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ELECTRON BEAM WAVES IN MICROWAVE TUBES*

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This paper presents a review of wave propagation along electron beams and of the interaction of these waves with the fields of microwave structures. It also provides the basis for a unified theory of microwave amplifiers with distributed interaction.

The small-signal power theorem for beams with zero curl of the generalized momentum is derived. This theorem and the better-known theorem for longitudinal beams are interpreted. The waves along longitudinal beams, cylindrical Brillouin beams, and Brillouin strip-beams in crossed fields are reviewed and their small-signal power flows are studied. The interaction of waves in a longitudinal-beam, traveling-wave tube is analyzed with the aid of Pierce's coupling-of-modes formalism. The small-signal power theorem is used in deciding whether or not exponential growth of a wave signifies gain. Finally, a variational principle is derived for longitudinal beams and beams with zero curl of the generalized momentum. For reasonable trial fields, the principle leads to Pierce's coupling-of-modes formalism, and can also be applied to study cases of stronger coupling than those analyzable by the coupling-of-modes theory. Equations of the magnetron amplifier are derived from the variational principle.

I. Introduction

One purpose of this paper is to present a unified theory for the various electron beam devices that use slow-wave structures with weak coupling between the beam and circuit fields (an approximation that is made in most analyses). We shall show explicitly that existing theories of the longitudinal-beam, traveling-wave tube and of the magnetron amplifier can be derived from a variational principle by physically appealing trial solutions. In the process of the derivation we shall review the work done by various authors on electron beam waves in microwave tubes, thus fulfilling another intention of this paper: to provide a brief tutorial review of important work in this field. Several regrettable omissions are necessary, for which the author offers his apologies.

In the interest of coherence and logical sequence of the present paper, we shall not follow the various theoretical developments in chronological order. Since for the understanding of tube operation, the power interchange between the electron beam and rf fields plays an important role, it is logical to start with a discussion of the small-

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signal power theorem. This theorem makes possible a study of power interchange in a manner that is consistent with the small-signal assumptions. From the power theorem we can learn when electron beam waves that grow exponentially with distance represent a gain mechanism, and when they correspond to attenuation. We shall limit our study of the small-signal power theorem to a form that is applicable to longitudinal beams and beams with zero curl of the generalized momentum. These are two important cases for which the theorem assumes a particularly simple form.

Next, we present a discussion of space-charge waves in a longitudinal electron beam, a cylindrical Brillouin beam, and a strip beam in crossed electric and magnetic fields. This discussion is limited to space-charge waves that propagate in the absence of a slow-wave circuit.

Pierce's theory of mode coupling is briefly reviewed and the role of the power theorem in its derivation is stressed. The coupling coefficients that appear in the coupling-of-modes equations are evaluated for a longitudinal beam tube under the assumption of weak coupling. A new approach is presented in Section VI which can be used to evaluate coupling coefficients even in cases of fairly strong coupling. Using the small-signal power theorem, we derive a variational principle for the propagation constant of waves in uniform slow-wave structures that contain longitudinal and Brillouin electron beams. This principle can be used to obtain the best possible values for the propagation constants when an approximate (trial) solution of the problem is known. The principle also serves to optimize any adjustable parameters in an approximate solution. The use of the variational principle to obtain the propagation constant from a trial solution consisting of a linear superposition of the fields of the slow waves of the unperturbed systems has the following advantages:

1. The problem of finding the (slow-wave) fields of a complicated, composite system is reduced to the problem of finding the fields of the simpler subsystems.
2. Under the circumstances, the best possible value of the propagation constant is obtained.
3. With the assumption of weak coupling between the beam and circuit, the method leads in all significant cases to the determinantal equations already obtained for the traveling-wave tube and magnetron amplifier. The coupling-of-modes formalism is shown to be derivable from the variational principle with a reasonable trial solution under the assumption of weak coupling. With the same trial solution, however, better solutions for the propagation constant can be obtained from the variational principle than from the coupling-of-modes theory if no approximations of weak coupling are introduced.
4. Through the variational principle a clear picture of the

approximations involved in the conventional analyses is presented.

II. The Small-Signal Power Theorem

The behavior of amplifiers in their linear region of operation is satisfactorily represented by linearized equations. The fundamental equations are linearized by assuming that the products of the amplitudes of the rf excitation in the electron beam are small as compared with the first-order terms, and can be disregarded. The velocity of the electrons is written in the form*

$$\bar{u}(r, t) = \bar{u}_0 + \bar{u} e^{j\omega t} \quad (1)$$

where \bar{u}_0 is the velocity of the electrons in the absence of an applied rf excitation. The kinetic energy of an electron at a particular instant of time t is $(m/2e) [\bar{u}_0 + \text{Re}(\bar{u} e^{j\omega t})]^2$. If we evaluate the time average of this kinetic energy over many electrons passing a given small cross section, we obtain $(m/2e) [\bar{u}_0 + (1/2)\bar{u} \cdot \bar{u}^*]$. Thus the time average of the kinetic energy involves terms of second order in \bar{u} . However, changes of this order in the time average velocity \bar{u}_0 are neglected in the small-signal theory. Hence, it seems that under the small-signal assumptions a study of kinetic energy and power relations is doomed to failure. This, however, is not so.

In order to recognize kinetic energy flow in a manner consistent with the small-signal assumption, we start with the fact that the electromagnetic power radiated by the beam can be evaluated from the small-signal solutions correctly within second order of the small-signal amplitudes (i. e., consistently with the small-signal theory). From the small-signal equations we can prove a relation between the electromagnetic power and products of the beam excitation amplitudes. This relation, the small-signal power theorem, can then be used to interpret the power interchange between the beam and the electromagnetic fields in a manner that is consistent with the small-signal assumptions.

Now let us consider the electromagnetic power delivered by (or supplied to) the electron beam. If we use the small-signal complex electric and magnetic field vectors \bar{E} and \bar{H} , the electromagnetic power P_e out of a volume τ enclosed by a surface S is given correctly within second order (of the small-signal amplitudes) by (see Fig. 1):

$$P_e = \frac{1}{2} \text{Re} \oint \bar{E} \times \bar{H}^* \cdot d\bar{S} \quad (2)$$

*To avoid a cumbersome notation later on, we do not use a subscript 1 to identify \bar{u} as a complex ac quantity. No confusion should arise; the time-dependent quantities will always be identified by the appropriate variables in the parentheses.

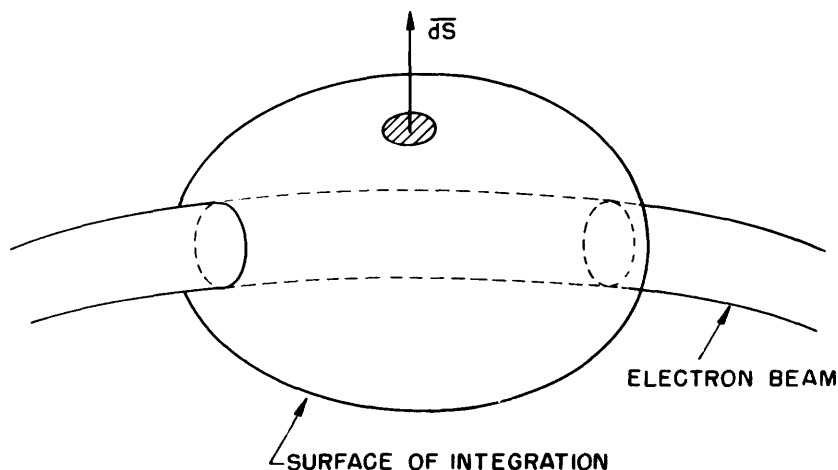


Fig.1 Electron beam and surface of integration.

The terms neglected in (2) are all of higher than second order in the small-signal amplitudes. They are: (a) the contribution of harmonics of the fundamental frequency; and (b) contribution from correction terms in the fundamental amplitudes that would be obtained from a higher-order theory.

We now turn to the derivation of the small-signal power theorem. This theorem has been proved for an entirely general case.^{1,2} But since the terms that appear in the general theorem have not yet been satisfactorily interpreted, we shall confine ourselves to two important special cases: the longitudinal electron beam, and the beam with zero curl of the generalized momentum.

A. Kinetic Power Theorem for Longitudinal Electron Beams

A longitudinal electron beam is an electron beam that is confined by an infinite magnetic field, with the time average and the time-varying velocities following the magnetic field lines. The field lines are not necessarily parallel.

The small-signal power theorem for longitudinal beams was first derived by L. J. Chu,³ who also first recognized the implications of a study of power flow based on small-signal assumptions. Details of the derivation are given in Appendix I of Ref. 4; they will not be repeated here. The method is analogous to the method used in deriving the power theorem for beams with zero curl of the generalized momentum, which is described later. The result is

$$\frac{1}{2} \operatorname{Re} \oint (\bar{\mathbf{E}} \times \bar{\mathbf{H}}^* + V \bar{\mathbf{J}}^*) \cdot d\bar{\mathbf{S}} = 0 \quad (3)$$

where \bar{J} is the complex ac current density

$$\bar{J} = \rho_0 \bar{u} + \rho \bar{u}_0 \quad (4)$$

in which ρ_0 is the time-average space-charge density, ρ is its complex amplitude, and

$$V = \frac{m}{e} \bar{u}_0 \cdot \bar{u} \quad (5)$$

the kinetic voltage. Here, \bar{u} is the small-signal Eulerian velocity. It is obtained by comparing the real, time-dependent velocity $\bar{u}(\bar{r}, t)$ at the point \bar{r} with the velocity $u_0(\bar{r})$ at the same point in the absence of an applied excitation.

$$\text{Re} \left[\bar{u}(\bar{r}) e^{j\omega t} \right] = \bar{u}(\bar{r}, t) - \bar{u}_0(\bar{r}) \quad (6)$$

We now turn to an interpretation of the power theorem.³ It shows that a net electromagnetic power flow out of the surface S must be balanced by a negative term $(1/2) \text{Re} \oint V \bar{J}^* \cdot d\bar{S}$. Thus Eq.(3) is a conservation theorem that traces the electromagnetic power delivered by a beam to changes in the flow of the quantity $(1/2) \text{Re} V \bar{J}^*$. If the beam enters the surface unexcited, we have $V = \bar{J} = 0$ on the entry cross section. If the beam delivers electromagnetic power, the quantity $(1/2) \text{Re} \oint V \bar{J}^* \cdot d\bar{S}$ integrated over the exit cross section must be negative. Now let us study the physical significance of the integral. For this purpose consider the flow of the quantity $V \bar{J}^*$ along a hose of current with walls everywhere parallel to \bar{J}_0 and \bar{J} . Let the vector cross section of the hose at some chosen point be $d\bar{a}$. The vector $d\bar{a}$ is assumed to be parallel to, and in the same direction as, \bar{u}_0 . Assume that $\bar{J} \cdot d\bar{a}$ and \bar{V} are out of phase at the chosen point along the hose. Then we have

$$\frac{1}{2} \text{Re} (\bar{J}^* \cdot d\bar{a} V) < 0 \quad (7)$$

a situation that exists at the exit cross section of a volume within which the beam delivers electromagnetic power.

Let us see what it means physically when V and $\bar{J} \cdot d\bar{a}$ are in phase opposition. When the amplitude of the ac current is at its positive extremum, the ac kinetic voltage has its extreme negative value. Since e is negative, we conclude from definition (5) that at this instant of time the amplitude of the velocity is at its positive extreme. When the ac amplitude of the velocity is positive the total velocity of the electrons passing the point at this particular instant of time is larger than it would be in the absence of an applied signal. Similarly, we find that the total rate of passage of electrons through

*We use the symbol $d\bar{S}$ for an element of a closed surface; the symbol $d\bar{a}$ for an element of area that does not pertain to a closed surface.

the hose at the same instant of time is smaller than it would be in the absence of an applied signal. This situation is illustrated schematically in Fig. 2. We show the distribution of the electrons and their

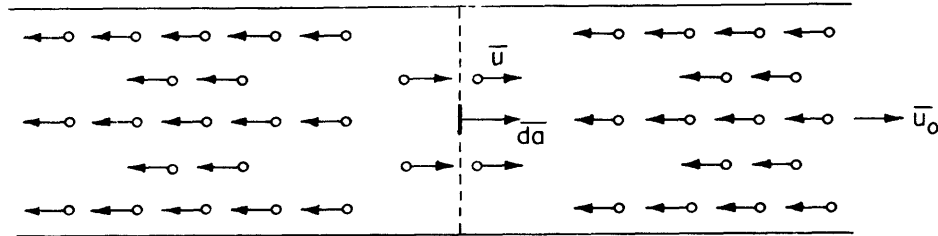


Fig.2 Small Signal velocities and densities in beam-hose with negative kinetic power.

velocities in a hose taken out of a beam with an excitation corresponding to negative kinetic power.

At a time half a period later, the rate of passage of electrons is highest, and the velocity has reached its minimum value. Thus at this later time, more electrons pass the cross section $d\bar{a}$ than in the absence of an rf signal. These electrons have a smaller velocity than they would have in the absence of a signal.

