A Dynamic Model of Competition

by

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Abstract

The Lotka-Volterra competition model has been extensively used in the study of technology interaction. It looks at the growth rate of a certain parameter of the interacting technologies through coupled nonlinear differential equations. The interaction is then modeled as a competition with a constant competition coefficient that adversely affects the growth rate. Various studies, however, have suggested that the interaction is not only pure competition and that other interactions are possible. These suggestions have remained mostly conceptual and descriptive – lacking a definite mathematical form of the interaction that can accommodate the suggested variations and the specific implication of those variations.

This thesis presents a specific form of the competition coefficient that depends on the cost and benefit of the competition to a particular technology. The cost and benefit functions are patterned after density-dependent (size) interactions in ecology. The resulting competition coefficient is not a constant but varies as the density of the competing technologies changes. Based on the variable coefficient, we extracted steady states and derived conditions of stability to analyze the dynamics of the competition. Results show that the model can provide a richer set of possibilities compared to the constant coefficient. It accommodates different modes of interactions such as symbiosis and predator-prey aside from pure competition in the steady state coexistence between technologies. It allows for shifts from one mode to another during the evolution of the technologies. Lastly, it provides modifications to strategies meant to achieve "winner-take-all" scenario coveted in business.

Thesis Supervisor: James M. Utterback
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"If I have been able to see further, it was only because I stood on the shoulders of giants."

- Isaac Newton

This thesis would not be possible without the support, encouragement, and constructive suggestions of Prof. James Utterback. His expertise in management and technology dynamics and willingness to help have kept me on course – ably guiding as I thread unknown paths, and gently prodding when the course runs dry.

I was fortunate to have resources that satiated my curiosity in nonlinear phenomena and ecological models. The books by Prof. Steven Strogatz (nonlinear dynamics) and Prof. James Murray (mathematical biology) stood out for being comprehensive and readable. It is rare that books of these types could teach by just reading them!

Deepest gratitude also goes to the people behind the System Design and Management (SDM) program: Pat Hale for providing the opportunity to be part of the 2008 cohort; Bill Foley, Amy Jordan, Dave Schultz and Chris Bates for always looking out for us; fellow cohorts notably Sy, Cynthia, Dev, Amar and Ben for the long hours working together and the priceless friendship!

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Last but definitely not the least, I want to thank my wife Eileen and son Kyle for their sacrifices while I'm away and their incessant encouragement that made all these possible. This thesis is dedicated to them.

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Chapter 1

Introduction

"It is not the strongest of the species that survives, nor the most intelligent...
but the one most responsive to change."
- Attributed to Charles Darwin

1.1 Competition Between Technologies

The dynamics of interaction between or among technologies have been discussed extensively in the literature[1]. The details of the dynamics can be overly complicated – with numerous players, each able to pursue unique strategies at different time frames. Different models have explained certain aspect of the dynamics of interaction. These models range from linear ordinary differential equations to coupled nonlinear partial differentials equations, discrete to continuous models and have yielded analytical and computational results.

Many of these models treat the interaction as competition where two or more technologies compete for the same general market and in the process inhibit each other's growth. Prominent among these models is the Lotka-Volterra competition model which will be the subject of this thesis. It considers the growth rate of a particular variable describing the evolution of the technology. This variable can be the number of

[1] See for instance (Abernathy & Utterback, 1978), (Utterback & Suarez, 1993), (Christensen, Innovator's Dilemma: When New Technologies Cause Great Firms to Fail, 1997), (Weil & Utterback, 2005)

units of the technology sold or the market size of the technology among other things. The diffusion of the technology as denoted by the growth rate of this variable is driven by its intrinsic growth rate (or growth rate in the absence of competition) but limited by its finite carrying capacity and by the extent of competition with other technologies. The interplay among these parameters is presented as coupled differential equations. The solution to these Lotka-Volterra equations synthesizes the dynamics of the interaction between or among the technologies.

The Lotka-Volterra competition model has been shown to be sufficiently general to encompass different kinds of models. Linear, exponential, Pearl, Gompertz, substitution models and oscillatory behaviors can all be matched by the competition model[2]. It is capable of depicting a wide range of dynamic behaviors such as the S-curve, network effects and oscillatory behaviors. It has been used to model different technologies replacing or substituting another technology - lead-free cans replacing soldered cans, tufted carpets replacing woolen carpets, ball point pens replacing fountain pens and nylon tire cord replacing rayon tire cord[3].

For all its successes, the model has been criticized by ecologists for the constancy of the competitive interaction as given by its constant coefficient[4]. An effort towards addressing this shortcoming is the introduction of the functional response in the interaction depending on the density (size of the population) of the competitors[5]. While acceptable in ecology, the modification, however, fall short in explaining certain technology interactions and business developments. The functional response can address the extent of competition but does not change the nature of interaction (e.g. still pure competition or predator-prey). This makes sense from the perspective of ecology where the lions eat the zebras and never vice-versa. It is, however, limiting in understanding technology interactions where the roles in the competition may change in time.

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[2] (Porter, Roper, Mason, Rossini, Banks, & Wiederholt, 1991)
[3] (Farrell, 1993)
[4] (Abrams, 1980)
[5] (Holling, 1966)

Several researchers have presented examples where interaction between technologies is not necessarily purely competitive. Instances where one technology has a positive effect on the growth rate of another technology (symbiosis), or one technology benefits at the expense of the other (predator-prey) have been cited[6]. The symbiotic interaction between computer software and hardware is a classic example of the former. The substitution of the bias ply-tires by radial-ply tires illustrates the latter.

Within the context of the Lotka-Volterra equations, the interactions can be classified to different modes depending on how it affects the growth rates of the interacting technologies. This classification is facilitated by changing the sign before the interaction term (which is assumed to be always positive). Pure competition would have negative signs for both (negative effect on both); predator-prey has a positive (predator – positive effect) – negative (prey – negative effect) combination; and symbiosis has both positive signs (positive effect for both)[7].

Not only are there different interactions possible aside from pure competition, these interactions can change from one mode to another in time. These temporal shifts can manifest in the technologies themselves, in the structure of the companies or in the industries for which these technologies are a part of. Take for instance the development of the hard disk drive for the personal computer market[8]. The incumbent 5.25-inch disk technology offered higher capacity while the 3.5-inch alternative was smaller and more energy efficient. The former was used in the mainstream desktop segment while the latter served the emerging market for portable computers. Thus, the two technologies were initially growing together but each limited to serving consumers in a different market segment. In time, however, the performance of the 3.5-inch drives had expanded from the portable segment to capture the low-end of the desktop segment.

Within the Lotka-Volterra context, such development can be modeled as symbiosis and later predator-prey. The mechanism of switching from one mode to another, for example from initially symbiotic relationship to eventually predator-prey, is, however,

[6] See for instance (Pistorius & Utterback, A Lotka-Volterra Model for Multi-mode Technological

[7] This classification scheme is borrowed from ecology (Odum, 1953)

[8] (Christensen, Innovator's Dilemma: When New Technologies Cause Great Firms to Fail, 1997)

lacking. What is needed is a definite mathematical form of the interaction that can accommodate the suggested transition.

Efforts have been made to consider variations in the Lotka-Volterra model. Variations of the parameters by considering sinusoidal dependency have been implemented to approximate phenomenological substitution models[9]. Effects of stochastic extensions of the equations were also studied[10]. These variations, though important, are confined to a particular mode. Examples beyond periodic variations or random fluctuations have been made to suggest a more dynamic pattern of shifts in the relation between two technologies from one mode of interaction to another[11].

This thesis provides a mechanism that would allow for the shifting of the relation between competing technologies. It presents a specific form of the competition coefficient that depends on the cost and benefit of the competition to the technologies involved. The cost and benefit functions are patterned after density-dependent (size) interactions in ecology. The resulting competition coefficient is not a constant but varies as the density of the competing technologies changes.

Based on the variable coefficient, we derived conditions of stability to analyze the dynamics and the implications of the competition. We compare the results to the case of constant coefficients. The results indicate that the model can provide a richer set of possibilities and a more dynamic model of competition. It presents different modes of coexistence and accommodates coexisting steady states with values larger than its carrying capacity. It allows for different forms of interactions such as symbiosis and predator-prey aside from pure competition during the evolution of the technologies. Lastly, it provides modifications to strategies meant to achieve "winner-take-all" scenario coveted in business[12].

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[9] (Bhargava, 1989)

[10] (Solomon, Richmond, Biham, & Malcai, 2004)

[11] See for instance (Pistorius & Utterback, Multi-mode Interaction among Technologies, 1997), (Modis, 1997)

[12] (Malhotra, Ku, & Murnighan, 2008)

1.2 Organization of the Thesis

Modeling the dynamics of a system of competing technologies involves three main tasks: defining the mathematical functions that govern the appropriate variables of the system; (2) collecting experimental data on these variables of the system; and (3) deciding on the values for the adjustable parameters in the mathematical functions of the model. The first two tasks are independent and would be sufficient for a separate thesis. They set the stage for the third task to provide specific predictions that can be extracted from the given mathematical formalism and the collected data set.

Given the time constraint within the SDM program, a conscious effort was made to focus on the first task - defining the mathematical form of the model that would best illustrate the competition between technologies. To go about this, we present the results of the current competition model to highlight the need for a new model and compare the results later. Chapter 2 presents the dynamics of the Lotka-Volterra competition model and its steady state (long-term) implications. The Lotka-Volterra framework, however, allows other interactions aside from pure competition. Chapter 3 discusses these interactions and serves as a reference of possible behavior that can be covered in our new competition model.

