Sym becomes a commutative, cocommutative Hopf algebra.

The following properties are fairly straightforward to verify from the definition of the coproduct given above.

Proposition 1.1.1

$$\Delta p_i = p_i \otimes 1 + 1 \otimes p_i \tag{1.10}$$

$$\Delta e_n = \sum_{i=0}^{n} e_i \otimes e_{n-i} \tag{1.11}$$

$$\Delta h_n = \sum_{i=0}^{n} h_i \otimes h_{n-i} \tag{1.12}$$

$$\Delta m_\lambda = \sum_{\lambda = \mu \cup \rho} m_\mu \otimes m_\rho \tag{1.13}$$

$$\Delta s_\lambda = \sum_{\mu} s_\mu \otimes s_{\lambda/\mu} \tag{1.14}$$

In the usual terminology associated with Hopf algebras, (1.10) says the the p_i's are *primitive*, and (1.11) and (1.12) say that the e_i's and h_i's are *divided powers*.

1.2 Stanley's Chromatic Symmetric Function

In this section, we will examine some of the details of Stanley's results concerning chromatic symmetric functions which we will need in later sections.

Recall that for any graph $\Gamma = (V, E)$, Stanley's chromatic symmetric function is defined to be

$$X_\Gamma = \sum_{\kappa \text{ proper}} x^\kappa \tag{1.15}$$

where the sum runs over all colorings $\kappa : V \to \mathbb{P}$ such that adjacent vertices are colored differently (*proper* colorings) and $x^\kappa = \prod_{v \in V} x_{\kappa(v)}$.

Stanley's power sum expansion of X_Γ in terms of the bond lattice of Γ will play an important role in the next section, and in Chapter 2, where it will allow us to express questions involving monomial immanants in terms of chromatic symmetric functions. In Chapter 3 we will generalize this result to voltage graphs.

Theorem 1.2.1 (Stanley [41, Thm. 2.6])

$$X_\Gamma = \sum_{\pi \in L_\Gamma} \mu(\hat{0}, \pi) p_{\text{type}(\pi)}.$$

Proof: If $\pi \in L_\Gamma$, let

$$X_\pi = \sum x^\kappa, \tag{1.16}$$

where the sum runs over all colorings $\kappa : V \to \mathbb{P}$ which are monochromatic on the blocks of π but color adjacent vertices in different blocks differently.

For any map $\kappa : V \to \mathbb{P}$, there is a unique $\sigma \in L_\Gamma$ such that κ is one of the colorings enumerated by X_σ (i.e. the blocks of σ are the connected components of the subgraphs on which κ is monochromatic). So, for any $\pi \in L_\Gamma$, the sum of all colorings which are monochromatic on the blocks of π is

$$p_{\text{type}(\pi)} = \sum_{\sigma \geq \pi} X_\sigma.$$

Möbius inversion implies that

$$X_\pi = \sum_{\sigma \geq \pi} p_{\text{type}(\sigma)} \mu(\pi, \sigma).$$

For $\hat{0} \in L_\Gamma$ (the partition with one vertex per block), $X_{\hat{0}} = X_\Gamma$, and the theorem follows. □

Stanley gives another power sum expansion of X_Γ in terms of subsets of the edges. The proof uses a similar Möbius inversion argument (in this case, just ordinary inclusion-exclusion).

Theorem 1.2.2 ([41, Thm. 2.5]) *If $S \subseteq E$, let $\lambda(S)$ denote the partition whose parts are the vertex sizes of the spanning subgraph of Γ with edge set S, then*

$$X_\Gamma = \sum_{S \subseteq E} (-1)^{|S|} p_{\lambda(S)}.$$

Some of the most interesting results and open questions about X_Γ concern the expansion in the elementary basis. Let

$$X_\Gamma = \sum_{\lambda \vdash d} c_\lambda e_\lambda.$$

Theorem 1.2.3 (Stanley [41, Thm. 3.3]) *Let $\operatorname{sink}(\Gamma, j)$ be the number of acyclic orientations of Γ with j sinks. Then*

$$\operatorname{sink}(\Gamma, j) = \sum_{\substack{\lambda \vdash d \\ \ell(\lambda) = j}} c_\lambda.$$

Another connection between X_Γ and the acyclic orientations of Γ is given by the following "reciprocity" result.

Theorem 1.2.4 (Stanley [41, Thm. 4.2])

$$\omega X_\Gamma = \sum_{(\mathfrak{o}, \kappa)} x^\kappa$$

where the sum runs over pairs $(\mathfrak{o}, \kappa)$ of acyclic orientations and colorings satisfying $\kappa(u) \leq \kappa(v)$ if (v, u) is an edge of $\mathfrak{o}$.

We will see later that these results have a common generalization (Theorem 1.3.9 below).

We will briefly outline Stanley's proof of Theorem 1.2.3, since we will make extensive use of some of the intermediate steps.

If P is any poset, let X_P denote $\sum x^\kappa$ where the sum runs over all strictly order preserving maps $\kappa : P \to \mathbb{P}$ and $x^k = \prod_{v \in P} x_{\kappa(v)}$. A proper coloring κ of Γ defines a unique acyclic orientation $\mathfrak{o}$ by directing a edge e with ends v and w towards the vertex with a smaller color. Then κ is one of the maps appearing in the definition of $X_{\overline{\mathfrak{o}}}$, where $\overline{\mathfrak{o}}$ denotes the transitive closure of $\mathfrak{o}$. Note that $\overline{\mathfrak{o}}$ is a poset. So it follows that

$$X_\Gamma = \sum_{\mathfrak{o}} X_{\overline{\mathfrak{o}}}, \tag{1.17}$$

where the sum runs over all acyclic orientations of Γ. It follows that X_Γ has the following expansion in the fundamental quasi-symmetric basis:

$$X_\Gamma = \sum_{\mathfrak{o}} \sum_{\alpha \in \mathcal{L}(\overline{\mathfrak{o}})} Q_{\overline{D(\alpha)}}, \tag{1.18}$$

where $\mathcal{L}(\overline{\mathfrak{o}})$ denotes the Jordan-Hölder set of $\overline{\mathfrak{o}}$, and $D(\alpha)$ denotes the descent set of a permutation. (Stanley shows that for any poset P, X_P has such a quasi-symmetric expansion in terms of $\mathcal{L}(P)$.)

Stanley defines the following map on QSym_d:

$$\varphi(Q_{S,d}) = \begin{cases} t(t-1)^i, & \text{if } S = \{i+1, i+2, \ldots, d-1\} \\ 0, & \text{otherwise.} \end{cases} \tag{1.19}$$

A crucial step in Stanley's proof of Theorem 1.2.3 is the following result.

Lemma 1.2.5 (Stanley [41]) *For any (finite) poset P, $\varphi(X_P) = t^m$, where m is the number of minimal elements of P.*

In particular, $\varphi(X_{\overline{\mathfrak{o}}}) = t^{\mathrm{sink}(\mathfrak{o})}$, where $\mathrm{sink}(\mathfrak{o})$ is the number of sinks of $\mathfrak{o}$. So Theorem 1.2.3 follows from applying φ to (1.17), and calculating that $\varphi(e_\lambda) = t^{\ell(\lambda)}$.

Although Stanley doesn't mention this, the following lemma can be seen by considering the characterization of the product on QSym discussed in Section 1.1.

Lemma 1.2.6 *φ is an algebra map on* QSym.

We close this section with a list of some of the results and conjectures involving the e-positivity and s-positivity of specific families of graphs, and a related condition considered by Stanley.

Theorem 1.2.7 (Gasharov [10]) *If P is a $(\mathbf{3}+\mathbf{1})$-free poset, then $X_{\mathrm{inc}(P)}$ is s-positive.*

Conjecture 1.2.8 ([42, Conj. 1.4]) *If Γ is clawfree (i.e. has no induced subgraph isomorphic to K_{13}), then X_Γ is s-positive.*

Conjecture 1.2.9 ([43, Conj. 5.5], [41, Conj. 5.1]) *If P is a $(\mathbf{3}+\mathbf{1})$-free poset, then $X_{\mathrm{inc}(P)}$ is e-positive.*

In [42], Stanley considers graphs Γ which have the property that if Γ has a stable partition of type λ and $\mu \leq \lambda$ in the dominance order, then Γ has a stable partition of type μ. He calls such graphs *nice*, and presents the following two results as evidence in favor of Conjecture 1.2.8.

Proposition 1.2.10 ([42]) *If Γ is s-positive, then Γ is nice.*

Proposition 1.2.11 ([42]) *A graph Γ and all of its induced subgraphs are nice if and only if Γ is clawfree.*

In the next section, we will present some analogous evidence in favor of Conjecture 1.2.9.

1.3 Various Images of X_Γ

In this section, we will consider the images of X_Γ under certain naturally defined algebra maps and discuss some related results.

Let Ψ^k denote the k-th *convolution power* of the identity map, i.e. the composition

$$\mathrm{Sym} \xrightarrow{\Delta^{\otimes k-1}} \mathrm{Sym}^{\otimes k} \xrightarrow{\text{product}} \mathrm{Sym},$$

where Δ is the coproduct discussed in Section 1.1. In particular, it follows from (1.10) that $\Psi^k(p_i) = kp_i$. If $F \in \mathrm{Sym}$, then F can be viewed as a symmetric function in $\{x_1, x_2, \dots, y_1, y_2, \dots, \dots, z_1, z_2, \dots\}$ (the union of k countable sets of variables). Then $\Psi^k(F)$ is the image of $F(x_1, x_2, \dots, y_1, y_2, \dots, \dots, z_1, z_2, \dots)$ under the (algebra) map induced by mapping all of the variables $x_i, y_i, \dots, z_i$ to x_i (for each i). It follows that Ψ^k is an algebra map. More generally, the convolution product of any two algebra maps on a commutative Hopf algebra is again an algebra map (see, for example, [31, App. A]).

Ψ^k also has a fairly elegant description in terms of the generating function for the e_i's (which can be easily seen from (1.3)):

$$\Psi^k(E(q)) = (E(q))^k. \tag{1.20}$$

We can define an algebra map $\Psi^{(t)} : \mathrm{Sym} \to \mathrm{Sym}_{\mathbb{Q}[t]}$ which specializes to Ψ^k when $t = k$ by letting $\Psi^{(t)} p_\lambda = t^{\ell(\lambda)} p_\lambda$. We will be particularly interested in the coefficients of e_λ in Ψ^k, so it will be useful to consider the polynomials $c_\lambda(t)$, defined via the expansion

$$\Psi^{(t)} X_\Gamma = \sum_\lambda c_\lambda(t) e_\lambda.$$

An interesting fact about these polynomials is given in the following proposition.

Proposition 1.3.1

$$\sum_\lambda c_\lambda(t) = (-1)^d \chi_\Gamma(-t)$$

Proof: If k is a positive integer, then it follows from (1.20) that the sum of the e_λ coefficients in $\Psi^k(e_i)$ is $\binom{k+i-1}{i}$. Viewed as polynomials in k, $\binom{k+i-1}{i} = (-1)^i \binom{-k}{i}$. It follows that for any $\mu \vdash d$, the sum of the e_λ coefficients in $\Psi^k(e_\mu)$ is given by $(-1)^d \prod_i \binom{-k}{\mu_i}$. Also, the specialization of e_μ at 1^k (i.e. $x_1 = x_2 = \dots = x_k = 1$, $x_{k+1} = x_{k+2} = \dots = 0$) is $\prod_i \binom{k}{\mu_i}$.

So we have, in fact, that for any homogeneous symmetric function F of degree d, the sum of the e_λ coefficients in $\Psi^k(F)$ can be found by specializing F at 1^k, replacing k by $-k$, and multiplying by $(-1)^d$. The equality of the polynomials in the statement of the proposition follows from the fact that they are equal whenever t is replaced by a positive integer. □

If we expand ωp_μ in the elementary basis:

$$\omega p_\mu = \sum_\lambda d_{\mu,\lambda} e_\lambda,$$

then $d_{\mu,\lambda} \neq 0$ if and only if λ is a *refinement* of μ (i.e. the multiset of parts of λ can be obtained by taking partitions of each part of μ), and if $d_{\mu,\lambda} \neq 0$, its sign is $(-1)^{\ell(\lambda)-\ell(\mu)}$. This can be seen from (1.3), or more explicitly from a geometric interpretation due to Stembridge (apply ω to the expansion in Proposition 2.1.8).

So the following lemma can be obtained from the power sum expansion of X_Γ in Theorem 1.2.1 (in particular from the fact that the coefficient of ωp_λ in X_Γ is positive when Γ has a connected partition of type λ and is zero otherwise).

Lemma 1.3.2 *The polynomial $c_\lambda(t)$ has alternating coefficients. The coefficient of t^i has sign $(-1)^{\ell(\lambda)-i}$. $c_\lambda(t)$ has degree less than or equal to $\ell(\lambda)$, with equality if and only if Γ has a connected partition of type λ.*

The preceding considerations lead naturally to the following result, which gives a condition which must be satisfied in order for a graph to be e-positive.

Proposition 1.3.3 *If X_Γ is e-positive, and Γ has a connected partition of type λ, then Γ has a connected partition of type μ for every μ which is a refinement of λ. In particular, a connected, e-positive graph must have a connected partition of every type.*

Proof: Since X_Γ is e-positive, $\Psi^k X_\Gamma$ must be e-positive for every positive integer k. This can be seen from (1.11) or (1.20). If, for some λ, $c_\lambda(t)$ had a negative leading coefficient, then for large k, $c_\lambda(k)$ would be negative. So, for each $\lambda \vdash d$, $c_\lambda(t)$ must either have a positive leading coefficient or be identically zero.

Suppose Γ has a connected partition of type $\tilde{\lambda}$, but for some refinement $\tilde{\mu}$ of $\tilde{\lambda}$, Γ has no connected partition of type $\tilde{\mu}$. Among the partitions which are refinements of $\tilde{\lambda}$ but are not the types of connected partitions of Γ, choose a maximal one (with respect to refinement), μ. Let λ be a partition which lies between μ and $\tilde{\lambda}$ in the refinement order, and covers μ (i.e. μ can be obtained from λ by splitting one of the parts of λ into two pieces). In particular, $\ell(\mu) = \ell(\lambda) + 1$. The maximality of μ implies that Γ has a connected partition of type λ.

We claim that the leading coefficient of $c_\mu(t)$ is negative. This follows from Lemma 1.3.2 and the remarks preceding it. The coefficient of $t^{\ell(\mu)}$ in $c_\mu(t)$ is zero since Γ has no connected partition of type μ. But the coefficient of $t^{\ell(\mu)-1} = t^{\ell(\lambda)}$ is nonzero (and hence negative) since the coefficient of ωp_λ in X_Γ is positive. This is a contradiction, so the proposition follows. □

Proposition 1.3.3 can be used to show that many graphs are *not* e-positive. For example, the "claw", K_{13}, has no connected partition of type $(2,2)$. Figure 1.1 depicts an example of a non-e-positive, clawfree graph given by Stanley [41]. This graph is connected, but has no connected partition of type $(3,3)$.

However, the converse of Proposition 1.3.3 is false. For the graph in Figure 1.2, for example, Γ has a connected partition of each type, but

$$X_\Gamma = 7e_7 + 11e_{61} + 3e_{52} + 17e_{43} + 4e_{511} + 10e_{421} + 10e_{331} - 3e_{322} + 4e_{3211} + e_{2221}.$$

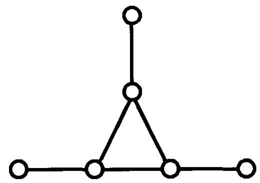

Figure 1.1: Stanley's example of a non-e-positive clawfree graph.

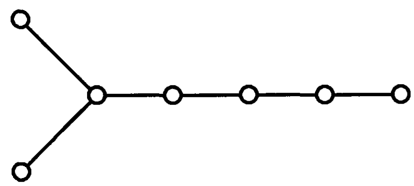

Figure 1.2: A non-e-positive graph with a connected partition of each type.

Proposition 1.3.3 shows that the property:

$$\text{If } \Gamma \text{ has a connected partition of type } \lambda \text{ and } \mu \text{ is a refinement of } \lambda, \text{ then } \Gamma \text{ has a connected partition of type } \mu. \tag{1.21}$$

is (at least roughly speaking) an analogue of "nice" (as discussed at the end of Section 1.2) for e-positivity. Since clawfree graphs are exactly those for which every induced subgraph is nice (see Proposition 1.2.11) and are conjectured to be s-positive, it's natural to at least *ask* whether a graph for which every induced subgraph satisfies (1.21) might have to be e-positive. When applied to all induced subgraphs, the property (1.21) can be simplified, and we can ask the following question instead.

Question 1.3.4 *Is there a non-e-positive graph with the property that any connected subgraph can be split into two blocks of arbitrary sizes?*

The following result shows that Conjecture 1.2.9 is at least consistent with Proposition 1.3.3, since any graph which has a Hamiltonian path trivially has a connected partition of each type. The fact that every indifference graph (an incomparability graph of a poset which is $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free) possesses a Hamiltonian path follows trivially from the characterization of these posets discussed in the introduction.

Proposition 1.3.5 *If P is a $(\mathbf{3}+\mathbf{1})$-free poset, then every connected subgraph of* $\operatorname{inc}(P)$ *has a Hamiltonian path.*

Proof: Note that a subgraph of $\operatorname{inc}(P)$ is the incomparability graph of an induced subposet of P, which is clearly $(\mathbf{3}+\mathbf{1})$-free if P is. Let P be a $(\mathbf{3}+\mathbf{1})$-free poset with n elements, and assume $\operatorname{inc}(P)$ is connected. By induction, we may assume that the proposition is true for smaller posets.

Let $\mathcal{P}$ be a path of maximal length in $\operatorname{inc}(P)$, and suppose that $\mathcal{P}$ does not contain every vertex. Then there is another vertex u which is adjacent to at least one vertex of $\mathcal{P}$.

By the inductive hypothesis, $\mathcal{P}$ must contain $n-1$ vertices, denote them by $v_1v_2\cdots v_{n-1}$. The ends, v_1 and v_{n-1}, cannot be adjacent to u, otherwise we could attach u at one of the ends to get a longer path. They also cannot be adjacent to each other, otherwise we could get a path $uv_j\cdots v_{n-1}v_1\cdots v_{j-1}$ for any v_j adjacent to u.

If v_j is any vertex which is adjacent to u, consider v_{j-1} and v_{j+1}. If v_{j+1} were adjacent to u, then $v_1\cdots v_j u v_{j+1}\cdots v_{n-1}$ would be a longer path. Similarly, v_{j-1} cannot be adjacent to u. So v_{j-1} and v_{j+1} are both comparable to u in P. If v_{j-1} and v_{j+1} were comparable to each other, then $\{u, v_{j-1}, v_{j+1}\}$ would be a chain of 3 elements, all incomparable to v_j, which would contradict the assumption that P is $(\mathbf{3}+\mathbf{1})$-free. So v_{j-1} and v_{j+1} must be adjacent.

Let $v_{j_1}, v_{j_2}, \ldots, v_{j_k}$ be all the vertices of $\mathcal{P}$ which are adjacent to u. By the argument above, the sequence obtained by omitting them, $v_1\cdots\widehat{v_{j_1}}\cdots\widehat{v_{j_2}}\cdots\widehat{v_{j_k}}\cdots v_{n-1}$, is a path in $\mathrm{inc}(P)$. The elements of this sequence must be either all less than u or all greater than u, otherwise two consecutive elements would have to be comparable (not adjacent). In particular, since every element of P which is comparable to u is in this sequence, u cannot lie in the middle of any three element chain. In other words, u is either minimal or maximal.

Now, since v_1 and v_{n-1} are comparable, one of them must lie in the middle of a three element chain with the other and u. Without loss of generality, assume this is v_{n-1}. The vertices $v_1, v_2, \ldots, v_{n-2}, u$ form a connected subgraph which must have a Hamiltonian path $\mathcal{P}'$ by the inductive hypothesis. Repeating the argument above with $\mathcal{P}'$ and v_{n-1} yields a contradiction. □

In light of this result, a natural question to ask is the following.

Question 1.3.6 *Is there a non-e-positive graph with the property that every connected subgraph has a Hamiltonian path?*

It *is* possible to have a non-e-positive graph Γ such that Γ itself has a Hamiltonian path. For example, the two non-s-positive graphs discussed in [42, Section 1] have connected components which are not e-positive but have Hamiltonian paths. In both of those examples, the graph contains a claw. We do not know of an example of a clawfree, non-e-positive graph which satisfies (1.21).

We will now consider the image of X_Γ under a more general algebra map. Let

$$\Theta : \mathrm{Sym} \to \mathrm{Sym}(x_1, x_2, x_3, \ldots, y_1, y_2, \ldots)$$

be the algebra map induced by first replacing the set of variables $\{x_i\}$ by variables indexed by $\mathbb{P}\times\mathbb{P}$, $\{x_{(i,j)}\}$, and then sending $x_{(i,j)}$ to x_iy_j. This sends the generating function $E(t) = \prod_i(1+x_it)$ to $\prod_{i,j}(1+x_iy_jt)$, so it follows from (1.5) that Θ is the algebra map induced by

$$e_k \mapsto \sum_{\lambda\vdash k} e_\lambda(x)m_\lambda(y) = \sum_{\lambda\vdash k} m_\lambda(x)e_\lambda(y).$$

Theorem 1.3.7 *The algebra map induced by* $e_k \mapsto \sum_{\lambda \vdash k} m_\lambda(x) e_\lambda(y)$ *sends* X_Γ *to*

$$\sum_{\pi \in L_\Gamma} X_\pi(y) X_{\Gamma|_\pi}(x) = \sum_{\pi \in \Pi(V)} \tilde{m}_{\text{type}(\pi)}(y) X_{\Gamma|_\pi}(x)$$

where X_π *is defined as in* (1.16).

Proof: Using the definition of Θ we began with, it follows that $\Theta(X_\Gamma)$ can be computed by considering the colorings of the vertices of Γ by ordered pairs (i,j) such that adjacent vertices are colored with different ordered pairs, and then associating the coloring with the monomial $\prod_v x_{\kappa_1(v)} y_{\kappa_2(v)}$, where κ_1 and κ_2 denote the projections of the coloring map into the first and second elements of the ordered pairs.

Consider all the proper colorings by ordered pairs where the second element of each pair is specified. If π is the partition of the vertices whose blocks are the sets with the same second element, then the choices of the first elements correspond exactly to the colorings enumerated by $X_{\Gamma|_\pi}(x)$. I.e. in order for the coloring to be proper, the first elements for adjacent vertices need to be different when the second elements are the same.

If we first choose an arbitrary partition of the vertices, π, and assign distinct second elements to each block, we obtain the equality of $\Theta(X_\Gamma)$ and the expression on the right.

If we first choose an arbitrary assignment of second elements, then (as in the proof of Theorem 1.2.1) there is a unique $\pi \in L_\Gamma$ such that this assignment corresponds to a coloring enumerated by X_π. As was noted in the proof of Theorem 1.2.1, the blocks of π are the connected components of the subgraphs on which the assignment is constant. In particular, if σ is the partition of the vertices whose blocks are the sets where the assignment of second elements is constant, then $\Gamma|_\pi$ and $\Gamma|_\sigma$ are actually equal. It follows that $\Theta(X_\Gamma)$ is equal to the expression on the left. □

It follows easily from the definitions that composing Θ with the specialization the y variables at 1^k yields the map Ψ^k. So specializing the y variables at (an arbitrary) 1^k in Theorem 1.3.7 yields the following formula for $\Psi^{(t)} X_\Gamma$.

Proposition 1.3.8

$$\Psi^{(t)} X_\Gamma = \sum_{\pi \in L_\Gamma} \chi_{\Gamma/\pi}(t) X_{\Gamma|_\pi}.$$

If we apply the map φ defined by (1.19) to the y variables in $\Theta(X_\Gamma)$, then Theorems 1.3.7 and 1.2.3 imply the following result.

Theorem 1.3.9 *The algebra map induced by* $e_k \mapsto \sum_{\lambda \vdash k} t^{\ell(\lambda)} m_\lambda$ *sends* X_Γ *to*

$$\sum_{\pi \in \Pi(V)} \sum_j \tilde{m}_{\text{type}(\pi)} t^j \,\text{sink}(\Gamma|_\pi, j),$$

where the first sum runs over arbitrary partitions of the vertices, and $\Gamma|_\pi$ *denotes the restriction to edges whose ends lie in the same block.*

Theorem 1.3.9 can be viewed as a common generalization of Theorems 1.2.3 and 1.2.4. If we set $t = 1$, the map above is just the standard involution ω, and it is not difficult to see that this interpretation of ωX_Γ is equivalent to Stanley's reciprocity result (Theorem 1.2.4). And if we evaluate the variables $(x_1, x_2, x_3, \ldots)$ in the image at $(1, 0, 0, \ldots)$, we recover Theorem 1.2.3. In Section 3.4, we will see that other specializations of Theorem 1.3.9 have interpretations in terms of certain hyperplane arrangements.

Since a product of monomial symmetric functions is monomial positive, Theorem 1.3.9 implies that certain nonnegative linear combinations of the c_λ's must be nonnegative (where $X_\Gamma = \sum c_\lambda e_\lambda$). We know of no such inequalities (satisfied for all graphs) other than what is implied by Theorem 1.3.9.

Chapter 2

Monomial Immanants

The bulk of this chapter will be devoted to discussing the following conjecture, and in particular, proving that it holds for some particular families of partitions, λ.

Conjecture 2.0.1 ([46, Conj. 5.2]) *Let $s_{[i,j]} \in \mathbb{Z}\mathfrak{S}_n$ denote the sum of all permutations of $\{i, i+1, \ldots, j\}$ and Π denote the set of all finite products of the $s_{[i,j]}$'s. Then $\forall \lambda \vdash n$, $\phi^\lambda(\pi) \geq 0$ for all $\pi \in \Pi$.*

It follows from results in Stembridge [46] that Conjecture 2.0.1 holds when $\lambda = (r^j), (n-1, 1)$, or $(2, 1^{n-2})$. In Sections 2.2, 2.4, and 2.5, we will show that this conjecture also holds in the cases when $\lambda = (i, j)$, $(r, 1^{n-r})$, or $(2^k, 1^\ell)$.

In Section 2.1, we will develop the tools that will be used in later sections. In particular we will discuss the lattice path interpretation of Jacobi-Trudi matrices which motivates our techniques. This interpretation was used by Gessel and Viennot to give a combinatorial proof of the Jacobi-Trudi identity, and by Greene to prove the following theorem.

Theorem 2.0.2 (Greene [20]) *$\chi^\lambda[H_{\mu/\nu}]$ is m-positive.*

These considerations show that Conjecture 2.0.1 implies the following corresponding statement for monomial immanants. This was Stembridge's motivation for making Conjecture 2.0.1.

Conjecture 2.0.3 ([46, Conj. 4.2(b)]) *$\phi^\lambda[H_{\mu/\nu}]$ is m-positive.*

In particular, our results show that Conjecture 2.0.3 holds when λ is in one of the families mentioned above. Section 2.1 also contains a proof that if Conjecture 2.0.1 holds for λ, then it holds for $(m\lambda_1, m\lambda_2, \ldots, m\lambda_k)$.

In Section 2.6, we will discuss a more general conjecture made by Stembridge concerning digraphs with certain path-intersection properties. In particular, we will show that this more general conjecture is also implied by Conjecture 2.0.1.

2.1 Skeletons

Rather than defining skeletons directly in terms of the $s_{[i,j]}$'s, we will use the following "geometric" definition.

2.1.1 Definition: *A* **skeleton** (of order n), S, *is a planar, finite, acyclic digraph (where multiple edges are allowed and distinguishable), together with linear orderings of its sources and sinks, which satisfies several conditions. The vertices of S consist of n sources of valence 1 (labelled $u_1, u_2, \ldots, u_n$ according to the ordering), n sinks of valence 1 (labelled $v_1, v_2, \ldots, v_n$ according to the ordering), and any number of "internal" vertices with in-valence = out-valence. There is an embedding $\iota = \iota_{\mathsf{x}} \times \iota_{\mathsf{y}} : S \to \mathbb{R}^2$ with the following properties:*

- *The embedding is planar (edges intersect only at vertices).*
- *There are real numbers $\mathsf{x}_1 < \mathsf{x}_2$ such that the sources are embedded on the line $\mathsf{x} = \mathsf{x}_1$ and the sinks are embedded on the line $\mathsf{x} = \mathsf{x}_2$. The y-coordinates of the sources, $\iota_{\mathsf{y}}(u_1), \iota_{\mathsf{y}}(u_2), \ldots, \iota_{\mathsf{y}}(u_n)$, form a strictly increasing sequence, as do the y-coordinates of the sinks.*
- *Every edge from w_1 to w_2 is embedded as a curve which moves strictly left-to-right from $\iota(w_1)$ to $\iota(w_2)$, i.e. the edge is embedded as the graph of a function on the interval $[\iota_{\mathsf{x}}(w_1), \iota_{\mathsf{x}}(w_2)]$.*

We will usually want to assume that the internal vertices are embedded with distinct x-coordinates, so that they are linearly ordered. We will also usually want to assume that the y-coordinates of u_i and v_i are equal. However, any embedding can be deformed to achieve either of these conditions, and the extra generality will be convenient.

For any skeleton S of order n, we can consider families, $\mathcal{F}$, of n paths whose union contains each edge of S exactly once. These conditions imply that each path begins at a source of S and ends at a sink. So, any such family defines a permutation $\sigma_{\mathcal{F}} \in \mathfrak{S}_n$ via $\sigma_{\mathcal{F}} : i \to j$ if some path in F goes from u_i to v_j. Let

$$\langle S \rangle = \sum_{\mathcal{F}} \sigma_{\mathcal{F}} \in \mathbb{Z}\mathfrak{S}_n \tag{2.1}$$

where the sum runs over all such families of paths, and let $Z[S]$ denote the image of $\langle S \rangle$ under the map $\sigma \mapsto p_{\mathsf{type}(\sigma)}$. We will call $Z[S]$ the *cycle indicator* of S.

In Section 2.1.2, we will show that for any skeleton, $\langle S \rangle$ factors as a product of $s_{[i,j]}$'s. Since $p_{\mathsf{type}(\sigma)} = \sum_{\lambda} \phi^{\lambda}(\sigma) h_{\lambda}$, it follows that the (conjectured) nonnegativity of ϕ^{λ} on products of $s_{[i,j]}$'s is equivalent to the following statement:

Conjecture 2.1.2 (Reformulation of Conjecture 2.0.1) *For any skeleton S, the cycle indicator $Z[S]$ is h-positive. Equivalently, $\omega Z[S]$ is e-positive.*

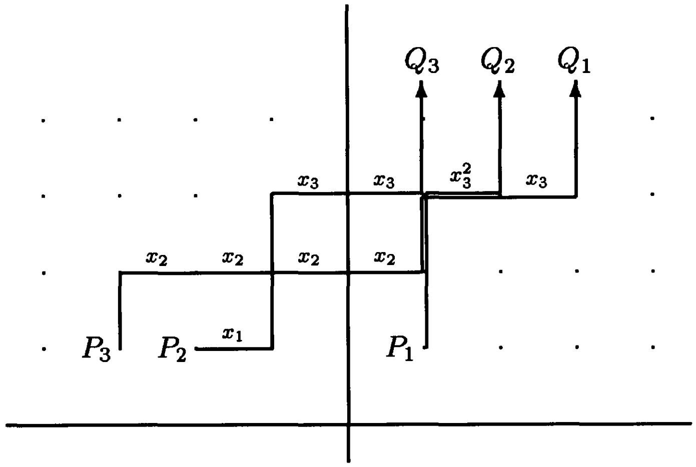

Figure 2.1: A family of lattice paths enumerated by $[H_{(4,4,4)/(2)}]$

2.1.1 Jacobi-Trudi Matrices and Lattice Paths

In this section, we will discuss an interpretation of the terms in

$$[H_{\mu/\nu}] = \sum_{\sigma \in \mathfrak{S}_n} \left(\prod_{i=1}^{n} h_{(\mu_i - i)-(\nu_{\sigma_i} - \sigma_i)} \right) \cdot \sigma^{-1} \tag{2.2}$$

as generating functions for certain families of lattice paths and the role these lattice paths play in the work of Gessel and Viennot [17], Goulden and Jackson [18], and Greene [20]. The exposition here is essentially equivalent to the treatment in [20].

Consider $\mathbb{Z} \times \mathbb{Z}$ as a digraph with edges from (i,j) to $(i,j+1)$ ("north") and from (i,j) to $(i+1,j)$ ("east"). Assign a weight of x_j to the "east" edges from (i,j) to $(i+1,j)$ when $j > 0$, and a weight of one to all of the other edges. Let the weight of a path be the product of the weights of its edges.

Let us consider paths from $(i,1)$ to $(i+k,\infty)$ (where this means infinite paths which go to infinity along the line $x = i+k$). The enumerator of the weights of such paths is

$$\sum_{i_1 \leq i_2 \leq \cdots \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_k} = h_k.$$

Let $P_i = (\nu_i - i, 1)$ and $Q_i = (\mu_i - i, \infty)$. It follows that the product

$$\prod_{i=1}^{n} h_{(\mu_i - i)-(\nu_{\sigma_i} - \sigma_i)}$$

enumerates families of n paths, where the i-th path goes from P_{σ_i} to Q_i. An example is shown in Figure 2.1. We will say that this family is *associated with* σ^{-1}. Since this product is the coefficient of σ^{-1}, we have

$$[H_{\mu/\nu}] = \sum_{\sigma \in \mathfrak{S}_n} (\text{Enumerator of sets of paths from } P_i \text{ to } Q_{\sigma_i}) \cdot \sigma \tag{2.3}$$

If $\mathcal{F}$ is a family of paths appearing in (2.3), let $\alpha = \alpha_{\mathcal{F}}$ denote the multiset of edges appearing in the union of the paths. Following Goulden and Jackson [18], we will call $\alpha_{\mathcal{F}}$ the *diagram* of $\mathcal{F}$. The crucial first step in Greene's proof of Theorem 2.0.2 is the analysis (due to Goulden and Jackson [18]) of the families of paths which have the same diagram, α. The terms in equation (2.3) can be regrouped according to their diagram to obtain

$$[H_{\mu/\nu}] = \sum_{\alpha} f_\alpha \langle \alpha \rangle \tag{2.4}$$

where f_α is the weight of α (the product of the weights of the edges, including multiplicities) and $\langle \alpha \rangle$ denotes the sum of the permutations associated with all families of paths whose diagram is α.

Each diagram α is associated with a subdigraph of $\mathbb{Z}\times\mathbb{Z}$ in the obvious way. Let $\tilde{\alpha}$ denote the digraph obtained from this as follows. Replace edges with multiple (distinguishable) edges according to how many times they appear in the multiset α and remove vertices with in-valence = out-valence = 1, joining the edges. Also add the vertices Q_i (they are now vertices, rather than limits of infinite paths). It is clear that the natural embedding of $\tilde{\alpha}$ in the plane can be deformed to match the conditions of Definition 2.1.1. So $\tilde{\alpha}$ is a skeleton.

Let $m(e)$ denote the multiplicity in α of the edge e and let $M(\alpha) = \prod_e m(e)!$ (the product running over all the edges). The families enumerated by $\langle\tilde{\alpha}\rangle$ each correspond to families enumerated by $\langle\alpha\rangle$, but the process of making the multiple edges distinguishable introduces a factor of $M(\alpha)$ so that $\langle\tilde{\alpha}\rangle = M(\alpha)\langle\alpha\rangle$.

Now it can be argued that $\langle\tilde{\alpha}\rangle$ factors as a product of $s_{[i,j]}$'s. We will postpone this argument until the next section (Proposition 2.1.6) in order to introduce more terminology.

Combining everything together, we obtain the expansion of Goulden and Jackson [18]

$$[H_{\mu/\nu}] = \sum_{\alpha} f_\alpha \frac{1}{M(\alpha)} \langle\tilde{\alpha}\rangle \tag{2.5}$$

where each $\langle\tilde{\alpha}\rangle$ is a product $s_{[i_1,j_1]} s_{[i_2,j_2]} \cdots s_{[i_k,j_k]}$.

This shows that Theorem 2.0.2 can be reduced to the following statement, conjectured by Goulden and Jackson and proved by Greene.