We have found that at the time when the number of electrons passing the cross section is less than the average number they have a larger velocity than they would have in the absence of an rf signal; at the time when more electrons pass the cross section they have a velocity smaller than the average. Thus the average (over all electrons passing the cross section) of the kinetic-energy flow through the surface is smaller than it would be in the absence of an applied signal. Accordingly, the occurrence of $(1/2) \text{Re}(V \bar{J}^* \cdot d\bar{a}) < 0$ can be interpreted as a decrease of the average kinetic energy flow below the flow in the absence of an rf signal. This finding is entirely in agreement with the fact that condition (7) accompanies a net delivery of electromagnetic power by the beam, according to Eq. (3).

Conversely, $(1/2) \text{Re}(V \bar{J}^* \cdot d\bar{a}) > 0$ can be interpreted as an increase of the time-average kinetic energy flow over the kinetic energy flow in the absence of an applied rf signal. We shall use the term "kinetic power" and the symbol P_k for the term $(1/2) \text{Re} \int (V \bar{J}^* \cdot d\bar{a})$.

The preceding argument shows that delivery of electromagnetic power can be traced to changes in the kinetic energy flow in a longitudinal beam in a way that is consistent with the assumptions of small-signal theory.

B. Small-Signal Power Theorem for Curl-Free Electron Beams

Electron beams that originate from a cathode on which there is no normal component of the dc magnetic field, have zero curl of the time-average generalized momentum \bar{p}_0

$$\bar{p}_0 = m\bar{u}_0 + e\bar{A}_0 \quad (8)$$

where \bar{A}_0 is the vector-potential of the magnetic field. If noise is disregarded and if the cathode is situated outside the field of the rf excitation, the curl of the alternating (time-varying) generalized momentum is also zero.²

$$\nabla \times \bar{p}_1 = \nabla \times (m\bar{u} + e\bar{A}) = 0 \quad (9)$$

where $\text{curl } \bar{A} = \bar{B}$, the complex ac magnetic field.

The force equation and the continuity equation for a flow with zero curl of the generalized momentum can be written as

$$j\omega\bar{u} + \nabla(\bar{u}_0 \cdot \bar{u}) - \frac{e}{m} \bar{E} = 0 \quad (10)$$

$$j\omega\bar{J} + \bar{u}_0 \nabla \cdot \bar{J} - j\omega\rho\bar{u} = 0 \quad (11)$$

Here we have

$$\bar{J} = \rho_0\bar{u} + \rho\bar{u}_0 \quad (12)$$

inside the electron beam. It is expedient to include in \bar{J} the surface current \bar{K} on the surface of the beam

$$\bar{K} = \rho_0\bar{r}_1 \cdot \bar{n}\bar{u}_0 \quad (13)$$

where \bar{n} is the normal to the beam boundary, and \bar{r}_1 is the small-signal displacement of an electron on the beam boundary. (The surface current (13) was first proposed by Hahn,⁵ later it was used by Rigrod and Lewis,⁶ Zitelli⁷ and others. The surface current is quite easily interpreted in "polarization variables" where it appears quite naturally.²) Thus, \bar{J} is given correctly everywhere by

$$\bar{J} = \rho_0\bar{u} + \rho\bar{u}_0 + \bar{K}\delta(S) \quad (14)$$

where $\delta(S)$ is a delta function which, when integrated over any flat pillbox on the beam surface, gives $\int \delta(S) d\tau = da$. (See Fig. 3) We shall now show that Eq. (11) is the correct continuity equation even when the surface current is contained in \bar{J} as given by (14). Introducing (14) into (11), we can neglect, within the current sheet, the terms $\rho_0\bar{u} + \rho\bar{u}_0$ and $-j\omega\rho\bar{u}$ as compared to the delta function. We have

$$\delta(S)j\omega\rho_0(\bar{r}_1 \cdot \bar{n})\bar{u}_0 + \bar{u}_0 \nabla \cdot \bar{J} = 0$$

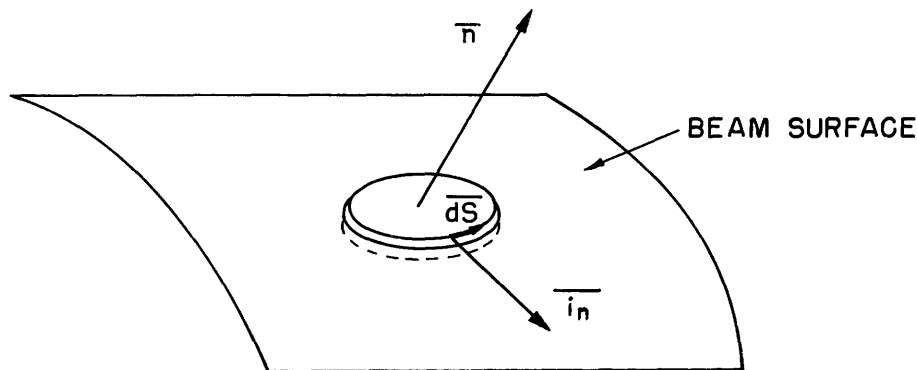


Fig.3 Pillbox on beam surface.

Cancelling the common factor \bar{u}_0 and integrating over the pillbox shown in Fig. 3, we have

$$\oint \bar{K} \cdot \bar{i}_n ds - \bar{J} \cdot \bar{n} = -j\omega \rho_0 \bar{r}_1 \cdot \bar{n} \quad (15)$$

Equation (15) is the correct continuity equation for the surface current. It states that the current lost to the pillbox by the surface current passing through its sides and by the volume current inside the beam passing through its endface, is equal to the time rate of decrease of the surface charge $\rho_0 \bar{r}_1 \cdot \bar{n}$. Introduction of the surface current in the continuity equation greatly simplifies the subsequent operations.

The small-signal Maxwell equations are

$$\nabla \times \bar{E} + j\omega \mu_0 \bar{H} = 0 \quad (16)$$

$$\nabla \times \bar{H} - j\omega \epsilon_0 \bar{E} - \bar{J} = 0 \quad (17)$$

Here again \bar{J} has to be interpreted by Eq. (14) in order to give the correct discontinuity of \bar{H} at the surface of the beam. Gauss's theorem can be applied to the resulting fields as if they were continuous, since the surface current can always be considered as distributed over a layer of small but finite thickness on the beam surface.

Dot-multiplying Eq. (16) by \bar{H}^* , the complex conjugate of Eq. (17) by $-\bar{E}$, Eq. (10) by $(-m/e)\bar{J}^*$, and the complex conjugate of Eq. (11) by $(m/e)\bar{u}$ we obtain, upon addition,

$$\begin{aligned} \nabla \cdot \left[\bar{E} \times \bar{H}^* + \left(\frac{m}{e} \bar{u} \cdot \bar{u} \right) \bar{J}^* \right] \\ + j\omega \left[\mu_0 \bar{H} \cdot \bar{H}^* + \frac{m}{e} \rho_0 |\bar{u}|^2 - \epsilon_0 \bar{E} \cdot \bar{E}^* \right] = 0 \end{aligned} \quad (18)$$

Integrating over a volume between two reference cross sections as shown in Fig. 4, and taking the real part, we have

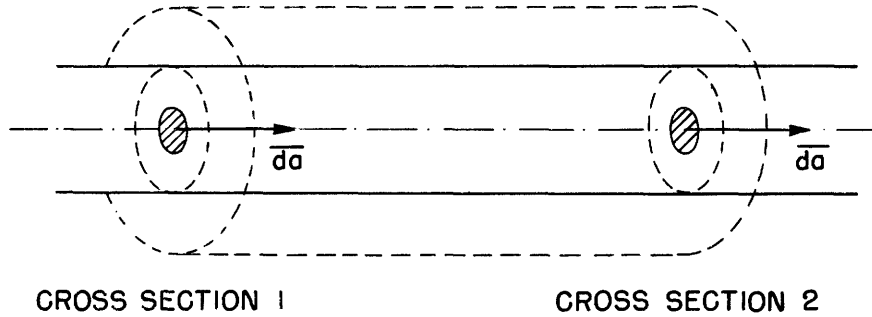


Fig.4 Surface of integration for Eq.(18) .

$$\begin{aligned} \operatorname{Re} \oint \bar{\mathbf{E}} \times \bar{\mathbf{H}}^* \cdot d\bar{\mathbf{S}} = & - \operatorname{Re} \int \left(\frac{m}{e} \bar{\mathbf{u}}_o \cdot \bar{\mathbf{u}} \right) \bar{\mathbf{J}}^* \cdot d\bar{\mathbf{a}} \Big|_{(2)} \\ & + \operatorname{Re} \int \left(\frac{m}{e} \bar{\mathbf{u}}_o \cdot \bar{\mathbf{u}} \right) \bar{\mathbf{J}}^* \cdot d\bar{\mathbf{a}} \Big|_{(1)} \end{aligned} \quad (19)$$

This is the form of the power theorem as first derived by J.W.Klüver.⁸ Equation (19) looks very much like the kinetic power theorem of Eq. (3); yet it implies a rather novel mechanism of power transfer.

In order to bring the mechanism of power transfer into evidence, we must consider how a change of energy transport in an electron beam is determined. In principle, this would be done as follows: An observer stands at a given reference cross section. He first studies the velocity (and potential energy) of a given group of electrons (the electrons originating from a given spot on the cathode, for example) in the absence of an applied signal. Then he determines the velocity of the same group of electrons as they pass the reference cross section in the presence of an applied signal. The observer would thus define a change of the electron velocity as

$$\operatorname{Re} (\bar{\mathbf{w}} e^{j\omega t}) = \bar{\mathbf{u}}(\bar{\mathbf{r}}, t) - \bar{\mathbf{u}}_o(\bar{\mathbf{r}}) + \operatorname{Re} (\bar{\mathbf{r}}_T e^{j\omega t}) \cdot \nabla \bar{\mathbf{u}}_o \quad (20)$$

where $\bar{\mathbf{r}}_T$ is the displacement in the cross-section of the group of electrons from their unperturbed point-of-passage through the cross-section. (See Fig. 5.) The velocity $\bar{\mathbf{w}}$ is the one to be used in a study of changes in kinetic energy in a beam system. The first two terms in Eq. (20) can be identified as the Eulerian velocity, $\operatorname{Re}(\bar{\mathbf{u}} e^{j\omega t})$ [compare Eq. (6)]. From Eq. (20) we thus have

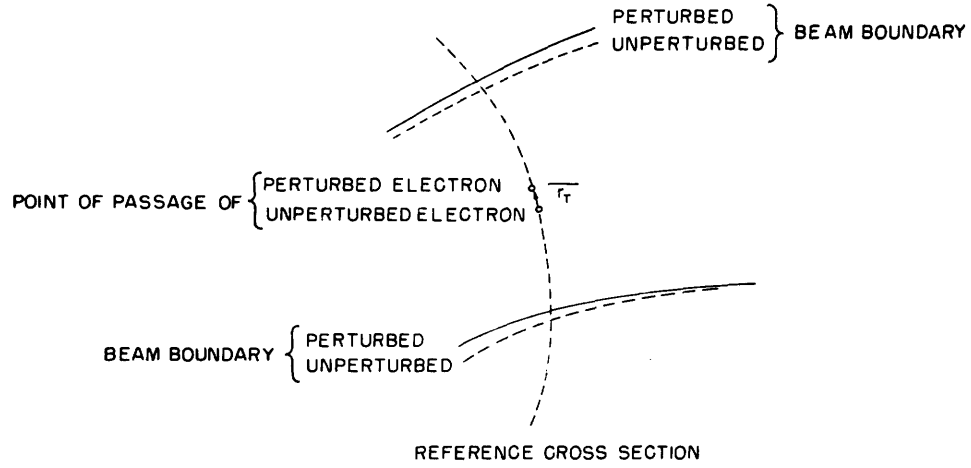


Fig.5 Definition of vector \bar{r}_T .

$$\bar{w} = \bar{u} + \bar{r}_T \cdot \nabla \bar{u}_0 \quad (21)$$

Now consider the quantity $(m/e)\bar{u}_0 \cdot \bar{u}$ which enters into the small-signal power theorem (19). Using Eq. (21), we have

$$\frac{m}{e} \bar{u}_0 \cdot \bar{u} = \frac{m}{e} \bar{u}_0 \cdot \bar{w} - \bar{r}_T \cdot \nabla \left(\frac{m}{e} \bar{u}_0^2 \right)$$

For a flow with $\nabla \times \bar{p}_0 = 0$ we have the force equation²

$$\nabla \frac{m}{e} \bar{u}_0^2 = \bar{E}_0$$

Combining the two equations results in

$$\frac{m}{e} \bar{u}_0 \cdot \bar{u} = \frac{m}{e} \bar{u}_0 \cdot \bar{w} - \bar{r}_T \cdot \bar{E}_0 \quad (22)$$

The first term on the right of Eq. (22) is the change in the kinetic energy associated with the longitudinal motion. The second term is the change in the potential energy of the beam caused by a transverse displacement in the dc electric field. The small-signal power, dP , in a hose of cross section $d\bar{a}$

$$dP = \frac{1}{2} \text{Re} \left[\left(\frac{m}{e} \bar{u}_0 \cdot \bar{w} \right) \bar{J}^* - \bar{r}_T \cdot \bar{E}_0 \bar{J}^* \right] \cdot d\bar{a} \quad (23)$$

thus consists of a kinetic and potential term. This power can be negative either because $m/e(\bar{u}_0 \cdot \bar{w}) \bar{J}^* \cdot d\bar{a}$ has a negative real part and the hose has lost kinetic energy flow as interpreted in connection with Eq. (3); or, the "potential power flow", $(1/2) \text{Re} \left[(-\bar{r}_T \cdot \bar{E}_0) \bar{J}^* \cdot d\bar{a} \right]$, is negative since the hose has moved, on the average, into regions of

lower potential energy (higher voltage, for a beam of negative charge). Figure 6 shows a sketch of a hose in a beam that has lost potential en-

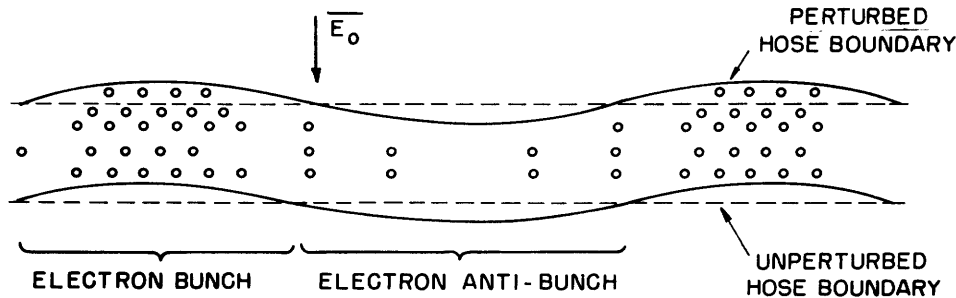


Fig.6 Electron densities in beam hose with negative "potential power".

ergy while possessing a negligible ac velocity excitation. This mechanism is at work in the magnetron amplifier,^{9,10,11} and in the helitron¹² (or spiratron¹³). It is quite remarkable that we were able to find an interpretation of this mechanism on the basis of the small-signal equations.