The gist of the thesis is in Chapter 4 where we define the specific form of the competition coefficient that can accommodate different types of interactions. The remainder of the chapter attempts to deduce the dynamics of the resulting model. It compares and contrast the results derived in Chapter 2 and Chapter 3. Chapter 5 concludes the thesis with a summary of the results and implications of the model. An effort was exerted to cover in general terms the possible next steps that need to be taken for the remaining tasks.

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Chapter 2

Lotka-Volterra Competition Model

"Winning isn't everything; it's the only thing."
- Henry Russell ("Red") Sanders

In this chapter, we are going to present the Lotka-Volterra competition model and review the implications of the model. We will highlight the attributes of the model that we would modify later, and cast the results of the model in a form that would facilitate comparison in succeeding chapters.

2.1 Competition Model

The Lotka-Volterra competition model looks at two or more technologies competing for the same general market and inhibiting each other's growth[13]. It considers the growth rate of a particular variable N_1 describing the evolution of the technology T_1. This variable can be the number of units of the technology sold or the market size of the technology. In the absence of competition, it is assumed that N_1 grows exponentially with intrinsic growth rate r_1. The increase, however, is limited by its carrying

[13] (Murray, 1989)

capacity denoted by K_1 resulting in a logistic growth[14]. The competition with another technology T_2 decreases further the growth by an amount proportional to the interaction of the defining variable of T_1 with T_2, i.e. $\propto N_1 N_2$. The proportionality constant is denoted as the competition coefficient c_{12} on N_1 due to N_2. As such c_{12} is the rate at which N_1 loses in the competition with N_2.

Similarly, T_2would be described by the growth rate of N_2 with intrinsic growth rate of r_2, carrying capacity K_2 and inhibited by competition with T_1 at a rate of c_{21}. With these attributes and parameters, the competition model then takes the form:

$$
\begin{aligned}
\frac{dN_1}{dt} &= \dot{N}_1 = r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) - \frac{c_{12}}{K_1} N_1 N_2 \\
\frac{dN_2}{dt} &= \dot{N}_2 = r_2 N_2 \left(1 - \frac{N_2}{K_2}\right) - \frac{c_{21}}{K_2} N_1 N_2
\end{aligned}
\tag{2.1}
$$

As originally presented, the parameters r_i, K_i, c_{ij} $(i,j = 1,2)$ are positive constants. In Chapter 4 we will make the case that the competition coefficient is not a constant, and specifically present a form that can shift from negative to positive values and vice-versa.

Note that it is possible to rescale the parameters in (2.1) (e.g. $n_i = \frac{N_i}{K_i}, \tau_i = r_i t$) to reduce the number of parameters and simplify the equations[15]. However, we purposely retained the explicit form of the equations since the parameters involved have real physical significance as can be seen in the succeeding discussion. Note also that the competition coefficient is scaled by the carrying capacity (i.e. $\frac{c_{ij}}{K_i}$) so that the results would be dimensionally consistent with the growth rate.

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[14] (Verhulst, 1838)

[15] See for instance (Morris & Pratt, 2003)

2.2 Critical Dynamics

The Lotka-Volterra competition model is a nonlinear differential equation with no simple closed-form analytic solution. However, one can completely understand the dynamics of the model by investigating its evolution in the phase plane. To go about this, we introduce some concepts that would be needed in our analysis. These concepts would be used again when we present other models of technology interactions. The details of these concepts are given in Appendix A.

These concepts include the equilibrium point, steady state, eigenvalues and stability. The points where there is no change in the growth rates of the pertinent variable N correspond to a steady state solution. These equilibrium points can be obtained by equating the Lotka-Volterra equations to zero (i.e. $\dot{N}_1 = 0, \dot{N}_2 = 0$) to yield the equilibrium. One can think of the equilibrium points as the pay-offs and the steady state solution as the long-term end state of the competition. Stability (or asymptotic stability)[16] is a characteristic of a system where small perturbations around the equilibrium points have only small effects and vanishes for long periods of time. The eigenvalue (denoted by the symbol λ) provides a means of classifying the stability of our system given the equilibrium points. For our purpose, we look for pay-offs of stable, lasting end states as opposed to transitory states. A necessary and sufficient condition for stability is that the real part of the eigenvalue $Re(\lambda) < 0$.

For the competition model, the equilibrium points, eigenvalues and conditions for stability are given in Table 2-1:

[16] We differentiate stability (asymptotic stability) with neutral stability. In the former, the perturbations vanish and the system is attracted back to the equilibrium point after long time period. In the latter, the perturbations permanently disturbed the system, remaining close to the original equilibrium point but not attracted to it.

Table 2-1. Equilibrium points and stability conditions: Competition model

Equilibrium Points (N_1^*, N_2^*)	Eigenvalues	Stable if
$(0,0)$	$\lambda_{1,2} = r_1, r_2$	Never
$(K_1, 0)$	$\lambda_1 = -r_1$ $\lambda_2 = r_2 - \frac{c_{21}K_1}{K_2}$	$c_{21} > \frac{r_2 K_2}{K_1}$
$(0, K_2)$	$\lambda_1 = -r_2$ $\lambda_2 = r_1 - \frac{c_{12}K_2}{K_1}$	$c_{12} > \frac{r_1 K_1}{K_2}$
$\left(\frac{r_2(K_1 r_1 - c_{12}K_2)}{r_1 r_2 - c_{12}c_{21}}, \frac{r_1(K_2 r_2 - c_{21}K_1)}{r_1 r_2 - c_{12}c_{21}}\right)$	$\lambda_{1,2} = \frac{-B \pm \sqrt{B^2 - 4C}}{2}$ $B = \frac{r_1}{K_1}N_1^* + \frac{r_2}{K_2}N_2^*$ $C = \left(\frac{r_1 r_2 - c_{12}c_{21}}{K_1 K_2}\right) N_1^* N_2^*$	$c_{12}c_{21} < r_1 r_2$

2.2.1 Case 1: Technologies are Similarly Situated

To understand the dynamics, let us consider the case where the competing technologies are similarly situated. This would correspond to two competing technologies with similar growth attributes and capacity potential. This means that $r_1 = r_2 = r; K_1 = K_2 = K$ and the competition coefficient is symmetric[17] i.e. $c_{12} = c_{21} = c$. We want to examine now their dynamics as their competitive interaction increases.

The equilibrium points and stability conditions are given by Table 2-2:

[17] In general, the competition coefficient is asymmetric, i.e. $c_{ij} \neq c_{ji}$. For example, with their difficulty in adapting threatening technology, established technologies do not threaten emerging technologies as much as other emerging technologies would.

Table 2-2. Competition equilibria and stability: $r_1 = r_2 = r; K_1 = K_2 = K ; c_{12} = c_{21} = c$

Equilibrium Points (N_1^*, N_2^*)	Stable if
(0,0)	Never
$(K, 0)$	$c \geq r$
$(0, K)$	$c \geq r$
$\left(\frac{Kr}{r+c}, \frac{Kr}{r+c}\right)$	$c < r$

For a given intrinsic growth rate r and competition coefficient c, the stability of the equilibrium points would require the competition coefficient c to be less than or greater than the growth rate r. It can be that for small values of c, $c < r$ which would make the equilibrium point $\left(\frac{Kr}{r+c}, \frac{Kr}{r+c}\right)$ stable. This means that both technologies coexist with equal steady state values $\frac{Kr}{r+c}$. As the competition heats up, c increases, eventually becoming greater than the growth rate r. This makes the equilibrium points $(K,0)$ and $(0,K)$ stable. These equilibrium points, however, meant the extinction of the other technology. This behavior is shown in the graph below:

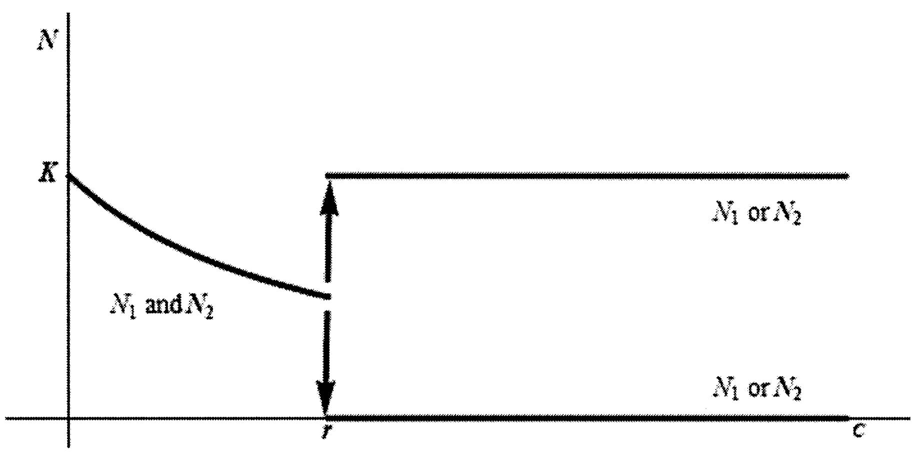


Figure 2-1. Equilibrium values for N as a function of the competition coefficient c

As Figure 2-1 shows, there is a sudden discontinuous transition at $c = r$. For low competition coefficient, the competing technologies share the market equally. However, for high competition, the market transitions into a "winner-take-all market"[18] condition in which one competitor grabs all market share, whereas the other gets nothing. Though we have presented only two competing technologies here, the condition has been shown to persist also for the model with more competitors[19].

