Moduli for Pairs of Elliptic Curves with Isomorphic N-torsion

by

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A.B. Sanskrit and Mathematics
Harvard University, 1993

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY
AT THE
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

JUNE 1998

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ABSTRACT

We study the moduli surface for pairs of elliptic curves together with an isomorphism between their N-torsion groups. The Weil pairing gives a "determinant" map from this moduli surface to $(\mathbf{Z}/N\mathbf{Z})^*$; its fibres are the components of the surface. We define spaces of modular forms on these components and Hecke correspondences between them, and study how those spaces of modular forms behave as modules for the Hecke algebra. We discover that the component with determinant -1 is somehow the "dominant" one; we characterize the difference between its spaces of modular forms and the spaces of modular forms on the other components using forms with complex multiplication. Finally, we show some simplifications that arise when N is prime, including a complete determination of such CM-forms, and give numerical examples.

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1 Introduction

Let $X_w(N)$ be the curve over $\mathbf{C}$ parameterizing elliptic curves together with a basis for their N-torsion that maps to some specified N'th root of unity under the Weil pairing.[1] It is Galois over the curve $X_w(1)$ with Galois group $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})/\{\pm 1\}$. Let $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$ act on the product surface $X_w(N) \times X_w(N)$ via the diagonal action; we can then form the quotient surface, which we shall denote by $X_{\simeq,1}(N)$. More generally, if ϵ is an element of $(\mathbf{Z}/N\mathbf{Z})^*$ and if $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$ acts on the first factor via the natural action but on the second factor via the automorphism

$$\theta_\epsilon \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & \epsilon^{-1} b \\ \epsilon c & d \end{pmatrix}$$

then we denote the quotient surface by $X_{\simeq,\epsilon}(N)$. And we set

$$X_{\simeq}(N) = \coprod_{\epsilon \in (\mathbf{Z}/N\mathbf{Z})^*} X_{\simeq,\epsilon}(N).$$

These surfaces can also be constructed in another fashion, as degenerate Hilbert modular surfaces: let $\mathfrak{H}$ be the upper half plane, with $\Gamma(1) = \mathrm{SL}_2(\mathbf{Z})$ acting on it via fractional linear

[1] This curve is traditionally denoted by $X(N)$; however, we have chosen to use the notation $X(N)$ to denote the (geometrically reducible) curve coming from the adelic mod N principal congruence subgroup, and have changed all notation accordingly.

transformations. Then $\Gamma(1) \times \Gamma(1)$ acts on $\mathfrak{H} \times \mathfrak{H}$; if we denote by $\Gamma_{\simeq,\epsilon}(N)$ the subgroup of $\Gamma(1) \times \Gamma(1)$ given by

$$\left\{ \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \,\middle|\, \begin{array}{rcll} a_1 & \equiv & a_2 & \pmod{N}, \\ b_1 & \equiv & \epsilon b_2 & \pmod{N}, \\ \epsilon c_1 & \equiv & c_2 & \pmod{N}, \\ d_1 & \equiv & d_2 & \pmod{N} \end{array} \right\}$$

then the quotient $\Gamma_{\simeq,\epsilon}(N) \backslash \mathfrak{H} \times \mathfrak{H}$ is an open subset of $X_{\simeq,\epsilon}(N)$, and if we denote by $\mathfrak{H}^*$ the space $\mathfrak{H} \coprod \mathbf{P}^1(\mathbf{Q})$ then $\Gamma_{\simeq,\epsilon}(N) \backslash \mathfrak{H}^* \times \mathfrak{H}^*$ is all of $X_{\simeq,\epsilon}(N)$.

The surface $X_{\simeq,\epsilon}(N)$ (or, more properly, the open subset given by using $\mathfrak{H} \times \mathfrak{H}$ instead of $\mathfrak{H}^* \times \mathfrak{H}^*$) is a coarse moduli space for triples (E_1, E_2, ϕ) where the E_i's are elliptic curves and ϕ is an isomorphism from $E_1[N]$ to $E_2[N]$ such that $\wedge^2\phi$ raises the Weil pairing to the ϵ'th power. The modular parameterization is given as follows: let $(\tau_1, \tau_2) \in \mathfrak{H} \times \mathfrak{H}$ and let E_i be the elliptic curve given by the lattice with basis $\{1, \tau_i\}$. Also, let e be an integer that reduces to ϵ mod N. We then have the map ϕ from $E_1[N]$ to $E_2[N]$ that sends τ_1/N to $e\tau_2/N$ and $1/N$ to $1/N$; it raises the Weil pairing to the ϵ'th power, the group of elements of $\Gamma(1) \times \Gamma(1)$ that preserve ϕ is the subgroup $\Gamma_{\simeq,\epsilon}(N)$ defined above, and every triple (E_1, E_2, ϕ) arises in this fashion.

Using this modular interpretation of these surfaces, we can think of them as Hilbert modular surfaces corresponding to the order $(\mathbf{Z} \times \mathbf{Z})_{\equiv(N)}$ of conductor N in $\mathbf{Z} \times \mathbf{Z}$, defined as

$$(\mathbf{Z} \times \mathbf{Z})_{\equiv(N)} = \{ (a, b) \in \mathbf{Z} \times \mathbf{Z} \mid a \equiv b \pmod{N} \}.$$

Let (E_1, E_2, ϕ) be a point on $X_{\simeq,\epsilon}(N)$. If we let H be the subgroup of $(E_1 \times E_2)[N]$ consisting of all points of the form $(x, \phi(x))$ then $E_1 \times E_2/H$ has real multiplication by $(\mathbf{Z} \times \mathbf{Z})_{\equiv(N)}$. This curve has a natural principal polarization iff $\epsilon = -1$ (c.f. Frey and Kani [3]): if A is an abelian variety and H is a subgroup of $A[N]$ then $(A/H)^\vee$ is isomorphic to $A^\vee/H^\vee$, where $H^\vee \subset A^\vee[N]$ is the set of points orthogonal to H under the Weil pairing, and in the case at hand we have $H = H^\vee$ iff $\epsilon = -1$. We shall see other reasons below why the surface $X_{\simeq,-1}(N)$ is the most important of the $X_{\simeq,\epsilon}(N)$'s to study; see Sections 5 and 6 in particular.

The above gives a construction for the surfaces $X_{\simeq,\epsilon}(N)$ over $\mathbf{C}$; since the moduli problem makes sense over $\mathbf{Q}$, there should be a construction of $X_{\simeq,\epsilon}(N)$ over $\mathbf{Q}$ as well. It is given as follows: let $X(N)$ be the moduli space of pairs (E, ϕ) where E is an elliptic curve and ϕ is an isomorphism of group schemes from $E[N]$ to $\mathbf{Z}/N\mathbf{Z} \times \mathbf{Z}/N\mathbf{Z}$. This curve is defined over $\mathbf{Q}$, it is Galois over $X(1)$ with Galois group $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})/\{\pm 1\}$, and all of those automorphisms are defined over $\mathbf{Q}$. Thus, the surface $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z}) \backslash X(N) \times X(N)$ is defined over $\mathbf{Q}$, where $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ acts via the diagonal action, and it is a moduli space for triples (E_1, E_2, ϕ) as above but without any condition on what ϕ does to the Weil pairing. This surface (which is our $X_{\simeq}(N)$) isn't connected, however: there is a map from it to $\mathrm{Aut}(\boldsymbol{\mu}_p) = (\mathbf{Z}/N\mathbf{Z})^*$ which sends (E_1, E_2, ϕ) to the ϵ such that ϕ raises the Weil pairing to the ϵ'th power. The fibre of that map over ϵ is then $X_{\simeq,\epsilon}(N)$, and it is defined over $\mathbf{Q}$.

The structure of the $X_{\simeq,\epsilon}(N)$'s as complex surfaces has been studied by Hermann in [7] and by Kani and Schanz in [8]; our $X_{\simeq,\epsilon}(N)$ is Hermann's $Y_{N,\epsilon^{-1}}$ and Kani and Schanz's $Z_{N,\epsilon^{-1}}$.[2] In particular, Kani and Schanz give explicit formulas and tables computing various invariants of the $X_{\simeq,\epsilon}(N)$'s, such as the dimensions of various cohomology groups. They also give explicit minimal desingularizations of the surfaces.

The goal of this paper is to study spaces of modular forms on the surfaces $X_{\simeq}(N)$ and $X_{\simeq,\epsilon}(N)$, and the interplay between the spaces on the various surfaces. In many ways, $X_{\simeq}(N)$ turns out to be the most natural object to study; to prove results on the $X_{\simeq,\epsilon}(N)$'s, one has to pass via $X_{\simeq}(N)$, using the surface $X_{\simeq,-1}(N)$ as a linchpin. In particular, we study how the $X_{\simeq,\epsilon}(N)$'s differ as ϵ varies; one might naively expect them all to be isomorphic, but it turns out that $X_{\simeq,\epsilon}(N)$ and $X_{\simeq,\epsilon'}(N)$ are isomorphic in general only if ϵ and ϵ' differ by a square. The surface $X_{\simeq,-1}(N)$ is somehow the most important of the surfaces $X_{\simeq,\epsilon}(N)$; we characterize the difference between spaces of modular forms on it and spaces of modular forms on the other $X_{\simeq,\epsilon}(N)$'s in terms of forms with complex multiplication. We also consider the case where N is prime and show that various simplifications occur there, allowing us to give a complete determination of the CM-forms that arise; we end by giving numerical examples of such forms.

While the study of the surfaces $X_{\simeq,\epsilon}(N)$ is interesting in its own right, they can also be seen as a first step towards exploring the wealth of level structures that should occur on higher-dimensional Hilbert modular varieties that don't have analogues on modular curves. In particular, if K is a real quadratic extension of $\mathbf{Q}$ and if p is a prime in $\mathbf{Q}$ that splits in K then we have level structures on the Hilbert modular surface associated to K that are completely analogous to the $X_{\simeq,\epsilon}(p)$ level structures; to the extent that the calculations in this paper are local, they should carry over to that situation as well.

I would like to thank Fred Diamond, Jordan Ellenberg, Steven Kleiman, and Barry Mazur for the help that they have given me while writing this; and the M.I.T. Department of Mathematics and the N.D.S.E.G. Fellowship Program for the support that they have provided.

2 Spaces of Modular Forms

Let $f\colon \mathfrak{H}\times\mathfrak{H}\to\mathbf{C}$ be a holomorphic function; let $\gamma=(\gamma_1,\gamma_2)$ be an element of $\mathrm{GL}_2^+(\mathbf{R})\times\mathrm{GL}_2^+(\mathbf{R})$, where $\mathrm{GL}_2^+(\mathbf{R})$ is the set of elements of $\mathrm{GL}_2(\mathbf{R})$ with positive determinant; and let $k=(k_1,k_2)$ be a pair of natural numbers. We define the function $f|_{k,\gamma}\colon \mathfrak{H}\times\mathfrak{H}\to\mathbf{C}$ by

$$f|_{k,\gamma}(z_1,z_2)=f(\gamma_1(z_1),\gamma_2(z_2))j(\gamma_1,z_1)^{-k_1}j(\gamma_2,z_2)^{-k_2}$$

where, if $\sigma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ is an element of $\mathrm{GL}_2^+(\mathbf{R})$, then $\sigma(z)=(az+b)/(cz+d)$ and

$$j(\sigma,z)=(ad-bc)^{-1/2}(cz+d).$$

[2]We replaced their ϵ by ϵ^{-1} to simplify the normalizations given in Section 7; since $X_{\simeq,\epsilon}(N)$ and $X_{\simeq,\epsilon^{-1}}(N)$ are isomorphic, this is an unimportant change.

We write $f|_\gamma$ instead of $f|_{k,\gamma}$ if k is clear from context.

Defining $\Gamma(1)$ to be $\mathrm{SL}_2(\mathbf{Z})$, we say that a subgroup Γ of $\Gamma(1) \times \Gamma(1)$ is a *congruence subgroup* if it contains the group $\Gamma_w(N) \times \Gamma_w(N)$ for some N, where $\Gamma_w(N)$ is defined to be the set of matrices in $\mathrm{SL}_2(\mathbf{Z})$ that are congruent to the identity mod N. A function $f\colon \mathfrak{H} \times \mathfrak{H} \to \mathbf{C}$ is a *modular form for* Γ *of weight* k if $f|_{k,\gamma} = f$ for all $\gamma \in \Gamma$ and if f is holomorphic at the cusps. To explain this latter condition, assume that $\Gamma_w(N) \times \Gamma_w(N) \subset \Gamma$. Then $f(z_1 + N, z_2) = f(z_1, z_2)$ for all $(z_1, z_2) \in \mathfrak{H} \times \mathfrak{H}$; so setting $q_1 = e^{2\pi\sqrt{-1}z_1/N}$, we can write

$$f(z_1, z_2) = \sum_{m \in \mathbf{Z}} c_m(f)(z_2) q_1^m$$

for some functions $c_m(f)$. If $c_m(f)$ is zero for all $m < 0$ and if a similar condition holds if we do a Fourier expansion in z_2, we say that f is *holomorphic at infinity.* And f is *holomorphic at all of the cusps* if, for all $\gamma \in \Gamma(1) \times \Gamma(1)$, $f|_{k,\gamma}$ is holomorphic at infinity.

A modular form is a *cusp form* if it vanishes at all of the cusps; that is to say, if whenever we take a Fourier expansion of $f|_{k,\gamma}$ in either variable as above, $c_0(f)$ is zero. We denote the space of all modular forms of weight k for Γ by $M_k(\Gamma)$; we denote the space of all cusp forms by $S_k(\Gamma)$.

The space $M_{(k_1,k_2)}(\Gamma_{\simeq,\epsilon}(N))$ is zero unless $k_1 - k_2$ is even: this follows from the fact that $\left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right)$ is in $\Gamma_{\simeq,\epsilon}(N)$.

If $\Gamma = \Gamma_1 \times \Gamma_2$, with each Γ_i a congruence subgroup of $\Gamma(1)$, then there is a natural map from $M_{k_1}(\Gamma_1) \otimes M_{k_2}(\Gamma_2)$ to $M_{(k_1,k_2)}(\Gamma_1 \times \Gamma_2)$ which sends $f_1 \otimes f_2$ to the function

$$(z_1, z_2) \mapsto f_1(z_1) f_2(z_2).$$

Furthermore, this map sends cusp forms to cusp forms. It is in fact an isomorphism in either the modular form or cusp form case:

Proposition 2.1. *If S is a subset of $\mathfrak{H}^*$ or $\mathfrak{H}^* \times \mathfrak{H}^*$ and Γ is a congruence subgroup of $\Gamma(1)$ or $\Gamma(1) \times \Gamma(1)$, let $M_k(\Gamma, S)$ be the set of forms in $M_k(\Gamma)$ that vanish on the points in S. Then for any congruence subgroups Γ_1 and Γ_2 of $\Gamma(1)$ and subsets S_1 and S_2 of $\mathfrak{H}^*$, the natural map*

$$M_{k_1}(\Gamma_1, S_1) \otimes M_{k_2}(\Gamma_2, S_2) \to M_{(k_1,k_2)}(\Gamma_1 \times \Gamma_2, (S_1 \times \mathfrak{H}^*) \cup (\mathfrak{H}^* \times S_2))$$

is an isomorphism.

Proof. We prove the Proposition by induction on the dimension of $M_{k_1}(\Gamma_1, S_1)$. Set $S = (S_1 \times \mathfrak{H}^*) \cup (\mathfrak{H}^* \times S_2)$, and assume that $\dim M_{k_1}(\Gamma_1, S_1)$ is zero. Let f be an element of $M_{(k_1,k_2)}(\Gamma_1 \times \Gamma_2, S)$. For any $z_2 \in \mathfrak{H}^*$, the function $z \mapsto f(z, z_2)$ is an element of $M_{k_1}(\Gamma_1, S_1)$, which is therefore zero, so f is the zero function.

Now assume that $\dim M_{k_1}(\Gamma_1, S_1)$ is positive, and let z_1' be an element of $\mathfrak{H}^*$ such that, setting $S_1' = S_1 \cup \{z_1'\}$,

$$\dim M_{k_1}(\Gamma_1, S_1') = \dim M_{k_1}(\Gamma_1, S_1) - 1.$$

Let $S' = (S_1' \times \mathfrak{H}^*) \cup (\mathfrak{H}^* \times S_2)$; we will construct a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M_{k_1}(\Gamma_1, S_1') \otimes M_{k_2}(\Gamma_2, S_2) \arrow[r] \arrow[d] & M_{k_1}(\Gamma_1, S_1) \otimes M_{k_2}(\Gamma_2, S_2) \arrow[r] \arrow[d] & M_{k_2}(\Gamma_2, S_2) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_{(k_1,k_2)}(\Gamma_1 \times \Gamma_2, S') \arrow[r] & M_{(k_1,k_2)}(\Gamma_1 \times \Gamma_2, S) \arrow[r] & M_{k_2}(\Gamma_2, S_2) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. This will prove our Proposition: the left vertical arrow is an isomorphism, by induction, and the right vertical arrow also is, so the middle vertical arrow is one as well.

The left horizontal arrows are the obvious injections. The right arrow on the top row sends $f_1 \otimes f_2$ to $f_1(z_1')f_2$; the definition of S_1' and the choice of z_1' shows that this makes the top row exact.

Similarly, we define the right arrow on the bottom row by having it send f to the function sending z to $f(z_1', z)$, which is in $M_{k_2}(\Gamma_2, S_2)$. This map is surjective: if we pick a function $f_1' \in M_{k_1}(\Gamma_1, S_1)$ such that $f_1'(z) = 1$ then we can get a splitting for this map by sending f_2 to the image of $f_1' \otimes f_2$ under the middle vertical arrow. The exactness of the bottom row then follows immediately from the definitions. □

Corollary 2.2. *Given any natural numbers k_1, k_2, and N, we have isomorphisms*

$$M_{(k_1,k_2)}(\Gamma_{\simeq,\epsilon}(N)) = (M_{k_1}(\Gamma_w(N)) \otimes M_{k_2}(\Gamma_w(N)))^{\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})}$$

and

$$S_{(k_1,k_2)}(\Gamma_{\simeq,\epsilon}(N)) = (S_{k_1}(\Gamma_w(N)) \otimes S_{k_2}(\Gamma_w(N)))^{\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})},$$

where $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$ acts on the first member of the tensor product in the natural fashion and on the second member via the automorphism θ_ϵ.

Proof. By the Proposition,

$$M_{(k_1,k_2)}(\Gamma_w(N) \times \Gamma_w(N)) = (M_{k_1}(\Gamma_w(N)) \otimes M_{k_2}(\Gamma_w(N)));$$

that $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$-invariants correspond to forms in $M_{(k_1,k_2)}(\Gamma_{\simeq,\epsilon}(N))$ follows from the definitions. The cusp form case is similar, setting S_1 and S_2 in the Proposition to be equal to $\mathbf{P}^1(\mathbf{Q})$. □

This allows us to express the dimension of the space $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(N))$ in terms of data given in Kani and Schanz [8]:

Corollary 2.3. *The dimensions of the spaces $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(N))$ and $H^2(X_{\simeq,\epsilon}(N), \mathcal{O}_{X_{\simeq,\epsilon}(N)})$ are equal, and they are also equal to the geometric genus of a desingularization of $X_{\simeq,\epsilon}(N)$.*

Proof. We have the equalities

$$\begin{aligned}\dim S_{(2,2)}(\Gamma_{\simeq,\epsilon}(N)) &= \dim(S_2(X_w(N)) \otimes S_2(X_w(N)))^{\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})} \\ &= \dim(H^1(X_w(N), \mathcal{O}_{X_w(N)}) \otimes H^1(X_w(N), \mathcal{O}_{X_w(N)}))^{\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})} \\ &= \dim H^2(X_w(N) \times X_w(N), \mathcal{O}_{X\times X})^{\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})} \\ &= \dim H^2(\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})\backslash(X_w(N) \times X_w(N)), \mathcal{O}_{\mathrm{SL}_2\backslash X\times X}) \\ &= \dim H^2(X_{\simeq,\epsilon}(N), \mathcal{O}_{X_{\simeq,\epsilon}(N)})\end{aligned}$$

by the fact that $S_2(X_w(N))$ is dual to $H^1(X_w(N), \mathcal{O}_{X_w(N)})$, the Künneth formula, and Kani and Schanz [9], Proposition 2.7 (which allows us to translate between invariants under a group and quotients by that group). The fact that this is equal to the geometric genus is part of Kani and Schanz [9], Proposition 3.1. □

Of course, this isn't too surprising: weight 2 cusp forms should correspond to holomorphic 2-forms.

If f is a modular form on $\Gamma_{\simeq,\epsilon}(N)$, it has a Fourier expansion

$$f(z_1, z_2) = \sum_{m_1,m_2\geq 0} c_{m_1,m_2}(f) q_1^{m_1} q_2^{m_2}$$

where $q_i = e^{2\pi\sqrt{-1}z_i/N}$. There is one thing that we can say immediately about the Fourier coefficients $c_{m_1,m_2}(f)$:

Proposition 2.4. *For all $f \in M_{(k_1,k_2)}(\Gamma_{\simeq,\epsilon}(N))$, the Fourier coefficient $c_{m_1,m_2}(f)$ is zero unless $\epsilon m_1 + m_2 \equiv 0 \pmod{N}$.*

Proof. Let e be an integer congruent to ϵ mod N and let

$$\gamma = \left(\begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \in \Gamma_{\simeq,\epsilon}(N).$$

Then

$$\begin{aligned} f &= f|_\gamma \\ &= \sum_{m_1,m_2\geq 0} c_{m_1,m_2}(f) e^{2\pi\sqrt{-1}m_1(z_1+e)/N} e^{2\pi\sqrt{-1}m_2(z_2+1)/N} \\ &= \sum_{m_1,m_2\geq 0} c_{m_1,m_2}(f) e^{2\pi\sqrt{-1}m_1z_1/N} e^{2\pi\sqrt{-1}m_2z_2/N} e^{2\pi\sqrt{-1}(m_1e+m_2)/N} \\ &= \sum_{m_1,m_2\geq 0} c_{m_1,m_2}(f) q_1^{m_1} q_2^{m_2} \zeta_N^{m_1e+m_2}\end{aligned}$$

where $\zeta_N = e^{2\pi\sqrt{-1}/N}$. But this implies that

$$c_{m_1,m_2}(f) = c_{m_1,m_2}(f)\zeta_N^{m_1e+m_2},$$

so $c_{m_1,m_2}(f)$ is zero unless $\epsilon m_1 + m_2 \equiv 0 \pmod{N}$. □

Thus, most of the Fourier coefficients are "missing". This turns out to make it natural to also study modular forms on the surface $X_{\simeq}(N)$, even when we are only interested in one of the individual $X_{\simeq,\epsilon}(N)$'s; we shall elaborate on this theme in Section 5.

One way to produce forms on $X_{\simeq,\epsilon}(N)$ is to consider forms on $X_{\simeq,\epsilon}(N/d)$ to be forms on $X_{\simeq,\epsilon}(N)$, for d a divisor of N. Such forms have Fourier coefficients c_{m_1,m_2} equal to zero unless d divides m_1 (and hence m_2, by Proposition 2.4). The converse is also true:

Theorem 2.5. *Let f be a modular form of weight k on $\Gamma_{\simeq,\epsilon}(N)$, and assume that, for some $d|N$, we have $c_{m_1,m_2}(f)=0$ unless $d|m_1$. Then f is an element of $M_k(\Gamma_{\simeq,\epsilon}(N/d))$.*

Proof. The fact that $c_{m_1,m_2}(f)=0$ unless $d|m_1$ is equivalent to having f be invariant under

$$\left(\begin{pmatrix}1 & N/d\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right).$$

Thus, we have to show that the smallest subgroup Γ containing both $\left(\left(\begin{smallmatrix}1 & N/d\\ 0 & 1\end{smallmatrix}\right), \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)\right)$ and $\Gamma_{\simeq,\epsilon}(N)$ is $\Gamma_{\simeq,\epsilon}(N/d)$. Furthermore, we can take the quotient by $\Gamma_w(N)\times\Gamma_w(N)$, and thus consider all matrices to be elements of $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$. If $\gamma=(\gamma_1,\gamma_2)$ is an element of Γ then

$$\gamma=(\gamma_1\cdot\theta_\epsilon^{-1}(\gamma_2^{-1}),1)\cdot(\theta_\epsilon^{-1}(\gamma_2),\gamma_2)$$

which expresses γ as an element of $(G\times\{\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)\})\cdot\Gamma_{\simeq,\epsilon}(N)$, where

$$G=\left\{\gamma_1\cdot\theta_\epsilon^{-1}(\gamma_2^{-1})\;\middle|\;(\gamma_1,\gamma_2)\in\Gamma\right\},$$

and conversely any element of $(G\times\{\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)\})\cdot\Gamma_{\simeq,\epsilon}(N)$ is also an element of Γ. But for that to be a subgroup of $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})\times\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$, it is necessary for G to be a normal subgroup of $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$. Thus, we have to show that the smallest normal subgroup of $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$ containing the matrix $\tau_{N/d}=\left(\begin{smallmatrix}1 & N/d\\ 0 & 1\end{smallmatrix}\right)$ is the kernel of the natural map from $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$ to $\mathrm{SL}_2(\mathbf{Z}/(N/d)\mathbf{Z})$. Furthermore, we can assume that d is a prime p, and by the Chinese remainder theorem we can assume that $N=p^l$ for some l.