Having derived the two forms of the power theorem, we are now ready to discuss their applications. Some of these have been published and will merely be listed here with the proper references.

1. An orthogonality condition can be derived for waves in uniform electron beams.²
2. Necessary and sufficient boundary conditions for obtaining a unique solution can be determined.²
3. The theory of lossless mode coupling¹⁴ is applicable to systems containing electron beams. While this theory has been used only for longitudinal electron beams,^{14,15} the theorem stated in Eq. (19) makes it applicable to beams with curl-free flow.
4. A variational principle can be found on the basis of the small signal power theorem.¹⁶
5. The amplifying and attenuating nature of waves in electron beams can be recognized from the power theorem without a laborious matching of boundary conditions.

In this paper we shall discuss only points 3, 4 and 5. Before we turn to the theory of lossless mode-coupling and the variational

principle, we shall review the waves propagating along electron beams in the absence of slow-wave structures. We shall consider the three better-known and more important cases: the longitudinal beam, the cylindrical Brillouin beam, and the Brillouin beam in crossed electric and magnetic fields.

III. Space-Charge Waves in Electron Beams

A. The Longitudinal Beam

The study of space-charge waves in electron beams started with W. C. Hahn's⁵ pioneering work. He and S. Ramo¹⁷ derived general expressions and introduced the idea of surface current for matching of the boundary conditions. Most of their attention, however, was devoted to the circular, cylindrical beam of radius b confined by an infinite magnetic field along the z -axis. We shall start with a review of the pertinent equations and results. The notation used here employs symbols that have come into widest usage and does not follow that of the original work. The force equation is

$$\left[j\beta_e + \frac{\partial}{\partial z} \right] V = E_z \quad (24)$$

where $\beta_e = \omega/u_0$ is the electronic propagation constant and $V = (m/e)\bar{u}_0 \cdot \bar{u}$ is the complex ac kinetic voltage. The continuity equation can be written in the form

$$\left[j\beta_e + \frac{\partial}{\partial z} \right] J = j \frac{|J_0|}{2V_0} \beta_e V \quad (25)$$

where $V_0 = |(m/2e)u_0^2|$ is the dc potential of the beam.

Equations (24) and (25) have a particularly simple solution if the beam is very thick $[(\omega/u_0)b \gg 1]$; then a solution exists with a negligible transverse dependence so that $\partial/\partial x \cong \partial/\partial y \approx 0$, and the only dependence to be considered is the z -dependence. The equation

$$\nabla \times \bar{H} = \bar{J} + j\omega\epsilon_0 \bar{E} \quad (26)$$

in conjunction with

$$\nabla \times \bar{E} = -j\omega\mu_0 \bar{H} \quad (27)$$

leads to a TEM solution that does not couple to the electrons, and is, therefore, uninteresting; and to a solution with only a z -component of the electric field that fulfills the equation

$$j\omega\epsilon_0 E_z + J_z = 0 \quad (28)$$

Introducing Eq. (28) into Eq. (24) we obtain

$$\left[j\beta_e + \frac{\partial}{\partial z} \right] V = - \frac{1}{j\omega\epsilon_0} J_z \quad (29)$$

We set

$$\frac{e}{m} \frac{\rho_0}{\epsilon_0} = \frac{e}{m} \frac{-|J_0|}{\epsilon_0 u_0} = \omega_p^2 \quad (30)$$

where ω_p is the plasma frequency. Further, defining the plasma propagation constant,

$$\beta_p = \frac{\omega_p}{u} \quad (31)$$

We can write for Eq. (29):

$$\left[j\beta_e + \frac{\partial}{\partial z} \right] V = j \frac{V_0}{2|J_0|} \frac{\beta_p^2}{\beta_e} J \quad (32)$$

Equations (25) and (32) can now be solved easily. They are reminiscent of transmission line equations with the exception that the operator $\partial/\partial z$ of a transmission line equation is replaced by the operator

$$j\beta_e + \frac{\partial}{\partial z}$$

For this reason the solutions of Eqs. (25) and (32) differ from those of transmission line theory by a multiplier $\exp(-j\beta_e z)$. The solutions are

$$V(z) = \left[V_+ e^{+j\beta_p z} + V_- e^{-j\beta_p z} \right] e^{-j\beta_e z} \quad (33)$$

$$J(z) = Y_0 \left[V_+ e^{+j\beta_p z} - V_- e^{-j\beta_p z} \right] e^{-j\beta_e z}$$

The "characteristic admittance" in units of mho/m^2 is

$$Y_0 = \frac{|J_0|}{2V_0} \frac{\omega}{\omega_p} \quad (34)$$

V_+ and V_- are integration constants. The wave with the subscript (+) has the propagation constant $\beta_e - \beta_p$ and thus has a phase velocity larger than u_0 (as long as $\beta_p < \beta_e$, a case always met in practice for reasons that will be explained later). This is the fast wave. The other wave has a phase velocity less than u_0 and is the slow wave. The kinetic power density $\bar{S}_k = (1/2) \text{Re} (\bar{V} \bar{J}^*)$ of the fast wave is positive; that of the slow wave is negative. Thus, a delivery of power by the beam along its length must cause a preponderant excitation of the slow space-charge wave.

The waves in the infinite beam provide the basis for the understanding of the space-charge waves in the finite circular-cylindrical beam. In order to find these waves, one has to solve the Maxwell equations (26) and (27) in conjunction with Eqs. (24) and (25), matching the boundary conditions on the surface of the beam. This has been done by various authors, the first discussion being due to Hahn⁵ and Ramo.¹⁷ An infinite number of waves is found in this manner; each wave has a particular radial and ϕ -dependence (ϕ is the angular coordinate in cylindrical coordinates). Here we shall be interested only in the cylindrically symmetric waves. If the space charge is not too large, a condition always met in practice, pairs of waves have a common radial dependence for both V and J . We shall denote a pair of such space-charge waves by the word "mode". The kinetic voltage and the ac current density of a particular mode have the radial dependence $J_0(p_m r)$, where J_0 is the Bessel function of zeroth order, and the value of the real parameter p_m changes from mode to mode. These p_m are found from a matching of boundary conditions. The mode with the lowest value of p_m , $p_m = p_0$, has the slowest radial variation and is usually the mode of greatest importance. It is, therefore, called the dominant space-charge mode. The field of a single wave of the dominant space-charge mode is sketched in Fig. 7.

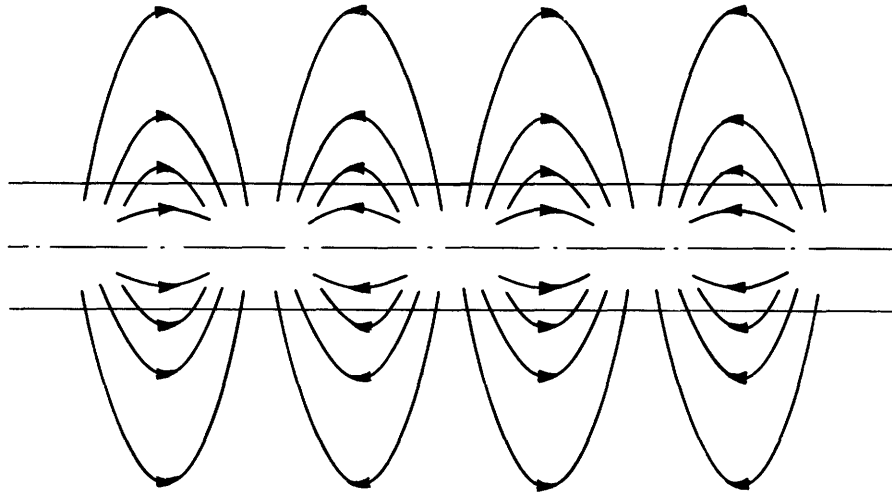


Fig.7 Sketch of field in single wave of dominant mode in longitudinal beam.

The results of the analysis that has been briefly outlined here can be conveniently explained *a posteriori* by physical reasoning, with the use of Eqs. (25) and (32) of the "one-dimensional" beam model. Equation (25) holds even in a beam of finite radius. Equation (32) was obtained from Eq. (24) the force equation, by substituting Eq. (28) for

the longitudinal electric field. Equation (28) does not hold for an excitation with non-zero radial variation. However, we can modify it in a simple manner to take into account such a radial variation (always present in a finite-beam geometry).

A finite transverse variation of the beam fields establishes a transverse electric field at the expense of the longitudinal field. Thus, for any particular mode, the longitudinal field corresponding to a given longitudinal current is less than it would be in the absence of a transverse dependence. The phase of the field with regard to the current is still the same as that resulting from Eq. (28), since the peak of the field is to be expected between the current bunches (or rather charge bunches, which, for small space charge, are essentially coincident with the current bunches). From this reasoning we obtain the force equation for the m^{th} mode

$$\left[j\beta_e + \frac{\partial}{\partial z} \right] V_m = j \frac{V_o}{2|J_o|} \frac{(r_m \beta_p)^2}{\beta_e} J_m \quad (35)$$

where $r_m (< 1)$ is the plasma frequency reduction factor (a real parameter). The kinetic voltage V_m and the current density J_m of this mode depend upon the radius r , as $J_o(p_m r)$, where J_o is the Bessel function of zeroth order. The plasma frequency reduction factor r_m is a function of $\beta_e b$ and b/a , (a and b are drift tube and beam radius). It is different for different modes. The faster the transverse variation of the mode, the smaller its space-charge reduction factor. The largest value of the reduction factor, r_o , pertains to the dominant space-charge wave. A plot of r_o for this mode is shown in Fig. 8. (The plot is taken from Ref. 19). The solutions of Eqs. (25) and (35) for V_m and J_m (now functions of r) are identical in form with Eqs. (33), except that β_p is replaced everywhere by $\beta_q = r_m \beta_p$, the reduced plasma propagation constant. Again, it is apparent that the kinetic power density of the fast wave is positive throughout the beam; that of the slow wave is negative. Each space-charge wave has also an electromagnetic power associated with it. However, the electromagnetic power is of the order β_q/β_e smaller than the kinetic power, as shown very elegantly by Louisell and Pierce.¹⁸ Thus, in most problems the electromagnetic power of the beam, as compared with the kinetic power, can be neglected.