Since there are two possible stable equilibria (i.e. $(K,0)$ or $(0,K)$) in the "winner-take-all market" above, the initial conditions determine which technology emerges the victor and which ends up the vanquished. To determine which conditions result in which scenario, a phase portrait can be constructed showing the trajectories for the different initial conditions[20]. To assist in the sketching of the trajectories, vector fields are drawn to indicate whether the flow is along the N_1 axis or N_2 axis. The trajectories for the different initial conditions would be tangent to the vector fields. The set of initial conditions where the trajectory ends in common equilibrium point is called its basin of attraction. Figure 2-2 below plots the trajectories and the basin of attraction for the conditions to a winner-take-all market with N_1 as the winner:

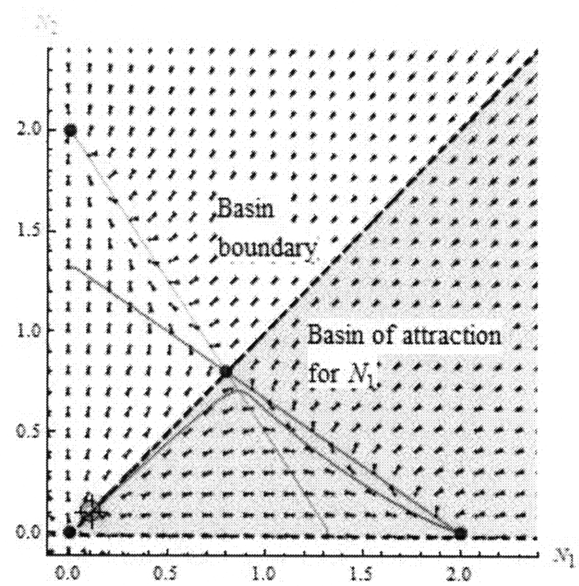
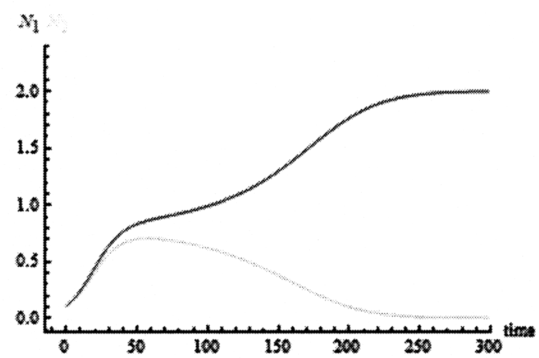


Figure 2-2. Basin of attraction for $(N_1, 0)$

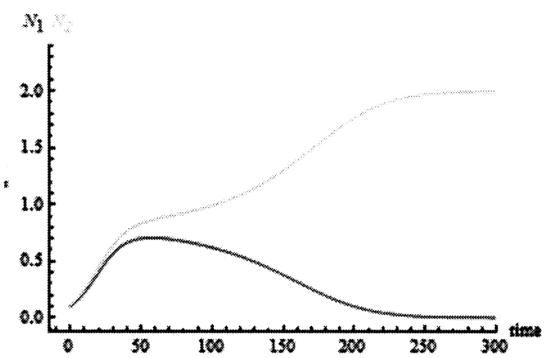
[18] (Frank & Cook, 1995)
[19] (Maurer & Huberman, 2003)
[20] See Appendix A for details on how to create the phase portrait

The figure shows two distinct regions (demarcated by the dashed-line) for which initial conditions lead to a different stable equilibrium (N_1 or N_2). In this situation, the demarcation (called basin boundary) is a 45° line embodying the similarity of the parameters of the competing technologies. In general, however, this demarcation is a curve owing to the differences in the parameters[21].

For this case, the better the initial condition is, the better it would fare in the competition. This is akin to the maxim that says "whoever has the deeper pocket wins". Examples of the time evolution of $N_1(t)$ and $N_2(t)$ (red for N_1 and green for N_2) for different initial conditions are illustrated in Figure 2-3:



(a) N_1 wins ($N_{1o} = 0.11$, N_{2o}=0.1)



(b) N_2 *wins* ($N_{1o} = 0.1$, N_{2o}=0.11)

Figure 2-3. Evolution of the competing technologies for different initial conditions N_{1o} and N_{2o}. (Parameter values: r_1=r_2=0.1, K_1=K_2=2, c_{12}=c_{21}=0.15)

An implication of the model is that the coexistence steady state is independent of the initial conditions. The detailed evolution may differ in time, but the end state would only depend on the carrying capacity and intrinsic growth rate. The initial conditions would only matter for winner-take-all scenarios but is not a factor if the technologies are going to coexist. In Chapter 4 we are going to present a model where there are different coexistence steady states and where the system would fall depends on the initial conditions.

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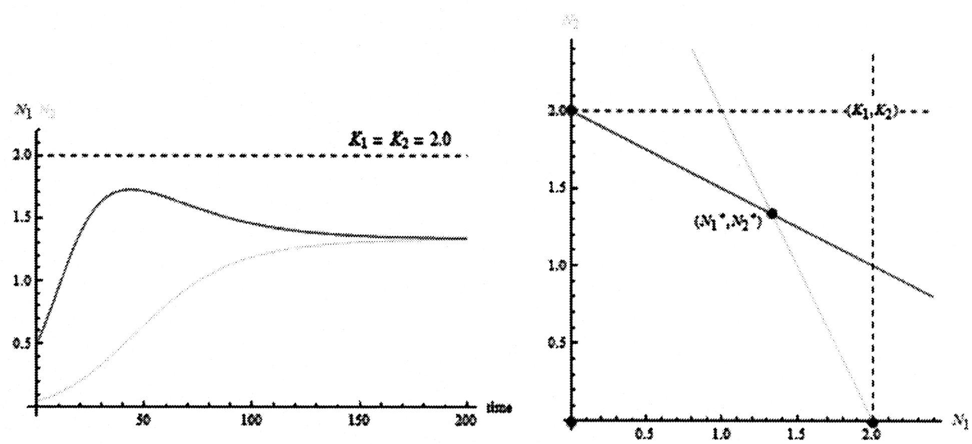
[21] (Strogatz, 1994)

Another implication of the model that is worth capturing is that the steady state values would never be higher than their respective carrying capacity. Figure 2-4 (a) illustrates the time evolution of $N_1(t)$ and $N_2(t)$ along with the carrying capacity (dashed line). Note that the values for any time are less than their respective carrying capacity. The coexistence steady state value is also less than its carrying capacity.

This can also be observed by looking at the nullclines for the Lotka-Volterra equations. The nullclines are the curves where either $\dot{N_1} = 0$ or $\dot{N_2} = 0$ and indicates whether the flow is along the N_1 axis or N_2 axis. From Figure 2-4 (b), the nullclines are lines slanted with negative slopes. Their intersection would provide the steady state value for the coexistence of the two technologies. This intersection would be at (N_1^*, N_2^*) which would have values less than their respective carrying capacity (i.e. $N_1^* < K_1, N_2^* < K_2$). Later, we will encounter interactions that would result in higher steady state value than the initial carrying capacity of all or one of the technologies.



(a) Time evolution **(b) Nullclines**

Figure 2-4. Competition coexistence steady state value. (a) Time evolution (b) Phase plane nullclines. (Parameter values: r_1=r_2=0.1, K_1=K_2=2, c_{12}=c_{21}=0.05)

2.2.2 **Case 2: Mature and New Technologies**

Let us now consider a more general case where the incumbent technology T_1 is already a mature technology with larger carrying capacity $K_1 = K$ but slower growth rate $r_1 = f_r r$. f_r is the fraction of the growth rate $r_2 = r$ of the incoming new technology T_2. T_2 has only a fraction f_k of T_1's carrying capacity, i.e. $K_2 = f_k K$. Let us again assume that the competitive interaction is symmetric. The results of Table 2-1 reduces to:

Table 2-3. Stability condition for mature technology - new technology competition

Equilibrium Points (N_1^*, N_2^*)	**Stable if**
(0,0)	Never
$(K, 0)$	$c > rf_k$
$(0, f_k K)$	$c > \frac{f_r r}{f_k}$
$\left(\frac{rK(f_r r - cf_k)}{f_r r^2 - c^2}, \frac{f_r rK(f_k r - c)}{f_r r^2 - c^2}\right)$	$c < \sqrt{f_r} r$

The strategies one can implement would be based on f_r and f_k. The combinations of these parameters would result in different steady state outcomes. The options and the corresponding pay-off(s) in terms of steady state values are illustrated in Figure 2-5. The N_1^* and N_2^* shown in the diagram correspond to the non-trivial equilibrium point $\left(N_1^* = \frac{rK(f_r r - cf_k)}{f_r r^2 - c^2}, N_2^* = \frac{f_r rK(f_k r - c)}{f_r r^2 - c^2}\right)$. For mature technology T_1, increasing the fractional intrinsic growth rate f_r while decreasing the fractional carrying capacity f_k (or increasing its carrying capacity K) would increase its chances of coveting the "winner-take-all" scenario. Conversely, the new entrant technology T_2 would benefit from a smaller f_r (or increased growth rate) and a large f_k (or increased carrying capacity K_2).