First, assume that $l=1$, so we want to show that the smallest normal subgroup G of $\mathrm{SL}_2(\mathbf{Z}/p\mathbf{Z})$ containing $\tau_1=\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ is the entire group. We first look at the image of G in $\mathrm{PSL}_2(\mathbf{Z}/p\mathbf{Z})$. If $p>3$ then this latter group is simple, so the image of G is all of $\mathrm{PSL}_2(\mathbf{Z}/p\mathbf{Z})$. If $p=3$ then this latter group is isomorphic to A_4 and τ_1 is an element of order 3; but since the only proper normal subgroups of A_4 contain only elements of order 1 and 2, we again have that the image of G is all of $\mathrm{PSL}_2(3)$. Similarly, if $p=2$, then $\mathrm{PSL}_2(\mathbf{Z}/2\mathbf{Z})$ is isomorphic to S_3 and τ_1 has order 2, so again our image must be all of $\mathrm{PSL}_2(\mathbf{Z}/2\mathbf{Z})$.

This implies that G must either be all of $\mathrm{SL}_2(\mathbf{Z}/p\mathbf{Z})$ or a subgroup of index two which projects onto all of $\mathrm{PSL}_2(\mathbf{Z}/p\mathbf{Z})$. But if $p=2$ then $\mathrm{SL}_2(\mathbf{Z}/2\mathbf{Z})=\mathrm{PSL}_2(\mathbf{Z}/2\mathbf{Z})$ so we're done; if $p=3$ then $\mathrm{SL}_2(\mathbf{Z}/3\mathbf{Z})$ has only two non-trivial one-dimensional representations, whose kernels are of index 3; and if $p>3$ then $\mathrm{SL}_2(\mathbf{Z}/p\mathbf{Z})$ has no non-trivial one-dimensional representations, so again has no subgroups of index 2. (See Fulton and Harris [4], Section 5.2 for the facts about $\mathrm{PSL}_2(\mathbf{Z}/p\mathbf{Z})$ and $\mathrm{SL}_2(\mathbf{Z}/p\mathbf{Z})$ used here.)

Finally, assume that $l > 1$, and that we have a normal subgroup G containing τ_q, where $q = p^{l-1}$. Note that q^2 is zero in $\mathbf{Z}/p^l\mathbf{Z}$, which greatly simplifies calculations. We then have to show that G contains all matrices of the form $\begin{pmatrix} 1+aq & bq \\ cq & 1+dq \end{pmatrix}$ with determinant 1; this condition on the determinant is equivalent to having a equal to $-d$ in $\mathbf{Z}/p\mathbf{Z}$.

First, we have all of the powers of τ_q, so we have the matrices $\left(\begin{smallmatrix} 1 & eq \\ 0 & 1 \end{smallmatrix}\right)$ for all e. Conjugating by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, we also have the matrices $\left(\begin{smallmatrix} 1 & 0 \\ fq & 1 \end{smallmatrix}\right)$ for all f, and multiplying those together, we have the matrices $\begin{pmatrix} 1 & eq \\ fq & 1 \end{pmatrix}$.

On the other hand, conjugating τ_q by $\left(\begin{smallmatrix} a & a-1 \\ 1 & 1 \end{smallmatrix}\right)$, we see that for any a we have a matrix $\begin{pmatrix} 1-aq & b'q \\ c'q & 1+aq \end{pmatrix}$ for some b', c'. But now if we have any matrix $\begin{pmatrix} 1+aq & bq \\ cq & 1-aq \end{pmatrix}$ that we wish to show is in G, it is enough to show that $\begin{pmatrix} 1+aq & bq \\ cq & 1-aq \end{pmatrix}\begin{pmatrix} 1-aq & b'q \\ c'q & 1+aq \end{pmatrix}$ is in G; and that matrix is of the form $\begin{pmatrix} 1 & eq \\ fq & 1 \end{pmatrix}$, hence in G by the previous paragraph. $\square$

We hope that the following stronger result is true:

Guess 2.6. *Let f be a modular form on $\Gamma_{\simeq,\epsilon}(N)$ such that $c_{m_1,m_2}(f) = 0$ unless $(m_i, N) > 1$. Then f can be written as a sum of modular forms f_j on $\Gamma_{\simeq,\epsilon}(N/p_j)$ where the p_j's are the prime divisors of N. Furthermore, if f is a cusp form then the f_j can be chosen to be cusp forms.*

Of course, Theorem 2.5 implies Guess 2.6 for N a prime power. They are both analogous to results proved as parts of Atkin-Lehner theory on the curves $X_1(N)$. (See Lang [10], Chapter VIII, in particular Theorem 3.1.) While we don't yet know how much of Atkin-Lehner theory on $X_1(N)$ carries over to the surfaces $X_{\simeq,\epsilon}(N)$, not all of it does: in particular, while there are operators

$$\iota_d \colon S_k(\Gamma_1(M)) \to S_k(\Gamma_1(N))$$

for each $d|N/M$, there is in some sense only one natural way to produce a form on $X_{\simeq,\epsilon}(N)$ from a form on $X_{\simeq,\epsilon}(M)$ for $M|N$. We shall give a precise statement and proof of this as Proposition 7.2.

We let $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ be the quotient of $S_k(\Gamma_{\simeq,\epsilon}(N))$ by the subgroup of forms f whose Fourier coefficients $c_{m_1,m_2}(f)$ are zero unless $(m_i, N) > 1$. In the $X_1(N)$ case, this would have the effect of replacing $S_k(\Gamma_1(N))$ by a space with the same Hecke eigenspaces but where each eigenspace is one-dimensional, generated by the newform in that eigenspace; we shall see in Theorem 5.6 that Hecke eigenspaces in $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ are also one-dimensional. Finally, we let

$$S_{k,\simeq}(N) = \prod_{\epsilon \in (\mathbf{Z}/N\mathbf{Z})^*} S_k(\Gamma_{\simeq,\epsilon}(N)),$$

and we let

$$\overline{S}_{k,\simeq}(N) = \prod_{\epsilon \in (\mathbf{Z}/N\mathbf{Z})^*} \overline{S}_k(\Gamma_{\simeq,\epsilon}(N)).$$

Note that in the definitions of $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ and $\overline{S}_{k,\simeq}(N)$ it's enough to assume that the Fourier coefficients are zero unless $(m_1, N) > 1$ (or unless $(m_2, N) > 1$), by Proposition 2.4.

Proposition 2.7. *The spaces* $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$ *and* $\overline{S}_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$ *are equal, as are the spaces* $S_{(2,2),\simeq}(p)$ *and* $\overline{S}_{(2,2),\simeq}(p)$.

Proof. We have to show that if f is an element of $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$ such that $c_{m_1,m_2}(f) = 0$ unless $p|m_1$ then f is zero. Theorem 2.5 implies that such an f is in fact a form on $\Gamma_{\simeq,\epsilon}(1)$. By Corollary 2.2, f can be considered to be an element of $S_2(\Gamma(1)) \otimes S_2(\Gamma(1))$. But $S_2(\Gamma(1))$ is zero, so f is zero. □

Proposition 2.8. *If* p *is a prime then*

$$\dim S_k(\Gamma_{\simeq,\epsilon}(p^l)) = \sum_{j=0}^{l} \dim \overline{S}_k(\Gamma_{\simeq,\epsilon}(p^j)).$$

Proof. This follows immediately from Theorem 2.5. □

Guess 2.6 would imply a similar statement for forms of arbitrary level.

3 Hecke Operators on $X_{\simeq,\epsilon}(N)$

Set

$$\Delta^*_{\simeq,\epsilon}(N) = \left\{ \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \;\middle|\; \begin{array}{l} a_i, b_i, c_i, d_i \in \mathbf{Z}, \\ a_i d_i - b_i c_i > 0, \\ (a_i d_i - b_i c_i, N) = 1, \\ \begin{array}{rcll} a_1 & \equiv & a_2 & \pmod N, \\ b_1 & \equiv & \epsilon b_2 & \pmod N, \\ \epsilon c_1 & \equiv & c_2 & \pmod N, \\ d_1 & \equiv & d_2 & \pmod N \end{array} \end{array} \right\}.$$

We can partition $\Delta^*_{\simeq,\epsilon}(N)$ into double $\Gamma_{\simeq,\epsilon}(N)$-cosets; each double coset is called a *Hecke operator*. They act on the spaces of modular forms as follows:

Let $\gamma = (\gamma_1, \gamma_2)$ be an element of $\Delta^*_{\simeq,\epsilon}(N)$, and let

$$\Gamma_{\simeq,\epsilon}(N)\gamma\Gamma_{\simeq,\epsilon}(N) = \coprod_j \Gamma_{\simeq,\epsilon}(N)\gamma_j$$

be a decomposition of the double coset generated by γ into left cosets. Then for a form f in $M_{(k_1,k_2)}(\Gamma_{\simeq,\epsilon}(N))$, we define

$$f|_{(k_1,k_2),\Gamma_{\simeq,\epsilon}(N)\gamma\Gamma_{\simeq,\epsilon}(N)} = \det(\gamma_1)^{(k_1/2)-1}\det(\gamma_2)^{(k_2/2)-1}\sum_j f|_{(k_1,k_2),\gamma_j}.$$

We see as in Shimura [14], Chapter 3, that $f|_{(k_1,k_2),\Gamma_{\simeq,\epsilon}(N)\gamma\Gamma_{\simeq,\epsilon}(N)}$ is an element of the space $M_{(k_1,k_2)}(\Gamma_{\simeq,\epsilon}(N))$, that cusp forms are transformed into cusp forms, and that the product of two Hecke operators is a sum of Hecke operators.

Let T_{n_1,n_2} be the operator given by the sum of the double cosets containing elements (γ_1,γ_2) where $\det(\gamma_i) = n_i$. This is zero unless $n_1 \equiv n_2 \pmod N$ and $(n_i, N) = 1$. Left coset representatives for it are given as follows:

Proposition 3.1. *Let (n_1, n_2) be a pair of positive integers that are congruent mod N and that are relatively prime to N. The set of elements of $\Delta^*_{\simeq,\epsilon}(N)$ that have determinant (n_1, n_2) then has the following left coset decomposition:*

$$\coprod_{\substack{a_1,a_2>0\\ a_i d_i=n_i\\ 0\le b_i<d_i}} \Gamma_{\simeq,\epsilon}(N)\left(\sigma_{a_1}\begin{pmatrix} a_1 & b_1N\\ 0 & d_1\end{pmatrix}, \sigma_{a_2}\begin{pmatrix} a_2 & b_2N\\ 0 & d_2\end{pmatrix}\right)$$

where, for $a \in (\mathbf{Z}/N\mathbf{Z})^*$, σ_a is any matrix in $\Gamma(1)$ that is congruent to $\left(\begin{smallmatrix} a^{-1} & 0\\ 0 & a\end{smallmatrix}\right)$ mod N.

Proof. First, note that the above cosets do indeed occur in T_{n_1,n_2}. Thus, we have to see that the representation is disjoint and that it gives us all of T_{n_1,n_2}. To lighten notation in the proof, we use the following convention: whenever we use an expression of the form $(x_i)_{i=1,2}$, we mean a pair of expressions (x_1, x_2).

To see disjointness (as left $\Gamma(1)\times\Gamma(1)$-cosets, hence as left $\Gamma_{\simeq,\epsilon}(N)$-cosets), assume that

$$\Gamma(1)\times\Gamma(1)\left(\sigma_{a_i}\begin{pmatrix} a_i & b_iN\\ 0 & d_i\end{pmatrix}\right)_{i=1,2} = \Gamma(1)\times\Gamma(1)\left(\sigma_{a_i'}\begin{pmatrix} a_i' & b_i'N\\ 0 & d_i'\end{pmatrix}\right)_{i=1,2}.$$

For this to be the case, we need the matrices

$$\sigma_{a_i}\begin{pmatrix} a_i & b_iN\\ 0 & d_i\end{pmatrix}\begin{pmatrix} a_i' & b_i'N\\ 0 & d_i'\end{pmatrix}^{-1}\sigma_{a_i'}^{-1}$$

to be elements of $\Gamma(1)$, which is true if and only if the matrices

$$\begin{pmatrix} a_i & b_iN\\ 0 & d_i\end{pmatrix}\begin{pmatrix} a_i' & b_i'N\\ 0 & d_i'\end{pmatrix}^{-1}$$

are. But that product is equal to

$$\frac{1}{n_i}\begin{pmatrix} a_id_i' & N(a_i'b_i - a_ib_i')\\ 0 & a_i'd_i\end{pmatrix}.$$

Since $a_i d_i = a'_i d'_i = n_i$, and $a_i, a'_i > 0$, the fact that the diagonal terms are integral forces $a_i = a'_i$ and $d_i = d'_i$. But then we need n_i to divide $Na_i(b_i - b'_i)$; since $(N, n_i) = 1$, this is equivalent to having $d_i | b_i - b'_i$, which forces $b_i = b'_i$ by our hypotheses on b_i and b'_i. Thus, the given cosets are indeed all disjoint from one another.

Now we have to show that they cover all of T_{n_1,n_2}. So let (δ_1, δ_2) be an element of $\Delta^*_{\simeq,\epsilon}(N)$ with determinant (n_1, n_2). By Shimura [14], Proposition 3.36, we can multiply δ_1 on the left by an element of $\Gamma(1)$ to get it into the form $\left(\begin{smallmatrix} a_1 & b_1 \\ 0 & d_1 \end{smallmatrix}\right)$, with $a_1 > 0$, $a_1 d_1 = n_1$, and $0 \le b_1 < d_1$. Subsequently multiplying it on the left by an element of the form $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ will put it into the form $\left(\begin{smallmatrix} a_1 & b_1 N \\ 0 & d_1 \end{smallmatrix}\right)$, but possibly with a different b_1. (We can still force b_1 to be in the range $0 \le b_1 < d_1$, however.) And since σ_{a_1} is an element of $\Gamma(1)$, we have shown that there is an element γ_1 of $\Gamma(1)$ such that $\gamma_1 \delta_1$ is of the form $\sigma_{a_1} \left(\begin{smallmatrix} a_1 & b_1 N \\ 0 & d_1 \end{smallmatrix}\right)$.

We can choose an element γ_2 of $\Gamma(1)$ such that (γ_1, γ_2) is in $\Gamma_{\simeq,\epsilon}(N)$: reduce γ_1 mod N, apply θ_ϵ to it, and lift it back to $\Gamma(1)$. Multiplying (δ_1, δ_2) on the left by (γ_1, γ_2), we can thus assume that δ_1 is of the form $\sigma_{a_1} \left(\begin{smallmatrix} a_1 & b_1 N \\ 0 & d_1 \end{smallmatrix}\right)$. But then the congruence relations force δ_2 to be congruent to the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & n_1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & n_2 \end{pmatrix} \pmod{N}.$$

Now that we have fixed δ_1 to be of the correct form, we still have to force δ_2 to be of the correct form, and we are only allowed to multiply δ_2 on the left by elements of $\Gamma_w(N)$. Thus, we need to find an element γ'_2 of $\Gamma_w(N)$ such that $\gamma'_2 \delta_2$ is of the form $\sigma_{a_2} \left(\begin{smallmatrix} a_2 & b_2 N \\ 0 & d_2 \end{smallmatrix}\right)$. However, δ_2 is in what Shimura calls Δ' (see Shimura [14], p. 68), so we can indeed find such a γ'_2 by Proposition 3.36 of Shimura [14]. □

From now on we follow the notational convention used in the above proof: whenever we use an expression of the form $(x_i)_{i=1,2}$, for any expressions x_i, we mean a pair of expressions (x_1, x_2). In particular, i will only be used to refer to an element of the set $\{1, 2\}$.

The action of the Hecke operators T_{n_1,n_2} descends to the spaces $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$:

Proposition 3.2. *If f is a form in $S_k(\Gamma_{\simeq,\epsilon}(N))$ such that $c_{m_1,m_2}(f) = 0$ unless $(N, m_i) > 1$ then $T_{n_1,n_2} f$ has the same property for all $n_1 \equiv n_2 \pmod{N}$.*

Proof. For $d|N$, define the operator i_d by

$$i_d(f) = \sum_{\substack{m_1, m_2 > 0 \\ d | m_1, m_2}} c_{m_1,m_2}(f) q_1^{m_1} q_2^{m_2}.$$

We then have the alternative definition of i_d as

$$i_d(f) = \frac{1}{d} \sum_{0 \le e < d} f \Big|_{\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & Ne/d \\ 0 & 1 \end{smallmatrix}\right)\right)},$$

since

$$\frac{1}{d}\sum_{0\le e<d} f\Big|\left(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&Ne/d\\0&1\end{smallmatrix}\right)\right) = \frac{1}{d}\sum_{e,m_1,m_2} c_{m_1,m_2}(f)q_1^{m_1}e^{2\pi\sqrt{-1}m_2(z_2+Ne/d)/N}$$
$$= \frac{1}{d}\sum_{e,m_1,m_2} c_{m_1,m_2}(f)q_1^{m_1}q_2^{m_2}e^{2\pi\sqrt{-1}m_2e/d}$$
$$= i_d(f).$$

(Note that if $d|m_2$ then also $d|m_1$, by Proposition 2.4.) By the principle of inclusion and exclusion, the statement that $c_{m_1,m_2}(f)=0$ unless $(N,m_i)>1$ is equivalent to having

$$f=\sum_{p|N} i_p(f) - \sum_{\substack{p_1,p_2|N\\ p_1<p_2}} i_{p_1p_2}(f)+\cdots,$$

and we want to show that if that is the case for f then it is also the case for $T_{n_1,n_2}f$. It is therefore enough to show that T_{n_1,n_2} commutes with any i_d. But

$$\begin{pmatrix}1 & eN/d\\ 0 & 1\end{pmatrix}\sigma_{a_2}\begin{pmatrix}a_2 & b_2N\\ 0 & d_2\end{pmatrix}$$

is congruent to

$$\sigma_{a_2}\begin{pmatrix}a_2 & b_2N\\ 0 & d_2\end{pmatrix}\begin{pmatrix}1 & en_2N/d\\ 0 & 1\end{pmatrix}$$

mod N, so by Proposition 3.1, commuting with T_{n_1,n_2} simply permutes the e's that occur in our alternate definition of i_d. $\square$

Proposition 3.2 would be an easy corollary to Guess 2.6.

Proposition 3.3. *For all $(\delta_1,\delta_2)\in\Delta^*_{\simeq,\epsilon}(N)$, the double cosets $\Gamma_{\simeq,\epsilon}(N)(\delta_1,\delta_2)\Gamma_{\simeq,\epsilon}(N)$ and $\Gamma_{\simeq,\epsilon}(N)(\delta_1^\iota,\delta_2^\iota)\Gamma_{\simeq,\epsilon}(N)$ are equal, where*

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}^\iota=\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}.$$

Proof. We need to find matrices (γ_1,γ_2) and (γ_1',γ_2') in $\Gamma_{\simeq,\epsilon}(N)$ such that

$$(\gamma_1\delta_1,\gamma_2\delta_2)=(\delta_1^\iota\gamma_1',\delta_2^\iota\gamma_2').$$

Since δ_1 and δ_1^ι have the same elementary divisors, we can choose a γ_1 and γ_1' that give us equality on the first coordinate. Now pick γ_2 and γ_2' such that (γ_1,γ_2) and (γ_1',γ_2') are in $\Gamma_{\simeq,\epsilon}(N)$. Then $\gamma_2\delta_2\equiv\delta_2^\iota\gamma_2' \pmod N$. But by Shimura [14], Lemma 3.29(1), we can then change γ_2 and γ_2' by elements of $\Gamma_w(N)$ so that $\gamma_2\delta_2=\delta_2^\iota\gamma_2'$, as desired. $\square$

We can define a Petersson inner product on the space of weight (k_1, k_2) cusp forms just as in the one-variable case:

$$\langle f, g\rangle = \int_{\Gamma_{\simeq,\epsilon}(N)\backslash\mathfrak{H}\times\mathfrak{H}} f(z_i)\overline{g(z_i)}y_1^{k_1-2}y_2^{k_2-2}\,dx_1\,dx_2\,dy_1\,dy_2$$

(where $z_i = x_i + \sqrt{-1}y_i$); then just as in Shimura [14], Formula (3.4.5), we see that the Hecke operators $\Gamma_{\simeq,\epsilon}(N)(\delta_1, \delta_2)\Gamma_{\simeq,\epsilon}(N)$ and $\Gamma_{\simeq,\epsilon}(N)(\delta_1^\iota, \delta_2^\iota)\Gamma_{\simeq,\epsilon}(N)$ are adjoint with respect to that inner product. Thus:

Corollary 3.4. *The $\mathbf{Z}$-algebra generated by the Hecke operators is a commutative algebra; the Hecke operators are self-adjoint with respect to the Petersson inner product on $S_k(\Gamma_{\simeq,\epsilon}(N))$ and simultaneously diagonalizable.*

Proof. The self-adjointness follows from Proposition 3.3 by the above discussion; the commutativity follows from Proposition 3.3 and Shimura [14], Proposition 3.8, and the simultaneous diagonalizability follows from the self-adjointness. □

The effect of Hecke operators on Fourier expansions is given as follows:

Proposition 3.5. *Let f be an element of $M_{(k_1,k_2)}(\Gamma_{\simeq,\epsilon}(N))$; if a is an element of $(\mathbf{Z}/N\mathbf{Z})^*$, let $f|_{\left(\sigma_a, \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)}$ have the Fourier expansion*

$$f|_{\left(\sigma_a, \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)}(z_1, z_2) = \sum_{m_1,m_2\geq 0} c_{a,m_1,m_2}q_1^{m_1}q_2^{m_2}.$$

If we set

$$T_{n_1,n_2}f(z_1, z_2) = \sum_{m_1,m_2\geq 0} d_{m_1,m_2}q_1^{m_1}q_2^{m_2}$$

then the d_{m_1,m_2}'s are given by

$$d_{m_1,m_2} = \sum_{\substack{a_1,a_2>0\\ a_i|(m_i,n_i)}} a_1^{k_1-1}a_2^{k_2-1}c_{(a_1/a_2),m_1n_1/a_1^2,m_2n_2/a_2^2}.$$

Proof. Let $f_a = f_{\left(\sigma_a, \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)}$. For any $a_1, a_2 \in (\mathbf{Z}/N\mathbf{Z})^*$, we have

$$f|_{(\sigma_{a_1}, \sigma_{a_2})} = f|_{\left(\sigma_{a_1/a_2}, \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)}.$$

Thus, using Proposition 3.1, we have:

$$\begin{aligned}
T_{n_1,n_2}f(z_1,z_2) &= n_1^{k_1/2-1}n_2^{k_2/2-1}\sum_{\substack{a_1,a_2>0\\ a_id_i=n_i\\ 0\le b_i<d_i}} f|\left(\sigma_{a_i}\left(\begin{smallmatrix} a_i & b_iN\\ 0 & d_i\end{smallmatrix}\right)\right)_{i=1,2}(z_1,z_2)\\
&= n_1^{k_1/2-1}n_2^{k_2/2-1}\sum_{a_i,b_i,d_i} f_{a_1/a_2}|\left(\left(\begin{smallmatrix} a_i & b_iN\\ 0 & d_i\end{smallmatrix}\right)\right)_{i=1,2}(z_1,z_2)\\
&= \sum_{\substack{m_1,m_2\ge 0\\ a_i,b_i,d_i}} c_{(a_1/a_2),m_1,m_2}\prod_{i=1,2} n_i^{k_i-1}d_i^{-k_i}e^{2\pi\sqrt{-1}m_i(a_iz_i+b_iN)/d_i}\\
&= \sum_{a_i,d_i,m_i,d_i|m_i} c_{(a_1/a_2),m_1,m_2}\prod_i n_i^{k_i-1}d_i^{-k_i+1}e^{2\pi\sqrt{-1}m_ia_iz_i/d_i}\\
&= \sum_{a_i,d_i,m_i} c_{(a_1/a_2),d_1m_1,d_2m_2}\prod_i n_i^{k_i-1}d_i^{-k_i+1}e^{2\pi\sqrt{-1}m_ia_iz_i}\\
&= \sum_{a_i,d_i,m_i} a_1^{k_1-1}a_2^{k_2-1}c_{(a_1/a_2),d_1m_1,d_2m_2}q_1^{m_1a_1}q_2^{m_2a_2}.
\end{aligned}$$

Comparing coefficients gives the desired result. □

Note that the matrices $\left(\sigma_a, \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)$ don't normalize $\Gamma_{\simeq,\epsilon}(N)$. This is why we have to introduce the functions f_a instead of simply diagonalizing $M_k(\Gamma_{\simeq,\epsilon}(N))$.