Let us summarize the relations for the dominant space-charge waves in a thin beam ($\beta_e b \ll 1$). Here, the ac current and kinetic voltage distributions are approximately uniform throughout the beam. For the beam power we have

$$P_b \cong \frac{1}{2} \text{Re}(V i^*) = P_k \quad (36)$$

where

$$i = \int J_z da \quad (37)$$

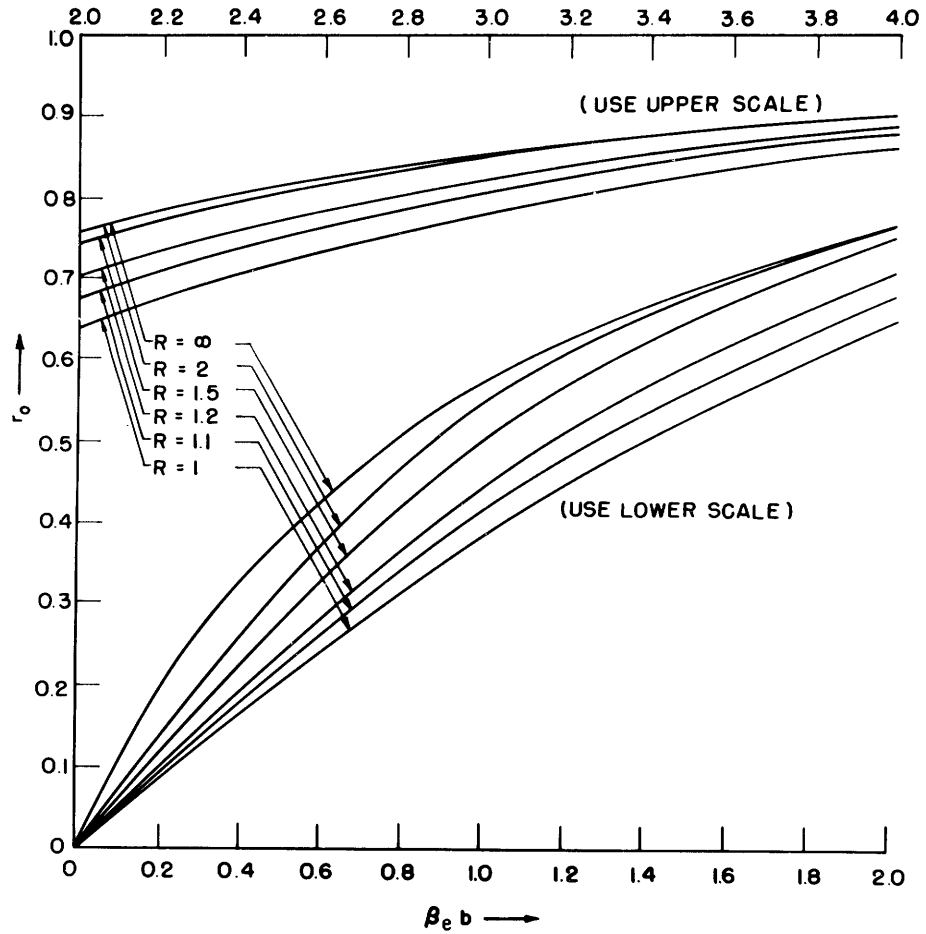


Fig.8 Plot of plasma frequency reduction factor r_0 for dominant space-charge mode vs $\beta_e b$. $R = a/b$.

integrated over a cross section of the beam, and

$$V = \left[V_+ e^{+j\beta_q z} + V_- e^{-j\beta_q z} \right] e^{-j\beta_e z} \quad (38)$$

$$i = \left[Y_0 V_+ e^{+j\beta_q z} - V_- e^{j\beta_q z} \right] e^{-j\beta_e z}$$

$$Y_0 = \frac{I_0}{2V_0} \frac{\beta_e}{\beta_q} \quad (39)$$

I_0 is the dc current in the beam. The similarity between these expressions and Eqs. (33) and (34) for the one-dimensional beam is unmistakable.

B. Space-Charge Waves in Cylindrical Brillouin Beams

The case of a Brillouin beam in a conducting drift-tube has been analyzed in detail by Zitelli⁷ and Rigrod and Lewis.⁶ They found solutions with zero curl of the generalized ac momentum. No space-charge-wave solutions with a finite curl of the generalized ac momentum have yet been found.* The solutions are of two types. There is an infinite number of solutions that have the propagation constants

$$\beta = \frac{\omega}{u_{oz}} \pm \frac{\omega_p}{u_{oz}} \quad (40)$$

independent of geometry. Here $\omega_p = \sqrt{(e/m)(\rho_o/\epsilon_o)}$, u_{oz} is the longitudinal dc velocity. These solutions do not have fields external to the beam. (if they did, their propagation constant would depend upon the geometry of the beam.) These solutions correspond to a "plasma oscillation" at the plasma frequency ω_p inside the beam with fields that are shielded to the outside of the beam by the accompanying surface charge on the beam surface, $\rho_o \bar{n} \cdot \bar{r}_1$. Since these solutions have no outside field, they cannot couple to the fields of an external structure unless the structure penetrates into the beam by means of an obstruction -- grids, for example. Similarly, an external structure without grids (the only kind considered here) cannot couple to these solutions.

We are more interested in the other mode solutions which have external fields. There is only one such mode with cylindrical symmetry that consists of two waves; it has zero ac space charge inside the beam. The field produced by this mode is caused entirely by the bulging and contracting of the beam surface. It is the gradient (at non-relativistic velocities) of a potential that fulfills Laplace's equation inside and outside the beam. *At practical (small) dc space-charge densities* the radial dependence of the longitudinal field and current inside the beam is of the form $I_0(\beta_e r)$, where β_e is the beam propagation constant ($\beta_e = \omega/u_{oz}$), I_0 is the modified Bessel function of zeroth order and first kind. Outside the beam, the longitudinal field decays away from the beam, the discontinuity in $\partial E_z/\partial r$, and, thus, in the radial field is provided by the surface charge $\rho_o \bar{n} \cdot \bar{r}_1$. Figure 9 shows a sketch of the lines of force of the ac field $\bar{E}(\bar{r})$ in such a surface wave. The bulging and contracting of the beam boundary is exaggerated in the figure. The fields are produced by equivalent negative and positive surface charges on the boundary of the unperturbed beam. These surface charges express the effect of the perturbations in the boundary. A small-signal field analysis does not give the details

*It is possible that such solutions violate the assumption of laminar flow. It would then be very difficult to find them.

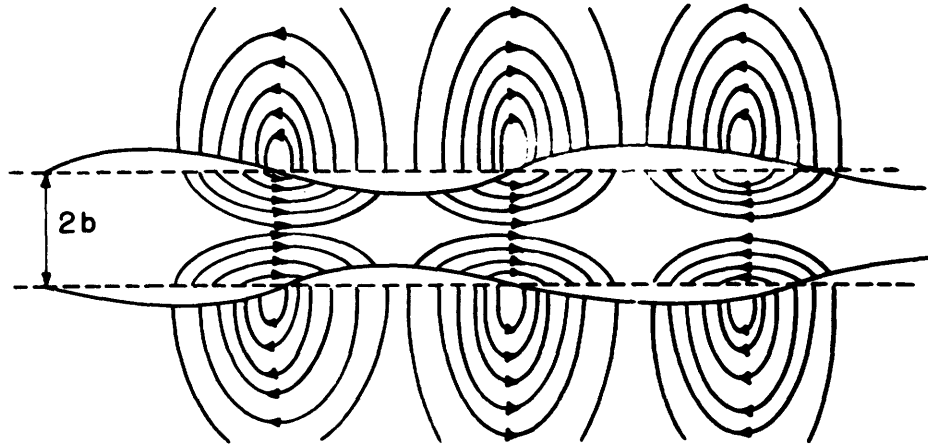


Fig.9 Sketch of fields of surface wave in Brillouin beam.

of the fields in the regions between the perturbed and unperturbed beam boundaries, and therefore no fields are drawn in those regions. (When computing the fields acting on the electrons that enter into those regions, one uses the values just inside the surface charge-carrying unperturbed beam boundary.) The ac surface current K_z is much larger than the ac volume current. This justifies the name "surface waves" by which we are going to denote these waves.

The propagation constants of the two waves of this mode are again given by

$$\beta = \beta_e \pm \beta_q \quad (41)$$

where β_q is a quantity less than

$$\beta_p = \frac{\omega}{u_{oz}}$$

and is a function of geometry. Thus, the z -dependence of this mode bears a strong resemblance to the z -dependence of the dominant space-charge mode of the longitudinal beam.

Let us now turn to the investigation of the power flow in these two waves (compare Eq. (19))

$$P_b = \frac{1}{2} \text{Re} \int \frac{m}{e} (\bar{u}_o \cdot \bar{u}) J_z^* da \quad (42)$$

A detailed study shows that for small space charge ($\beta_q \ll \beta_e$) the contribution to the power from the volume current is negligible when compared to the power of the surface current. We have, approximately,

$$P_b = \frac{1}{2} \operatorname{Re} (V_z i_z^*) \quad (43)$$

where

$$i_z = K_z 2\pi b \quad (44)$$

for a beam of radius b , and

$$V_z = \frac{m}{e} \bar{u}_o \cdot \bar{u} \Big|_{r=b} \quad (45)$$

These quantities have the z -dependence

$$V_z = \left[V_+ e^{j\beta_q z} + V_- e^{-j\beta_q z} \right] e^{-j\beta_e z} \quad (46)$$

$$i_z = Y_o \left[V_+ e^{j\beta_q z} - V_- e^{-j\beta_q z} \right] e^{-j\beta_e z}$$

where

$$Y_o = \frac{|I_o|}{2V_o} \frac{\beta_e}{\beta_q} \left[\frac{2I_1(\beta_e b)}{\beta_e b I_o(\beta_e b)} \right] \quad (47)$$

(I_o is the dc current in the beam.) When the beam is thin, the last factor is unity. Then Eqs. (43), (46), and (47) are in perfect agreement with those for the thin longitudinal beam.

C. Space-Charge Waves in a Brillouin Beam in Crossed Fields

We shall now briefly consider the waves that propagate along a strip beam in free space, infinite in the x -direction, moving in the z -direction, and of thickness θ in the y -direction. The method of analysis and the fundamental equations for this case were first presented by Buneman.²⁰ Macfarlane and Hay²¹ gave a detailed analysis and presented various curves for the propagation constant as functions of frequency. These curves are quite involved for thick beams with a large velocity spread. But the thin beam results are often adequate for the coupling-of-modes theory and the variational principle applied to magnetron amplifiers. These are much more easily obtained from a "thin beam analysis" first outlined by Gould.¹¹ Here, we shall summarize two methods and their results for later application in magnetron amplifier studies.

a. We start with the adiabatic approximation in which the forces of inertia are neglected. Then a balance between the electric and magnetic forces is required:

$$\bar{E} + \bar{u} \times \bar{B}_0 = 0$$

When this approximation is made, we obtain two solutions that are consistent with the approximation. For a thin beam, far away from any conductors, we find the propagation constants

$$\beta = \frac{\omega}{u_0} (1 \pm jSD) \quad (48)$$

where u_0 is the dc velocity of the center of the beam and

$$SD = \frac{-\sigma_0}{2\epsilon_0 B_0 u_0} \quad (49)$$

where $\sigma_0 = \rho_0 \theta$ is the dc charge per unit beam area (in the x-z plane) and B_0 is the dc magnetic field. The ac space-charge density for these solutions is zero, not unlike the solutions in the cylindrical Brillouin beam found by Rigrod and Lewis.

b. The same result for the propagation constant can be obtained from a one-dimensional beam model used by Pierce¹⁰ in his magnetron amplifier analysis. Pierce neglected space-charge forces. Gould¹¹ took space-charge forces into account while, in essence, retaining Pierce's beam model. In this model we consider the beam to be infinitesimally thin and to consist of a single layer of electrons. The time-average velocity of this layer is u_0 . The beam experiences a transverse displacement y_1 (y_1 has the same meaning as \bar{r}_T in Eq. (20)) and a longitudinal current i_z . The displacement and the longitudinal current produce electric fields that can be computed from electrostatic considerations. The fields, in turn, are introduced into the force equation. Four propagation constants result:

$$\beta = \frac{\omega}{u_0} (1 + jD\delta) \quad (50)$$

where $D\delta$ for a beam far from conducting boundaries is approximately given by

$$jD\delta = \pm \frac{\omega_c}{\omega} \quad (51)$$

and

$$D\delta = \pm SD \quad (52)$$

$\omega_c = |(e/m)B_0|$ is the cyclotron frequency, and SD is given by Eq. (49). The approximation used to get the roots (52) holds, provided

$$SD \ll \frac{\omega}{\omega_c} \quad (53)$$

The waves corresponding to (52) are identical with solutions (48). The waves corresponding to (50) are cyclotron waves with a circular rotation of the electrons in a reference frame moving with the velocity u_0 , negligibly affected by space-charge forces. The waves (52) are the "diocotron waves".²² They correspond to the breakup of a thin beam as caused by the mutual repulsion forces between the "valleys" and "hills" in a corrugated strip beam.^{23, 24} In the coupling-of-modes theory of the magnetron amplifier these waves are particularly important, because the circuit wave is usually adjusted to be close to synchronism with u_0 and thus couples most strongly to the diocotron waves whose phase velocity is equal to u_0 . For this reason we shall study the small-signal power of these waves in greater detail.

We use the small-signal expression in the form of Eq. (23) because an Eulerian velocity is meaningless in the case of an infinitesimally thin strip beam. Indeed, the Eulerian velocity is obtained from the difference between the actual electron velocity and the velocity of the unperturbed electrons at the same point. If the beam executes a transverse motion, it enters space within which there are no electrons in the absence of a perturbation and thus, strictly speaking, no Eulerian velocity can be defined.* Thus, having stated the nature of the velocity to be used, we now conclude that the transverse displacement y_1 is related to w_y by $[j\omega + u_0 (d/dz)] y_1 = w_y$. Using Eq. (48) for the diocotron waves, we have

$$y_1 = \pm \frac{w_y}{\omega SD} .$$

In a beam far from conducting boundaries, the longitudinal velocity w_z can be shown to be equal in magnitude to w_y . Thus, using the above expression to compare the magnitude of the two terms in the general expression for the small-signal power (23), we have

$$\left| \frac{m}{e} u_0 w_z \right| = \left| \frac{m}{e} u_0 \omega SD y_1 \right| .$$

Now compare this with

$$\left| y_1 E_0 \right| = \left| \frac{m}{e} u_0 y_1 \omega_c \right|$$

Using inequality (53), we find that the contribution to the power from the kinetic term is much less than that of the potential term. We can

*The application of Eq.(23) to a beam consisting of a single layer of electrons is somewhat strained. However, a small-signal power theorem was derived for such beams.¹ Its application to the present case leads to the same result.

compute the small-signal power entirely on the basis of the potential term. Since $E_0 = -u_0 B_0 = (m/e) u_0 \omega_c$, this is

$$P_b \approx -\frac{1}{2} \operatorname{Re} \left(\frac{m}{e} u_0 \omega_c y_1 i_z^* \right). \quad (54)$$

It is clear that a single diocotron wave cannot carry any power; the power would grow (or decay) exponentially, in violation of power conservation. Power is thus carried only in the presence of both the growing and decaying waves. A detailed evaluation of the expression (54) confirms this.

