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**WORKING PAPER**  
**ALFRED P. SLOAN SCHOOL OF MANAGEMENT**

AN EXTENSION OF THE MARKOWITZ PORTFOLIO SELECTION MODEL  
TO INCLUDE VARIABLE TRANSACTIONS' COSTS, SHORT SALES,  
LEVERAGE POLICIES AND TAXES *[Pogue]*

388-69A

~~Revised~~  
G. A. Pogue

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## Introduction

The approach to portfolio selection upon which most of the current academic work in this area is based was developed by H. M. Markowitz and presented in a 1952 paper.<sup>1</sup> Since that time many extensions to Markowitz's basic approach have been suggested by various authors attempting to explain the asset-holding behavior of individuals or develop normative rules for asset choice.

In much of this work a standard set of assumptions about the securities markets continually reappears. These assumptions relate to the costs involved in revising an existing portfolio to obtain another which is more desirable in terms of revised expectations about future security prices. The assumptions relate to two types of portfolio transactions' costs; the brokerage fees involved in exchanging portfolio assets and price effects resulting from asset illiquidities.

Current portfolio selection models generally ignore the brokerage fees involved in revising an existing portfolio. The result of this assumption is that frequent portfolio revisions may occur which are not justified relative to the resulting brokerage fees. Small changes in expectations about a particular security can result in transactions which would not occur if the broker's fees for purchasing or selling that asset were considered.

The second cost relates to the liquidity of portfolio assets. It is usually assumed that assets are perfectly liquid, that is, convertible without delay into currency at full market value, in any quantity. This as-

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<sup>1</sup>Harry M. Markowitz, "Portfolio Selection, The Journal of Finance, March 1952.





sumption is challenged by many institutional investors. Depending on the nature of the security involved, institutional investors contend that substantial unfavorable price spreads can result in attempts to buy or sell large quantities of stock. If volume related price effects exist, then portfolio selection models which neglect these costs can produce portfolio turnover rates which are non-optimal in terms of the price-spread transactions costs involved. This consideration is of particular importance to large institutional investors.<sup>1</sup>

In addition to these assumptions regarding portfolio transactions costs, a restricted set of investment alternatives is usually considered. Excluded are short sales and liability holdings, including secured margin loans and other types of unsecured debt. Substantial use of these techniques by individuals and financial institutions<sup>2</sup> exists in the capital markets.

When the set of investment alternatives is expanded to include short sales and liabilities, the resulting set of efficient portfolios will generally dominate the set created in their absence. Thus, for a given risk level, portfolios selected under the expanded set of investment alternatives will have expected returns which are equal to or greater than the portfolios selected under the usual restrictions.

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<sup>1</sup>While empirical evidence indicates the existence of price effects for large transactions, they are generally smaller than the effects hypothesized by many institutional investors. The question of impact on the market of large blocks of stock is currently receiving the attention of a number of researchers and institutions, including the Securities and Exchange Commission.

<sup>2</sup>For example, a set of investment companies, usually designated as hedge funds, make particular use of these procedures.



Finally, there is the question of taxes on portfolio capital gains and dividend income. When capital gains taxes are considered, transactions produced by a model which ignores taxes may no longer be optimal. The effect of differential tax rates on capital gains and dividend income is a factor which is relevant when portfolios are selected or revised.

The purpose of this paper is to consider a number of these generally neglected issues. The Markowitz model will be extended to include the investor's expectations regarding the two components of portfolio transactions costs, brokerage charges and price effects associated with large volume transactions. The model will include short sale and liability alternatives, as well as a treatment of the tax problem.

#### Investor Preferences and Subjective Beliefs

The following assumptions about investor preferences and subjective prior beliefs regarding security returns are required.

- A 1. The investor attempts to maximize his expected utility of terminal wealth, in the von Neumann-Morgenstern sense. Here terminal wealth is considered to be identical to the market value of the investor's portfolio at the end of his planning horizon.
- A 2. The investors planning horizon consists of a single period. The investment strategy involves selection of an optimal portfolio at the beginning of the period which will be held unchanged to the terminal date.
- A 3. The investor is assumed to be risk averse. The investor's marginal utility of wealth is assumed to be everywhere non-negative and a decreasing function of wealth.



In addition, one of the following assumptions is made.

- B 1. The investor's subjective prior joint distribution of one-period security returns is multivariate normal. It then follows that the distributions of portfolio returns will be normal as well.<sup>1</sup>
- B 2. The subjective distribution of one-period security returns are such that the returns on feasible portfolios will be normally distributed.<sup>2</sup>
- B 3. The investor's utility function can be well approximated by a quadratic function in the range of portfolio returns.<sup>3</sup>

<sup>1</sup>One-period security return,  $\hat{R}_j$ , is a linear transformation of terminal security value  $\tilde{M}_j$ , where  $\tilde{M}_j = \tilde{P}_j + \tilde{D}_j$  and

$$\begin{aligned}\tilde{P}_j &= \text{Terminal market price of security } j. \\ \tilde{D}_j &= \text{dividends paid during the period.}\end{aligned}$$

Thus, the investors one-period expected utility maximization problem can be defined in terms of one-period portfolio return,  $\hat{R}_p$ , as well as in terms of terminal portfolio value  $\tilde{M}_p$ . Similarly, if the security returns,  $\hat{R}_j$ ,  $j=1, \dots, N$  are jointly normally distributed, the terminal security values,  $\tilde{M}_j$ , will be as well.

<sup>2</sup>This requirement is potentially considerably less restrictive than that implied by assumption B 1. It is probably most applicable in the case of large institutional investors, who hold many securities in their portfolios, (e.g., a hundred or more) none of which contributes in a major way to the distribution of total portfolio return. This condition relies on a generalization of the central limit theorem to random variables which are not identically or independently distributed. In the case of independently (but not identically) distributed random variables, we can rely on Lindeberg's generalization of the central limit theorem (See W. Feller, An Introduction to Probability Theory and Its Application, Vol II, pp. 256-257). For the more realistic case of non-independence the limit theorems become more complex and, as a practical matter, the question of portfolio normality is probably best investigated via simulation.

<sup>3</sup>Along with this assumption, it will also be necessary to assume the existence of means and standard deviations for the investor's prior distributions of one-period security returns.





Conditions B 1 and B 2 place restrictions on the investor's subjective probability distributions. Condition B 3 places parametric restrictions on his utility of return function. Tobin<sup>1</sup> has shown that when one of these assumptions is valid, the investor's preference for portfolios can be determined solely on the basis of the one-period means and standard deviations of return. The optimal portfolio will be a member of the mean-standard deviation efficient set, where an efficient portfolio must satisfy the following criteria: (1) if any other portfolio provides a lower standard deviation of one period return, it must also have a lower expected return; and (2) if any other portfolio has greater expected return, it must also have greater standard deviation of return.

The following are the major notational symbols used throughout the paper.