In particular, the following is true:

Corollary 3.6. *Let $f \in M_k(\Gamma_{\simeq,\epsilon}(N))$ be a simultaneous eigenform for all of the Hecke operators. Then if $\lambda_{m_1,m_2}(f)$ is the eigenvalue for T_{m_1,m_2}, we have*

$$c_{m_1,m_2}(f) = \lambda_{m_1,m_2}(f)c_{1,1}(f).$$

□

Unfortunately, this Corollary isn't quite as useful as one might hope, since the above coefficients are all zero by Proposition 2.4 unless $\epsilon = -1$! However, in that situation, we do get the following result:

Corollary 3.7. *If f and g are elements of $S_k(\Gamma_{\simeq,-1}(N))$ that are eigenfunctions for all T_{n_1,n_2}'s with the same eigenvalues then, considered as elements of $\overline{S}_k(\Gamma_{\simeq,-1}(N))$, they differ by a multiplicative constant.*

Proof. By Proposition 2.4 and Corollary 3.6, if $c = c_{1,1}(f)/c_{1,1}(g)$ then $c_{m_1,m_2}(f - cg)$ is zero unless $(m_i, N) > 1$. □

This can be restated as follows: let $\overline{\mathbf{T}}_{k,\epsilon}(N)$ be the $\mathbf{C}$-algebra of endomorphisms of $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ generated by the Hecke operators T_{n_1,n_2} for $n_1 \equiv n_2 \pmod N$. Then:

Proposition 3.8. *The space $\overline{S}_k(\Gamma_{\simeq,-1}(N))$ is a free module of rank one over $\overline{\mathbf{T}}_{k,-1}(N)$.*

Proof. By Corollary 3.4, we can find a basis for $\overline{S}_k(\Gamma_{\simeq,-1}(N))$ consisting of simultaneous eigenforms for all of the elements of $\overline{\mathbf{T}}_{k,-1}(N)$. Furthermore, by Corollary 3.7, no two of those eigenforms have the same eigenvalues. This implies our Proposition. □

Similarly, we define $\mathbf{T}^*_{k,\epsilon}(N)$ to be the $\mathbf{C}$-algebra of endomorphisms of $S_k(\Gamma_{\simeq,\epsilon}(N))$ generated by the Hecke operators T_{n_1,n_2} for $n_1 \equiv n_2 \pmod N$. Proposition 2.7 tells us that the spaces $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$ and $\overline{S}_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$ are equal; thus, the above Proposition has the following Corollary:

Corollary 3.9. *The space $S_{(2,2)}(\Gamma_{\simeq,-1}(p))$ is a free module of rank one over $\mathbf{T}^*_{(2,2),-1}(p)$.* □

With a little bit more care, we can use the above techniques to prove similar facts for $\epsilon = -k^2$ instead of just $\epsilon = -1$. (This isn't too surprising, since $X_{\simeq,-1}(N)$ and $X_{\simeq,-k^2}(N)$ are isomorphic.) They are in fact true for arbitrary ϵ; the proof demands different techniques, and will be given as Theorem 5.6. It does seem, however, that $X_{\simeq,-1}(N)$ is somehow the "dominant" $X_{\simeq,\epsilon}(N)$; see Sections 5 and 6 for further discussion of this matter.

Finally, we let $\mathbf{T}^*_{\equiv}(N)$ denote the free polynomial algebra over $\mathbf{C}$ with variables T_{n_1,n_2} for every pair n_1,n_2 of positive integers that are relatively prime to N and congruent mod N. This algebra acts on the spaces $S_k(\Gamma_{\simeq,\epsilon}(N))$ and $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ for all k and ϵ; its image in the endomorphism rings of those spaces gives us the algebras $\mathbf{T}^*_{k,\epsilon}(N)$ and $\overline{\mathbf{T}}_{k,\epsilon}(N)$ that we defined above.

4 Hecke Operators on $X_{\simeq}(N)$

The Hecke operators T_{n_1,n_2} defined above have the following modular interpretation: let (E_1, E_2, ϕ) be a point of $X_{\simeq,\epsilon}(N)$, and let $\pi_i\colon E_i \to E'_i$ be maps of elliptic curves of degree n_i, where $(n_i, N) = 1$. Then ϕ induces a map from $E'_1[N]$ to $E'_2[N]$ which is an isomorphism of group schemes; T_{n_1,n_2} sends our point to the sum of all points (E'_1, E'_2, ϕ) that arise in such a fashion. Why, then, do we impose the restriction that n_1 be congruent to n_2 mod N? The answer is that, if $\pi\colon E \to E'$ is a map of degree n (with $(n, N) = 1$) then π doesn't preserve the Weil pairing:

$$\begin{aligned}(\pi x, \pi y) &= (x, \pi^\vee \pi y)\\ &= (x, [n]y)\\ &= (x, y)^n.\end{aligned}$$

So if ϕ raises the Weil pairing to the ϵ'th power then, if we push it forward via maps of order n_i as above, the resulting map raises the Weil pairing to the $\epsilon n_2/n_1$ power. This explains why we had to assume that $n_1 \equiv n_2 \pmod N$ for the Hecke operators to act on the surfaces $X_{\simeq,\epsilon}(N)$. However, we should have Hecke operators T_{n_1,n_2} for arbitrary n_i with $(n_i, N) = 1$ which act on the surface $X_{\simeq}(N)$.

The above considerations, when translated into matrices, lead us to the following definition: for any ϵ, ϵ' in $(\mathbf{Z}/N\mathbf{Z})^*$, set

$$\Delta^*_{\simeq,\epsilon,\epsilon'}(N) = \left\{ \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \;\middle|\; \begin{array}{l} a_i, b_i, c_i, d_i \in \mathbf{Z}, \\ a_i d_i - b_i c_i > 0, \\ (a_i d_i - b_i c_i, N) = 1, \\ \quad a_1 \;\equiv\; a_2 \qquad (\mathrm{mod}\ N), \\ \quad b_1 \;\equiv\; \epsilon' b_2 \qquad (\mathrm{mod}\ N), \\ \quad \epsilon c_1 \;\equiv\; c_2 \qquad (\mathrm{mod}\ N), \\ \quad \epsilon d_1 \;\equiv\; \epsilon' d_2 \qquad (\mathrm{mod}\ N) \end{array} \right\}.$$

It is obvious from the definitions that $\Delta^*_{\simeq,\epsilon,\epsilon} = \Delta^*_{\simeq,\epsilon}$ and one easily checks that

$$\Delta^*_{\simeq,\epsilon,\epsilon'} \cdot \Delta^*_{\simeq,\epsilon',\epsilon''} \subset \Delta^*_{\simeq,\epsilon,\epsilon''}.$$

These facts imply in particular that $\Delta^*_{\simeq,\epsilon,\epsilon'}$ is invariant under multiplication by $\Gamma_{\simeq,\epsilon}(N)$ on the left and by $\Gamma_{\simeq,\epsilon'}(N)$ on the right; thus, $\Delta^*_{\simeq,\epsilon,\epsilon'}$ can be partitioned into Hecke operators that send forms on $X_{\simeq,\epsilon}(N)$ to forms on $X_{\simeq,\epsilon'}(N)$. For any n_1 and n_2 with $(n_i, N) = 1$ and with $\epsilon n_1 \equiv \epsilon' n_2 \pmod N$, we define the Hecke operator T_{n_1,n_2} to be the sum of the double cosets $\Gamma_{\simeq,\epsilon}(N)(\gamma_1,\gamma_2)\Gamma_{\simeq,\epsilon'}(N)$ occurring in $\Delta^*_{\simeq,\epsilon,\epsilon'}$ for which $\det(\gamma_i) = n_i$. This does depend on ϵ, but it has a natural set of left coset representatives that is independent of ϵ:

Proposition 4.1. *Let n_1 and n_2 be positive integers that are relatively prime to N, and let ϵ and ϵ' be elements of $(\mathbf{Z}/N\mathbf{Z})^*$ such that $\epsilon n_1 \equiv \epsilon' n_2 \pmod N$. Then the set of elements of $\Delta^*_{\simeq,\epsilon,\epsilon'}(N)$ that have determinant (n_1, n_2) has the following left coset decomposition:*

$$\coprod_{\substack{a_1,a_2>0 \\ a_i d_i = n_i \\ 0 \le b_i < d_i}} \Gamma_{\simeq,\epsilon}(N) \left(\sigma_{a_1} \begin{pmatrix} a_1 & b_1 N \\ 0 & d_1 \end{pmatrix}, \sigma_{a_2} \begin{pmatrix} a_2 & b_2 N \\ 0 & d_2 \end{pmatrix} \right)$$

where, for $a \in (\mathbf{Z}/N\mathbf{Z})^*$, σ_a is any matrix that is congruent to $\left(\begin{smallmatrix} a^{-1} & 0 \\ 0 & a \end{smallmatrix}\right)$ mod N. Furthermore, the above left cosets are also disjoint as $\Gamma(1) \times \Gamma(1)$ cosets.

Proof. The proof is the same as the proof of Proposition 3.1. $\square$

Recall that we defined

$$S_{k,\simeq}(N) = \prod_{\epsilon \in (\mathbf{Z}/N\mathbf{Z})^*} S_k(\Gamma_{\simeq,\epsilon}(N))$$

and made a similar definition for $\overline{S}_{k,\simeq}(N)$. Also, if $\mathbf{f}$ is an element of $S_{k,\simeq}(N)$, we write $\mathbf{f}_\epsilon$ for its ϵ'th component. We then define Hecke operators T_{n_1,n_2} acting on the space $S_{k,\simeq}(N)$ by setting $(T_{n_1,n_2}\mathbf{f})_\epsilon = T_{n_1,n_2}(\mathbf{f}_{\epsilon n_2/n_1})$; Proposition 4.1 shows that that action "looks the same" for all ϵ. The following Proposition shows that the action of these Hecke operators descends to the spaces $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$, and hence allows us to similarly define an action of them on the space $\overline{S}_{k,\simeq}(N)$:

Proposition 4.2. *If f is a form in $S_k(\Gamma_{\simeq,\epsilon}(N))$ such that $c_{m_1,m_2}(f) = 0$ unless $(N, m_i) > 1$ then $T_{n_1,n_2}f$ has the same property for all n_i relatively prime to N.*

Proof. The proof is the same as the proof of Proposition 3.2. □

The action on Fourier expansions is also as expected from Proposition 3.5, with the same proof:

Proposition 4.3. *Let f be an element of $M_{(k_1,k_2)}(\Gamma_{\simeq,\epsilon}(N))$; if a is an element of $(\mathbf{Z}/N\mathbf{Z})^*$, let $f|_{\left(\sigma_a, \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)}$ have the Fourier expansion*

$$f|_{\left(\sigma_a, \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)}(z_1, z_2) = \sum_{m_1,m_2 \geq 0} c_{a,m_1,m_2} q_1^{m_1} q_2^{m_2}.$$

If we set

$$T_{n_1,n_2} f(z_1, z_2) = \sum_{m_1,m_2 \geq 0} d_{m_1,m_2} q_1^{m_1} q_2^{m_2}$$

then the d_{m_1,m_2}'s are given by

$$d_{m_1,m_2} = \sum_{\substack{a_1,a_2>0 \\ a_i|(m_i,n_i)}} a_1^{k_1-1} a_2^{k_2-1} c_{(a_1/a_2),m_1n_1/a_1^2,m_2n_2/a_2^2}.$$

□

This Proposition (or Proposition 4.1, which it is a corollary of) allows us to translate theorems about forms on $X_w(N)$ into theorems about forms on $X_{\simeq}(N)$: if f is a form on some $X_{\simeq,\epsilon}(N)$ and we have a Hecke operator T_{n_1,n_2}, we can consider f to be form on $X_w(N) \times X_w(N)$ and apply $T_{n_1} \times T_{n_2}$ to it there. This gives us a form on $X_w(N) \times X_w(N)$; but by Proposition 4.1, that has the same effect as directly applying the T_{n_1,n_2} that we have defined above to f considered as a form on $X_{\simeq,\epsilon}(N)$, so our resulting form, which is a priori only a form on $X_w(N) \times X_w(N)$, is really a form on $X_{\simeq,\epsilon n_1/n_2}(N)$. Thus, the fact that the Hecke operators T_n (with $(n, N) = 1$) on $X_w(N)$ commute implies that our Hecke operators T_{n_1,n_2} commute. Similarly, we can define a Petersson inner product on $S_{k,\simeq}(N)$ by taking the orthogonal direct sum of the inner products on the $S_k(\Gamma_{\simeq,\epsilon}(N))$'s; our Hecke operators are then normal with respect to that inner product because the Hecke operators on $X_w(N)$ are.

It is frequently useful to encapsulate this relation between forms on $X_{\simeq}(N)$ and forms on $X_w(N)$ by defining a map $\overline{\Sigma}\colon \overline{S}_{k,\simeq}(N) \to \overline{S}_{k_1}(\Gamma_w(N)) \otimes \overline{S}_{k_2}(\Gamma_w(N))$ which sends $\mathbf{f} \in \overline{S}_{k,\simeq}(N)$ to $\sum_{\epsilon\in(\mathbf{Z}/N\mathbf{Z})^*} \mathbf{f}_\epsilon$. By $\overline{S}_{k_i}(\Gamma_w(N))$ we mean $S_{k_i}(\Gamma_w(N))/V$ where V is the space of forms $f \in S_{k_i}(\Gamma_w(N))$ such that $c_m(f) = 0$ unless $(m, k_i) > 1$; it is a module over the Hecke algebra generated by the operators T_n with $(n, N) = 1$, and its eigenspaces for that algebra are one-dimensional. The following two Propositions then sum up the discussion of the previous paragraph:

Proposition 4.4. *The map from $S_{k,\simeq}(N)$ to $S_{k_1}(\Gamma_w(N)) \otimes S_{k_2}(\Gamma_w(N))$ that sends a form* $\mathbf{f}$ *to* $\sum_{\epsilon\in(\mathbf{Z}/N\mathbf{Z})^*}\mathbf{f}_\epsilon$ *commutes with the action of Hecke operators. It descends to an injection* $\overline{\Sigma}\colon \overline{S}_{k,\simeq}(N) \to \overline{S}_{k_1}(\Gamma_w(N)) \otimes \overline{S}_{k_2}(\Gamma_w(N))$*; if* $\mathbf{f} \in \overline{S}_{k,\simeq}(N)$ *then*

$$\mathbf{f}_\epsilon = \sum_{\substack{m_1,m_2>0 \\ \epsilon m_1+m_2\equiv 0 (\mathrm{mod}\, N) \\ (m_i,N)=1}} c_{m_1,m_2}(\overline{\Sigma}\mathbf{f})q_1^{m_1}q_2^{m_2}.$$

Proof. The only parts that remain to be proved are that the last map is an injection and that $\mathbf{f}_\epsilon$ can be recovered in the given manner. First, we note that, for all m_1, m_2 with $(m_i, N) = 1$,

$$c_{m_1,m_2}(\overline{\Sigma}\mathbf{f}) = \sum_{\epsilon\in(\mathbf{Z}/N\mathbf{Z})^*} c_{m_1,m_2}(\mathbf{f}_\epsilon).$$

But Proposition 2.4 says that $c_{m_1,m_2}(\mathbf{f}_\epsilon) = 0$ unless $\epsilon \equiv -m_2/m_1 \pmod N$; $c_{m_1,m_2}(\overline{\Sigma}\mathbf{f})$ therefore equals $c_{m_1,m_2}(\mathbf{f}_{-m_2/m_1})$. This together with Proposition 2.4 immediately implies our formula for $\mathbf{f}_\epsilon$. And if $\overline{\Sigma}\mathbf{f} = 0$ then this implies that, for all ϵ and for all m_i such that $\epsilon \equiv -m_2/m_1 \pmod N$, $c_{m_1,m_2}(\mathbf{f}_\epsilon)$ is zero. But that implies that $\mathbf{f}_\epsilon = 0$ by using Proposition 2.4 again. □

Proposition 4.5. *The* $\mathbf{Z}$*-algebra generated by the Hecke operators* T_{n_1,n_2} *acting on* $S_{k,\simeq}(N)$ *is a commutative algebra; the Hecke operators are normal with respect to the Petersson inner product on* $S_{k,\simeq}(N)$ *and simultaneously diagonalizable.*

Proof. This follows from the above reduction of these facts to facts about forms on $X_w(N)$ and from Shimura [14], Theorem 3.41. □

Let $\mathbf{f}$ be an element of $S_{k,\simeq}(N)$, and let m_1 and m_2 be integers relatively prime to N. We define $c_{m_1,m_2}(f)$ to be equal to $c_{m_1,m_2}(\mathbf{f}_{-m_2/m_1})$. We also make the same definition for $\mathbf{f} \in \overline{S}_{k,\simeq}(N)$. If we set $f = \sum_{\epsilon\in(\mathbf{Z}/N\mathbf{Z})^*}\mathbf{f}_\epsilon$ then f is a form on $X_w(N) \times X_w(N)$, and $c_{m_1,m_2}(\mathbf{f}) = c_{m_1,m_2}(f)$, by Proposition 2.4, as noted in the proof of Proposition 4.4.

Proposition 4.6. *Let* $\mathbf{f}$ *be an element of* $S_{k,\simeq}(N)$*; for* $a \in (\mathbf{Z}/N\mathbf{Z})^*$*, let* $\mathbf{f}_a$ *be defined by*

$$(\mathbf{f}_a)_\epsilon = \mathbf{f}_{(a^{-2}\epsilon)}|_{\left(\sigma_a, \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)}.$$

Then for all n_1, n_2 *with* $(n_i, N) = 1$ *and for all* m_1, m_2 *with* $(m_i, N) = 1$, *we have*

$$c_{m_1,m_2}(T_{n_1,n_2}\mathbf{f}) = \sum_{\substack{a_1,a_2>0 \\ a_i|(m_i,n_i)}} a_1^{k_1-1}a_2^{k_2-1}c_{m_1n_1/a_1^2,m_2n_2/a_2^2}(\mathbf{f}_{a_1/a_2}).$$

Proof. This is a corollary of Proposition 4.3. □

We define $\mathbf{T}^*(N)$ to be the free polynomial algebra over $\mathbf{C}$ with generators T_{n_1,n_2} for each pair n_1, n_2 of positive integers that are relatively prime to N. We define $\mathbf{T}^*_{k,\simeq}(N)$ to be its image in the endomorphism ring of $S_{k,\simeq}(N)$; we define $\overline{\mathbf{T}}_{k,\simeq}(N)$ to be its image in the endomorphism ring of $\overline{S}_{k,\simeq}(N)$.

Corollary 4.7. *If $\mathbf{f} \in S_{k,\simeq}(N)$ is a simultaneous eigenform for all Hecke operators T_{n_1,n_2} in $\mathbf{T}^*_{k,\simeq}(N)$ with eigenvalues $\lambda_{n_1,n_2}(\mathbf{f})$ then, for all m_1 and m_2 with $(m_i, N) = 1$, we have*

$$c_{m_1,m_2}(\mathbf{f}) = \lambda_{m_1,m_2}(\mathbf{f})c_{1,1}(\mathbf{f}).$$

□

Thus, if $\mathbf{f}$ is a non-zero element of $\overline{S}_{k,\simeq}(N)$ that is an eigenform for all the T_{n_1,n_2}'s then $c_{1,1}(\mathbf{f})$ is also non-zero; we call such an $\mathbf{f}$ a *normalized eigenform* if $c_{1,1}(\mathbf{f}) = 1$.

Corollary 4.8. *The space $\overline{S}_{k,\simeq}(N)$ is a free module of rank one over $\overline{\mathbf{T}}_{k,\simeq}(N)$.*

Proof. By Proposition 4.5, we can find a basis for $\overline{S}_{k,\simeq}(N)$ consisting of simultaneous eigenforms for all elements of $\overline{\mathbf{T}}_{k,\simeq}(N)$; the previous Corollary shows that the eigenspaces are one-dimensional, implying this Corollary. □

Corollary 4.9. *The space $S_{(2,2),\simeq}(p)$ is a free module of rank one over $\mathbf{T}^*_{(2,2),\simeq}(p)$.*

Proof. This follows from Corollary 4.8 and Proposition 2.7. □

We should also mention a special class of operators that are contained in our Hecke algebras $\mathbf{T}^*_{k,\simeq}(N)$. Given elements ϵ and a of $(\mathbf{Z}/N\mathbf{Z})^*$, we have

$$(1,\sigma_a)^{-1}\Gamma_{\simeq,\epsilon}(N)(1,\sigma_a) = \Gamma_{\simeq,a^{-2}\epsilon}(N).$$

The action of $(1,\sigma_a)$ therefore gives an isomorphism from $S_k(\Gamma_{\simeq,\epsilon}(N))$ to $S_k(\Gamma_{\simeq,a^{-2}\epsilon}(N))$, denoted by $\langle a\rangle$. However, the action is the same if we multiply $(1,\sigma_a)$ by $\left(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}a&0\\0&a\end{smallmatrix}\right)\right)$; but if we consider it as an operator on $X_w(N)\times X_w(N)$, as in the discussion before Proposition 4.4, then this, up to a constant, is the product of the identity with the Hecke operator $T(a,a)$. By Shimura [14], Theorem 3.24(4), $T(a,a)$ is in the $\mathbf{Q}$-algebra generated by the $T(n)$'s, so $\langle a\rangle$ is in $\mathbf{T}^*_{k,\simeq}(N)$. Thus:

Proposition 4.10. *For all $a \in (\mathbf{Z}/N\mathbf{Z})^*$, the operator $\langle a\rangle$ given by the action of $(1,\sigma_a)$ is an isomorphism from $S_k(\Gamma_{\simeq,\epsilon}(N))$ to $S_k(\Gamma_{\simeq,a^{-2}\epsilon}(N))$; furthermore, it is contained in $\mathbf{T}^*_{k,\simeq}(N)$.* □

As with the operators T_{n_1,n_2}, $\langle a\rangle$ extends to the spaces $S_{k,\simeq}(N)$ and $\overline{S}_{k,\simeq}(N)$ via the definition $(\langle a\rangle\mathbf{f})_\epsilon = \langle a\rangle(\mathbf{f}_{a^2\epsilon})$.

5 Relationships between the Spaces $\overline{S}_{k,\simeq}(N)$, $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$, and $\overline{S}_k(\Gamma_{\simeq,-1}(N))$

When trying to prove that Hecke eigenspaces in $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ are one-dimensional, we ran into problems because forms are "missing" Fourier coefficients: in particular, they don't have a $(1,1)$ Fourier coefficient unless $\epsilon \equiv -1 \pmod N$, so we couldn't simply use Corollary 3.6. However, the space $\overline{S}_{k,\simeq}(N)$ doesn't have that problem, and there is a natural projection map from $\overline{S}_{k,\simeq}(N)$ to $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$. This gives us a replacement for the missing Fourier coefficients; it also gives us a framework for seeing how the spaces $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ differ (as $\mathbf{T}^*_{\equiv}(N)$-modules) as ϵ varies.

The key Lemma here is the following:

Lemma 5.1. *The space $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ has a basis consisting of simultaneous $\overline{\mathbf{T}}_{k,\epsilon}(N)$-eigenforms f that are of the form $\mathbf{f}_\epsilon$ for simultaneous $\overline{\mathbf{T}}_{k,\simeq}(N)$-eigenforms $\mathbf{f} \in \overline{S}_{k,\simeq}(N)$.*

Proof. If $\mathbf{f} \in \overline{S}_{k,\simeq}(N)$ is a $\overline{\mathbf{T}}_{k,\simeq}(N)$-eigenform then it is certainly an eigenform for those Hecke operators T_{n_1,n_2} where $n_1 \equiv n_2 \pmod N$; its ϵ-component $\mathbf{f}_\epsilon$ is therefore an eigenform for those operators as well. The Lemma then follows from the fact that $\overline{S}_{k,\simeq}(N)$ has a basis of eigenforms, by Proposition 4.5. $\square$

It is possible for two different $\overline{\mathbf{T}}_{k,\simeq}(N)$-eigenforms in $\overline{S}_{k,\simeq}(N)$ to project to the same $\overline{\mathbf{T}}_{k,\epsilon}(N)$-eigenform in $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$; we shall discuss this in Theorem 5.3. Also, some eigenforms in $\overline{S}_{k,\simeq}(N)$ project to zero for some choices of ϵ: see the comments after the proof of the following Proposition and Section 6. We shall state a slightly stronger version of this Lemma as Corollary 5.8.