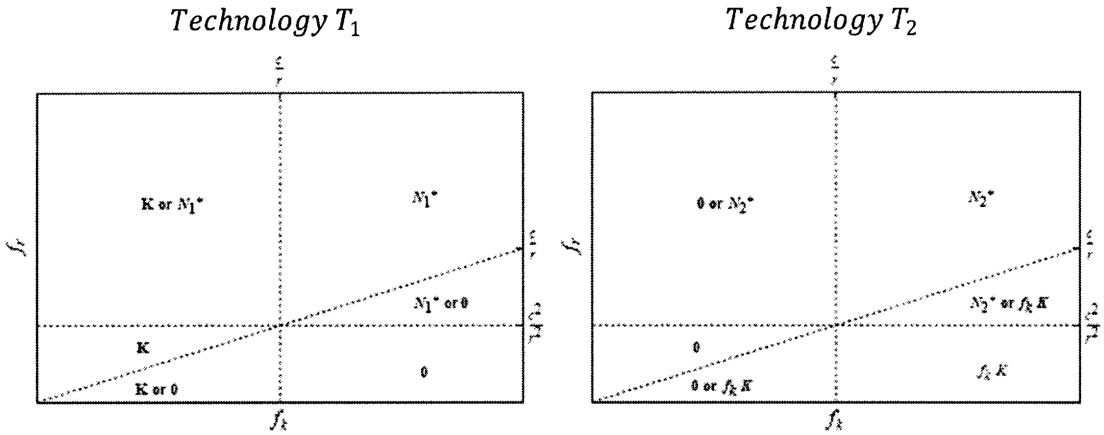


Figure 2-5. Strategy options and resulting steady state values for technologies T_1 and T_2 based on fractional growth rate f_r and carrying capacity f_k

As can be seen in the diagrams, there is at least one stable equilibrium, but there are never more than two. If there are two stable equilibria, the initial conditions determine into which of the two the system will fall.

An observation however, is that competitive dynamics need not drive competitors out of the market. Examples are available that show established technologies not necessarily completely destroyed but surviving – albeit in niche markets or a market distinct from that threatened. The competition between the mainframes and the personal computer did not result in the extinction of the mainframes[22]. It has been driven towards a niche market coexisting with the much larger personal computer market. This attribute is something that a new model has to capture.

[22] (Gilbert, 2003)

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Chapter 3

Lotka-Volterra Interaction Framework

"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is."

- John von Neumann

There are different models on the interaction of technologies[23] and the Lotka-Volterra competition model is just but one of them. Implicit however, in the form of the Lotka-Volterra model is a simple framework to define consistently other modes of interactions. This chapter covers these modes of interactions and their representative dynamics. The results will serve as a reference when we consider the variable competition coefficient in the succeeding chapter.

[23] See for instance (Agarwal, Sarkar, & Echambadi, 2002), (Lenox, Rockart, & Lewin, 2007) and the references cited in there.

3.1 Classification of Interactions

A classification of the interactions possible can be made based on the effect of the interaction on the growth rates of the interacting technologies. Implicit in this framework is the positive nature of the coefficients c_{12} and c_{21} in the Lotka-Volterra coupled differential equations. This assumption will be relaxed later when we develop the variable competition coefficient in the succeeding chapter. Within this framework, the effect can be readily discerned by noting the combination of the signs preceding the interaction coefficients c_{12} and c_{21} (which were previously confined to competition interaction), i.e.:

$$\begin{aligned} \dot{N}_1 &= r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) \pm \frac{c_{12}}{K_1} N_1 N_2 \\ \dot{N}_2 &= r_2 N_2 \left(1 - \frac{N_2}{K_2}\right) \pm \frac{c_{21}}{K_2} N_1 N_2 \end{aligned} \tag{3.1}$$

For example, pure competition mode would just be one of the interactions possible wherein each technology has negative effect on the other's growth rate. This will be shown in (3.1) as $-c_{12}$ and $-c_{21}$ respectively with $c_{12}, c_{21} > 0$. A summary of the possible modes of interactions is illustrated in Table 3-1:

Table 3-1. Mode of interaction based on the combination of the signs preceding the interaction coefficient

		Effect of N_i on N_j's Growth Rate	
	$\pm C_{ij}$	+	-
Effect of N_j on N_i's Growth Rate	+	**Symbiosis**	**Predator(N_i) - Prey(N_j)**
	-	**Predator(N_j) - Prey(N_i)**	**Pure Competition**

Different dynamics can be obtained from the different modes of interactions[24]. We will review these dynamics to provide a reference for the results we are going to obtain later.

[24] See for instance (May, 1973)

3.2 Predator-Prey

3.2.1 Original Predator-Prey

One of the earliest ecological models used in the study of technological evolution were inspired by the works of Volterra[25]. Volterra proposed differential equations for the growth of predators and prey. He assumed that the absence of any prey for sustenance results in an exponential death rate of the predator. The presence of preys contributes to the predator's growth rate by an amount proportional to the available prey as well as to the size of the predator population. Denoting as N_1 the population of the predators and N_2 that of the prey, this leads to the predator equation:

$$\dot{N}_1 = -r_1 N_1 + c_{12} N_1 N_2, \qquad r_1, c_{12} > 0 \tag{3.2}$$

where r_1 is the death rate and c_{12} is a measure on the effect of the presence of the prey N_2 to the predator N_1.

For the prey, Volterra considered an unbounded growth in a Malthusian[26] way governed by its intrinsic growth rate. The presence of predators reduces the growth rate of the prey by an amount proportional to the predator's and prey's population. This leads to the prey equation:

$$\dot{N}_2 = r_2 N_2 - c_{21} N_1 N_2, \qquad r_2, c_{21} > 0 \tag{3.3}$$

where r_2 is a constant corresponding to the intrinsic growth rate and c_{21} is measure of the effect of the predator N_1 on the prey N_2. The equations were also studied by Lotka[27] in the context of chemical kinetics and hence are called Lotka-Volterra model for predator-prey interactions.

[25] (Volterra, 1931)
[26] (Malthus, 1798)
[27] (Lotka, 1920)

3.2.2 **Modified Predator-Prey**

The original predator-prey equations as given by (3.2) and (3.3) were a model for predator sharks and prey fishes. This analogy has serious limitations as a long-term model for technological interactions[28]. For one, the predator is dependent on the existence of the prey. For technology interactions, this is more of an exception than the rule[29]. What is more prevalent is that technologies exist independent of other competing technologies though their growth may benefit or be inhibited by the interaction with other technologies.

Another limitation is the neutral stability of the results[30]. The results imply an oscillatory interaction where perturbations to the interaction would just move the system to another orbit. This orbit can have larger or smaller amplitude than the original one but in phase with it. If we are going to consider the growth of a technology in terms of the number of units, this would be limited by the number of its eventual users. Data on interaction between technologies have failed to provide an example of long-term oscillations between technologies. It does not have the regularity of the oscillations or the permanence of the interaction.

These limitations are addressed by considering positive intrinsic growth rates but limited by their respective carrying capacities $K_1$and K_2. The resulting equations then become:

$$\begin{aligned} \dot{N_1} &= r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) + \frac{c_{12}}{K_1} N_1 N_2 \\ \dot{N_2} &= r_2 N_2 \left(1 - \frac{N_2}{K_2}\right) - \frac{c_{21}}{K_2} N_1 N_2 \end{aligned} \tag{3.4}$$

From here on, predator-prey interaction would mean the modified predator-prey interaction as given by (3.4). Evident from the form of the coupled differential equations is that the predator technology T_1 benefits from the interaction by an amount

[28] (Samuelson, 1971)

[29] One can cite the case for infrastructure vs. services base competition (Hayashi, 2005). In this case, service providers use the infrastructure provided by the infrastructure providers while competing with them on the delivery of services.

[30] (Murray, 1989)

$\frac{c_{12}}{K_1}N_1N_2$. The growth of prey technology T_2 on the other hand is inhibited by an amount $\frac{c_{21}}{K_2}N_1N_2$.

3.2.3 Critical Dynamics

Given the coupled differential equations in (3.4), we extract the general dynamics for predator-prey interactions. The equilibrium points and stability conditions are tabulated in Table 3-2:

Table 3-2. Equilibrium points and stability conditions: Predator-prey interaction

Equilibrium Points (N_1^*, N_2^*)	**Eigenvalues**	**Stable if**
(0,0)	$\lambda_{1,2} = r_1, r_2$	Never
$(K_1, 0)$	$\lambda_1 = -r_1$ $\lambda_2 = r_2 - \frac{c_{21}K_1}{K_2}$	$c_{21} > \frac{r_2K_2}{K_1}$
$(0, K_2)$	$\lambda_1 = -r_2$ $\lambda_2 = r_1 + \frac{c_{12}K_2}{K_1}$	Never
$\left(\frac{r_2(r_1K_1+c_{12}K_2)}{r_1r_2+c_{12}c_{21}}, \frac{r_1(r_2K_2-c_{21}K_1)}{r_1r_2+c_{12}c_{21}}\right)$	$\lambda_{1,2} = \frac{-B \pm \sqrt{B^2-4C}}{2}$ $B = \frac{r_1}{K_1}N_1^* + \frac{r_2}{K_2}N_2^*$ $C = \left(\frac{r_1r_2+c_{12}c_{21}}{K_1K_2}\right)N_1^*N_2^*$	$c_{21} < \frac{r_2K_2}{K_1}$

From Table 3-2 there are only two stable steady states: the extinction of prey technology T_2 or the coexistence of both technologies. Comparison of the predation coefficient c_{21} to the combination of the parameters r_2, K_2 and K_1 determines which steady state the system would fall into. This result is independent of the initial conditions of the technologies.