- $N$  = number of securities in the universe considered.
- $\tilde{P}_j$  = the price of security  $j$  at the end of the planning horizon.
- $\tilde{D}_j$  = the dividends paid on security  $j$  during the time horizon.
- $\tilde{M}_j$  = the terminal market value of security  $j$   
 $= \tilde{P}_j + D_j$
- $\hat{M}_j$  = the mean of the investor's prior distribution for  $\tilde{M}_j$   
 $= \hat{P}_j + \hat{D}_j$
- $\sigma_{jj}$  = the variance of the investor's distribution for  $\tilde{M}_j$ ,  
 $= E(\tilde{M}_j - \hat{M}_j)^2$ .
- $\sigma_{jj'}$  = the covariance between  $\tilde{M}_j$  and  $\tilde{M}_{j'}$ ,  
 $= E(\tilde{M}_{j'} - \hat{M}_{j'}) (\tilde{M}_j - \hat{M}_j)$ .

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<sup>1</sup>James Tobin, "Liquidity Preferences as Behavior Toward Risk," Review of Economic Studies, (Feb 1958) pp. 65-86.



$X_j$  = the number of shares of security  $j$  held during the investment period.

$X_j(0)$  = the number of shares of security held prior to the investment period (before the portfolio is revised).

$P_j(0)$  = the price of security  $j$  at the beginning of the investment period.

For compactness of notation, the following vector quantities are defines.

$\underline{X}'$  = the revised portfolio vector,

$$= (X_1, \dots, X_N).$$

$\underline{X}'(0)$  = the initial portfolio vector,

$$= (X_1(0), \dots, X_N(0)).$$

$\tilde{\underline{M}}$  = the vector of terminal security values,

$$= (\tilde{M}_1, \dots, \tilde{M}_N).$$

$\underline{P}(0)$  = the vector of initial security prices,

$$= (P_1(0), \dots, P_N(0)).$$

$\underline{\Sigma}$  = the covariance matrix of security terminal values,

$$= \|\sigma_{jj'}\|$$

$$= \|E(\tilde{M}_j - \hat{M}_j)(\tilde{M}_{j'} - \hat{M}_{j'})\|$$

$$\begin{matrix} j=1, \dots, N \\ j'=1, \dots, N \end{matrix}$$

Thus the investor's estimate of the portfolio market value at the end of the investment period is given by

$$\hat{M}_p = \sum_{j=1}^N X_j \hat{M}_j = \underline{X}^1 \hat{\underline{M}}$$

$$= \sum_{j=1}^N X_j (\hat{P}_j + \hat{D}_j) = \underline{X}^1 (\hat{\underline{P}} + \hat{\underline{D}})$$



The variance of the investor's prior distribution of portfolio return is

$$\begin{aligned} V_p &= \sum_{j=1}^N \sum_{j'=1}^N x_j x_{j'} \sigma_{jj'} \\ &= \underline{X} \underline{\Sigma} \underline{X} \end{aligned}$$

The efficient pairs  $(\hat{M}_p, V_p)$  and the corresponding portfolio vectors  $\underline{X}$  which yield them are determined by solving the problem

$$\begin{aligned} \text{Max } Z &= \Theta \underline{X}^1 \underline{M} - \underline{X}^1 \underline{\Sigma} \underline{X} \\ \text{for all } \Theta &\geq 0 \end{aligned}$$

subject to the set of resource, policy and legal restrictions which are relevant for the investor. In the model developed in this paper, most of the constraints are linear functions of the decision variables,  $X_1, \dots, X_N$ , and thus can be summarized as

$$\underline{AX} \leq \underline{B}$$

where  $\underline{A}$  = a matrix of resource utilization coefficients  
 $\underline{B}$  = a vector of resource limitation or other activity constraints.

We now proceed to develop the form of the vectors A and B via the consideration of transactions costs, taxes and various types of investment and financing alternatives.

### Transactions Costs

As previously discussed, security transactions costs are considered as comprising of two parts, an asset exchange or brokerage fee and a liquidity or marketability cost.





## (a) Brokerage Fees

For ease of exposition of the total model, only the case of proportional brokerage fees will be considered in the main text. A formulation of the volume discount case is presented in appendix A.

Let  $c_j$  = the fraction of the pershare auction market price which must be paid in brokerage fees to transact one share of security  $j$ .

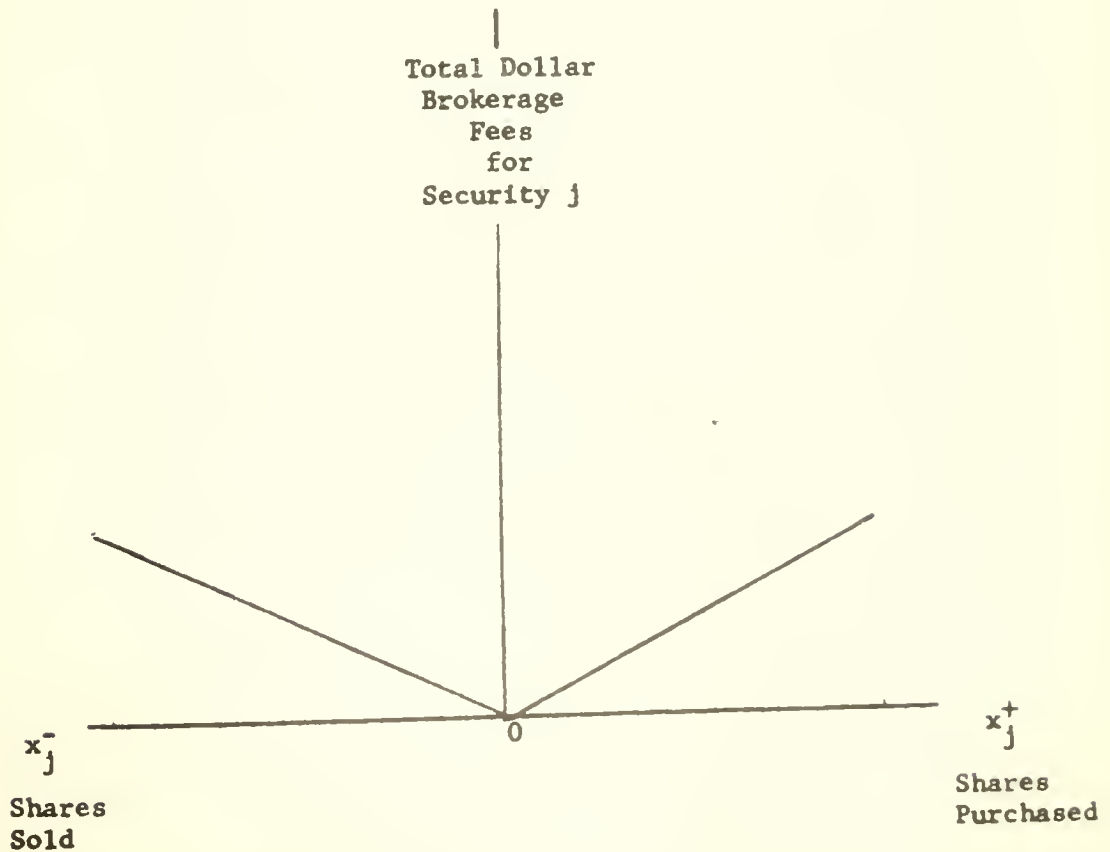
Therefore the cost of purchasing  $x_j^+$  shares of security  $j$  is given  $c_j x_j^+$ .