Proposition 5.2. *If $f \in \overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ is a $\overline{\mathbf{T}}_{k,\epsilon}(N)$-eigenform then there is an $\overline{\mathbf{T}}_{k,-1}(N)$-eigenform $g \in \overline{S}_k(\Gamma_{\simeq,-1}(N))$ such that $c_{m_1,m_2}(g) = \lambda_{m_1,m_2}(f)$ for all $m_1 \equiv m_2 \pmod N$.*

Proof. By Lemma 5.1, there is an eigenform $\mathbf{f} \in \overline{S}_{k,\simeq}(N)$ such that $\lambda_{m_1,m_2}(\mathbf{f}) = \lambda_{m_1,m_2}(f)$ for all $m_1 \equiv m_2 \pmod N$. (We might a priori not be able to assume that $\mathbf{f}_\epsilon = f$; however, f is a linear combination of eigenforms projecting from $\overline{S}_{k,\simeq}(N)$, so those eigenforms must have the same eigenvalues as f.) We can assume that $\mathbf{f}$ is normalized. We then set $g = \mathbf{f}_{-1}$; it is a normalized eigenform contained in $\overline{S}_k(\Gamma_{\simeq,-1}(N))$, and $\lambda_{m_1,m_2}(g) = \lambda_{m_1,m_2}(\mathbf{f}) = \lambda_{m_1,m_2}(f)$. But Corollary 3.6 then tells us that $c_{m_1,m_2}(g) = \lambda_{m_1,m_2}(f)$. $\square$

Define $\overline{K}'_{k,\epsilon}(N)$ to be the subspace of $\overline{S}_k(\Gamma_{\simeq,-1}(N))$ generated by eigenforms whose eigenvalues are those of an eigenform in $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$; define $\overline{K}_{k,\epsilon}(N)$ to be the subspace of $\overline{S}_k(\Gamma_{\simeq,-1}(N))$ generated by eigenforms which do *not* arise in such a fashion. The Hecke algebra $\overline{\mathbf{T}}_{k,\epsilon}(N)$ is isomorphic to the image of $\overline{\mathbf{T}}_{k,-1}(N)$ in the endomorphism ring of $\overline{K}'_{k,\epsilon}(N)$: both actions are diagonalizable, so the rings are isomorphic iff the same eigenvalues occur, which is the case by the definition of $\overline{K}'_{k,\epsilon}(N)$ and by Proposition 5.2. In fact, the spaces $\overline{K}'_{k,\epsilon}(N)$ and $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ are isomorphic as $\mathbf{T}^*_{\equiv}(N)$-modules, because the eigenspaces in

$\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ are one dimensional; we shall prove this fact later as Theorem 5.6. Thus, $\overline{K}_{k,\epsilon}(N)$ measures the difference between $\overline{S}_k(\Gamma_{\simeq,-1}(N))$ and $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$; we shall study this space in Section 6.

Since the proof of Proposition 5.2 involved lifting eigenforms in $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ to eigenforms in $\overline{S}_{k,\simeq}(N)$, we'd like to see how ambiguous the choice of such a lifting is. The following Theorem answers that question:

Theorem 5.3. *Let* $\mathbf{f}$ *be an eigenform in* $\overline{S}_{k,\simeq}(N)$, *and let* $H \subset (\mathbf{Z}/N\mathbf{Z})^*$ *be the set of* ϵ *such that* $\mathbf{f}_{-\epsilon} \neq 0$. *Then:*

1. *H is a subgroup of* $(\mathbf{Z}/N\mathbf{Z})^*$.
2. *H depends only on* $\mathbf{f}_{-1}$.
3. *Every element of* $(\mathbf{Z}/N\mathbf{Z})^*/H$ *has order one or two.*
4. *If* $\mathbf{g}$ *is another eigenform in* $\overline{S}_{k,\simeq}(N)$ *then* $\mathbf{g}_{-1} = \mathbf{f}_{-1}$ *if and only if there is a character* χ *on H such that* $\mathbf{g}_{-\epsilon} = \chi(\epsilon)\mathbf{f}_{-\epsilon}$ *for all* $\epsilon \in H$.

First, we prove two Lemmas that we shall need during the proof of the Theorem.

Lemma 5.4. *Let* $\mathbf{f}$ *be an eigenform in* $\overline{S}_{k,\simeq}(N)$ *and* ϵ *an element of* $(\mathbf{Z}/N\mathbf{Z})^*$ *such that* $\mathbf{f}_\epsilon \neq 0$. *For any positive integers* m_1 *and* m_2 *there exist positive integers* n_1 *and* n_2 *such that* $\epsilon n_1 + n_2 \equiv 0$ *(mod N)*, $(n_i, m_i) = 1$ *for* $i \in \{1, 2\}$, *and* $c_{n_1,n_2}(\mathbf{f}_\epsilon) \neq 0$.

Proof. By Proposition 4.4, $\overline{\Sigma}\mathbf{f}$ is an eigenform in $\overline{S}_{k_1}(\Gamma_w(N)) \otimes \overline{S}_{k_2}(\Gamma_w(N))$. Since the eigenspaces in $\overline{S}_{k_i}(\Gamma_w(N))$ are one-dimensional, there must exist $f_i \in \overline{S}_{k_i}(\Gamma_w(N))$ such that $\overline{\Sigma}\mathbf{f} = f_1 \otimes f_2$.

For any $\epsilon' \in (\mathbf{Z}/N\mathbf{Z})^*$, set

$$f_{i,\epsilon'} = \sum_{\substack{n>0 \\ n\equiv\epsilon' (\mathrm{mod}\, N)}} c_n(f_i)q^n.$$

It is also an element of $\overline{S}_{k_i}(\Gamma_w(N))$. (This follows easily from Shimura [14], Proposition 3.64.) Then

$$\mathbf{f}_\epsilon = \sum_{\epsilon'\in(\mathbf{Z}/N\mathbf{Z})^*} f_{1,\epsilon'} \otimes f_{2,-\epsilon\epsilon'},$$

by Proposition 4.4.

Since $\mathbf{f}_\epsilon \neq 0$, there exists $\epsilon' \in (\mathbf{Z}/N\mathbf{Z})^*$ such that $f_{1,\epsilon'}$ and $f_{2,-\epsilon\epsilon'}$ are both nonzero. By Lang [10], Theorem VIII.3.1, there exist n_i such that $(n_i, Nm_i) = 1$ and that $c_{n_1}(f_{1,\epsilon'})$ and $c_{n_2}(f_{2,-\epsilon\epsilon'})$ are both non-zero. But Proposition 4.4 then implies that $c_{n_1,n_2}(\mathbf{f}_\epsilon) \neq 0$, as desired. □

Lemma 5.5. *Let $\mathbf{f} \in \overline{S}_{k,\simeq}(N)$ be an eigenform such that, for some ϵ, $\mathbf{f}_{-\epsilon}$ is non-zero. Then for all j, $\mathbf{f}_{-\epsilon^j}$ is non-zero. In particular, $\mathbf{f}_{-1/\epsilon}$ is non-zero.*

Proof. We can assume that $\mathbf{f}$ is a normalized eigenform. Since $\mathbf{f}_{-\epsilon}$ is non-zero, there is some coefficient $\lambda = c_{m_1,m_2}(\mathbf{f})$ that is non-zero, where $(m_i, N) = 1$ and $\epsilon m_1 \equiv m_2 \pmod N$. We therefore have $T_{m_1,m_2}(\mathbf{f}) = \lambda \mathbf{f}$, by Corollary 4.7, so for all $\epsilon' \in (\mathbf{Z}/N\mathbf{Z})^*$,

$$\begin{aligned}\lambda \mathbf{f}_{-\epsilon'} &= (T_{m_1,m_2}\mathbf{f})_{-\epsilon'} \\ &= T_{m_1,m_2}(\mathbf{f}_{-\epsilon' m_2/m_1}) \\ &= T_{m_1,m_2}(\mathbf{f}_{-\epsilon'\epsilon}).\end{aligned}$$

In particular, setting $\epsilon' = \epsilon^j$, we see that

$$\lambda \mathbf{f}_{-\epsilon^j} = T_{m_1,m_2}(\mathbf{f}_{-\epsilon^{j+1}}),$$

so if $\mathbf{f}_{-\epsilon^j}$ is non-zero then, since λ also is, $\mathbf{f}_{-\epsilon^{j+1}}$ is as well, and we have our Lemma by induction. □

Proof of Theorem 5.3. We can assume that $\mathbf{f}$ is a normalized eigenform. To show that H is a subgroup, let ϵ_1 and ϵ_2 be elements of H. Thus, there exist $n_{1,i}$ and $n_{2,i}$ (for $i = 1, 2$) such that $c_{n_{1,i},n_{2,i}}(\mathbf{f}_{-\epsilon_i})$ is non-zero; by Lemma 5.4, we can assume that $(n_{1,1}, n_{1,2}) = (n_{2,1}, n_{2,2}) = 1$, and by Proposition 2.4, $\epsilon_i n_{1,i} \equiv n_{2,i} \pmod N$.

By Corollary 4.7, $c_{n_{1,i},n_{2,i}}(\mathbf{f}) = \lambda_{n_{1,i},n_{2,i}}(\mathbf{f})$. But

$$\lambda_{n_{1,1}n_{1,2},n_{2,1}n_{2,2}}(\mathbf{f}) = \lambda_{n_{1,1},n_{2,1}}(\mathbf{f})\lambda_{n_{1,2},n_{2,2}}(\mathbf{f}),$$

by our assumption that $(n_{i,1}, n_{i,2}) = 1$, and is therefore non-zero, as is the corresponding Fourier coefficient of $\mathbf{f}$. This is a Fourier coefficient of $\mathbf{f}_\epsilon$ for

$$\begin{aligned}\epsilon &\equiv -(n_{2,1}n_{2,2}/n_{1,1}n_{1,2}) \\ &\equiv -(n_{2,1}/n_{1,1})(n_{2,2}/n_{1,2}) \\ &\equiv -\epsilon_1\epsilon_2.\end{aligned}$$

Thus, $\epsilon_1\epsilon_2 \in H$, so H is a subgroup of $(\mathbf{Z}/N\mathbf{Z})^*$.

To see that every element of $(\mathbf{Z}/N\mathbf{Z})^*/H$ has order one or two, pick $a \in (\mathbf{Z}/N\mathbf{Z})^*$ and let $\mathbf{f} \in \overline{S}_{k,\simeq}(N)$ be an eigenform. Then $(\langle a\rangle \mathbf{f})_{-1} = \langle a\rangle(\mathbf{f}_{-a^2})$. Since $\langle a\rangle$ is an invertible operator contained in $\overline{\mathbf{T}}_{k,\simeq}(N)$, by Proposition 4.10, the fact that $\mathbf{f}_{-1} \neq 0$ implies that $(\langle a\rangle \mathbf{f})_{-1} \neq 0$ as well, so so $\mathbf{f}_{-a^2} \neq 0$ and $a^2 \in H$.

To show that H depends only on $\mathbf{f}_{-1}$, it's enough to prove the last part of the Theorem. We shall prove that if $\mathbf{g}$ is an eigenform such that $\mathbf{g}_{-1} = \mathbf{f}_{-1}$ then there is a character χ on H such that $\mathbf{g}_{-\epsilon} = \chi(\epsilon)\mathbf{f}_{-\epsilon}$; the converse (i.e. that $\mathbf{g}$'s constructed in that fashion are eigenforms) follows easily from the definition of $T_{n_1,n_2}\mathbf{f}$ as $(T_{n_1,n_2}\mathbf{f})_\epsilon = T_{n_1,n_2}(\mathbf{f}_{\epsilon n_2/n_1})$.

Thus, assume that we have normalized eigenforms $\mathbf{f}$ and $\mathbf{g}$ such that $\mathbf{f}_{-1} = \mathbf{g}_{-1}$; let ϵ be an element of H, so $\mathbf{f}_{-\epsilon} \neq 0$. By Lemma 5.5, $\mathbf{f}_{-(1/\epsilon)}$ is also non-zero. There then exist m_1

and m_2 relatively prime to N such that $m_1 \equiv \epsilon m_2 \pmod N$ and $c_{m_1,m_2}(\mathbf{f}) \neq 0$. Therefore, $\lambda_{m_1,m_2}(\mathbf{f})$ is also non-zero. And

$$\begin{aligned}\lambda_{m_1,m_2}(\mathbf{f})\mathbf{f}_{-\epsilon} &= (T_{m_1,m_2}\mathbf{f})_{-\epsilon} \\ &= T_{m_1,m_2}(\mathbf{f}_{-\epsilon m_2/m_1}) \\ &= T_{m_1,m_2}(\mathbf{f}_{-1}) \\ &= T_{m_1,m_2}(\mathbf{g}_{-1}) \\ &= \lambda_{m_1,m_2}(\mathbf{g})\mathbf{g}_{-\epsilon}.\end{aligned}$$

Since $\lambda_{m_1,m_2}(\mathbf{f})$ and $\mathbf{f}_{-\epsilon}$ are both non-zero, this implies that $\lambda_{m_1,m_2}(\mathbf{g})$ and $\mathbf{g}_{-\epsilon}$ are also both non-zero, and that if we define $\chi(\epsilon) = \lambda_{m_1,m_2}(\mathbf{f})/\lambda_{m_1,m_2}(\mathbf{g})$ (for any choice of m_i such that $m_1 \equiv \epsilon m_2 \pmod N$ and such that $c_{m_1,m_2}(\mathbf{f}_{-1/\epsilon}) \neq 0$) then $\mathbf{g}_{-\epsilon} = \chi(\epsilon)\mathbf{f}_{-\epsilon}$, as desired. We then only have to show that χ is a character, not just a function; that follows by using the same arguments that we used to show that H was a subgroup, using the multiplicativity of λ_{m_1,m_2} and Lemma 5.4. □

We now have all the tools necessary to prove that the spaces $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ are free of rank one over $\overline{\mathbf{T}}_{k,\epsilon}(N)$ for all $\epsilon \in (\mathbf{Z}/N\mathbf{Z})^*$.

Theorem 5.6. *For all $\epsilon \in (\mathbf{Z}/N\mathbf{Z})^*$, all of the $\overline{\mathbf{T}}_{k,\epsilon}(N)$-eigenspaces in $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ are one-dimensional, and the space $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ is a free module of rank one over $\overline{\mathbf{T}}_{k,\epsilon}(N)$.*

Proof. Pick a $\overline{\mathbf{T}}_{k,\epsilon}(N)$-eigenspace in $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$. By Lemma 5.1, it has a basis consisting of eigenforms of the form $\mathbf{f}_\epsilon$ where $\mathbf{f}$ is a normalized eigenform in $\overline{S}_{k,\simeq}(N)$. Thus, we need to show that if $\mathbf{f}$ and $\mathbf{g}$ are normalized eigenforms in $\overline{S}_{k,\simeq}(N)$ such that $\mathbf{f}_\epsilon$ and $\mathbf{g}_\epsilon$ are in the same eigenspace then $\mathbf{f}_\epsilon$ and $\mathbf{g}_\epsilon$ are in fact constant multiples of each other. However, $\lambda_{n_1,n_2}(\mathbf{f}_\epsilon) = \lambda_{n_1,n_2}(\mathbf{f}) = c_{n_1,n_2}(\mathbf{f})$, for all $n_1 \equiv n_2 \pmod N$, so the fact that $\mathbf{f}_\epsilon$ and $\mathbf{g}_\epsilon$ have the same eigenvalues simply means that $\mathbf{f}_{-1}$ and $\mathbf{g}_{-1}$ are equal. Theorem 5.3 then implies that $\mathbf{f}_\epsilon$ and $\mathbf{g}_\epsilon$ are multiples of each other. Thus, the eigenspaces are one-dimensional, and $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ is indeed a free $\overline{\mathbf{T}}_{k,\epsilon}(N)$-module of rank one. □

The basic idea behind the proof of Theorem 5.6 is that, if we have a form in $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$, we can use Lemma 5.1 to fill in the Fourier coefficients that are forced to vanish by Proposition 2.4. Of course, it's often easiest just to work with $\overline{S}_{k,\simeq}(N)$ and $X_{\simeq}(N)$ directly. As usual, we have the following Corollary:

Corollary 5.7. *For all $\epsilon \in (\mathbf{Z}/p\mathbf{Z})^*$, the space $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$ is a free module of rank one over $\mathbf{T}^*_{(2,2),\epsilon}(p)$.*

Proof. This follows from Theorem 5.6 and Proposition 2.7. □

We also have the following slight strengthening of Lemma 5.1:

Corollary 5.8. *For every eigenform $f \in \overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ there exists an eigenform $\mathbf{f} \in \overline{S}_{k,\simeq}(N)$ such that $\mathbf{f}_\epsilon = f$.*

Proof. By Lemma 5.1, $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ has a basis consisting of such eigenforms. Since the eigenspaces are one-dimensional, however, every eigenform must be a multiple of one of those basis elements. □

And, finally, we have the facts that $\overline{K}'_{k,\epsilon}(N)$ and $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ are isomorphic as $\mathbf{T}^*_{\equiv}(N)$-modules and a geometric consequence of that fact:

Corollary 5.9. *For all $\epsilon \in (\mathbf{Z}/N\mathbf{Z})^*$, $\overline{S}_k(\Gamma_{\simeq,-1}(N))$ is isomorphic to $\overline{K}_{k,\epsilon}(N) \oplus \overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ as a module over $\mathbf{T}^*_{\equiv}(N)$.*

Proof. By definition, $\overline{S}_k(\Gamma_{\simeq,-1}(N)) = \overline{K}_{k,\epsilon}(N) \oplus \overline{K}'_{k,\epsilon}(N)$. But $\overline{K}'_{k,\epsilon}(N)$ is a $\mathbf{T}^*_{\equiv}(N)$-module that is a direct sum of one-dimensional spaces corresponding to the Hecke eigenvalues occurring in $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$; the Corollary then follows from Theorem 5.6. □

Corollary 5.10. *If N is a power of a prime then the geometric genus of (a desingularization of) $X_{\simeq,\epsilon}(N)$ is maximized when $\epsilon = -1$.*

Proof. Corollary 2.3 and Proposition 2.8 allow us to reduce this Corollary to showing that, for all ϵ and for all $M|N$, the dimension of $\overline{S}_{(2,2)}(\Gamma_{\simeq,-1}(M))$ is at least as large as the dimension of $\overline{S}_{(2,2)}(\Gamma_{\simeq,\epsilon}(M))$. This in turn follows directly from the above Corollary. □

This Corollary is in fact true for all $N \leq 30$, as can be seen by examining the tables at the end of Kani and Schanz [8]. Guess 2.6 would imply this Corollary for all natural numbers N, since in that case Proposition 2.8 would be true for all N.

6 The Hecke Kernel

In the previous Section, we saw that, for all $\epsilon \in (\mathbf{Z}/N\mathbf{Z})^*$, we can write $\overline{S}_k(\Gamma_{\simeq,-1}(N))$ as $\overline{K}_{k,\epsilon}(N) \oplus \overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$. Thus, the key to understanding modular forms in all of the $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$'s is to understand the space $\overline{S}_k(\Gamma_{\simeq,-1}(N))$; once we have that, we then need to understand its subspaces $\overline{K}_{k,\epsilon}(N)$. The goal of the present section is to study those subspaces, which we call "Hecke kernels". Note that Corollary 5.10 gives us a geometric interpretation of these spaces in some situations.

We first give the alternate following characterizations of forms in $\overline{K}_{k,\epsilon}(N)$:

Proposition 6.1. *Let f be an eigenform in $\overline{S}_k(\Gamma_{\simeq,-1}(N))$ and let ϵ be an element of $(\mathbf{Z}/N\mathbf{Z})^*$. The following are equivalent:*

1. *f is in $\overline{K}_{k,\epsilon}(N)$.*
2. *For any or all eigenforms $\mathbf{f} \in \overline{S}_{k,\simeq}(N)$ such that $\mathbf{f}_{-1} = f$, $\mathbf{f}_\epsilon = 0$.*
3. *For all n_1, n_2 such that $\epsilon n_1 + n_2 \equiv 0 \pmod{N}$, $T_{n_1,n_2} f = 0$.*
4. *For all m_1, m_2, n_1, and n_2 with $n_1 m_1 \equiv n_2 m_2 \pmod{N}$, $\epsilon n_1 + n_2 \equiv 0 \pmod{N}$, and $(n_i, m_i) = 1$ for $i \in \{1, 2\}$, we have $c_{n_1 m_1, n_2 m_2}(f) = 0$.*

Proof. We can assume f is a normalized eigenform. First we, show the equivalence between 1 and 2: let $\mathbf{f}$ be an eigenform in $S_{k,\simeq}(N)$ such that $\mathbf{f}_{-1} = f$, which we can find by Corollary 5.8. By Theorem 5.3, $\mathbf{f}_\epsilon$ only depends on the choice of $\mathbf{f}$ up to a non-zero constant multiple. If $\mathbf{f}_\epsilon \neq 0$ then $\mathbf{f}_\epsilon$ is an eigenform in $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ whose eigenvalues are the same as those of f, hence are the same as the Fourier coefficients of f, so f isn't in $\overline{K}_{k,\epsilon}(N)$. Conversely, if f isn't in $\overline{K}_{k,\epsilon}(N)$ then there exists an eigenform $g \in \overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ whose eigenvalues are the Fourier coefficients of f. Corollary 5.8 allows us to pick an eigenform $\mathbf{g} \in \overline{S}_{k,\simeq}(N)$ such that $\mathbf{g}_\epsilon = g$; multiplying it (and g) by a constant factor, we can assume that $\mathbf{g}$ is a normalized eigenform. Then $\mathbf{g}_\epsilon$ and $\mathbf{g}_{-1}$ have the same eigenvalues, so $\mathbf{g}_{-1}$ is a multiple of f, by our assumption on g; $\mathbf{g}$ therefore gives us an eigenform in $\overline{S}_{k,\simeq}(N)$ such that $\mathbf{g}_{-1} = f$ and $\mathbf{g}_\epsilon \neq 0$, as desired. By Theorem 5.3, this is independent of the choice of $\mathbf{g}$, justifying our use of the phrase "any or all".

Next we show that 2 and 3 are equivalent. Thus, we have normalized eigenforms $f \in \overline{S}_k(\Gamma_{\simeq,-1}(N))$ and $\mathbf{f} \in \overline{S}_{k,\simeq}(N)$ such that $f = \mathbf{f}_{-1}$ and we want to show that $\mathbf{f}_\epsilon = 0$ iff, for all n_1 and n_2 such that $\epsilon n_1 + n_2 \equiv 0 \pmod N$, $T_{n_1,n_2} f = 0$. First assume that $\mathbf{f}_\epsilon = 0$. By Lemma 5.5, $\mathbf{f}_{1/\epsilon} = 0$. Then for all n_i as above,

$$\begin{aligned}
T_{n_1,n_2} f &= T_{n_1,n_2}(\mathbf{f}_{-1}) \\
&= (T_{n_1,n_2}\mathbf{f})_{-n_1/n_2} \\
&= (T_{n_1,n_2}\mathbf{f})_{1/\epsilon} \\
&= \lambda_{n_1,n_2}(\mathbf{f})\mathbf{f}_{1/\epsilon} \\
&= 0.
\end{aligned}$$

Conversely, if $T_{n_1,n_2} f = 0$ for all n_i with $\epsilon n_1 + n_2 \equiv 0 \pmod N$ then the above series of equalities shows that $\lambda_{n_1,n_2}(\mathbf{f})\mathbf{f}_{1/\epsilon}$ is always zero, or equivalently (by Corollary 4.7), $c_{n_1,n_2}(\mathbf{f})\mathbf{f}_{1/\epsilon} = 0$. If $\mathbf{f}_\epsilon \neq 0$ then there exist such n_i such that $c_{n_1,n_2}(\mathbf{f}) \neq 0$; thus, $\mathbf{f}_{1/\epsilon} = 0$, so $\mathbf{f}_\epsilon$ is zero after all, by Lemma 5.5.

Next we show that 3 implies 4. Assume that, for all n_1 and n_2 with $\epsilon n_1 + n_2 \equiv 0 \pmod N$, $T_{n_1,n_2} f = 0$. Then, for all m_1 and m_2 with $(m_i, n_i) = 1$, we have $T_{m_1 n_1, m_2 n_2}(f) = T_{m_1,m_2}(T_{n_1,n_2}(f)) = 0$, so in particular that is true for m_i with $(m_i, n_i) = 1$ and with $m_1 n_1 \equiv m_2 n_2 \pmod N$. But Corollary 3.6 then implies that $c_{m_1 n_1, m_2 n_2}(f) = 0$.