Examples of the dynamics are shown in Figure 3-1. The first case (a) depicts the time evolution of the coexistence of both technologies provided that the predation coefficient c_{21} on T_2 is less than $\frac{r_2 K_2}{K_1}$. The second case (b) illustrates the instance where the predator technology extinguishes the prey technology.

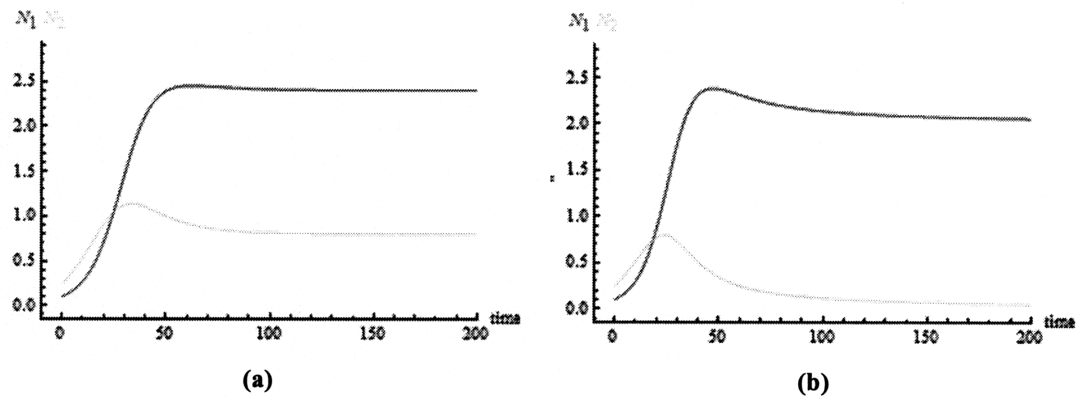
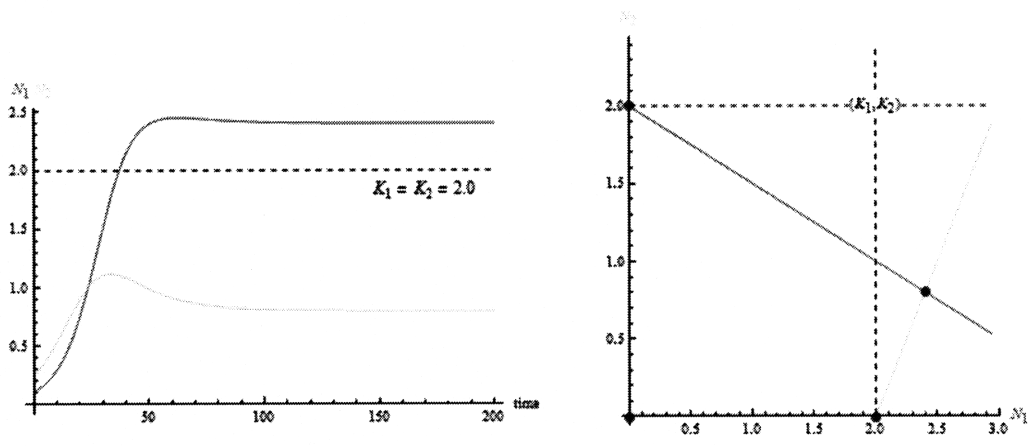


Figure 3-1. Predator-prey sample dynamics: (a) Technologies coexisting; (b) Only N_1 Survives. Parameter values: r_1=r_2=0.1, K_1=K_2=2, c_{12}=c_{21}=c

The coexistence steady state for the predator-prey interaction provides a behavior different from the competition model. Compared to the competition model, the nullclines have opposite orientation of the slopes (positive for the red N_1 and negative for the green N_2). The intersection of these nullclines is the coexistence steady state value. Since each nullcline needs to pass through its carrying capacity, the intersection results in a value higher than the carrying capacity for the predator technology (i.e. $N_1^* > K_1$) but less than for the prey technology (i.e. $N_2^* < K_2$). This is illustrated in Figure 3-2 where the dashed line correspond to the carrying capacity (in this case $K_1 = K_2 = 2$) which is less than the steady state value of 2.4. This means that the benefit to the predator technology in the interaction with the prey technology accrues resulting in greater carrying capacity than if there were no interaction present.



(a) Time evolution **(b) Nullclines**

Figure 3-2. Predator-prey coexistence steady state value. (a) Time evolution (b) Phase plane nullclines. (Parameter values: $r_1=r_2=0.1$, $K_1=K_2=2$, $c_{12}=c_{21}=0.05$)

3.3 Symbiosis

Symbiosis results when both technologies benefit from the relation. This means that the respective growth rates of the interacting technologies are enhanced by the relation. If one looks at the context of mature and emerging technologies, mature technologies are often optimized with regard to its key parameters after a new technology emerges. This "sailing-ship" effect has been observed in different instances and result in the growth of the mature technology[31]. The emerging technology on the other hand, may benefit from the infrastructure that was built for the mature technology or from the efforts that have been made to open the market.

The Lotka-Volterra equations for the symbiotic interaction are given by:

$$\begin{aligned} \dot{N}_1 &= r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) + \frac{c_{12}}{K_1} N_1 N_2 \\ \dot{N}_2 &= r_2 N_2 \left(1 - \frac{N_2}{K_2}\right) + \frac{c_{21}}{K_2} N_1 N_2 \end{aligned} \qquad (3.5)$$

[31]See for instance (Pistorius & Utterback, Multi-mode Interaction among Technologies, 1997)

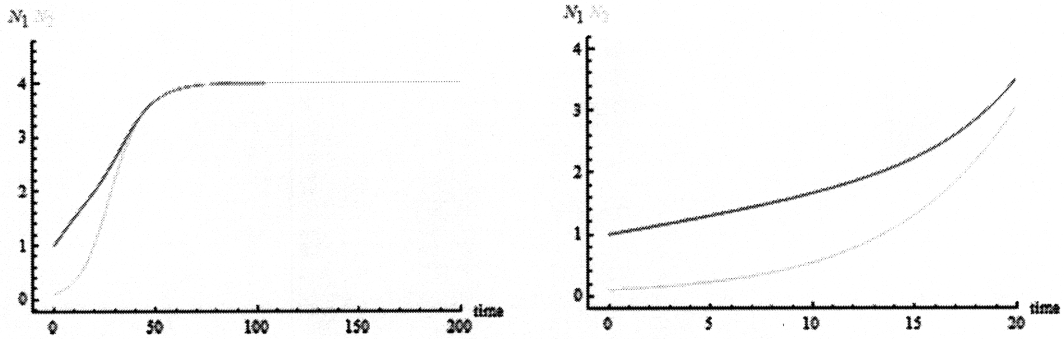
3.3.1 **Critical Dynamics**

The equilibrium points and stability conditions for the coupled differential equations given in (3.5) are tabulated in Table 3-3:

Table 3-3. Equilibrium points and stability conditions: Symbiosis

Equilibrium Points (N_1^*, N_2^*)	**Eigenvalues**	**Stable if**
$(0,0)$	$\lambda_{1,2} = r_1, r_2$	Never
$(K_1, 0)$	$\lambda_1 = -r_1$ $\lambda_2 = r_2 + \frac{c_{21}K_1}{K_2}$	Never
$(0, K_2)$	$\lambda_1 = -r_2$ $\lambda_2 = r_1 + \frac{c_{12}K_2}{K_1}$	Never
$\left(\frac{r_2(K_1 r_1 + c_{12}K_2)}{r_1 r_2 - c_{12}c_{21}}, \frac{r_1(K_2 r_2 + c_{21}K_1)}{r_1 r_2 - c_{12}c_{21}}\right)$	$\lambda_{1,2} = \frac{-B \pm \sqrt{B^2 - 4C}}{2}$ $B = \frac{r_1}{K_1}N_1^* + \frac{r_2}{K_2}N_2^*$ $C = \left(\frac{r_1 r_2 - c_{12}c_{21}}{K_1 K_2}\right) N_1^* N_2^*$	$c_{12}c_{21} < r_1 r_2$

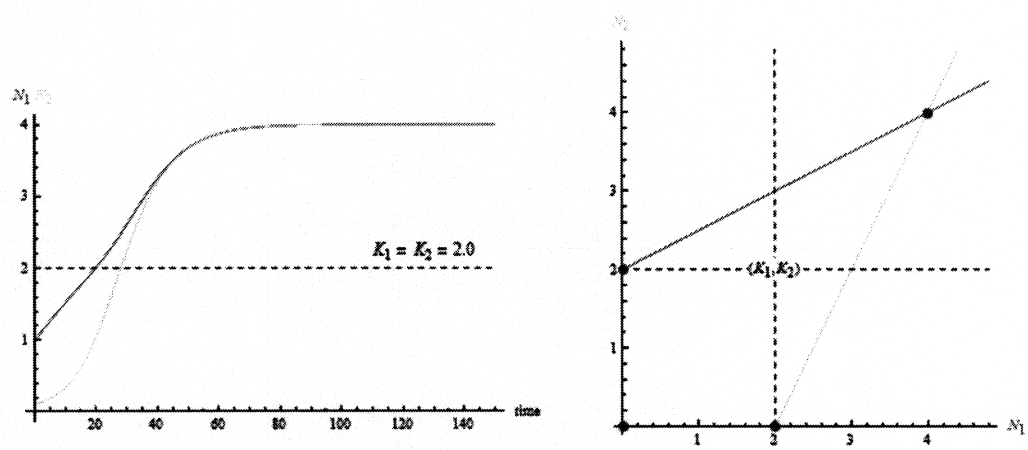
In the case of symbiosis, there is only one stable equilibrium point. This is achieved when the product of the symbiotic coefficients are less than the product of the intrinsic growth rates. When this condition is not met, the technologies grow without bounds. Examples of symbiotic dynamics are shown in Figure 3-3:



(a) **(b)**

Figure 3-3. Symbiosis sample dynamics: (a) Technologies coexisting ($c_{12}c_{21} < r_1r_2$); (b) Unbounded growth. Parameter values: r_1=r_2=0.1, K_1=K_2=2, c_{12}=c_{21}=c

Whereas only one technology benefits in the predator-prey interaction, symbiosis has both technologies enhanced by their interaction. This is represented by nullclines of positive slopes and an intersection at a point larger than their carrying capacity. This results in non-trivial coexistence values greater than their respective carrying capacity when there is no interaction (i.e. $N_1^* > K_1, N_2^* > K_2$). This is illustrated in Figure 3-4:



(a) Time evolution **(b) Nullclines**

Figure 3-4. Symbiosis coexistence steady state value. (a) Time evolution (b) Phase plane nullclines. (Parameter values: *r1=r2=0.1*, K_1=K_2=2, c_{12}=c_{21}=0.05)

3.4 Multi-mode Interaction

A multi-mode model has been proposed where the interaction changes from one mode to another[32]. This can facilitate description of dynamics where say initially the interaction between technologies was initially symbiosis and later turned to pure competition. However, as mentioned early on, there was no mechanism presented to model the transition from one mode to another and the proposal remained descriptive.

This thesis extends the competition model to capture different modes in the competition between technologies. Instead of constraining the competition coefficient to a constant, the coefficient can take on positive or negative values depending on some conditions. The different interaction modes identified in the Lotka-Volterra classification scheme are incorporated based on the combination of the sign of the competition coefficients c_{12} and c_{21} in the competition model, i.e.:

$$\begin{aligned} \dot{N}_1 &= r_1 N_1\left(1-\frac{N_1}{K_1}\right)-\frac{c_{12}(N_2)}{K_1}N_1N_2 \\ \dot{N}_2 &= r_2 N_2\left(1-\frac{N_2}{K_2}\right)-\frac{c_{21}(N_1)}{K_2}N_1N_2 \end{aligned} \tag{3.6}$$

where $c_{12}(N_2)$ and $c_{21}(N_1)$ can be positive or negative. The competition coefficient $c_{ij}(N_j)$ captures the impact of the competition with N_j on N_i. By comparing the sign of the competition coefficient, the Lotka-Volterra interaction classification can be modified in the context of competition as tabulated in Table 3-4:

Table 3-4. Lotka-Volterra classification framework based on the sign of the competition coefficient

		Sign of $C_{21}(N_1)$	
		+	-
Sign of $C_{12}(N_2)$	+	**Pure Competition**	**Predator(N_2) - Prey(N_1)**
	-	**Predator(N_1) - Prey(N_2)**	**Symbiosis**

[32] (Pistorius & Utterback, The Death Knell of Mature Technologies, 1995), (Pistorius & Utterback, A Lotka-Volterra Model for Multi-mode Technological Interaction: Modeling Competition, Symbiosis, and Predator-Prey Modes, 1996), (Pistorius & Utterback, Multi-mode Interaction among Technologies, 1997)

In the following chapter, we will provide a mechanism that would enable the switching from one mode to another. Specifically, it presents a form of the competition coefficient that can switch from positive to negative values and vice versa depending on the size of the variable N. We believe this is the first instance to implement the Lotka-Volterra multi-mode framework within the context of technology interaction.

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Chapter 4

Variable Competition Coefficient

"The only constant is change."
- Heraclitus

Whereas the previous chapter showed dynamics exclusive to a particular mode of interaction, this chapter presents a Lotka-Volterra competition model with variable competition coefficient that integrates the different dynamics. It uses the concept of cost and benefit to model the variation of the competition coefficient. It then utilized the results of the previous chapter to dissect the different modes of interaction possible in the evolution of the competition.

4.1 Variation in the competition

The assumption of a constant coefficient clearly simplifies the dynamics of competition between technologies. We posit that in reality, the competition between technologies changes from one mode to another depending on several drivers. These drivers may include the elaborate strategy to enter a market segment, the constraints within a technology, or the influence of regulation to enforce a particular interaction.

Broadly, what we are after is a competition that can exhibit variations in the interaction. For simplicity, we can consider the competition between an incumbent technology and an emerging new technology. We contend that when an emerging technology has a small size[33], the interaction with an incumbent technology is generally symbiotic. However, as the emerging technology grows the interaction eventually shifts to a more intense competition. A possible sketch of the behavior of the competition coefficient is illustrated in Figure 4-1:

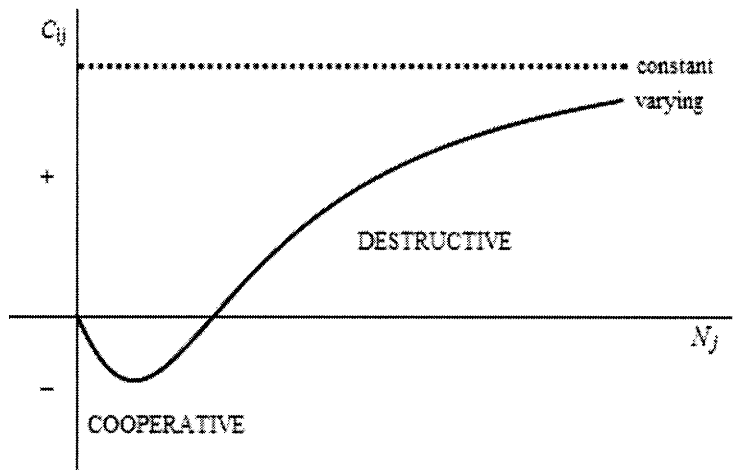

Figure 4-1. Varying competition coefficient (solid line) and constant coefficient (dashed line)

It is no coincidence that the profile we have sketched is similar to a company life cycle. The characteristic growth patterns of companies correspond to patterns of cash generation and usage. However, instead of using time as the independent variable, we take the size of the technology as the proxy. This has the added benefit of dynamically moving "forward" and 'backward" whereas time can only move forward. We are going to exhaust the analogy to these patterns when we weave the mathematical form of the competition coefficient from the benefits the competition provides and the costs it entails.

[33] We liberally use "size" to mean any of the pertinent variables describing the evolution of the technology, e.g. "market size", "number of units sold", etc.

4.1.1 **Cooperative Competition**

Figure 4-1 illustrates a case where initially the competition is cooperative but as the size N_j of technology T_j increases, the competition switches to destructive competition. When the competition coefficient is negative, it shows up as a positive interaction term in the Lotka-Volterra equation. This provides a "cooperative" effect where the relation enhances the growth rate of the particular technology. Cooperative competition may manifest when technologies work together for part of technology development or access to marketplace. The concept has become popular and the new term "co-opetion" has been coined to describe it[34].

The degree to which the market was expanded by an innovative technology is a factor that affects the variation of the competition. It has been shown that disruptive technologies that expand markets will almost always come from outside the industry[35]. In industries where the market is not yet well established, symbiosis is often the initial dominant interaction between technologies. When the market becomes well established, symbiosis decreases and the companies aligned to a particular technology become more competitive[36].

It is not uncommon for one proponent of a technology to be initially "cooperative" to another company whose technology may be a competitor if in doing so provides more benefit to it. An example that comes to mind is the relationship of Yahoo! and Google with regards to search technologies. Yahoo was an early investor of Google and used Google's search engine before it bought Inktomi and competed head-on with Google[37]. Many traditional pharmaceutical companies collaborate with new entrants to adapt the technology and build their competencies. The license of Humulin, a human insulin based on recombinant DNA, by Genentech to Eli Lily is an example of this[38].

[34] (Brandenburger & Nalebuff, 1996)
[35] (Utterback, Mastering the Dynamics of Innovation, 1994)
[36] (Tisdell, 2004)
[37] (Iyer, Lee, & Venkatraman, 2006). An aside, though they have been fierce competitors since then, there are rounds of talks for possible partnership in search advertising.
[38] (Rothaermel, 2000)

4.1.2 **Destructive Competition**

On the other hand, when the competition coefficient is positive, the impact is destructive – it inhibits the growth of the technology. Within the Lotka-Volterra classification scheme, the technology may be the prey in predator-prey interaction or one of the competitors in the pure competition mode. Either scenario inhibits the growth of the technology.

It was mentioned earlier the possibility of a lower performing (in the traditional metric) technology to attack from below before competing head-on with the established technology in the mainstream market[39]. Such approach is not limited to an "attack from below" (lower performance, lower cost, higher ancillary benefits) but other combinations have resulted in one technology eroding the share of the other technology[40]. The substitution of the vinyl album by the compact disc technology (higher performance, lower cost, higher ancillary benefits), film camera by digital cameras (lower performance, higher cost, higher ancillary benefits) and the slide rule by the electronic calculator (higher performance, higher cost, lower ancillary benefits) are just but a few of the examples. Taken together, these examples provide a picture of an incumbent technology providing a new entrant with a market to grow and later preyed by it in the mainstream market.