We shall also need the relation between the longitudinal current i_z and the transverse displacement y_1 . This relation is best expressed in terms of an important quantity i_y defined by

$$i_y = j\omega \sigma_0 \ell y_1 \quad (55)$$

where ℓ is the width of the beam in the x-direction. We then have for a beam far from any conductors:

$$\begin{aligned} i_y &= +i_z && \text{for wave growing in positive } z \text{ direction} \\ i_y &= -i_z && \text{for wave decaying in positive } z \text{ direction} \end{aligned} \quad (56)$$

IV. Pierce's Theory of Mode Coupling

The small-signal power theorem provided the basis for Pierce's coupling-of-modes analysis of longitudinal beam tubes. We shall briefly review the theory of mode coupling, its advantages, and its disadvantages in its present formulation. We shall consider the simplest case of coupling between two approximately synchronous waves, a circuit wave and the slow-beam wave of a thin, circular cylindrical, longitudinal electron beam. (See Fig. 10) This is the case of a traveling-wave tube with a large value of Pierce's QC parameter. A slightly

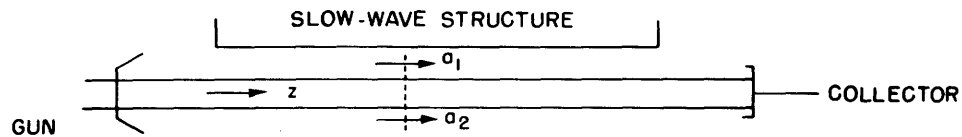


Fig.10 Schematic of beam tube. Arrows on the a's indicate direction of phase velocity.

different notation from that employed in the original papers^{14,15} is used.

The fields of the (unperturbed) slow-beam wave in the absence of the circuit are largest in the beam, and decay exponentially (like

modified Bessel functions of the second kind) outside the beam. In a similar way, the field of a (unperturbed) slow wave on the lossless circuit in the absence of the beam reaches a peak near the circuit and, in general, decays rapidly with distance from the circuit. The circuit field is relatively weak at the position of the beam, and vice versa. We should, therefore, expect that the propagation along the combined circuit-beam structure could be described as a perturbation of the two (or more) waves along the unperturbed circuit and the unperturbed beam. Denote the amplitude of the circuit wave by a_1 , and the amplitude of the beam wave by a_2 . Amplitudes a_1 and a_2 are so normalized that $\pm |a_1|^2$ and $-|a_2|^2$ are the powers carried by either wave. The sign of the circuit power is determined by the nature of the wave. A wave traveling in approximate synchronism with the beam waves on a forward-wave circuit has positive power (i. e., power traveling in the direction of the dc beam velocity); a wave on a backward-wave circuit has negative power. Amplitudes a_1 and a_2 can be related to the more familiar quantities, that is, to the circuit field at the beam, E_1 , and Pierce's K parameter and to the characteristic beam admittance Y_0 , and beam current i_2 in the fast wave. Indeed, the power P_1 on the circuit is

$$P_1 = \pm |a_1|^2 = \pm \frac{|E_1|^2}{2\beta_e^2 K} \quad (57)$$

and the power in the fast wave (if we neglect the electromagnetic power term) is

$$P_2 = -|a_2|^2 = -\frac{1}{2} \frac{1}{Y_0} |i_2|^2 \quad (58)$$

By an arbitrary phase adjustment we can set

$$a_1 = \frac{E_1}{\beta_e \sqrt{2K}} \quad (59)$$

$$a_2 = \frac{i_2}{\sqrt{2Y_0}} \quad (60)$$

Denote the unperturbed propagation constants of the two waves by β_1 and β_2 . Then, in the absence of coupling, (complete separation of circuit and beam) we have

$$\frac{da_1}{dz} = -j\beta_1 a_1 \quad (61)$$

$$\frac{da_2}{dz} = -j\beta_2 a_2 \quad (62)$$

If coupling is introduced between beam and circuit, the presence of one wave affects the rate of change of the amplitude of the other. If the coupling is weak, the coefficients $j\beta_1$ and $j\beta_2$ will not be affected, and the coupling is properly taken into account by setting

$$\begin{aligned} \frac{da_1}{dz} &= -j\beta_1 a_1 + c_{12} a_2 \\ \frac{da_2}{dz} &= c_{21} a_1 - j\beta_2 a_2 \end{aligned} \quad (63)$$

Further information about the coupling coefficients c_{12} and c_{21} can be obtained from power considerations. Indeed, if the coupling is weak, the total power is equal to the sum of the powers as computed on the basis of the unperturbed systems

$$P = \pm |a_1|^2 - |a_2|^2$$

Since the system is governed by a conservation law, the small-signal power theorem, we must have

$$\frac{dP}{dz} = \pm \left[\frac{da_1^*}{dz} a_1 + a_1^* \frac{da_1}{dz} \right] - \left[\frac{da_2^*}{dz} a_2 + a_2^* \frac{da_2}{dz} \right] = 0$$

Introducing Eqs. (63) into the foregoing equation we find

$$a_2^* (\pm c_{12}^* - c_{21}) a_1 + \text{complex conjugate} = 0$$

This relation has to be fulfilled for an arbitrary choice of initial conditions, i. e., arbitrary a_1 and a_2 . Thus, we must have

$$c_{21} = \pm c_{12}^* \quad (64)$$

Introducing condition (64) into Eqs. (63), assuming a dependence $\exp(-\gamma z)$, and solving for γ results in

$$\gamma = \frac{1}{2} j (\beta_1 + \beta_2) \pm \frac{1}{2} j \sqrt{(\beta_1 - \beta_2)^2 \mp 4 |c_{12}|^2} \quad (65)$$

The upper sign applies for a forward-wave circuit, the lower sign for a backward-wave circuit. It should be noted that this difference in sign arose from power considerations. It was pointed out quite generally by Pierce,¹⁴ with the aid of an equation like Eq. (65), that coupling between waves with equal sign of power (both waves carry positive

or negative power) never leads to exponential growth or decay, whereas coupling between waves of power with opposite sign leads to growth or decay at, or near, synchronism.

We also can obtain quantitative information about the coupling coefficient c_{21} from power considerations. Thus, consider the rate of change of the power P_2 in the beam, dP_2/dz ,

$$\frac{dP_2}{dz} = - \left[\frac{da_2^*}{dz} a_2 + a_2^* \frac{da_2}{dz} \right] = - a_2^* c_{21} a_1 - a_2 c_{21}^* a_1^* .$$

This power has to be provided by the field of the circuit E_1 that works against the beam current density \bar{J}_2 . If the beam is thin, the current density \bar{J}_2 and the longitudinal component of the field E_1 are uniform across the beam, and thus

$$\frac{dP_2}{dz} = \frac{1}{4} (E_1 i_2^* + E_1^* i_2) \quad \text{where} \quad i_2 = \int_{\text{cross section}} \bar{J}_2 \, d\bar{a}$$

Using Eqs. (59) and (60) and combining the last two equations, we have

$$-c_{21} = \frac{1}{4} E_1 i_2^* = \beta_e \sqrt{Y_0 K} / 2 \quad (66)$$

Using Eq. (39) and the fact that in Pierce's notation

$$C^3 = \frac{I_0}{4V_0} K$$

we have

$$c_{21}^2 = \frac{\beta_e^3 C^3}{2\beta_q}$$

which checks with Gould's expression.¹⁵ (Gould obtained this value by comparing the results obtained from the theory of mode-coupling with Pierce's original determinantal equation.)

Let us now summarize the advantages and disadvantages of the coupling-of-modes analysis derived here. There are the following advantages.

1. The solution of a composite system is reduced to the solution of two simpler subsystems.
2. From power considerations the general nature of the coupled waves can be predicted without even setting up the equations of coupling. It has the following disadvantages.

1. The approximations made in deriving the coupling equations are not entirely clear.

2. The analysis fails for more strongly coupled systems.

The variational principle to be derived in the sequel removes the first-mentioned disadvantage and leads, in the case of weak coupling, to the coupled-mode equations. Since it can also be applied to cover cases of stronger coupling, it does not suffer to the same extent from disadvantage (2).

V. Other Applications of the Power Theorem

In the presence of a slow-wave structure the beam waves couple to the slow circuit-waves, and a gain results if, in the process, the beam delivers electromagnetic power to the structure. When this happens, we conclude from the small-signal power theorem that the small-signal beam power must become more and more negative. This means that the circuit has to couple predominantly to a wave carrying negative kinetic power.

Considerations of this kind explain many known phenomena. Thus, it is known that for a maximum gain in traveling-wave tubes with appreciable space charge (large QC), the beam voltage has to be higher than the voltage corresponding to equality between beam velocity and circuit wave velocity. This phenomenon can be interpreted as follows: The slow space-charge wave carries negative power. In order to achieve gain we have to couple predominantly to this wave. If the space charge is large, the phase velocity of the slow wave is appreciably less than the beam velocity. In order to favor coupling to the slow wave, its phase velocity has to be brought close to synchronism with the circuit wave. This is done by raising the beam voltage above the value corresponding to synchronism between the beam velocity and phase velocity of circuit wave.

The power theorem also can be used to decide whether or not an electron beam corresponds to a gain mechanism or to attenuation. Exponential dependence upon distance of a wave can signify gain if, and only if,

(a) the growth occurs in the direction of the electromagnetic power flow, and

(b) the small-signal beam power becomes increasingly negative in the direction of the dc beam velocity.

Condition (a) states that an exponential growth of power corresponds to gain only when it leads to a spatial rate of increase (rather than decay) of the power. Condition (b) states the requirement that the (initially weakly excited) beam must lose kinetic energy imparted to the beam by the dc power supply (and not by an initial modulation).

To understand the implications of these conditions consider a simple example where an exponential growth of power does not signify gain. Consider interaction between the fast wave of a longitudinal

beam in synchronism with a wave of a backward-wave structure. The coupling to any other wave is disregarded. The fast wave carries positive kinetic power, the wave on the backward-wave circuit has negative electromagnetic power corresponding to an electromagnetic power flow from collector end to gun end. (See Fig. 10.) From Pierce's coupling-of-modes formalism we conclude that the coupling leads to a pair of waves, one growing in the direction of the dc beam velocity, the other growing in the opposite direction. It is this latter wave that has growth in the direction of the electromagnetic power flow and thus meets condition (a). Yet, it cannot lead to gain. The wave has an electromagnetic and a kinetic power flow as shown in Fig. 11, the two

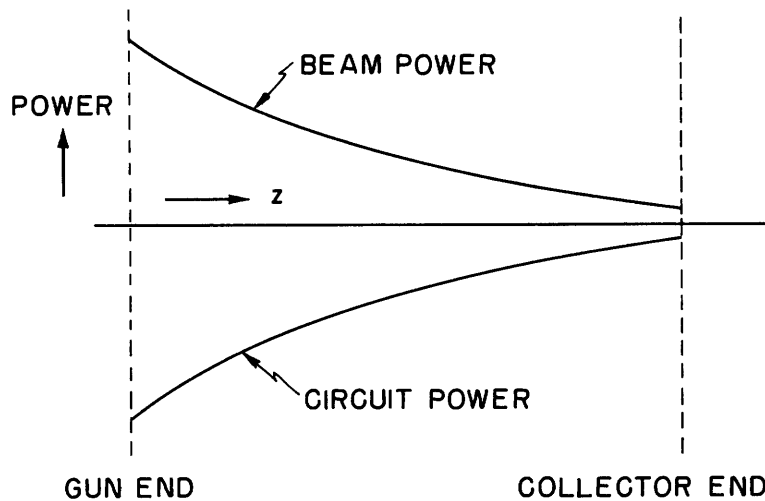


Fig. 11 The power flow in waves resulting from coupling between backward-circuit wave and fast beam wave.

power flows adding up to zero by the requirement of conservation of the generalized power. From the figure it is clear that the beam is losing positive kinetic power rather than acquiring negative kinetic power. In the light of our interpretation of the power theorem, *negative* kinetic power increasing in the direction of the dc beam velocity corresponds to a slow-down of an initially weakly excited beam; decreasing positive kinetic power corresponds to extraction of power supplied to the beam by an initial rf excitation. Only the former mechanism corresponds to gain.