The brokerage transactions' cost curve is illustrated in figure 1.

FIGURE 1

**BROKERAGE TRANSACTIONS' COST CURVE**

**Proportionate Case**





(b) Marketability Costs

The difficulty in purchasing or selling a given quantity of stock in a specified period is generally considered to be related to the liquidity of the auction market, which, for a specific security can be measured in terms of the "normal" trading volume of the stock. A particular transaction which represents 10-20% of the average trading volume in a given period can, in most cases, be more easily transacted than a trade which represents many times the normal auction market volume. The additional expense results from the costs of informing additional purchasers or sellers about the current unusual opportunities that exist and offering them inducements to re-balance their portfolios, which can consist of favorable price spreads and/or payment of any brokerage fees resulting from the trade. In addition, in relation to purchases of large blocks of a stock, some additional incentive may be required to induce individuals with capital gains liabilities to provide their shares.

In this model, for each security, we use the expected normal trading volume as a metric with which to relate expected marketability costs to volume of shares traded.<sup>1</sup> Note that since an expected transactions' costs curve is being defined for each security, the investor can incorporate any expectations he may hold regarding the special ease or difficulty of trading large volumes of a particular stock.

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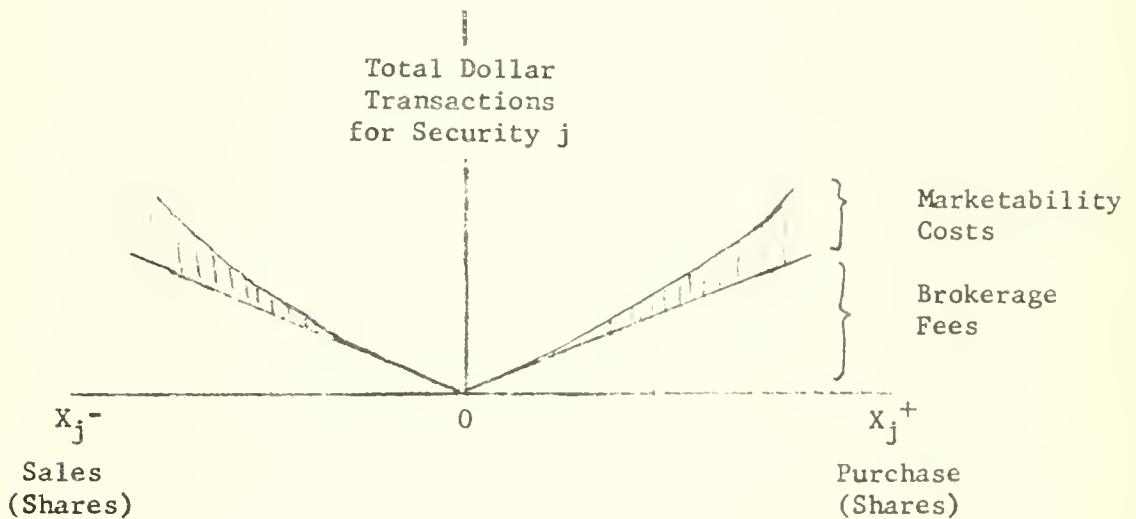
<sup>1</sup>Additional measures of the relative size of a transaction could be used instead of the proportion of "normal" trading volume. An example is the percentage of stock outstanding represented by the trade.



The type of total transactions' cost curve used in the model is illustrated in Figure 2(a). The shaded area represents the investor's expectation of the costs that will be necessary to purchase or sell a given volume of shares of security  $j$ , in addition to brokerage fees.

Figure 2a

TOTAL TRANSACTIONS' COSTS CURVE



In Figure 2(b), the above curve has been approximated by a piecewise linear representation. The change points for the marginal transactions' costs rates (i.e., the slopes of the linear segments) occur when purchases or sales of security  $j$  amount to specified percentages of the expected normal trading volume for that security.









We can now define the number of shares of security  $j$  traded in terms of purchases or sales in the linear segments of the cost curve.

The number of shares of security  $j$  traded

$$\begin{aligned} &= X_j - X_j(0) \\ &= x_j^+ - x_j^- \\ &= \sum_{i=1}^{m^+} x_{ji}^+ - \sum_{i=1}^{m^-} x_{ji}^- \end{aligned}$$

The transactions' costs incurred

$$= \sum_{j=1}^{m^+} r_{ji}^+ x_{ji}^+ + \sum_{j=1}^{m^-} r_{ji}^- x_{ji}^-$$

The transactions costs will be included in the budget equation, (described below) reducing the amount of resources available for reinvestment in a revised portfolio.

Additionally, we require that each of the transaction's variables

$x_{ji}^+$  and  $x_{ji}^-$  be upper bounded

$$\begin{aligned} x_{ji}^+ &\leq \hat{x}_{ji}^+ & i=1, \dots, m^+ \\ x_{ji}^- &\leq \hat{x}_{ji}^- & i=1, \dots, m^- \end{aligned}$$

Because of the convexity of the transactions' cost curve, we need not be concerned about the possibility that  $x_{j \ i+1}^+ \geq 0$  while  $x_{ji}^+ \leq \hat{x}_{ji}^+$ . This condition will not arise because higher segments of the curve are more costly in terms of transactions' costs.



### Taxes

The investor is assumed to be interested in the terminal market value of his portfolio, net of income taxes on dividend income received during the period and capital gains on portfolio appreciation. Also, when portfolio revisions are made at the beginning of the investment period, capital gains tax liabilities (or credits) will result from the realization of gains (or losses) on the securities traded.<sup>1</sup>

Define  $P_j(A)$  = the average purchase price of the investor's initial holding of security  $j$

Let  $T_c$  = the investor's margin tax rate on capital gains

$T_I$  = the investor's marginal tax rate on income.

When the initial portfolio,  $\underline{X}(0)$ , is revised, the cash flow resulting from capital gains or losses on securities held is given by

$$T_c \sum_{j=1}^N x_j^- [P_j(0) - P_j(A)]$$

where  $x_j^-$  is the number of shares of security  $j$  which are sold. This term will be included in the budget equation discussed below.

The market value of the terminal portfolio, net of tax liabilities

is given by

$$\begin{aligned} \tilde{M}_P^T = & \sum_{j=1}^N X_j \tilde{P}_j + \left[ \sum_{j=1}^N X_j \tilde{D}_j \right] (1-T_I) - T_c \left[ \sum_{j=1}^N X_j (\tilde{P}_j - P_j(0)) \right. \\ & \left. + \sum_{j=1}^N (X_j(0) - X_j^-) (P_j(0) - P_j(A)) \right] \end{aligned}$$

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<sup>1</sup>For simplicity all capital gains are assumed to be long term. Extension of the model to include short term gains is straightforward.