Theorem 2.1.3 ([20]) *For any intervals $[i_\ell, j_\ell] \subseteq [n]$ and any irreducible character χ^λ,*

$$\chi^\lambda(s_{[i_1,j_1]} \cdots s_{[i_k,j_k]}) \geq 0.$$

As was mentioned in the introduction, Greene derives Theorem 2.1.3 by proving the following stronger result.

Theorem 2.1.4 ([20, Thm. 1.3]) *If $[i,j] \subseteq [n]$, let $\rho_\lambda(s_{[i,j]})$ denote the matrix representing $s_{[i,j]}$ in Young's seminormal representation of $\mathfrak{S}_n$, indexed by λ. Then every entry of the matrix $\rho_\lambda(s_{[i,j]})$ is nonnegative.*

Gessel and Viennot's [17] combinatorial proof of the Jacobi-Trudi identity is equivalent to the following observations. The sign character vanishes on any product $s_{[i_1,j_1]}\cdots s_{[i_k,j_k]}$ in which one on the intervals has more than one element. Hence, the surviving contributions to $\det H_{\mu/\nu}$ come from non-intersecting families of paths. Finally, non-intersecting families of paths correspond directly to semi-standard tableaux of shape μ/ν (by filling the ith row with the weights along the ith path).

2.1.5 Remarks: Our language differs somewhat from that used in [18,20]. Greene uses the term "skeleton" for α, rather than $\widetilde{\alpha}$. (He attributes the terminology to John Stembridge.)

The reader will note that Definition 2.1.1 includes digraphs which cannot be obtained as $\widetilde{\alpha}$'s via the construction above. However, the $\widetilde{\alpha}$'s contain a certain amount of "redundancy" — for example, in the skeleton associated with the set of paths in Figure 2.1, the vertices at $(1,2)$ and $(1,3)$ could be combined without changing any essential properties.

Goulden and Jackson [18] define some *reduction operations* to simplify α, and it can be shown that the digraphs they obtain as reduced diagrams are exactly the skeletons of Definition 2.1.1 which have no vertices with in-valence = out-valence = 1.

2.1.2 Elementary Operations on Skeletons

The reason for our lengthy definition of "skeleton" is that it makes certain properties transparent.

Gluing and factoring

Two skeletons S_1 and S_2 of order n can be "glued together" to form a new skeleton $S_1 \diamond S_2$ by identifying each sink v_i^2 of S_2 with the source u_i^1 of S_1, i.e. S_1 goes on the right of S_2. Let S_1S_2 denote the result of removing the vertices of $S_1 \diamond S_2$ where the gluing took place and replacing the pairs of edges which entered and left these vertices by single edges. Then S_1S_2 is also a skeleton of order n. Note that adding or removing vertices with in-valence = out-valence = 1 does not affect the associated permutations.

Likewise, given an embedding of a skeleton, S, as in Definition 2.1.1, we may assume that the internal vertices are linearly ordered according to their x-coordinates (i.e. we may assume that these x-coordinates are distinct). For any consecutive pair of them, w_1 and w_2, we can factor S as a product S_1S_2 by introducing nodes at the places where some vertical line lying between w_1 and w_2 intersects edges, and then "ungluing" the left and right portions at these nodes. This process can be continued to factor S as a product of skeletons which have only one internal vertex.

Now, if $S = S_1S_2$, a family of paths $\mathcal{F}$ enumerated by $\langle S\rangle$ determines families $\mathcal{F}_1$ and $\mathcal{F}_2$ for S_1 and S_2, respectively. The permutation $\sigma_{\mathcal{F}}$ is just the composition $\sigma_{\mathcal{F}_1}\circ\sigma_{\mathcal{F}_2}$. (This is the reason we chose to put S_1 to the right of S_2.) It is also clear that any pair $\mathcal{F}_1$, $\mathcal{F}_2$ can be glued together to get a family for the whole. In other words, choosing a family of paths which uses each edge exactly once is the same as (independently) choosing, at each internal vertex, the pairs of edges which follow each other. So we have that

$$\langle S_1 \diamond S_2 \rangle = \langle S_1 S_2 \rangle = \langle S_1 \rangle \langle S_2 \rangle, \tag{2.6}$$

the product taken in $\mathbb{Z}\mathfrak{S}_n$.

Now, if S has only one internal vertex, then the indices of the sources incident to it form an interval of integers $[i,j] = \{i, i+1, \ldots, j\}$, and the same is true for the sinks. This follows from the planarity of the embedding in Definition 2.1.1. It also follows that the intervals for the sources and sinks are the same (the sources and sinks below these intervals are paired by the edges not incident to the internal vertex, and similarly for those above the intervals). It is clear that $\langle S \rangle$ is just the sum in $\mathbb{Z}\mathfrak{S}_n$ of all permutations of the elements of $[i,j]$, i.e. $s_{[i,j]}$.

We can put all of these ideas together to get the following proposition.

Proposition 2.1.6 *For any skeleton S of order n, there are intervals $[i_1, j_1], \ldots, [i_k, j_k] \subseteq [n]$ such that $\langle S \rangle = s_{[i_1,j_1]} \cdots s_{[i_k,j_k]}$.*

Figure 0.1 in the introduction depicts a skeleton for which $\langle S \rangle$ factors $s_{[3,4]} s_{[1,3]} s_{[2,4]}$.

The argument given above uses essentially the same ideas as Goulden and Jackson's derivation of (2.5).

Subskeletons

It will be useful to consider two different notions of a "subskeleton" embedded within a skeleton. If S_0 is a subdigraph of S, we will call S_0 a *generalized subskeleton* of S if it satisfies the conditions:

1. The sources of S_0 are sources of S, and the sinks of S_0 are sinks of S.

2. For each of the remaining vertices of S_0, in-valence = out-valence.

S_0 inherits an ordering on its sinks and sources from S, as well as an embedding, and condition 2 is equivalent to requiring S_0 to be a skeleton. We will use the term *subskeleton* for those generalized subskeletons whose sources and sinks have the same set of indices.

If S_0 is a generalized subskeleton, let the *complement* $S \setminus S_0$ denote the subdigraph whose edges complement those of S_0. It is clear that $S \setminus S_0$ is a generalized subskeleton and is a subskeleton when S_0 is. If $\mathcal{P}$ is a set of paths in S which begin in the sources and end in the sinks whose union contains no edge more than once, then clearly the subdigraph, $S|_{\mathcal{P}}$, obtained by taking the edges used by paths in $\mathcal{P}$, is a generalized subskeleton of S.

We will call a subskeleton *connected* if the underlying graph (i.e. forgetting the directions of the edges) is connected.

Given two skeletons S_1 and S_2 of orders k and ℓ, we can form a new skeleton by taking the digraph which is the disjoint union of S_1 and S_2, and labeling the sinks and sources of S_1 and S_2 (in order) by $\{1, \ldots, k\}$ and $\{k+1, \ldots, l+\ell\}$, respectively. This *disjoint union*, $S_1 \sqcup S_2$, can be embedded in the plane by embedding S_2 above S_1. Some simple observations are that each skeleton is the disjoint union of its connected components (the maximal connected subskeletons) and that

$$Z[S_1 \sqcup S_2] = Z[S_1]Z[S_2].$$

In particular, the conjecture that $Z[S]$ is h-positive can be reduced to the case of connected skeletons.

In Sections 2.1.5 and 2.3, we will consider certain partitions of S into generalized subskeletons and subskeletons. In particular, our choice of language in this matter is motivated by the analogy between connected subgraphs and connected subskeletons which will be explored in Section 2.3.

Symmetries

There are also a few symmetries of skeletons which should be mentioned in this section of "elementary observations."

Proposition 2.1.7 *For any skeletons S and S':*

1. *If S^{op} is the digraph obtained by reversing the edges of S, then S^{op} is a skeleton and $Z[S^{\mathrm{op}}] = Z[S]$.*

2. *If $S^{\text{up-down}}$ is the same digraph, but with the orders of the sinks and sources reversed, then $S^{\text{up-down}}$ is a skeleton and $Z[S^{\text{up-down}}] = Z[S]$.*

3. *$Z[SS'] = Z[S'S]$.*

Proof: Embeddings for S^{op} and $S^{\text{up-down}}$ are given by reflecting an embedding for S in the y-axis and x-axis, respectively. The statements about the cycle indicators follow from the fact that cycle types are not changed by taking inverses or conjugating by the permutation $(i \mapsto n+1-i)$. The last statement is similar $(\tau\sigma = \tau(\sigma\tau)\tau^{-1})$. □

2.1.3 Interpreting $\phi^\lambda\langle\text{Skeleton}\rangle$

Stembridge gives the following characterization of ϕ^λ. For the sake of completeness, we will repeat his proof here.

Proposition 2.1.8 ([46, Prop. 1.1]) *For any $\sigma \in \mathfrak{S}_n$, regard the cycles of σ as a directed graph, and let $r_\lambda(\sigma)$ be the number of subgraphs which are isomorphic to a disjoint union of directed paths of vertex cardinalities $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then*

$$\phi^\lambda(\sigma) = \epsilon_\lambda(-1)^\sigma r_\lambda(\sigma).$$

Proof: We have

$$p_{\text{type}(\sigma)} = \sum_{\lambda \vdash n} \phi^{\lambda}(\sigma) h_{\lambda},$$

so it suffices to compute the expansion of p_μ in the complete basis. The generating functions for power sums and complete symmetric functions are related by the following (see (1.3)).

$$\sum_{n \geq 1} \frac{p_n}{n} t^n = \log \left(\sum_{r \geq 0} h_r t^r \right) = \sum_{r \geq 1} \frac{(-1)^{r-1}}{r} \left(\sum_{j \geq 1} h_j t^j \right)^r$$

If λ has k parts, then the coefficient of $t^{|\lambda|} h_\lambda$ in the right hand side is $(-1)^{k-1}/k$ times the number of permutations of λ (i.e. the number of distinct ordered k-tuples of integers which are rearrangements of the parts of λ). So the coefficient of h_λ in the expansion of p_n is $(-1)^{k-1} n/k$ times the number of permutations of λ.

If σ consists of a single cycle of length n, then $k r_\lambda(\sigma)$ is n times the number of permutations of λ. To see this, note that $k r_\lambda(\sigma)$ counts the number of ways to divide the n-cycle into directed paths of vertex sizes given by λ and to choose one of the paths. Likewise, n times the number of permutations of λ counts the number of ways to choose a starting point and divide the n-cycle into directed paths with one of them beginning at the chosen starting point. So to see that the proposition holds for a single n-cycle, we note that in this case, $\epsilon_\lambda (-1)^\sigma = (-1)^{n-k}(-1)^{n-1} = (-1)^{k-1}$.

For an arbitrary permutation with type$(\sigma) = \mu$, let $\sigma^{(i)}$ denote the corresponding cycle of σ of length μ_i. Then applying the proposition to each cycle, we obtain:

$$p_\mu = \prod_i p_{\mu_i} = \prod_i \left(\sum_{\lambda^{(i)} \vdash \mu_i} (-1)^{\ell(\lambda^{(i)})-1} r_{\lambda^{(i)}}(\sigma^{(i)}) h_{\lambda^{(i)}} \right) =$$

$$= \sum_{\lambda \vdash n} (-1)^{\ell(\lambda)-\ell(\mu)} \left(\sum_{\substack{\lambda^{(i)} \vdash \mu_i \\ \cup \lambda^{(i)} = \lambda}} \prod_i r_{\lambda^{(i)}}(\sigma^{(i)}) \right) h_\lambda.$$

The inner sum in the last expression counts the number of ways to divide the parts of λ into partitions of the appropriate sizes, and then choose directed paths for each cycle with vertex sizes given by these partitions. Clearly the total of all these is $r_\lambda(\sigma)$. So the proposition follows (noting that $\epsilon_\lambda(-1)^\sigma = (-1)^{\ell(\lambda)-\ell(\mu)}$). □

Let cyl(S) denote the digraph obtained from S by identifying each source u_i with the sink v_i. As the name suggests, cyl(S) can be embedded on a cylinder. Namely, take an embedding of S as in Definition 2.1.1 where the y-coordinates of u_i and v_i are equal, and glue the vertical lines containing the sources and sinks together to form the cylinder.

As before, let $\mathcal{F}$ be a set of n paths which begin in the sources of S and end in the sinks, and whose union contains each edge exactly once. Consider the subset of paths, $\mathcal{P}$, corresponding to a cycle of length j in $\sigma_{\mathcal{F}}$, i.e. $\mathcal{P}$ consists of paths from u_{i_1} to v_{i_2}, from u_{i_2}

to v_{i_3}, and so on until we get to a path from u_{i_j} to v_{i_1}. $\mathcal{P}$ corresponds to a closed trail (a closed walk with no edge used more than once) in the digraph cyl(S).

Conversely, consider any closed trail in cyl(S). Clearly, the trail determines a set of paths, $\mathcal{P}$, in S such that the labels of the sources and sinks used in $\mathcal{P}$ are the same. The remarks made in the previous section imply that $S|_{\mathcal{P}}$ and its complement $S \setminus (S|_{\mathcal{P}})$ are subskeletons of S. In particular, we can choose a family of paths on the complement to get a family for all of S.

By abuse of notation, we will refer to a closed trail of cyl(S), or the associated family, $\mathcal{P}$, as a *cycle (of length j) of S*. Since the digraph underlying S is acyclic, this should not create any confusion. Also note that if $\mathcal{P}$ is a cycle of length j for S, then the corresponding trail in cyl(S) is embedded on the cylinder as a closed curve with winding number j.

The gluing operation described above can be used to construct a digraph consisting of infinitely many copies of S glued together:

$$S^\infty = \cdots \diamond S \diamond S \diamond S.$$

Recall that the gluing operator, $\diamond$, places the first factor on the right, so S^∞ extends infinitely to the right. We will assume that we have some fixed embedding of S, and S^∞ is embedded using copies of the embedding of S.

We will say that edges in the ith copy of S (counting from the left) lie in the ith *column* of S^∞. We will use the term "end-nodes" to denote the vertices of S^∞ which come from sinks and sources of S. Label every copy of these with the numbers $\{1, 2, \ldots, n\}$ according to their order in S (i.e. increasing y-coordinates in the embedding). Also, number the "columns" of these from left to right, starting with zero, so that the edges in the ith column lie between the $(i-1)$st and ith columns of end-nodes.

Consider a cycle of S as defined above, viewed as a set of paths: P_1 (from u_{i_1} to v_{i_2}), P_2 (from u_{i_2} to v_{i_3}), and so on up to P_j (from u_{i_j} to v_{i_1}). For each path, P_k, we can uniquely lift this cycle to an infinite path in S^∞ which begins at (a copy of) u_{i_k} in the 0th column of end-nodes whose restriction to any column is one of these paths. Namely, this infinite path contains a copy of P_k in column 1, a copy of P_{k+1} in column 2, and so on (P_1 following P_k).

Now consider a family of paths, $\mathcal{F}$, which contributes to $\langle S \rangle$ and a subgraph of $\sigma_{\mathcal{F}}$ which is isomorphic to a disjoint union of directed paths of vertex cardinalities $\lambda_1, \lambda_2, \ldots, \lambda_k$ (i.e. one of those counted by $r_\lambda(\sigma_{\mathcal{F}})$). For each of those directed paths, lift the cycle it is contained in to one starting with u_i, where i is the source of this directed path in $\sigma_{\mathcal{F}}$. In this way, we can construct a family of k paths in S^∞. The families of paths obtained in this way are completely characterized by the following conditions:

1. Each path begins at a source of S^∞ and extends infinitely to the right.

2. For each edge, e, of S, at least one of the copies of e in S^∞ is used in one of the paths.

3. If e_1 and e_2 are edges of S, and a copy of e_1 is followed by a copy of e_2 in one of the paths, then every copy of e_1 appearing in any of the paths is followed by a copy of e_2.

Clearly, the family $\mathcal{P}$ of paths in S^∞ determines $\mathcal{F}$ and the subgraph $\sigma_\mathcal{F}$. Let $\mathfrak{P}(S)$ denote the set of these families of paths, and let $\mathfrak{P}_\lambda(S)$ denote the set of families coming from a particular λ.

Given a family of paths, $\mathcal{P} \in \mathfrak{P}(S)$, we can recover $\lambda = \lambda(\mathcal{P})$ by looking at the distance (i.e. the number of columns) along each path before it passes through an end-node with the same label as one which begins one of the paths. Equivalently, if we delete every edge from our set of paths which is not the leftmost copy of the corresponding edge of S, we will be left with a set of k paths with lengths (measured in columns) given by λ. It will prove useful to consider this set of "truncated" paths later on, denote it by $T(\mathcal{P})$.

We can also recover the number of cycles in the corresponding permutation (paths corresponding to the same cycle are "shifted" copies of each other in the obvious sense). Denote the number of cycles by $C(\mathcal{P})$. Note that the sign of the contribution made by this family to the coefficient of h_λ in $Z[S]$ (i.e. $\epsilon_\lambda(-1)^\sigma$) is just $(-1)^{\#\mathcal{P}-C(\mathcal{P})}$.

So, in this language, Proposition 2.1.8 says that

$$Z[S] = \sum_{\mathcal{P}\in\mathfrak{P}(S)} (-1)^{\#\mathcal{P}-C(\mathcal{P})} h_{\lambda(\mathcal{P})}, \tag{2.7}$$

or equivalently,

$$\phi^\lambda\langle S\rangle = \sum_{\mathcal{P}\in\mathfrak{P}_\lambda(S)} (-1)^{\#\mathcal{P}-C(\mathcal{P})} \tag{2.8}$$

Given a set of paths $\mathcal{P} \in \mathfrak{P}(S)$, we could "shift" all of these paths to the left by j columns to obtain a new set of paths $\mathcal{P}'$, i.e. we could look at the restriction of these paths to the columns $j+1, j+2, \ldots$, which is just a copy of S^∞ again. Obviously, $\mathcal{P}$ and $\mathcal{P}'$ have the same underlying set of cycles. In terms of the the directed paths which form a subgraph of $\sigma_\mathcal{F}$ as discussed above, this operation corresponds to shifting all of the directed paths along the cycles of $\sigma_\mathcal{F}$. So $\lambda(\mathcal{P}) = \lambda(\mathcal{P}')$.

In particular an element $\mathcal{P} \in \mathfrak{P}(S)$ is determined by the set of underlying cycles and the set of labels of the end-nodes which appear in the jth column of end-nodes, for any j. Also note that the set of underlying cycles is determined by a bijection from the edge set of S, $E(S)$, to itself, i.e. map e_1 to e_2 if copies of e_1 are followed by copies of e_2 in the paths of $\mathcal{P}$.

2.1.4 Surgery at an Intersection

Suppose that we have a set of paths $\mathcal{P} \in \mathfrak{P}(S)$ such that two of these paths, P_1 and P_2, intersect at some (internal) vertex v in the jth column. Let $\tilde{e}_1$ and $\tilde{f}_1$ be the edges of P_1 which enter and leave (respectively) the vertex v, and let e_1 and f_1 be the corresponding edges of S. Define e_2 and f_2 similarly for the path P_2.

Let $\mathcal{P}'$ be the unique set of paths in S^∞ defined by

- Each copy of e_1 is followed by a copy of f_2 and each copy of e_2 is followed by a copy of f_1.

- For any edge e of S other than e_1, e_2, each copy of e is followed by a copy of whatever edge it is followed by in $\mathcal{P}$.

- $\mathcal{P}'$ has the same set of end-nodes appearing in the jth column of end-nodes as $\mathcal{P}$ does.

Let us refer to this process as "doing surgery on $\mathcal{P}$ at $\tilde{e}_1, \tilde{e}_2$." It's easy to see that $\mathcal{P}' \in \mathfrak{P}(S)$.

Clearly $\mathcal{P}'$ also has an intersection of two paths at v in column j, with the same edges, $\tilde{e}_1, \tilde{e}_2$, entering v. If we do surgery on $\mathcal{P}'$ at $\tilde{e}_1, \tilde{e}_2$, we get $\mathcal{P}$ again.

Consider the sets of cycles underlying $\mathcal{P}$ and $\mathcal{P}'$. If e_1 and e_2 (as above) lie in different cycles for $\mathcal{P}$, then they lie in the same cycle for $\mathcal{P}'$ (and vice-versa). I.e., for $\mathcal{P}'$, e_1 is followed by f_2, then the cycle continues as it did in $\mathcal{P}$ until it reaches e_2, which is followed by f_1, and then the cycle continues as it did in $\mathcal{P}$ until it reaches e_1 again. The other cycles are unaffected by the surgery. In particular, the number of cycles changes by exactly one in going from $\mathcal{P}$ to $\mathcal{P}'$, but the number of paths remains the same, so the associated sign changes.

This surgery idea is useful because it preserves λ. In order to see this, we need to look more closely at which edges are used in which column of S^∞. Consider the paths of $\mathcal{P}$, beginning at the $(j-1)$st column of end-nodes (i.e. the column of end-nodes which are sources for the copy of S containing the intersection where the surgery occurs). In other words, delete all the edges to the left of this. Now, for each edge of S, delete all copies of it which are not leftmost among what remains. The comments made in the previous section imply that the result is a set of paths with lengths (measured in columns) given by $\lambda(\mathcal{P})$. Do the same thing for $\mathcal{P}'$. Let Q be one of these truncated paths for $\mathcal{P}$, and let Q' be the truncated path for $\mathcal{P}'$ which contains the same end-node in the jth column of end-nodes (by definition, $\mathcal{P}$ and $\mathcal{P}'$ use the same end-nodes in this column). The portions of Q and Q', if any, which lie in columns $j+1, j+2, \dots$ cannot contain copies of e_1 or e_2 (all such copies in these columns have been deleted), and hence must be identical. In particular, Q and Q' have the same length. So we have $\lambda(\mathcal{P}) = \lambda(\mathcal{P}')$.

For future reference, we summarize this discussion:

Lemma 2.1.9 *If $\mathcal{P}, \mathcal{P}' \in \mathfrak{P}(S)$ are obtained from each other by doing surgery at some intersection, then $\lambda(\mathcal{P}) = \lambda(\mathcal{P}')$, and $(-1)^{\#\mathcal{P}-c(\mathcal{P})} = -(-1)^{\#\mathcal{P}'-c(\mathcal{P}')}$.*

Obviously, the lemma suggests that the most natural way to try to prove that $Z[S]$ is h-positive (see (2.7)) would be to try to carefully choose intersections so that these surgeries could be used to pair each negative contribution to a positive one. It is possible to do this on $\mathfrak{P}_\lambda(S)$ for certain choices of λ. In fact, we will see that when λ is a rectangle, a hook, or a partition with exactly two parts, such a pairing can be constructed by using the leftmost intersection (resolving ambiguities at intersections of three or more paths in some canonical way). This includes all of the cases which are shown in [46].

However, doing surgery at a leftmost intersection can cause new intersections to appear to the left, as in Figure 2.2, for example. In this case, the set of paths in the upper picture has a positive sign, and will not be paired with anything.

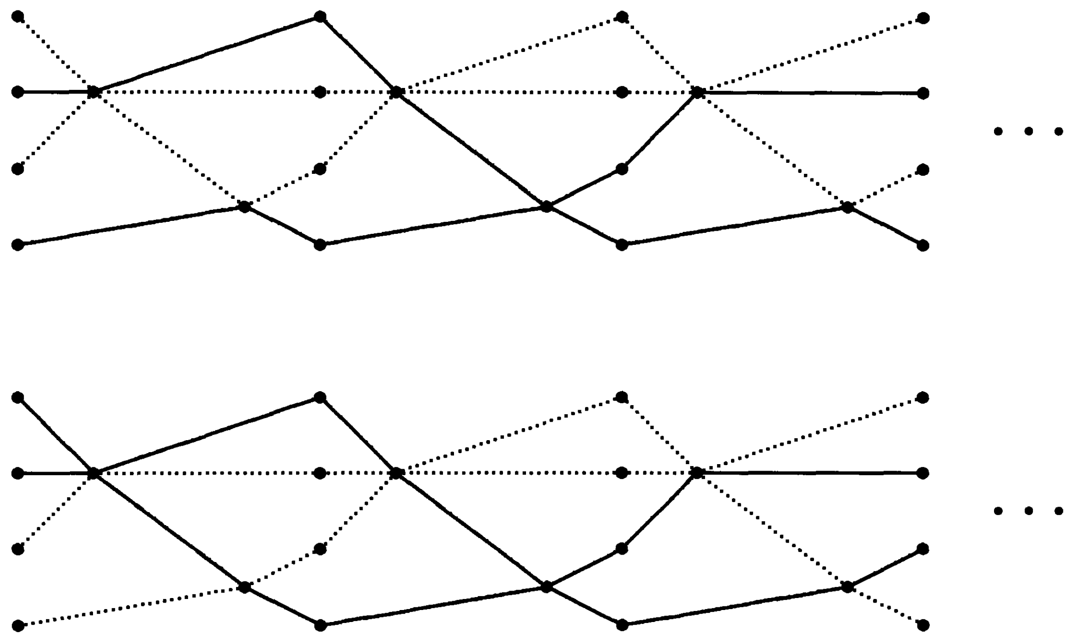

Figure 2.2: In general, surgery need not preserve leftmost intersections.

Of course, in order to make any use of these surgery ideas, we need the set of paths to contain intersections. The following lemma will be fundamental to the remainder of our discussion.

Lemma 2.1.10 *If two distinct paths $P_1, P_2 \in \mathcal{P} \in \mathfrak{P}(S)$ have the same underlying cycle of length j, then P_1 and P_2 must intersect, and their leftmost intersection must occur within the first $j-1$ columns of S^∞. Note that j must be at least 2. In particular, any $\mathcal{P}$ with a negative sign must have an intersection.*

Proof: Let i_0 denote the smallest label of the end-nodes appearing in the underlying cycle of P_1 and P_2, and let j_1, j_2 denote the column numbers of the first occurrence of this end-node in the paths P_1, P_2, respectively. We can assume $j_1 < j_2$ (if they were equal, P_1 and P_2 would be the same path). Since both paths begin repeating after j columns, we also have $0 \leq j_1 < j_2 \leq j$. If we look at the column of end-nodes labelled j_1 in the planar embedding of S^∞, we see that at this point, the path P_2 is above the path P_1, since the path P_1 is at the lowest end-node to occur in either path. Similarly, in the column numbered j_2, P_1 must be above P_2. It follows that for some consecutive pair $(i-1, i)$, the path P_2 is above P_1 in the $(i-1)$st column of end-nodes and below P_1 in the ith column of end-nodes. In particular, the paths must intersect in the ith column of S^∞ (i.e. in between these columns of end-nodes).

A similar argument (using the largest label of the end-nodes) implies that there is a column of S^∞ (within the first j columns) where P_2 begins below P_1 and ends above it. This column also contains an intersection, and clearly this column is different from the one obtained above.

Since there are (at least) two columns among the first j which contain intersections, the leftmost intersection must occur within the first $j-1$ columns. The last assertion follows, since any $\mathcal{P}$ with no intersections must have the same number of paths and cycles, and so $(-1)^{\#\mathcal{P}-c(\mathcal{P})} = +1$. □

Recall from Section 2.1.3 that $T(\mathcal{P})$ is the set of paths resulting from removing from $\mathcal{P}$ all edges which are not the leftmost copies of the corresponding edges of S. Intersections among these are considerably more well-behaved than the general case. If we do surgery on $\mathcal{P}$ at some intersection in $T(\mathcal{P})$ in the jth column, then the set of edges used by $\mathcal{P}$ in each of the columns $1, 2, \ldots, j$ remains unchanged. The argument is basically the same as the proof that λ is not affected by surgery. The edges used in column j remain the same, since the surgery is defined to fix the end-nodes at the right side of this column (the jth column of end-nodes) and the only changes in the underlying set of paths on S are that some edges are rearranged between the two paths whose intersection is being considered. The paths in columns $1, 2, \ldots, j-1$ actually remain exactly the same, since the set of end-nodes used in the the $(j-1)$st column of end nodes remains unchanged (these are determined by the set of edges used in column j) and the only edges affected by the surgery are assumed to appear for the first time in column j.

In particular, if we have a set of paths $\mathcal{P}$ which contains an intersection in the truncated part, $T(\mathcal{P})$, then look at the leftmost vertex which contains an intersection in $T(\mathcal{P})$, and let $\tilde{e}_1$ and $\tilde{e}_2$ denote the highest pair of edges of $T(\mathcal{P})$ entering this vertex (the embedding gives an ordering of the edges into this vertex). If $\mathcal{P}'$ is the result of doing surgery on $\mathcal{P}$ at $\tilde{e}_1, \tilde{e}_2$, then $\mathcal{P}'$ also has an intersection in the truncated part $T(\mathcal{P}')$, and this same pair of edges is the leftmost and highest such pair (the only changes among which edges are used where have occurred to the right of the column where the surgery happened). So doing this surgery defines a sign-reversing (and λ preserving) involution on the sets of paths, $\mathcal{P}$, which have an intersection in the truncated part, $T(P)$, and so we have

Proposition 2.1.11 *The total contribution to $Z[S]$ coming from sets of paths, $\mathcal{P} \in \mathfrak{P}(S)$, which have an intersection in the truncated part, $T(\mathcal{P})$, is zero.*

An easy corollary is Stembridge's [46] result that nonnegativity holds when λ is a rectangle. His proof is based on a matrix identity discussed below (Theorem 2.1.15).

Corollary 2.1.12 *If $\lambda = (r^j)$ (a rectangle), then $\phi^\lambda\langle S\rangle$ is the number of sets of paths $\mathcal{P} \in \mathfrak{P}_\lambda(S)$ which have no intersections.*

Proof: If $\mathcal{P} \in \mathfrak{P}_\lambda(S)$, then all of the truncated paths in $T(\mathcal{P})$ have a length of r columns. So *every* end-node in the rth column of end-nodes which is used by the paths in $\mathcal{P}$ is also an end-node used to begin one of the paths. It follows that shifting all of the paths to the left by r columns doesn't change anything. In particular, $\mathcal{P}$ has intersections if and only if $T(\mathcal{P})$ has intersections. Lemma 2.1.10 tells us that the sets of paths without intersections each make a positive contribution, so the result follows from the previous proposition. □

Another corollary of Proposition 2.1.11 is that the non-vanishing of $\phi^\lambda\langle S\rangle$ places restrictions on the valences of vertices of S.

Corollary 2.1.13 *If S has an internal vertex whose in-valence (= out-valence) is larger than λ_1, then $\phi^\lambda\langle S\rangle = 0$.*

Proof: Consider any $\mathcal{P} \in \mathfrak{P}_\lambda(S)$. The truncated paths, $T(\mathcal{P})$ are completely contained in the first λ_1 columns (since λ_1 is the largest part of λ), and they contain exactly one copy of each edge of S. In particular, there must be a column in which $T(\mathcal{P})$ contains at least two edges which enter the given internal vertex (there are more such edges than available columns). So $T(\mathcal{P})$ always contains an intersection, and the result follows. □

2.1.14 Remark: It can be seen from Greene's explicit computation of the matrices $\rho_\lambda(s_{[i,j]})$ in his proof of Theorem 2.1.4 that $\chi^\lambda\langle S\rangle = 0$ when S has an internal vertex whose in-valence (= out-valence) is larger than λ_1. Corollary 2.1.13 can be derived from this using the triangularity between the monomial and Schur function bases of Sym.

2.1.5 Divisibility Considerations

Stembridge's analysis of ϕ^λ when λ is a rectangle is based on the following matrix identity.

Theorem 2.1.15 ([46, Thm. 2.8]) *If $\lambda = (r^j)$, then*

$$\phi^\lambda[A] = \sum \det A(I_1|I_2)\det A(I_2|I_3)\cdots\det A(I_r|I_1),$$

where the sum runs over ordered partitions $(I_1, I_2, \ldots, I_r)$ of $[n]$ into disjoint j-sets, and $A(I|J)$ denotes the submatrix of A obtained by selecting the rows indexed by I and the columns indexed by J.

Note that summand in the above expression could be expressed as

$$\det(A(I_1|I_2)A(I_2|I_3)\cdots A(I_r|I_1)).$$

Our main result in this section will be to show that there is an analogue of this result for skeletons in which the sign character (i.e. the determinant) can be replaced by an arbitrary monomial character. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $m\lambda = (m\lambda_1, m\lambda_2, \ldots, m\lambda_k)$. We will show that $\phi^{m\lambda}\langle S\rangle$ can be expressed as a sum of ϕ^λ applied to skeletons, in a manner which is analogous to Theorem 2.1.15.

Consider a set of paths $\mathcal{P} \in \mathfrak{P}_{m\lambda}(S)$. If a particular end-node is used in the ith and jth columns of end-nodes, and not in between, then, by the usual argument, $j - i$ is a part of $m\lambda$. It follows that if i and j are any columns of end-nodes where this particular end-node is used, then $j - i$ is a multiple of m. So, if we let I_i denote the set of labels of end-nodes which are used by $\mathcal{P}$ in columns $j \equiv i \mod m$, then $I_0, I_1, \ldots, I_{m-1}$ are disjoint. Clearly their union is $[n]$.

Let $\mathcal{F}$ denote the family of paths in S which corresponds to $\mathcal{P}$, as in the definition of $\mathfrak{P}$ in Section 2.1.3. Clearly each path of $\mathcal{F}$ begins at a source with a label in I_i and ends at a sink with label in I_{i+1} (with the convention that $I_m = I_0$). Let $\mathcal{F}_i$ denote the paths beginning in I_i, and let S_i denote the subdigraph of S defined by taking all the edges used in the paths of $\mathcal{F}_i$. Clearly each S_i is a skeleton - in the language of Section 2.1.2, it's what we called a generalized subskeleton of S.

Gluing these together into $\widetilde{S} = S_{m-1} \diamond S_{m-2} \diamond \cdots \diamond S_0$ gives a subskeleton of $S \diamond S \diamond \cdots \diamond S$ (m copies of S). In terms of the embeddings, each S_i is embedded in the $(i+1)$st copy of S counting from the left, and the sources and sinks of $\widetilde{S}$ have the same set of labels (I_0). Each edge of S is used in only one path in $\mathcal{F}$, and hence in only one S_i. (Let us use the phrase "partition of S" to indicate a collection of subdigraphs where each edge appears in a unique subdigraph.) It follows that the families $\mathcal{F}$ (which yield these S_i's) can be chosen by independently choosing families for each S_i, and these correspond to choosing families, $\widetilde{\mathcal{F}}$ for $\widetilde{S}$.

Now, $(\widetilde{S})^\infty$ can be embedded in the plane so that each copy of $\widetilde{S}$ is embedded in (the images of) m consecutive copies of S in the embedding of S^∞. Moreover, the set of infinite paths, $\widetilde{\mathcal{P}}$, which corresponds to $\widetilde{\mathcal{F}}$ is just $\mathcal{P}$, i.e. exactly the same sequences of edges are used in the paths.

The reason for setting all of this up is that $\lambda(\mathcal{P}) = m\lambda(\widetilde{\mathcal{P}})$, (i.e. $\lambda(\widetilde{\mathcal{P}}) = \lambda$). To see this, note that the truncated sets of paths, $T(\widetilde{\mathcal{P}})$ and $T(\mathcal{P})$, are also equal. The lengths of these paths in terms of columns give $\lambda(\widetilde{\mathcal{P}})$ and $\lambda(\mathcal{P})$, but each column of $(\widetilde{S})^\infty$ corresponds to m columns of S^∞, and so the result follows. We also need to remark that the signs associated with $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ are the same, since the numbers of paths and underlying cycles are the same. Grouping the terms of (2.8) together according to the S_i's, we obtain the following:

Proposition 2.1.16

$$\phi^{m\lambda}\langle S\rangle = \sum \phi^{\lambda}\langle S_{m-1} \diamond S_{m-2} \diamond \cdots \diamond S_0\rangle,$$

where the sums runs over all ordered partitions $(S_0, S_2, \ldots, S_{m-1})$ *of* S *into generalized subskeletons for which the labels of the sinks of* S_i *are the same as the labels of the sources of* S_{i+1} *(with the convention* $S_m = S_0$*).*

An obvious corollary is that nonnegativity of ϕ^λ on skeletons implies that $\phi^{m\lambda}$ is also nonnegative on skeletons. Moreover, since there is actually a direct correspondence between the sets of paths, if there is a pairing between the negative contributions and a subset of the positive contributions for λ which is defined in terms of surgeries at intersections, then this pairing gives such a pairing for $m\lambda$ (directly).