VI. The Variational Principle

In the preceding discussion, the small-signal kinetic power theorem was derived and interpreted, and the various waves that can exist along an electron beam were studied. The coupling-of-modes

formalism discussed in Section IV combined the two concepts and, with the aid of an example, arrived at an intuitively appealing set of equations for the traveling-wave tube with large QC. This section is devoted to the derivation of a variational principle that will put the coupling-of-modes formalism on a firmer basis. It will also make possible the analysis of systems with an interaction between beam and circuit stronger than that consistent with the coupling-of-modes formalism of Section IV.

We shall give a brief derivation of the variational principle for curl-free electron beams in uniform, lossless, microwave structures. The variational principle for longitudinal beams is little different from that for curl-free beams.

Suppose we are interested in a solution with a z -dependence of the form $\exp(-\gamma z)$ propagating along a uniform electron beam in a uniform lossless slow-wave structure. As an example, consider a cylindrical electron beam inside a cylindrical sheet helix. The entire system is surrounded by a conducting envelope. (See Fig. 12.) We

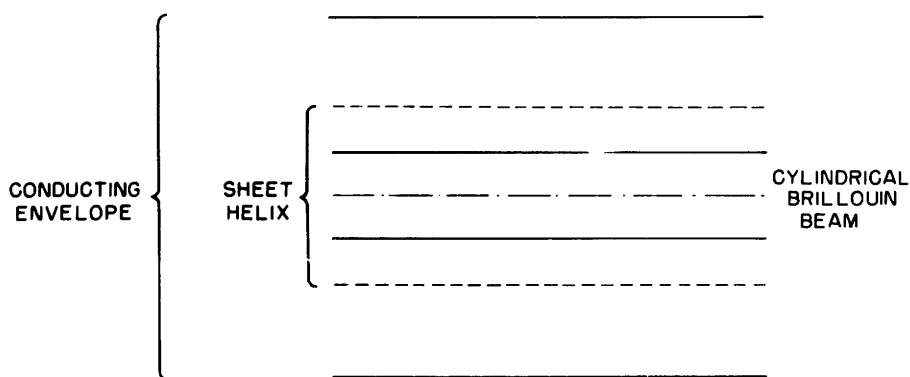


Fig.12 Section of sheet-helix traveling-wave tube with cylindrical Brillouin beam.

shall later make use of the concepts "beam system", "slow-wave system", and "composite system". In our example, the "beam system" is the beam inside the conducting envelope with the helix removed, and the "slow-wave system" refers to the helix inside the same envelope with the beam removed. The "composite system" is the system consisting of helix, beam, and envelope. Denote the transverse dependence of the electric field of the solution in the composite system, with the z -dependence $\exp(-\gamma z)$ by $\bar{E}_+(x, y)$, that of the magnetic field by $\bar{H}_+(x, y)$, that of the current density by $\bar{J}_+(x, y)$, and that of the Eulerian velocity by $\bar{u}_+(x, y)$. Introduction of the expressions

$$\bar{E}(x, y, z) = e^{-\gamma z} \bar{E}_+, \text{ etc.}$$

into Maxwell's equations (17) and (16), and into Eqs. (10) and (11) results in

$$\nabla_{\mathbf{T}} \times \bar{\mathbf{E}}_+ + j\omega\mu_0 \bar{\mathbf{H}}_+ = \gamma \bar{\mathbf{i}}_z \times \bar{\mathbf{E}}_+ \quad (67)$$

$$\nabla_{\mathbf{T}} \times \bar{\mathbf{H}}_+ - j\omega\epsilon_0 \bar{\mathbf{E}}_+ - \bar{\mathbf{J}}_+ = \gamma \bar{\mathbf{i}}_z \times \bar{\mathbf{H}}_+ \quad (68)$$

$$j\omega \bar{\mathbf{u}}_+ + \nabla_{\mathbf{T}} (\bar{\mathbf{u}}_0 \cdot \bar{\mathbf{u}}_+) - \frac{e}{m} \bar{\mathbf{E}}_+ = \gamma \bar{\mathbf{u}}_0 \cdot \bar{\mathbf{u}}_+ \quad (69)$$

$$j\omega \bar{\mathbf{J}}_+ + \bar{\mathbf{u}}_0 \cdot \nabla_{\mathbf{T}} \bar{\mathbf{J}}_+ - j\omega \rho_0 \bar{\mathbf{u}}_+ = \gamma \bar{\mathbf{u}}_0 \bar{\mathbf{i}}_z \cdot \bar{\mathbf{J}}_+ \quad (70)$$

where

$$\nabla_{\mathbf{T}} = \bar{\mathbf{i}}_x \frac{\partial}{\partial x} + \bar{\mathbf{i}}_y \frac{\partial}{\partial y} \quad (71)$$

Now, the following statement can be proved¹⁶ for any uniform system that is lossless in the general sense that it fulfills a conservation theorem of the form of Eq. (18):

A solution of the system of the form $\exp(-\gamma z)$ implies the existence of a solution $\exp(\gamma^* z)$.

If γ is pure imaginary, this statement is trivial. If γ is complex, however, this means that the existence of an exponentially growing wave implies the existence of an exponentially decaying wave of equal and opposite decrement.

Denote the transverse dependences of the solution with the dependence $\exp(+\gamma^* z)$ by $\bar{\mathbf{E}}_-(x, y)$, $\bar{\mathbf{H}}_-(x, y)$, $\bar{\mathbf{J}}_-(x, y)$, and $\bar{\mathbf{u}}_-(x, y)$. These vector fields fulfill Eqs. (67) to (70) with γ replaced by $-\gamma^*$. By dot-multiplying Eq. (67) by $\bar{\mathbf{H}}_-^*$, Eq. (68) by $-\bar{\mathbf{E}}_-^*$, Eq. (69) by $\bar{\mathbf{J}}_-^*$, and Eq. (70) by $(m/e)\bar{\mathbf{u}}_-^*$, and integrating over the cross section, we obtain, upon adding and solving for γ ,

$$\begin{aligned} \gamma = & \left\{ \frac{1}{4} \int \left[(\nabla_{\mathbf{T}} \times \bar{\mathbf{E}}_+ + j\omega\mu_0 \bar{\mathbf{H}}_+) \cdot \bar{\mathbf{H}}_-^* - (\nabla_{\mathbf{T}} \times \bar{\mathbf{H}}_+ \right. \right. \\ & - j\omega\epsilon_0 \bar{\mathbf{E}}_+ - \bar{\mathbf{J}}_+) \cdot \bar{\mathbf{E}}_-^* + (j\omega \frac{m}{e} \bar{\mathbf{u}}_+ + \nabla_{\mathbf{T}} \frac{m}{e} \bar{\mathbf{u}}_0 \cdot \bar{\mathbf{u}}_+ \\ & - \bar{\mathbf{E}}_+) \cdot \bar{\mathbf{J}}_-^* + (j\omega \bar{\mathbf{J}}_+ + \bar{\mathbf{u}}_0 \cdot \nabla_{\mathbf{T}} \bar{\mathbf{J}}_+ \\ & \left. \left. - j\omega \rho_0 \bar{\mathbf{u}}_+) \cdot \frac{m}{e} \bar{\mathbf{u}}_-^* \right] da \right\} \left\{ \frac{1}{4} \int \bar{\mathbf{i}}_z \cdot \left[\bar{\mathbf{E}}_+ \times \bar{\mathbf{H}}_-^* \right. \right. \\ & \left. \left. + \bar{\mathbf{E}}_-^* \times \bar{\mathbf{H}}_+ + \frac{m}{e} \bar{\mathbf{u}}_-^* \cdot \bar{\mathbf{u}}_0 \bar{\mathbf{J}}_+ + \frac{m}{e} (\bar{\mathbf{u}}_+ \cdot \bar{\mathbf{u}}_0) \bar{\mathbf{J}}_-^* \right] da \right\}^{-1} \quad (72) \end{aligned}$$

This formidable expression is simplified considerably when applied to

the coupling-of-modes formalism. For the moment let us note its variational character. It can be shown, with the aid of Eqs. (67) through (70) and the corresponding equations for the (-) solution, that (72) is a variational expression, provided the electric and magnetic fields inside the system are continuous. Thus, suppose that we introduce into Eq. (72) expressions for the quantities \bar{E}_\pm , \bar{H}_\pm , \bar{J}_\pm , and \bar{u}_\pm , the quantities \bar{E}'_\pm , \bar{H}'_\pm , \bar{J}'_\pm , and \bar{u}'_\pm , which are not solutions of Eqs. (67) to (70), but deviate from the correct solutions as follows

$$\left. \begin{aligned} \bar{E}'_\pm &= \bar{E}_\pm + \delta \bar{E}_\pm \\ \bar{H}'_\pm &= \bar{H}_\pm + \delta \bar{H}_\pm \\ \bar{J}'_\pm &= \bar{J}_\pm + \delta \bar{J}_\pm \\ \bar{u}'_\pm &= \bar{u}_\pm + \delta \bar{u}_\pm \end{aligned} \right\} \quad (73)$$

If the deviations $\delta \bar{E}_\pm$ and $\delta \bar{H}_\pm$ are continuous functions of x and y , if the tangential component of $\delta \bar{E}_\pm$ vanishes on conductors, and if $\delta \bar{J}_\pm$ and $\delta \bar{u}_\pm$ vanish outside the beam boundary, the error in γ obtained from Eq. (72) is of higher order than first in these deviations. Thus, Eq. (72) is indeed a variational expression.

If the \bar{H} -fields in the region of interest experience a discontinuity, such as along a sheet helix, the numerator of the variational expression (72) has to be supplemented by the term $-\oint \bar{E}_+ \cdot \bar{K}_-^* ds$, where \bar{K}_- is the surface current pertaining to the solution $\exp(\gamma^* z)$ and causing the discontinuity. The integral is carried over the helix contour in the x - y plane. Note, that for the *correct* solution this term is zero. The stipulation, then, is that the trial solutions used in (72) fulfill electric boundary conditions on the conducting envelope (not necessarily on the sheet helix), that the tangential E -field of the trial solution be continuous across the sheet helix, and that the H -field tangential to the sheet helix experience only a discontinuity in its component normal to the wires of the sheet. A similar term in the numerator can serve to extend the validity of the variational principle to trial solutions that do not fulfill the boundary conditions on the conducting envelope. Then \bar{K}_- in the integral $\oint \bar{E}_+ \cdot \bar{K}_-^* ds$ gives the current of the (-) solution in the envelope; the contour integral is carried over the contour of the envelope.

The corresponding variational expression for the longitudinal beam is derived in a similar manner. We set up the equations for the (+) solution and the (-) solution, i. e., waves with the z -dependences $\exp(-\gamma z)$ and $\exp(\gamma^* z)$, respectively. Those for the (+) solution are, for example

$$\nabla_T \times \bar{E}_+ + j\omega\mu_0 \bar{H}_+ = \gamma \bar{i}_z \times \bar{E}_+ \quad (74)$$

$$\nabla_{\mathbf{T}} \times \bar{\mathbf{H}}_+ - j\omega\epsilon_0 \bar{\mathbf{E}}_+ - \bar{\mathbf{J}}_+ = \gamma \bar{\mathbf{i}}_z \times \bar{\mathbf{H}}_+ \quad (75)$$

$$j \frac{\omega}{u_0} V_+ - E_{z+} = \gamma V_+ \quad (76)$$

$$j \frac{\omega}{u_0} \bar{\mathbf{J}}_+ - j \frac{|J_0|}{2V_0} \frac{\omega}{u_0} V_+ \bar{\mathbf{i}}_z = \gamma \bar{\mathbf{J}}_+ \quad (77)$$

Here $\bar{\mathbf{J}}_+$ is a z-directed vector. By proper dot-multiplication of Eqs. (74) to (77) by the corresponding quantities of the (-) solution we obtain the variational principle

$$\begin{aligned} \gamma = & \left\{ \frac{1}{4} \int \left[(\nabla_{\mathbf{T}} \times \bar{\mathbf{E}}_+ + j\omega\mu_0 \bar{\mathbf{H}}_+) \cdot \bar{\mathbf{H}}_-^* - (\nabla_{\mathbf{T}} \times \bar{\mathbf{H}}_+ \right. \right. \\ & - j\omega\epsilon_0 \bar{\mathbf{E}}_+ - \bar{\mathbf{J}}_+) \cdot \bar{\mathbf{E}}_-^* + (j \frac{\omega}{u_0} V_+ - E_{z+}) J_{z-}^* \\ & \left. \left. + (j \frac{\omega}{u_0} J_{z+} - j \frac{|J_0|}{2V_0} \frac{\omega}{u_0} V_+) V_-^* \right] da \right\} \quad (78) \\ & \left\{ \frac{1}{4} \int \left[\bar{\mathbf{E}}_+ \times \bar{\mathbf{H}}_-^* + \bar{\mathbf{E}}_-^* \times \bar{\mathbf{H}}_+ + \bar{\mathbf{V}}_-^* \bar{\mathbf{J}}_+ + V_+ \bar{\mathbf{J}}_-^* \right] \cdot \bar{\mathbf{i}}_z da \right\}^{-1} \end{aligned}$$

Again, the numerator has to be supplemented by $-\oint \bar{\mathbf{E}}_+ \cdot \bar{\mathbf{K}}_-^* ds$ if a sheet helix is considered which causes discontinuities of the fields, or if the trial solutions do not satisfy electric boundary conditions on the envelope.