The first term is the market value of the terminal portfolio. The second term is the net of taxes dividend income received during the investment period. The first part of the third term represents capital gains taxes due on security appreciation during the investment period. The second part of the term represents capital gains taxes on unrealized appreciation in the starting portfolio.

Recalling that

$$X_j - X_j(0) = x_j^+ - x_j^-,$$

the above expression can be simplified to give

$$\begin{aligned} \widetilde{M}_P^T &= (1-T_c) \left[ \sum_{j=1}^N X_j \widetilde{P}_j \right] + (1-T_I) \left[ \sum_{j=1}^N X_j \widetilde{D}_j \right] \\ &\quad + T_c \left[ \sum_{j=1}^N (X_j - x_j^+) P_j(A) + \sum_{j=1}^N x_j^+ P_j(0) \right] \\ &= \underline{X}^1 [(1-T_c)\underline{\widetilde{P}} + (1-T_I)\underline{\widetilde{D}}] + T_c [(\underline{X} - \underline{X}^+) \underline{P}(A) + \underline{X}^+ \underline{P}(0)] \end{aligned}$$

### Short Sales

The allowance for short sales can be incorporated by defining an additional set of  $N$  securities which are simply short positions in the original securities.

Define  $X_{N+j}$ ,  $j=1, \dots, N$  as the number of shares of security  $j$  held short during the investment period.

The return on a share of security  $j$ ,  $\widetilde{R}_j$ , and the return on a share of security  $j$  sold short,  $\widetilde{R}_{N+j}$ , have the following relationships



$$\text{Expected Return} \\ E(\tilde{R}_j) = \frac{1}{2} E(\tilde{R}_{N+j})$$

$$\text{Variance of Return} \\ \sigma^2(\tilde{R}_j) = \sigma^2(\tilde{R}_{N+j})$$

Pair Wise Correlations Between Returns

$$\rho(\tilde{R}_j, \tilde{R}_{N+j}) = -1$$

When the investor sells securities short, the proceeds of the short sale (hereafter referred to as "the deposit") are retained by the broker affecting the sale until the short position is closed out. The deposit must be adjusted as market prices change so that its value is equal to the market value of the securities sold short. In addition to the deposit, the investor must provide collateral to the broker equal to the market value of the securities sold short.<sup>1</sup> For ease of future exposition, any short positions are assumed to be collateralized at the beginning of the investment period with cash.<sup>2</sup> The investor earns no interest on the deposit held by the broker (the credit balance in his short account) but earns interest at the rate on broker's loans,  $r_m$ , on collateral held by the broker (the credit balance in his margin account.)

Define  $C(0)$  as the amount of deposit and collateral balances held before the portfolio revision,<sup>3</sup> and  $C(1)$  as the required balance after

<sup>1</sup>Short sales on margin are considered in the next section.

<sup>2</sup>The model can easily be generalized to allow for collateral in the form of unencumbered securities.

<sup>3</sup>Since the model is a discrete and not continuous time period model,  $C(0)$  will equal the deposit and collateral balance existing after the previous portfolio revision, i.e., one investment period ago. Given security prices have adjusted during the period, the existing short positions may thus be under or over collateralized prior to the current revision. Therefore, one of the functions of the current revision is to adjust the deposit and collateral balances on existing, as well as for new short positions. If prices have fallen during the period, funds can be withdrawn from collateral balances for investment purposes, and vice versa.



the portfolio is revised

$$C(1) = 2.0 \sum_{j=N+1}^{2N} X_j P_j(0)$$

In the budget equation a term equal to  $C(1) - C(0)$  must be included to allow for the absorption or generation of portfolio cash due to changes in the deposit and collateral requirements when the portfolio is revised.

The investor's balance sheet (see Exhibit 1) now includes liabilities equal to the amount of his short position.

Exhibit 1

BALANCE SHEET AT BEGINNING  
INVESTMENT PERIOD  
(After Portfolio Revision)

Assets	Liabilities
Portfolio Market Value  $\sum_{j=1}^N X_j P_j(0)$	Short Position  $\sum_{j=N+1}^{2N} X_j P_j(0)$
Collateral & Deposits C(1)	Net Worth NW(1)

Portfolio Debt -- Secured Loans

If the investor's portfolio policies permit, additional funds can be obtained by making portfolio purchases and short sales on margin. If the investor purchases securities on margin, a specified portion of the purchase price is advanced and the remainder borrowed from the broker.



The securities purchased become collateral for the loan and must be left with the broker. The collateral, however, remains the investor's property and he is entitled to any dividends which are paid on the stock. The broker is compensated via an interest charge on the amount of the loan (i.e., on the debit balance in the investor's margin account).

The minimum portion of the purchase price that the investor may provide is determined by the Federal Reserve Board of Governors. This proportion, called the initial margin applies only to the day of purchase. A maintenance margin applies to the security after the day of the transaction. Minimum maintenance margins, which are lower than initial margins for listed securities, are determined by the registered security exchanges. Individual brokers, however, have the freedom to raise the maintenance margin requirement, which is often done for low-priced securities or securities considered to be speculative. For security purchases the margin can be expressed as

$$\begin{aligned} \text{Margin} &= \frac{\text{Value of collateral} - \text{debit balance}}{\text{Value of Collateral}} \\ &= \frac{\text{Margin Account Equity Balance}}{\text{Value of the Collateral}} \end{aligned}$$

Margin requirements also apply to short sales. When shares are sold short on margin, only a specified fraction of the collateral need be deposited with the broker. The collateral deposited is credited to the investor's margin account and credits allowed by the broker can be used to offset interest charges on funds borrowed to buy other securities on margin. Since nothing is really being borrowed from the broker in the short sale case (the investor is simply putting up less than 100%





collateral for the borrowed security) no interest is charged on this de facto loan from the broker.

The margin existing on short positions is given by

$$\text{Margin} = \frac{\text{Deposit with Broker} - \text{Market Value of Short Position} + \text{Collateral}}{\text{Market Value of Short Position}}$$

If the deposit balance with the broker is defined as always being equal to the market value of the borrowed securities (and the value of the investor's collateral correspondingly adjusted), then

$$\text{Margin} = \frac{\text{Value of Collateral}}{\text{Market Value of Shares Borrowed}}$$

Minimum initial and maintenance margin requirements apply to short sales in a manner identical to security purchases.

Define  $\beta_j^I$  = the initial margin requirement for purchases of security  $j$ ,  $j=1, \dots, 2N^1$

$\beta_j^M$  = the maintenance margin requirement for shares<sub>2</sub> previously held of security  $j$ ,  $j=1, \dots, 2N^2$

<sup>1</sup>Recall that a purchase of security  $j$  for  $j=N+1, \dots, 2N$  is a short sale. Conversely, sale of security  $j$  for the same range corresponds to the covering of a short position.

Some margin requirements as of May 1969 are given below.