Finally, we show that 4 implies 2, so let f be a normalized eigenform such that all such coefficients $c_{m_1 n_1, m_2 n_2}(f)$ are zero, and let $\mathbf{f} \in \overline{S}_{k,\simeq}(N)$ be a lift of f. Assume that $\mathbf{f}_\epsilon \neq 0$. Thus, there exist n_1 and n_2 with $c_{n_1,n_2}(\mathbf{f}) \neq 0$, or, equivalently, $\lambda_{n_1,n_2}(\mathbf{f}) \neq 0$. Then for all m_1 and m_2 with $(m_i, n_i) = 1$ and with $m_1 n_1 \equiv m_2 n_2 \pmod N$, or equivalently $(1/\epsilon)m_1 + m_2 \equiv 0 \pmod N$,

$$\begin{aligned}
0 &= \lambda_{m_1 n_1, m_2 n_2}(\mathbf{f}) \\
&= \lambda_{m_1,m_2}(\mathbf{f})\lambda_{n_1,n_2}(\mathbf{f}),
\end{aligned}$$

so $\lambda_{m_1,m_2}(\mathbf{f}) = 0$ for all m_i with $(m_i, n_i) = 1$ and $(1/\epsilon)m_1 + m_2 \equiv 0 \pmod N$. By Lemma 5.4, $\mathbf{f}_{1/\epsilon} = 0$; by Lemma 5.5, $\mathbf{f}_\epsilon = 0$, a contradiction. Thus 4 implies 2. □

For an arbitrary form in $\overline{K}_{k,\epsilon}(N)$, it is necessary for those coefficients specified in part 4 of Proposition 6.1 to vanish. The following Proposition shows that even more coefficients of elements of $\overline{K}_{k,\epsilon}(N)$ vanish:

Proposition 6.2. *For all a and ϵ in $(\mathbf{Z}/N\mathbf{Z})^*$, the spaces $\overline{K}_{k,\epsilon}(N)$ and $\overline{K}_{k,a^2\epsilon}(N)$ are equal.*

Proof. Let f be an eigenform in $\overline{K}_{k,\epsilon}(N)$; we want to show that f is in $\overline{K}_{k,a^2\epsilon}(N)$. Let $\mathbf{f}$ be a lift of it to $\overline{S}_{k,\simeq}(N)$. By Proposition 6.1, $\mathbf{f}_\epsilon = 0$. Thus, $(\langle a^{-1}\rangle\mathbf{f})_{a^2\epsilon} = \langle a^{-1}\rangle(\mathbf{f}_\epsilon)$ is also zero. But by Proposition 4.10, $\langle a^{-1}\rangle$ is in $\overline{\mathbf{T}}_{k,\simeq}(N)$, so $(\langle a^{-1}\rangle\mathbf{f})$ is a multiple of $\mathbf{f}$, which is non-zero since $\langle a^{-1}\rangle$ is invertible. Thus, $\mathbf{f}_{a^2\epsilon} = 0$, so f is in $\overline{K}_{k,a^2\epsilon}(N)$, by Proposition 6.1. □

Thus, if $\mathbf{f} \in \overline{S}_{k,\simeq}(N)$ is a normalized eigenform such that $\mathbf{f}_\epsilon$ is zero for some ϵ, or equivalently that $\mathbf{f}_{-1}$ is in $\overline{K}_{k,\epsilon}(N)$, then $\mathbf{f}_{a^2\epsilon}$ is also zero for all $a \in (\mathbf{Z}/N\mathbf{Z})^*$. So if we let $f = \overline{\Sigma}\mathbf{f}$ then lots of the Fourier coefficients of f are zero. This leads one to suspect that f might be related to forms with complex multiplication, where we define an eigenform g on $X_w(N)$ to have *complex multiplication* if there exists a non-trivial character ϕ such that $\phi(p)\lambda_p(g) = \lambda_p(g)$ (or, equivalently, $\lambda_p(g) = 0$ unless $\phi(p) = 1$) for all primes p in a set of density one, where $\lambda_p(g)$ is the T_p-eigenvalue for g. (This is as in Ribet [12], §3, except that we don't require g to be a newform.) We also say that g is a *CM-form.* It is indeed the case that such forms are linked to elements of the Hecke kernel:

Theorem 6.3. *An eigenform f is in $\overline{K}_{(k_1,k_2),\epsilon}(N)$ if and only if there exist eigenforms $f_i \in S_{k_i}(X(N))$ such that, for all $n_1 \equiv n_2 \pmod N$ with $(n_i, N) = 1$,*

$$c_{n_1,n_2}(f) = c_{n_1}(f_1)c_{n_2}(f_2)$$

and such that the f_i have complex multiplication by some character ϕ such that $\phi(-\epsilon) = -1$. Furthermore, $\overline{K}_{(k_1,k_2),\epsilon}(N)$ is spanned by such forms.

Proof. Let $k = (k_1, k_2)$, and let $f \in \overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$ be an eigenform. Pick an eigenform $\mathbf{f} \in \overline{S}_{k,\simeq}(N)$ such that $\mathbf{f}_{-1} = f$ and let H be the subgroup of $\epsilon' \in (\mathbf{Z}/N\mathbf{Z})^*$ such that $\mathbf{f}_{-\epsilon'} \neq 0$, as in Theorem 5.3. By Proposition 4.4, $\overline{\Sigma}\mathbf{f}$ is an eigenform in $\overline{S}_{k_1}(\Gamma_w(N)) \otimes \overline{S}_{k_2}(\Gamma_w(N))$; but eigenspaces in that latter space are one-dimensional, so $\overline{\Sigma}\mathbf{f} = f_1 \otimes f_2$, where $f_i \in \overline{S}_{k_1}(\Gamma_w(N))$ is an eigenform. We wish to relate f's being an element of $\overline{K}_{k,\epsilon}(N)$, i.e. having $\mathbf{f}_\epsilon = 0$, to the f_i's being CM-forms.

For all m_1 and m_2 with $(m_i, N) = 1$, $c_{m_1,m_2}(\mathbf{f}) = c_{m_1}(f_1)c_{m_2}(f_2)$. If $\epsilon' \notin H$, i.e. $\mathbf{f}_{-\epsilon'} = 0$, then, for all m_i such that $\epsilon' m_1 \equiv m_2 \pmod N$, $c_{m_1,m_2}(\mathbf{f}) = 0$, so $c_{m_1}(f_1) = 0$ or $c_{m_2}(f_2) = 0$. Since the f_i are eigenforms, their first Fourier coefficients are non-zero; thus, setting $m_2 = 1$, $c_{m_1}(f_1) = 0$ for $m_1 \equiv 1/\epsilon' \pmod N$ where $\epsilon' \notin H$. Since H is a subgroup, this means that $c_{m_1}(f_1) = 0$ for $m_1 \notin H$ (where we project m_1 to an element of $(\mathbf{Z}/N\mathbf{Z})^*$). Similarly, $c_{m_2}(f_2) = 0$ for $m_2 \notin H$.

First, assume that $f \in \overline{K}_{k,\epsilon}(N)$, i.e. that $\mathbf{f}_\epsilon = 0$, or that $-\epsilon \notin H$. Pick a non-trivial character ϕ of $(\mathbf{Z}/N\mathbf{Z})^*$ that is trivial on H and such that $\phi(\epsilon) \neq -1$. The previous paragraph shows that f_1 and f_2 both have complex multiplication by ϕ. By part 3 of Theorem 5.3, ϕ has order two; thus, $\phi(-\epsilon) = -1$, as desired.

Conversely, assume that there exists a character ϕ such that the forms f_i have complex multiplication by ϕ and such that $\phi(-\epsilon) = -1$. Pick m_1 and m_2 such that $\epsilon m_1 + m_2 \equiv 0 \pmod{N}$. Then $-\epsilon \equiv m_2/m_1 \pmod{N}$; since $\phi(-\epsilon) = -1$, either $\phi(m_1)$ or $\phi(m_2)$ is not equal to one. Thus, either $c_{m_1}(f_1)$ or $c_{m_2}(f_2)$ is zero, so $c_{m_1,m_2}(\mathbf{f}) = 0$. This is true for all such m_i, so $\mathbf{f}_\epsilon = 0$, i.e. $f \in \overline{K}_{k,\epsilon}(N)$.

Finally, the fact that $\overline{K}_{k,\epsilon}(N)$ is spanned by such forms follows from the fact that it has a basis of eigenforms, which is obvious from the definition of $\overline{K}_{k,\epsilon}(N)$. □

For p prime we define $K_{\simeq}(p)$ to be the subspace $\overline{K}_{(2,2),\epsilon}(p)$ of $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$ for any $\epsilon \in (\mathbf{Z}/p\mathbf{Z})^*$ such that $-\epsilon$ is non-square, where we identify $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$ with $\overline{S}_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$ by Proposition 2.7. (For this to make sense, we should assume that $p \neq 2$; since $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(2))$ is zero for all ϵ, this isn't very important.) This is independent of the choice of ϵ by Proposition 6.2; its dimension is the difference between the geometric genera of $X_{\simeq,-1}(p)$ and $X_{\simeq,\epsilon}(p)$, by Corollary 5.10. We shall give an explicit basis for this space in Sections 9 and 10.

7 The Adelic Point of View

As we have seen in Section 4, to get a satisfactory theory of Hecke operators, we had to consider the surface $X_{\simeq}(N)$, not just the surfaces $X_{\simeq,\epsilon}(N)$. In fact, to even construct the surfaces $X_{\simeq,\epsilon}(N)$ (at least when working over $\mathbf{Q}$), we passed via the surface $X_{\simeq}(N)$, as mentioned in the Introduction. To explain these facts, it helps to look at $X_{\simeq}(N)$ from the adelic point of view. Thus, we review some of definitions from that theory and explain their relevance to our context. For references, see Diamond and Im [2], Section 11.

Let $\mathbf{A}^\infty$ denote the finite adeles, i.e. the restricted direct product of the fields $\mathbf{Q}_p$ with respect to the rings $\mathbf{Z}_p$. Let U be an open compact subgroup of $\mathrm{GL}_2(\mathbf{A}^\infty)$. We define the curve Y_U to be $\mathrm{GL}_2^+(\mathbf{Q})\backslash(\mathfrak{H} \times \mathrm{GL}_2(\mathbf{A}^\infty))/U$. Here, $\mathrm{GL}_2^+(\mathbf{Q})$ is the set of matrices in $\mathrm{GL}_2(\mathbf{Q})$ with positive determinant, acting on $\mathfrak{H}$ via fractional linear translations and on $\mathrm{GL}_2(\mathbf{A}^\infty)$ via the injection $\mathbf{Q} \hookrightarrow \mathbf{A}$; U acts trivially on $\mathfrak{H}$ and acts on $\mathrm{GL}_2(\mathbf{A}^\infty)$ via multiplication on the right. This defines Y_U as a non-compact curve over the complex numbers; it has a canonical compactification X_U given by adding a finite number of cusps. The curves X_U and Y_U in fact have canonical models over $\mathbf{Q}$ which are irreducible; over $\mathbf{C}$, however, the number of their components is given by the index of $\det U$ in $\hat{\mathbf{Z}}^\times$. If U and U' are open compact subgroups of $\mathrm{GL}_2(\mathbf{A}^\infty)$ and if g is an element of $\mathrm{GL}_2(\mathbf{A}^\infty)$ such that $g^{-1}Ug \subset U'$ then multiplication by g on the right gives a map $g^*\colon X_U \to X'_U$; it descends to the models over $\mathbf{Q}$.

We define a *cusp form of weight k on X_U* to be a function $\mathbf{f}\colon \mathfrak{H} \times \mathrm{GL}_2(\mathbf{A}^\infty) \to \mathbf{C}$ such that

1. $\mathbf{f}(z, g)$ is a holomorphic function in z for fixed g.

2. $\mathbf{f}(\gamma z, \gamma g) = j(\gamma, z)^{-k}\mathbf{f}(z, g)$ for all $\gamma \in \mathrm{GL}_2^+(\mathbf{Q})$.

3. $\mathbf{f}(z, gu) = \mathbf{f}(z, g)$ for all $u \in U$.

4. $\mathbf{f}(z, g)$, considered as a function in z, vanishes at infinity for all g.

We denote by $S_k(U)$ the space of all such forms. If $g^{-1}Ug \subset U'$ then we get a map $g_*\colon S_k(U') \to S_k(U)$ by defining $(g_*\mathbf{f})(z, h)$ to be $\mathbf{f}(z, hg)$.

Each U-double coset in $\mathrm{GL}_2(\mathbf{A}^\infty)$ gives a Hecke operator, which acts on $S_k(U)$. If $U = \mathrm{GL}_2(\mathbf{Z}_p) \times U^p$ then the Hecke operator T_p is generated by the elements of $\mathrm{M}_2(\mathbf{Z}_p)$ whose determinant is in $p\mathbf{Z}_p^\times$; defining the Hecke operator S_p to be the double coset generated by $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$ in the $\mathrm{GL}_2(\mathbf{Q}_p)$ component, the ring of Hecke operators consisting of those double cosets generated by elements in $\mathrm{GL}_2(\mathbf{Q}_p)$ is generated by T_p and $S_p^{\pm 1}$.

If we define $S_k(\mathbf{C})$ to be the direct limit of the $S_k(U)$'s as U gets arbitrarily small then the above maps g_* make this into an admissible representation of $\mathrm{GL}_2(\mathbf{A}^\infty)$; the original spaces $S_k(U)$ can be recovered from that representation by taking its U-invariants. The main fact that we need is the following adelic analogue of Atkin-Lehner theory:

Theorem 7.1 (Strong Multiplicity One). *If π and π' are two irreducible constituents of $S_k(\mathbf{C})$ such that π_p and π'_p are isomorphic for almost all p then π and π' are equal. (Not just isomorphic.) Furthermore, if $\mathbf{f}$ and $\mathbf{f}'$ are elements of π and π' then this is the case iff $\mathbf{f}$ and $\mathbf{f}'$ have the same eigenvalues for almost all T_p and S_p; in this case, they have the same eigenvalues for all p such that $\mathbf{f} \in S_k(U)$ for some U of the form $\mathrm{GL}_2(\mathbf{Z}_p) \times U^p$.* □

The subgroups that we shall be concerned with are

$$U_w(N) = \left\{ g \in \mathrm{GL}_2(\hat{\mathbf{Z}}) \;\middle|\; g \equiv \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$

and

$$U(N) = \left\{ g \in \mathrm{GL}_2(\hat{\mathbf{Z}}) \;\middle|\; g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}.$$

These define the modular curves $X_w(N)$ and $X(N)$, respectively. The modular interpretation of $X(N)$ is given as follows: for each $\epsilon \in (\mathbf{Z}/N\mathbf{Z})^*$, choose a matrix $g_\epsilon \in \mathrm{GL}_2(\hat{\mathbf{Z}})$ congruent to $\left(\begin{smallmatrix} \epsilon^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ mod N. The strong approximation theorem for GL_2 implies that every point in $Y(N)$ has a representative of the form (z, g_ϵ) for some unique choice of ϵ; we let this point correspond to the elliptic curve $\mathbf{C}/\langle z, 1\rangle$ together with the basis for its N-torsion given by $(\epsilon z/N, 1/N)$. We then have an action of $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ on $X(N)$ that sends a matrix $\overline{g} \in \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ to the map $(g^{-1})^*\colon X(N) \to X(N)$, where g is any lifting of $\overline{g}$ to $\mathrm{GL}_2(\hat{\mathbf{Z}})$; it has the modular interpretation of preserving the elliptic curve and having $\overline{g}$ act on the basis for its N-torsion on the left.

Note that, in contrast, the action of $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$ on $X_w(N)$ can't easily be defined adelically; this is one reason why it's hard to define such an action over $\mathbf{Q}$, and thus why we find it convenient to use the curves $X(N)$ rather than $X_w(N)$ at times. However, with a bit of care it is possible to use the action of $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ on $X(N)$ to extract information about the action of $\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})$ on $X_w(N)$; we shall do this in Section 9.

Now we turn to the surfaces $X_{\simeq}(N)$. Definitions similar to the above go through, replacing $\mathfrak{H} \times \mathrm{GL}_2(\mathbf{A}^\infty)$ by $\mathfrak{H} \times \mathfrak{H} \times \mathrm{GL}_2(\mathbf{A}^\infty) \times \mathrm{GL}_2(\mathbf{A}^\infty)$ and putting in two copies of everything else. We then recover our surfaces $X_{\simeq}(N)$ and spaces $S_{k,\simeq}(N)$ of cusp forms by using the following subgroup:

$$U_{\simeq}(N) = \left\{ (g_1, g_2) \in \mathrm{GL}_2(\hat{\mathbf{Z}}) \times \mathrm{GL}_2(\hat{\mathbf{Z}}) \;\middle|\; g_1 \equiv g_2 \pmod{N} \right\}.$$

The above definitions of Hecke operators pass over immediately to our situation; in particular, it is easy to check that T_{p_1,p_2} is $T_{p_1} \times T_{p_2}$ (for $(p, N) = 1$) and $\langle p \rangle$ is $1 \times S_p$ (again for $(p, N) = 1$; note that $S_p \times 1$ is $\langle p^{-1} \rangle$). Using these definitions, we also easily see that that, as claimed,

$$X_{\simeq}(N) = \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z}) \backslash (X(N) \times X(N)),$$

where $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ acts diagonally with the action given above.

In contrast with this situation, there does *not* exist a subgroup $U_{\simeq,\epsilon}(N)$ that would allow us to define $X_{\simeq,\epsilon}(N)$ in the same way. This explains why we couldn't naturally define a Hecke operator T_{n_1,n_2} acting on $X_{\simeq,\epsilon}(N)$ unless $n_1 \equiv n_2 \pmod{N}$, and why we have to go to a bit of work to define those surfaces over $\mathbf{Q}$. Of course, it isn't hard to see which points on $X_{\simeq}(N)$ are on $X_{\simeq,\epsilon}(N)$ for some ϵ: they are the points that have a representative of the form (z_1, z_2, g_1, g_2) with $g_i \in \mathrm{GL}_2(\hat{\mathbf{Z}})$ and with $\det g_1 \equiv \epsilon \det g_2 \pmod{N}$. And if we are given $\mathbf{f} \in S_k(U_{\simeq}(N)) = S_{k,\simeq}(N)$, we can recover $\mathbf{f}_\epsilon$ from it by letting

$$\mathbf{f}_\epsilon(z_1, z_2) = \mathbf{f}(z_1, z_2, 1, g_\epsilon).$$

With these definitions in hand, we can show that there is no obvious way to map forms in $S_{k,\simeq}(N/d)$ to forms on $S_{k,\simeq}(N)$ other than composing an automorphism of $S_{k,\simeq}(N/d)$ with the natural injection. We consider the "obvious" maps to be maps of the form g_*.

Proposition 7.2. *The only $g \in \mathrm{GL}_2(\mathbf{A}^\infty) \times \mathrm{GL}_2(\mathbf{A}^\infty)$ such that $g^{-1}U_{\simeq}(N)g \subset U_{\simeq}(N/d)$ are those in $Z(\mathrm{GL}_2(\mathbf{A}^\infty) \times \mathrm{GL}_2(\mathbf{A}^\infty)) \cdot U_{\simeq}(N/d)$, where $Z(G)$ denotes the center of G.*

Proof. This is a local computation, so we can replace $\mathrm{GL}_2(\mathbf{A}^\infty) \times \mathrm{GL}_2(\mathbf{A}^\infty)$ by $\mathrm{GL}_2(\mathbf{Q}_p) \times \mathrm{GL}_2(\mathbf{Q}_p)$, $U_{\simeq}(N)$ by $U_{\simeq}(p^k) \cap (\mathrm{GL}_2(\mathbf{Q}_p) \times \mathrm{GL}_2(\mathbf{Q}_p))$, and $U_{\simeq}(N/d)$ by a similar statement with p^j in place of p^k (for some $j \leq k$). Assume that $g = (g_1, g_2)$. One easily sees that the only matrices h in $\mathrm{GL}_2(\mathbf{Q}_p)$ such that $h^{-1}\mathrm{GL}_2(\mathbf{Z}_p)h \subset \mathrm{GL}_2(\mathbf{Z}_p)$ are in $Z(\mathrm{GL}_2(\mathbf{Q}_p)) \cdot \mathrm{GL}_2(\mathbf{Z}_p)$; this handles the case $j = k = 0$. Also, since $U_{\simeq}(p^j) \subset \mathrm{GL}_2(\mathbf{Z}_p) \times \mathrm{GL}_2(\mathbf{Z}_p)$ and since any matrix in $\mathrm{GL}_2(\mathbf{Z}_p)$ is the first coordinate of a matrix in $U_{\simeq}(p^k)$, we can assume that, after multiplying them by an element of $Z(\mathrm{GL}_2(\mathbf{Q}_p) \times \mathrm{GL}_2(\mathbf{Q}_p))$, the g_i are both in $\mathrm{GL}_2(\mathbf{Z}_p)$. We then have to show that $g_1 \equiv g_2 \pmod{p^j}$. Multiplying both g_1 and g_2 by g_1^{-1}, we can even assume that g_1 is the identity matrix.

Thus, we have to characterize those $g_2 \in \mathrm{GL}_2(\mathbf{Z}_p)$ such that, for all $(h_1, h_2) \in \mathrm{GL}_2(\mathbf{Z}_p) \times \mathrm{GL}_2(\mathbf{Z}_p)$ such that $h_1 \equiv h_2 \pmod{p^k}$, $h_1 \equiv g_2^{-1} h_2 g_2 \pmod{p^j}$. Since $j \leq k$, h_1 and h_2 both reduce to the same matrix matrix $h \in \mathrm{GL}_2(\mathbf{Z}/(p^j)\mathbf{Z})$, and our assumption is then that, after reducing mod p^j, g_2 normalizes h. Thus, the image of g_2 mod p^j is in the center of $\mathrm{GL}_2(\mathbf{Z}/(p^j)\mathbf{Z})$; so, after multiplying g_2 by an element of $Z(\mathrm{GL}_2(\mathbf{Z}_p))$, g_1 and g_2 are congruent mod p^j, i.e. $(g_1, g_2) \in Z(\mathrm{GL}_2(\mathbf{Z}_p) \times \mathrm{GL}_2(\mathbf{Z}_p)) \cdot U_{\simeq}(p^j)$, as desired. □

Finally, we note the following simple fact about Hecke operators:

Proposition 7.3. *Let G equal $\mathrm{GL}_2(\mathbf{A}^\infty)$ or $\mathrm{GL}_2(\mathbf{A}^\infty) \times \mathrm{GL}_2(\mathbf{A}^\infty)$, and let $U = \prod_p U_p$ be a compact open subgroup of G. If g_1 and g_2 are elements of G such that, for some choice of primes $p_1 \neq p_2$, the p'th component of g_i is the identity unless $p = p_i$, then the Hecke operators Ug_1U and Ug_2U commute.*

Proof. We can write Ug_iU as a disjoint union of left cosets $Ug_{i,j}$ where the p'th component of $g_{i,j}$ is the identity unless $p = p_i$. The Proposition then follows from the fact that $g_{1,j}$ and $g_{2,j'}$ commute. □

8 Hecke Operators Dividing the Level

In defining our Hecke operators T_{n_1,n_2} above, we have assumed that $(n_i, N) = 1$; this has led to a theory that is exactly parallel to the theory of Hecke operators T_n on $X_w(N)$ with $(n, N) = 1$. Indeed, they look exactly the same locally when considered adelically (other than the obvious fact that we have to index them by two integers instead of one). When considering Hecke operators T_{n_1,n_2} with $(n_i, N) > 1$, the situation becomes much more delicate. We can restrict ourselves to considering double cosets generated by matrices in $\mathrm{GL}_2(\mathbf{Q}_p) \times \mathrm{GL}_2(\mathbf{Q}_p)$ for $p|N$ prime; however, in contrast to the situation for $(p, N) = 1$, the algebra generated by such double cosets no longer has an obvious, small set of generators and is no longer commutative. This problem arises in the modular curve case; there, it is traditional to restrict oneself to a smaller algebra of double cosets, hoping to find an algebra which is large enough to have useful operators in it but small enough to be tractable. The goal of this section is to define such an algebra in our case and to begin studying its properties.

For purposes of this section, p will be a prime dividing N, and p^j will be the highest power of p that divides N.

We define $\Delta_{\simeq}(N)$ to be the set of matrices $(g_1, g_2) \in \mathrm{M}_2(\hat{\mathbf{Z}}) \times \mathrm{M}_2(\hat{\mathbf{Z}})$ such that $g_1 \equiv g_2 \pmod{N}$, and we define the Hecke algebra $\mathbf{T}_{k,\simeq}(N)$ to be the algebra of endomorphisms of $S_{k,\simeq}(N)$ generated by double cosets contained in $\Delta_{\simeq}(N)$. Furthermore, for n_1 and n_2 positive integers and for $\epsilon \in (\mathbf{Z}/N\mathbf{Z})^*$, we define the Hecke operator $T_{n_1,n_2,\eta}$ to be the set of double cosets of matrices $(g_1, g_2) \in \Delta_{\simeq}(N)$ such that the ideal generated by $\det g_i$ is $n_i\hat{\mathbf{Z}}$ and such that $\eta(\det g_1)/n_1 \equiv (\det g_2)/n_2 \pmod{N}$. (Note that $(\det g_i)/n_i$ is in $\hat{\mathbf{Z}}^\times$.) If $(n_i, N) = 1$ then this is zero unless $\eta = n_1/n_2$, in which case we recover our old operator

T_{n_1,n_2}. If $p|N$, however, there may be multiple η such that $T_{p^{j_1},p^{j_2},\eta}$ is nonzero; note that it is always zero unless either $j_1 = j_2$ or both j_i are at least as large as j.

When translating this back to our earlier point of view, we find that the set of Hecke operators that send forms in $S_k(\Gamma_{\simeq,\epsilon}(N))$ to form in $S_k(\Gamma_{\simeq,\epsilon'}(N))$ are given by taking double cosets in the following set:

$$\Delta_{\simeq,\epsilon,\epsilon'} = \left\{ \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \right) \middle| \begin{array}{llll} a_i, b_i, c_i, d_i \in \mathbf{Z}, & & & \\ a_i d_i - b_i c_i > 0, & & & \\ a_1 & \equiv & a_2 & \pmod N, \\ b_1 & \equiv & \epsilon' b_2 & \pmod N, \\ \epsilon c_1 & \equiv & c_2 & \pmod N, \\ \epsilon d_1 & \equiv & \epsilon' d_2 & \pmod N \end{array} \right\}.$$

This is the same as the definition of $\Delta^*_{\simeq,\epsilon,\epsilon'}$ except that we remove the condition on the determinant; it contains $T_{n_1,n_2,\eta}$ iff $\eta = \epsilon'/\epsilon$. This has the following modular interpretation: if we let $\eta = \epsilon'/\epsilon$ then $T_{n_1,n_2,\eta}$ sends a triple (E_1, E_2, ϕ) to the sum of all triples (E_1', E_2', ϕ') such that there exist isogenies $\pi_i \colon E_i \to E_i'$ of degree n_i such that the following diagram commutes:

$$\begin{array}{ccc} E_1[N] & \xrightarrow{\phi} & E_2[N] \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ E_1'[N] & \xrightarrow[\phi']{} & E_2'[N] \end{array}.$$

We define $\mathbf{T}_{k,\epsilon}(N)$ to be the algebra of endomorphisms of $S_k(\Gamma_{\simeq,\epsilon}(N))$ generated by the $T_{n_1,n_2,1}$'s.