2.1.17 Remark: It it natural to ask if there is a matrix identity analogue of Proposition 2.1.16 for a general λ. Unfortunately, there doesn't seem to be. An analogous method (with weighted paths) can be used to show that as a polynomial in the entries of A, the sum $\sum \phi^\lambda[(A(I_1|I_2)A(I_2|I_3)\cdots A(I_r|I_1)]$ (over the same partitions as in Theorem 2.1.15) is equal to $\phi^{r\lambda}[A]$ plus some terms involving products of the entries of A which don't correspond to

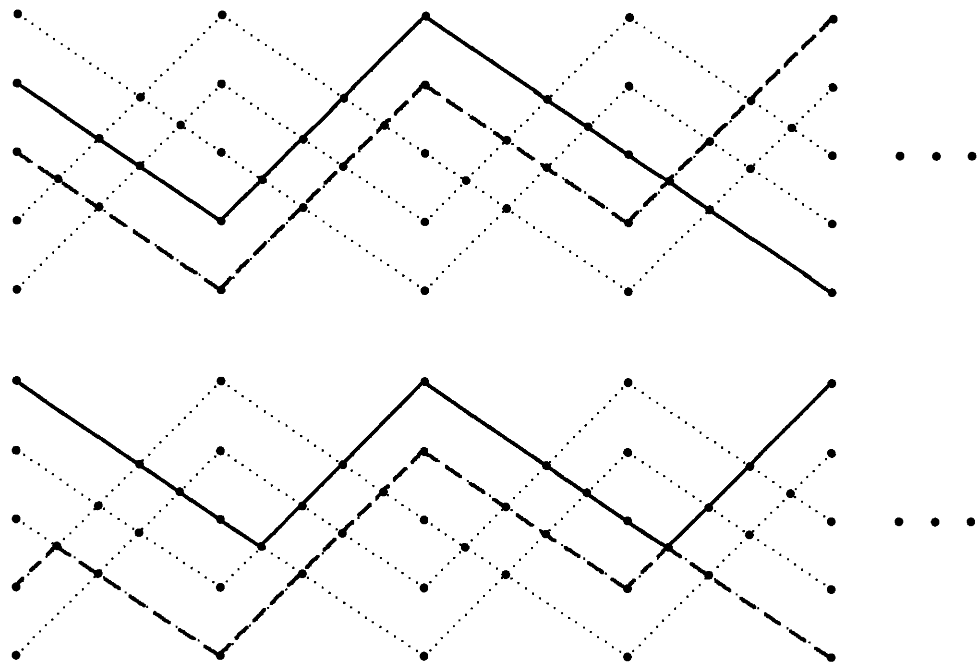

Figure 2.3: A pair for $\lambda = (3, 2)$

rook placements. This is significantly less interesting. The essential difference is that the determinant will kill these extra terms, but ϕ^λ need not (in general).

2.2 The Case of Two Paths

Since any surgery at an intersection of paths in some $\mathcal{P} \in \mathfrak{P}(S)$ involves a pair of paths, the case in which there are only two paths is the most basic one, from the perspective of these methods.

Lemma 2.2.1 (The "Widening" Lemma) *Let $\mathcal{P} \in \mathfrak{P}_{(i,n-i)}(S)$ be a family (a pair) of paths which have the same underlying cycle. Let $\mathcal{P}'$ denote the result of doing surgery at the leftmost intersection of $\mathcal{P}$ (such an intersection must exist by Lemma 2.1.10). Then the leftmost intersection of $\mathcal{P}'$ occurs at the vertex where the surgery took place. Moreover, among the portions of these paths to the left of this vertex, the uppermost one for $\mathcal{P}'$ lies (weakly) above the uppermost one for $\mathcal{P}$ (in the embedding), and the lowermost one for $\mathcal{P}'$ lies (weakly) below the lowermost one for $\mathcal{P}$.*

Proof: Let P_1 denote the path with the uppermost source in the embedding, and P_2 denote the other one. Let $\tilde{e}_1$ and $\tilde{e}_2$ be the edges of P_1 and P_2 (respectively) which enter the leftmost vertex where there is an intersection. Denote this vertex by v. Note that the portion of P_1 to the left of v lies strictly above the corresponding portion of P_2, since the paths would otherwise intersect to the left of v.

Let P_1', P_2' denote the paths of $\mathcal{P}'$ which contain $\tilde{e}_1, \tilde{e}_2$, respectively. Let Q denote the portion of P_1 up to v, and let Q' denote the corresponding portion of P_1'. We want to claim that Q' lies weakly above Q in the embedding.

Let $\tilde{f}_1$ and $\tilde{f}_2$ be the edges which follow $\tilde{e}_1$ and $\tilde{e}_2$ (respectively) in $\mathcal{P}$. Let e_1, e_2, f_1, f_2 denote the corresponding edges of S, as in the definition of surgery (Section 2.1.4).

If Q and Q' are not equal, then Q must contain one of the edges f_1, f_2 (as we move to the left from the intersection, the surgery only changes things which contain copies of these edges). If Q contained f_1, it would have to also contain a copy of e_1 other than the one at the end (f_1 leaves an internal vertex, so it can't be at the left end of Q). But if Q contained two copies of e_1, then the leftmost intersection would occur after the underlying cycle started repeating, which would contradict Lemma 2.1.10. So we may assume that Q contains a copy of f_2 and contains no copies of f_1.

We can construct Q' as the limit of a sequence $Q = Q_1, Q_2, Q_3, \ldots$ as follows. If Q_i contains a copy of e_2 followed by a copy of f_2, replace the portion of Q_i up to and including the rightmost such copy of e_2 by the "tail" of Q of the appropriate length, and call the result Q_{i+1}. In other words, the difference between the underlying cycles in $\mathcal{P}$ and $\mathcal{P}'$ is that for $\mathcal{P}'$, the copies of f_2 must be preceded by e_1 instead of e_2, and similarly for copies of f_1. The other "rules" of what precedes something are the same. In forming Q_{i+1} from Q_i, we look at the rightmost place where f_2 is not preceded by e_1 and fix it by preceding it with e_1 and a path which follows the "rules" for $\mathcal{P}$ (i.e. the tail of Q). In this way we eventually get back to the left edge of S^∞, and we have "fixed" all the "rules" by then. So we eventually get Q'.

At each step, the portion of Q_i up to and including the rightmost copy of e_2 which is followed by f_2 is a path which follows the "rules" of $\mathcal{P}$ and ends in e_2, i.e. it's a tail of the portion of P_2 up to v (since the length of Q_i is exactly the same as this portion of P_2, it's actually a proper tail). And it gets replaced by the tail of Q (which is the corresponding portion of the upper path in $\mathcal{P}$) of the appropriate length. So part of Q_i is replaced by something lying above this part in the embedding. So, in particular, Q' lies (weakly) above Q.

The analogous argument shows that the portion of P_2' to the left of v lies (weakly) below the corresponding portion of P_2. In particular, the portions of P_1', P_2' considered here can't intersect each other, except at v, so v is also the leftmost intersection in $\mathcal{P}'$. □

Figure 2.3 shows an example of the surgery in the "Widening" Lemma. In this case, both of the paths leading into the leftmost intersection are altered by the surgery. One of the paths is drawn as a dotted line to make it clear how the paths go through the intersection.

An immediate conseqence of Lemma 2.2.1 is that the negative contributions to $\phi^{(i,n-i)}\langle S\rangle$ can be paired with a subset of the positive contributions. And so we have the following proposition.

Proposition 2.2.2 *If S is any skeleton, $\phi^{(i,n-i)}\langle S\rangle \geq 0$.*

Stembridge [46, Thm. 6.4] gives a proof that $\phi^{(n-1,1)}\langle S\rangle \geq 0$. His proof is actually carried out in a more general framework which we will discuss in detail in Section 2.6.

Although stated in somewhat different language, the pairing between negative and positive contributions to $\phi^{(n-1,1)}\langle S\rangle$ obtained above is identical to the pairing used in Stembridge's proof, which also relies on a "surgery" argument. So our methods can be viewed as a generalization of those used by Stembridge. The shape $(n-1,1)$ is also a hook, and the proof of the hook case in Section 2.4 will also specialize to give the same pairing for $\lambda=(n-1,1)$.

It will be useful in Sections 2.3.1 and 2.4 to have a more detailed description of what happens when we do the surgery in the "Widening" Lemma for the $\lambda=(n-1,1)$ case.

Corollary 2.2.3 *Let $\mathcal{P}=\{P_1,P_2\}\in\mathfrak{P}_{(n-1,1)}(S)$ be a pair of paths with the same underlying cycle, and assume that P_2 can be obtained by "shifting" P_1 to the left by one column (see p. 52), or equivalently that the truncated part of P_1 in $T(\mathcal{P})$ has a length of $n-1$ columns. Let $\mathcal{P}'=\{P_1',P_2'\}$ be the result of doing surgery at the leftmost intersection of $\mathcal{P}$ (numbered so that P_1' and P_1 enter the vertex containing the intersection via the same edge). Then the underlying cycles of P_1' and P_2' have lengths $n-1$ and 1, respectively. Moreover, the portions of P_1 and P_1' up to the leftmost intersection are identical.*

Proof: Lemma 2.1.10 implies that the leftmost intersection of $\mathcal{P}$ occurs within the first $n-1$ columns. If $n=2$, all of the statements are trivial. So assume that $n>2$. Then $\mathcal{P}'$ contains two paths whose underlying cycles have lengths of 1 and $n-1\neq 1$. Since the leftmost intersection occurs within the first $n-1$ columns, the portion of the path of $\mathcal{P}'$ whose underlying cycle has length $n-1$ contains no edge of S more than once. It follows that this portion of the path remains fixed while doing surgery at the intersection to recover $\mathcal{P}$ (i.e. in the notation of the proof of Lemma 2.2.1, copies of f_1,f_2 do not appear in this portion of the path in question). This implies that the path whose underlying cycle has length $n-1$ is P_1' and that the portions of P_1 and P_1' up to the leftmost intersection are identical. □

Corollary 2.2.4 *If S contains at most two internal vertices, $\omega Z[S]$ is e-positive.*

Proof: If S has j connected components of order one (i.e. edges from a source to a sink), then $\omega Z[S]=e_1^j\cdot\omega Z[S']$, where S' is obtained by removing these. So we may assume that every path in S passes through an internal vertex. So any set of paths $\mathcal{P}\in\mathfrak{P}(S)$ with three or more paths must contain an intersection in the first column. By Proposition 2.1.11, the total contribution coming from such sets of paths is zero. Hence the only λ's which could have a non-zero coefficient are those with 1 or 2 parts. Hence the result follows from Proposition 2.2.2 and the case when $\lambda=(n)$, which is trivial. □

Corollary 2.2.4 can also be obtained as a special case of Stanley's result [41, Cor. 3.6] that if Γ is a graph whose vertices can be partitioned into two disjoint cliques, then X_Γ is e-positive.

2.3 ωZ[Skeleton] and Chromatic Symmetric Functions

In this section, we will explore the relationship between the cycle indicators of skeletons and chromatic symmetric functions of graphs. A result of Stanley and Stembridge [43] asserts that the e-positivity of the incomparability graphs of posets which are $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free is equivalent to the h-positivity of $Z[S]$ for certain skeletons. We will show that if S is any skeleton, $\omega Z[S]$ can be written as a *sum* of chromatic symmetric functions of incomparability graphs, and we will explore some of the properties of these graphs. Unfortunately, this doesn't reduce the problem, since the individual graphs obtained here can definitely fail to be e-positive.

We will say that a skeleton is *connected* when the graph obtained by ignoring the directions on the edges is connected. It is not difficult to see that a skeleton of order n is connected if and only if it has a cycle (as in Section 2.1.3) of length n. Surgery ideas clearly imply that a cycle of maximal length in a connected skeleton must pass through each edge. Conversely, if S is not connected, then planarity implies that some component of S has labels $[1, i]$ $(i < n)$ for its sources and sinks, and then clearly S cannot have a cycle of length n.

Let S be a skeleton of order n. If the edge set of S is the disjoint union of the edge sets of a collection $\{S_1, S_2, \ldots, S_k\}$ of connected subskeletons, we will call this collection a *partition of S into connected subskeletons.* Note that the sources and sinks of each S_i must have the same sets of indices. Given such a partition τ, let $\rho(\tau) \vdash n$ denote the partition of n whose parts are the orders of the S_i's.

Let $P(S)$ denote the set of partitions of S into connected subskeletons, partially ordered by refinement. $P(S)$ has a unique maximal element whose blocks are the connected components of S, but will usually have many minimal elements. Denote the unique maximal element by $\hat{1}$.

Let $I(S)$ denote the set of minimal elements of $P(S)$, and consider any $\pi \in I(S)$. By using a shifted copy and surgery at the intersection guaranteed by Lemma 2.1.10, we can "split" any cycle of S into two smaller cycles. So the comments above imply that each block of π consists of a single path, P_i, from one of the beginning nodes u_i to the corresponding ending node v_i. In particular, the number of elements in $I(S)$ is the coefficient of the identity permutation in $\langle S \rangle \in \mathbb{Z}\mathfrak{S}_n$.

We can define a graph $\Gamma(\pi)$ on $[n]$ by putting an edge between i and j if the ith and jth paths of π intersect. Equivalently, $\Gamma(\pi)$ is the incomparability graph of the poset P on $[n]$ defined by $i <_P j$ when $i <_{\mathbb{P}} j$ and the ith and jth paths of π do not intersect (i.e. the jth path lies strictly above the ith path in the embedding). Also note that (obviously) the ordering of the sources and sinks in S gives a linear extension of P.

In order to understand the relationship between S and these graphs, we will need to consider intervals of the form $[\pi, \hat{1}] \subseteq P(S)$, where π is minimal. For each fixed $\pi \in I(S)$, the partitions of S into connected subskeletons which lie in this interval are completely determined by the induced partitions of $[n]$ coming from the labels of the sources and sinks of each subskeleton. To see this, consider some $\tau \in [\pi, \hat{1}]$ (i.e. $\tau \geq \pi$ in $P(S)$). The block of

τ which contains the source and sink labelled i must contain the path P_i described above, since π is a refinement of τ. But every edge of S lies in one of these paths, so the edges used in each block of τ are determined by the sources and sinks of that block (and π, of course).

Moreover, for this fixed π, the partitions $\tau \in [\pi, \hat{1}]$ correspond exactly to the partitions of $[n]$ which are connected partitions of the graph $\Gamma(\pi)$. This follows from the fact that the internal vertices of a block of τ can be viewed as intersections of the paths, $\{P_i\}$, used to define $\Gamma(\pi)$. In other words, for any minimal π, the interval $[\pi, \hat{1}]$ is isomorphic to $L_{\Gamma(\pi)}$, the lattice of contractions (or bond lattice) of $\Gamma(\pi)$.

In particular, the Möbius function of $P(S)$ strictly alternates in sign, i.e. for any pair, $\sigma \leq \tau$,

$$(-1)^{|\sigma|-|\tau|}\mu_{P(S)}(\sigma, \tau) > 0. \tag{2.9}$$

This follows since the same is true for bond lattices (as was noted in the introduction), and $\mu_{P(S)}(\sigma, \tau) = \mu_{\Gamma_\pi}(\sigma, \tau)$ for any minimal π which is comparable to σ (the Möbius function on (σ, τ) only depends on the poset structure of the interval $[\sigma, \tau]$).

The following lemma is an analogue for skeletons of Stanley's expansion of X_Γ in terms of the Möbius function of L_Γ (Theorem 1.2.1).

Lemma 2.3.1 *For any connected skeleton S_0, let $\mathcal{C}(S_0)$ denote the number of cycles of S_0 of maximal length (= the order of S_0). Then for any connected skeleton S_0 and any skeleton S, we have*

$$\mathcal{C}(S_0) = \sum_{\pi \in I(S_0)} |\mu(\pi, \hat{1})| \tag{2.10}$$

$$\omega Z[S] = \sum_{\substack{\pi \in I(S), \tau \in P(S) \\ \pi \leq \tau}} \mu(\pi, \tau) p_{\rho(\tau)}. \tag{2.11}$$

Proof: Both statements are trivial if the skeletons have order 1. We will show that if S is any skeleton and (2.10) holds for the *proper* (i.e. not the whole of S) connected subskeletons of S, then (2.11) holds for S. We will also show that for a connected skeleton S, (2.11) implies (2.10). It follows by induction on the order of the skeletons that both statements hold in general. We will need some preliminary observations.

$Z[S]$ is defined to be $\sum_{\mathcal{F}} p_{\text{type}(\sigma_{\mathcal{F}})}$, where the sum runs over the families of paths appearing in (2.1). The cycles of $\sigma_{\mathcal{F}}$ correspond to cycles of S as discussed in Section 2.1.3. The comments at the beginning of this section imply that a set of cycles (using each edge exactly once) defines a partition of S into connected subskeletons. If we combine the terms corresponding to the same partition of S and apply the involution ω, we obtain

$$\omega Z[S] = \sum_{\tau \in P(S)} \left(\prod_{S_i \in \tau} \mathcal{C}(S_i) \right) \epsilon_{\rho(\tau)} p_{\rho(\tau)}. \tag{2.12}$$

Now consider the contribution of a given τ to the right hand side of (2.11). Suppose that τ has k blocks, $S_1, S_2, \ldots, S_k$. Choosing a partition of S into connected subskeletons which is a refinement of τ is obviously equivalent to choosing partitions for each S_i. The poset structure on the *order ideal* generated by τ (i.e. the elements of $P(S)$ which are less than or equal to τ) is just the direct product of the posets $P(S_1)$, $P(S_2), \ldots, P(S_k)$. It then follows from the product theorem for Möbius functions (see [40, Prop. 3.8.2]) that

$$\sum_{\substack{\pi \in I(S) \\ \pi \leq \tau}} \mu(\pi,\tau) = \prod_{i=1}^{k} \left(\sum_{\pi \in I(S_i)} \mu_{P(S_i)}(\pi, \hat{1}_{S_i}) \right). \tag{2.13}$$

Now we are ready to prove the inductive steps. First, let S be any skeleton of order n, and assume that (2.10) holds for the proper connected subskeletons of S. Note that the computation of the sign of the Möbius function in (2.9) implies that if π is minimal, then the sign of $\mu(\pi,\tau)$ is $\epsilon_{\rho(\tau)}$. If τ has more than one part, then our inductive hypothesis (applied to each part) implies that the expression in (2.13) is equal to the contribution of τ in (2.12). This implies that the coefficients of p_λ on each side of (2.11) are equal when λ has 2 or more parts. If S is not connected, then this includes all of the non-zero coefficients. So assume S is connected. We need to show that the coefficients of $p_{(n)}$ on each side of (2.11) are equal. It suffices to show that the *sum* of all of the p_λ coefficients on each side are equal. The sum of the p_λ coefficients in $Z[S]$ is just the sum of the signs of the permutations associated with S, i.e. 1 if S has order one, and zero otherwise (S is connected). Likewise, for each minimal π,

$$\sum_{\tau \in [\pi, \hat{1}]} \mu(\pi,\tau) = \begin{cases} 1 & \pi = \hat{1}, \\ 0 & \text{otherwise.} \end{cases}$$

This can be taken as the definition of the Möbius function. For a connected S, $\hat{1}$ is minimal if and only if S has order one.

The second inductive step is fairly trivial. If S is a connected skeleton of order n which satisfies (2.11), then taking the absolute values of the coefficients of $p_{(n)}$ on the left and right sides of (2.11) yields (2.10). □

Theorem 2.3.2 $\omega Z[S] = \sum_{\pi \in I(S)} X_{\Gamma(\pi)}$.

Proof: Given that the interval $[\pi, \hat{1}]$ in $P(S)$ is isomorphic to $L_{\Gamma(\pi)}$, this follows immediately by comparing the expansions in (2.11) and Theorem 1.2.1. □

So immediately we have that the (conjectured) h-positivity of $Z[S]$ is equivalent to the e-positivity of a sum of chromatic symmetric functions of graphs. In particular, everything which can be said about the coeeficients in the elementary expansion of an arbitrary X_Γ can be applied here. For example, Theorem 1.2.3 yields the following corollary.

Corollary 2.3.3 *For any skeleton S of order n and any $k \in \mathbb{P}$,*

$$\sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k}} \phi^\lambda \langle S \rangle = \sum_{\pi \in I(S)} \mathrm{sink}(\Gamma(\pi), k).$$

We know of no other way to prove the nonnegativity of the expression appearing on the left hand side.

There is a "canonical" element $\pi_0 \in P(S)$ which corresponds to choosing the identity in each factor. $\Gamma(\pi_0)$ is an indifference graph, but in general, $\Gamma(\pi)$ has no special properties. In fact, it is fairly easy to see that every incomparability graph appears as a $\Gamma(\pi)$ for some skeleton.

Theorem 2.3.2 can be viewed as a generalization of the result of Stanley and Stembridge [43] that the e-positivity of indifference graphs is equivalent to a special case of the monomial immanant conjectures. For the skeletons which correspond to indifference graphs, it can be shown that all of the $\Gamma(\pi)$'s are equal to the corresponding indifference graph.

2.3.1 Cycles and Acyclic Orientations

The main result of this section will be a bijective proof of the $k = 1$ case of Corollary 2.3.3. Let S be a skeleton of order n. The elements of $\mathfrak{P}_{(n)}(S)$ are liftings to S^∞ of cycles of S whose length is n. (Technically speaking, we ought to say that the elements of $\mathfrak{P}_{(n)}(S)$ are families consisting of these single paths.) The sign associated with each element of $\mathfrak{P}_{(n)}(S)$ is positive, so $\phi^\lambda\langle S\rangle$ just counts the number of them. We may assume that S is a connected skeleton of order n (if S is not connected, there are no such cycles).

We will actually construct a bijection between the lifted cycles which begin at the source u_i and pairs $(\pi, \mathfrak{o})$, where $\pi \in I(S)$ and $\mathfrak{o}$ is an acyclic orientation of $\Gamma(\pi)$ whose unique sink is i. The enumerative consequences of this stronger bijection are implied by Corollary 2.3.3 and a result of Greene and Zaslavsky. We will discuss this in Remark 2.3.4 below.

Let $\mathfrak{I}_i(S)$ denote the set of paths in S which begin at the source labelled i and end at the sink labelled i, and let $\mathfrak{I}(S) = \bigcup \mathfrak{I}_i(S)$.

Given an element of $\mathfrak{P}_{(n)}(S)$, we will construct a corresponding acyclic orientation of one of the graphs by first giving a sequence of elements of $\mathfrak{I}(S)$ such that these paths correspond to a minimal $\pi \in P(S)$ (and hence a graph $\Gamma(\pi)$) and the order in which they appear in the sequence is a linear extension of the acyclic orientation with a unique sink which we desire. We will then show that exactly one linear extension of each of these acyclic orientations can be obtained via our algorithm.

First, we describe an "extraction" map which takes a path $P \in \mathfrak{P}_{(n)}(S)$ and yields an element $Q \in \mathfrak{I}(S)$ and a path $P' \in \mathfrak{P}_{(n-1)}(S\backslash Q)$, when $n > 1$. Let $P_1 = P$ and let P_2 denote the result of "shifting" P to the left by one column, as in Section 2.1.3. By Corollary 2.2.3, if we do surgery at the leftmost intersection of P_1 and P_2 (say at the vertex v), we obtain paths P_1' and P_2' such that the cycles underlying these have length $n-1$ and 1, respectively. Let $P' = P_1'$ and $Q = P_2'$. Corollary 2.2.3 tells us moreover that the portions of P' and P up to v are identical.

The sequence of elements of $\mathfrak{I}(S)$ will be constructed *backwards*, using this extraction process. Namely "extract" Q_n from P, then proceed with the path P', extracting Q_{n-1}, and so on. In this way we get a sequence $Q_1, Q_2, \ldots, Q_n$. These define a minimal partition, π, of S into connected subskeletons. The order they appear in induces an acyclic orientation of the corresponding graph, $\Gamma(\pi)$, such that the order they appear in is a linear extension of the closure of the acyclic orientation. (This is true for any linear ordering of the vertices of a graph.)

Q_1 is clearly a sink in this acyclic orientation. We claim that it is the only sink. To see this, note that the union of the paths $Q_1, \ldots, Q_i$ is connected (since at one point in the construction there was a cycle of length i on this union), and this implies that Q_i is not a sink. I.e., it must intersect one of the preceding paths, and hence there is an edge (in $\Gamma(\pi)$) connecting it to the path it intersects, and this edge is directed out of Q_i in the acyclic orientation.

Note that if P begins at u_i, then so does P', and so on. In particular, Q_1 begins at u_i. So we have a map from paths in $\mathfrak{P}_{(n)}(S)$ which begin at u_i to pairs $(\pi, \mathfrak{o})$, where $\pi \in I(S)$ and $\mathfrak{o}$ is an acyclic orientation of $\Gamma(\pi)$ whose unique sink is i. We want to show that this map is a bijection.

First, we consider what conditions are necessary to allow a path $Q \in \mathfrak{I}(S)$ to be "inserted" into a path P' in S^∞ which is a lifting of a cycle of $S \setminus Q$ (of arbitrary length) to form a path P, in such a way that the extraction map described above yields Q and P' when applied to P. Let Q^∞ denote the lifting of Q to S^∞ (i.e. a copy of Q in each column). What is required is that performing surgery at the leftmost intersection of P' and Q^∞ (say at the vertex v of S^∞) yields a pair of paths whose leftmost intersection occurs at v. If this is the case, it follows from Corollary 2.2.3 that one of these paths is identical to P' up to the vertex v — this path is the P that we want.

Let P'_T denote the portion of P' up to v. If v is in the first column of S^∞, then trivially this intersection remains leftmost while doing the surgery. Otherwise, let v' denote the corresponding vertex one column to the left of v. When we do the surgery on Q^∞ and P' at v, P'_T remains fixed and the portion of Q^∞ to the left of v' is replaced by a copy of the portion of P'_T starting in the second column and continuing up to v. (We are not concerned with what happens to the right of v.) So v remains the leftmost intersection if and only if this new copy of part of P'_T does not intersect P'_T.

Let P'' denote the path obtained by shifting P' one column to the left. Suppose the leftmost intersection of P'' and P' occurs at the vertex w of S^∞. Let w' denote the corresponding vertex located one column to the right. Since P'' passes through w, P' must pass through w'. Let $\mathcal{B}(P')$ denote the portion of P' up to w'. The statement that the copy of P'_T shifted one column to the left does not intersect the original P'_T (considered above) is equivalent to saying that P'_T is contained in $\mathcal{B}(P')$, i.e. that Q^∞ intersects $\mathcal{B}(P')$. (If Q^∞ intersects $\mathcal{B}(P')$ at w', then it intersects it at w, since Q^∞ passes through the same underlying vertices of S in each column.) So Q can be "inserted" into P' if (and only if) Q^∞ intersects $\mathcal{B}(P')$. (If the cycle underlying P' has a length of one, then let $\mathcal{B}(P_1)$ consist of the portion of P' contained in the first column.)

If Q can be and is inserted into P' to obtain P, then $\mathcal{B}(P)$ consists of P'_T plus a path one column long consisting of the portion of Q^∞ from v up to the corresponding vertex one column to the right. In particular, $\mathcal{B}(P)$ contains a copy of each edge of Q.

Now suppose we have chosen a $\pi \in I(S)$ and an acyclic orientation of $\Gamma(\pi)$, $\mathfrak{o}$, with a unique sink (corresponding to a path $Q_1 \in \mathfrak{I}(S)$). We will show that the paths corresponding to the vertices of $\Gamma(\pi)$ can be given an ordering $Q_1, Q_2, \ldots, Q_n$ such that the "extraction" process described above can be reversed. In other words, so that we can construct a sequence $Q_1 = P_1, P_2, \ldots, P_n = P$ by "inserting" Q_i into P_{i-1} to obtain P_i, and the map $(\pi, \mathfrak{o}) \mapsto P$ is the inverse of the map constructed above.

After the ith stage of the construction, we will have chosen the elements $Q_1, \ldots, Q_i$ in such a way that these form an order ideal of $\bar{\mathfrak{o}}$ (the poset on the vertices of $\Gamma(\pi)$ which is the transitive closure of $\mathfrak{o}$). We also will have constructed a path $P_i \in S^\infty$ which uses the union of the edges appearing in $Q_1, \ldots, Q_i$. After this stage of the construction, we will examine a set of potential candidates (call it V_{i+1}) for the next vertex of $\Gamma(\pi)$, such that each $Q_\alpha \in V_{i+1}$ could be inserted into P_i. Furthermore, we will choose one of these to insert in such a way that it will still be possible to insert the remaining candidates (if any) into P_{i+1}, and we will carry these remaining candidates over into V_{i+2}.

The list of candidates, V_{i+1}, that we will consider is the set of vertices of $\Gamma(\pi)$ which are minimal in $\bar{\mathfrak{o}}$ among the vertices not included so far $(Q_1, \ldots, Q_i)$. After the first stage, the minimal elements in V_2 are all paths which intersect $Q_1 = P_1 = \mathcal{B}(P_1)$, and so any of these could be inserted into P_1.

Now consider an arbitrary stage of the construction in which each of the minimal (in $\bar{\mathfrak{o}}$) elements which have yet to be inserted (i.e. the elements of V_{i+1}) could be inserted into P_i. So each of these represents a path Q_α such that Q_α^∞ intersects $\mathcal{B}(P_i)$. Since Q_α^∞ passes through the same set of underlying vertices of S in each column, it intersects $\mathcal{B}(P_i)$ if and only if $\mathcal{B}(P_i)$ contains a copy of one of these underlying vertices. Since these minimal elements must be incomparable in $\bar{\mathfrak{o}}$, the paths Q_α corresponding to these do not intersect each other, and hence have no vertices in common. So there is a well-defined linear ordering of the elements of V_{i+1} according to the leftmost intersections of the Q_α's with $\mathcal{B}(P_i)$: $Q_{\alpha_1}, \ldots, Q_{\alpha_j}$. We will insert Q_{α_j} into P_i to form P_{i+1}. The remarks made above imply that $\mathcal{B}(P_{i+1})$ consists of the portion of $\mathcal{B}(P_i)$ up its leftmost intersection with $Q_{\alpha_j}^\infty$ (which is assumed to occur to the right of the corresponding intersections for the other elements of V_{i+1}) followed by a path one column long whose edges are the edges appearing in Q_{α_j} (in an appropriate order, of course). In particular, the remaining elements of V_{i+1} intersect $\mathcal{B}(P_{i+1})$ (and they are still minimal among what has yet to be inserted). Any *new* minimal elements must correspond to paths which are comparable to Q_{α_j} (i.e. intersect it), otherwise they would have been minimal before. So these share some vertex with Q_{α_j}, and hence the liftings to S^∞ must intersect $\mathcal{B}(P_{i+1})$.

So each of the elements of V_{i+2} can be inserted into P_{i+1}. By induction, we can proceed and insert all of the elements, obtaining an inverse for our "extraction" procedure defined above.

2.3.4 Remark: Our construction gives a bijective proof that the number of paths $P \in \mathfrak{P}_{(n)}(S)$ which begin at the source u_i is equal to the number of pairs $(\pi, \mathfrak{o})$, where $\pi \in I(S)$ and $\mathfrak{o}$ is an acyclic orientation of $\Gamma(\pi)$ whose unique sink is i. There is a "cyclic action" on $\mathfrak{P}_{(n)}(S)$, since there are n possible liftings (corresponding to the n possible starting points) for each cycle of S of length n. This shows that the number of paths $P \in \mathfrak{P}_{(n)}(S)$ which begin at the source u_i is independent of i.

A result of Greene and Zaslavsky [21, Thm. 7.3] says that for any graph Γ and vertex v, the number of acyclic orientations of Γ with a unique sink at v is independent of the choice of v.

So it actually follows from Corollary 2.3.3, that there is an equality for each i. However, we note that the cyclic action on $\mathfrak{P}_{(n)}(S)$ does not (generally speaking) preserve the corresponding graph under the bijection constructed above.

2.4 The Hook Case

Proposition 2.4.1 *If $\lambda = (r, 1^{n-r})$, then $\phi^\lambda\langle S\rangle$ is nonnegative for any skeleton S. Moreover, there is a pairing between the negative contributions and a subset of the positive contributions, given by doing surgery at the leftmost intersection in the cases when this intersection remains leftmost after the surgery.*

Proof: For any $\mathcal{P} \in \mathfrak{P}_\lambda(S)$, $T(\mathcal{P})$ consists of one path of length r and $n - r$ paths of length one (lengths measured in columns). In particular, the only possible intersections among these are in the first column. So it follows from Proposition 2.1.11 that the sum of the contributions coming from sets of paths with intersections in the first column is zero. This is actually always true, and a trivial consequence of the arguments leading to Proposition 2.1.11. We can assume $r > 1$, otherwise ϕ^λ is just the sign character.

We will need to make quite a few preliminary observations before we can show how to "cancel" the remaining negative contributions. Consider a set of paths $\mathcal{P} \in \mathfrak{P}_\lambda(S)$ which has no intersections in the first column. Recall from Section 2.1.3 that $\mathcal{P}$ comes from a family of paths, $\mathcal{F}$, in S and a subgraph of the associated permutation $\sigma = \sigma_{\mathcal{F}}$. When λ is $(r, 1^{n-r})$, this subgraph consists of a directed path with r vertices, and $n - r$ isolated vertices.

If one of the cycles of σ contained only isolated vertices and more than one of them, then there would be an intersection in the first column. This follows, for example, by applying Corollary 2.1.12 (really just the special case corresponding to the determinant) to the paths in $\mathcal{P}$ whose underlying cycle corresponds to this cycle of σ. Since we have assumed there are no intersections in the first column, it follows that σ has exactly one cycle of length greater than one. This cycle contains the directed path of (vertex) length r along with (possibly) some isolated vertices.

Let i_1 denote the first vertex of this path of length r in (the graph of) σ, and let $i_2, i_3, \ldots, i_k$ denote the (possibly empty) list of isolated vertices in the same cycle, ordered so that, under σ, $i_k \mapsto i_{k-1} \mapsto \cdots \mapsto i_2 \mapsto i_1$. Let $P \in \mathcal{P}$ be the path with begins at the source labelled i_1, and let $Q_2, \ldots, Q_k$ be the paths in $\mathcal{F}$ which begin at the sources labelled

$i_2, \dots, i_k$. Then the paths in $\mathcal{P}$ whose underlying cycle corresponds to the nontrivial cycle of σ are the concatenations: $P_1 = P$, $P_2 = Q_2P$, $P_3 = Q_3Q_2P, \dots$, and $P_k = Q_kQ_{k-1}\cdots Q_2P$. In particular, since there are no intersections in the first column, $Q_2, Q_3, \dots, Q_k$ do not intersect each other.

It follows from Lemma 2.1.10 that any pair of the paths $P_1, \dots, P_k$ must intersect (if $k > 1$, of course). We claim that the leftmost vertex which contains an intersection among these contains an intersection of P_1 with P_2. The sequence $i_1, i_2, \dots, i_k$ is monotonic (if we had $i_{j+1} > i_j < i_{j-1}$ or $i_{j+1} < i_j > i_{j-1}$, then Q_{j+1} would intersect Q_j). Without loss of generality, assume that the sequence is increasing. Then the portion of P_1 in the first column is below the corresponding portion of P_2 in the embedding, and so on. Clearly, the paths must maintain this ordering as they move to the right, until they reach an intersection. In particular, the leftmost vertex to contain an intersection must contain one between P_i and P_{i+1} for some i. If we had $i \neq 1$, then P_{i-1} and P_i would intersect one column to the left of the intersection between P_i and P_{i+1} (i.e. "shifting" P_i to the left by one column yields P_{i-1}, etc.). Also note that P_1 and P_2 are the *only* paths among the P_i's which intersect at this vertex, otherwise "shifting" P_2 and the third path, say P_i, to the left by one column would show that there was an intersection between P_1 and P_{i-1} in the previous column.