Destructive competition does not necessarily end in the complete destruction of the technology. As long as the intrinsic growth rate is higher than its competition rate, the technology will survive. What the previous competition model predicts, however, is that the coexistence of both technologies would always result in a steady state value less than their respective initial carrying capacities. We have shown earlier that in a predator-prey interaction, there would be an "increase" in the carrying capacity of the predator technology accrued from the interaction with the prey technology. The variable competition coefficient reflects this behavior.

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[39] (Christensen, Innovator's Dilemma: When New Technologies Cause Great Firms to Fail, 1997)
[40] (Utterback & Acee, Disruptive Technologies: An Expanded View, 2005)

4.2 Form of the Competition Coefficient

An earlier section has alluded to the company life cycle as a pattern for the competition coefficient. A specific form of the competition coefficient can be obtained base on the cost it entails to compete and the benefits or rewards the competition would bring about. There might be different factors to consider for the cost and benefit but for our purpose we consider the case where the cost and benefit varies depending on the size N of the competing technologies. As in the earlier models, this size attribute can be the number of units sold or the extent of its market share. The analysis follows similar treatment in ecology where shifts from beneficial to detrimental roles in the association between species have been observed based on their population density[41].

4.2.1 Benefit Function

For purposes of illustration, let us consider an emerging market. In the early stage of an emerging market, the perceived opportunity is large. No technology is dominant and there is substantial uncertainty in the market as well as in the technology. During this fluid phase, the rate of experimentation and innovation grows[42]. For the competing technologies there is not that much benefit for an intense competition at this early stage. As the uncertainty on the technology and market potential are resolved, the stakes become higher and the benefits clearer. This benefit grows as its size increases reaching a threshold value where all the addressable market of technology T_i have been captured. A function that would have these attributes is:

$$Benefit\ for\ T_j\ competing\ with\ T_i\ = \frac{\alpha_i N_j^2}{\gamma_i^2 + N_j^2} \tag{4.1}$$

The coefficients α_i and γ_i are properties of T_i that modulate the impact of competition with T_j. α_i is the maximum extent of the benefit T_j can derive in competing with T_i while $\gamma_i$dictates the value of N_j where the rate of increase of the benefits are increas-

[41] (Hernandez, Dynamics of Transitions between Population Interactions: A Nonlinear Interaction alpha-Function Defined, 1998), (Hernandez & Barradas, Variation in the Outcome of Population Interactions: Bifurcations and Catastrophes, 2003)
[42] (Utterback, Mastering the Dynamics of Innovation, 1994)

ing the fastest (at $\frac{\gamma_i}{\sqrt{3}}$). The behavior of the benefit function and its associated parameters are illustrated in Figure 4-2:

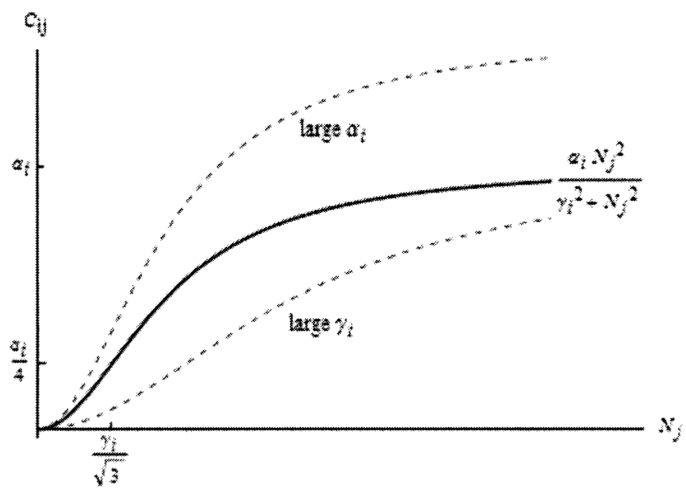


Figure 4-2. Benefit function and associated parameters

Increasing the value of α_i results in higher loss to T_i in the interaction with T_j. This redounds to a higher benefit for T_j. Increasing γ_i on the other hand delays the full impact of the competition at larger value of N_j.

4.2.2 Cost Function

In conjunction with the benefits, the competing technologies bear the costs of competition for a share in the market. Substantial risks are borne by the companies to explore the technology, develop the market and compete with other players. Relative to the initial size of the technology, the cost of competition is high at its nascent stage. This cost reaches a maximum when the size of the technology is γ_i. The cost then declines as the technology increases its traction with the market. Similar observation has been made where the advantage of size relates to greater market power and efficient scale[43]. These attributes can be modeled by the function:

[43] (Agarwal, Sarkar, & Echambadi, 2002)

$$Cost\ for\ T_j\ competing\ with\ T_i = \frac{\beta_i N_j}{\gamma_i^2 + N_j^2} \tag{4.2}$$

Similar to the benefit function, the coefficients β_i and $\gamma_i$are properties of technology T_i related to competition with technology T_j. β_i is proportional to the maximum cost it would entail to compete with T_i in the market. γ_i on the other hand specifies the size N_j where that cost is maximum. The behavior of the cost function and its associated parameters are illustrated in Figure 4-3:

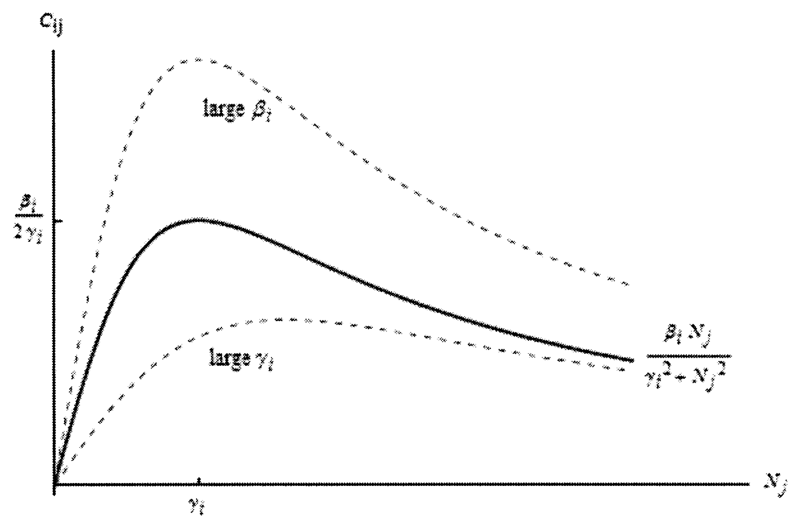


Figure 4-3. Cost function and associated parameters

The larger β_i is, the higher the cost impact to T_j the competition would T_i entails. Increasing γ_i would lessen the impact and at the same time delay to higher value of N_j.

4.2.3 Variable Competition Coefficient

The competition coefficient will be defined as the difference of the benefit and the cost function. The competition would be cooperative as long as the cost to "aggressively" compete is much higher than the benefit it provides. The competition coefficient would then take the form:

$$
\begin{aligned}
c_{ij} &= Benefit\ for\ T_j\ competing\ with\ T_i \\
&\qquad\qquad - Cost\ for\ T_j\ competing\ with\ T_i \\
&= \frac{\alpha_i N_j^2 - \beta_i N_j}{\gamma_i^2 + N_j^2}
\end{aligned} \tag{4.3}
$$

The behavior of the resulting competition coefficient is given in Figure 4-4:

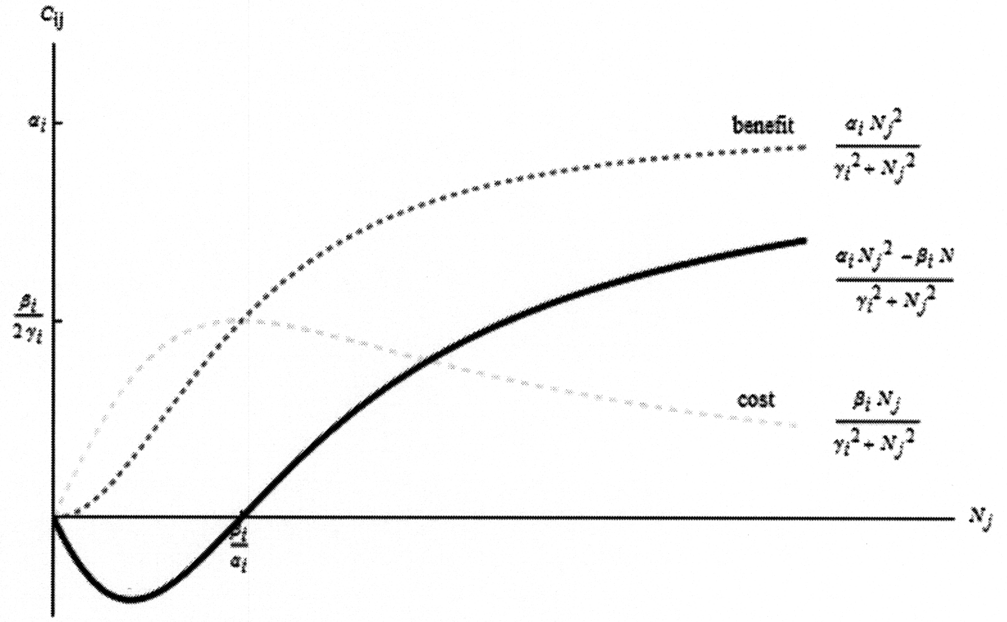


Figure 4-4. Variable competition coefficient

As can be seen from Figure 4-4, the competition coefficient would be equal to the constant $c_{ij} = \alpha_i$ as $N_j \to \infty$ or as N_j grows very large. For N_j less than $\frac{\beta_i}{\alpha_i}$, the impact of the competition is cooperative. For T_j the cost of competing with T_i outweigh the benefits and hence is cooperative to it. Beyond $\frac{\beta_i}{\alpha_i}$, however, the attitude of T_j shifts to destructive knowing that more benefits can be obtained in the competition with T_i.