The operators T_{n_1,n_2} for $(n_i, N) > 1$ are a good deal more difficult to study than the T_{n_1,n_2} for $(n_i, N) = 1$. It's harder to get coset representatives, and it's impossible to get a complete set of upper-triangular coset representatives, which makes studying the action of these operators via Fourier coefficients much more difficult. The case $T_{p,p,1}$ already begins to illustrate some of the difficulties and surprising features that appear:

Proposition 8.1. *Let p be a prime dividing N, let ϵ be an element of $(\mathbf{Z}/N\mathbf{Z})^*$, and let e be an integer congruent to ϵ mod N. Then the set of elements of $\Delta_{\simeq,\epsilon,\epsilon}(N)$ that have determinant (p,p) has the following left coset decomposition:*

$$\left(\coprod_{\substack{0 \le b < p \\ 0 \le k < p}} \Gamma_{\simeq,\epsilon}(N) \left(\begin{pmatrix} 1 & eb \\ 0 & p \end{pmatrix}, \begin{pmatrix} 1 & b + kN \\ 0 & p \end{pmatrix} \right) \right) \coprod \left(\coprod_{0 \le l < p} \Gamma_{\simeq,\epsilon}(N) \left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} p & 0 \\ lN & 1 \end{pmatrix} \right) \right).$$

Proof. Let (δ_1, δ_2) be an element of $\Delta_{\simeq,\epsilon,\epsilon}(N)$ with determinant (p,p). By multiplying γ_1 on the left by an element of $\Gamma(1)$, we can assume that δ_1 is either of the form $\left(\begin{smallmatrix} 1 & eb \\ 0 & p \end{smallmatrix}\right)$ or of the form $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. This forces δ_2 to be congruent to $\left(\begin{smallmatrix} 1 & b \\ 0 & p \end{smallmatrix}\right)$ or to $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Assume that δ_1 is $\left(\begin{smallmatrix} 1 & eb \\ 0 & p \end{smallmatrix}\right)$; the other case is similar and easier. We can further change δ_2 by multiplication on the left by an element of $\Gamma_w(N)$; we have to show that, by doing so, we can get to exactly one of our putative representatives.

Assume that

$$\delta_2 = \begin{pmatrix} 1 + a'N & b + b'N \\ c'N & p + d'N \end{pmatrix}.$$

The fact that its determinant is equal to p is equivalent to the statement that

$$d' + a'p + a'd'N - bc' - b'c'N = 0. \tag{8.1}$$

I claim that there is a unique k between 0 and p such that δ_2 and $\left(\begin{smallmatrix} 1 & b+kN \\ 0 & p \end{smallmatrix}\right)$ generate the same left $\Gamma_w(N)$-coset, or equivalently such that

$$\begin{aligned} &\begin{pmatrix} 1 + a'N & b + b'N \\ c'N & p + d'N \end{pmatrix} \begin{pmatrix} 1 & b + kN \\ 0 & p \end{pmatrix}^{-1} \\ &\qquad = \begin{pmatrix} 1 + a'N & -kN/p - a'bN/p - a'kN^2/p + b'N/p \\ c'N & 1 - bc'N/p - c'kN^2/p + d'N/p \end{pmatrix} \end{aligned}$$

is an element of $\Gamma_w(N)$. Using the fact that $p|N$, this reduces to the pair of equations

$$b' - k - a'b \equiv 0 \pmod p \tag{8.2}$$

$$d' - bc' \equiv 0 \pmod p. \tag{8.3}$$

But (8.3) is an immediate consequence of (8.1); and we can choose a unique k mod p such that (8.2) is satisfied. This proves our desired existence and uniqueness. □

We'd like to use Proposition 8.1 to determine the action of $T_{p,p}$ on Fourier coefficients. Unfortunately, the matrices $\left(\begin{smallmatrix} p & 0 \\ lN & 1 \end{smallmatrix}\right)$ aren't upper triangular for $l \neq 0$, which causes problems in understanding how they affect Fourier expansions. We could perhaps get around that by introducing some sort of operator which encapsulates the effect of those matrices (just as we introduced the action of the σ_a's when considering the action of Hecke operators in Sections 3 and 4); however, it's not clear that doing so would be useful. Instead, we shall simply note the following fact:

Lemma 8.2. *Let f be an element of $S_k(\Gamma_{\simeq,\epsilon}(N))$, let p be a prime dividing N, and let*

$$g = \sum_{0 \leq l < p} f|_{\left(\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} p & 0 \\ lN & 1 \end{smallmatrix}\right)\right)}.$$

Then $c_{m_1,m_2}(g) = 0$ unless $p|m_i$.

Proof. This is equivalent to showing that g is invariant under the action of the matrix $\left(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&N/p\\0&1\end{smallmatrix}\right)\right)$. Thus, for each l, we need to find a unique l' such that

$$\begin{pmatrix}p&0\\lN&1\end{pmatrix}\begin{pmatrix}1&N/p\\0&1\end{pmatrix}\begin{pmatrix}p&0\\l'N&1\end{pmatrix}^{-1}$$

is in $\Gamma_w(N)$. But this matrix is equal to

$$\begin{pmatrix}1-l'N^2/p & N\\ lN/p-l'N/p-ll'N^3/p^2 & 1+lN^2/p\end{pmatrix},$$

so choosing $l'=l$ works. □

Using this, we can get partial information about the Fourier coefficients of $T_{p,p}(f)$:

Proposition 8.3. *Let f be an element of $S_k(\Gamma_{\simeq,\epsilon}(N))$, and let p be a prime dividing N. Then if $(m_i,p)=1$, we have*

$$c_{m_1,m_2}(T_{p,p,1}(f))=\begin{cases}c_{pm_1,pm_2}(f) & \text{if } \epsilon m_1+m_2\equiv 0 \pmod N,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Let $k=(k_1,k_2)$. By Lemma 8.2, we can ignore the matrices $\left(\left(\begin{smallmatrix}p&0\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}p&0\\lN&1\end{smallmatrix}\right)\right)$. Thus, we have to determine the Fourier expansion of g, where g is defined as

$$\begin{aligned}g(z_1,z_2)&=p^{k_1/2-1}p^{k_2/2-1}\sum_{\substack{0\le b<p\\0\le k<p}}f_{\left(\left(\begin{smallmatrix}1&eb\\0&p\end{smallmatrix}\right),\left(\begin{smallmatrix}1&b+kN\\0&p\end{smallmatrix}\right)\right)}(z_1,z_2)\\&=p^{-2}\sum_{\substack{0\le b,k<p\\0<m_1,m_2<\infty}}c_{m_1,m_2}(f)e^{2\pi\sqrt{-1}m_1(z_1+eb)/pN}e^{2\pi\sqrt{-1}m_2(z_2+b+kN)/pN}\\&=p^{-1}\sum_{b,m_1,p|m_2}c_{m_1,m_2}(f)e^{2\pi\sqrt{-1}m_1(z_1+eb)/pN}e^{2\pi\sqrt{-1}m_2(z_2+b)/pN}.\end{aligned}$$

By Proposition 2.4, $c_{m_1,m_2}(f)$ is zero unless $\epsilon m_1+m_2\equiv 0 \pmod N$. Thus, $p|m_2$ implies that $p|m_1$. Letting $m_i'=m_i/p$, we then have N/p dividing $em_1'+m_2'$, and

$$\begin{aligned}g(z_1,z_2)&=p^{-1}\sum_{b,m_i'}c_{pm_1',pm_2'}(f)q_1^{m_1'}q_2^{m_2'}e^{2\pi\sqrt{-1}b(em_1'+m_2')/N}\\&=\sum_{\epsilon m_1'+m_2'\equiv 0 (\mathrm{mod}\, N)}c_{pm_1',pm_2'}(f)q_1^{m_1'}q_2^{m_2'},\end{aligned}$$

which is what we wanted to prove. □

Thus, $T_{p,p,1}$ pulls out the Hecke coefficients that are multiples of p, and makes sure that the ones that Proposition 2.4 forces to be zero are zero.

I don't know a good set of coset representatives for arbitrary T_{n_1,n_2}'s. However, as Proposition 8.3 showed, we don't need a complete set of coset representatives to get partial information on how the Hecke operators acts on Fourier coefficients. A partial set of representatives for the operators $T_{p^{j_1},p^{j_2},\eta}$ is given by the following Proposition:

Proposition 8.4. *Let p be a prime dividing N, let ϵ and ϵ' be elements of $(\mathbf{Z}/N\mathbf{Z})^*$, let j_1 and j_2 integers such that $\epsilon' p^{j_1} \equiv \epsilon p^{j_2} \pmod{N}$, and let e' be an integer congruent to ϵ' mod N. Then the set of elements of $\Delta_{\simeq,\epsilon,\epsilon'}(N)$ that have determinant (p^{j_1}, p^{j_2}) can be written as $S \coprod T$ where S has the following left coset decomposition:*

$$\coprod_{\substack{b\in\mathbf{Z}/p^{j_1}\mathbf{Z}\\ k\in\mathbf{Z}/p^{j_2}\mathbf{Z}}} \Gamma_{\simeq,\epsilon}(N)\left(\begin{pmatrix}1 & e'b\\ 0 & p^{j_1}\end{pmatrix}, \begin{pmatrix}1 & b+kN\\ 0 & p^{j_2}\end{pmatrix}\right),$$

where we can choose b and k to be elements of an arbitrary set of integer representatives for the $\mathbf{Z}/p^{j_i}\mathbf{Z}$'s, and where T is such that, considered as an operator sending modular forms to functions, then for all $f \in S_k(\Gamma_{\simeq,\epsilon}(N))$, we have $c_{m_1,m_2}(f|_T) = 0$ unless $p|m_i$.

Proof. Let (δ_1, δ_2) be an element of $\Delta_{\simeq,\epsilon,\epsilon}$ of determinant (p^{j_1}, p^{j_2}). By multiplying δ_1 on the left by an element of $\Gamma(1)$, we can assume that it is equal to $\left(\begin{smallmatrix} a & e'b\\ 0 & d\end{smallmatrix}\right)$, where $ad = p^{j_1}$. Furthermore, the choice of a, b, and d is unique up to changing b by a multiple of d. We define S and T by saying that $(\delta_1, \delta_2) \in S$ if a is equal to 1 (hence $d = p^{j_1}$) and $(\delta_1, \delta_2) \in T$ otherwise; we then have to show that S and T have the desired properties.

First assume that $a = 1$, so that $\delta_1 = \left(\begin{smallmatrix} 1 & e'b\\ 0 & p^{j_1}\end{smallmatrix}\right)$. In that case, δ_2 is equal to $\left(\begin{smallmatrix} 1+a'N & b+b'N\\ c'N & p^{j_2}+d'N\end{smallmatrix}\right)$ for some a', b', c', and d', and we have to show that δ_2 is in the same left $\Gamma_w(N)$ coset as $\left(\begin{smallmatrix} 1 & b+kN\\ 0 & p^{j_2}\end{smallmatrix}\right)$ for a unique $k \in \mathbf{Z}/p^{j_2}\mathbf{Z}$. The fact that δ_2 has determinant p^{j_2} is equivalent to the statement that

$$d' + a'p^{j_2} + a'd'N - bc' - b'c'N = 0. \tag{8.4}$$

We want the matrix

$$\begin{aligned}\begin{pmatrix}1+a'N & b+b'N\\ c'N & p^{j_2}+d'N\end{pmatrix}&\begin{pmatrix}1 & b+kN\\ 0 & p^{j_2}\end{pmatrix}^{-1}\\ &= \begin{pmatrix}1+a'N & -kN/p^{j_2} - a'bN/p^{j_2} - a'kN^2/p^{j_2} + b'N/p^{j_2}\\ c'N & 1 - bc'N/p^{j_2} - c'kN^2/p^{j_2} + d'N/p^{j_2}\end{pmatrix}\end{aligned}$$

to be an element of $\Gamma_w(N)$. This reduces to the pair of equations

$$b' - k - a'b - a'kN \equiv 0 \pmod{p^{j_2}} \tag{8.5}$$

$$d' - bc' - c'kN \equiv 0 \pmod{p^{j_2}}. \tag{8.6}$$

There is a unique $k \in \mathbf{Z}/p^{j_2}\mathbf{Z}$ such that (8.5) is true: the coefficient for k in the equation is $-(1+a'N)$, which is invertible in $\mathbf{Z}/p^{j_2}\mathbf{Z}$ since $p|N$ and $\mathbf{Z}/p^{j_2}\mathbf{Z}$ is a local ring. We need to show that, for that k, (8.6) is also true. We shall show this by induction, by showing that

$$d' - bc' - c'kN \equiv (-a'N)^l(d' - bc' - c'kN) \pmod{p^{j_2}} \tag{8.7}$$

for all l: since $p|N$, $(-a'N)^l$ is congruent to zero mod p^{j_2} for l sufficiently large, so this implies (8.6). Furthermore, (8.7) is trivially true for $l = 0$; for the induction step, we then have to show that

$$d' - bc' - c'kN \equiv (-a'N)(d' - bc' - c'kN) \pmod{p^{j_2}}. \tag{8.8}$$

But

$$\begin{aligned} d' - bc' - c'kN &\equiv b'c'N - a'd'N - c'kN && \text{by (8.4)} \\ &\equiv b'c'N - a'd'N - c'N(b' - a'b - a'kN) && \text{by (8.5)} \\ &\equiv -a'd'N + a'bc'N + a'c'kN^2 \\ &= (-a'N)(d' - bc' - c'kN), \end{aligned}$$

proving (8.8).

Now assume that $\delta_1 = \left(\begin{smallmatrix} a & e'b \\ 0 & d \end{smallmatrix}\right)$ with $p|a$. In that case, (δ_1, δ_2) must be an element of T. So we just have to show that, for all $f \in S_k(\Gamma_{\simeq,\epsilon}(N))$, $c_{m_1,m_2}(f|_T)$ is zero unless $p|m_i$. To do this, we have to show that the action of T is preserved under right multiplication by $\left(\left(\begin{smallmatrix} 1 & N/p \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)$. Thus, we'll be done if we can show that

$$\delta_1 \begin{pmatrix} 1 & N/p \\ 0 & 1 \end{pmatrix}$$

is congruent to δ_1 mod N. But that is equal to

$$\begin{pmatrix} a & e'b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & N/p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & e'b + aN/p \\ 0 & d \end{pmatrix}$$

which is congruent to δ_1 mod N by our assumption that $p|a$. □

As with Proposition 8.3, this gives us partial information about $T_{p^{j_1},p^{j_2},\eta}$'s action on Fourier coefficients:

Proposition 8.5. *Let p be a prime dividing N, let ϵ and ϵ' be elements of $(\mathbf{Z}/N\mathbf{Z})^*$, let j_1 and j_2 integers such that $\epsilon p^{j_1} \equiv \epsilon' p^{j_2} \pmod{N}$, let $\eta = \epsilon'/\epsilon$, and let f be an element of $S_k(\Gamma_{\simeq,\epsilon}(N))$. Then if $(m_i, p) = 1$, we have*

$$c_{m_1,m_2}(T_{p^{j_1},p^{j_2},\eta}(f)) = \begin{cases} c_{p^{j_1}m_1}(f) & \textit{if } \epsilon' m_1 + m_2 \equiv 0 \pmod{N}, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. We need to study $f|S$, where S is the operator given by Proposition 8.4. Let q be the greatest common divisor of p^{j_1} and N, which is also equal to the g.c.d. of p^{j_2} and N by our assumption on the j_i's. Then the integers $b' + Nb''$, for $0 \le b' < q$ and $0 \le b'' < p^{j_1}/q$, provide a complete set of representatives for $\mathbf{Z}/p^{j_1}\mathbf{Z}$. Also, we simply take the integers k between 0 and p^{j_2} as our representatives for $\mathbf{Z}/p^{j_2}\mathbf{Z}$. Then, ignoring coefficients not prime to p, we have

$$\begin{aligned}
T_{p^{j_1},p^{j_2},\eta} f(z_1, z_2) &= p^{j_1(k_1/2-1)+j_2(k_2/2-1)} \sum_{\substack{0\le b'<q\\ 0\le b''<p^{j_1}/q\\ 0\le k<p^{j_2}}} f\Big|\left(\begin{pmatrix} 1 & e'b'+Ne'b'' \\ 0 & p^{j_1}\end{pmatrix}, \begin{pmatrix} 1 & b'+Nb''+Nf \\ 0 & p^{j_2}\end{pmatrix}\right)(z_1, z_2) \\
&= p^{-(j_1+j_2)} \sum_{\substack{0\le b'<q\\ 0\le b''<p^{j_1}/q\\ 0\le k<p^{j_2}\\ 0<m_1,m_2<\infty}} c_{m_1,m_2}(f) e^{2\pi\sqrt{-1}\left(m_1 \frac{z_1+e'b'+Ne'b''}{p^{j_1}N} + m_2 \frac{z_2+b'+N(b''+k)}{p^{j_2}N}\right)} \\
&= p^{-j_1} \sum_{\substack{b',b'',m_1,m_2\\ p^{j_2}|m_2}} c_{m_1,m_2}(f) e^{2\pi\sqrt{-1}\left(m_1 \frac{z_1+e'b'+Ne'b''}{p^{j_1}N} + m_2 \frac{z_2+b'+Nb''}{p^{j_2}N}\right)}
\end{aligned}$$

Write m_2 as $p^{j_2}m_2'$. Then since $c_{m_1,m_2}(f)$ is zero unless $\epsilon m_1 + m_2 \equiv 0 \pmod N$ and since $p^{j_2}|m_2$, we can write m_1 as qm_1'. Also,

$$e^{2\pi\sqrt{-1}m_2Nb''/p^{j_2}N} = e^{2\pi\sqrt{-1}m_2'b''} = 1,$$

so we can ignore that term. The above is therefore equal to

$$\begin{aligned}
&p^{-j_1} \sum_{b',b'',m_1',m_2'} c_{qm_1',p^{j_2}m_2'}(f) e^{2\pi\sqrt{-1}(qm_1'(z_1+e'b'+Ne'b'')/p^{j_1}N+m_2'(z_2+b')/N)} \\
&\qquad = q^{-1} \sum_{\substack{b',m_1',m_2'\\ (p^{j_2}/q)|m_1'}} c_{qm_1',p^{j_2}m_2'}(f) e^{2\pi\sqrt{-1}(qm_1'(z_1+e'b')/p^{j_1}N+m_2'(z_2+b')/N)}.
\end{aligned}$$

We can then write m_1' as $(p^{j_1}/q)m_1''$, or equivalently $m_1 = p^{j_1}m_1''$. Since we can assume that $\epsilon m_1 + m_2 \equiv 0 \pmod N$ and since $\epsilon p^{j_1} \equiv \epsilon' p^{j_2} \pmod N$, we have $\epsilon' m_1'' + m_2' \equiv 0 \pmod{N/q}$. So the above is equal to

$$q^{-1} \sum_{b',m_1'',m_2'} c_{p^{j_1}m_1'',p^{j_2}m_2'}(f) q_1^{m_1''} q_2^{m_2'} e^{2\pi\sqrt{-1}(e'm_1''+m_2')b'/N} = \sum_{\substack{m_1'',m_2'\\ N|(m_1''+\epsilon' m_2')}} c_{p^{j_1}m_1'',p^{j_2}m_2'}(f) q_1^{m_1''} q_2^{m_2'},$$

as desired. $\square$

This allows us to determine the image of $T_{p^{j_1},p^{j_2},\eta} f$ in $\overline{S}_k(\Gamma_{\simeq,\epsilon'}(N))$ for any form $f \in S_k(\Gamma_{\simeq,\epsilon}(N))$. (Note that $T_{p^{j_1},p^{j_2},\eta}$ is not well defined as a map from $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$, just as a map from $S_k(\Gamma_{\simeq,\epsilon}(N))$.)

9 The Case of Prime Level

In this Section, we discuss facts that are special to the case of weight $(2,2)$ forms on prime level. The main fact here is that we can ignore Fourier coefficients that are multiples of p, as stated in Proposition 2.7; this in turn implies that certain spaces of cusp forms are free of rank one over their Hecke algebras. This is encapsulated in the following Theorem:

Theorem 9.1. *For all primes p, the natural map from $S_{(2,2),\simeq}(p)$ to $\overline{S}_{(2,2),\simeq}(p)$ is an isomorphism; identifying those spaces, the algebras $\mathbf{T}_{(2,2),\simeq}(p)$, $\mathbf{T}^*_{(2,2),\simeq}(p)$, and $\overline{\mathbf{T}}_{(2,2),\simeq}(p)$ are all equal as algebras of endomorphisms of $S_{(2,2),\simeq}(p)$. Similarly, for all $\epsilon \in (\mathbf{Z}/p\mathbf{Z})^*$ the natural map from $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$ to $\overline{S}_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$ is an isomorphism; identifying those spaces, the algebras $\mathbf{T}_{(2,2),\epsilon}(p)$, $\mathbf{T}^*_{(2,2),\epsilon}(p)$, and $\overline{\mathbf{T}}_{(2,2),\epsilon}(p)$ are equal as algebras of endomorphisms of $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$. The spaces $S_{(2,2),\simeq}(p)$ and $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$ are free of rank one over $\mathbf{T}_{(2,2),\simeq}(p)$ and $\mathbf{T}_{(2,2),\epsilon}(p)$, respectively.*

Proof. The claimed isomorphisms of spaces are Proposition 2.7. We give the proof of the first set of equalities of algebras; the proof of the second set proceeds in exactly the same fashion. The last sentence then follows from the previous ones by Corollaries 4.9 and 5.7.

The fact that $\mathbf{T}^*_{(2,2),\simeq}(p)$ and $\overline{\mathbf{T}}_{(2,2),\simeq}(p)$ are equal follows from Proposition 2.7. Corollary 4.9 says that $\overline{S}_{(2,2),\simeq}(p)$ is a free rank one module over $\mathbf{T}^*_{(2,2),\simeq}(p)$. Furthermore, Proposition 7.3 implies that the operators T_{n_1,n_2} with $(n_i,p) = 1$ commute with Hecke operators associated to double cosets of $\mathrm{GL}_2(\mathbf{A}^\infty) \times \mathrm{GL}_2(\mathbf{A}^\infty)$ generated by element of $\mathrm{GL}_2(\mathbf{Z}_p) \times \mathrm{GL}_2(\mathbf{Z}_p)$. But if R is a commutative ring and M a free rank one R-module then the only R-module endomorphisms of M are given by multiplication by an element of R. Thus, those extra operators are contained in $\mathbf{T}^*_{(2,2),\simeq}(p)$, so $\mathbf{T}_{(2,2),\simeq}(p) = \mathbf{T}^*_{(2,2),\simeq}(p)$. □

This implies that, for example, there is an expression for $T_{p,p,1}$ in terms of the operators T_{n_1,n_2} with $n_1 \equiv n_2 \pmod p$ and $(n_i,p) = 1$ as operators on $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$; it would be interesting to find a natural such expression. In the rest of this Section, we shall present some general calculations that lead us towards methods for calculating the spaces $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$; in the next Section, we shall give some explicit constructions of forms.

Since

$$S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p)) = (S_2(\Gamma_w(p)) \otimes S_2(\Gamma_w(p)))^{\mathrm{SL}_2(\mathbf{Z}/p\mathbf{Z})},$$

to understand $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$ we should understand the representation theory of $\mathrm{SL}_2(\mathbf{Z}/p\mathbf{Z})$ on $S_2(\Gamma_w(p))$. Since $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ acts trivially on $S_2(\Gamma_w(p))$, we can look at the representation theory of $\mathrm{PSL}_2(\mathbf{F}_p)$ instead. We shall start by considering arbitrary weights and levels, and adding the assumptions of weight 2 and level p as it becomes convenient.

The basic fact about representations of groups on spaces of cusp forms is the Strong Multiplicity One Theorem. This tells us how to pick out the irreducible representations of $\mathrm{GL}_2(\mathbf{A}^\infty)$ that are contained in $S_k(\mathbf{C})$: they are just the Hecke eigenspaces. Taking $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$-invariants, this breaks up $S_k(U(N))$ into smaller subrepresentations of

$\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$. (Of course, these smaller subrepresentations may not be irreducible as representations of $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$.) To apply this, we need to relate $S_k(U(N))$ and its eigenspaces to spaces that we understand better.

First we recall that $\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma_w(N) \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \subset \Gamma_1(N^2)$. This allows us to pass from forms on $X_w(N)$ to forms on $X_1(N^2)$: the image of $S_k(\Gamma_w(N))$ is the direct sum of the spaces $S_k(\Gamma_0(N^2), \chi)$ where χ is a character on $(\mathbf{Z}/N\mathbf{Z})^*$. A form $f = \sum c_m q^m$, where $q = e^{2\pi\sqrt{-1}z/N}$, gets sent to a form with the same Fourier expansion except that q is now equal to $e^{2\pi\sqrt{-1}z}$. Furthermore, if ψ is a character on $(\mathbf{Z}/N\mathbf{Z})^*$ then the form f_ψ, which is defined to have Fourier expansion $\sum c_m \psi(m) q^m$, is still a form in $S_k(\Gamma_w(N))$, by Shimura [14], Proposition 3.64.