Now let $R_1, R_2, \dots, R_{k'}$ denote the paths in $\mathcal{F}$ which correspond to the isolated vertices of σ in cycles of length one. Note that the path in $\mathcal{P}$ which begins with R_i is just R_i repeated in each column. Let us denote this path by R_i^∞. Since there are no intersections in the first column, the R_i's do not intersect each other and do not intersect the Q_i's. So the leftmost intersection of R_i^∞ with any other path (if such an intersection exists) must be with $P_1 = P$, and this intersection is *strictly* to the left of any other intersections involving R_i^∞, otherwise a "shifting" argument would give an intersection to the left. In particular, a leftmost intersection of this form cannot occur at the same vertex as the leftmost intersection between P_1 and P_2 described above.

Now we are (finally) in a position to define a sign-reversing involution on a subset of the sets of paths $\mathcal{P} \in \mathfrak{P}_\lambda(S)$ with no intersection in the first column. This subset will include all the terms with a negative sign, and hence the proposition will follow.

Let v be the leftmost vertex of S^∞ which contains an intersection among the paths of $\mathcal{P}$. The arguments above tell us that at v, $P = P_1$ intersects either P_2 or one of the R_i^∞'s (in particular, there are exactly two paths through v). Let $\mathcal{P}'$ denote the result of doing surgery at this intersection. If v is the leftmost intersection of $\mathcal{P}'$, let the involution map $\mathcal{P}$ to $\mathcal{P}'$, otherwise $\mathcal{P}$ is not in the domain of the involution. (The comments we made while defining surgery in Section 2.1.4 imply that this is an involution.)

So it suffices to show that if v is not the leftmost intersection of $\mathcal{P}'$, then $\mathcal{P}$ has a positive sign. Let j denote the number of the column containing v. Let $\widehat{v}$ denote the internal vertex of S which corresponds to v.

First consider the case where v contained an intersection between P_1 and P_2. We claim that in this case, v contains the leftmost intersection of $\mathcal{P}'$ (the result of the surgery). Corollary 2.2.3 (the more detailed "widening" lemma for $(n-1,1)$) implies that the portion of P_2 up to v remains fixed by the surgery. It follows that the portions of $P_3, \ldots, P_k$ (if we have these) in the first j columns also remain after doing the surgery, since the edges of S involved in the surgery do not appear in $Q_3, \ldots, Q_k$. We can assume as above that the sequence $i_1, i_2, \ldots, i_k$ of labels of the sources of $P_1, P_2, \ldots, P_k$ is increasing. Then, in columns 1 through $j-1$ and the portion of the jth column up to v, P_1 lies below P_2 which lies below P_3, and so on. The "Widening" Lemma tells us that the surgery replaces the portion of P_1 up to v with a (weakly) lower path. The relevant portions of the other paths do not change, so the leftmost intersection among the paths which have changed is at v. The lower path which replaces P_1 has an underlying cycle of length 1, and the edges of S used in this cycle are just the edges appearing in P_1 between the copy of $\widehat{v}$ in column $j-1$ and the copy in column j (i.e. v). If this path were to intersect one of the R_i^∞'s to the left of v, then (a copy of) this intersection would occur in each column (since both paths have underlying cycles of length 1), and hence there would have been an intersection between this R_i^∞ and the portion of P_1 lying between the copies of $\widehat{v}$ in columns $j-1$ and j. So if the leftmost intersection occurs between P_1 and P_2, then it "remains leftmost" after the surgery.

Now consider the case where v contained an intersection of P_1 with one of the R_i^∞'s, say $R_{i_0}^\infty$. In this case, it might actually happen that the result of doing the surgery yields a set of paths with an intersection to the left of v. But we will show that if $k > 1$, then v is the leftmost intersection of the result after the surgery. Here k is the number of paths whose underlying cycle is the same as P_1, i.e. the same k as above. Note that if $k = 1$, the set of paths has one underlying cycle per path, so it has a positive sign. So assume $k > 1$. In particular, there is a P_2.

Let P_T denote the portion of P_1 in $T(\mathcal{P})$, i.e. the path in $T(\mathcal{P})$ of length r. Note that P_1 is just the infinite concatenation $P_T Q_k Q_{k-1} \cdots Q_1 P_T \cdots$. Since $R_{i_0}^\infty$ does not intersect the Q_i's, the portion of P_1 up to v must be contained within P_T. Write this portion of P_1 as a concatenation $Q_0 P_0$ where Q_0 is the portion lying in the first column.

The changes in the first j columns (i.e. changes in which edges are used in which column) made by doing the surgery are simply that a copy of P_0 replaces the portion of $R_{i_0}^\infty$ up to the copy of $\widehat{v}$ in column $j-1$. Consider the leftmost intersection in $\mathcal{P}'$, and suppose this intersection occurs to the left of v. Then it must occur at an intersection of this (new) copy of P_0 with something. If this copy of P_0 were to intersect one of the R_i^∞'s which remains in $\mathcal{P}'$, then copy of P_0 in $P_1 \in \mathcal{P}$ would have a corresponding intersection one column to the right (but still to the left of v). Likewise, if this new copy of P_0 were to intersect one of $P_1, \ldots, P_k$, then the leftmost of such intersections would have to occur between the copy of P_0 and P_1 (this follows from the same argument which showed that the leftmost intersection among $P_1, \ldots, P_k$ was an intersection of P_1 with P_2). But in that case, the copy of P_0 in P_1 would have a corresponding intersection, one column to the right, but still to the left of v.

So we have shown that the sets of paths $\mathcal{P} \in \mathfrak{P}_\lambda(S)$ for which the leftmost intersection is **not** preserved by doing surgery at it is a subset of those with $k = 1$ (the sets of paths with no intersections at all also have $k = 1$). As we remarked above, these sets all have positive signs. So, doing surgery at the leftmost intersection (in the cases where it remains leftmost) gives a pairing between the negative contributions to $\phi^\lambda\langle S\rangle$ and a subset of the positive ones, showing that $\phi^\lambda\langle S\rangle \geq 0$ (when λ is a hook). □

An application of Proposition 2.1.16 gives us the following generalization.

Corollary 2.4.2 *For any skeleton S, $\phi^\lambda\langle S\rangle \geq 0$ when λ is of the form (kr, k^j).*

2.5 The $(2^k, 1^\ell)$ Case

If $\mathcal{P} \in \mathfrak{P}_{(2^k,1^\ell)}(S)$ for some k and ℓ, then the truncated part $T(\mathcal{P})$ contains paths of lengths 1 and 2 (columns). Let $\mathcal{F}$ be the corresponding family of paths in S. $\mathcal{F}$ contains $k+\ell$ paths which appear (in the paths $\mathcal{P}$) in the first column of S^∞, call these $P_1, P_2, \ldots, P_{k+\ell}$ (the order doesn't matter). $\mathcal{F}$ also contains k paths which appear for the first time in the second column (i.e. the portions of the paths of $T(\mathcal{P})$ which lie in the second column). Call these $Q_1, \ldots, Q_k$.

The sum of all the contributions coming from sets of paths with intersections in $T(\mathcal{P})$ is zero (by Proposition 2.1.11), so we may restrict our considerations to sets with no such intersections. In other words, the P_i's do not intersect each other, and the Q_i's do not intersect each other.

Consider any internal vertex, v, of S. If v has an in-valence of more than two, then two P_i's or two Q_i's would pass through v. In that case we would have $\phi^{(2^k,1^\ell)}\langle S\rangle = 0$. This is a special case of Corollary 2.1.13. And since internal vertices of in-valence=out-valence=1 can be removed without changing $\langle S\rangle$, we may assume for the remainder of the discussion that every internal vertex of S has in-valence=out-valence=2.

For any skeleton S with in-valence=out-valence=2 at each internal vertex, let ZigZag(S) denote the digraph obtained from S as follows. Replace each internal vertex v by two vertices v_{left}, v_{right}. Replace each edge by a corresponding edge which is directed into v_{left} if the original was directed into an internal vertex v and is directed out of w_{right} if the original was directed out of an internal vertex w. The ends of edges incident to sources and sinks are not affected.

Clearly an embedding of ZigZag(S) in the plane can be derived from one for S by making changes within a small neighborhood of each internal vertex, while maintaining planarity. I.e. v_{left} and v_{right} are very close together with v_{left} on the left. Geometrically, we can think about this operation as "cutting" each internal vertex in half with the left half getting the in-edges and the right half getting the out-edges. In particular, the planarity implies that if we ignore the directions on the edges, the embedded image of ZigZag(S) is a set of non-intersecting curves whose ends (if any) lie in the sources and sinks of S.

Now, we can also do the completely analogous operation on cyl(S) (the digraph in which the source and sink with the same label have been identified, as in Section 2.1.2), by "cutting" the vertices of cyl(S) corresponding to the internal vertices of S, or equivalently, by identifying the appropriate vertices of ZigZag(S). Call the result cyl(ZigZag(S)). Clearly,

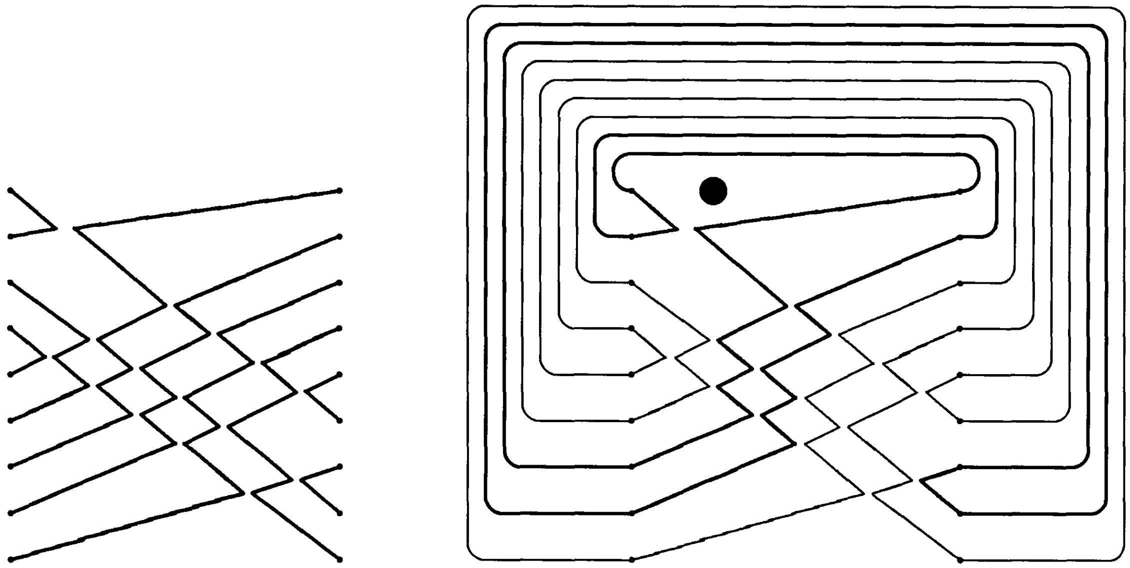

Figure 2.4: On the left, an example of ZigZag(S) (all corners are vertices). On the right, the corresponding cyl(ZigZag(S)), "flattened" onto a punctured plane.

cyl(ZigZag(S)) has an embedding on the cylinder. The interesting thing to notice is that if we ignore the vertices and the directions on the edges, then the embedding of cyl(ZigZag(S)) in the cylinder is a family of simple closed curves which don't intersect each other.

In particular, each of these simple closed curves has a winding number of zero or one around the cylinder. In other words, we can "flatten" the cylinder into a punctured plane while preserving the fact that these are simple closed curves. As simple closed curves in the plane, these have a well-defined "inside" and "outside" (the Jordan Curve Theorem), and for each of them, the puncture point is either inside (winding number one) or outside (winding number zero).

Figure 2.4 depicts an example of ZigZag(S) and the corresponding cyl(ZigZag(S)) (with the cylinder "flattened" into a punctured plane). In this example, there is one curve with winding number one and one with winding number zero (draw slightly bolder).

Of course, these topological ideas are really just a convenience, but our main result in this section would be (at best) awkward to state without them.

Theorem 2.5.1 *If S has an internal vertex of in-valence greater than two, then $\phi^{(2^k,1^\ell)}\langle S\rangle$ vanishes. Otherwise, define* cyl(ZigZag(S)) *as above, and let w_0 be the number of curves in the embedding with winding number zero, and let w_1 be the number of curves in the embedding with winding number one. Then*

$$\phi^{(2^k,1^\ell)}\langle S\rangle = \delta_{\ell,w_1} 2^{w_0}.$$

Let us refer to the connected components of ZigZag(S) as "zigzags." By abuse of notation, we will also use the word "zigzags" for the subdigraphs of S with the corresponding edges. It turns out that the problem of enumerating the contributions of sets of paths $\mathcal{P} \in \mathfrak{P}_{(2^k,1^\ell)}(S)$ which have no intersections in $T(\mathcal{P})$ can be completely restated in terms of these zigzags.

Lemma 2.5.2 *For any set of paths $\mathcal{P} \in \mathfrak{P}_{(2^k,1^\ell)}(S)$ (any k and ℓ) with no intersections in $T(\mathcal{P})$, color the edges of S which appear in the first column of $T(\mathcal{P})$ blue and the edges appearing in the second column red. Then we have the following:*

1. *A zigzag which has "ends" at a source and sink of S contains an odd number of edges. The remaining zigzags (which may have ends which are both sources or both sinks of S, or which might have no ends at all) have an even number of edges.*

2. *For every choice of $\mathcal{P}$ as above, as we move along a zigzag, the colors of the edges encountered alternate between blue and red. If a red edge is incident to a source or sink of S then the edge incident to the sink or source (respectively) with the same label must be blue.*

3. *Conversely, any such coloring of the edges of S comes from a set of paths $\mathcal{P} \in \mathfrak{P}_{(2^k,1^\ell)}(S)$ (for some k and ℓ) with no intersection in $T(\mathcal{P})$.*

4. *Color the sources and sinks of S according to the color of the (unique) edge which is incident to them. Then the sign associated with $\mathcal{P}$ is given by $(-1)^{k+a+b}$, where a is the number of pairs $i < j$ such that the ith source is red and the jth source is blue, and b is the number of pairs $i < j$ such that the ith sink is red and the jth sink is blue.*

Proof: The first statement follows from the fact that as we move along the curve underlying a zigzag, we alternate between moving with the direction of the edges and against the direction of the edges (at each vertex, the edges are both directed in or both directed out).

The second statement is also trivial: Two edges in a row of the same color would correspond to an intersection in $T(\mathcal{P})$. A red edge which is incident to a source is the first edge in the second column of a path in $T(\mathcal{P})$, and the edge preceding it in this path is in the first column and hence blue. A red edge incident to a sink is the last edge in a path of $T(P)$ and must be followed (in $\mathcal{P}$) by the first edge of a path (this is just the definition of $T(\mathcal{P})$).

Conversely, if we are given such a coloring, then the edges colored blue form a set of nonintersecting paths in S (for each internal vertex, exactly one of the in-edges and one of the out-edges is blue since the coloring alternates along all of the zigzags). And the red edges also form a set of nonintersecting paths in S. The collection of all these paths defines a family $\mathcal{F}$ of paths from the sources to the sinks which use each edge exactly once. If we take the liftings of the underlying cycles to S^∞ which place the blue paths in the first column, we get a set of paths $\mathcal{P}$ for some λ. The condition on the red edges incident to sources implies that all of the red paths appear in the second column, and the condition on the sinks implies that these are followed by blue paths in the third column. Deleting edges to get $T(\mathcal{P})$ is just removing every blue edge in the second column or later, and the remainder of the path to the right. $T(\mathcal{P})$ is contained within the first two columns, hence $\lambda = (2^k, 1^\ell)$.

The sign associated with $\mathcal{P}$ is $\epsilon_\lambda(-1)^\sigma$, where $(-1)^\sigma$ is the sign of the corresponding permutation (such that the paths of $\mathcal{F}$ go from the ith source to the σ_ith sink). Clearly, $\epsilon_{(2^k,1^\ell)} = (-1)^k$, so the last statement amounts to showing that $(-1)^{a+b}$ is the sign of σ.

The sign of σ is $(-1)^{\mathrm{inv}(\sigma)}$, where $\mathrm{inv}(\sigma)$ is the number of *inversions* of σ, i.e. the number of pairs $i < j$ such that $\sigma_i > \sigma_j$. Since the paths of the same color don't intersect each other, any inversion corresponds to a pair of paths of different colors. Consider the number of inversions involving a fixed red path from u_i (the ith source) to v_{σ_i} (the σ_ith sink). Consider the blue sources with labels larger than i, and the blue sinks with labels larger than σ_i. Let a_i and b_i denote the numbers of these, respectively. Since the blue paths don't intersect, the jth largest blue source is connected by a blue path to the jth largest blue sink. So if $a_i > b_i$, there are b_i blue paths which begin and end above the red path in question, and there are an additional $a_i - b_i$ blue paths which begin above the red path and end below it. Similarly, if $b_i > a_i$, then there are $b_i - a_i$ inversions involving the red path. Taking the product over all the red paths, we obtain $(-1)^\sigma = (-1)^{\sum a_i + b_i} = (-1)^{a+b}$. □

The idea behind our proof of Theorem 2.5.1 will be to show that $\phi^{(2^k,1^\ell)}\langle S\rangle$ is actually a topological invariant of the set of underlying curves in cyl(ZigZag(S)). In particular, we will show that there are "simplification" operations which can be applied to ZigZag(S) without changing the problem, and that these operations can be used to reduce the problem to a trivial case. In order to define these operations, we will actually need to work with a slightly more general situation.

Consider an ordinary (undirected) graph Γ which is a disjoint union of paths and cycles, and a planar embedding $\iota = \iota_{\mathsf{x}} \times \iota_{\mathsf{y}} : \Gamma \to \mathbb{R}^2$ with the following properties:

1. The image is contained between two vertical lines $\mathsf{x} = \mathsf{x}_1$ and $\mathsf{x} = \mathsf{x}_2$ ($\mathsf{x}_1 < \mathsf{x}_2$), with n vertices $u_1, \ldots, u_n$ lying on $\mathsf{x} = \mathsf{x}_1$ and n vertices $v_1, \ldots, v_n$ lying on $\mathsf{x} = \mathsf{x}_2$, and everything else lying strictly between the lines.

2. $\iota_{\mathsf{y}}(u_i) = \iota_{\mathsf{y}}(v_i)$ and $\iota_{\mathsf{y}}(u_1) < \iota_{\mathsf{y}}(u_2) < \cdots < \iota_{\mathsf{y}}(u_n)$

3. $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ is the set of vertices of Γ of valence one.

4. If a component of Γ is a path with an even number of edges, then both "ends" are embedded on the same vertical line. If a component is a path with an odd number of edges, then the ends are embedded on different vertical lines.

Given such a graph and embedding, consider the set of edge-colorings of Γ using the colors red and blue which color edges adjacent to a common vertex differently, and for which u_i and v_i are not both adjacent to red edges. If ℓ has the same parity as n, let $\psi_\ell(\Gamma, \iota) = (-1)^{(n-\ell)/2} \sum (-1)^{a+b}$ where the sum runs over all such colorings which have ℓ of the pairs u_i, v_i both adjacent to blue edges, and a is the number of pairs $i < j$ such that u_i is adjacent to a red edge and u_j is adjacent to a blue edge, and b is the number of pairs $i < j$ such that v_i is adjacent to a red edge and v_j is adjacent to a blue edge.

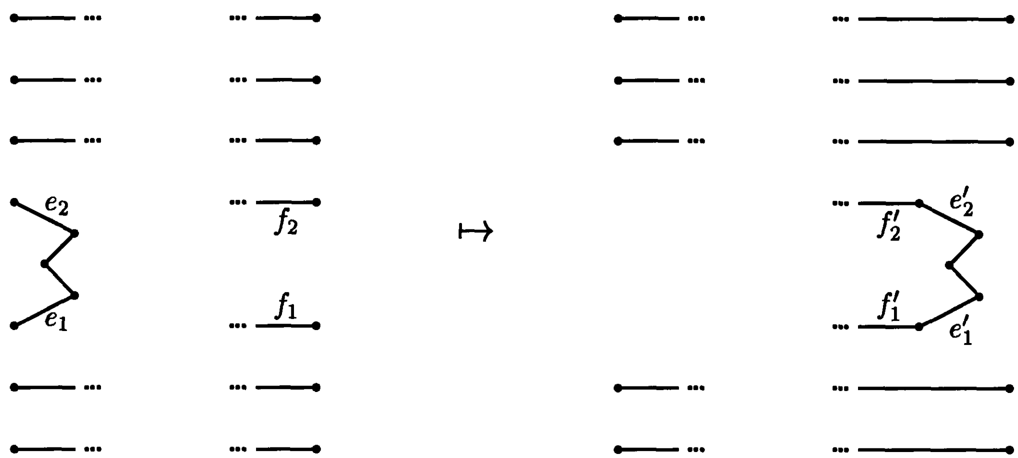

Figure 2.5: Simplification of Zigzags

If Γ is the graph underlying ZigZag(S) and ι is the embedding discussed above, then Lemma 2.5.2 says that $\psi_\ell(\Gamma, \iota) = \phi^{(2^k,1^\ell)}\langle S\rangle$.

Now we can describe the "simplification" operations which were alluded to previously. Let Γ and ι satisfy the properties discussed above. If there is a path in Γ joining two of the u_i's, then there must be a path joining a consecutive pair, u_{i_0} and u_{i_0+1}. This follows from the planarity of the embedding — if a path joins u_i and u_{i+j}, then any vertices $u_{i+1}, \ldots, u_{i+j-1}$ must also come in pairs joined by paths.

We can move the path joining u_{i_0} and u_{i_0+1} over to the other side to get a new graph and embedding (Γ', ι'), as is depicted in Figure 2.5. This new graph has $n-2$ vertices embedded on each of the vertical lines containing its "ends," and it is elementary to check that it satisfies the properties listed above.

Lemma 2.5.3 $\psi_\ell(\Gamma, \iota) = \psi_\ell(\Gamma', \iota')$.

Proof: Let e_1, e_2, f_1, and f_2 denote the edges of Γ adjacent to u_{i_0}, u_{i_0+1}, v_{i_0}, and v_{i_0+1}, respectively. Let e'_1, e'_2, f'_1, and f'_2 denote the corresponding edges of Γ'.

All of the colorings considered in the definition of $\psi_\ell(\Gamma', \iota')$ correspond to colorings considered for Γ. These have the "same ℓ," since the edges e'_1, f'_1 are not both colored blue, and the same is true for e'_2 and f'_2. Fix one of these colorings. Since the path joining u_{i_0} and u_{i_0+1} has an even number of edges, e_1 and e_2 have different colors, and the same is true for f_1 and f_2. Consider the pairs $i < j$ used to compute the sign associated with this coloring, i.e. the pairs such that u_i is adjacent to a red edge and u_j is adjacent to a blue edge or v_i is adjacent to a red edge and v_j is adjacent to a blue edge. In addition to the pairs corresponding to pairs for Γ', there are new pairs involving u_{i_0}, u_{i_0+1}, v_{i_0}, and v_{i_0+1}. We get a new pair for each u_j and v_j with $j > i_0+1$ which are adjacent to blue edges, a new pair for each u_i and u_i with $i < i_0$ which are adjacent to red edges, and one more pair involving *only* u_{i_0}, u_{i_0+1}, v_{i_0}, and v_{i_0+1}, i.e. (i_0, i_0+1). We claim that there are an odd number of new pairs. For each $j > i_0+1$ we either get two new pairs if u_j and v_j are both adjacent to blue edges, or one new pair otherwise (they cannot both be adjacent to red edges). So the parity of the number of new pairs is the same as the parity of the total number of u_i's and v_j's adjacent to red edges in Γ' plus one (for the pair (i_0, i_0+1)). But it follows from the definitions of the colorings that we're considering that there are always an equal number of u_i's adjacent to red edges and v_j's adjacent to red edges. So the number of new pairs is odd. Also, $(-1)^{(n-\ell)/2} = -(-1)^{(n-2-\ell)/2}$, so the colorings contributing to $\psi_\ell(\Gamma', \iota')$ contribute the same sign to $\psi_\ell(\Gamma, \iota)$.

In addition to the colorings considered above, there are colorings contributing to $\psi_\ell(\Gamma, \iota)$ with e_1 and f_1 both colored blue or e_2 and f_2 both colored blue. As we noted above, e_1 and e_2 have different colors, since the path joining them has an even number of edges. We can pair the colorings with e_1, f_1 both blue (and hence e_2 red and f_2 blue) with the colorings where e_2, f_2, and f_1 are blue and e_1 is red, by reversing the color of every edge in the path containing e_1 and e_2 while keeping everything else the same. The signs associated with these colorings are the opposite — in the latter case, $(i_0, i_0 + 1)$ is a pair contributing to the sign, and in the former case it is not, and the number of pairs involving i_0 and $i_0 + 1$ and something else are not affected (just replace i_0 with $i_0 + 1$ in those pairs and vice versa). So these other colorings make a net contribution of zero to $\psi_\ell(\Gamma, \iota)$, and we have $\psi_\ell(\Gamma, \iota) = \psi_\ell(\Gamma', \iota')$. □

The following lemma computes the values of the trivial cases.

Lemma 2.5.4 *If (Γ, ι) are as described above, and there is no path of Γ joining two of the u_i's, then*

$$\psi_\ell(\Gamma, \iota) = \delta_{n,\ell} 2^w,$$

where w denotes the number of components of Γ which are cycles.

Proof: If no path of Γ joins two u_i's, then each u_i is joined to some v_j. It follows from the planarity conditions on Γ that u_i must be joined to v_i. The path joining u_i and v_i must have an odd number of edges, so in any coloring contributing to any of the $\psi_\ell(\Gamma, \iota)$'s, the edges adjacent to u_i and v_i must be colored the same, and hence blue. It follows from the definition that for each such coloring, $\ell = n$ and the contribution of the coloring is positive. These conditions determine the color of each edge in the paths of Γ, and the cycles of Γ can be colored in any way which alternates along each cycle. So there are 2^w colorings to consider, each making a positive contribution to $\psi_\ell(\Gamma, \iota)$. □

We can now give a proof of the main result.

Proof of Theorem 2.5.1: As we noted at the beginning of this section, $\phi^{(2^k, 1^\ell)}\langle S\rangle$ vanishes if S has any internal vertices of in-valence=out-valence more than two. Otherwise, we can construct ZigZag(S) and cyl(ZigZag(S)) as described above. The simplification operations described above can be applied until we get to the trivial case covered by Lemma 2.5.4. To prove Theorem 2.5.1 we only need to note that for each of the simplification operations which we apply to (the graph underlying) ZigZag(S), there is a corresponding operation on the graph underlying cyl(ZigZag(S)) and these operations don't affect the winding numbers of any of the curves. In the trivial case, each of the paths connecting u_i to v_i corresponds to a curve with winding number one and the cycles correspond to curves of winding number zero. □

The following corollary follows immediately from Proposition 2.1.16 and Theorem 2.5.1.

Corollary 2.5.5 *For any skeleton S, $\phi^\lambda\langle S\rangle \geq 0$ when λ is of the form $((2k)^i, k^j)$.*

2.6 More General Digraphs

In [46, Section 6], Stembridge considers generalizations of Jacobi-Trudi matrices coming from acyclic digraphs whose paths are required to have certain intersection properties. In this section we will discuss these digraphs and show that the corresponding matrices, like the Jacobi-Trudi matrices, can be expressed in terms of skeletons.

Let $D = (V, E)$ be an acyclic digraph. If $u, v \in V$, let $\mathfrak{P}(u, v)$ denote the set of paths from u to v in D.

Let $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ be ordered n-tuples of vertices. Stembridge says that $\mathbf{u}$ and $\mathbf{v}$ are *D-compatible* if

$$i < j,\ k > l \implies \text{every path in } \mathfrak{P}(u_i, v_k) \text{ intersects every path in } \mathfrak{P}(u_j, v_l). \tag{2.14}$$

The language used here comes from an earlier paper of Stembridge, [44], where the emphasis is on fixing the digraph and allowing $\mathbf{u}$ and $\mathbf{v}$ to vary. We will want to vary the digraph, and it will be more convenient to say that $(D, \mathbf{u}, \mathbf{v})$ is a *compatible triple* when (2.14) holds.

Assume the edges of D are given weights in a set of indeterminates $\{z_e \mid e \in E\}$, and define the weight, wt, of a multiset of edges to be the product of the weights of the edges. Also, if $u, v \in V$, let

$$a(u, v) = \sum_{P \in \mathfrak{P}(u,v)} \mathrm{wt}(P).$$

If $(D, \mathbf{u}, \mathbf{v})$ is a compatible triple, let

$$A(D, \mathbf{u}, \mathbf{v}) = [a(u_j, v_i)]_{1 \leq i,j \leq n}. \tag{2.15}$$

The matrix which Stembridge associates with $(D, \mathbf{u}, \mathbf{v})$ is the transpose of this. This definition works better with our convention of using σ^{-1} in the definition of $[A]$ (see p. 15). The two differences in conventions "cancel" so that $[A(D, \mathbf{u}, \mathbf{v})]$ is identical to what Stembridge denotes as $[A(\mathbf{u}, \mathbf{v})]$.

Note that if $(D, \mathbf{u}, \mathbf{v})$ is a compatible triple, then so is $(D^{\mathrm{op}}, \mathbf{v}, \mathbf{u})$, where D^{op} is the result of reversing the directions of the edges of D. And $A(D^{\mathrm{op}}, \mathbf{v}, \mathbf{u})$ is the transpose of $A(D, \mathbf{u}, \mathbf{v})$.

As Stembridge points out, Jacobi-Trudi matrices can be viewed as a special case of this construction. In Section 2.1.1, we discussed a digraph on $\mathbb{Z} \times \mathbb{Z}$, together with n-tuples,

$(P_1, \ldots, P_n)$ and $(Q_1, \ldots, Q_n)$, which were used in the work of Gessel and Viennot [17], Goulden and Jackson [18], and Greene [20]. These form a compatible triple, and the matrix in (2.15) becomes the Jacobi-Trudi matrix if the weights are specialized as we discussed in Section 2.1.1.

Stembridge makes the following conjecture, which generalizes Conjecture 2.0.3.

Conjecture 2.6.1 ([46, Conj. 6.3]) *If $(D, \mathbf{u}, \mathbf{v})$ is a compatible triple, $\phi^\lambda[A(D, \mathbf{u}, \mathbf{v})]$ is monomial positive.*

He also shows that this conjecture holds in the cases when $\lambda = (r^\ell)$ (including (n) and (1^n)), when $\lambda = (21^{n-2})$, and when $\lambda = (n-1, 1)$.

We actually need some assumptions in order for monomial immanants of $A(D, \mathbf{u}, \mathbf{v})$ to be well-defined. For simplicity, we will assume D is finite (see Remark 2.6.6 below).

Using the "$[A]$" notation defined in (0.1), we can write

$$[A(D, \mathbf{u}, \mathbf{v})] = \sum_{\sigma \in \mathfrak{S}_n} \left(\prod_{i=1}^{n} a(u_i, v_{\sigma_i}) \right) \cdot \sigma. \tag{2.16}$$

The coefficient of σ in the above enumerates sets of paths such that the ith path begins at u_i and ends at v_{σ_i}. As in the case of Jacobi-Trudi matrices, we can combine the terms for sets of paths whose union uses the same multiset of edges and write

$$[A(D, \mathbf{u}, \mathbf{v})] = \sum_{\alpha} \mathrm{wt}(\alpha) \langle \alpha \rangle, \tag{2.17}$$

where the sum runs over all multisets, α, of edges, and $\langle \alpha \rangle$ denotes the sum of the permutations associated with the sets of paths with weight $\mathrm{wt}(\alpha)$ (with the use of indeterminates, specifying $\mathrm{wt}(\alpha)$ is equivalent to specifying α).

There are some small complications here which don't come up in the Jacobi-Trudi case. Namely, we have not assumed that the vertices in the n-tuples $\mathbf{u}$ and $\mathbf{v}$ are distinct, so the set of paths does not necessarily determine the "associated" permutation. (As an extreme case, we might have $u_1 = u_2 = \cdots = u_n$ and $v_1 = v_2 = \cdots = v_n$.) Also, we might have $u_i = v_j$ for some i, j and trivial paths with no edges connecting them. There is, however, a simple way to avoid these complications. Let D' be the digraph obtained from D by adding $2n$ new vertices, $u'_1, u'_2, \ldots, u'_n, v'_1, v'_2, \ldots, v'_n$ and edges directed from u'_i to u_i and from v_i to v'_i. If we specialize the weights of the new edges to 1, then $A(D', \mathbf{u}', \mathbf{v}')$ becomes $A(D, \mathbf{u}, \mathbf{v})$. This is immediate since the paths in $\mathfrak{P}(u_i, v_j)$ exactly correspond to those in $\mathfrak{P}(u'_i, v'_j)$. So we can (and will) assume that $\mathbf{u}$ consists of n distinct sources of valence 1 and that $\mathbf{v}$ consists of n distinct sinks of valence 1.

Given a multiset of edges α which appears in (2.17), we can construct a digraph $\tilde{\alpha}$ as follows. Start with the subdigraph of D induced by the edges in α. Replace each edge with multiple (distinguishable) edges according to their multiplicity in α. Given the assumption at the end of the previous paragraph and the fact that the edges of $\tilde{\alpha}$ can be partitioned into paths which start at some u_i and end at some v_j, it follows that each vertex of $\widetilde{\alpha}$ either has in-valence = out-valence or is one of the u_i's or v_j's. Of course, $\widetilde{\alpha}$ contains all of the u_i's and v_j's. Note that $(\widetilde{\alpha}, \mathbf{u}, \mathbf{v})$ is also a compatible triple, since removing or doubling edges doesn't affect (2.14).

Arguments analogous to those in the Jacobi-Trudi case allow us to write

$$[A(D, \mathbf{u}, \mathbf{v})] = \sum_{\alpha} \mathrm{wt}(\alpha) \frac{1}{M(\alpha)} \langle \widetilde{\alpha} \rangle, \tag{2.18}$$

where $M(\alpha)$ is defined as in Section 2.1.1, and $\langle \widetilde{\alpha} \rangle$ is the sum of the permutations, $\sigma_{\mathcal{F}}$, induced by families $\mathcal{F}$ of n paths in $\widetilde{\alpha}$ which use each edge exactly once. Note that since we have assumed D is finite, $\widetilde{\alpha}$ is also finite.

Let us use the term *special* (for lack of a better word) to describe the finite compatible triples $(D, \mathbf{u}, \mathbf{v})$ for which the vertices in $\mathbf{u}$ are distinct sources of valence 1, the vertices of $\mathbf{v}$ are distinct sinks of valence 1, and the remaining vertices have in-valence = out-valence. Note that all of these are actually obtained as $\widetilde{\alpha}$'s above.

Let us call a special compatible triple $(D, \mathbf{u}, \mathbf{v})$ *reducible* when $\langle D \rangle \in \mathbb{Z}\mathfrak{S}_n$ can be written as a sum, $\sum \langle D_i \rangle$, of two or more such things. Otherwise, we will say that $(D, \mathbf{u}, \mathbf{v})$ is *irreducible*. Since the coefficient of each σ in $\langle D \rangle$ is a nonnegative integer (and it's not hard to see that at least one of them must be positive), it follows that for any special compatible triple, $\langle D \rangle$ can be written as a sum $\sum \langle D_i \rangle$, where each D_i is irreducible.

Our main result in this section will be the following theorem.

Theorem 2.6.2 *If $(D, \mathbf{u}, \mathbf{v})$ is an irreducible special compatible triple, then it is also a skeleton.*

We will prove Theorem 2.6.2 by showing that an embedding of D in the plane with the properties given in Definition 2.1.1 can be constructed "one vertex at a time." In the following two lemmas, we will show that the irreducibility of a special compatible triple implies the existence of pairs of paths with a specified set of intersections.