#### MINIMUM INITIAL MARGINS

Listed Stocks & Short Sales,  $\beta_j^I = 80\%$

Listed Bonds Convertible into Stocks,  $\beta_j^I = 60\%$

#### MINIMUM MAINTENANCE MARGINS

Listed Stocks,  $\beta_j^M = 25\%$

Short Sales,  $\beta_j^M = 30\%$

Over-the-counter securities can be purchased on a cash basis only, thus

$$\beta_j^I = \beta_j^M = 1.0$$



Define  $M(1)$  and  $M(0)$  as the total amounts of brokers' loans held before and after the portfolio revision (at the beginning of the investment period).

$$M(1) = M^I(1) + M^M(1)$$

where

$$\begin{aligned} M^I(1) &= M_L^I(1) + M_S^I(1) \\ &\leq \sum_{j=1}^{2N} (1 - \beta_j^I) x_j^+ P_j(0) \\ M^M(1) &= M_L^M(1) + M_S^M(1) \\ &\leq \sum_{j=1}^{2N} (1 - \beta_j^M) (x_j - x_j^+) P_j(0) \end{aligned}$$

A term equal to  $M(1) - M(0)$  must be included in the budget equation to represent the source or use of portfolio funds resulting from the change in brokers' loans outstanding resulting from portfolio revision.

#### Portfolio Debt -- Unsecured Loans

The investor may be able to obtain additional funds for portfolio investment via unsecured liabilities. An example would be unsecured bank loans. These additional liabilities would be secured only by the general assets of the investor's portfolio and would depend upon his solvency at the end of the investment period for repayment. The amount of funds available from this source, as well as the amount of margin loans he can obtain, will be related to his creditors' estimates of his ability to



repay.<sup>1</sup>

Let  $B(1)$  = the amount of unsecured loans to be held during the investment period

$B(0)$  = the original amount of unsecured debt held (before portfolio revision).

The investor's budget equation will thus contain a term  $B(1) - B(0)$  to account for the funds flows resulting from changes in the unsecured debt level when the portfolio is revised.

The investor's balance sheet after portfolio revision is shown in Exhibit 2. The portfolio cash balance is incorporated into the portfolio as security  $N$ .<sup>2</sup>

It is assumed that the secured margin loan  $M(1)$  and the unsecured bank loan  $B(1)$  are held for the duration of the investment period. The investor's creditors are assumed to limit the amount of credit offered such that the probability of the investor's terminal net worth being less than zero is virtually zero. Thus, the investor, with probability close to one, will have sufficient cash and unencumbered securities to fully meet his portfolio liabilities.

The net worth of the portfolio at the end of the investment period,  $\tilde{NW}$ , is given below.

<sup>1</sup>For some investors, such registered investment companies, the limits on the amount of portfolio liabilities that can be held at any time are much more explicit. The Investment Companies Act of 1940, for example, limits the liabilities of mutual funds to one half of the net asset value (net worth) of the portfolio.

<sup>2</sup>While  $X_{2N}$  will always be identically equal to zero, the variable is retained for convenience of notation.



Exhibit 2

## BALANCE SHEET AT BEGINNING OF INVESTMENT PERIOD

(After Portfolio Revision)

Assets	Liabilities
Cash Balance $X_N$	Short Position $\sum_{j=N+1}^{2N} X_j P_j(0)$
Portfolio Market Value $\sum_{j=1}^{N-1} X_j P_j(0)$	Secured Margin Loans $M(1)$
Collateral & Deposits for Short Positions $C(1)$	Unsecured Loans $B(1)$
	Net Worth $NW(1)$





$$\begin{aligned} \widetilde{NW} &= (1-T_C) \sum_{j=1}^N (X_j - X_{N+j}) \widetilde{P}_j + (1-T_I) \sum_{j=1}^N (X_j - X_{N+j}) \widetilde{D}_j \\ &+ T_C \left[ \sum_{j=1}^N (X_j - X_j^+ - X_{N+j} + X_{N+j}^+) P(A) + \sum_{j=1}^N (X_j^+ - X_{N+j}^+) P_j(0) \right] \\ &+ (1 + \frac{r_m}{2}) C(1) - (1 + r_m) M(1) - (1 + r_B) B(1) \end{aligned}$$

$$\begin{aligned} \widetilde{NW} &= (\underline{X}_L - \underline{X}_S)' [(1-T_C) \widetilde{P} + (1-T_I) \widetilde{D}] + T_C [(\underline{X}_L - \underline{X}_S - \underline{X}_L^+ + \underline{X}_S^+) P(A) + (\underline{X}_L^+ - \underline{X}_S^+) P(0)] \\ &+ (1 + \frac{r_m}{2}) C(1) - (1 + r_m) M(1) - (1 + r_B) B(1) \end{aligned}$$

where  $r_m$  = after-tax cost of brokers' loans

$r_B$  = the after-tax cost of unsecured loans

$\underline{X}_L$  = revised portfolio vector (long positions)

$\underline{X}_S$  = revised portfolio vector (short positions)

The following notation is defined to obtain a compact expression for the variance of  $\widetilde{NW}$ .

$$\text{Let } \sigma_{jj}^T = E[(1-T_C) \widetilde{P}_j + (1-T_I) \widetilde{D}_j]^2$$

$$\sigma_{jj'}^T = E[(1-T_C) \widetilde{P}_j + (1-T_I) \widetilde{D}_j][(1-T_C) \widetilde{P}_{j'} + (1-T_I) \widetilde{D}_{j'}]$$

$$\Sigma^T = \|\sigma_{jj'}^T\|$$

where

$$j=1, \dots, N$$

$$j'=1, \dots, N$$



The variance of the investors terminal net worth is given by

$$\sigma^2(\tilde{NW}) = (\underline{X}_L - \underline{X}_S)' \underline{V}^{-1} (\underline{X}_L - \underline{X}_S)$$

It is assumed that creditors will not supply additional funds unless they believe the investor's terminal net worth will be positive with a specified high degree of certainty. This implies that the relationship which limits the amount of debt the investor can obtain from all sources is of the form

$$P(\tilde{NW} \leq 0) \leq \epsilon$$

where  $\epsilon$  is a value close to zero. Additional credit will be available up to the point at which the above relationship becomes binding<sup>1</sup> (i.e. an equality).

This probabilistic constraint imposed by the creditors on the Investor's portfolio actions can be

<sup>1</sup>In order that the ~~maximum~~ amount of liabilities available to the investor equal that predicted by the model, the creditors would have to have similar views regarding terminal security values as the investor. If this is not the case, then more or less debt funds will actually be available, the amount depending on the creditors' views about security performance.

The amount of credit available will also depend upon the specification of  $\epsilon$ , a quantity which depends upon the degree of creditor risk aversion. An extension to this model would be to relate the rates charged on brokers' and unsecured loans to the risk of default, i.e., to the probability  $\epsilon$  that the investors' terminal wealth will be less than zero.