VII. Coupling-of-Modes Theory as Derived from the Variational Principle

We shall now show how the coupling-of-modes theory, and with it the existing theories of the traveling-wave tube and magnetron amplifier, can be derived from the variational principle.

A variational principle is useful in obtaining values for propagation constants if we have good approximate solutions for the transverse dependence of the fields inside the system. Consider a thin longitudinal beam in a sheet helix to which we apply the variational principle (78) supplemented with the term $-\oint \bar{\mathbf{E}}_+ \cdot \bar{\mathbf{K}}_-^* ds$ in the numerator. If the phase velocity of the slow circuit wave with the propagation constant β_1 is close to one of the beam waves, say the slow beam wave with the propagation constant β_2 , then reasonable trial functions for the (+) are

$$\left. \begin{aligned}
 \bar{E}_+ &= a_1 \bar{E}_1 + a_2 \bar{E}_2 \\
 \bar{H}_+ &= a_1 \bar{H}_1 + a_2 \bar{H}_2 \\
 \bar{J}_+ &= a_2 \bar{J}_2 \\
 \bar{u}_+ &= a_2 \bar{u}_2
 \end{aligned} \right\} \quad (79)$$

where the subscript 1 indicates the transverse dependences of the circuit fields, and the subscript 2 indicates the transverse dependences of the beam fields, current density, and velocity. These are all normalized in a manner that will be discussed shortly. The a 's are adjustable parameters. For the trial functions for the (-) solution it is reasonable to use another linear combination of fields 1 and 2:

$$\left. \begin{aligned}
 \bar{E}_- &= b_1 \bar{E}_1 + b_2 \bar{E}_2 \\
 \bar{H}_- &= b_1 \bar{H}_1 + b_2 \bar{H}_2 \\
 \bar{J}_- &= b_2 \bar{J}_2 \\
 \bar{u}_- &= b_2 \bar{u}_2
 \end{aligned} \right\} \quad (80)$$

The transverse dependences of waves 1 and 2 satisfy Eqs. (74) to (77) with γ replaced by $\gamma_1 (= j\beta_1)$ and $\gamma_2 (= j\beta_2)$, respectively. Using this fact and introducing the trial functions (79) and (80) into the variational principle (78) we obtain

$$\gamma = \frac{b_1^* H_{11} a_1 + b_1^* H_{12} a_2 + b_2^* H_{21} a_1 + b_2^* H_{22} b_2}{b_1^* P_{11} a_1 + b_1^* P_{12} a_2 + b_2^* P_{21} a_1 + b_2^* P_{22} a_2} \quad (81)$$

where

$$\left. \begin{aligned}
 P_{11} &= \frac{1}{2} \operatorname{Re} \int \bar{E}_1 \times \bar{H}_1^* \cdot \bar{i}_z \, da \\
 P_{12} &= \frac{1}{4} \int (\bar{E}_1^* \times \bar{H}_2 + \bar{E}_2 \times \bar{H}_1^*) \cdot \bar{i}_z \, da = P_{21}^* \\
 P_{22} &= \frac{1}{2} \operatorname{Re} \int (\bar{E}_2 \times \bar{H}_2^* + V_2 \bar{J}_2^*) \cdot \bar{i}_z \, da
 \end{aligned} \right\} \quad (82)$$

These are the self- and cross-powers of waves 1 and 2. For the H's we have

$$\begin{aligned}
 H_{11} &= \gamma_1 P_{11} \\
 H_{12} &= \gamma_2 P_{12} - \frac{1}{4} \oint \bar{\mathbf{E}}_2 \cdot \bar{\mathbf{K}}_1^* ds \\
 H_{21} &= \gamma_1 P_{21} - \frac{1}{4} \oint \bar{\mathbf{E}}_1 \cdot \bar{\mathbf{J}}_2^* da \\
 H_{22} &= \gamma_2 P_{22}
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} H_{11} \\ H_{12} \\ H_{21} \\ H_{22} \end{aligned}} \right\} \quad (83)$$

By integration by parts of the integral expression for H_{21} , we can further show that

$$H_{21} = -H_{12}^* \quad (84)$$

The variational expression (81) contains a_1 , a_2 , b_1^* , and b_2^* as adjustable parameters. Since γ is an extremum for the correct solution, we obtain the best value for γ if we find the extremum for γ as given by (81). Differentiation with regard to b_1^* and b_2^* leads to the two equations,

$$\begin{aligned}
 \gamma P_{11} a_1 + \gamma P_{12} a_2 &= H_{11} a_1 + H_{12} a_2 \\
 \gamma P_{21} a_1 + \gamma P_{22} a_2 &= H_{21} a_1 + H_{22} a_2
 \end{aligned}
 \quad (85)$$

The solution for γ from the homogeneous equations (85) are obtained from the determinantal equation

$$\begin{vmatrix}
 \gamma P_{11} - H_{11} & \gamma P_{12} - H_{12} \\
 \gamma P_{21} - H_{21} & \gamma P_{22} - H_{22}
 \end{vmatrix} = 0 \quad (86)$$

We first note that both γ and $-\gamma^*$ are solutions of Eq. (86). This can be easily demonstrated by taking the complex conjugate of the entire determinant and taking into account the specific expressions (82), (83), and (84) for the P 's and H 's. This fact is of importance when studying the equations that result when the variational expression (81) is differentiated with regard to a_1 and a_2 . Then equations result that are of the form of Eqs. (85) but now contain the b^* 's instead of the a 's. We can easily show that these equations yield solutions identical with those of Eq. (85), using the fact that solutions γ , and $-\gamma^*$ always must appear in pairs. Thus, we may concentrate entirely on the solutions of Eqs. (85).

Equations (85) can be simplified if the coupling between the beam and circuit is weak. In this case, the propagation constant γ deviates from γ_1 or γ_2 only by a term of the order of the coupling parameter $|H_{12}|$ in the region of γ values within which appreciable interaction occurs. Neglecting terms involving products of P_{12} and H_{12} , we

have

$$\gamma P_{12} - H_{12} \cong \gamma_2 P_{12} - H_{12} = \frac{1}{4} \oint \bar{E}_2 \cdot \bar{K}_1^* ds \quad (87)$$

$$\gamma P_{21} - H_{21} \cong \gamma_1 P_{21} - H_{21} = \frac{1}{4} \oint \bar{E}_1 \cdot \bar{J}_2^* da \quad (88)$$

Using these expressions in Eq. (85) we obtain

$$\gamma P_{11} a_1 = \gamma_1 P_{11} a_1 - \frac{1}{4} \oint \bar{E}_2 \cdot \bar{K}_1^* ds a_2 \quad (89)$$

$$\gamma P_{22} a_2 = \gamma_2 P_{22} a_2 - \frac{1}{4} \oint \bar{E}_1 \cdot \bar{J}_1^* da a_1$$

By proper normalization we can adjust the self-power of the slow beam wave, P_{22} , to -1. Correspondingly, we can adjust the self-power of the circuit wave to the value of ± 1 , where the lower sign pertains to a backward wave circuit. We obtain for the preceding equations

$$\gamma a_1 = \gamma_1 a_1 + \frac{1}{4} \oint \bar{E}_2 \cdot \bar{K}_1^* ds a_2 \quad (90)$$

$$\gamma a_2 = \gamma_2 a_2 + \frac{1}{4} \oint \bar{E}_1 \cdot \bar{J}_2^* ds a_1$$

If we note that multiplication by $-\gamma$ corresponds to differentiation with respect to z we have reduced Eqs. (90) to the coupling Eqs. (63). The coupling coefficient c_{21} , Eq. (66), checks with that of the second of Eqs. (90). Relation (64) is not fulfilled in an obvious way. We may show however that in the case of weak coupling this relation is indeed satisfied; that is, we have

$$\frac{1}{4} \oint \bar{E}_2 \cdot \bar{K}_1^* ds \cong -\frac{1}{4} E_1^* i_2 = -\frac{1}{4} \int \bar{E}_1^* \cdot J_2 da \quad (91)$$

Indeed, from Eqs. (84) and (83) we have

$$\begin{aligned} H_{12} &= \gamma_2 P_{12} - \frac{1}{4} \oint \bar{E}_2 \cdot \bar{K}_1^* ds \\ &= -H_{21}^* = -\gamma_1^* P_{12} + \frac{1}{4} \int \bar{E}_1^* \cdot J_2 da \end{aligned}$$

or

$$\frac{1}{4} \oint \bar{E}_2 \cdot \bar{K}_1^* ds = -\frac{1}{4} \int \bar{E}_1^* \cdot J_2 da + (\gamma_1^* + \gamma_2) P_{12} \quad (92)$$

But γ_1 and γ_2 are imaginary and can differ only by a term of the order of the coupling coefficient $|H_{12}|$ in the range in which interaction occurs. Hence, the last term in the preceding equation is negligible.

Thus, we have shown that Eq. (91) is satisfied within our degree of approximation. Accordingly, we have proved that Pierce's coupling-of-modes formalism is a consequence of a variational principle into which we have introduced a trial solution that consists of a superposition of the beam fields and the slow-wave fields of the circuit wave. Pierce's formalism implies an additional approximation that neglects all terms of second order in the coupling coefficients. Equation (85) obtained from the variational principle is thus applicable to coupling stronger than the coupling that is permitted in the coupling-of-modes theory.

Next, let us consider the application of the variational principle when interaction between more than two waves is of importance. For this purpose we shall consider a curl-free beam (e.g., a solid cylindrical Brillouin beam, or a strip beam in crossed electric and magnetic fields) and shall use the corresponding variational principle (72). If the phase velocity of the slow circuit wave is close to the phase velocities of a group of beam waves, say $(n-1)$ such waves, then it is reasonable to suppose that the transverse dependences of the fields, beam currents, and so forth, are well represented by a linear superposition of the transverse dependences of the slow wave of the "slow-wave system" and the $(n-1)$ waves of the "beam system". Denote the fields of the circuit wave by the subscript 1, the beam waves by the subscripts 2 to n . Suppose that all beam waves have pure imaginary propagation constants except the last two, which may be a pair of growing and decaying diocotron waves. We have

$$\bar{E}_+ = \sum_{i=1}^n a_i \bar{E}_i \quad (93)$$

The trial solution for the H-field is

$$\bar{H}_+ = \sum_{i=1}^n a_i \bar{H}_i \quad (94)$$

For the current, it is

$$\bar{J}_+ = \sum_{i=2}^n a_i \bar{J}_i \quad (95)$$

and for the velocity, it is

$$\bar{u}_+ = \sum_{i=2}^n a_i \bar{u}_i \quad (96)$$

The (-) trial functions are chosen analogously with coefficients b_1 to b_n . These trial fields are then introduced into Eq. (72). The result is best written in matrix form. We define

$$\underline{\underline{a}} = \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_n \end{bmatrix} \quad \underline{\underline{b}} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{bmatrix} \quad (97)$$

Further, we denote by a dagger (\dagger) the operation of taking the complex conjugate transpose of a matrix. We then obtain from Eq. (72):

$$\gamma = \frac{\underline{\underline{b}}^\dagger \underline{\underline{H}} \underline{\underline{a}}}{\underline{\underline{b}}^\dagger \underline{\underline{P}} \underline{\underline{a}}} \quad (98)$$

The matrices $\underline{\underline{H}}$ and $\underline{\underline{P}}$ are obtained from an evaluation of the various integral expressions in the numerator and denominator of Eq. (72). The element P_{ij} of $\underline{\underline{P}}$ is the cross-power between the i th and j th wave. Since power orthogonality² exists between any two unperturbed beam waves (with the exception of the finite cross-power between the two diocotron waves), we have $P_{ij} = 0$ for $i \neq 1, j \neq 1$, with the exception of $P_{n,n-1}$ and $P_{n-1,n}$, for which $P_{n,n-1} = P_{n-1,n}^* \neq 0$. We further note that the self-power of the diocotron waves is equal to zero. Thus, the P -matrix has the simple form

$$\underline{\underline{P}} = \begin{bmatrix} P_{11} & P_{12} & P_{13} & \dots & P_{1(n-1)} & P_{1n} \\ P_{21} & P_{22} & 0 & \dots & 0 & 0 \\ P_{31} & 0 & P_{33} & \dots & 0 & 0 \\ \cdot & & & \dots & & \\ \cdot & & & \dots & & \\ P_{(n-1)1} & 0 & 0 & & 0 & P_{(n-1)n} \\ P_{n1} & 0 & 0 & & P_{n(n-1)} & 0 \end{bmatrix} \quad (99)$$

Further, we can easily show that

$$\underline{\underline{P}}^\dagger = \underline{\underline{P}} \quad (100)$$

In the $\underline{\underline{H}}$ -matrix many of its elements are equal to zero because of the power orthogonality. Defining the matrix of propagation constants

$$\underline{\underline{\Gamma}} = \text{diag} (\gamma_1, \gamma_2, \dots, \gamma_{n-1}, \gamma_n) \quad (101)$$

where

$$\gamma_n = -\gamma_{n-1}^* \quad \text{with } \text{Re}(\gamma_n) \neq 0$$

we may write \underline{H} conveniently in the form

$$\underline{H} = \underline{P} \underline{\Gamma} + \underline{C} \quad (102)$$

where

$$C = -\frac{1}{4} \begin{bmatrix} 0 & \int E_2 \cdot K_1^* ds & \int E_2 \cdot K_1^* ds & \dots & \int E_{n-1} \cdot K_1^* ds & \int E_n \cdot K_1^* ds \\ \int E_1 \cdot J_2^* da & 0 & 0 & & 0 & 0 \\ \int E_1 \cdot J_3^* da & 0 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \int E_1 \cdot J_{n-1}^* da & 0 & 0 & & 0 & 0 \\ \int E_1 \cdot J_n^* da & 0 & 0 & & 0 & 0 \end{bmatrix} \quad (103)$$