We now turn to producing forms contained in $S_k(U(N))$. A form $\mathbf{f} \in S_k(U(N))$ is a function from $\mathfrak{H} \times \mathrm{GL}_2(\mathbf{A}^\infty)$ to $\mathbf{C}$ with those properties listed in Section 7; it then follows easily that if, for $\epsilon \in (\mathbf{Z}/N\mathbf{Z})^*$, we define $\mathbf{f}_\epsilon$ by setting $\mathbf{f}_\epsilon(z) = \mathbf{f}(z, g_\epsilon)$ (where g_ϵ is a matrix in $\mathrm{GL}_2(\hat{\mathbf{Z}})$ that is congruent to $\left(\begin{smallmatrix} \epsilon^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ mod N) then each of the $\mathbf{f}_\epsilon$'s is a form in $S_k(\Gamma_w(N))$. By the Strong Approximation Theorem, a choice of such $\mathbf{f}_\epsilon$'s determines $\mathbf{f}$ uniquely. Thus, we can think of forms on $S_k(U(N))$ as $\phi(N)$-tuples of forms on $S_k(\Gamma_w(N))$.

This allows us to determine the Hecke eigenspaces in $S_k(U(N))$. The dimension of $S_k(U(N))$ is $\phi(N)$ times the dimension of $S_k(\Gamma_w(N))$, so the hope is that each eigenform on $S_k(\Gamma_w(N))$ will somehow give us $\phi(N)$ different eigenforms on $S_k(U(N))$. This is indeed what happens, as we shall see in Proposition 9.4:

Lemma 9.2. *Let $\mathbf{f}$ be an element of $S_k(U(N))$ and let q be a prime not dividing N. Then, for all $\epsilon \in (\mathbf{Z}/N\mathbf{Z})^*$, $(T_q\mathbf{f})_\epsilon = T_q(\mathbf{f}_{\epsilon q})$ and $(S_q\mathbf{f})_\epsilon = S_q(\mathbf{f}_{\epsilon q^2})$.*

Proof. This follows from tracing through the definitions; alternately one can use the modular interpretation of points on $X(N)$ and Hecke operators together with the fact that if $\pi : E \to E'$ is an isogeny of degree N then $(\pi x, \pi y)_{E'} = (x, y)_E^n$, where $(,)_E$ denotes the Weil pairing. □

Corollary 9.3. *Let $g \in S_k(\Gamma_w(N))$ be an eigenform, with eigenvalues $\{a_q, \chi(q)\}$ (for T_q and S_q respectively, as q varies over primes not dividing N). Let ψ be a character of $(\mathbf{Z}/N\mathbf{Z})^*$. Then the form $\mathbf{f}(g, \psi) \in S_k(U(N))$ defined by $\mathbf{f}(g, \psi)_\epsilon = \psi(\epsilon) g$ is an eigenform with eigenvalues $\{\psi(q) a_q, \psi^2(q)\chi(q)\}$.*

Proof. Write $\mathbf{f}$ for $\mathbf{f}(g, \psi)$. By the Lemma,

$$\begin{aligned}(T_q\mathbf{f})_\epsilon &= T_q(\mathbf{f}_{\epsilon q}) \\ &= T_q(\psi(\epsilon q) g) \\ &= \psi(q)\psi(\epsilon) a_q g \\ &= \psi(q) a_q \mathbf{f}_\epsilon.\end{aligned}$$

The calculation for S_q proceeds in exactly the same manner. □

This allows us to produce a basis of eigenforms for $S_k(U(N))$ in terms of a basis of eigenforms for $S_k(\Gamma_w(N))$:

Proposition 9.4. *Let $\{g_j\}$ be a basis of eigenforms for $S_k(\Gamma_w(N))$. Then the set of forms $\{\mathbf{f}(g_j, \psi)\}$, as g_j varies over elements of the basis and ψ varies over characters of $(\mathbf{Z}/N\mathbf{Z})^*$, give a basis of eigenforms for $S_k(U(N))$. Every set $\{a_q, \chi(q)\}$ of eigenvalues for T_q and S_q (as q runs over primes not dividing N) that occurs in $S_k(U(N))$ occurs in $S_k(\Gamma_w(N))$. A basis for the set of eigenforms in $S_k(U(N))$ with eigenvalues $\{a_q, \chi(q)\}$ is given by taking the forms $\mathbf{f}(g, \psi)$ where ψ varies over the characters of $(\mathbf{Z}/N\mathbf{Z})^*$ and where, once ψ is fixed, g varies over a basis for those eigenforms in $S_k(\Gamma_w(N))$ which have eigenvalues $\{a_q\psi^{-1}(q), \chi(q)\psi^{-2}(q)\}$.*

Proof. Assume that we have an expression of linear dependence involving the forms $\mathbf{f}(g_j, \psi)$. Looking at the first coordinate, the fact that the forms $\{g_j\}$ form a basis for $S_k(\Gamma_w(N))$ implies that we can assume that our relation involves only forms $\mathbf{f}(g, \psi)$ for some fixed form g. But those forms are linearly independent since characters are linearly independent. This gives us $\phi(N) \cdot \dim S_k(\Gamma_w(N))$ forms; but that's the dimension of $S_k(U(N))$, so those forms give a basis for $S_k(U(N))$ that consists of eigenforms.

Every set of eigenvalues on $S_k(U(N))$ is therefore of the form $\{\psi(q)a_q, \psi^2(q)\chi(q)\}$, where $\{a_q, \chi(q)\}$ is the set of eigenvalues of a form $g \in S_k(\Gamma_w(N))$, by Corollary 9.3. But those are the eigenvalues of g_ψ, which is also an eigenform in $S_k(\Gamma_w(N))$. The last statement of the Proposition follows in a similarly direct manner from the first paragraph of the proof and Corollary 9.3. $\square$

To restate the last sentence of the above Proposition: assume that $g \in S_k(\Gamma_w(N))$ is a newform with eigenvalues $\{a_p, \chi(p)\}$. A basis for the eigenforms in $S_k(U(N))$ with those eigenvalues is given by the forms $\mathbf{f}(g_{\psi^{-1}}, \psi)$ together with the forms $\mathbf{f}(h, \psi)$ where h runs over oldforms with the same eigenvalues as $g_{\psi^{-1}}$.

Let us now fix $k = 2$ and $N = p$ prime. We may assume that $p > 5$, since $S_2(\Gamma_w(p))$ is zero otherwise. Pick a set $A = \{a_q, \chi(q)\}$ of eigenvalues. Let $g \in S_2(\Gamma_w(p))$ be a newform with those eigenvalues; we wish to calculate the dimension of the space S_A of forms in $S_2(U(p))$ with eigenvalues A. For each character ψ, we can produce an element of S_A all of whose components are multiples of $g_{\psi^{-1}}$; this gives us $(p-1)$ forms. Furthermore, when $g_{\psi^{-1}}$ is an oldform, we can produce extra forms. Since $S_2(\Gamma(1))$ is zero, we can produce at most one extra form for each ψ this way: this happens when the eigenvalues $\{a_q\psi^{-1}(q), \chi(q)\psi^{-2}(q)\}$ occur in $S_2(\Gamma_1(p))$.

For how many ψ does an extra form arise in this way? By the Strong Multiplicity one theorem, studying S_A reduces to the study of irreducible representations of $\mathrm{GL}_2(\mathbf{A}^\infty)$ and their $U(p)$-invariants. Factoring those representations, we have to study irreducible representations of $\mathrm{GL}_2(\mathbf{Q}_q)$ and their $U(p)_q$-invariants. If $q \neq p$ then $U(p)_q = \mathrm{GL}_2(\mathbf{Z}_q)$; since the space of $\mathrm{GL}_2(\mathbf{Z}_q)$ invariants of an irreducible representation of $\mathrm{GL}_2(\mathbf{Q}_q)$ is either zero- or one-dimensional, we can therefore concentrate on the irreducible representations of

$\mathrm{GL}_2(\mathbf{Q}_p)$, and in particular calculating the dimension of their $U(p)_p$-invariants, where

$$U(p)_p = \left\{ g \in \mathrm{GL}_2(\mathbf{Z}_p) \;\middle|\; g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{p} \right\}.$$

A description of the irreducible representations of $\mathrm{GL}_2(\mathbf{Q}_q)$ is given in Diamond and Im, Section 11.2; their classification breaks them up into *principal series*, *special*, and *supercuspidal* representations. From the description of principal series representations given there, one easily calculates that the space of $U(p)_p$-invariants of a principal series representation is either zero- or $(p+1)$-dimensional and that the space of invariants of a special representation is either zero- or p-dimensional. Furthermore, Diamond and Im state that the conductor of a supercuspidal representation is at least p^2, which means that no oldforms can occur among the $g_{\psi^{-1}}$'s, so by the discussion in the previous paragraph, the dimension of the space of invariants must be $(p-1)$-dimensional. Thus, we have two, one, or no extra dimensions of oldforms arising in the principal series, special, and supercuspidal cases, respectively.

Let us now turn towards the space $S_2(\Gamma_w(p))$. The group $\mathrm{PSL}_2(\mathbf{F}_p)$ acts on this space; we wish to determine its irreducible representations. Since this action is not given adelically, we can't just apply the theory of irreducible $\mathrm{GL}_2(\mathbf{A}^\infty)$-representations and the Strong Multiplicity One Theorem to get the answer. However, we can use the adelic action to get information about this representation as follows: let g be an element of $S_2(\Gamma_w(p))$ and let $\mathbf{f}$ be an element of $S_2(U(p))$ such that $\mathbf{f}_1 = g$. Let $\overline{\gamma}$ be an element of $\mathrm{PSL}_2(\mathbf{F}_p)$ and let γ be an element of $\mathrm{GL}_2(\mathbf{Z}_p)$ projecting to it. Then $\overline{\gamma}$ sends g to $(\gamma_*^{-1} g)_1$, as can be seen by tracing through the definitions. (Note that we need to make the action contravariant, since the action $g \mapsto g|_\gamma$ is; thus, it isn't surprising that we have to act by γ_*^{-1}.) In particular, we get representations of $\mathrm{PSL}_2(\mathbf{F}_p)$ on $S_2(\Gamma_w(p))$ by projecting the representations given in the previous paragraphs down to their first coordinate.

The map from $S_2(U(p))$ to $S_2(\Gamma_w(p))$ sending $\mathbf{f}$ to $\mathbf{f}_1$ is injective unless there is a ψ such that $g = g_\psi$, by Proposition 9.4, i.e. unless g is a CM-form, in which case all of the forms in the representation are CM-forms, and the dimension of the representation in $S_2(\Gamma_w(p))$ is half of the dimension of the representation in $S_2(U(p))$. Thus, we have decomposed $S_2(\Gamma_w(p))$ as a direct sum of representations that are either of dimension $p-1$, p, $p+1$, $(p-1)/2$, or $(p+1)/2$.

These representations may not be irreducible, however. Most of the time, they do turn out to be irreducible; we can see this by looking at the character table of $\mathrm{PSL}_2(\mathbf{F}_p)$. The dimensions of the irreducible representations of $\mathrm{PSL}_2(\mathbf{F}_p)$ are 1, $p-1$, p, $p+1$, and either $(p-1)/2$ (if $p \equiv 3 \pmod 4$) or $(p+1)/2$ (if $p \equiv 1 \pmod 4$). Furthermore, the only one-dimensional representation of $\mathrm{PSL}_2(\mathbf{F}_p)$ is the trivial one, which doesn't occur in $S_2(\Gamma_w(p))$ (since that would be equivalent to having a form that is invariant under $\mathrm{PSL}_2(\mathbf{F}_p)$, i.e. a form in $S_2(\Gamma(1))$). There are no 2-dimensional representations, either, so by comparing dimensions, we see that the representations that we have constructed above are either trivial or the direct sum of two representations of dimension $(p-1)/2$ or $(p+1)/2$.

To make the situation more concrete, we first consider the case where $p \equiv 1 \pmod 4$. In this case, the character table of $\mathrm{PSL}_2(\mathbf{F}_p)$ is

	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1&\omega\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}x&0\\0&x^{-1}\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}x&\omega y\\y&x\end{smallmatrix}\right)$
U	1	1	1	1	1
V	p	0	0	1	-1
W_α	$p+1$	1	1	$\alpha(x)+\alpha(x^{-1})$	0
X_β	$p-1$	-1	-1	0	$-(\beta(\zeta)+\beta(\zeta^p))$
W'	$\frac{p+1}{2}$	$\frac{1+\sqrt{p}}{2}$	$\frac{1-\sqrt{p}}{2}$	$\frac{\alpha_{sq}(x)+\alpha_{sq}(x^{-1})}{2}$	0
W''	$\frac{p+1}{2}$	$\frac{1-\sqrt{p}}{2}$	$\frac{1+\sqrt{p}}{2}$	$\frac{\alpha_{sq}(x)+\alpha_{sq}(x^{-1})}{2}$	0

This is in Fulton and Harris [4], Section 5.2; ω is a non-square element of $\mathbf{F}_p$, $\zeta = x + y\sqrt{\omega}$, α is a character of $\mathbf{F}_p^*$ whose square isn't the identity, β is a character of the elements of norm one in $\mathbf{F}_{p^2}^*$, and α_{sq} is the non-identity character of $\mathbf{F}_p^*$ whose square is the identity.

If two irreducible representations R_1 and R_2 both occur in $S_2(\Gamma_w(p))$, we want to see how many forms they contribute to $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$, i.e. the dimension of forms in $R_1 \otimes (R_2 \circ \theta_\epsilon)$ that are fixed under $\mathrm{PSL}_2(\mathbf{F}_p)$. Let χ_i be the character of R_i; then we need to calculate $\langle \chi_1 \otimes (\chi_2 \circ \theta_\epsilon), 1\rangle = \langle \chi_1, \overline{\chi_2 \circ \theta_\epsilon}\rangle$. But all of the characters above are real; furthermore, they are invariant under composition with θ_ϵ unless the representation is W' or W'' and ϵ is a non-square, in which case the characters of W' and W'' get swapped. Thus, this is almost always $\langle \chi_1, \chi_2\rangle$, which is 1 if $\chi_1 = \chi_2$ and 0 otherwise; however, if ϵ is a non-square and $\chi_2 = \chi_{W'}$ (resp. $\chi_{W''}$) then we get 1 if $\chi_1 = \chi_{W''}$ (resp. $\chi_{W'}$) and zero otherwise.

In particular, the only contribution to the dimension of $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$ that depends on ϵ is the contribution that comes from the representations W' and W'' occurring in $S_2(\Gamma_w(p))$. Assume that W' occurs n' times and that W'' occurs n'' times. In that case, we see that those representations combine to contribute $(n')^2 + (n'')^2$ to the dimension of $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$ if ϵ is a square and $2n'n''$ if ϵ is a non-square. The difference of these two numbers is $(n' - n'')^2$; since -1 is a square, the dimension is maximized when $\epsilon = -1$, as predicted by Corollary 5.10.

This is a bit misleading, however, because in this case n' and n'' are equal, so the dimension of $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$ is the same for all ϵ. We can see this by calculating n' and n'' using Ligozat [11], Proposition II.1.3.2.1: the characters of W' and W'' only differ in matrices that are conjugate to $\left(\begin{smallmatrix}1&*\\0&1\end{smallmatrix}\right)$, and the only place that such matrices occur in the formula given there is in the term $\sum_{a \bmod p} \chi(\left(\begin{smallmatrix}1&a\\0&1\end{smallmatrix}\right))$, which equals $(p+1)/2$ both for $\chi = \chi_{W'}$ and $\chi = \chi_{W''}$.

As a corollary, this implies that there are no CM-forms in $S_2(\Gamma_w(p))$ for $p \equiv 1 \pmod 4$. For if there were such a form g, it would generate an irreducible representation $R_g \subset S_2(\Gamma_w(p))$, all of whose elements would be CM-forms; there would then be a form in $R_g \otimes (R_g \circ \theta_{-1})$ that is invariant under $\mathrm{PSL}_2(\mathbf{F}_p)$. But such a form would be a CM-form in $S_{(2,2)}(\Gamma_{\simeq,-1}(p))$, so Theorem 6.3 would then imply that the dimension of $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$ for ϵ a non-square is strictly smaller than the dimension of $S_{(2,2)}(\Gamma_{\simeq,-1}(p))$, contradicting our calculations above.

Let us now turn to the case where $p \equiv 3 \pmod 4$. The character table of $\mathrm{PSL}_2(\mathbf{F}_p)$ is then

	$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}1&\omega\\0&1\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}x&0\\0&x^{-1}\end{smallmatrix}\right)$	$\left(\begin{smallmatrix}x&\omega y\\y&x\end{smallmatrix}\right)$
U	1	1	1	1	1
V	p	0	0	1	-1
W_α	$p+1$	1	1	$\alpha(x)+\alpha(x^{-1})$	0
X_β	$p-1$	-1	-1	0	$-(\beta(\zeta)+\beta(\zeta^p))$
X'	$\frac{p-1}{2}$	$\frac{-1+\sqrt{-p}}{2}$	$\frac{-1-\sqrt{-p}}{2}$	0	$\frac{-(\beta_{sq}(\zeta)+\beta_{sq}(\zeta^p))}{2}$
X''	$\frac{p-1}{2}$	$\frac{-1-\sqrt{-p}}{2}$	$\frac{-1+\sqrt{-p}}{2}$	0	$\frac{-(\beta_{sq}(\zeta)+\beta_{sq}(\zeta^p))}{2}$

The notation is as before, but now β_{sq} is the unique non-trivial character of the elements of norm one in $\mathbf{F}_{p^2}^*$ whose square is trivial.

This time, all of the characters are real except for the characters of X' and X''; and all are invariant under composition with θ_ϵ except for the characters of X' and X'', whose characters get swapped. So the only contribution to the dimension of $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$ that depends on ϵ is that that comes from representations isomorphic to X' or X''; if they occur n' and n'' times, respectively, then they contribute $2n'n''$ to the dimension of $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$ when ϵ is a square and $(n')^2+(n'')^2$ when ϵ is a non-square. Thus, $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$ is largest when ϵ is a non-square, and when ϵ is a square, the dimension shrinks by $(n'-n'')^2$. Since -1 is not a square, this again agrees with Corollary 5.10.

This time, however, $n'-n''$ is non-zero. We can't calculate it as easy as we calculated it in the previous case, because the method used there calculates the number of times a representation occurs plus the number of times that its complex conjugate occurs, and here the character is no longer totally real. Instead, we refer to Hecke [6], where he proves that the difference is equal to the class number $h(-p)$ of $\mathbf{Q}(\sqrt{-p})$. Thus,

$$\dim S_{(2,2)}(\Gamma_{\simeq,-1}(p)) - \dim S_{(2,2)}(\Gamma_{\simeq,1}(p)) = h(-p)^2.$$

As before, this implies that there are exactly $h(-p)\cdot(p-1)/2$ CM-forms contained in $S_2(\Gamma_w(p))$; they have been constructed by Hecke in [5]. We shall review his construction in Section 10, and use them to write down the Hecke kernel $K_{\simeq}(p)$ explicitly. We shall also show how to use the theory outlined in this Section to perform explicit calculations of spaces $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p))$ for small primes.

To recap:

Theorem 9.5. *If p is a prime congruent to 1 mod 4 then there are no CM-forms contained in $S_2(\Gamma_w(p))$ and the Hecke kernel $K_{\simeq}(p)$ is zero. If $p>3$ is congruent to 3 mod 4 then there are $h(-p)\cdot(p-1)/2$ CM-forms contained in $S_2(\Gamma_w(p))$ and $K_{\simeq}(p)$ has dimension $(h(-p))^2$, where $h(-p)$ is the class number of $\mathbf{Q}(\sqrt{-p})$.* □

The existence of those representations consisting of CM-forms (or, more precisely, the fact that there are $h(-p)$ of them) is the only really interesting bit of arithmetic information in $S_2(\Gamma_w(p))$, considered as an abstract representation of $\mathrm{PSL}_2(\mathbf{F}_p)$, for any prime p. (Of course, there's always lots of arithmetic information contained in the cusp forms themselves, just not in the space considered solely as a representation.) To see this, note that for any irreducible representation V of $\mathrm{PSL}_2(\mathbf{F}_p)$, if we write n_V to refer to the multiplicity of V in $S_2(\Gamma_w(p))$, then we can easily calculate $n_V + n_{\overline{V}}$, where $\overline{V}$ is the complex conjugate of V, using Ligozat [11], Proposition III.1.3.2.1, as mentioned above. The answer turns out to be a polynomial in p (essentially; the actual polynomial that you get depends on the value of $p \pmod{24}$). However, $V = \overline{V}$ unless V equals X' or X''. Thus, the only delicate question here is finding $n_{X'}$ and $n_{X''}$; or equivalently, to find $n_{X'} - n_{X''}$, since $n_{X'} + n_{X''}$ is easy to determine.

10 Examples

$X_{\simeq,-1}(7)$

The first $X_{\simeq,\epsilon}(p)$ to have a non-zero $(2,2)$-cusp form is $X_{\simeq,-1}(7)$, as can be seen by looking at Table 1 in Kani and Schanz [8] (and using Corollary 2.3 above); in fact, we see that $\dim S_{(2,2)}(\Gamma_{\simeq,-1}(7)) = 1$. We can explicitly determine a non-zero form in this space as follows:

Conjugating $\Gamma_w(7)$ by $\left(\begin{smallmatrix} 7 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we can consider $X_w(7)$ to lie between the curves $X_0(49)$ and $X_1(49)$. The former is an elliptic curve (after choosing a base point); its L-series gives rise to a weight two cusp form

$$f(z) = \sum_{m>0} c_m q^m$$

on $X_0(49)$ and $X_w(7)$. (Here, $q = e^{2\pi\sqrt{-1}z}$ if we are thinking of f as a form on $X_0(49)$ and $q = e^{2\pi\sqrt{-1}z/7}$ if we are thinking of f as a form on $X_w(7)$.) If χ is a non-trivial character on $(\mathbf{Z}/7\mathbf{Z})^*$ such that $\chi(-1) = 1$ then the functions

$$f_\chi(z) = \sum_{m>0} c_m \chi(m) q^m$$

and

$$f_{\chi^2}(z) = \sum_{m>0} c_m \chi^2(m) q^m$$

are also modular forms in $S_2(\Gamma_w(7))$, by Shimura [14], Proposition 3.64; since the latter space is three-dimensional, $\{f, f_\chi, f_{\chi^2}\}$ forms a basis for it. For $n \in (\mathbf{Z}/7\mathbf{Z})^*$, we have $f_\chi|_{\sigma_a} = \chi^2(a) f_\chi$ and $f_{\chi^2}|_{\sigma_a} = \chi(a) f_{\chi^2}$.

To produce an element of $S_{(2,2)}(\Gamma_{\simeq,-1}(7))$, we have to find a form contained in $S_2(\Gamma_w(7)) \otimes S_2(\Gamma_w(7))$ that is fixed by $\mathrm{PSL}_2(\mathbf{F}_7)$ (acting on the second factor via θ_{-1}). For our form to be fixed by the matrices (σ_a, σ_a), it has to be of the form

$$a_0 \cdot f \otimes f + a_1 \cdot f_\chi \otimes f_{\chi^2} + a_2 \cdot f_{\chi^2} \otimes f_\chi.$$

And for our form to be fixed by the matrix $\left(\left(\begin{smallmatrix}1&-1\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\right)$, we must have $a_0 = a_1 = a_2$. (We shall carry out this argument more carefully in the proof of Theorem 10.1.) However, those constraints leave us with only a one-dimensional space of possible cusp forms, and since $S_{(2,2)}(\Gamma_{\simeq,-1}(7))$ is non-empty, we see that it must be generated by the form

$$g = \frac{1}{3}(f \otimes f + f_\chi \otimes f_{\chi^2} + f_{\chi^2} \otimes f_\chi) = \sum_{m_1 \equiv m_2 \pmod 7} c_{m_1} c_{m_2} q_1^{m_1} q_2^{m_2},$$

where the c_i's are the coefficients of f as above.

Now that we've got our form g in hand, we'd like to relate it to some of our general theorems about forms in $\overline{S}_k(\Gamma_{\simeq,\epsilon}(N))$. Note that g has lots of Fourier coefficients that are zero: not only is $c_{m_1,m_2}(g)$ zero unless $m_1 \equiv m_2 \pmod 7$, but it's also zero unless the m_i's are squares mod 7. This follows from the fact that the elliptic curve $X_0(49)$ has complex multiplication by $\mathbf{Q}(\sqrt{-7})$. By Proposition 6.1, our form is therefore in $K_{\simeq}(7)$; indeed, $S_{(2,2)}(\Gamma_{\simeq,1}(7))$ is trivial.