Lemma 2.6.3 *If $(D, \mathbf{u}, \mathbf{v})$ is an irreducible special compatible triple and w is an internal vertex of D with out-valence > 1, then there are paths P_1 and P_2 in D which begin at the sources and end at the sinks and intersect at w and only at w.*

Proof: Let e be any edge which is directed into w, and let $f_1, f_2, \ldots, f_k$ denote the complete list of edges directed out of w. Each set of paths in D from the sources to the sinks which use each edge exactly once (i.e. the sets of paths which contribute to $\langle D \rangle$) contains exactly one path which includes e and one of the f_i's. If we combine the sets of paths in which the same f_i follows e, then these exactly correspond to the sets of paths for the digraph D_i, formed by adding a new vertex of in-valence=out-valence=1 and letting e be directed into and f_i be directed out of this vertex instead of w. D_i has the same sets and sources as D, and the permutations associated with the corresponding sets of paths are exactly the same. So we have

$$\langle D \rangle = \langle D_1 \rangle + \langle D_2 \rangle + \cdots + \langle D_k \rangle.$$

Since $(D, \mathbf{u}, \mathbf{v})$ is irreducible and $k > 1$, it must be that at least one of the triples $(D_i, \mathbf{u}, \mathbf{v})$ is not a compatible triple. That implies that there are integers $i < j, k > l$ and paths Q_1 (from u_i to v_k) and Q_2 (from u_j to v_l) which do not intersect in D_i (so that they violate (2.14)). The corresponding paths, P_1 and P_2, in D must intersect, and the only changes have occurred at w, so they intersect only at w. □

Lemma 2.6.4 *If $(D, \mathbf{u}, \mathbf{v})$ is an irreducible special compatible triple and w_1, w_2 are distinct internal vertices of D with in-valence = out-valence > 1, then there are sinks v_i, v_j and paths P_1 from w_1 to v_i and P_2 from w_2 to v_j which do not intersect. There are also nonintersecting paths from the sources to w_1, w_2.*

Proof: Suppose $(D, \mathbf{u}, \mathbf{v})$, and w_1, w_2 are as in the statement, but every path from w_1 to a sink intersects every path from w_2 to a sink. First note that a trivial consequence of Lemma 2.6.3 is that there is a path, P from w_1 to a sink which does not contain w_2 (i.e. w_2 cannot lie in both of the paths).

Let Q_1 and Q_2 be paths from w_2 to the sinks which intersect only at w_2, which exist by applying Lemma 2.6.3 to w_2 (and taking the portions of the paths from w_2 to the sinks). By our supposition, P must intersect both of these. Let w_0 be the first vertex on P which appears in either of the Q_i's, say it appears in Q_1. Since w_2 is not contained in P, $w_2 \neq w_0$. It follows that the portion of P between w_1 and w_0 does not intersect Q_2. Also, the portion of Q_1 from w_0 to a sink does not intersect Q_2. Concatenating these portions, we obtain a path from w_1 to a sink which does not intersect Q_2. This contradicts our assumption. The statement about the sources can be proven with an analogous argument or by noting that $(D^{\mathrm{op}}, \mathbf{v}, \mathbf{u})$ is also an irreducible special compatible triple. □

Proof of Theorem 2.6.2: Let $(D, \mathbf{u} = (u_1, \ldots, u_n), \mathbf{v} = (v_1, \ldots, v_n))$ be an irreducible special compatible triple. We wish to prove that it is also a skeleton. We can assume without loss of generality that D has no vertices with in-valence = out-valence = 1.

Let $\mathsf{x}_1 < \mathsf{x}_2$ be real numbers. We will construct an embedding of D into the region of the plane between the vertical lines $\mathsf{x} = \mathsf{x}_1$ and $\mathsf{x} = \mathsf{x}_2$ in a sequence of steps.

Suppose that at some stage of the construction we have a sequence of distinct edges $e_1, e_2, \ldots, e_n$ of D, and another digraph D' formed by adding an n-tuple of (distinct) vertices $\mathbf{v}' = (v_1', \ldots, v_n')$ with in-valence = out-valence =1 and replacing e_i with two edges forming a path $e_i' v_i' e_i''$. Furthermore, assume that the subgraph, S, of D' obtained by taking the edges lying in paths from $\mathbf{u}$ to $\mathbf{v}'$ is a skeleton (whose ordered sets of sources and sinks are $\mathbf{u}$ and $\mathbf{v}'$), and that we have an embedding of S between the lines $\mathsf{x} = \mathsf{x}_1$ and $\mathsf{x} = \mathsf{x}_2$ as in Definition 2.1.1.

We can visualize this situation as a drawing of D in the plane with $u_1, \ldots, u_n$ on the line $\mathsf{x} = \mathsf{x}_1$ with increasing y-coordinates, the portion of D drawn in between the lines is planar with all edges moving strictly left to right, the edges $e_1, \ldots, e_n$ cross the line $\mathsf{x} = \mathsf{x}_2$ (at points with increasing y-coordinates), and the rest of D is drawn to the right of $\mathsf{x} = \mathsf{x}_2$ with no assumptions there. Clearly, we can start such a construction by taking the edges incident to the sources.

The internal vertices of D which do not appear is S are partially ordered by $w_1 \leq w_2$ if there is a path from w_1 to w_2. If w_0 is a minimal element in this ordering, then all of the edges directed into w_0 are in the list $e_1, \ldots, e_n$. If the indices of the edges directed into w_0 are an interval, $[i, j]$, then clearly we can "pull" w_0 across the line $\mathsf{x} = \mathsf{x}_2$, and get another stage in which the edges directed out of w_0 replace the edges directed into w_0 in our list, and S is replaced by a skeleton isomorphic to the product $s_{[i,j]}S$ (i.e. $s_{[i,j]}$ to the right of S).

Now we claim that given w_0 as above, we can modify the current stage of our construction so that the edges into w_0 do form an interval. Note that if two edges e_i and e_j in our list above share a tail (i.e. are both directed out of the same vertex), then we can exchange their positions in the list while preserving S and its embedding, i.e. if e''_i is directed into w_i and e''_j is directed into w_j, then replace these by edges directed from v'_i to w_j and from v'_j to w_i, while preserving the "left halves" e'_i and e'_j.

The only obstacle that can get in the way of using these exchanges to make the edges into w_0 an interval is if there are edges $e_{i_1}, e_{i_2}, e_{i_3}$ such that $i_1 < i_2 < i_3$, e_{i_1} and e_{i_3} are directed into w_0, e_{i_2} is directed into some other vertex w_1, and the tail of e_{i_2} is different from the tails of $e_{i_1}$and e_{i_3}. We claim that this is a contradiction. By Lemma 2.6.4, there are non-intersecting paths P_0, from w_0 to some v_k, and P_1, from w_1 to some v_l. If $k > l$, use Lemma 2.6.4 again to get non-intersecting paths Q_0, from some v_i to the tail of e_{i_1}, and Q_1, from some v_j to the tail of e_{i_2}. Since S is a skeleton, it follows that $i < j$. The concatenations $Q_0 e_{i_1} P_0$ and $Q_1 e_{i_2} P_1$ are a non-intersecting pair of paths whose existence contradicts (2.14). If $k < l$, a similar contradiction can be found using e_{i_2} and e_{i_3}.

In this manner, we can get a sequence of stages in which the "skeleton part," S, gains an internal vertex between consecutive stages. Since D is finite, eventually we get a stage in which there are no more internal vertices to add to S. Now we claim that at this stage, the sequence of edges $e_1, \ldots, e_n$ can be reordered via the method described above so that the ith edge is directed into the sink v_i (if this is the case, then clearly we can deform the embedding so that the v_i's lie on $\mathsf{x} = \mathsf{x}_2$ with increasing y-coordinates). The argument for this is the same as the one in the previous paragraph, replacing w_0 and w_1 by sinks and forgetting about P_0 and P_1. □

We can combine Theorem 2.6.2 with Greene's result that irreducible characters are nonnegative on skeletons (Theorem 2.1.3) to obtain the following.

Corollary 2.6.5 *If $(D, \mathbf{u}, \mathbf{v})$ is a (finite) compatible triple and χ^λ is an irreducible character of $\mathfrak{S}_n$, then $\chi^\lambda[A(D, \mathbf{u}, \mathbf{v})]$ is monomial positive.*

2.6.6 Remark: The assumption that D is finite can be relaxed somewhat. All that we actually need is that the $\widetilde{\alpha}$'s constructed above are finite.

Technically speaking, Jacobi-Trudi matrices are not actually a special case of Stembridge's definition, but rather a limit of such things, since the "vertices" $(Q_1, \ldots, Q_n)$ are limits of infinite paths rather than vertices of the digraph. Clearly, the results here can be applied to any situation which can be viewed as a similar limit of finite cases.

Chapter 3

Signed and Voltage Graphs

3.1 Signed and Voltage Graphs

In this section, we will briefly cover the definition of a voltage graph and the generalizations of Zaslavsky's chromatic polynomials to this situation, as outlined by Zaslavsky in [54].

If Γ is a graph, we will assume that directions are assigned to the edges in some arbitrary fashion, and that e^{-1} denotes e with the direction reversed. Let G be a finite group.

A *voltage graph* (with group G) is a pair $\Phi = (\Gamma, \varphi)$ where Γ is a graph and φ (the *voltage*) is a map $E(\Gamma) \to G$. By convention $\varphi(e^{-1}) = \varphi(e)^{-1}$.

If $G = \mathbb{Z}/2\mathbb{Z}$, then Φ is called a *signed graph*, and following Zaslavsky's notation, we will usually denote these by Σ.

We will consider colorings $\kappa : V \to (\mathbb{P} \times G) \cup \{0\}$. A coloring is called *proper* when for each edge e from v to w, $\kappa(w) \neq \kappa(v)\varphi(e)$. (The zero color is fixed by G; $0g = 0$.) Colorings whose image lie in $\mathbb{P} \times G$ are called *zero-free.*

Zaslavsky's chromatic polynomials for voltage graphs also come in unbalanced and balanced (zero-free) versions:

$$\chi^u_\Phi(n|G| + 1) = \text{number of proper colorings using } ([n] \times G) \cup \{0\} \tag{3.1}$$
$$\chi^b_\Phi(n|G|) = \text{number of proper colorings using } [n] \times G. \tag{3.2}$$

In [54], Zaslavsky notes that most of the algebraic properties of the chromatic polynomials of signed graphs extend to the case of voltage graphs, but that there is no orientation theory for voltage graphs in general.

Likewise, many of our constructions here can be carried out for voltage graphs, and we will begin by considering this level of generality. In Section 3.4 we will restrict ourselves to signed graphs and concentrate on results which are related to the corresponding hyperplane arrangements.

3.2 Chromatic Functions for Signed and Voltage Graphs

Given a coloring $\kappa : V \to (\mathbb{P} \times G) \cup \{0\}$, let us write $x^\kappa = \prod_v x_{\kappa(v)}$. Then the direct analogues of Stanley's definition (1.15) for voltage graphs would be:

$$X^u_\Phi = \sum x^\kappa, \qquad X^b_\Phi = \sum x^\kappa \tag{3.3}$$

where the sums run over all proper colorings and zero-free proper colorings, respectively. These are formal series in variables $x_{(j,g)}$, with $j \in \mathbb{P}, g \in G$, together with x_0 in the unbalanced case. When $G = \mathbb{Z}/2\mathbb{Z}$, we will write $x_{\pm j}$ for $x_{(j,\pm 1)}$. In general, let us write $|(j,g)| = j$ and $|x_{(j,g)}| = x_j$. Also, if F is any formal series in the variables $x_{(j,g)}$ and x_0, write $|F|$ for the image obtained by replacing each $x_{(j,g)}$ with x_j.

It is obvious from the definitions that $|X^u_\Phi| \in \mathbb{Q}[x_0] \otimes \text{Sym}$ and $|X^b_\Phi| \in \text{Sym}$, i.e. permutations of $\mathbb{P}$ do not affect which colorings are proper. These functions have some nice properties which are not shared by X^u_Φ, X^b_Φ. For example, they are unaffected by switching (see (3.7)).

3.2.1 The spaces where X^u_Φ, X^b_Φ live.

Let G be a finite group and consider the set of formal series with rational coefficients in the variables $\{x_{(j,g)} \,|\, j \in \mathbb{P}, g \in G\}$ for which setting $x_{(j,g)} = 0$, $j > N$ yields a polynomial. These are acted on by permutations of $\mathbb{P}$ and, for each $i \in \mathbb{P}$, there is an action of G by left multiplication on $\{x_{(i,g)} \,|\, g \in G\}$. Let us denote by G-Sym the set of those series which are invariant under all of these actions, or equivalently, under the appropriate actions of the wreath products $G \wr \mathfrak{S}_n$.

Since $\kappa(w) \neq \kappa(v)\varphi(e)$ if and only if $g\kappa(w) \neq g\kappa(v)\varphi(e)$ for all $g \in G$ (and these equations are also not affected by the "$\mathbb{P}$ part" of the coloring), $X^u_\Phi \in \mathbb{Q}[x_0] \otimes \text{G-Sym}$ and $X^b_\Phi \in \text{G-Sym}$.

If $G = \{1\}$, then G-Sym can be identified with Sym in the obvious way. When $G = \mathbb{Z}/2\mathbb{Z}$, let us denote G-Sym by HSym. We will spend the rest of this section describing some bases of G-Sym.

Let $P(G)$ denote the set of polynomials with rational coefficients in the variables $\{x_g \,|\, g \in G\}$ which are invariant under the action of G by left multiplication (on the indices). In particular, $P(\{1\}) = \mathbb{Q}[x_1]$ and $P(\mathbb{Z}/2\mathbb{Z}) = \text{Sym}(x_{+1}, x_{-1})$.

Let us fix an ordering of the elements of G, say $G = \{g_1, g_2, \ldots, g_{|G|}\}$. There is a spanning set for $P(G)$ indexed by ordered $|G|$-tuples of nonnegative integers,

$$f_{(a_1,\ldots,a_{|G|})} = \sum_{g \in G} x^{a_1}_{(gg_1)} x^{a_2}_{(gg_2)} \cdots x^{a_{|G|}}_{(gg_{|G|})}. \tag{3.4}$$

G acts on the $|G|$-tuples in the obvious way, and the polynomials given by (3.4) only depend on the G-orbit of the $|G|$-tuple. So, a basis for $P(G)$ can be obtained by taking one of the expressions above from each orbit.

In particular, $f_{(1,0,\ldots,0)} = \sum_{g\in G} x_g$ spans the degree 1 elements of $P(G)$. Let us call this $e_1(G)$.

If $\{f_\alpha\}_{\alpha\in I}$ is any homogeneous basis of $P(G)$, then G-Sym has bases indexed by multisets of elements of I. One such basis, an analogue of the augmented monomial basis of Sym, is given by

$$\widetilde{\mathfrak{m}}_{\{f_1,f_2,\ldots,f_k\}} = \sum_{(j_1,j_2,\ldots,j_k)\text{ distinct}} f_1(j_1)f_2(j_2)\cdots f_k(j_k),$$

where $f(j)$ denotes the image of f under the map induced by $x_g \mapsto x_{(j,g)}$, and the sum is over k-tuples of distinct positive integers.

Let us write $\{f_1^{c_1},\ldots,f_k^{c_k}\}$ to denote the multiset with f_1 appearing c_1 times, etc. When we use this notation, it is implied that $f_1,\ldots,f_k$ are distinct. Since $P(G)$ is an algebra, this notation is ambiguous. We will write (f^m) when we want the m-th power of f to be an element of the multiset.

An analogue of the monomial basis is given by

$$\mathfrak{m}_{\{f_1^{c_1},\ldots,f_k^{c_k}\}} = \frac{1}{c_1!c_2!\cdots c_k!}\widetilde{\mathfrak{m}}_{\{f_1^{c_1},\ldots,f_k^{c_k}\}}.$$

To see that the "$\mathfrak{m}$-basis" and "$\widetilde{\mathfrak{m}}$-basis" are actually bases of G-Sym, first note that the span of these is independent of which homogeneous basis of $P(G)$ we use. So assume that we are using the expressions given in (3.4). In this case, if we fix a monomial in the variables $x_{(j,g)}$ with a nonzero coefficient in $\mathfrak{m}_{\{f_1^{c_1},\ldots,f_k^{c_k}\}}$, then $\mathfrak{m}_{\{f_1^{c_1},\ldots,f_k^{c_k}\}}$ and $\widetilde{\mathfrak{m}}_{\{f_1^{c_1},\ldots,f_k^{c_k}\}}$ are constant multiples of the sum of all the monomials that can be obtained by acting on the fixed monomial by G and permutations of $\mathbb{P}$. It follows that these span G-Sym, and that their expansions in monomials have disjoint support. So they give a basis of G-Sym.

Let $\mathfrak{p}_f = \mathfrak{m}_{\{f\}} = \widetilde{\mathfrak{m}}_{\{f\}}$. Then an analogue of the power sum basis is given by

$$\mathfrak{p}_{\{f_1,f_2,\ldots,f_k\}} = \mathfrak{p}_{f_1}\mathfrak{p}_{f_2}\cdots\mathfrak{p}_{f_k}. \tag{3.5}$$

Since

$$\widetilde{\mathfrak{m}}_{\{f_1\}}\widetilde{\mathfrak{m}}_{\{f_2\}}\cdots\widetilde{\mathfrak{m}}_{\{f_k\}} = \widetilde{\mathfrak{m}}_{\{f_1,f_2,\ldots,f_k\}} + (\text{monomials with fewer than } k \text{ distinct } \mathbb{P} \text{ indices}),$$

it follows that the expressions in (3.5) also give a basis of G-Sym.

If $P(G)$ happens to be a polynomial algebra with a set of generators $\{b_1,b_2,\ldots\}$ (with each b_i a homogeneous polynomial in the x_g's), then G-Sym is a polynomial algebra on the set of $\mathfrak{m}$'s with multisets of elements from $\{b_1,b_2,\ldots\}$.

To see this, first note that G-Sym is a polynomial algebra on the set

$$\{\mathfrak{p}_f = \mathfrak{m}_{\{f\}} \mid f \text{ is a monomial in the } b_i\text{'s}\}, \tag{3.6}$$

since the monomials in the b_i's give a homogeneous basis of $P(G)$. If $q_1,q_2 \in P(G)$, it's easy to check that

$$\widetilde{\mathfrak{m}}_{\{(q_1q_2)\}} = \widetilde{\mathfrak{m}}_{\{q_1\}}\widetilde{\mathfrak{m}}_{\{q_2\}} - \widetilde{\mathfrak{m}}_{\{q_1,q_2\}}.$$

It follows that each $\mathfrak{p}_f$ in (3.6) is a polynomial in the $\tilde{\mathfrak{m}}$'s whose multisets consist of b_i's. And so every element of G-Sym is a polynomial in the set of $\mathfrak{m}$'s with multisets of elements from $\{b_1, b_2, \ldots\}$. The algebraic independence of these follows from the fact that there are the same number of these in each degree as the $\mathfrak{p}_f$'s considered above.

The bases of G-Sym obtained in this way can be thought of as analogues of the elementary basis, since when $G = \{1\}$, $P(G)$ is a polynomial algebra in $e_1(G) = x_1$, and

$$\mathfrak{m}_{\{e_1(G)^k\}} = \mathfrak{m}_{\{e_1(G),\ldots,e_1(G)\}} = e_k.$$

More generally, we can embed Sym into G-Sym via $e_k \mapsto \mathfrak{m}_{\{e_1(G)^k\}}$ (even if $P(G)$ is not a polynomial algebra). This is just the map induced by $x_j \mapsto \sum_{g\in G} x_{(j,g)}$. It is not hard to see that, under this map,

$$|X_\Phi^b| \mapsto \sum_{\nu:V\to G} X_{\Phi^\nu}^b, \tag{3.7}$$

where Φ^ν denotes the voltage graph with voltage $\varphi^\nu(e) = \nu(v)^{-1}\varphi(e)\nu(w)$ if e is an edge directed from v to w. In Zaslavsky's terminology, Φ is said to be *switched* by ν (see [53]).

More about the HSym case

The constructions above depend on choosing bases of $P(G)$. Since

$$P(\mathbb{Z}/2\mathbb{Z}) = \mathrm{Sym}(x_{+1}, x_{-1}),$$

there are several obvious choices of basis.

It seems that for our purposes, the best analogues of the monomial and power sum bases are obtained by taking $\{f_\alpha\}_{\alpha\in I}$ to be the augmented monomial basis of $\mathrm{Sym}(x_{+1}, x_{-1})$, indexed by partitions with one or two parts, which can be viewed as unordered pairs $\{a, b\}$ of nonnegative integers, with $a + b > 0$:

$$\tilde{m}_{\{a,b\}} = x_{+1}^a x_{-1}^b + x_{+1}^b x_{-1}^a.$$

Let us simplify notation by writing $\mathfrak{p}_{\{a,b\}}$ for $\mathfrak{p}_{\tilde{m}_{\{a,b\}}}$, and similarly for the monomial analogues. Note that

$$|\tilde{\mathfrak{m}}_{\{\{a_1,b_1\},\ldots,\{a_k,b_k\}\}}| = 2^k \tilde{m}_{((a_1+b_1),\ldots,(a_k+b_k))} \tag{3.8}$$

$$\text{and } |\mathfrak{p}_{\{a,b\}}| = 2p_{(a+b)}. \tag{3.9}$$

If we view $\mathrm{Sym}(x_{+1}, x_{-1})$ as a polynomial algebra in e_1, e_2, then HSym is a polynomial algebra in

$$\mathfrak{e}_{(a,b)} = \mathfrak{m}_{\{e_1^a, e_2^b\}} = \frac{1}{a!b!} \sum_{(i_1,\ldots,i_a,j_1,\ldots,j_b)\text{ distinct}} (x_{i_1} + x_{-i_1}) \cdots (x_{i_a} + x_{-i_a})(x_{j_1}x_{-j_1}) \cdots (x_{j_b}x_{-j_b}), \tag{3.10}$$

where (a, b) is an ordered pair of nonnegative integers with $a+b > 0$. The basis consisting of products of these has the nice property that if Γ is an ordinary graph, then $X^{b}_{+\Gamma}$ is positive in this basis if and only if X_Γ is e-positive. In fact, $X^{b}_{+\Gamma}$ is the image of X_Γ under the map induced by $e_k \mapsto \sum_{a+2b=k} \mathfrak{e}_{(a,b)}$.

Unfortunately, unlike in the unsigned case, the sum of the coefficients in this basis can be negative.

Analogues of the Quasi-symmetric Functions

In order to state the reciprocity results for X^{u}_{Σ}, we will need an appropriate analog of QSym. Most of the results in this section can be generalized to an arbitrary group, but we will only need to consider the $\mathbb{Z}/2\mathbb{Z}$ case.

Let HQSym denote the span of

$$\mathfrak{M}_{((a_1,b_1),(a_2,b_2),\dots,(a_k,b_k))} = \sum_{0<i_1<i_2<\cdots<i_k} x_{i_1}^{a_1} x_{-i_1}^{b_1} \cdots x_{i_k}^{a_k} x_{-i_k}^{b_k}. \tag{3.11}$$

Here a_j, b_j are nonnegative integers with $a_j + b_j > 0$, and $i_j \in \mathbb{P}$. These expressions are obviously linearly independent. Now, $\tilde{\mathfrak{m}}_{\{\{a_1,b_1\},\dots,\{a_k,b_k\}\}}$ is the sum of the $\mathfrak{M}$'s over the $2^k k!$ ways to order the set of pairs and the elements of each pair, so HSym $\subseteq$ HQSym.

Let HQSym_d denote the subspace of HQSym consisting of homogeneous functions of degree d. There is a spanning set for HQSym_d, given by

$$\mathfrak{Q}_{S,(\epsilon_1,\dots,\epsilon_d)} = \sum_{\substack{0<a_1\le a_2\le\cdots\le a_d \\ a_i<a_{i+1} \text{ if } i\in S}} x_{\epsilon_1 a_1} \cdots x_{\epsilon_d a_d}, \tag{3.12}$$

where $S \subseteq [d-1]$ and $\epsilon_i \in \{\pm 1\}$. These are not independent, but they will, nevertheless, serve as analogues of the fundamental quasi-symmetric basis.

Let $\mathbb{Q}(\mathbb{Z}/2\mathbb{Z})^d$ denote the group algebra of $(\mathbb{Z}/2\mathbb{Z})^d$. (We actually only need to consider the linear structure of $\mathbb{Q}(\mathbb{Z}/2\mathbb{Z})^d$.) We will denote the basis elements of $\mathbb{Q}(\mathbb{Z}/2\mathbb{Z})^d$ by $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_d)$. There is an obvious surjection

$$\theta : \mathrm{QSym}_d \otimes \mathbb{Q}(\mathbb{Z}/2\mathbb{Z})^d \to \mathrm{HQSym}_d,$$

given by

$$\theta : Q_S \otimes (\epsilon_1, \dots, \epsilon_d) \mapsto \mathfrak{Q}_{S,(\epsilon_1,\dots,\epsilon_d)}.$$

If $S = \{i_1 < i_2 < \cdots < i_{k-1}\} \subseteq [d-1]$, then

$$\theta : M_{S,d} \otimes (\epsilon_1, \dots, \epsilon_d) \mapsto \mathfrak{M}_{((a_1,b_1),(a_2,b_2),\dots,(a_k,b_k))},$$

where $a_1 + b_1 = i_1$, $a_2 + b_2 = i_2 - i_1$, and so on, and

$$a_j = \text{the number of } +1\text{'s in } \epsilon_{i_{j-1}+1}, \dots, \epsilon_{i_j}$$
$$b_j = \text{the number of } -1\text{'s in } \epsilon_{i_{j-1}+1}, \dots, \epsilon_{i_j},$$

(with the conventions $i_0 = 0, i_k = d$). It follows that the kernel of θ is spanned by the differences $M_S \otimes \boldsymbol{\epsilon} - M_S \otimes \boldsymbol{\epsilon}'$, where $\boldsymbol{\epsilon}'$ is obtained from $\boldsymbol{\epsilon}$ by permutations within the subsequences $\epsilon_{i_{j-1}+1}, \dots, \epsilon_{i_j}$.

Lemma 3.2.1 *The involution given by* $\gamma(\mathfrak{Q}_{S,(\epsilon_1,\dots,\epsilon_d)}) = \mathfrak{Q}_{[d-1]-S,(\epsilon_1,\dots,\epsilon_d)}$ *is well-defined on* HQSym.

Proof: The lemma amounts to showing that the involution $\omega \otimes 1$ on $\mathrm{QSym}_d \otimes \mathbb{Q}(\mathbb{Z}/2\mathbb{Z})^d$ preserves the kernel of θ. By (1.9), we have

$$\omega \otimes 1 : M_S \otimes \boldsymbol{\epsilon} = (-1)^{d-1}(-1)^{|S|} \sum_{T \subseteq S} M_T \otimes \boldsymbol{\epsilon}.$$

Consider the subsequences of $\boldsymbol{\epsilon}$ to which permutations can be applied to generate the kernel. If $T \subseteq S$, then the subsequences for T contain the subsequences for S. So the permutations which can be applied for S can also be applied for each T appearing on the right hand side, and the result follows. □

Note that the $\mathfrak{Q}_{S,(+1,\dots,+1)}$ form the basis of a subspace which we can identify with QSym, and $\gamma = \omega$ there.

We will also need to work with $\mathbb{Q}[x_0] \otimes \mathrm{HQSym}$. Its homogeneous component of degree d is spanned by either of the sets

$$\tilde{\mathfrak{Q}}_{S,(\epsilon_1,\dots,\epsilon_d)} = \sum_{\substack{0 \le a_1 \le a_2 \le \cdots \le a_d \\ a_i < a_{i+1} \text{ if } i \in S,\ 0 < a_1 \text{ if } 0 \in S}} x_{\epsilon_1 a_1} \cdots x_{\epsilon_d a_d} \tag{3.13}$$

$$\widetilde{\mathfrak{M}}_{S,(\epsilon_1,\dots,\epsilon_d)} = \sum_{\substack{a_i < a_{i+1} \text{ if } i \in S,\ 0 < a_1 \text{ if } 0 \in S \\ a_i = a_{i+1} \text{ if } i \notin S,\ 0 = a_1 \text{ if } 0 \notin S}} x_{\epsilon_1 a_1} \cdots x_{\epsilon_d a_d}, \tag{3.14}$$

where $S \subseteq [0, d-1] = \{0, 1, \dots, d-1\}$. Clearly, $\tilde{\mathfrak{Q}}_{S,\boldsymbol{\epsilon}} = \sum_{T \supseteq S} \widetilde{\mathfrak{M}}_{T,\boldsymbol{\epsilon}}$.

If $S = \{i_1, i_2, \dots, i_{k-1}\} \subseteq [0, d-1]$, then S defines a deconcatenation of $\epsilon_1, \dots, \epsilon_d$ into subsequences in a similar manner to what was considered above:

$$\underbrace{\epsilon_1, \dots, \epsilon_{i_1}}, \underbrace{\epsilon_{i_1+1}, \dots, \epsilon_{i_2}}, \dots, \underbrace{\epsilon_{i_{k-1}+1}, \dots, \epsilon_d}, \text{ if } i_1 > 0, \tag{3.15}$$

$$\underbrace{\epsilon_1, \dots, \epsilon_{i_2}}, \dots, \underbrace{\epsilon_{i_{k-1}+1}, \dots, \epsilon_d}, \text{ if } i_1 = 0. \tag{3.16}$$

The relations satisfied by the $\widetilde{\mathfrak{M}}_{S,\boldsymbol{\epsilon}}$'s are generated by permutations of the elements of each of the subsequences above and arbitrary changes of the elements in the first sequence in (3.15). The reason for the arbitrary changes there are that this sequence gives the factor of $x_{\epsilon_1 0} \cdots x_{\epsilon_{i_1} 0} = x_0^{i_1}$.

Replacing S by a subset (equivalently, joining consecutive subsequences) is well-defined modulo these relations, so the argument in Lemma 3.2.1 generalizes to give the following result.

Lemma 3.2.2 *The involution given by* $\tilde{\gamma}(\tilde{\mathfrak{Q}}_{S,(\epsilon_1,\dots,\epsilon_d)}) = \tilde{\mathfrak{Q}}_{\overline{S},(\epsilon_1,\dots,\epsilon_d)}$ *is well-defined on* $\mathbb{Q}[x_0] \otimes \mathrm{HQSym}$. *(Where* $\overline{S} = \{0, 1, \dots, d-1\} - S$.*)*

3.3 Expansions

In this section, we will consider various "expansions" of X_Φ^u and X_Φ^b, and the versions, $|X_\Phi^u|$ $|X_\Phi^b|$, obtained by "forgetting" the "G" part of the indices of the variables.

The following straightforward result shows that the unbalanced chromatic functions can be expressed in terms of the balanced ones. This directly generalizes the balanced expansion formula of Zaslavsky [55, Thm 1.1].

Proposition 3.3.1

$$X_\Phi^u = \sum_{\substack{W \subseteq V \\ W^c \text{ stable}}} x_0^{|W^c|} X_{\Phi|_W}^b,$$

where a stable subset of the vertices is a subset in which none of the vertices are connected by edges.

Proof: This follows (trivially) from observing that the vertices colored zero cannot be connected by any edges (of any voltage). □

For an ordinary graph Γ, and any group G, Γ can be viewed as a voltage graph, $+\Gamma$, by assigning a voltage of 1 to each edge. In this case, the proper colorings just need to assign different colors to adjacent vertices, so $X_{+\Gamma}^b$ is just the image of X_Γ under the isomorphism between Sym and $\mathrm{Sym}(\{x_{(j,g)} \mid j \in \mathbb{P}, g \in G\})$. The following proposition follows from the interpretation of Ψ^k discussed at the beginning of Section 1.3.

Proposition 3.3.2

$$|X_{+\Gamma}^b| = \Psi^{|G|} X_\Gamma.$$

There are expansions of X_Φ^u and X_Φ^b in terms of the analogues of the power sum and monomial bases which are fairly straightforward generalizations of the corresponding results for ordinary graphs, although there are some complications.

Let $f_\Phi = \sum x^\kappa$ where the sum runs over all maps $\kappa : V \to G$ such that $\kappa(w) \neq \kappa(v)\varphi(e)$ if e is an edge from v to w. Equivalently, take the image of X_Φ^u under $x_{j,g} \mapsto \delta_{j,1} x_g$. Then clearly f_Φ is an element of $P(G)$.

The following proposition can be viewed as an analogue of the monomial expansion of X_Γ mentioned as Proposition 1 in the Introduction.

Proposition 3.3.3

$$X_\Phi^b = \sum_{\pi \in \Pi(V)} \widetilde{\mathfrak{m}}_{\{f_{\pi_1}, \ldots, f_{\pi_k}\}}$$

where the sum runs over all partitions $\pi = \{\pi_1, \ldots, \pi_k\}$ of the vertices, and f_{π_i} is $f_{\Phi|_{\pi_i}}$, as defined above.

Proof: Given a proper, zero-free coloring $\kappa : V \to \mathbb{P} \times G$, let π be the partition of the vertices whose blocks are the sets on which $|\kappa|$ is constant (where $|(j,g)| = j$). It's clear that the "G" part of the restriction of κ to each block π_i is a coloring enumerated by f_{π_i}, and conversely, any such colorings on each block, along with an assignment of distinct elements of $\mathbb{P}$ to each block, will define a proper coloring κ. But it follows directly from the definition that these are enumerated by $\tilde{\mathfrak{m}}_{\{f_{\pi_1},\dots,f_{\pi_k}\}}$. □

Although the expansion in Proposition 3.3.3 is not given in terms of a basis, the expressions $\tilde{\mathfrak{m}}_{\{f_1,\dots,f_k\}}$ are linear in each entry f_i, so it's fairly easy to get an expansion in a basis if there is some natural basis of $P(G)$ in which the f_{π_i}'s can be expanded. For example, the following expansion for signed graphs is just an unraveling of the definitions.

Corollary 3.3.4

$$X^b_\Sigma = \sum_{\{\{A_1,B_1\},\dots,\{A_k,B_k\}\}} \tilde{\mathfrak{m}}_{\{\{|A_1|,|B_1|\},\dots,\{|A_k|,|B_k|\}\}},$$

where the sum runs over collections which satisfy the following conditions: (i) the sets (of vertices) $A_1, \dots, A_k, B_1, \dots, B_k$ are pairwise disjoint and their union is V, (ii) each union $A_i \cup B_i$ is nonempty, (iii) any edges between two vertices in the same A_i (or B_i) must have a negative sign, and (iv) any edges between elements of A_i and B_i must have a positive sign.

In order to state analogues of the power sum expansions of X_Γ, we will first need to consider a few definitions.

Voltages can be assigned to walks in Γ by taking products. Similarly, a circuit has a well-defined voltage up to conjugation. A set of edges $S \subseteq E$ is called *balanced* if every circuit in S has voltage 1.

We will denote the poset of *balanced flats* of Φ by P^b_Φ. This can be viewed as the set of the balanced subsets S of the edges such that for any edge e which is not in S, e does not lie in a balanced circuit of $S \cup \{e\}$. Note that for an ordinary graph Γ (i.e. $G = \{1\}$), P^b_Γ can be identified with the bond lattice of Γ.

If $\kappa : V \to (\mathbb{P} \times G)$ is any (zero-free) coloring, then let $I(\kappa)$ denote the set of edges for which $\kappa(w) = \kappa(v)\varphi(e)$, and call this the set of impropriety of κ. One of Zaslavsky's fundamental results is the following.

Lemma 3.3.5 ([54]) *If $\kappa : V \to (\mathbb{P} \times G)$ is any (zero-free) coloring, then the set of impropriety, $I(\kappa)$, is a balanced flat of Γ.*

Zaslavsky uses this result to give the following "algebraic" definitions of the balanced chromatic polynomial.

Theorem 3.3.6 ([54]) *If $S \subseteq E$, let $c(S)$ denote the number of connected components of the spanning subgraph of Γ (the graph underlying Φ) with edge set S. Then*

$$\chi^b_\Phi(t) = \sum_{\substack{S \subseteq E \\ balanced}} (-1)^{|S|} t^{c(S)} = \sum_{S \subseteq P^b_\Phi} \mu(\emptyset, S) t^{c(S)},$$

where μ *is the Möbius function of* P_Φ^b.

If S is any balanced set of edges, we can obtain all of the zero-free colorings of Φ whose set of impropriety contains S as follows. For each component of the spanning subgraph of Γ (the graph underlying Φ), choose a vertex v and color this with some arbitrary (j, g). If an edge e of S goes from v to w, then color w with $\kappa(v)\phi(e)$ (this is necessary for e to be an improper edge). Continuing in this manner, all the vertices can be assigned colors. The importance behind S being balanced is that the color we assign to a vertex is independent of the path we take to reach it (all paths from v to w using edges in S have the same voltage).