For a discussion of probabilistic constraints, see Charnes, A. and Cooper, W. W. "Chance Constrained Programming," Management Science, Oct. 1959.



converted to a deterministic equivalent<sup>1</sup> under each of the assumptions made earlier about the joint distributions of terminal security values. Under assumptions B1 and B2 the distribution of terminal portfolio net worth will be normally distributed. Thus, from normal probability tables we can determine a value  $k$  such that,

$$P[\tilde{NW} \leq E[\tilde{NW}] + k\sigma(NW)] = \epsilon$$

Thus the condition that  $P[\tilde{NW} \leq 0] \leq \epsilon$  is equivalent to the condition that

$$E(\tilde{NW}) + k\sigma(\tilde{NW}) \geq 0$$

where for  $\epsilon$  small  $k$  will be negative. Under assumption B3, where only the means, variances and covariances of security returns are specified, we use Tchebysheff's extended lemma<sup>2</sup> to obtain a deterministic equivalent of the probabilistic constraint.

By Tchebysheff's lemma

$$P\left[\frac{\tilde{NW} - E(\tilde{NW})}{\sigma(\tilde{NW})} \leq k\right] \leq \frac{1}{1+k^2}$$

where  $k < 0$ .

<sup>1</sup>For a discussion of the transformation of stochastic constraints to deterministic equivalents see, Charnes, A. and Cooper W. W., "Deterministic Equivalents for Optimizing and Satisfying under Chance Constraints," Operations Research, Jan.-Feb. 1963.

<sup>2</sup>Harold Cramer, Mathematical Methods of Statistics, Princeton University Press, (1946), p. 256, Exercise 5.



Take

$$\epsilon = \frac{1}{1+k^2}$$

$$k = - \left[ \frac{1-\epsilon}{\epsilon} \right]^{1/2}$$

and it is seen that any portfolio satisfying

$$E(\tilde{NW}) + k \sigma(\tilde{NW}) \geq 0, \quad k < 0$$

will also satisfy the original probability constraint.

Thus, in each of the three cases the deterministic equivalent of the probabilistic constraint has the following form

$$\begin{aligned} & (\underline{X}_L - \underline{X}_S) \left[ (1-T_c) \hat{P} + (1-T_I) \hat{D} \right] + T_c \left[ (\underline{X}_L - \underline{X}_S - \underline{X}_L^+ + \underline{X}_S^+) \underline{P}(A) + (\underline{X}_L^+ - \underline{X}_S^+) \underline{P}(0) \right] \\ & + \left(1 + \frac{r_m}{2}\right) C(1) - (1+r_B) M(1) - (1+r_B) B(1) + k \left[ (\underline{X}_L - \underline{X}_S) \sqrt{T(\underline{X}_L - \underline{X}_S)} \right]^{1/2} \geq 0 \\ & (k < 0) \end{aligned}$$

which is a convex function in the decision variables  $X_L, X_S, X_L^+, X_S^+$ .

With the exception of this constraint, the model developed in this paper can be specified as a quadratic programming problem. With the addition of this constraint, which is quadratic (after transferring terms and squaring both sides), the model falls into a more general class of convex programming problems. While convex programming codes exist which can handle problems for several securities, their computational efficiencies are markedly inferior to quadratic programming codes, which in reasonable amounts of time can handle several hundred securities. In





many practical cases, sufficient additional policy and legal restrictions on portfolio liabilities may exist such that this constraint will generally be non-binding. In cases where no liabilities exist, it can be ignored. In cases where some liabilities exist, the properties of the solution vector, obtained by ignoring the constraint, could be examined via simulation to determine if the constraint were violated. If violated, subsidiary restrictions of portfolio liabilities could be tightened and the process repeated.<sup>1</sup>

#### Portfolio Budget Constraint

The budget constraint insures the balancing of sources and uses of funds when the portfolio is revised.

Let  $X_N(0)$  = initial cash balance

$X_N$  = cash balance after portfolio revision

$F(0)$  = exogenous cash flows, which are to be optimally invested (or disbursed) when the portfolio is revised. This could include dividends accumulated from the previous investment period.

The derivation of the cash balance after revision is shown in Exhibit 3.

Cash generated by selling borrowed shares (item 5) is simultaneously absorbed by increases in required deposits with the broker (item 6).

Similarly, when short positions are covered, the required collateral and deposit balances are reduced, generating cash.

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<sup>1</sup>Approximation methods for dealing with this constraint within a quadratic programming framework will be presented in a future paper.



Exhibit 3CALCULATION OF REVISED PORTFOLIO  
CASH BALANCE

No.	Component Symbol	Description
1	$+X_N(0)$	Opening Cash Balance
2	$+F(0)$	Exogenous Cash Flows
3	$-\sum_{j=1}^{2N} \left[ \sum_{i=1}^{m+} r_{ji}^+ x_{ji}^+ + \sum_{i=1}^{m-} r_{ji}^- x_{ji}^- \right]$	Transactions' Costs
4	$-\sum_{j=1}^{N-1} (X_j - X_j(0)) P_j(0)$	Security Purchases (sales)
5	$+\sum_{j=N+1}^{2N-1} (X_j - X_j(0)) P_j(0)$	Short Sales (or coverage of short positions)
6	$-(C(1) - C(0))$	Increase in Deposit and Collateral Balance
7	$+(M(1) - M(0))$	Increase in Brokers' Loans
8	$+(B(1) - B(0))$	Increase in Unsecured Debt
9	$-T_C \sum_{j=1}^N (X_j^- - X_{N+j}^-) (P_j(0) - P_j(A))$	Capital Gains Taxes



The Single Period Portfolio Selection Model --  
Summary of Equations

The model for maximizing the investors expected utility of terminal net worth can now be summarized.

Select a portfolio of assets and liabilities  $\underline{X}$  where

$$\underline{X} = \begin{bmatrix} \underline{X}_L \\ \underline{X}_S \\ B(1) \\ M(1) \end{bmatrix}$$

to maximize

$$Z = \Theta E(\tilde{NW}(\underline{X})) - \sigma^2(\tilde{NW}(\underline{X})) \quad \Theta \geq 0$$

where

$$E(\tilde{NW}(\underline{X})) = (\underline{X}_L - \underline{X}_S)' \left( (1-T_c)\hat{P} + (1-T_I)\hat{D} \right) + T_c [(X_L - X_S - X_L^+ + X_S^+)P(A) + (X_L^+ - X_S^+)P(0)] \\ + (1 + \frac{r_m}{2})C(1) - (1+r_m)M(1) - (1+r_B)B(1)$$

$$\sigma^2(\tilde{NW}(\underline{X})) = (\underline{X}_L - \underline{X}_S)' \hat{\Delta}^{-1} (\underline{X}_L - \underline{X}_S)$$

Subject to

1. Budget Equation

$$\sum_{j=1}^{2N} \left[ x_j - x_{N+j} - x_j^{(0)} + x_{N+j}^{(0)} \right] P_j^{(0)} \\ + \sum_{j=1}^{2N} \left[ \sum_{i=1}^{m+} r_{ji}^+ x_{ji}^+ + \sum_{i=1}^{m-} r_{ji}^- x_{ji}^- \right] \\ + F(0)$$



$$\begin{aligned}
& - [C(1) - C(0)] \\
& + [M(1) - M(0)] \\
& + [B(1) - B(0)] \\
& - T_c \left[ \sum_{j=1}^N (x_j^- - x_{N+j}^-) (P_j(0) - P_j(A)) \right] = 0
\end{aligned}$$