And by integration by parts of the off-diagonal elements of \underline{H} , we can show that $H_{1j} = -H_{j1}^*$. Further, since the diagonal elements of \underline{H} are all pure imaginary, we can summarize these relations as

$$\underline{H} = -\underline{H}^\dagger \quad (104)$$

Differentiating the variational expression (98) with regard to the elements of \underline{a} , we get the coupling-of-modes equation

$$\gamma \underline{P} \underline{a} = \underline{H} \underline{a} \quad (105)$$

Using Eq. (102), we can write Eq. (105) in an alternate form,

$$\gamma \underline{a} = \underline{\Gamma} \underline{a} + \underline{P}^{-1} \underline{C} \underline{a} \quad (106)$$

The values of γ are given by the determinantal equation

$$\det(\gamma \underline{P} - \underline{H}) = 0 \quad (107)$$

Taking the complex conjugate of the determinantal equation and using Eqs. (100) and (104) we can prove that $-\gamma^*$ is a solution of (107) if γ is a solution.

$$\det(\gamma^* \underline{P}^\dagger - \underline{H}^\dagger) = \det(\gamma^* \underline{P} + \underline{H}) = 0$$

Again, as in the case of the two waves, we can show that differentiation of Eq. (98) by the elements of \underline{a} leads to equations for the b_i identical with those of Eq. (105), except that γ is replaced by $-\gamma^*$. Since we have just demonstrated that these γ solutions always appear

in pairs, we see that the equation obtained by differentiation of (98) with regard to \underline{a} is identical with Eq. (105). Finally, for weak coupling, when products of the first-order coupling terms in the matrix $\underline{P}^{-1} \underline{C}$ can be neglected, we can replace \underline{P} , after proper normalization, by

$$\underline{P} = \begin{bmatrix} \pm 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \pm 1 & 0 & & 0 & 0 \\ 0 & 0 & \pm 1 & & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & & 0 & 1 \\ 0 & 0 & 0 & & 1 & 0 \end{bmatrix} \quad (108)$$

The signs in \underline{P} have to be chosen to correspond to the sign of the power flow pertaining to a particular wave. For the coupling equation (106) using the fact that \underline{P} of Eq. (108) satisfies the condition $\underline{P} = \underline{P}^{-1}$, we obtain

$$\gamma \underline{a} = \underline{\Gamma} \underline{a} + \underline{P} \underline{C} \underline{a} \quad (109)$$

which is much simpler than (106) because of the simple character of $\underline{\Gamma}$, \underline{C} , and \underline{P} (of Eq. (108)). Using Eqs. (101), (102), (104), and the form of the \underline{P} -matrix, Eq. (108), we can also show that

$$\underline{C} + \underline{C}^\dagger = 0 \quad (110)$$

This completes the general discussion of the variational principle as applied to obtaining a coupling-of-modes formalism. We have already given an illustration of this formalism in the two-wave analysis for the longitudinal beam traveling-wave tube. The preceding discussion was devoted to the curl-free beam. It is therefore worth while to discuss what differences (if any) there are between the cylindrical Brillouin beam and the longitudinal-beam traveling-wave tube.

Two-Wave Analysis of a Traveling-Wave Tube with Cylindrical Brillouin Beam

If we compare Eqs. (90) with the coupling Eqs. (109) when applied to a case of two waves with $P_{11} = \pm 1$, $P_{22} = -1$, and \underline{C} given by the 2×2 matrix in the upper left-hand corner of (103), we find that the equations check in every respect. We further note the complete correspondence between the beam equations (38) and (39) for a thin longitudinal beam and (46) and (47) for a thin Brillouin beam. We thus conclude that Eq. (66) for c_{21} holds for the thin Brillouin beam as well.

It is not difficult to extend the present formalism to a thick

cylindrical Brillouin beam.

Magnetron Amplifier

We shall now briefly outline the application of the variational principle to the problem of finding the determinantal equation for the forward-wave magnetron amplifier.

We shall start with the coupling-of-modes equation (109) and use a trial solution that consists of a circuit wave and of the two diocotron waves of the thin beam. The magnetron sole is assumed to be far removed (see Fig. 13). Further, weak coupling between circuit and

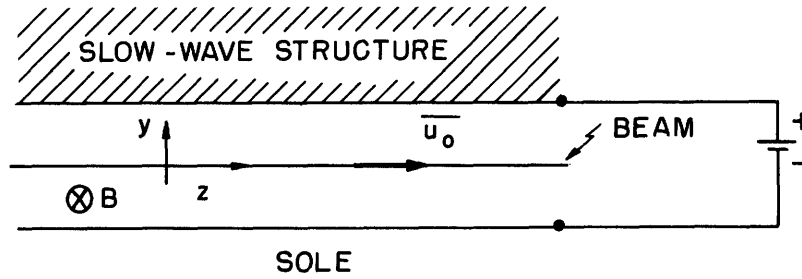


Fig.13 Schematic of magnetron amplifier.

beam will be assumed (i. e., a relatively large distance between circuit and beam) so that we can use as trial solutions the expressions for the diocotron waves in free space developed in Section III C. We shall consider a tube of width ℓ in the x-direction. Following Pierce, we again define a circuit impedance by

$$K = \frac{|E_{1z}|^2}{2\beta_e^2 P} \quad (111)$$

where E_{1z} is the circuit-wave field along the z-axis at the position of the beam. The y-component of the circuit field for a forward circuit wave is related to the z-component by

$$E_{1y} = j\alpha E_{1z} \quad (112)$$

If the sole is far removed from the circuit, as assumed here, $\alpha = 1$. The coupling coefficient C_{21} is (see Eq. (103))

$$C_{21} = -\frac{1}{4} \int \bar{E}_1 \cdot \bar{J}_2^* da \quad (113)$$

where \bar{J}_2 is the current in the *growing* diocotron wave. We have, in general,

$$C_{21} = -\frac{1}{4} \left[\int E_{1z} J_{2z}^* da + \int E_{1y} J_{2y}^* da \right] \quad (114)$$

J_{2z} includes the surface current. If the beam is thin, an average displacement y of the beam* produces a surface current $yu_o\rho_o$ on top of the beam, $-yu_o\rho_o$ on the bottom of the beam. If the field E_{1z} varies through the beam of thickness θ , the *surface current* gives a finite contribution to the first integral in (114) which is

$$-\frac{1}{4} \left[yu_o^* \rho_o \theta \frac{\partial E_{1z}}{\partial y} \right] = -\frac{1}{4} \left[yu_o^* \sigma_o \frac{\partial E_{1z}}{\partial y} \right] \quad (115)$$

The contribution of the *volume current* to $\int E_{1z} J_{2z}^* da$ is obtained by assuming E_{1z} to be constant through the beam. A similar assumption can be made concerning E_{1y} and $J_{2y} = \rho_o w_{y2}$. We thus obtain for C_{21}

$$C_{21} = -\frac{1}{4} \left[E_{1z} i_{z2}^* + y_2^* \frac{\partial E_{1z}}{\partial y} u_o \sigma_o \ell + E_{1y} \sigma_o \ell w_{y2}^* \right] \quad (116)$$

with $i_z = \int J_z da$. This expression can be simplified by noting that in a slow wave the curl of \bar{E}_1 is zero. We thus have

$$\frac{\partial \bar{E}_{1z}}{\partial y} e^{-\gamma_1 z} = \frac{\partial}{\partial z} (E_{1y} e^{-\gamma_1 z}) = -\gamma_1 E_{1y} e^{-\gamma_1 z} \quad (117)$$

The transverse velocity is

$$w_{y2} = (j\omega - u_o \gamma) y_2 \quad (118)$$

and γ is close to γ_1 . Introducing (112), (117), and (118) (with $\alpha=1$) in Eq. (116), we obtain

$$C_{21} = -\frac{1}{4} E_{1z} \left[i_{z2}^* + j i_{y2}^* \right]$$

where (see Eq. (55)) $i_{y2} = j\omega \sigma_o \ell y_2$. Further, since for the growing diocotron wave

$$i_{y2} = +i_{z2} \quad (56)$$

we have

$$C_{21} = -\frac{1}{4} E_{1z} i_{z2}^* \left[1 + j \right] \quad (119)$$

Now, let us so normalize E_{1z} that the circuit power associated with

*We shall omit the subscript 1 indicating small-signal amplitude. A subscript 1 now indicates a quantity pertaining to the circuit wave.

it is unity. We thus have from Eq. (111)

$$E_{1z} = \beta_e \sqrt{2K} \quad (120)$$

Further, computing the cross-power term associated with the wave amplitude $a_2 a_3^*$ and $a_3 a_2^*$ with the aid of Eq. (55) we obtain the following values for the elements P_{23} and P_{32} of \underline{P} :

$$\begin{aligned} P_{32} &= -\frac{1}{4} \frac{m}{e} u_o \omega_c (y_2 i_{z3}^* + y_3 i_{z2}^*) \\ &= +\frac{1}{4} \left| \frac{2V_o}{I_o} \right| \frac{\omega_c}{\omega} j (i_{y2} i_{z3}^* - i_{y3} i_{z2}^*) \end{aligned} \quad (121)$$

$$P_{23} = \frac{1}{4} \left| \frac{2V_o}{I_o} \right| \frac{\omega_c}{\omega} j (i_{y3} i_{z2}^* - i_{y2} i_{z3}^*) \quad (122)$$

We successfully normalize $P_{32} = P_{23} = 1$ if we take into account Eqs. (56) and set

$$i_{z3} = +j i_{z2} \quad (123)$$

and

$$i_{z2} = \sqrt{2 \left| \frac{I_o}{2V_o} \right| \frac{\omega}{\omega_c}} \quad (124)$$

We thus obtain for the coupling coefficient (119)

$$C_{21} = -\frac{\beta_e}{2} \sqrt{\left| \frac{I_o}{2V_o} \right|} \frac{\omega}{\omega_c} K (1+j) = -\frac{\beta_e}{2} D(1+j) \quad (125)$$

in the notation of Ref. 11. The coupling coefficient C_{31} follows from its definition with the aid of (123), (124), and the fact that for the decaying diocotron wave, $i_{y3} = -i_{z2}$.

$$C_{31} = -\frac{1}{4} E_{1z} i_{z3}^* [1-j] = +\frac{\beta_e}{2} Dj(1-j) \quad (126)$$

With the following definitions (compare Eqs. (50) through (52)):

$$\left. \begin{aligned} -\gamma &= -j\beta_e (1 + j\delta D) \\ -\gamma_1 &= -j\beta_e (1 + bD) \\ -\gamma_2 &= -j\beta_e (1 + jDS) \\ -\gamma_3 &= -j\beta_e (1 - jDS) \end{aligned} \right\} \quad (127)$$

and noting that according to Eq. (110), $C_{21} = -C_{12}^*$, $C_{31} = -C_{13}^*$, we obtain from Eq. (109) the determinantal equation

$$\begin{vmatrix} +(\delta + jb) & +\frac{1}{2}(1 - j) & +\frac{j}{2}(1 + j) \\ +\frac{j}{2}(1 - j) & +(\delta - S) & 0 \\ -\frac{1}{2}(1 + j) & 0 & +(\delta + S) \end{vmatrix} = 0 \quad (128)$$

or

$$(\delta + jb)(\delta^2 - S^2) = \delta.$$

This equation checks with Eq. (25) of Ref. 11 applied to a forward wave, with $g = 0$. But, the parameter g should be set equal to zero anyway, when considering a beam that is far from the sole and circuit as assumed here in order to be consistent with the approximation $\alpha \cong 1$.

Conclusion

Pierce's coupling-of-modes description of distributed amplifiers is, without doubt, very appealing. This description reduces the solution of a composite system to a perturbation of the simpler subsystems, for which solutions are more easily obtained. It uses the concept of power that is basic to an understanding of the interaction between a beam and a circuit. The coupling of modes theory has found limited application since no method had been given for the evaluation of the coupling coefficients, in particular in cases of fairly strong coupling. The variational principle discussed here removes this difficulty. It leads to coupling equations, which, in the case of weak coupling, yield the results obtained by an intuitive method using power considerations. The coupled-wave equations that are obtained from the variational principle are the result of an *optimization of the propagation constant* of the solution. Thus, the coupled-wave equations resulting from the variational principle give the best value for the gain of the tube that can be obtained from a knowledge of the unperturbed waves along the component systems (beam and microwave structure). Finally, a variational principle for any physical quantity permits us to obtain results that are closer and closer to the true value of the quantity as more and more adjustable parameters are introduced. It thus allows the use of any trial solution (not necessarily a superposition of the unperturbed waves) with a sufficient number of adjustable parameters.

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