For each component, the other colorings are obtained by choosing a different j and g. Choosing a different g amounts to multiplying the color of each vertex in this component by some $h \in G$ (on the left). So for each component, the sum of the monomials corresponding to these colorings is of the form $\mathfrak{p}_f = \mathfrak{m}_{\{f\}}$ where f is one of the basis elements of $P(G)$ defined in (3.4). If we let $F(S)$ denote the multiset consisting of the basis elements for each component, then the standard Möbius inversion argument implies that we have the following expansions in terms of our analogue of the power sum basis.

Theorem 3.3.7

$$X_\Phi^b = \sum_{\substack{S \subseteq E(\Phi) \\ S \text{ is balanced}}} (-1)^{|S|} \mathfrak{p}_{F(S)} = \sum_{S \in P_\Phi^b} \mu(\emptyset, S) \mathfrak{p}_{F(S)}$$

We can obtain slightly more elegant and explicit results by "forgetting" about the "G" part of the colorings. These results can be viewed as analogues of Theorems 1.2.2 and 1.2.1.

Corollary 3.3.8

$$|X_\Phi^b| = \sum_{\substack{S \subseteq E(\Phi) \\ S \text{ is balanced}}} (-1)^{|S|} |G|^{c(S)} p_{\lambda(S)} = \sum_{S \in P_\Phi^b} \mu(\emptyset, S) |G|^{c(S)} p_{\lambda(S)},$$

where $\lambda(S)$ *denotes the partition whose parts are the vertex sizes of the components of the spanning subgraph with edge set* S, *and* $c(S)$ *is the number of components.*

3.4 X_Σ^u, X_Σ^b in terms of Hyperplane Arrangements

In this section, we will describe the expansions of the chromatic functions for signed graphs in terms of the analogues of the elementary symmetric functions and fundamental quasi-symmetric functions. These expansions reflect properties of the hyperplane arrangement associated with Σ. For an ordinary graph, Γ, we will show that information about various hyperplane arrangements associated with Γ can be obtained from X_Γ.

As we mentioned in the introduction, Zaslavsky [52] interprets signed graph colorings in terms of certain hyperplane arrangements. Assume that Σ is a signed graph on d vertices,

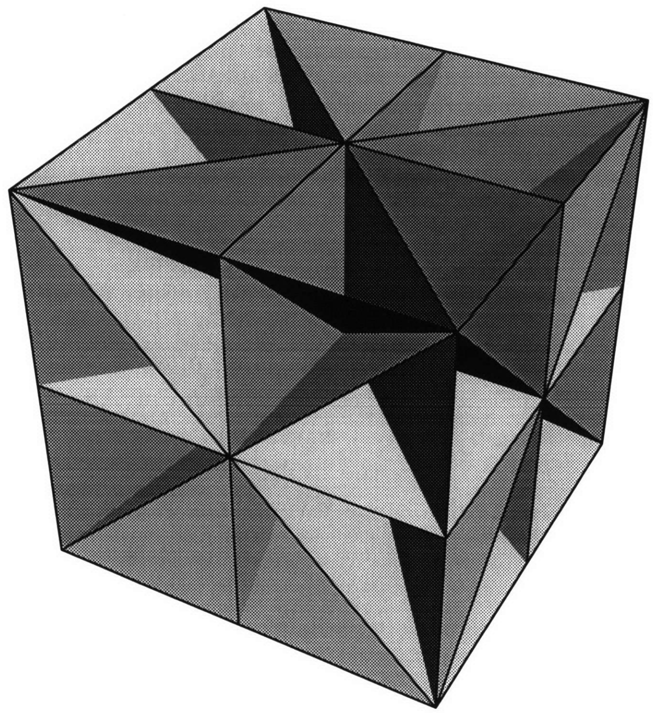

Figure 3.1: The hyperplanes $\mathcal{B}_3^*$.

labelled by $[d]$. Then Σ is associated with an arrangement in $\mathbb{R}^d$ by including the hyperplane $\mathsf{x}_i = \epsilon \mathsf{x}_j$ if there is an edge (i, j) with sign ϵ. The set of these hyperplanes will be denoted by $H[\Sigma]$. It follows immediately from the definitions that proper colorings of Σ exactly correspond to those points in $\mathbb{R}^d$ which have integer coordinates and lie in the complement of $H[\Sigma]$.

With this definition, positive loops correspond to the whole space ($\mathsf{x}_i = \mathsf{x}_i$), which is considered to be a "degenerate hyperplane" in Zaslavsky's treatment. In our considerations here, we can just assume that Σ has no positive loops (otherwise, there are no proper colorings).

Negative loops correspond to the coordinate hyperplanes. It is sometimes useful to consider the result of adding these to some arrangement, so following Zaslavsky, we will denote by Σ° the signed graph obtained from Σ by adding a negative loop at each vertex.

Let $\mathcal{B}_d^*$ denote the set of hyperplanes dual to the elements of the root system, $\mathcal{B}_d$, of the hyperoctahedral group, $\mathfrak{B}_d$. Then $H[\Sigma]$ is a subarrangement of $\mathcal{B}_d^*$, and every subarrangement can obtained in this way. The hyperplane arrangement $\mathcal{B}_3^*$ is shown in Figure 3.1.

Let $\mathcal{A}_{d-1}^*$ denote the set of hyperplanes of the form $\mathsf{x}_i = \mathsf{x}_j$, $(i \neq j)$. These correspond to positive edges which are not loops, and the duals of these in $\mathcal{B}_d$ can be identified with the root system of $\mathfrak{S}_d$. Ordinary graphs can be viewed as signed graphs (with a sign of +1 for each edge). These then correspond to subarrangements of $\mathcal{A}^*_{d-1}$. Note that in this case, the regions into which $\mathbb{R}^d$ is divided by the hyperplanes exactly correspond to the regions into which the first orthant is divided.

If $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_d) \in \mathbb{Z}^d \subseteq \mathbb{R}^d$, let $x^\kappa = x_{\kappa_1} x_{\kappa_2} \cdots x_{\kappa_d}$. Denote the lattice points in the complement of the hyperplane arrangement $H[\Sigma]$ by $H[\Sigma]^c$. Then it is immediate from Zaslavsky's definition of $H[\Sigma]$ that

$$X^u_\Sigma = \sum_{\kappa \in H[\Sigma]^c} x^\kappa .$$

We could take this as the definition of X^u_Σ. This definition has also been suggested by Bruce Sagan (personal communication) as a hyperoctahedral analogue of Stanley's chromatic symmetric function. Blass and Sagan [4] use the lattice points in $H[\Sigma]^c$ to prove some results concerning the chromatic polynomials χ^u_Σ and some generalizations to subspace arrangements and other Weyl hyperplane arrangements.

We can identify the elements of the hyperoctahedral group, $\mathfrak{B}_d$, with "signed permutations," $\alpha = (\epsilon_1\sigma_1, \epsilon_2\sigma_2, \ldots, \epsilon_d\sigma_d)$, $(\sigma \in \mathfrak{S}_d, \quad \epsilon_i \in \{\pm 1\})$. We can then identify these signed permutations with points in $\mathbb{Z}^d \subseteq \mathbb{R}^d$. Note that there is exactly one such point in the interior of each Weyl chamber. We will employ the convention of multiplying elements of $\mathfrak{B}_d$ as follows:

$$(\epsilon_1\sigma_1, \ldots, \epsilon_d\sigma_d)(\epsilon'_1\sigma'_1, \ldots, \epsilon'_d\sigma'_d) = (\epsilon'_1\epsilon_{\sigma'_1}\sigma(\sigma'_1), \ldots, (\epsilon'_d\epsilon_{\sigma'_d}\sigma(\sigma'_d)).$$

The usual definition of the descent set of an element of $\mathfrak{B}_d$ is

$$D(\alpha) = \{i \in \{0, 1, \ldots, d-1\} \mid \epsilon_i\sigma_i > \epsilon_{i+1}\sigma_{i+1}, \text{ or } (i = 0 \text{ and } 0 > \epsilon_i\sigma_i)\}.$$

We now have the terminology to state the following generalization of equation (1.18):

Theorem 3.4.1 *If R is a region of $H[\Sigma]$ and α is a signed permutation contained in $-R$, then*

$$\sum_{\kappa \in R} x^\kappa = \sum_{\beta \in \mathfrak{B}_d \cap R} \tilde{\mathfrak{Q}}_{D(\alpha\beta^{-1}),(\epsilon_1,\ldots,\epsilon_d)}$$

where $\epsilon_i = \operatorname{sign}(\beta_i^{-1})$, and $\alpha\beta^{-1}$ denotes the usual group operations in $\mathfrak{B}_d$.

Proof: Fix a signed permutation $\alpha = (\delta_1\sigma_1, \ldots, \delta_d\sigma_d) \in \mathfrak{B}_d \cap (-R)$, and let $\kappa = (\kappa_1, \cdots, \kappa_d)$ be a lattice point in the interior of R. Define η_i by

$$\eta_i = \begin{cases} \operatorname{sign}(\kappa_i) & \text{if } \kappa_i \neq 0 \\ \delta_i & \text{if } \kappa_i = 0 \end{cases} \tag{3.17}$$

There is a unique permutation $\tau \in \mathfrak{S}_d$ satisfying the following:

$$\begin{aligned} |\kappa_i| < |\kappa_j| &\implies \tau_i^{-1} < \tau_j^{-1} \\ |\kappa_i| = |\kappa_j| \text{ and } \eta_i\delta_i\sigma_i < \eta_j\delta_j\sigma_j &\implies \tau_i^{-1} < \tau_j^{-1} \end{aligned} \tag{3.18}$$

Let $\beta = \beta(\kappa) = (\eta_1\tau_1^{-1}, \dots, \eta_d\tau_d^{-1})$. With the conventions described above,

$$\beta^{-1} = (\eta_{\tau_1}\tau_1, \dots, \eta_{\tau_d}\tau_d) \text{ and } \alpha\beta^{-1} = (\eta_{\tau_1}\delta_{\tau_1}\sigma_{\tau_1}, \dots, \eta_{\tau_d}\delta_{\tau_d}\sigma_{\tau_d}).$$

Let $\epsilon_i = \eta_{\tau_i}$, and consider the d-tuples $(\eta_{\tau_1}a_1, \dots, \eta_{\tau_d}a_d)$ satisfying

$$\begin{aligned} &a_i \leq a_{i+1}, \\ &a_i < a_{i+1}, \text{ if } \eta_{\tau_i}\delta_{\tau_i}\sigma_{\tau_i} > \eta_{\tau_{i+1}}\delta_{\tau_{i+1}}\sigma_{\tau_{i+1}} \\ &\quad 0 < a_1, \text{ if } 0 > \eta_{\tau_1}\delta_{\tau_1}\sigma_{\tau_1}. \end{aligned} \tag{3.19}$$

Note that the sum of the monomials corresponding to these is $\tilde{\Omega}_{D(\alpha\beta^{-1}),(\epsilon_1,\dots,\epsilon_d)}$.

Choose such a d-tuple and let $\kappa_i' = \eta_i a_{\tau_i^{-1}}$. Note that this just permutes the coordinates of the d-tuple. Now we want to claim that, for each κ' obtained in this way, $\beta(\kappa') = \beta$ and that each κ' is an point in the interior of R. The theorem follows from these claims. (β is the κ' obtained when $a_i = i$.)

First consider the construction of $\beta(\kappa')$. If $\kappa_i' \neq 0$, then clearly $\operatorname{sign}(\kappa_i') = \eta_i$. If $\kappa_i' = 0$, then $a_{\tau_i^{-1}} = 0$, so $0 = a_1 = \cdots = a_{\tau_i^{-1}}$. It follows from the conditions on the a_i's that

$$0 < \eta_{\tau_1}\delta_{\tau_1}\sigma_{\tau_1} < \cdots < \eta_i\delta_i\sigma_i. \tag{3.20}$$

In particular, $\eta_i = \delta_i$. If $|\kappa_i'| < |\kappa_j'|$, then $\tau_i^{-1} < \tau_j^{-1}$, since the sequence $a_1, \dots, a_d$ is weakly increasing. Finally, if $|\kappa_i'| = |\kappa_j'|$ then the equality $a_{\tau_i^{-1}} = a_{\tau_j^{-1}}$ implies that there must be a sequence of ascents between the τ_i^{-1}th and τ_j^{-1}th positions of $\alpha\beta^{-1}$, so that if $\tau_j^{-1} < \tau_i^{-1}$,

$$\eta_j\delta_j\sigma_j > \eta_{\tau_{(\tau_j^{-1}+1)}}\delta_{\tau_{(\tau_j^{-1}+1)}}\sigma_{\tau_{(\tau_j^{-1}+1)}} > \cdots > \eta_i\delta_i\sigma_i.$$

It follows that τ is the unique permutation satisfying (3.18) for κ'. The η_i and τ corresponding to κ' are the same as those for κ, and so $\beta(\kappa') = \beta(\kappa)$.

Now we will verify that each κ' obtained above lies in the interior of R. We know that

$$-\alpha = (-\delta_1\sigma_1, \dots, -\delta_d\sigma_d) \text{ and } \kappa = (\eta_1|\kappa_1|, \dots, \eta_d|\kappa_d|$$

are in the interior of R. We will also need the fact that R is an intersection of half-spaces coming from hyperplanes in $\mathcal{B}_d^*$.

Consider the following three sequences.

$$\begin{array}{ccccccccc} 0 & \leq & |\kappa_{\tau_1}| & \leq & |\kappa_{\tau_2}| & \leq & \cdots & \leq & |\kappa_{\tau_d}| \\ 0 & \leq & |\kappa'_{\tau_1}| & \leq & |\kappa'_{\tau_2}| & \leq & \cdots & \leq & |\kappa'_{\tau_d}| \\ 0 & & \eta_{\tau_1}\delta_{\tau_1}\sigma_{\tau_1} & & \eta_{\tau_2}\delta_{\tau_2}\sigma_{\tau_2} & & \cdots & & \eta_{\tau_d}\delta_{\tau_d}\sigma_{\tau_d} \end{array}$$

For every equality which occurs in one of the upper two sequences there is an ascent at the corresponding position in the lower sequence. For κ, this follows from the definitions of η_i and τ, and for κ', it follows from the definition (3.19) (since $\kappa'_{\tau_i} = a_i$).

Any half-space coming from a hyperplane in $\mathcal{B}_d^*$ can be defined using an inequality in one of the forms:

$$s_1 \mathrm{x}_{\tau_i} < s_2 \mathrm{x}_{\tau_j} \tag{3.21}$$

$$\text{or } 0 < s_2 \mathrm{x}_{\tau_j}, \tag{3.22}$$

where $i < j$ and $s_1, s_2 \in \{\pm 1\}$.

Consider an inequality of the form (3.21) which is satisfied by κ and $-\alpha$. Then we have

$$s_1 \eta_{\tau_i} |\kappa_{\tau_i}| < s_2 \eta_{\tau_j} |\kappa_{\tau_j}|. \tag{3.23}$$

It follows that $s_2 \eta_{\tau_j} = +1$ (using the fact that $|\kappa_{\tau_k}|$ is an increasing sequence).

Case 1: $s_1 \eta_{\tau_i} = +1$. Then (3.23) says that $|\kappa_{\tau_i}| < |\kappa_{\tau_j}|$, and κ' satisfies (3.21) if and only if the same is true for κ'. If $|\kappa'_{\tau_i}| = |\kappa'_{\tau_j}|$, then it follows that $\eta_{\tau_i}\delta_{\tau_i}\sigma_{\tau_i} < \eta_{\tau_j}\delta_{\tau_j}\sigma_{\tau_j}$. But then $-s_1\delta_{\tau_i}\sigma_{\tau_i} > -s_2\delta_{\tau_j}\sigma_{\tau_j}$ (since $s_1\eta_{\tau_i} = +1$ and $s_2\eta_{\tau_j} = +1$), so that $-\alpha$ does not satisfy (3.21).

Case 2: $s_1\eta_{\tau_i} = -1$. Then (3.23) simply says that $|\kappa_{\tau_j}| > 0$, and κ' satisfies (3.21) if and only if the same is true for κ'. If $\kappa'_{\tau_j} = 0$, then

$$0 < \eta_{\tau_1}\delta_{\tau_1}\sigma_{\tau_1} < \eta_{\tau_2}\delta_{\tau_2}\sigma_{\tau_2} < \ldots < \eta_{\tau_j}\delta_{\tau_j}\sigma_{\tau_j}. \tag{3.24}$$

In particular, $\eta_{\tau_i} = \delta_{\tau_i}$ and $\eta_{\tau_j} = \delta_{\tau_j}$. So, $-s_1\delta_{\tau_i}\sigma_{\tau_i} = \sigma_{\tau_i}$, and $-s_2\delta_{\tau_j}\sigma_{\tau_j} = -\sigma_{\tau_j}$. This contradicts the assumption that $-\alpha$ satisfies (3.21), i.e. $-s_1\delta_{\tau_i}\sigma_{\tau_i} < -s_2\delta_{\tau_j}\sigma_{\tau_j}$.

Finally, we need to consider half-spaces defined by inequalities of the form (3.22). κ can satisfy this only if $s_2\eta_{\tau_j} = +1$. And so it follows that κ' satisfies (3.22) if and only if $|\kappa'_{\tau_j}| > 0$. If $\kappa'_{\tau_j} = 0$, then we have (3.24) (as above). Again, we have that $\eta_{\tau_j} = \delta_{\tau_j}$, and it follows that $-s_2\delta_{\tau_j}\sigma_{\tau_j} < 0$ which contradicts the assumption that $-\alpha$ satisfies (3.22). □

As mentioned in section 3.2.1, these $\tilde{\mathfrak{Q}}$'s are not linearly independent, and the terms appearing in Theorem 3.4.1 do depend on the choice of α.

The pairs (R, κ) where κ is a lattice point in the closure of R correspond exactly to what Zaslavsky [54] calls *compatible pairs* of acyclic orientations and colorings. Stanley's reciprocity result (Theorem 1.2.4) generalizes to the present context in a particularly nice way:

Theorem 3.4.2 *If R is a region of $H[\Sigma]$ and α is a signed permutation contained in $-R$, then*

$$\sum_{\kappa \in \text{closure}(R)} x^{\kappa} = \sum_{\beta \in \mathfrak{B}_d \cap R} \tilde{\mathfrak{Q}}_{\overline{D(\alpha\beta^{-1})},(\epsilon_1,\ldots,\epsilon_d)}$$

where $\epsilon_i = \mathrm{sign}(\beta_i^{-1})$, *and* $\overline{S} = [0, d-1] - S$, *or equivalently,*

$$\sum_{(R,\kappa)} x^{\kappa} = \tilde{\gamma}(X_{\Sigma}^{u}),$$

where the sum runs over pairs of regions R *in* $H[\Sigma]$ *and lattice points* κ *in the closure of* R, *and* $\tilde{\gamma}$ *denotes the involution defined in Lemma 3.2.2.*

Proof: Clearly the second part follows from the first and Theorem 3.4.1.

The proof of the first statement is very similar to the proof of Theorem 3.4.1. Fix a signed permutation $\alpha = (\delta_1\sigma_1, \ldots, \delta_d\sigma_d) \in \mathfrak{B}_d \cap (-R)$, and let $\kappa = (\kappa_1, \cdots, \kappa_d)$ be a lattice point in the closure of R.

$\beta = \beta(\kappa) = (\eta_1\tau_1^{-1}, \ldots, \eta_d\tau_d^{-1})$ is defined as follows. Let η_i be given by

$$\eta_i = \begin{cases} \mathrm{sign}(\kappa_i) & \text{if } \kappa_i \neq 0 \\ -\delta_i & \text{if } \kappa_i = 0, \end{cases} \tag{3.25}$$

and let $\tau \in \mathfrak{S}_d$ be the unique permutation satisfying:

$$\begin{aligned} |\kappa_i| < |\kappa_j| &\implies \tau_i^{-1} < \tau_j^{-1} \\ |\kappa_i| = |\kappa_j| \text{ and } \eta_i\delta_i\sigma_i > \eta_j\delta_j\sigma_j &\implies \tau_i^{-1} < \tau_j^{-1} \end{aligned} \tag{3.26}$$

And, of course, in this case, we will consider the d-tuples $(\eta_{\tau_1}a_1, \ldots, \eta_{\tau_d}a_d)$ satisfying

$$\begin{aligned} &a_i \leq a_{i+1}, \\ &a_i < a_{i+1}, \text{ if } \eta_{\tau_i}\delta_{\tau_i}\sigma_{\tau_i} < \eta_{\tau_{i+1}}\delta_{\tau_{i+1}}\sigma_{\tau_{i+1}} \\ &\quad 0 < a_1, \text{ if } 0 < \eta_{\tau_1}\delta_{\tau_1}\sigma_{\tau_1}. \end{aligned}$$

The sum of the monomials corresponding to these is $\tilde{\mathfrak{Q}}_{\overline{D(\alpha\beta^{-1})},(\epsilon_1,\ldots,\epsilon_d)}$, where $\epsilon_i = \eta_{\tau_i} = \mathrm{sign}(\beta^{-1})$, as before. As in Theorem 3.4.1, we will consider the points $\kappa'_i = \eta_i a_{\tau_i^{-1}}$.

The proof that $\beta(\kappa') = \beta(\kappa)$ is entirely analogous to the argument in the previous theorem. We will omit the details.

In this case, when we consider the sequences

$$\begin{array}{ccccccccc} 0 & \leq & |\kappa_{\tau_1}| & \leq & |\kappa_{\tau_2}| & \leq & \cdots & \leq & |\kappa_{\tau_d}| \\ 0 & \leq & |\kappa'_{\tau_1}| & \leq & |\kappa'_{\tau_2}| & \leq & \cdots & \leq & |\kappa'_{\tau_d}| \\ 0 & & \eta_{\tau_1}\delta_{\tau_1}\sigma_{\tau_1} & & \eta_{\tau_2}\delta_{\tau_2}\sigma_{\tau_2} & & \cdots & & \eta_{\tau_d}\delta_{\tau_d}\sigma_{\tau_d}, \end{array}$$

we see that each equality in one of the upper two sequences corresponds to a descent at the corresponding position in the lower sequence.

Suppose we have an inequality

$$s_1\mathrm{x}_{\tau_i} \leq s_2\mathrm{x}_{\tau_j}$$

defining a closed half-space containing R, with $i < j$ and $s_1, s_2 \in \{\pm 1\}$, which is satisfied by κ and $-\alpha$, but not by κ'. Then we have, more explicitly,

$$s_1\eta_{\tau_i}|\kappa_{\tau_i}| \leq s_2\eta_{\tau_j}|\kappa_{\tau_j}| \tag{3.27}$$
$$-s_1\delta_{\tau_i}\sigma_{\tau_i} \leq -s_2\delta_{\tau_j}\sigma_{\tau_j} \tag{3.28}$$
$$s_1\eta_{\tau_i}|\kappa'_{\tau_i}| > s_2\eta_{\tau_j}|\kappa'_{\tau_j}|. \tag{3.29}$$

It follows from (3.29) (and the fact that $a_i \leq a_{i+1}$) that $(s_1\eta_{\tau_i}, s_2\eta_{\tau_j})$ must be $(+1, -1)$ or $(-1, -1)$.

Case 1: $(s_1\eta_{\tau_i}, s_2\eta_{\tau_j}) = (-1, -1)$. Then, together, (3.27) and $|\kappa_{\tau_i}| \leq |\kappa_{\tau_j}|$ imply that $|\kappa_{\tau_i}| = |\kappa_{\tau_j}|$. It follows that $\eta_{\tau_i}\delta_{\tau_i}\sigma_{\tau_i} > \eta_{\tau_j}\delta_{\tau_j}\sigma_{\tau_j}$, and hence that $-s_1\delta_{\tau_i}\sigma_{\tau_i} > -s_2\delta_{\tau_j}\sigma_{\tau_j}$ which directly contradicts (3.28).

Case 2: $(s_1\eta_{\tau_i}, s_2\eta_{\tau_j}) = (+1, -1)$. Then (3.27) implies that $|\kappa_{\tau_j}| = 0$. It follows that

$$0 > \eta_{\tau_1}\delta_{\tau_1}\sigma_{\tau_1} > \eta_{\tau_2}\delta_{\tau_2}\sigma_{\tau_2} > \ldots > \eta_{\tau_j}\delta_{\tau_j}\sigma_{\tau_j}.$$

In particular, $0 > s_1\delta_{\tau_1}\sigma_{\tau_1} > -s_2\delta_{\tau_2}\sigma_{\tau_2}$, which says that the left side of (3.28) is positive while the right side is negative.

The half-spaces $0 \leq s\mathrm{x}_{\tau_j}$ can be handled similarly. □

It seems that the best way to give a hyperplane interpretation and generalization of Stanley's result about $\mathrm{sink}(\Gamma, j)$ (Theorem 1.2.3 above) involves adding the coordinate hyperplanes, i.e. looking at $H[\Sigma^\circ]$. This is because, for an ordinary graph, we can determine the number of sinks of an acyclic orientation by looking at the intersection of the corresponding region with the first orthant and counting the number of faces which lie in coordinate hyperplanes.

We can also include some information about which orthant a given region lies in. Namely, if R is a region of $H[\Sigma^\circ]$, let $\mathrm{neg}(R)$ be the number of negative coordinates in an interior point of R and let $\mathrm{sink}(R)$ denote the number of faces of R which lie in coordinate hyperplanes.

Let ψ be given by

$$\psi : \mathfrak{Q}_{S,(\epsilon_1,\ldots,\epsilon_d)} \mapsto \begin{cases} t(t-1)^i s^j, & \text{if } S = \{i+1, i+2, \ldots, d-1\} \\ & \text{and } j \text{ of the } \epsilon_i\text{'s are } -1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.30}$$

Lemma 3.4.3 *ψ is a well-defined algebra map on* HQSym.

Proof: A trivial consequence of our computation of the quotient of the kernel of θ (see p.85) is that any map defined on $\mathrm{QSym}_d \otimes \mathbb{Q}(\mathbb{Z}/2\mathbb{Z})^d$ which is invariant under *all* permutations of the $(\epsilon_1, \ldots, \epsilon_d)$ part passes to a well-defined map on the quotient, HQSym.

The fact that ψ is an algebra map follows from the fact that Stanley's φ (defined in (1.19)) is an algebra map, and the fact that $x_{+j} \mapsto 1$, $x_{-j} \mapsto s$ (for $j \in \mathbb{P}$) defines an algebra map on HQSym. □

Since we are considering $H[\Sigma^\circ]$, the interior of a given region, R, is contained within a single orthant. By reflecting the region in various coordinate planes, we can "move" it into the first quadrant, to get a region R' there. An expansion in $\mathfrak{Q}_{S,\epsilon}$'s for the enumerator of the points in the interior of R can be obtained from the expansion of the enumerator of R' in QSym by attaching an appropriate ϵ to each term. In particular, the following result actually follows from the proof of Theorem 1.2.3.

Proposition 3.4.4

$$\psi(X_\Sigma^b) = \psi(X_{\Sigma^\circ}^u) = \sum s^{\mathrm{neg}(R)} t^{\mathrm{sink}(R)},$$

where the sum runs over the regions for $H[\Sigma^\circ]$.

It follows from the remarks made in Section 3.2.1 and a straightforward calculation that

$$\psi : \mathfrak{M}_{((a_1,b_1),(a_2,b_2),\dots,(a_k,b_k))} \mapsto (-1)^{d-k}(1-(1-t)^{a_1+b_1})s^{b_1+b_2+\cdots+b_k}.$$

And it follows from directly comparing the appropriate definitions that $\mathfrak{e}_{(a,b)}$ is the sum of all $\mathfrak{M}_{((a_1,b_1),\dots,(a_k,b_k))}$'s for which b of the ordered pairs are $(1,1)$ and the remaining a of the ordered pairs are either $(0,1)$ or $(1,0)$. Summing the values of ψ on all these, and doing a bit of simplifying, we can obtain the following (not particularly elegant) result:

Proposition 3.4.5 *The algebra map induced by*

$$\mathfrak{e}_{(a,b)} \mapsto (-s)^b(1+s)^a\left[t\binom{a+b}{a} - t(t-1)\binom{a+b-1}{a}\right], \tag{3.31}$$

where $\mathfrak{e}_{(a,b)}$ *is as defined in section 3.2.1, maps*

$$X_\Sigma^b = X_{\Sigma^\circ}^u \mapsto \sum s^{\mathrm{neg}(R)} t^{\mathrm{sink}(R)},$$

where the sum runs over the regions for $H[\Sigma^\circ]$.

Consequences for Ordinary Graphs

For an ordinary graph, Γ, considered as a signed graph, $+\Gamma$, we can compose $e_k \mapsto \sum_{a+2b=k} \mathfrak{e}_{(a,b)}$ (which sends X_Γ to $X_{+\Gamma}^b$) with the map (3.31) to obtain a much simpler result:

Proposition 3.4.6 *The algebra map induced by*

$$e_k \mapsto (1+s^k)t + (s+s^2+\cdots+s^{k-1})t^2$$

sends X_Γ *to* $\sum s^{\mathrm{neg}(R)} t^{\mathrm{sink}(R)}$, *where the sum runs over the regions for* $H[+\Gamma^\circ]$.

This proposition implies that certain nonnegative combinations of the c_λ are positive (where $X_\Gamma = \sum c_\lambda e_\lambda$). If we set $s = 0$, we obtain Theorem 1.2.3.

If we set $t = 1$, we obtain that $e_k \mapsto (1 + s + \cdots + s^k)$ sends X_Γ to $\sum s^{\mathrm{neg}(R)}$. It can be seen (using (1.3)) that this map is the same as the algebra map induced by $\omega p_k \mapsto (1 + s^k)$. So it follows from the ωp-positivity of X_Γ that the image has nonnegative coefficients.

However, a curious consequence is that for an e-positive Γ, the coefficients of this generating function, $\sum s^{\mathrm{neg}(R)}$, are unimodal. This is not true in general: for example, the "claw" K_{13} gives

$$8 + 13s + 12s^2 + 13s^3 + 8s^4.$$

There are several possible variations of these ideas. The strongest result of this kind for ordinary graphs seems to be Theorem 1.3.9. Proposition 3.4.6 can be obtained by evaluating the variables in Theorem 1.3.9 at $(1, s, 0, 0, \ldots)$.

We can also get a nice result for the "sign-symmetric" signed graph $\pm\Gamma^\circ$, obtained by replacing each edge of Γ with two edges, one positive and one negative and adding a negative loop at each vertex.

Proposition 3.4.7 *The algebra map induced by*

$$e_k \mapsto t(1 + s)^k \tag{3.32}$$

sends X_Γ *to* $\sum s^{\mathrm{neg}(R)} t^{\mathrm{sink}(R)}$, *where the sum runs over the regions for* $H[\pm\Gamma^\circ]$.

Proof: The proper colorings of $\pm\Gamma^\circ$ are just the zero-free colorings for which taking the absolute values of the colors gives a proper coloring of Γ. It follows that $X^u_{\pm\Gamma^\circ}$ can be obtained from X_Γ via the algebra map induced by $x_i \mapsto (x_i + x_{-i})$, or equivalently, $e_k \mapsto \mathfrak{e}_{(k,0)}$ (see (3.10)). Composing this with (3.31) yields the map (3.32). □

Part II

The Decomposition of Hochschild Cohomology and Gerstenhaber Operations

Chapter 4

Shuffles and Free Lie Algebras

The results in this chapter and the next were obtained in collaboration with Nantel Bergeron, and are taken from the article:

> N. Bergeron and H. L. Wolfgang, The decomposition of Hochschild cohomology and Gerstenhaber operations, *Journal of Pure and Applied Algebra* **104** (1995), pp. 243–265.
>

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4.1 Eulerian Idempotents and the Decomposition of (Co)Homology

In this section, we recall the main results about the idempotents $e_n^{(k)}$ and the decomposition of the Hochschild (co)homology.

As in the introduction, let A be a commutative, unital algebra over a field K of characteristic zero, and M a symmetric A-bimodule. Define $\mathfrak{B}_n A = A \otimes A^{\otimes n}$, where all tensors are taken over K. We will denote $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ by the shorthand, $a_0[a_1, \dots, a_n]$. We have that $\mathfrak{B}_* A$ is a complex with boundary map $\partial = \partial_n : \mathfrak{B}_n A \to \mathfrak{B}_{n-1} A$, given by

$$\begin{aligned}\partial_n(a[a_1,\dots,a_n]) \quad = \quad & aa_1[a_2,\dots,a_n] + \\ & \sum_{i=1}^{n-1}(-1)^i a[a_1,\dots,a_i a_{i+1},\dots,a_n] + \\ & (-1)^n a_n a[a_1,\dots,a_{n-1}].\end{aligned}$$

Note that $\mathfrak{B}_n A$ is a symmetric A-bimodule via multiplication on the left A factor, and ∂ is an A-bimodule map. Since A is commutative and M is a symmetric A-bimodule, it follows that the Hochschild homology $H_*(A, M)$ is the homology of $\mathfrak{B}_* A \otimes_A M$, and $H^*(A, M)$ is the homology of $\mathrm{Hom}_A(\mathfrak{B}_* A, M) \cong \mathrm{Hom}_K(A^{\otimes *}, M)$ (See [1, 13, 15, 31]). We can identify $\mathfrak{B}_n A \otimes_A M$ with $C_n(A, M) = M \otimes A^{\otimes n}$, and $\mathrm{Hom}_A(\mathfrak{B}_* A, M)$ with $C^*(A, M) = \mathrm{Hom}_K(A^{\otimes *}, M)$. Note that $C_0(A, M)$ and $C^0(A, M)$ can be identified with M in a natural way, and $\partial_1 = 0$ implies that $H_0(A, M) \cong H^0(A, M) \cong M$. We will be concerned mostly with the case $M = A$.

Let $\mathfrak{S}_n$ denote the symmetric group on n elements, and let $\mathbb{Q}[\mathfrak{S}_n]$ denote the group algebra. We define a (left) action of $\mathbb{Q}[\mathfrak{S}_n]$ on $A^{\otimes n}$ by letting $\sigma \in \mathfrak{S}_n$ act on $(a_1, a_2, \dots, a_n)$ by $(a_{\sigma_1^{-1}}, a_{\sigma_2^{-1}}, \dots, a_{\sigma_n^{-1}})$. This can be extended to an action on $\mathfrak{B}_n A = A \otimes A^{\otimes n}$ by letting $\mathbb{Q}[\mathfrak{S}_n]$ act on the right factor, i.e. $\sigma(a[a_1, \dots, a_n]) = a[a_{\sigma_1^{-1}}, a_{\sigma_2^{-1}}, \dots, a_{\sigma_n^{-1}}]$.

The *Eulerian idempotents* $e_n^{(k)} \in \mathbb{Q}[\mathfrak{S}_n]$ can be defined in a number of ways. The simplest definition is a generating function due to Garsia [8]:

$$\sum_{k=1}^{n} e_n^{(k)} x^k = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} (x - d(\sigma))(x - d(\sigma) + 1) \cdots (x - d(\sigma) + n - 1) \,\mathrm{sgn}(\sigma) \sigma,$$

where $d(\sigma) = \mathrm{Card}\{i : \sigma_i > \sigma_{i+1}\}$ is the number of *descents* of σ.

In [13, 30] we find that

$$\mathrm{id} = e_n^{(1)} + e_n^{(2)} + \cdots + e_n^{(n)} \tag{4.1}$$
$$e_n^{(i)} e_n^{(j)} = \delta_{ij} e_n^{(i)}, \tag{4.2}$$

where $\delta_{ij} = 0$ if $i \neq j$ and 1 if $i = j$. That is, the $e_n^{(k)}$ are orthogonal idempotents. And moreover,

$$\partial_n e_n^{(k)} = e_{n-1}^{(k)} \partial_n.$$

Combining these properties, we have that

$$\mathfrak{B}_* A = \oplus_k e_*^{(k)} \mathfrak{B}_* A$$

is a decomposition into subcomplexes. This shows

Theorem 4.1.1 ([13, 30])

$$H_n(A, M) = \bigoplus_k H_{k,n-k}(A, M),$$

$$H^n(A, M) = \bigoplus_k H^{k,n-k}(A, M),$$

where

$$\begin{aligned} H_{k,n-k}(A, M) &= e_n^{(k)} H_n(A, M) \cong H_n(M \otimes e_*^{(k)} A^{\otimes *}), \\ H^{k,n-k}(A, M) &= H^n(A, M) e_n^{(k)} \cong H_n(\mathrm{Hom}_K(e_*^{(k)} A^{\otimes *}, M)), \\ H_{0,0}(A, M) &= H_0(A, M), \quad \textit{and} \quad H^{0,0}(A, M) = H^0(A, M). \end{aligned}$$

We follow the notation of Gerstenhaber and Schack [13–15] in indexing the components of the decomposition. This notation differs from that found elsewhere, but it will allow us to state some of our results a little more easily.