$X_{\simeq,-1}(p)$ for $p \equiv 3 \pmod 4$

The above may look like a general recipe for producing forms on $X_{\simeq,\epsilon}(p)$ out of forms on $X_0(p^2)$, but it isn't. To see why, note that the transition involved two steps: matching up characters, which involved checking invariance under the matrices (σ_a, σ_a), and making sure that certain Fourier coefficients were zero, which involved checking invariance under the matrices $\left(\left(\begin{smallmatrix}1&\epsilon\\0&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\right)$. Thus, we checked that our putative form is invariant under the subgroup $B(p)$ of upper-triangular matrices, not all of $\mathrm{PSL}_2(\mathbf{F}_p)$. The reason why we could get away with that above was that we knew a lot about $S_2(\Gamma_w(7))$ and that the dimension of $S_{(2,2)}(\Gamma_{\simeq,-1}(7))$ is 1.

Fortunately, all is not lost for more general p. To see why, we have to look at the equation

$$S_{(2,2)}(\Gamma_{\simeq,\epsilon}(p)) = (S_2(\Gamma_w(p)) \otimes S_2(\Gamma_w(p)))^{\mathrm{PSL}_2(\mathbf{F}_p)}$$

more closely. Let ρ_1 and ρ_2 be irreducible representations occurring in $S_2(\Gamma_w(p))$, and let χ_i be the character of ρ_i. Then the representation $\rho_1 \otimes (\rho_2 \circ \theta_\epsilon)$ occurs in $S_2(\Gamma_w(p)) \otimes S_2(\Gamma_w(p))$; the dimension of the space of elements in it fixed by $\mathrm{PSL}_2(\mathbf{F}_p)$ is just

$$\langle \chi_1 \cdot (\chi_2 \circ \theta_\epsilon), 1_{\mathrm{PSL}_2(\mathbf{F}_p)} \rangle_{\mathrm{PSL}_2(\mathbf{F}_p)} = \langle \chi_1, \overline{\chi_2 \circ \theta_\epsilon} \rangle_{\mathrm{PSL}_2(\mathbf{F}_p)}.$$

Since we assumed that the ρ_i's were irreducible, this equals one if $\chi_1 = \overline{\chi_2 \circ \theta_\epsilon}$ and zero otherwise.

Assume that it is in fact the case that $\chi_1 = \overline{\chi_2 \circ \theta_\epsilon}$. If ρ_1 is also irreducible considered as a representation of $B(p)$, then we shall also have

$$\langle \chi_1 \cdot (\chi_2 \circ \theta_\epsilon), 1_{B(p)} \rangle_{B(p)} = \langle \chi_1, \overline{\chi_2 \circ \theta_\epsilon} \rangle_{B(p)} = 1.$$

But this says that there's only a one-dimensional space of vectors in $\rho_1 \otimes \rho_2$ that is fixed by $B(p)$, and since there is also a one-dimensional space of vectors in $\rho_1 \otimes \rho_2$ that is fixed by $\mathrm{PSL}_2(\mathbf{F}_p)$, they must be the same space. Thus, under the hypothesis that our representation is irreducible when considered as a representation of $B(p)$, we can test to see whether an element of $\rho_1 \otimes \rho_2$ is a cusp form on $X_{\simeq,\epsilon}(p)$ simply by making sure that it is invariant under (σ_n, σ_n) and $\left(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&\epsilon\\0&1\end{smallmatrix}\right)\right)$.

To make this concrete, assume that p is congruent to 3 (mod 4) but not equal to 3 and that $\epsilon = -1$. The character table for $\mathrm{PSL}_2(\mathbf{F}_p)$ is given in Section 9; checking the non-trivial characters listed there, we see that X' and X'' remain irreducible when restricted to $B(p)$. Thus, if we can produce representations isomorphic to X' or X'' in $S_2(\Gamma_w(p))$, we'll be able to explicitly write down forms in $S_{(2,2)}(\Gamma_{\simeq,-1}(p))$. We saw that there should be $h(-p)$ such representations coming from CM-forms; they would be good ones to look for.

Fortunately, those representations are produced in Hecke [5]. They are defined as follows: let I be an integral ideal in $\mathbf{Q}(\sqrt{-p})$ with norm A and let ρ be an element of I. We define a theta series as follows:

$$\theta_H(z; \rho, I, \sqrt{-p}) = \sum_{\substack{\mu \in I \\ \mu \equiv \rho (\operatorname{mod} I\sqrt{-p})}} \mu e^{2\pi\sqrt{-1}z\frac{\mu\overline{\mu}}{pA}},$$

where $\overline{\mu}$ is the complex conjugate of μ. We easily verify the following facts: for $\rho_1 \equiv \rho_2 \pmod{I\sqrt{-p}}$,

$$\theta_H(z; \rho_1, I, \sqrt{-p}) = \theta_H(z; \rho_2, I, \sqrt{-p});$$

for all ρ, I, we have

$$\theta_H(z; -\rho, I, \sqrt{-p}) = -\theta_H(z; \rho, I, \sqrt{-p});$$

and if λ is an element of K such that λI is also an integral ideal then

$$\theta_H(z; \lambda\rho, \lambda I, \sqrt{-p}) = \lambda\theta_H(z; \rho, I, \sqrt{-p}).$$

Letting V_I be the vector space generated by the functions $\theta_H(z; \rho, I, \sqrt{-p})$ for $\rho \in I$, the above shows that V_I only depends on the ideal class of I and that it is generated by setting $\rho = j\alpha$ where α is a fixed element of $I \backslash I\sqrt{-p}$ and j is an integer with $1 \leq j \leq (p-1)/2$.

By Hecke [5], Satz 8, these θ_H's are in fact modular functions of weight 2 on $\Gamma_w(p)$. By Hecke [5] §4, Formulas I and II, the spaces V_I are preserved by the operations $z \mapsto z+1$ and $z \mapsto -1/z$; since the matrices $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ generate $\mathrm{PSL}_2(\mathbf{F}_p)$, this implies that V_I is a representation of $\mathrm{PSL}_2(\mathbf{F}_p)$. One checks that the representation is non-zero and that it is in fact an X' by means of Hecke [5], Satz 7. This gives us our desired $h(-p)$ different copies of X'.

Now that we have our representations, we follow the same program as in the $X_{\simeq,-1}(7)$ case:

Theorem 10.1. *Let p be a prime congruent to 3 mod 4. For each ideal class of $\mathbf{Q}(\sqrt{-p})$, fix an integral ideal I in that class and an element α_I of I that's not contained in $I\sqrt{-p}$. Let*

$$f_I = \sum_{a\in(\mathbf{Z}/p\mathbf{Z})^*} \theta_H(z; a\left(\frac{a}{p}\right)\alpha_I, I, \sqrt{-p})$$

have the Fourier expansion

$$f_I(z) = \sum_{m>0} c_{I,m} q^m,$$

where $q = e^{2\pi\sqrt{-1}z/p}$. If I_1 and I_2 are (not necessarily distinct) ideal classes then the function

$$f_{I_1,I_2}(z_1,z_2) = \sum_{m_1\equiv m_2 \pmod p} c_{I_1,m_1} c_{I_2,m_2} q_1^{m_1} q_2^{m_2}.$$

is an element of $S_{(2,2)}(\Gamma_{\simeq,-1}(p))$ contained in $K_{\simeq}(p)$; furthermore, the f_{I_1,I_2}'s give a basis for $K_{\simeq}(p)$ as I_1 and I_2 vary over the ideal classes of $\mathbf{Q}(\sqrt{-p})$.

Proof. First, we verify that the forms f_I are indeed CM-forms. By definition,

$$c_m(\theta_H(z;\rho,I,\sqrt{-p})) = \sum_{\substack{\mu\in I \\ \mu\equiv\rho(\mathrm{mod}\, I\sqrt{-p}) \\ \mu\bar{\mu}=mA}} \mu,$$

where A is the norm of I But $\mu\bar{\mu}$ is a square mod p for all μ in the ring of integers of $\mathbf{Q}(\sqrt{-p})$, as is A, so c_m is zero unless m is a square mod p. Thus, every element of V_I is invariant under twisting by the quadratic character of $(\mathbf{Z}/p\mathbf{Z})^*$, hence a CM-form. We have therefore produced $h(-p)$ different irreducible representations consisting of CM-forms; Theorem 9.5 shows that those are all such representations.

By the above discussion and the discussion in Section 9, a basis for $K_{\simeq}(p)$ is therefore given by picking a non-zero element of $V_{I_1}\otimes V_{I_2}$ invariant under $\mathrm{PSL}_2(\mathbf{F}_p)$ for each pair (I_1,I_2). By the irreducibility of these representations under $B(p)$, to check whether or not a form is invariant under $\mathrm{PSL}_2(\mathbf{F}_p)$ it's enough to check whether or not it is invariant under the matrices σ_d (for $(d,p)=1$) and $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$.

First, pick an ideal class I. By Hecke [5], Satz 7, we have

$$\theta_H(z;\rho,I,\sqrt{-p})|_{\sigma_d} = \theta_H(z; a\left(\frac{a}{p}\right)\rho, I, \sqrt{-p})$$

where $ad\equiv 1 \pmod p$. Therefore, the form f_I defined in the statement of the Theorem is indeed invariant under the matrices σ_d. If χ is a character of $(\mathbf{Z}/p\mathbf{Z})^*$, let

$$f_{I,\chi} = \sum_{m>0} \chi(m) c_{I,m} q^m.$$

This is also in V_I:

$$f_{I,\chi}(z) = \chi^{-1}(\alpha\overline{\alpha}/A) \sum_{a\in(\mathbf{Z}/p\mathbf{Z})^*} \chi^2(a)\theta_H(z; a\left(\frac{a}{p}\right)\alpha, I, \sqrt{-p}).$$

(The point of that formula is that we can pull out the Fourier coefficients of f_I whose indices are congruent to some fixed element of $(\mathbf{Z}/p\mathbf{Z})^*$ by taking a suitable linear combination of the forms $f_I|_{\left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right)}$; combining the resulting forms appropriately gives us $f_{I,\chi}$.) Shimura [14], Proposition 3.64 implies that $f_{I,\chi}|_{\sigma_d} = \chi^2(d)f_{I,\chi}$. Thus, $f_{I,\chi}$ and $f_{I,\chi'}$ are linearly independent unless $\chi^2 = (\chi')^2$; if we restrict χ by assuming that $\chi(-1) = 1$ then the forms $f_{I,\chi}$ are linearly independent, and hence form a basis for V_I. Thus, the elements of $V_{I_1} \otimes V_{I_2}$ that are invariant under the matrices σ_d are the linear combinations of the forms $f_{I_1,\chi} \otimes f_{I_2,\chi^{-1}}$ as χ varies over the characters of $(\mathbf{Z}/p\mathbf{Z})^*/\{\pm 1\}$.

There is then exactly one linear combination of those forms which is invariant under the matrix $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$. Forms invariant under that matrix are characterized by Proposition 2.4: they have $c_{m_1,m_2} = 0$ unless $m_1 \equiv m_2 \pmod p$. Let ψ be a primitive character of $(\mathbf{Z}/p\mathbf{Z})^*/\{\pm 1\}$, and define

$$f_{I_1,I_2} = \frac{2}{p-1}\sum_{j=1}^{\frac{p-1}{2}} f_{I_1,\psi^j} \otimes f_{I_2,\psi^{-j}}.$$

Then

$$\begin{aligned} c_{m_1,m_2}(f_{I_1,I_2}) &= \frac{2}{p-1}\sum_{j=1}^{\frac{p-1}{2}} c_{m_1}(f_{I_1,\psi^j})c_{m_2}(f_{I_2,\psi^{-j}}) \\ &= \frac{2}{p-1}\sum_j \psi^j(m_1)\psi^{-j}(m_2)c_{m_1,I_1}c_{m_2,I_2} \\ &= \frac{2}{p-1}\left(\sum_j \psi^j(m_1/m_2)\right)c_{m_1,I_1}c_{m_2,I_2}. \end{aligned}$$

But ψ is a character of order $(p-1)/2$, so $\sum_j \psi^j(m_1/m_2)$ is zero unless m_1 and m_2 project to the same element of $(\mathbf{Z}/p\mathbf{Z})^*/\{\pm 1\}$, i.e. unless $m_1 \equiv \pm m_2$; in that case, the sum is $(p-1)/2$. Since -1 is a non-square and since $c_{m_i,I_i} = 0$ if m_i is a non-square mod p, $c_{m_1,I_1}c_{m_2,I_2}$ is zero if $m_1 \equiv -m_2 \pmod p$, so in fact the sum is zero unless $m_1 \equiv m_2 \pmod p$. Thus, the f_{I_1,I_2} that we have defined here is the same as the one defined in the statement of the Theorem, and is invariant under the matrices σ_d and $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$. It is therefore an element of $S_{(2,2)}(\Gamma_{\simeq,-1}(p))$ as desired. $\square$

$X_{\simeq,-1}(11)$

The next prime level to consider is level 11. Table 1 in Kani and Schanz [8] and Corollary 2.3 show us that

$$\dim S_{(2,2)}(\Gamma_{\simeq,\epsilon}(7)) = \begin{cases} 3 & \text{if } -\epsilon \text{ is a square} \\ 2 & \text{otherwise.} \end{cases}$$

By Corollary 5.9, to understand the structure of $S_{(2,2)}(\Gamma_{\simeq,\epsilon}(11))$ as a $\mathbf{T}_{(2,2),\epsilon}(11)$-module, it's enough to give a basis of eigenforms for $S_{(2,2)}(\Gamma_{\simeq,-1}(11))$ and to say which of those eigenforms are in $K_{\simeq}(11)$. The latter question is answered by Theorem 10.1; since $h(-11) = 1$, $K_{\simeq}(11)$ is one-dimensional, agreeing with our dimension count above. The rest of this section will be devoted to finding the other two eigenforms contained in $S_{(2,2)}(\Gamma_{\simeq,-1}(11))$.

The first step is to have a basis of eigenforms for $S_2(\Gamma_w(11))$ and to understand the latter as a $\mathrm{PSL}_2(\mathbf{F}_{11})$-representation. Fortunately, Ligozat [11] provides a fairly complete answer to this question. The space $S_2(\Gamma_w(11))$ is 26-dimensional and decomposes into a sum of three irreducible representations: an 11-dimensional one (isomorphic to the representation we called V in Section 9), a 10-dimensional one (isomorphic to X_β where β is a character of order three), and a 5-dimensional one (isomorphic to X'). The 5-dimensional one is made up of the CM-forms in $S_2(\Gamma_w(11))$; we have discussed that in the previous example. There is also one $\mathrm{PSL}_2(\mathbf{F}_{11})$-invariant vector in $V \otimes (V \circ \theta_{-1})$ and one in $X_\beta \otimes (X_\beta \circ \theta_{-1})$; our goal is to determine those vectors, which are the eigenforms that we are looking for.

Let $f_1(z) = \eta^2(z/11)\eta^2(z)$ and let $f_2(z) = \eta(z/11)\eta^2(z)\eta(11z)$. If we let

$$\begin{aligned}
g_1(z) &= f_1(11z) \\
g_2(z) &= f_1(z) - f_1(11z) \\
g_3(z) &= -2f_2(z) - 3T_2 f_2(z) - 2T_4 f_2(z) - T_8 f_2(z) \\
g_4(z) &= 2f_2(z) - T_4 f_2(z) - T_8 f_2(z) \\
g_5(z) &= -2f_2(z) + T_4 f_2(z) - T_8 f_2(z)
\end{aligned}$$

then the forms g_i are all eigenforms with trivial character; if ψ is the quadratic character of $\mathbf{F}_{11}^*$ then $g_{2,\psi} = g_3$, $g_{4,\psi} = g_5$, and vice-versa. (This is in the first part of Ligozat [11]; we have chosen our g_2 so that $c_{1,1}(g_2) = 0$.) Furthermore, all eigenspaces in $S_2(\Gamma_w(11))$ are one-dimensional except for the one spanned by g_1 and g_2. Thus, if χ is a primitive character of $\mathbf{F}_{11}^*$ then the 11-dimensional representation is spanned by g_1 and by the g_{2,χ^j}'s and the 10-dimensional representation is spanned by the $g_{4,\chi}$'s.

As in the previous example, we can easily determine those forms in $V \otimes (V \circ \theta_{-1})$ that are invariant under the Borel subgroup $B(11)$ of upper-triangular matrices fairly easily. Unfortunately, that space is no longer one-dimensional: it's three-dimensional, with a basis given by the following forms:

$$h_1 = g_1 \otimes g_1$$
$$h_2 = \frac{1}{10} \sum_{j=1}^{10} g_{2,\chi^j} \otimes g_{2,\chi^{-j}}$$
$$h_3 = \frac{1}{10} \sum_{j=1}^{10} g_{2,\chi^j} \otimes g_{2,\chi^{5-j}}.$$

Thus, we have to find which linear combination of these forms is invariant under $\mathrm{PSL}_2(\mathbf{F}_{11})$ or, equivalently (given that they're invariant under $B(11)$), invariant under the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Let h_4 be the form that we are looking for, and let $\mathbf{h} \in \overline{S}_{k,\simeq}(11)$ be an eigenform such that $h_4 = \mathbf{h}_{-1}$. By Proposition 4.4, $\overline{\Sigma}\mathbf{h}$ is also an eigenform; but

$$h_4 = \sum_{\substack{m_1, m_2 > 0 \\ m_1 \equiv m_2 \pmod{11}}} c_{m_1,m_2}(\overline{\Sigma}\mathbf{h}) q_1^{m_1} q_2^{m_2}.$$

In other words, the projection of h_4 into $\overline{S}_2(\Gamma_w(11)) \otimes \overline{S}_2(\Gamma_w(11))$ is given by taking one of the eigenforms in the latter space and stripping away the Fourier coefficients whose indices aren't congruent mod 11. However, no linear combination of h_2 and h_3 arises from an eigenform in that manner other than h_2 and h_3 themselves. Thus, either $h_4 = c \cdot h_1 + h_2$ or $h_4 = c \cdot h_1 + h_3$ for some constant c. (We have to allow an arbitrary multiple of h_1 because h_1 is zero as an element of $\overline{S}_2(\Gamma_w(11)) \otimes \overline{S}_2(\Gamma_w(11))$.) We shall therefore test such forms to see whether they are invariant under $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

The basic fact that we shall use is the transformation law for η under that matrix: for all $z \in \mathfrak{H}$,

$$\eta(-1/z) = (-iz)^{1/2} \eta(z),$$

where we take the branch of the square root that is positive on positive real numbers. (This is Apostol [1], Theorem 3.1.) Using this, we see that $f_1|_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)}(z) = -11 f_1(11z)$ and that $f_2|_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)}(z) = -f_2(z)$. Since the action of $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and the action of Hecke operators commute by Proposition 7.3, this shows that $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ sends g_1 to $-\frac{1}{11} g_1 - \frac{1}{11} g_2$, it sends g_2 to $-\frac{120}{11} g_1 + \frac{1}{11} g_2$, and it sends g_3, g_4, and g_5 to their negatives.

Unfortunately, it's not so easy to see what $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ does to twists of the g_i's. The saving grace is that it sends eigenforms to eigenforms and that $\sigma_a \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \sigma_{a^{-1}}$, so it sends eigenforms with character χ^k to eigenforms with characters χ^{-k}. Thus, if we diagonalize $S_2(\Gamma_w(11)) \otimes S_2(\Gamma_w(11))$ with respect to the action of the matrices $\sigma_a \otimes \sigma_{a'}$ then it's sufficient to show that there's only one choice for h whose trivial component (under that diagonalization) is preserved by the action of $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Assume first that $h_4 = c \cdot h_1 + h_3$. Then the trivial component of h_4 is $c \cdot g_1 \otimes g_1 + (1/10)(g_2 \otimes g_3 + g_3 \otimes g_2)$, which transforms under $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ to

$$\frac{c}{121}(g_1 + g_2) \otimes (g_1 + g_2) + \frac{1}{1210}(120g_1 - g_2) \otimes g_3 + \frac{1}{1210}g_3 \otimes (120g_1 - g_2).$$

This includes terms of the form $g_1 \otimes g_3$ and $g_3 \otimes g_1$, which didn't occur in the original trivial component, so it's impossible for that to be fixed by the action of $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Now assume that $h_4 = c \cdot h_1 + h_2$. The trivial component is now $c \cdot g_1 \otimes g_1 + (1/10)(g_2 \otimes g_2 + g_3 \otimes g_3)$; this transforms to

$$\frac{c}{121}(g_1 + g_2) \otimes (g_1 + g_2) + \frac{1}{1210}(120g_1 - g_2) \otimes (120g_1 - g_2) + \frac{1}{10}g_3 \otimes g_3$$
$$= (\frac{c}{121} + \frac{1440}{121})g_1 \otimes g_1 + (\frac{c}{121} - \frac{12}{121})g_1 \otimes g_2 + (\frac{c}{121} - \frac{12}{121})g_2 \otimes g_1 + (\frac{c}{121} + \frac{1}{1210})g_2 \otimes g_2 + \frac{1}{10}g_3 \otimes g_3.$$

This is our original trivial component iff $c = 12$; thus, the normalized eigenform arising from the 11-dimensional representation is

$$h_4 = 12g_1 \otimes g_1 + \frac{1}{10}\sum_{j=1}^{10} g_{1,\chi^j} \otimes g_{2,\chi^{-j}}.$$

We now turn to finding the normalized eigenform h_5 that arises from the 10-dimensional representation. If we take invariants under $B(11)$, we find a two-dimensional subspace, and we see as above that h_5 is one of the following forms:

$$h_6 = \frac{1}{10}\sum_{j=1}^{10} g_{4,\chi^j} \otimes g_{4,chi^{-j}}$$
$$h_7 = \frac{1}{10}\sum_{j=1}^{10} g_{4,\chi^j} \otimes g_{4,chi^{5-j}}.$$

Unfortunately, we can't eliminate either of the forms by looking at the trivial component of the representation (under its diagonalization with respect to the matrices $\sigma_a \otimes \sigma_{a'}$), since those components turn out to be invariant under the action of $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Thus, we look at the $\chi^2 \otimes \chi^{-2}$-component instead. The form $g_{4,\chi}|\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ must be an eigenform with character χ, so it is either a multiple of $g_{4,\chi^{-1}}$ or a multiple of $g_{4,\chi^{-6}}$. Calculations using GP/PARI show that it equals $c \cdot g_{4,\chi^{-1}}$ for some constant c. Similarly, $g_{4,\chi^6}|\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = c' \cdot g_{4,\chi^{-6}}$ for some constant $c' \neq c$.[3]

[3]While these numerical calculations are only approximate, and thus they give us only an approximate value for c and c', the approximations are good enough to make it clear that $g_{4,\chi}|\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is a multiple of $g_{4,\chi^{-1}}$ rather than of $g_{4,\chi^{-6}}$ and that $c \neq c'$. It presumably wouldn't be at all difficult to prove those facts rigorously using appropriate error estimates.

The $\chi^{-2} \otimes \chi^2$-component of h_6 is

$$\frac{1}{10}(g_{4,\chi^{-1}} \otimes g_{4,\chi} + g_{4,\chi^{-6}} \otimes g_{4,\chi^6}).$$

This gets sent under $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ to

$$\frac{1}{10}(\frac{1}{c} g_{4,\chi} \otimes c g_{4,\chi^{-1}} + \frac{1}{c'} g_{4,\chi^6} \otimes c' g_{4,\chi^{-6}}) = \frac{1}{10}(g_{4,\chi} \otimes g_{4,\chi^{-1}} + g_{4,\chi^6} \otimes g_{4,\chi^{-6}}).$$

But this is the $\chi^2 \otimes \chi^{-2}$-component of h_6. The $\chi^{-2} \otimes \chi^2$-component of h_7 is

$$\frac{1}{10}(g_{4,\chi^{-1}} \otimes g_{4,\chi^6} + g_{4,\chi^{-6}} \otimes g_{4,\chi}).$$

This gets sent under $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ to

$$\frac{1}{10}(\frac{1}{c} g_{4,\chi} \otimes c' g_{4,\chi^{-1}} + \frac{1}{c'} g_{4,\chi^6} \otimes c g_{4,\chi^{-1}}) = \frac{1}{10}(\frac{c'}{c} g_{4,\chi} \otimes g_{4,\chi^{-1}} + \frac{c}{c'} g_{4,\chi^6} \otimes g_{4,\chi^{-6}}).$$

Since $c \neq c'$, this is not equal to the $\chi^2 \otimes \chi^{-2}$-component of h_7. Thus, the normalized eigenform h_5 that we are looking for is h_6.

Finally, we note that h_4 is an eigenform for $T_{11,11,1}$ with eigenvalue 12: since $T_{11,11,1}$ commutes with the action of operators in $\mathbf{T}^*_{\equiv}(11)$ by Proposition 7.3, $T_{11,11,1}h_4$ must be a $\mathbf{T}^*_{\equiv}(11)$-eigenform with the same eigenvalues as h_4, hence a multiple of h_4. Proposition 8.5 shows that its leading coefficient must be $c_{11,11}(h_4) = 12$; since h_4 is normalized, it is therefore an eigenform for $T_{11,11,1}$ with eigenvalue 12. The fact that $c_{11,11}(h_5) = 0$ similarly shows that $T_{11,11,1}h_5 = 0$; we also see that $T_{p,p,1}f_{I_1,I_2} = 0$ for the forms f_{I_1,I_2} constructed in Theorem 10.1.

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