## 2. Collateral Requirements

$$C(1) = 2 \sum_{j=N+1}^{2N} X_j P_j(0)$$

## 3. Transactions Cost Curve Constraints

$$X_j - X_j(0) = \sum_{i=1}^{m^+} x_{ji}^+ - \sum_{i=1}^{m^-} x_{ji}^- \quad j=1, \dots, 2N$$

$$x_{ji}^+ \leq \hat{x}_{ji}^+ \quad i=1, \dots, m^+ \\ j=1, \dots, 2N$$

$$x_{ji}^- \leq \hat{x}_{ji}^- \quad i=1, \dots, m^- \\ j=1, \dots, 2N$$

## 4. Margin Loans Restrictions

$$M(1) \leq \sum_{j=1}^{2N} (1 - \beta_j^I) x_j^+ P_j(0) + \sum_{j=1}^{2N} (1 - \beta_j^M) (X_j - x_j^+) P_j(0)$$

## 5. Portfolio Liabilities Constraint

$$E(\widetilde{NW}(X)) - K \sigma(\widetilde{NW}(X)) \geq 0, \quad K < 0$$





## 6. Lower Bound Restrictions

$$\begin{array}{lll}
 X_j & \geq & 0 & j=1, \dots, 2N \\
 x_{ji}^+ & \geq & 0 & j=1, \dots, 2N, \quad i=1, \dots, m^+ \\
 x_{ji}^- & \leq & 0 & j=1, \dots, 2N, \quad i=1, \dots, m^- \\
 M(1) & \geq & 0 \\
 B(1) & \geq & 0
 \end{array}$$

The Efficient Frontier After Transactions' Costs

By systematically varying the parameter  $\theta$  from 0 to  $\infty$ , the efficient set of portfolios for the investor's expectations about security performance can be generated. The portfolios are efficient after the transactions' costs involved in modifying the starting portfolio,  $X_0(0)$ , to obtain a new portfolio,  $X_\theta$ , which maximizes the investor's expected utility of terminal net worth.

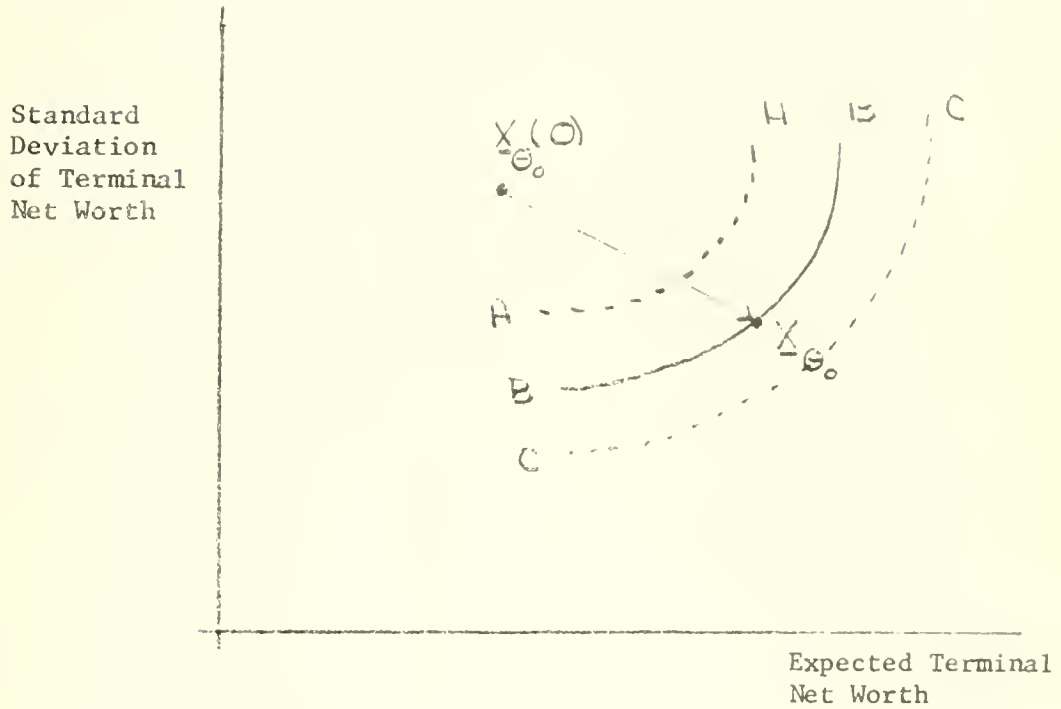
The efficient frontier of portfolios which can include short sales and debt liabilities will dominate portfolios restricted from holding liabilities. A typical situation is illustrated in Figure 3, where the curve BB represents the efficient frontier including liabilities and AA represents the efficient frontier excluding liabilities. The curve CC represents the frontier BB, with the exception that transactions' costs have been ignored (assumed equal to zero). Curve CC dominates BB, but is an unrealizable alternative, due to the existence of transactions' costs for real transactions.

The shape and point of tangency with the efficient frontier of the investors indifference curve depends on the specification of the form of his utility function. If his risk preferences have not changed (i.e.,



Figure 3

## THE EFFICIENT FRONTIER AFTER TRANSACTIONS' COSTS



- AA - efficient frontier with no portfolio liabilities
- BB - efficient frontier with portfolio liabilities
- CC - efficient frontier neglecting transactions' costs



he seeks the same rate of exchange between portfolio risk and expected return,  $\theta_0$ , as before) he will move from his existing portfolio  $X_{\theta_0}(0)$  to the portfolio  $X_{\theta_0}$  which is on the efficient frontier (see figure 3). If his preferences have changed, the efficient frontier contains a portfolio which is optimal for him,<sup>1</sup> considering the costs of shifting to it from his existing portfolio.

### Summary

In this paper an extended version of Markowitz's portfolio selection model has been presented. The Markowitz model has been extended to include consideration of several factors which are important in real world investment decisionmaking. These are (a) transactions' costs, including brokerage fees and volume related marketability costs; (b) short sales; (c) margin loans for security purchases and short sales; (d) unsecured portfolio debt and its relationship to the probability of insolvency to the investor.

In a later paper the general model discussed here will be specialized and applied to the portfolio management problem faced by mutual fund management. Examples of the use of the model in managing a portfolio over a series of investment periods will be presented.

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<sup>1</sup>For examples of the effects of short sales and margin loans on the two asset (plus cash) efficient frontier, see Donald D. Hester, "Efficient Portfolios with Short Sales and Margin Holdings," Chapter 3 in Risk Aversion and Portfolio Choice, Edited by Donald D. Hester and James Tobin, Cowles Foundation Monograph Number 19, John Wiley and Sons, 1967.



## Appendix A

### Brokerage Fees - General Case

Prior to December 5, 1968, the non-member commission rates charged by members of the New York, American and other major stock exchanges was, for a given security, directly proportional to the number of shares traded.<sup>1</sup> Since that time a volume discount has been introduced which applies to the portion of a transaction above 1000 shares for securities selling below \$90 per share. For securities below \$90 per share in price, the fee per "round lot" trading unit (100 shares) is less per hundred shares above 1000 shares than below. Within these respective ranges the commission charge per hundred shares remains fixed. Table 1 summarizes, on a percentage basis, commissions on 100 share transactions for securities at various prices.<sup>2</sup>

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<sup>1</sup>Fee differentials associated with odd lot trading have been ignored.