For future reference, let us set

$$\begin{aligned} C_{k,n-k}(A,M) &= e_n^{(k)} C_n(A,M) \cong M \otimes e_n^{(k)} A^{\otimes n} \\ C^{k,n-k}(A,M) &= C^n(A,M) e_n^{(k)} \cong \mathrm{Hom}_K(e_n^{(k)} A^{\otimes n}, M) \end{aligned}$$

Note that the action of $\mathbb{Q}[\mathfrak{S}_n]$ on $C^n(A,M)$ is on the right, given by

$$(f\sigma)(a_1,\ldots,a_n) = f(\sigma(a_1,\ldots,a_n)).$$

4.2 Free Lie Algebras

Let $\mathcal{A} = \{a_1, a_2, \ldots, a_f\}$ be a finite alphabet, and let $\mathbb{Q}\langle\mathcal{A}\rangle$ denote the free associative algebra generated by $\mathcal{A}$, i.e. the vector space spanned by *words* with *letters* in $\mathcal{A}$. Then $\mathbb{Q}\langle\mathcal{A}\rangle$ is a Lie algebra under the bracket product defined on words by $[u,v] = uv - vu$. We say that w is a *bracketing of letters* if w is a letter or if $w = [u,v]$ where u and v are bracketings of letters. Let $\mathrm{Lie}\langle\mathcal{A}\rangle$ be the span of these, i.e. the sub-Lie algebra generated by $\mathcal{A}$. Then $\mathrm{Lie}\langle\mathcal{A}\rangle$ can be identified with the free Lie algebra generated by $\mathcal{A}$ and $\mathbb{Q}\langle\mathcal{A}\rangle$ with the enveloping algebra. One version of the Poincaré-Birkhoff-Witt theorem implies that $\mathbb{Q}\langle\mathcal{A}\rangle$ is isomorphic to the symmetric algebra of $\mathrm{Lie}\langle\mathcal{A}\rangle$.

Reutenauer [33] introduces the idempotents $\rho_n^{(k)}$ and shows that they give the projection maps for the natural decomposition of the symmetric algebra of $\mathrm{Lie}\langle\mathcal{A}\rangle$. The version of these results that we will use first appears in Garsia [8] (see also [9]). We begin with some definitions. Let $\mathbb{Q}[\mathcal{A}^n]$ denote the span of words of length n. There is a right action of $\mathbb{Q}[\mathfrak{S}_n]$ on $\mathbb{Q}[\mathcal{A}^n]$ defined on words by $(w_1 w_2 \ldots w_n)\sigma = w_{\sigma_1} w_{\sigma_2} \ldots w_{\sigma_n}$.

Consider the *symmetrized product*

$$(P_1, P_2, \ldots, P_k)^S = \frac{1}{k!} \sum_{\sigma \in S_k} P_{\sigma_1} P_{\sigma_2} \cdots P_{\sigma_k}$$

where $P_i \in \mathbb{Q}\langle\mathcal{A}\rangle$. Let

$$\mathrm{HS}_k = \mathrm{Span}\left\{(P_1, P_2, \ldots, P_k)^S : P_i \in \mathrm{Lie}\langle\mathcal{A}\rangle\right\}$$

and let $\mathrm{HS}_k^n = \mathrm{HS}_k \cap \mathbb{Q}[\mathcal{A}^n]$. One of the main results of [8] is

Theorem 4.2.1 *The idempotents $\rho_n^{(k)} \in \mathbb{Q}[\mathfrak{S}_n]$, defined by*

$$\sum_{k=1}^{n} \rho_n^{(k)} x^k = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} (x - d(\sigma))(x - d(\sigma) + 1) \cdots (x - d(\sigma) + n - 1)\sigma,$$

give projections into HS_k^n*:*

$$\mathbb{Q}[\mathcal{A}^n] = \bigoplus_{k\geq 0} \mathrm{HS}_k^n$$
$$\mathbb{Q}[\mathcal{A}^n]\rho_n^{(k)} = \mathrm{HS}_k^n$$

As mentioned in the introduction, it is known (see [9]) that the idempotents $\rho_n^{(k)}$ and $e_n^{(k)}$ are mapped to each other by the automorphism $\theta : \mathbb{Q}[\mathfrak{S}_n] \to \mathbb{Q}[\mathfrak{S}_n]$ induced by $\sigma \mapsto (-1)^\sigma \sigma$.

For convenience, let us set $\mathrm{HS}_{\leq j} = \oplus_{k\leq j} \mathrm{HS}_k$. We will need the following well known result.

Lemma 4.2.2 *For* $P_i \in \mathrm{Lie}\langle\mathcal{A}\rangle$,

$$P_1 P_2 \cdots P_k = (P_1, P_2, \ldots, P_k)^S + E$$

with $E \in \mathrm{HS}_{\leq k-1}$.

Of particular importance to us is the case where $\mathcal{A} = \{1, 2, \ldots, m\}$. Then, for $n \leq m$, a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ can be considered as a word in $\mathbb{Q}\langle\mathcal{A}\rangle$. So we can consider $\mathbb{Q}[\mathfrak{S}_n]$ as a subspace of $\mathbb{Q}\langle\mathcal{A}\rangle$. Then define

$$\mathrm{SHS}_k^n = \mathbb{Q}[\mathfrak{S}_n] \cap \mathrm{HS}_k^n\,.$$

Note that this is independent of m. The right action is such that $(\sigma_1\sigma_2\ldots\sigma_n)\tau = \sigma\circ\tau$, i.e. the right action is just right multiplication. We then have immediately,

Corollary 4.2.3

$$\begin{aligned}\mathbb{Q}[\mathfrak{S}_n]\rho_n^{(k)} &= \mathrm{SHS}_k^n\\ \mathbb{Q}[\mathfrak{S}_n]e_n^{(k)} &= \theta(\mathrm{SHS}_k^n)\end{aligned}$$

Now, $\mathbb{Q}[\mathfrak{S}_n]$ acts on the right of the complex for Hochschild cohomology, so we have an alternative expression for the decomposition:

$$C^{k,n-k}(A,M) = C^n(A,M)e_n^{(k)} = C^n(A,M)\mathbb{Q}[\mathfrak{S}_n]e_n^{(k)} = C^n(A,M)\theta(\mathrm{SHS}_k^n). \tag{4.3}$$

4.3 Generalized Harrison Homology

In this section we study more closely the components of the decomposition in Theorem 4.1.1. It is known [13] that the components $H^{1,n-1}(A,M)$ and $H_{1,n-1}(A,M)$ are the Harrison (co)homology groups of A with coefficients in M. We will generalize this construction to describe all the components of the decomposition. This section is taken from Wolfgang [50]. Theorem 4.3.6 has been independently obtained by Ronco [35] and by Sletsjøe [38].

We define a *composition* of n as a k-tuple of positive integers, $p = (p_1, p_2, \dots, p_k)$, such that $p_1 + p_2 + \dots + p_k = n$. We refer to k as the number of *parts* of p, and we denote this number by $\ell(p)$. We will use the shorthand $p \models n$ for "p is a composition of n." For $\sigma \in \mathfrak{S}_n$, we define the *descent set* of σ as $D(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$, and for $p \models n$, we define $S(p) = \{p_1, p_1 + p_2, \dots, p_1 + p_2 + \dots + p_{k-1}\}$. Note that if $D(\sigma) \subseteq S(p)$, then $\sigma(1) < \sigma(2) < \dots < \sigma(p_1)$ and $\sigma(p_1 + 1) < \dots < \sigma(p_1 + p_2)$ and so on. Let $X_p = \sum_{D(\sigma)\subseteq S(p)} \sigma$ and $\widetilde{X}_p = \sum_{D(\sigma)\subseteq S(p)} \operatorname{sgn}(\sigma)\sigma$.

When $\mathbb{Q}[\mathfrak{S}_n]$ acts on the left of $A^{\otimes n}$ as in Section 4.1, $\widetilde{X}_p$ corresponds to the usual *signed shuffle* operation, i.e.

$$\widetilde{X}_p(a_1, a_2, \dots, a_n) = (a_1, \dots, a_{p_1}) \widetilde{\sqcup\!\sqcup} (a_{p_1+1}, \dots, a_{p_1+p_2}) \widetilde{\sqcup\!\sqcup} \cdots \widetilde{\sqcup\!\sqcup} (a_{p_1+\dots+p_{k-1}+1}, \dots, a_n).$$

Example 4.3.1

$$\begin{aligned}\widetilde{X}_{(2,2)}(a,b,c,d) &= (a,b,c,d) - (a,c,b,d) + (a,c,d,b) \\ &\quad +(c,a,b,d) - (c,a,d,b) + (c,d,a,b).\end{aligned}$$

Let us write

$$\mathrm{TA} = K \oplus A \oplus A^{\otimes 2} \oplus \dots \oplus A^{\otimes n} \oplus \cdots$$

and set $\mathrm{TA}_n = A^{\otimes n}$, $\mathrm{TA}_n^{(k)} = e_n^{(k)} A^{\otimes n}$, and $\mathrm{TA}^{(k)} = \oplus_n \mathrm{TA}_n^{(k)}$. Then $\widetilde{\sqcup\!\sqcup}$ defines a graded commutative product on TA, i.e. for $w \in A^{\otimes m}, v \in A^{\otimes n}$, $w \widetilde{\sqcup\!\sqcup} v = (-1)^{mn} v \widetilde{\sqcup\!\sqcup} w$. We can extend this to an operation on $A \otimes \mathrm{TA} = \mathfrak{B}_* A$ by

$$a[a_1, \dots, a_n] \widetilde{\sqcup\!\sqcup} b[b_1, \dots, b_m] = ab[(a_1, \dots, a_n) \widetilde{\sqcup\!\sqcup} (b_1, \dots, b_m)]. \tag{4.4}$$

We denote by Sh_n^k the span over $\mathbb{Q}$ of the elements $\widetilde{X}_p \sigma$ such that $\ell(p) = k$ and $\sigma \in \mathfrak{S}_n$. That is

$$\mathrm{Sh}_n^k = \mathbb{Q}[\widetilde{X}_p \sigma : \ell(p) = k,\ \sigma \in \mathfrak{S}_n] \subseteq \mathbb{Q}[\mathfrak{S}_n].$$

We will refer to the elements of Sh_n^k as k-shuffles. Note that $\mathrm{Sh}_n^1 = \mathbb{Q}[\mathfrak{S}_n]$. If we write IA for the *augmentation ideal*,

$$\mathrm{IA} = A \oplus A^{\otimes 2} \oplus \dots \oplus A^{\otimes n} \oplus \cdots,$$

then $\mathrm{Sh}_*^k(\mathrm{TA}) = (\mathrm{IA})^{\widetilde{\sqcup\!\sqcup} k}$, i.e. $\mathrm{Sh}_n^k A^{\otimes n} = (\mathrm{IA})^{\widetilde{\sqcup\!\sqcup} k} \cap A^{\otimes n}$. And similarly,

$$\mathrm{Sh}_*^k(\mathfrak{B}_* A) = A \otimes \mathrm{Sh}_*^k(\mathrm{TA}) = (A \otimes \mathrm{IA})^{\widetilde{\sqcup\!\sqcup} k}.$$

We note that

$$\mathrm{Sh}_n^{l+1} \subseteq \mathrm{Sh}_n^l$$

since an $(l+1)$-shuffle can be expanded as a linear combination of l-shuffles. (Here we are using the fact that we defined the Sh_n^k to be right ideals in $\mathbb{Q}[\mathfrak{S}_n]$.) Moreover, we have the interesting fact that the map ∂ is a derivation for the signed shuffles.

Proposition 4.3.2 (see [31])

$$\begin{aligned}\partial(a[a_1,\dots,a_n]\,\widetilde{\text{ш}}\,b[b_1,\dots,b_m]) &= \partial(a[a_1,\dots,a_n])\,\widetilde{\text{ш}}\,(b[b_1,\dots,b_m]) + \\ &\quad (-1)^n(a[a_1,\dots,a_n])\,\widetilde{\text{ш}}\,\partial(b[b_1,\dots,b_m]).\end{aligned}$$

Together with the fact that $\partial(a[a_1]) = 0$, this implies that

$$\partial\,\mathrm{Sh}_n^k(A) \subseteq \mathrm{Sh}_{n-1}^k(A),$$

where $\mathrm{Sh}_n^k(A) = \mathrm{Sh}_n^k\,\mathfrak{B}_n A$. Hence, the $\mathrm{Sh}_*^k(A)$ give a filtration of $\mathfrak{B}_* A$ by subcomplexes,

$$\mathfrak{B}_* A = \mathrm{Sh}_*^1(A) \supseteq \mathrm{Sh}_*^2(A) \supseteq \mathrm{Sh}_*^3(A) \supseteq \cdots .$$

It follows that

$$\begin{aligned} M \otimes \mathrm{Sh}_*^k A^{\otimes *} &\cong \mathrm{Sh}_*^k(A) \otimes_A M \\ \text{and } \mathrm{Hom}_K(\mathrm{Sh}_*^k A^{\otimes *}, M) &\cong \mathrm{Hom}_A(\mathrm{Sh}_*^k(A), M)\end{aligned}$$

give filtrations of $C_*(A, M)$ and $C^*(A, M)$ by subcomplexes.

Harrison homology is defined to be the homology of the complex $\mathfrak{B}_* A/\mathrm{Sh}_*^2(A)$, and Harrison cohomology is the homology of the complex of cochains vanishing on $\mathrm{Sh}_*^2 A^{\otimes *}$. Barr [1] shows that these are summands of Hochschild (co)homology by showing that (in the notation used here) $e_n^{(2)} + \cdots + e_n^{(n)}$ is an idempotent projecting $\mathfrak{B}_* A$ onto $\mathrm{Sh}_*^2(A)$. It follows that

$$\mathfrak{B}_n A = e_n^{(1)} \mathfrak{B}_n A \oplus \mathrm{Sh}_n^2(A) \tag{4.5}$$

and hence there is a natural isomorphism between Harrison (co)homology and the first components of the decomposition of Hochschild (co)homology.

Proposition 4.3.3 *As right ideals of* $\mathbb{Q}[\mathfrak{S}_n]$,

$$\mathrm{Sh}_n^{k+1} = \ker\left(\sum_{r=1}^{k} e_n^{(r)}\right). \tag{4.6}$$

Proof: The following lemma is a special case of a result in [9].

Lemma 4.3.4 ([9]) *If* $p \models n$, *and* $\ell(p) > r$ *then* $\rho_n^{(r)} X_p = 0$.

Applying θ, this gives us that $e_n^{(r)} \widetilde{X}_p = 0$ if $\ell(p) = k+1$ and $r \leq k$. Hence

$$\mathrm{Sh}_n^{k+1} \subseteq \ker\left(\sum_{r=1}^{k} e_n^{(r)}\right).$$

Now we note that

$$\ker\left(\sum_{r=1}^{k} e_n^{(r)}\right) = \operatorname{Im}\left(\sum_{s=k+1}^{n} e_n^{(s)}\right)$$

since the $e_n^{(r)}$ are orthogonal idempotents and $e_n^{(1)} + \cdots + e_n^{(n)} = 1$. So to get equality in (4.6), it suffices to show that $e_n^{(s)} \in \mathrm{Sh}_n^{k+1}$ for $s \geq k+1$. We recall an expression for the $\rho_n^{(k)}$ found in Garsia [8]:

Proposition 4.3.5 *[8]*

$$\rho_n^{(k)} = \sum_{m=k}^{n} (-1)^{m-k} \frac{s(m,k)}{m!} \sum_{\substack{p \models n \\ \ell(p)=k}} X_p,$$

where $s(m,k)$ denotes the Stirling numbers of the first kind.

Applying θ, we obtain $e_n^{(k)}$ as a linear combination of k-shuffles. This proves Proposition 4.3.3. □

Theorem 4.3.6

$$\bigoplus_{r=1}^{k} H_{r,n-r}(A,M) \cong H_n(C_*(A,M)/\mathrm{Sh}_*^{k+1}(A) \otimes_A M) \tag{4.7}$$

$$\bigoplus_{r=1}^{k} H^{r,n-r}(A,M) \cong H_n(C^*(A,M)/\mathrm{Hom}_K(\mathrm{Sh}_*^{k+1} A^{\otimes *}, M)) \tag{4.8}$$

$$H_{k,n-k}(A,M) \cong H_n(\mathrm{Sh}_*^{k}(A) \otimes_A M/\mathrm{Sh}_*^{k+1}(A) \otimes_A M) \tag{4.9}$$

$$H^{k,n-k}(A,M) \cong H_n(\mathrm{Hom}_K(\mathrm{Sh}_*^{k} A^{\otimes *}, M)/\mathrm{Hom}_K(\mathrm{Sh}_*^{k+1} A^{\otimes *}, M)) \tag{4.10}$$

Proof: From Proposition 4.3.3, we have

$$\left(\sum_{r=1}^{k} e_n^{(r)}\right) C_*(A,M) \cong C_*(A,M)/\mathrm{Sh}_*^{k+1}(A) \otimes_A M,$$

$$\text{and } C^*(A,M)\left(\sum_{r=1}^{k} e_n^{(r)}\right) \cong C^*(A,M)/\mathrm{Hom}_K(\mathrm{Sh}_*^{k+1} A^{\otimes *}, M).$$

Taking the homology, we obtain (4.7) and (4.8). The expressions for the individual components, (4.9) and (4.10), follow from the orthogonality of the $e_n^{(k)}$. □

4.3.7 Remark: Loday [30] shows that the decomposition of Theorem 4.1.1 is valid for any functor $\Delta^{op} \to K$-**Module** which factors through the category **Fin′** of the sets $[n] = \{0, 1, 2, \ldots, n\}$ with morphisms $f\colon [n] \to [m]$ such that $f(0) = 0$. Theorem 4.3.6 relies only on the identity (4.6). If we let $\mathbb{Q}[\mathbf{Fin}']$ be the algebra of morphisms of **Fin′**, the identity (4.6) was shown inside $\mathbb{Q}[\mathfrak{S}_n] \subseteq \mathbb{Q}[\mathbf{Fin}']$. Hence Theorem 4.3.6 is also valid for any functor $\Delta^{op} \to K$-**Module** which factors through the category **Fin′**.

4.4 The Dual Poincaré-Birkhoff-Witt Theorem

In this section, we study the shuffle powers of $\mathrm{TA}^{(1)} = e_*^{(1)}\,\mathrm{TA}$ and $e_*^{(1)}\mathfrak{B}_*A$. In particular, we will relate the idempotents $e_n^{(k)}$ to a dual, graded version of the Poincaré-Birkhoff-Witt theorem due to Hain [23].

To shorten the notation, let us write C_n for $C_n(A, A) = \mathfrak{B}_nA$ and $C_{(k)}$ for $\oplus_n e_n^{(k)} C_n$. Note that $H_* = H_*(A, A)$ is the homology of C_*.

Gerstenhaber and Schack [15] show that the shuffle product on C_* is bigraded in the sense that

$$C_{j,m-j} \,\widetilde{\text{ш}}\, C_{k,n-k} \subseteq C_{j+k,m+n-j-k}. \tag{4.11}$$

They obtain this from the fact that $s_n^{(2)} = \sum_{i=1}^n 2^i e_n^{(i)}$ is an algebra map on TA. The same proof shows that $\mathrm{TA}_n^{(i)} \,\widetilde{\text{ш}}\, \mathrm{TA}_m^{(j)} \subseteq \mathrm{TA}_{n+m}^{(i+j)}$. In particular,

$$(\mathrm{TA}^{(1)})^{\widetilde{\text{ш}}k} \subseteq \mathrm{TA}^{(k)}. \tag{4.12}$$

We will show that these spaces are actually equal.

We have the following generalization of (4.5):

Proposition 4.4.1

$$\mathrm{Sh}_*^k\,\mathrm{TA} = (\mathrm{TA}^{(1)})^{\widetilde{\text{ш}}k} \oplus \mathrm{Sh}_*^{k+1}\,\mathrm{TA} \tag{4.13}$$

$$\mathrm{Sh}_*^k(A) = (C_{(1)})^{\widetilde{\text{ш}}k} \oplus \mathrm{Sh}_*^{k+1}(A) \tag{4.14}$$

Proof: For $k = 1$, both equations follow from Proposition 4.3.3. Note that in this case, (4.14) is just (4.5). For the general case, $\mathrm{Sh}_*^k\,\mathrm{TA}$ is spanned by elements of the form $w = v_1 \,\widetilde{\text{ш}} \cdots \widetilde{\text{ш}}\, v_k$, with $v_i \in \mathrm{IA}$. From the $k = 1$ case, we can write $v_i = v_i' + v_i''$, with $v_i' \in \mathrm{TA}^{(1)}$ and $v_i'' \in \mathrm{Sh}_*^2\,\mathrm{TA}$. Expanding w in terms of these yields $v_1' \,\widetilde{\text{ш}} \cdots \widetilde{\text{ш}}\, v_k' + ((k+1)\text{-shuffles})$. To see that the sum is direct, note that, by (4.12), $e_n^{(k)}$ fixes the elements of $(\mathrm{TA}^{(1)})^{\widetilde{\text{ш}}k}$, and by Proposition 4.3.3, $e_n^{(k)}$ vanishes on $\mathrm{Sh}_*^{k+1}\,\mathrm{TA}$. The second equation follows by tensoring the first with A. □

The comments in the above proof about $e_n^{(k)}$, together with Proposition 4.3.3, give us the following theorem.

Theorem 4.4.2

$$\mathrm{TA}^{(k)} = (\mathrm{TA}^{(1)})^{\widetilde{\sqcup\!\sqcup} k} \tag{4.15}$$

$$C_{(k)} = (C_{(1)})^{\widetilde{\sqcup\!\sqcup} k}. \tag{4.16}$$

It is also possible to obtain $\mathrm{TA}^{(k)} \subseteq (\mathrm{TA}^{(1)})^{\widetilde{\sqcup\!\sqcup} k}$ by considering the construction of the $e_n^{(k)}$ in Loday [31]. With a coproduct induced by "deconcatenation", TA becomes a graded commutative bialgebra, and so there is a convolution product on the endomorphisms of TA. Loday defines $e_*^{(k)}$ as the k-th convolution power of $e_*^{(1)}$ (up to a constant). Writing out the definitions explicitly, one obtains that $e_*^{(k)}(a_1, \ldots, a_n)$ is given by

$$\frac{1}{k!} \sum_{1 \le i_1 < i_2 < \cdots < i_{k-1} < n} e_*^{(1)}(a_1, \ldots, a_{i_1}) \widetilde{\sqcup\!\sqcup}\, e_*^{(1)}(a_{i_1+1}, \ldots, a_{i_2}) \widetilde{\sqcup\!\sqcup} \cdots \widetilde{\sqcup\!\sqcup}\, e_*^{(1)}(a_{i_{k-1}+1}, \ldots, a_n).$$

For any graded vector space $V = V_1 \oplus V_2 \oplus \cdots$, the *graded symmetric algebra* of V is defined to be

$$\Lambda V = S(V_{\text{even}}) \otimes E(V_{\text{odd}}),$$

where S and E give the usual (ungraded) symmetric and exterior algebras, and V_{even} and V_{odd} have the obvious meanings. Then ΛV is the free graded-commutative algebra generated by V. Note that ΛV possesses two natural gradings, one coming from the number of factors and one coming from the sum of the V-degrees of the factors.

It is clear that the shuffle product induces a well defined map $\Lambda(\mathrm{TA}^{(1)}) \to \mathrm{TA}$. In [23], Hain constructs the idempotent $e_n^{(1)}$, and shows the following remarkable result.

Proposition 4.4.3 ([23]) *The shuffle product induces an isomorphism*

$$\Lambda(\mathrm{TA}^{(1)}) \cong \mathrm{TA}\,.$$

He presents this as a dual, graded version of the Poincaré-Birkhoff-Witt theorem. If A is finite dimensional, this follows by dualizing the usual PBW theorem.

Note that it follows from (4.15) that this map is onto.

$\Lambda(\mathrm{TA}^{(1)})$ has a direct sum decomposition whose k-th term is spanned by products of k elements of $\mathrm{TA}^{(1)}$. It follows from (4.15) that the shuffle product maps this k-th term to $\mathrm{TA}^{(k)} = e_*^{(k)}\,\mathrm{TA}$. So the idempotents $e_n^{(k)}$ bear the same relationship to this dual, graded PBW theorem as the $\rho_n^{(k)}$ bear to the PBW theorem discussed in Section 4.2.

The shuffle product on C_* induces a shuffle product on the homology $H_* = H_*(A, A)$. It follows from (4.11) that this product is bigraded. Let us write $H_{(1)}$ for the homology of $C_{(1)}$, i.e. $H_{(1)}$ is the sum of the Harrison homology groups. Then the shuffle product induces a map

$$\Lambda(H_{(1)}) \to H_*. \tag{4.17}$$

This map, however, might be neither injective nor surjective. In fact, we will see in Section 5.3 that for the dual numbers, the shuffle product of any two elements of the Harrison homology is zero.

4.4.4 Remark: The statements above about the decomposition of TA rely only on the vector space structure of A. In particular, these comments are valid in the context of Hain [23], i.e. bar constructions on CDG (commutative differential graded) algebras. Hain shows that the map in Proposition 4.4.3 is an isomorphism of DG algebras. So the comments above yield a Hodge-type decomposition of the homology of such a bar construction, which generalizes the results of [23]. Note that, in contrast to what happens for Hochschild homology, Hain shows that the analog of (4.17) is an isomorphism.

Chapter 5

Gerstenhaber Operations

5.1 Gerstenhaber operations on $H^*(A, A)$ and ideals

We now focus our attention on $H^*(A, A)$ for A a commutative algebra. In this section we construct ideals for the cup product and Lie bracket on $H^*(A, A)$. Recall that $H^* = H^*(A, A)$ is the homology of $C^* = C^*(A, A) = \mathrm{Hom}_K(A^{\otimes *}, A)$. Let us write $f \sim g$ when f and g differ by a coboundary. For $f^n \in C^n$ and $g^m \in C^m$, define $f^n \cup g^m \in C^{n+m}$ by

$$f^n \cup g^m(a_1, \ldots, a_{n+m}) = f^n(a_1, \ldots, a_n)g^m(a_{n+1}, \ldots, a_{n+m}).$$

The important properties of this product are that it induces a product on the cohomology, and that the induced product is graded commutative:

$$f^n \cup g^m \sim (-1)^{nm} g^m \cup f^n, \tag{5.1}$$

as was shown by Gerstenhaber [11].

Gerstenhaber [11] defines, for $f^n \in C^n$ and $g^m \in C^m$ a *composition product* $f^n \bar{\circ} g^m \in C^{n+m-1}$ as follows: For $i = 1, \ldots, n$, let

$$(f^n \circ_i g^m)(a_1, \ldots, a_{n+m-1}) = f^n(a_1, \ldots, a_{i-1}, g^m(a_i, \ldots, a_{i+m-1}), a_{i+m}, \ldots, a_{n+m-1}).$$

If $m = 0$, the above definition holds, with $g^m()$ interpreted as a fixed element of A, and if $n = 0$, $f^n \circ_i g^m$ is defined to be 0. Then let $f^n \bar{\circ} g^m = \sum_{i=1}^{n} (-1)^{(i-1)(m-1)} f^n \circ_i g^m$. As Gerstenhaber points out, if f and g are cocycles, then $f \bar{\circ} g$ need not be a cocycle. However, defining $[f^n, g^m] = f^n \bar{\circ} g^m - (-1)^{(n-1)(m-1)} g^m \bar{\circ} f^n$ yields a well-defined graded Lie product on the cohomology. Note that the grading is by degree, which is the dimension -1, i.e.

$$[f^n, g^m] \sim -(-1)^{(n-1)(m-1)}[g^m, f^n].$$

Let us write $C^{(k)}$ for $\oplus_n C^{k, n-k}(A, A)$ and similarly for $H^{(k)}$. In [13], Gerstenhaber and Schack the behavior of the decomposition under the cup and bracket products. For the

Harrison components, they obtain

$$[C^{(1)}, C^{(1)}] \subseteq C^{(1)} \text{ and hence } [H^{(1)}, H^{(1)}] \subseteq H^{(1)} \tag{5.2}$$
$$(H^{(1)})^{\cup k} \subseteq H^{(k)}, \tag{5.3}$$

leading them to ask if these operations are graded with respect to the decomposition, i.e. does one have $H^{(j)} \cup H^{(k)} \subseteq H^{(j+k)}$ and $[H^{(j)}, H^{(k)}] \subseteq H^{(j+k-1)}$? They also show that $\oplus_{r\geq 1} H^{*,r}$ is an ideal of H^* for the cup product by exhibiting it as the kernel of a natural map $H^*(A,A) \to H^*_{\mathrm{CE}}(A,A)$, the codomain being the Lie algebra cohomology of A considered as a trivial Lie algebra.

In [14, 15], they conjecture that the cup product is not generally graded with respect to the decomposition, but that $\mathcal{F}_q = \oplus_{r\geq q} H^{*,r}$ form a decreasing filtration of H^* by ideals, possibly with $\mathcal{F}_p \cup \mathcal{F}_q \subseteq \mathcal{F}_{p+q}$.

We will show $H^{(j)} \cup H^{(k)} \subseteq H^{(\leq j+k)}$ and $[H^{(j)}, H^{(k)}] \subseteq H^{(\leq j+k-1)}$, where $H^{(\leq k)} = \oplus_{j\leq k} H^{(j)}$. This implies the conjecture above and that furthermore, the $\mathcal{F}_q$ are ideals for the Lie bracket and $[\mathcal{F}_p, \mathcal{F}_q] \subseteq \mathcal{F}_{p+q}$. In Section 5.3 we will give an example where neither product is graded.

Assume for the moment that $n, m \geq 1$. We will deal with the behavior of C^0 separately in Proposition 5.1.16. As in Section 4.2, we can regard $\mathbb{Q}[\mathfrak{S}_n]$ and $\mathbb{Q}[\mathfrak{S}_m]$ as subspaces of $\mathbb{Q}\langle\mathcal{A}\rangle$ for the alphabet $\mathcal{A} = \{1, 2, \ldots, n+m\}$. Recall from (4.3) that

$$C^{k,n-k} = C^n\theta(\mathrm{SHS}^n_k),$$

where SHS^n_k is spanned by those symmetrized products of k elements of $\mathrm{Lie}\langle\mathcal{A}\rangle$ which lie in $\mathbb{Q}[\mathfrak{S}_n] \subseteq \mathbb{Q}\langle\mathcal{A}\rangle$. Using this characterization of the decomposition, we can study the cup and bracket products in terms of some operations on words.

We begin with the cup product. If $v \in \mathfrak{S}_m$ and $k \leq n$, we define $v{\uparrow}^k$ to be the word obtained from v by adding k to each letter of v, e.g. $312{\uparrow}^4 = 756$. Extend this definition linearly to all of $\mathbb{Q}[\mathfrak{S}_m]$. A straightforward application of the definitions shows

Proposition 5.1.1 *For $f \in C^n$, $g \in C^m$, $w \in \mathbb{Q}[\mathfrak{S}_n]$, $v \in \mathbb{Q}[\mathfrak{S}_m]$,*

$$(f\theta(w)) \cup (g\theta(v)) = (f \cup g)\theta(w.(v{\uparrow}^n)),$$

where $w.(v{\uparrow}^n)$ is the concatenation of the two words.

Modulo the coboundaries, we can obtain a somewhat stronger result.

Lemma 5.1.2

$$f \cup g \sim (f \cup g)\theta(n+1 \ldots n+m\, 1 \ldots n).$$

Proof: This follows from (5.1) and the commutativity of A. □

Corollary 5.1.3 *For $f \in C^n$, $g \in C^m$, $w \in \mathbb{Q}[\mathfrak{S}_n]$, $v \in \mathbb{Q}[\mathfrak{S}_m]$,*

$$(f\theta(w)) \cup (g\theta(v)) \sim (f \cup g)\theta((w, (v{\uparrow}^n))^S),$$

and more generally,

$$(f_1\theta(w_1)) \cup \cdots \cup (f_k\theta(w_k)) \sim (f_1 \cup \cdots \cup f_k)\theta((w_1, w_2{\uparrow}^{|w_1|}, \ldots, w_k{\uparrow}^{|w_1 \ldots w_{k-1}|})^S).$$

Proposition 5.1.4 *If $f \in C^{(j)}, g \in C^{(k)}$, then $f \cup g \in C^{(\leq j+k)}$.*

Proof: Since $C^{(j)} = \oplus_n C^n\theta(\mathrm{SHS}_j^n)$, it suffices to take $f = f'\theta(w)$, $g = g'\theta(v)$ where $w \in \mathrm{SHS}_j^n$, $v \in \mathrm{SHS}_k^m$. Note that $v{\uparrow}^n \in \mathrm{HS}_k$, and thus $w.(v{\uparrow}^n) \in \mathrm{HS}_{\leq j+k}$ by Lemma 4.2.2. □

Similarly, Corollary 5.1.3 can be used to give a new proof of (5.3).

We now turn to the composition and bracket products. For $w \in \mathfrak{S}_n$ and $v \in \mathfrak{S}_m$, we define $\Phi_j(w,v) \in \mathfrak{S}_{n+m-1}$ to be the word obtained from w by substituting the following for the letters of w:

$$\begin{array}{ccccccc} 1 & \ldots & j-1 & j & j+1 & \ldots & n \\ 1 & \ldots & j-1 & v{\uparrow}^{(j-1)} & j+m & \ldots & n+m-1 \end{array} \tag{5.4}$$

Extend this definition bilinearly to allow $w \in \mathbb{Q}[\mathfrak{S}_n]$ and $v \in \mathbb{Q}[\mathfrak{S}_m]$.