<sup>2</sup>To obtain the total fees associated with a transaction, state stock transfer taxes and the Securities and Exchange Commission transfer fee must be added. These fees are based on the selling price of the stock and are directly proportional to the number of shares traded, thus are easily incorporated.





Table 1

## NON-MEMBER COMMISSION RATES

Price of Stock Per Share	Commission of Stock Round Lot Transactions Below 1,000 Shares	As Percentage Price Portion of Transaction Above 1,000 Shares
\$200	0.30	0.30
\$100	0.49	0.49
\$ 80	0.59	0.53
\$ 50	0.88	0.51
\$ 30	1.13	0.60
\$ 10	1.70	0.90
\$ 3	3.00	1.83

Let  $x_j^+$  = the number of shares of security  $j$  purchased  $j=1, \dots, N$ .

$x_{j1}^+$  = the number of shares purchased at the higher fee rates  
( $x_{j1} \leq 1000$  shares)

$x_{j2}^+$  = the number of shares purchased in excess of 1000 shares  
( $x_{j1}^+$  must equal 1000 before  $x_{j2}^+$  can be greater than zero.)

$$x_j^+ = x_{j1}^+ + x_{j2}^+$$

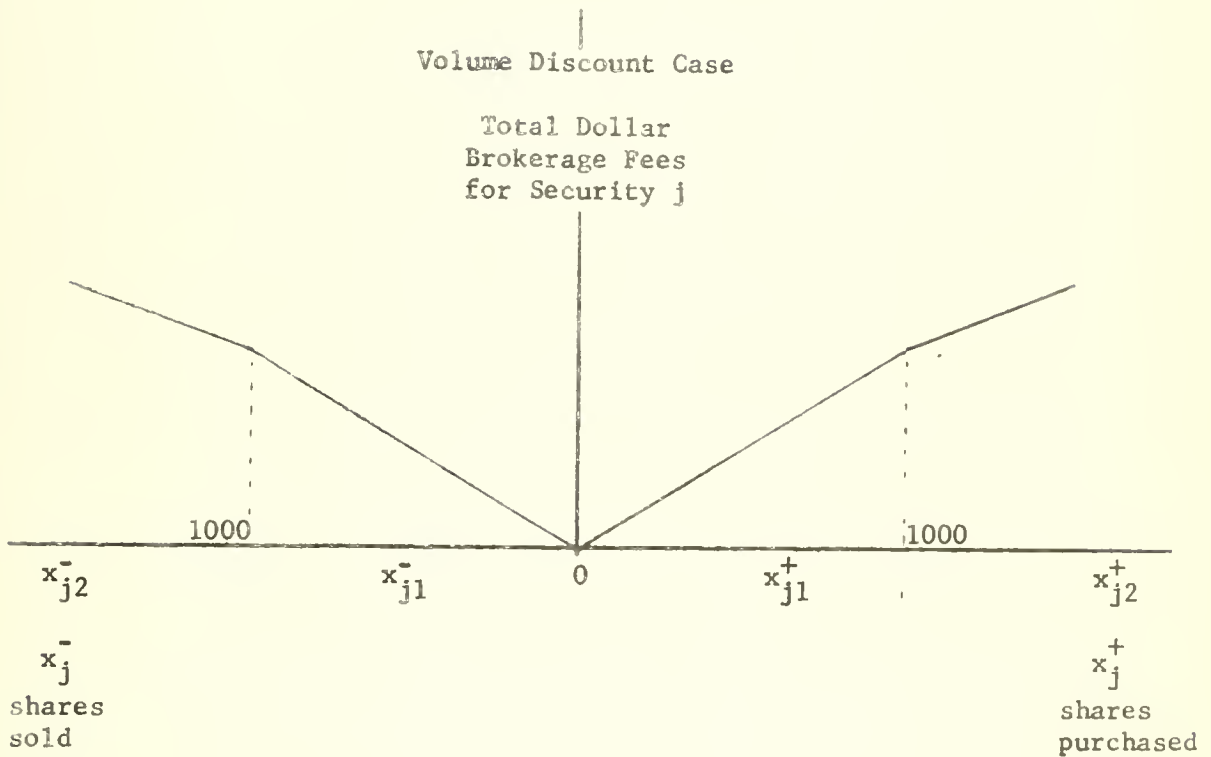
Similarly quantities,  $x_j^-$ ,  $x_{j1}^-$  and  $x_{j2}^-$  can be defined for share sales.

The brokerage transactions cost curve illustrated in Figure 1(a).



Figure 1(a)

## BROKERAGE TRANSACTIONS' COST CURVE



If the above approach to defining the brokerage transactions' cost curve is to be meaningful, a means must be derived to insure that  $x_{j1}^+$  will equal 1,000 shares before  $x_{j2}^+$  takes on non-zero values. In other words, care must be taken to insure that the portfolio selection model executes the first 1,000 shares of a transaction at the higher commission rates before transacting at the lower rates which apply only to the portion of an order above 1,000 shares.

To accomplish this, define an integer valued variable,  $Z_j^+$ , which can take on the values 0 and 1.

$$\text{Let } Z_j^+ = 0 \quad \text{if } x_{j1}^+ < 1000$$

$$Z_j^+ = 1 \quad \text{if } x_{j1}^+ \geq 1000$$



Let  $M$  be an extremely large number which exceeds the maximum feasible value of any  $x_{j2}^+$ : Then the constraints

$$\begin{aligned}x_{j2}^+ &\leq Z_j^+ M \\x_{j2}^+ &\geq 0\end{aligned}$$

will insure that whenever  $x_{j1}^+ < 1000$ , implying  $Z_j^+ = 0$ , then  $x_{j2}^+$  will equal 0. Whenever  $x_{j1}^+ \geq 1000$  then  $Z_j^+ = 1$  and thus  $x_{j2}^+$  can be greater than zero (and effectively unbounded).

Hence, it only remains to find a constraint which insures that  $Z_j^+$  takes on the correct values. The constraint

$$Z_j^+ \leq \frac{x_{j1}^+}{1000}$$

will insure the desired result. Whenever  $x_{j1}^+$  is less than 1000,  $Z_j^+$  must be equal to zero. A remaining problem arises when  $x_{j1}^+$  is greater than 1000, leaving  $Z_j^+$  free to be either 0 or 1. Fortunately, this problem is automatically taken care of by the economics of the situation. Since the brokerage commission rate is lower for  $x_{j2}^+$  than  $x_{j1}^+$ ,  $x_{j1}^+$  will never exceed 1000 shares.  $Z_j^+$  will always equal 1 whenever  $x_{j1}^+ = 1000$  as this permits additional transactions beyond the 1000 shares amount to occur at the lower rate. A parallel analysis exists for share sales.

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