Example 5.1.5

$$\Phi_2(3124, 21) = 41325$$

An elementary computation gives

Lemma 5.1.6 *For $w \in \mathfrak{S}_n$ and $v \in \mathfrak{S}_m$, the sign of $\Phi_j(w,v) \in \mathfrak{S}_{n+m-1}$ is given by*

$$(-1)^w(-1)^v(-1)^{(w_j^{-1}-1)(m-1)}(-1)^{(j-1)(m-1)}.$$

Proposition 5.1.7 *For $f \in C^n$, $g \in C^m$, $w \in \mathbb{Q}[\mathfrak{S}_n]$, $v \in \mathbb{Q}[\mathfrak{S}_m]$,*

$$(f\theta(w))\bar{\circ}(g\theta(v)) = \sum_j (-1)^{(j-1)(m-1)}(f \circ_j g)\theta(\Phi_j(w,v)).$$

Proof: By linearity, it suffices to consider $w \in \mathfrak{S}_n, v \in \mathfrak{S}_m$. First consider

$$(fw \circ_k gv)(a_1, \ldots, a_{n+m-1}) = fw(a_1, \ldots, a_{k-1}, gv(a_k, \ldots, a_{k+m-1}), a_{k+m}, \ldots, a_{n+m-1}).$$

If $j = w_k$, then the result of the actions of w and v will yield

$$f(b_1, \ldots, b_{j-1}, g(b_j, \ldots, b_{j+m-1}), b_{j+m}, \ldots, b_{n+m-1}),$$

where $(b_1, \ldots, b_{n+m-1})$ is some permutation of $(a_1, \ldots, a_{n+m-1})$. If we group the variables together into blocks according to the arguments of f,

$$\underbrace{b_1}_{1}, \ldots, \underbrace{b_{j-1}}_{j-1}, \underbrace{b_j, \ldots, b_{j+m-1}}_{j}, \underbrace{b_{j+m}}_{j+1}, \ldots, \underbrace{b_{n+m-1}}_{n},$$

then the numbers $w_1, \ldots, \widehat{j}, \ldots, w_n$ (where $j = w_k$ is omitted) give the blocks where $a_1, \ldots, a_{k-1}, a_{k+m}, \ldots, a_{n+m-1}$ appear, respectively. The variables $a_k, \ldots, a_{k+m-1}$ appear in the block j, with positions governed by v. The substitution (5.4) comes from relating the block numbers to the indices of b_i. This shows that $fw \circ_k gv = (f \circ_j g)\Phi_j(w, v)$. Applying the signs of the permutations from Lemma 5.1.6, we obtain

$$\begin{aligned} (f\theta(w))\bar{\circ}(g\theta(v)) &= \sum_j (-1)^{(w_j^{-1}-1)(m-1)}(-1)^w(-1)^v(fw) \circ_{w_j^{-1}} (gv) \\ &= \sum_j (-1)^{(j-1)(m-1)}(f \circ_j g)\theta(\Phi_j(w, v)). \end{aligned}$$

□

Lemma 5.1.8

$$[\mathrm{HS}_k, \mathrm{Lie}\langle\mathcal{A}\rangle] \subseteq \mathrm{HS}_k$$

Proof: It is not difficult to show that

$$[(P_1, P_2, \ldots, P_k)^S, Q] = \sum_{i=1}^{k} (P_1, \ldots, [P_i, Q], \ldots, P_k)^S.$$

The result follows immediately. □

Lemma 5.1.9 *If $w \in \mathrm{SHS}_1^n$ and $v \in \mathrm{SHS}_k^m$, then $\Phi_j(w, v) \in \mathrm{SHS}_k^{n+m-1}$.*

Proof: $w \in \mathrm{Lie}\langle\mathcal{A}\rangle$, so we can assume WLOG that w is a bracketing of letters. The letter j appears only once in this bracket expression, so w can be obtained by starting with j and successively bracketing with elements of $\mathrm{Lie}\langle\mathcal{A}\rangle$ not containing j. The substitution (5.4) replaces j with an element of HS_k and these Lie elements with other Lie elements. So the result follows from Lemma 5.1.8 by induction. □

Lemma 5.1.10 *If $w \in \mathrm{SHS}_k^n$ and $v \in \mathrm{SHS}_\ell^m$, then $\Phi_j(w, v) \in \mathrm{SHS}_{\le k+\ell-1}^{n+m-1}$.*

Proof: It suffices to consider the case $w = (P_1, P_2, \ldots, P_k)^S$ and $v = (Q_1, Q_2, \ldots, Q_\ell)^S$, where $P_i, Q_i \in \mathrm{Lie}\langle\mathcal{A}\rangle$. Note that the letters appearing in the different P_i are disjoint. In particular, the letter j appears in only one of the P_i. WLOG, say j appears in P_1. We can write

$$\Phi_j(w, v) = (\widetilde{P}_1, \widetilde{P}_2, \dots, \widetilde{P}_k)^S$$

where $\widetilde{P}_i$ is the result of applying the substitution (5.4) to P_i. Now, $\widetilde{P}_2, \dots, \widetilde{P}_k \in \mathrm{Lie}\langle\mathcal{A}\rangle$ since, in these, each letter has been replaced by another letter. By the argument in Lemma 5.1.9, $\widetilde{P}_1 \in \mathrm{HS}_\ell$. So $\Phi_j(w, v)$ is a linear combination of products of $(k + \ell - 1)$ elements of $\mathrm{Lie}\langle\mathcal{A}\rangle$. Then Lemma 4.2.2 implies that it is in $\mathrm{HS}_{\leq k+\ell-1}$. □

Lemma 5.1.11 *If $w \in \mathrm{SHS}_k^n$ and $v \in \mathrm{SHS}_1^m$, then $\Phi_j(w, v) \in \mathrm{SHS}_k^{n+m-1}$.*

Proof: When we take $\ell = 1$ in the above proof, we get $\widetilde{P}_1 \in \mathrm{HS}_1 = \mathrm{Lie}\langle\mathcal{A}\rangle$. Then $\Phi_j(w, v) \in \mathrm{HS}_k$. □

Taking Proposition 5.1.7 and Lemma 5.1.10 yields

Theorem 5.1.12

$$\begin{aligned} C^{(j)}\bar{\circ}C^{(k)},\ [C^{(j)}, C^{(k)}] &\subseteq C^{(\leq j+k-1)} \\ \textit{and hence} \quad [H^{(j)}, H^{(k)}] &\subseteq H^{(\leq j+k-1)}. \end{aligned}$$

Similarly, using Lemmas 5.1.9 and 5.1.11 yields stronger results for the Harrison components:

Theorem 5.1.13

$$\begin{aligned} C^{(1)}\bar{\circ}C^{(k)},\ C^{(k)}\bar{\circ}C^{(1)},\ [C^{(1)}, C^{(k)}] &\subseteq C^{(k)} \\ \textit{and hence} \quad [H^{(1)}, H^{(k)}] &\subseteq H^{(k)} \end{aligned}$$

5.1.14 Remark: A K-linear map $\mu : A \otimes A \to A$ defines an associative product on A if and only if $[\mu, \mu] = 2\mu\bar{\circ}\mu = 0$. And μ is commutative if and only if μ is a Harrison cochain. Gerstenhaber [11] shows that the coboundary map can be expressed as $\delta f = [f, -\mu]$, where μ is the algebra product of A. So Theorem 5.1.13 can be viewed as a generalization of the fact that the subspaces $C^{(k)}$ are actually subcomplexes.

5.1.15 Remark: The fact (5.2) that $C^{(1)}$ and $H^{(1)}$ are closed under the graded Lie bracket shows that C^* and H^* are graded Lie modules over $C^{(1)}$ and $H^{(1)}$, respectively. Theorem 5.1.13 shows that the decompositions $C^* = \oplus C^{(k)}$ and $H^* = \oplus H^{(k)}$ are direct sums of $C^{(1)}$-modules and $H^{(1)}$-modules.

For the sake of completeness, we consider also the products involving $H^0 = C^0$:

Proposition 5.1.16 *For $f \in C^{(k)}, g \in C^0$, $f \cup g \in C^{(k)}$, and $[f, g] \in C^{(k-1)}$.*

Proof: Recall that $C^0 \cong A$. Taking the cup product with g is just multiplication by an element of A, and it is clear that the decomposition of C^* is a decomposition into A-modules.

The bracket product is a little more interesting. Take, by convention, $\Phi_j(w, \emptyset)$ to be the word obtained from w by substituting for the letters of w:

$$\begin{array}{cccccccc} 1 & \dots & j-1 & j & j+1 & \dots & n \\ 1 & \dots & j-1 & \text{(omit)} & j & \dots & n-1 \end{array}$$

Then the proof Proposition 5.1.7 is still valid. Here $\emptyset$ is the "empty word", i.e. the identity of the concatenation algebra $\mathbb{Q}\langle\mathcal{A}\rangle$. For $w \in \text{Lie}\langle\mathcal{A}\rangle$, "omitting" a letter from w yields $\emptyset$ if $|w| = 1$, and 0 otherwise. E.g. $ab - ba \mapsto a - a = 0$. So if $w \in \text{SHS}^n_k$, then $\Phi_j(w, \emptyset) \in \text{SHS}^{n-1}_{k-1}$. Since $[f, g] = f\bar{\circ}g$ for $g \in C^0$, this yields the result. □

Corollary 5.1.17 *$\mathcal{F}_p = \oplus_{r \geq p} H^{*,r}$ are ideals for the cup and bracket products, with $\mathcal{F}_p \cup \mathcal{F}_q \subseteq \mathcal{F}_{p+q}$ and $[\mathcal{F}_p, \mathcal{F}_q] \subseteq \mathcal{F}_{p+q}$.*

Proof: If $f \in H^{i,r}, g \in H^{j,t}$ with $r \geq p, t \geq q$, then $f \cup g \in \oplus_{k \leq i+j} H^{k, i+r+j+t-k}$, by Proposition 5.1.4. Now for each k in this sum, $k \leq i + j$ implies that $i + r + j + t - k \geq r + t \geq p + q$. So $f \cup g \in \mathcal{F}_{p+q}$. Similarly, Theorem 5.1.12 implies that $[f, g] \in \mathcal{F}_{p+q}$. Taking $q = 0$ shows that $\mathcal{F}_p$ is an ideal for either product. □

5.1.18 Remark: It is possible to obtain many of the results of this section, including Corollary 5.1.17, by considering shuffles instead of symmetrized products of Lie elements, i.e. using the fact that $H^{(\leq k)}$ is the homology of the cochains vanishing on $(k+1)$-shuffles. There are two problems with this approach. The first is that one loses all information about the individual components of $f \cup g$ and $[f, g]$. The second is that it would be difficult to see why the Harrison cochains have stronger properties, e.g. Theorem 5.1.13.

5.1.19 Remark: A recent paper by Sletsjøe [38] incorrectly asserts that the Gerstenhaber operations are graded with respect to the decomposition. Using methods similar to those discussed in the previous remark, he correctly proves the equivalent of Proposition 5.1.4 and Theorem 5.1.12, i.e. if $f \in C^{(j)}$, $g \in C^{(k)}$, then $f \cup g \in C^{(\leq j+k)}$ and $[f, g] \in C^{(\leq j+k-1)}$. He then assumes (incorrectly) that $f \cup g$ and $[f, g]$ are cohomologous to their projections into $C^{(j+k)}$ and $C^{(j+k-1)}$, respectively.

From our point of view, the "error terms" projecting into the smaller components should seem fairly arbitrary. This is because writing a product of k elements of $\text{Lie}\langle\mathcal{A}\rangle$ is terms of the HS_j generally gives nonzero projections into each of $\text{HS}_1, \dots, \text{HS}_k$. It turns out that the error terms are not *completely* arbitrary. In the next section, we will see that the error terms for $f \cup g$ and $[f, g]$ which survive at the level of cohomology are in the components of the same parity as $(j + k)$ and $(j + k - 1)$, respectively. It seems that there are no other restrictions on the error terms, but it is difficult to make this statement precise.

In Section 5.3, we will give specific examples where the error terms do not vanish at the level of cohomology. This shows that the two main theorems of [38] are false.

5.2 $\mathbb{Z}/2\mathbb{Z}$-gradings for Gerstenhaber Operations

In [13], Gerstenhaber and Schack show that if we define $H^{(\text{even})}(A,A) = H^{(0)}(A,A) \oplus H^{(2)}(A,A) \oplus \cdots$, and similarly for $H^{(\text{odd})}$, then this yields a $\mathbb{Z}/2\mathbb{Z}$-grading for the cup product, i.e. $H^{(\text{even})} \cup H^{(\text{even})} \subseteq H^{(\text{even})}$, and so on. In this section, we will generalize this result to give $\mathbb{Z}/2\mathbb{Z}$-gradings for any algebra with an involution. There are analogous shifted gradings for the bracket product.

Suppose A is a (not necessarily commutative) algebra with involution, i.e. there is a linear map $(a \mapsto \overline{a})$ with $\overline{ab} = \overline{b}\,\overline{a}$ and $\overline{\overline{a}} = a$. In particular, a commutative algebra A possesses the trivial involution ($\overline{a} = a$).

Loday [31] defines a decomposition of the Hochschild homology $H_*(A,A)$ into two parts. We will construct an analogous decomposition of the cohomology.

As in the commutative case, the Hochschild cohomology $H^n = H^n(A,A)$ is the homology of the complex $C^n = C^n(A,A) = \operatorname{Hom}_K(A^{\otimes n}, A)$ with boundary given by $\delta f = [f, -\mu]$ where μ is the algebra product of A (See Remark 5.1.14). The definitions and basic properties of the cup and bracket products stated at the beginning of Section 5.1 also hold in the non-commutative case, in particular we still have graded commutativity of the cup product.

We define an operation ω_n on C^n by

$$(f\omega_n)(a_1, \ldots, a_n) = \overline{f(\overline{a_n}, \ldots, \overline{a_1})}.$$

For $f \in C^n, g \in C^m$,

$$\begin{aligned}(f\omega_n) \cup (g\omega_m)(a_1, \ldots, a_{n+m}) &= \overline{f(\overline{a_n}, \ldots, \overline{a_1})}\ \overline{g(\overline{a_{n+m}}, \ldots, \overline{a_{n+1}})} \\ &= \overline{g(\overline{a_{n+m}}, \ldots, \overline{a_{n+1}}) f(\overline{a_n}, \ldots, \overline{a_1})} \\ &= ((g \cup f)\omega_{n+m})(a_1, \ldots, a_{n+m}).\end{aligned}$$

Similarly, a straightforward computation shows

$$(f\omega_n) \circ_j (g\omega_m) = (f \circ_{n-j+1} g)\omega_{n+m-1}.$$

In order to get an operation which behaves well with respect to the bracket product, we need to introduce some signs. Let $y_n = (-1)^{n(n+1)/2}\omega_n$. Then simple computations give

Lemma 5.2.1

$$(fy_n) \cup (gy_m) = (-1)^{mn}(g \cup f)y_{n+m} \tag{5.5}$$

$$[(fy_n), (gy_m)] = -[f,g]y_{n+m-1}. \tag{5.6}$$

Since $(y_n)^2 = \text{id}$, the mutually orthogonal maps $(1 \pm y_n)/2$ are projections onto the ± 1 eigenspaces of y_n. Hence we have a decomposition $C^n = C^n_+ \oplus C^n_-$, where $C^n_\pm$ are the eigenspaces of eigenvalue ± 1. From (5.6), we obtain

$$[C^*_+, C^*_+] \subseteq C^*_-,\ [C^*_+, C^*_-] \subseteq C^*_+,\ [C^*_-, C^*_-] \subseteq C^*_-. \tag{5.7}$$

For the product map $\mu \in C^2$, we have

$$(\mu\omega_2)(a,b) = \overline{\overline{b}\overline{a}} = ab = \mu(a,b).$$

So $\mu y_2 = -\mu$. In particular, $-\mu \in C^*_-$. Since $\delta f = [f, -\mu]$, (5.7) implies that $C^*_\pm$ are subcomplexes. So we obtain a decomposition of cohomology:

Theorem 5.2.2

$$H^n = H^n_+ \oplus H^n_-,$$

where $H^*_\pm$ are the homologies of $C^*_\pm$.

The cup and bracket products are $\mathbb{Z}/2\mathbb{Z}$-graded and "shifted" $\mathbb{Z}/2\mathbb{Z}$-graded for the decomposition in the sense that

Theorem 5.2.3

$$H^*_+ \cup H^*_+, H^*_- \cup H^*_- \subseteq H^*_+, \quad H^*_+ \cup H^*_- \subseteq H^*_-. \tag{5.8}$$
$$[H^*_+, H^*_+], [H^*_-, H^*_-] \subseteq H^*_-, \quad [H^*_+, H^*_-] \subseteq H^*_+ \tag{5.9}$$

Proof: (5.8) follows immediately from (5.5) and the graded commutativity of the cup product at the level of homology. (5.9) follows from (5.7) by taking the homology. □

We now consider the case when A is a commutative algebra with trivial involution ($\overline{a} = a$). Then y_n is given by an element of $\mathbb{Q}[\mathfrak{S}_n]$. Gerstenhaber and Schack [13] show that

$$\frac{1 + y_n}{2} = e_n^{(2)} + e_n^{(4)} + \cdots.$$

Corollary 5.2.4

$$\begin{aligned} H^{(\text{even})} \cup H^{(\text{even})}, H^{(\text{odd})} \cup H^{(\text{odd})} &\subseteq H^{(\text{even})}, \\ H^{(\text{even})} \cup H^{(\text{odd})} &\subseteq H^{(\text{odd})}. \\ [H^{(\text{even})}, H^{(\text{even})}], [H^{(\text{odd})}, H^{(\text{odd})}] &\subseteq H^{(\text{odd})}, \\ [H^{(\text{even})}, H^{(\text{odd})}] &\subseteq H^{(\text{even})} \end{aligned}$$

The results for the cup product are in [13]. While they do not state these results for the bracket product, they do show (5.6) in this case.

5.3 Counterexamples

In this section, we provide counterexamples to show that the natural map (4.17) $\Lambda(H_{(1)}) \to H_*$ need be neither injective nor surjective and that neither of the Gerstenhaber operations need be graded with respect to the decomposition of cohomology.

The Hochschild homology of the *dual numbers* $\mathcal{D} = K[x]/\langle x^2 \rangle$ is well known:

Proposition 5.3.1

$$H_n(\mathcal{D},\mathcal{D}) \textit{ is spanned by } \begin{cases} 1[x,\dots,x] & n \textit{ odd} \\ x[x,\dots,x] & n \textit{ even}, n>0 \end{cases}$$

Corollary 5.3.2

$$H_n(\mathcal{D},\mathcal{D}) = H_{k,n-k}(\mathcal{D},\mathcal{D}) \textit{ where } k = \left\lfloor \frac{n+1}{2} \right\rfloor .$$

Proof: This follows from the fact that $\mathfrak{S}_n$ acts trivially on the basis above (see [15,30]). A more direct proof can be obtained from observing that $[x,\dots,x]$ is a nonzero multiple of either $[(x,x)\widetilde{\sqcup\!\sqcup}\cdots\widetilde{\sqcup\!\sqcup}(x,x)]$ or $[(x,x)\widetilde{\sqcup\!\sqcup}\cdots\widetilde{\sqcup\!\sqcup}(x,x)\widetilde{\sqcup\!\sqcup}x]$, depending on the parity. Then the result follows from (4.15) since x and (x,x) are fixed by $e_1^{(1)} = \mathrm{id}$ and $e_2^{(1)} = (\mathrm{id}+(12))/2$, respectively. □

Proposition 5.3.3

$$H_{(1)}(\mathcal{D},\mathcal{D})\widetilde{\sqcup\!\sqcup}H_{(1)}(\mathcal{D},\mathcal{D}) = 0.$$

Proof: The Harrison homology is spanned by $1[x]$ and $x[x,x]$. One has immediately that

$$\begin{aligned} 1[x]\widetilde{\sqcup\!\sqcup}1[x] &= 1[x\widetilde{\sqcup\!\sqcup}x] = 0 \\ x[x,x]\widetilde{\sqcup\!\sqcup}x[x,x] &= x^2[(x,x)\widetilde{\sqcup\!\sqcup}(x,x)] = 0 \\ \text{and } 1[x]\widetilde{\sqcup\!\sqcup}x[x,x] &= x[x,x,x] = \partial(\tfrac{1}{2}[x,x,x,x]). \end{aligned}$$

So all of these products vanish in H_*. □

So the dual numbers provide the example promised in Section 4.4 of an algebra for which the map (4.17) $\Lambda(H_{(1)}) \to H_*$ is neither injective nor surjective.

In order to show that the cup and bracket products need not be graded with respect to the decomposition, we will construct an algebra for which $H^{2,0} \cup H^{2,0} \not\subseteq H^{4,0}$ and $[H^{2,0}, H^{2,0}] \not\subseteq H^{3,0}$.

For any commutative algebra A and symmetric bimodule M, $\partial e_n^{(n)} = 0$ implies that $H^{n,0}$ coincides with the cocycles in $C^{n,0}$. Gerstenhaber and Schack [15] show that $H^{n,0}(A,M)$ consists of the skew multiderivations, i.e. those skew cochains which are a derivation in each argument. We will construct an algebra with a skew multiderivation $f \in H^{2,0}(A, A)$, for which $f \cup f$ and $[f, f]$ have nontrivial projections into $H^{2,2}$ and $H^{1,2}$ respectively.

For convenience, we will denote $f(a, b)$ as a pairing $\{a, b\}$ and consider commutative algebras with a pairing which is a skew multiderivation. It is elementary to check that for such an algebra the projection of $f \cup f$ onto $C^{2,2}$ is given by

$$(f \cup f)e_4^{(2)}(a, b, c, d) = \frac{2}{3}\{a, b\}\{c, d\} + \frac{1}{3}\{a, c\}\{b, d\} - \frac{1}{3}\{a, d\}\{b, c\}.$$

The projection of $[f, f]$ onto $C^{1,2}$ is given by

$$[f, f]e_3^{(1)}(a, b, c) = -\frac{2}{3}\{a, \{b, c\}\} + \frac{4}{3}\{b, \{c, a\}\} - \frac{2}{3}\{c, \{a, b\}\}.$$

We begin by taking the free object, F, generated by taking products and pairings of the symbols x, y, z, w, subject (only) to the relations that F is commutative and that $\{-, -\}$ is a skew multiderivation.

Since $\{-, -\}$ is a derivation in each variable, F is generated as an algebra by product-free pairings of x, y, z, w, e.g. $\{x, \{y, z\}\}$. In fact, if we take a set of representatives of these pairings modulo skew-symmetry, then F is freely generated as a commutative algebra by these. This gives a grading on F as a polynomial algebra. There is also a grading coming from the number of pairings, i.e. such that $\{x, y\}\{z, w\}$ and $\{x, \{y, z\}\}$ have degree 2. And F is multigraded by the x, y, z, w-degrees.

Now let $A = F/J$, where J is the smallest algebra ideal containing xy, yz, zw which is also an ideal for the pairing (so that $\{-, -\}$ is well-defined on the quotient). Note that J splits into homogeneous components with respect to any of the notions of degree on F.

Proposition 5.3.4 *For A and $f(a, b) = \{a, b\}$ as above, $(f \cup f)e_4^{(2)}$ is not a coboundary.*

Proof: For any $g \in C^3$,

$$(\delta g)(x, y, z, w) = xg(y, z, w) + wg(x, y, z),$$

since $xy = yz = zw = 0$ in A. So it suffices to show that

$$(f \cup f)e_4^{(2)}(x, y, z, w) = \frac{2}{3}\{x, y\}\{z, w\} + \frac{1}{3}\{x, z\}\{y, w\} - \frac{1}{3}\{x, w\}\{y, z\} \tag{5.10}$$

is not in the algebra ideal generated by x and w. Lifting the problem to F, we need to show that no element of J is of the form $(f \cup f)e_4^{(2)}(x, y, z, w) + xu + wv$. By the remarks above about homogeneity, it suffices to consider the elements of J lying in the component of F involving one product, two pairings, and exactly one each of x, y, z, w, i.e. the component spanned by $\{x, y\}\{z, w\}$, $\{x, z\}\{y, w\}$, $\{x, w\}\{y, z\}$, and the twelve terms $x\{y, \{z, w\}\}$, etc. The elements of J lying in this component are spanned by $\{z, \{w, xy\}\}$, $\{w, \{z, xy\}\}$, $\{\{z, w\}, xy\}$, and the analogous expressions generated by the products yz and zw. This reduces the problem to a simple linear algebra question which can be verified directly. □

Proposition 5.3.5 *For A, f as above, $[f,f]e_3^{(1)}$ is not a coboundary.*

Proof: For any $g \in C^2$,

$$(\delta g)(x,y,z) = xg(y,z) - zg(x,y).$$

Here it suffices to show that

$$-\frac{2}{3}\{x,\{y,z\}\} + \frac{4}{3}\{y,\{z,x\}\} - \frac{2}{3}\{z,\{x,y\}\}$$

is not contained in the algebra ideal generated by x and z.

As in the previous proposition, we can lift the problem to F and take a homogeneous component. But the component involved here is the one involving two pairings and exactly one each of x, y, z. This component is spanned by $\{x,\{y,z\}\}, \{y,\{z,x\}\}, \{z,\{x,y\}\}$, and clearly intersects neither J nor the algebra ideal generated by x and z. □

5.3.6 Remark: There are simpler counterexamples for the bracket product. For example $A = K[x,y,z]/ < x,y,z >^2$ with $f(x,y) = f(x,z) = f(y,z) = z$. Part of the difficulty with finding simple counterexamples for the cup product comes from the fact that the map

$$(a,b,c,d) \mapsto \{a,b\}\{c,d\} - \{a,d\}\{b,c\}$$

is always the coboundary of $g(a,b,c) = \{a,\{b,c\}\}$.

Notation

$\mathbb{P}$	the set of positive integers	
$\mathbb{Q}$	the set of rational numbers	
$\mathbb{R}$	the set of real numbers	
$\mathbb{Z}$	the set of integers	
$[n]$	the set of integers $\{1, 2, \ldots, n\} \subseteq \mathbb{P}$	
$[i, j]$	the set of integers $\{i, i+1, \ldots, j\} \subseteq \mathbb{Z}$ (usually $\subseteq \mathbb{P}$)	
$\mathfrak{S}_n$	the symmetric group on n elements	
$p \models n$	$p = (p_1, p_2, \ldots, p_k)$ is a composition of n: $p_i \in \mathbb{P}, p_1 + p_2 + \cdots + p_k = n$	
$\lambda \vdash n$	λ is a partition of n: a composition satisfying $\lambda_1 \geq \cdots \geq \lambda_k$	
$\ell(p)$, $\ell(\lambda)$	the number of parts in p, λ	
$\lvert p\rvert$, $\lvert\lambda\rvert$	$p_1 + p_2 + \cdots + p_k$, $\lambda_1 + \lambda_2 + \cdots + \lambda_k$	
ϵ_λ	$(-1)^{\lvert\lambda\rvert - \ell(\lambda)}$	31
Sym	the ring of symmetric functions in $\{x_1, x_2, \ldots\}$ with coefficients in $\mathbb{Q}$	29
$e_\lambda, h_\lambda, p_\lambda, m_\lambda, \widetilde{m}_\lambda, s_\lambda$	elements of the elementary, complete, power sum, monomial, augmented monomial, and Schur function bases of Sym	29
QSym	the ring of quasi-symmetric functions in $\{x_1, x_2, \ldots\}$ with coefficients in $\mathbb{Q}$	32
$Q_{S,d}$	an element of the fundamental basis of QSym_d	32
Γ	a graph	
L_Γ	the lattice of contractions of Γ	12
X_Γ	Stanley's chromatic symmetric function	11, 34
$[A]$	$\sum_{\sigma \in \mathfrak{S}_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \cdot \sigma^{-1}$	15
χ^λ	an irreducible character of $\mathfrak{S}_n$	15
ϕ^λ	a "monomial" character of $\mathfrak{S}_n$	16
$H_{\mu/\nu}$	a Jacobi-Trudi matrix	16, 45
$s_{[i,j]}$	the sum (in $\mathbb{Z}\mathfrak{S}_n$) of all permutations of $[i, j]$	17
S	a skeleton	44
$\langle S \rangle$	the sum of the permutations associated with S	44
$Z[S]$	the cycle indicator of S	44
Σ	a signed graph	20
Φ	a voltage graph	81

θ	the automorphism $\mathbb{Q}[\mathfrak{S}_n] \to \mathbb{Q}[\mathfrak{S}_n]$ induced by $\sigma \mapsto (-1)^\sigma \sigma$	104
$\widetilde{\sqcup\!\sqcup}$	the signed shuffle product	105
Sh_n^k	k-shuffles (in $\mathbb{Q}[\mathfrak{S}_n]$)	105
$\rho_n^{(k)}$	one of the Reutenauer idempotents	103
$e_n^{(k)}$	one of the Eulerian idempotents	102
$H_{k,n-k}(A, M)$	the kth component of the decomposition of Hochschild homology	102
$H^{k,n-k}(A, M)$	the kth component of the decomposition of Hochschild cohomology	102
$\cup, [\ ,\ ]$	the cup product and Lie bracket defined on $H^*(A, A)$	111
$\mathcal{A}$	an alphabet	103
$\mathbb{Q}\langle\mathcal{A}\rangle$	the free associative algebra generated by $\mathcal{A}$	103
$\mathrm{Lie}\langle\mathcal{A}\rangle$	the free Lie algebra generated by $\mathcal{A}$	103

Bibliography

[1] M. Barr, Harrison homology, Hochschild homology and triples, *J. Algebra* **8** (1968), 314–323.

[2] N. Bergeron and H. L. Wolfgang, The decomposition of Hochschild cohomology and Gerstenhaber operations, *J. Pure Appl. Algebra* **104** (1995), 243–265.

[3] G. D. Birkhoff, A determinant formula for the number of ways of coloring a map, *Ann. Math.* **14** (1912), 42–46.

[4] A. Blass and B. E. Sagan, Characteristic and Ehrhart polynomials, preprint dated October 14, 1995.

[5] H. Cartan and S. Eilenberg, *Homological Algebra,* (Princeton University Press, Princeton, 1956).

[6] R. A. Dean and G. Keller, Natural partial orders, *Canad. J. Math.* **20** (1968), 535–554.

[7] P. C. Fishburn, *Interval Orders and Interval Graphs,* (Wiley, New York, 1985).

[8] A. M. Garsia, Combinatorics of the free Lie algebra and the symmetric group, in: P. H. Rabinwitz and E. Zehnder, eds., *Analysis, Et Cetera, Research Papers Published in Honor of Jurgen Moser's 60th Birthday.* (Academic Press, New York, 1990), 309–382.

[9] A. M. Garsia and C. Reutenauer, A decomposition of Solomon's descent algebra, *Adv. in Math.* **77** (1989), 189–262.

[10] V. Gasharov, Incomparability graphs of (3+1)-free posets are s-positive, preprint dated December 3, 1993.

[11] M. Gerstenhaber, The cohomology structure of an associative ring. *Ann. Math.* **78** (1963) 267–288.

[12] M. Gerstenhaber and S. D. Schack, Simplicial cohomology is Hochschild cohomology, *J. Pure Appl. Algebra* **30** (1983), 143–156.

[13] M. Gerstenhaber and S. D. Schack, A Hodge-type decomposition for commutative algebra cohomology. *J. Pure Appl. Algebra* **48** (1987), 229–247.

[14] M. Gerstenhaber and S. D. Schack, Algebraic cohomology and deformation theory, in: M. Hazewinkel and M. Gerstenhaber, eds., *Deformation Theory of Algebras and Structures and Applications.* (Kluwer, Dordrecht, 1988), 11–264.

[15] M. Gerstenhaber and S. D. Schack, *The shuffle bialgebra and the cohomology of commutative algebras, J. Pure Appl. Algebra* **70** (1991), 263–272.

[16] I. M. Gessel, Multipartite P-partitions and inner products of skew Schur functions, in:

C. Greene, ed., *Combinatorics and Algebra*, Contemporary Mathematics Series, Vol. 34, (Amer. Math. Soc., Providence, RI, 1984), 289–301.

[17] I. M. Gessel and G. X. Viennot, Determinants and plane partitions, preprint.

[18] I. P. Goulden and D. M. Jackson, Immanants of combinatorial matrices, *J. Algebra* **148** (1992), 305–324.

[19] C. Greene, Acyclic orientations (Notes), in: M. Aigner, ed., *Higher Combinatorics,* (Reidel, Dordrecht, 1977), 65–68.

[20] C. Greene, Proof of a conjecture on immanants of the Jacobi-Trudi matrix, *Linear Algebra Appl.* **171** (1992), 65–79.

[21] C. Greene and T. Zaslavsky, On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-Radon partitions, and orientations of graphs, *Trans. Amer. Math. Soc.* **280** (1983), 97–126.

[22] M. Haiman, Hecke algebra characters and immanant conjectures, *J. Amer. Math. Soc.* **6** (1993), 569–595.

[23] R. M. Hain, On the indecomposable elements of the bar construction, *Proc. Amer. Math. Soc.* **98** (1986), 312–316.

[24] F. Harary, On the notion of balance of a signed graph, *Michigan Math. J.* **2** (1953–1954), 143–146.

[25] D. K. Harrison, Commutative algebras and cohomology, unpublished version of [26][1].

[26] D. K. Harrison, Commutative algebras and cohomology, *Trans. Amer. Math. Soc.* **104** (1962), 191–204.

[27] G. Hochschild, On the cohomology groups of an associative algebra, *Ann. Math.* **46** (1945), 58–67.

[28] G. Hochschild, On the cohomology theory for associative algebras, *Ann. Math.* **47** (1946), 568–579.

[29] E. Lieb, Proofs of some conjectures on permanents, *J. Math. Mech.* **16** (1966), 127–134.

[30] J.-L. Loday, Opérations sur l'homologie cyclique des algèbres commutatives, *Invent. Math.* **96** (1989), 205–230.

[31] J.-L. Loday, *Cyclic Homology, Grundlehren Math. Wiss. 301* (Springer, Berlin–Heidelberg–New York, 1992).

[32] I. G. Macdonald, *Symmetric Functions and Hall Polynomials,* Second Edition, (Oxford Univ. Press, Oxford, 1995).

[33] C. Reutenauer, Theorem of Poincaré-Birkhoff-Witt and symmetric group representations equal to Stirling numbers, in: G. Labelle and P. Leroux, eds., *Combinatoire Énumérative, Lecture Notes in Mathematics 1234* (Springer, Berlin–Heidelberg–New York, 1986), 267–284.

[34] C. Reutenauer, *Free Lie Algebras.* (Oxford, 1993).

[35] M. Ronco, On the Hochschild homology decompositions. *Comm. Alg.* **21** (1993), 4699–4712.

[1] The reasons for listing this as a separate reference are discussed on page 23.

[36] G.-C. Rota, On the foundations of combinatorial theory I. Theory of Möbius functions, *Z. Wahrscheinlichkeitstheorie* **2** (1964), 340–368.

[37] I. Schur, Über endliche Gruppen und Hermitesche Formen, *Math. Z.* **1** (1918), 184–207.

[38] A. B. Sletsjøe, Products in the decomposition of Hochschild cohomology. *Comm. Alg.* **22** (1994), 1899–1911.

[39] R. P. Stanley, Acyclic orientations of graphs, *Discrete Math.* **5** (1973), 171–178.

[40] R. P. Stanley, *Enumerative Combinatorics,* Vol. 1, (Wadsworth & Brooks/Cole, Pacific Grove, CA, 1986).

[41] R. P. Stanley, A symmetric function generalization of the chromatic polynomial of a graph, *Adv. Math.* **111** (1995), 166–194.

[42] R. P. Stanley, Graph colorings and related symmetric functions: ideas and applications, preprint dated 1 February 1995.

[43] R. P. Stanley and J. R. Stembridge, On immanants of Jacobi-Trudi matrices and permutations with restricted position, *J. Combin. Theory Ser. A* **62** (1993), 261–279.

[44] J. R. Stembridge, Nonintersecting paths, pfaffians, and plane partitions, *Adv. Math.* **83** (1990), 96–131.

[45] J. R. Stembridge, Immanants of totally positive matrices are nonnegative, *Bull. London Math. Soc.* **23** (1991), 422–428.

[46] J. R. Stembridge, Some conjectures for immanants, *Can. J. Math.* **44** (1992), 1079–1099.

[47] G. X. Viennot, Heaps of pieces, I: Basic definitions and combinatorial lemmas, in: G. Labelle and P. Leroux, eds., *Combinatoire énumérative, Lecture Notes in Mathematics 1234* (Springer, Berlin–Heidelberg–New York, 1986), 321–350.

[48] D. J. A. Welsh, *Matroid Theory,* (Academic Press, London, 1976).

[49] H. Whitney, A logical expansion in mathematics, *Bull. Amer. Math. Soc.* **38** (1932), 572–579.

[50] H. L. Wolfgang, *Eulerian Idempotents and the Decomposition of Hochschild Homology,* Senior thesis, Harvard (1993)

[51] T. Zaslavsky, *Facing Up to Arrangements: Face-Count Formulas for Partitions of Space by Hyperplanes, Mem. Amer. Math. Soc. No. 154,* (Amer. Math. Soc., Providence, RI, 1975).

[52] T. Zaslavsky, The geometry of root systems and signed graphs, *Am. Math. Monthly* **88** (1981), 88–105.

[53] T. Zaslavsky, Signed graphs, *Disc. Appl. Math.* **4** (1982), 47–74. Erratum, ibid., **5** (1983), 248.

[54] T. Zaslavsky, Signed graph coloring, *Disc. Math.* **39** (1982), 215–228.

[55] T. Zaslavsky, Chromatic invariants of signed graphs, *Disc. Math.* **42** (1982), 287–312.

[56] T. Zaslavsky, How colorful the signed graph?, *Disc. Math.* **52** (1984), 279–284.

[57] T. Zaslavsky, Orientation of signed graphs, *Europ. J. Combinatorics* **12** (1991), 361–375.

[58] A. V. Zelevinsky, *Representations of Finite Classical Groups, Lecture Notes in Mathematics 869* (Springer, Berlin–Heidelberg–New York, 1981).