Using the form of the competition coefficient, the Lotka-Volterra competition equations then take the form:

$$
\begin{aligned}
\dot{N}_1 &= r_1 N_1\left(1-\frac{N_1}{K_1}\right)-\left(\frac{\alpha_1 N_2^2-\beta_1 N_2}{\gamma_1^2+N_2^2}\right)\frac{N_1 N_2}{K_1} \\
\dot{N}_2 &= r_2 N_2\left(1-\frac{N_2}{K_2}\right)-\left(\frac{\alpha_2 N_1^2-\beta_2 N_1}{\gamma_2^2+N_1^2}\right)\frac{N_1 N_2}{K_2}
\end{aligned}
\tag{4.4}
$$

With the model given by the above coupled nonlinear differential equations, what remains to be done is to deduce the dynamics of the model.

4.3 Critical Dynamics

Similar to the previous chapter, we are going to look into the dynamics of the model by analyzing the behavior of the system at its equilibrium points. The equilibrium points and their stability are given in Table 4-1:

Table 4-1. Equilibrium points and stability conditions: Variable Competition

Equilibrium Points (N_1^*, N_2^*)	Eigenvalues	Stable if
(0,0)	$\lambda_{1,2} = r_1, r_2$	Never
$(K_1, 0)$	$\lambda_1 = -r_1$ $\lambda_2 = r_2 + \frac{K_1^2(-K_1\alpha_2 + \beta_2)}{K_1^2 + \gamma_2^2}$	$\alpha_2 > \frac{r_2 K_2}{K_1}\left(1 + \frac{\gamma_2^2}{K_1^2}\right) + \frac{\beta_2}{K_1}$
$(0, K_2)$	$\lambda_1 = -r_2$ $\lambda_2 = r_1 + \frac{K_2^2(-K_2\alpha_1 + \beta_1)}{K_2^2 + \gamma_1^2}$	$\alpha_1 > \frac{r_1 K_1}{K_2}\left(1 + \frac{\gamma_1^2}{K_2^2}\right) + \frac{\beta_1}{K_2}$
Solution to the equations: $r_1\left(1 - \frac{N_1}{K_1}\right)$ $-\left(\frac{\alpha_1 N_2 - \beta_1}{\gamma_1^2 + N_2^2}\right)\frac{N_2^2}{K_1}$ $= 0$ $r_2\left(1 - \frac{N_2}{K_2}\right)$ $-\left(\frac{\alpha_2 N_1 - \beta_2}{\gamma_2^2 + N_1^2}\right)\frac{N_1^2}{K_2}$ $= 0$	$\lambda_{1,2} = \frac{-B \pm \sqrt{B^2 - 4C}}{2}$ $B = \frac{r_1}{K_1}N_1^* + \frac{r_2}{K_2}N_2^*$ $C = \frac{r_1 r_2}{K_1 K_2}N_1^* N_2^*$- $((\partial_2 c_{12})N_2^* +$ $c_{12})((\partial_1 c_{21})N_1^* +$ $c_{21})N_1^* N_2^*$	$\frac{r_1 r_2}{K_1 K_2}$ $-\left((\partial_2 c_{12})N_2^*\right.$ $\left.+ c_{12}\right)\left((\partial_1 c_{21})N_1^* + c_{21}\right)$ > 0 where $\partial_j c_{ij}$ $= \frac{\partial}{\partial N_j}\left(\frac{\alpha_i N_j^2 - \beta_i N_j}{\gamma_i^2 + N_j^2}\right)_{N_j^*}$

The trivial equilibrium points (those involving an extinction of at least one technology) for the competition model are also present for the variable coefficient. The other equilibrium points can be extracted as solutions to the nullcline equations written in the table. These equilibrium points will be our entry towards understanding the behavior of the model. We are going to tackle each of these sets of equilibrium points and discuss the unique dynamics that becomes available with a variable coefficient.

4.3.1 Modification in the Condition for "Winner-Take-All" Scenario

The equilibrium points $(K_1, 0)$ and $(0, K_2)$ pertains to a winner-take-all scenario where one competitor grabs all market shares and the other gets nothing. The constant coefficient model had similar results as well. The stability condition, however, differs by an amount related to the cost and benefit the competition entails. A summary of the comparison is tabulated in Table 4-2:

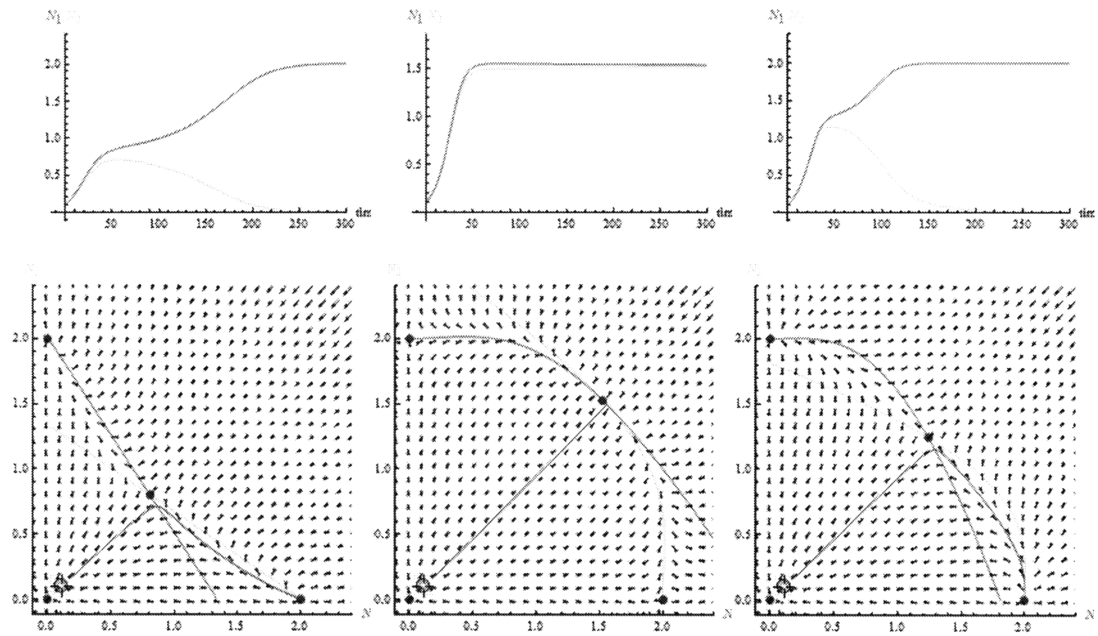
Table 4-2. Comparison of Winner-Take-All Conditions

Constant Coefficient $c_{ij} = constant$		Variable Coefficient $c_{ij} = \frac{\alpha_i N_j^2 - \beta_i N_j}{(\gamma_i^2 + N_j^2)K_i}$	
Equilibrium Points	Stable If	Equilibrium Points	Stable If
$(K_1, 0)$	$c_{21} > \frac{r_2 K_2}{K_1}$	$(K_1, 0)$	$\alpha_2 > \frac{r_2 K_2}{K_1}\left(1 + \frac{\gamma_2^2}{K_1^2}\right) + \frac{\beta_2}{K_1}$
$(0, K_2)$	$c_{12} > \frac{r_1 K_1}{K_2}$	$(0, K_2)$	$\alpha_1 > \frac{r_1 K_1}{K_2}\left(1 + \frac{\gamma_1^2}{K_2^2}\right) + \frac{\beta_1}{K_2}$

The constant coefficient consistently underestimates the condition for winner-take-all scenario by an amount:

$$\Delta_i = \frac{r_j K_j \gamma_j^2}{K_i^3} + \frac{\beta_j}{K_i} \tag{4.5}$$

For simplicity, let us consider the case where the technologies are similarly situated, which means $r_1 = r_2 = r$, $K_1 = K_2 = K$, $\beta_1 = \beta_2 = \beta$, $\gamma_1 = \gamma_2 = \gamma$. For $\beta = 0$ and $\gamma = 0$, we revert back to the constant coefficient results where $\alpha > r$. For non-zero values however, it becomes $\alpha > r + \Delta$ where the difference $\Delta = \frac{r\gamma^2}{K^2} + \frac{\beta}{K}$. This means that for one to emerge a winner and at the same time obliterate the other, the competition coefficient (in the sense of constant coefficient) has to be larger than the intrinsic growth rate r by an amount given by Δ. As such, if the competition between technologies behaves where the relation varies as we have described, the winner-take-all strategies presented in Chapter 2 would fall short. This situation is illustrated in Figure 4-5:



(a) Constant coefficient: c = 0.15 | **(b) Variable coefficient: $\alpha = 0.15, \beta = 0.1, \gamma = 2$** | **(c) Variable coefficient: $\alpha = 0.3, \beta = 0.1, \gamma = 2$**

Figure 4-5. Comparison of Winner-Take-All scenario: (a) Constant coefficient (b) Variable coefficient (c) Variable coefficient – modified. Parameter values: r_1=r_2=0.1, K_1=K_2=2

The changes made to the competition coefficient results in a different dynamics. The time evolution (top row) and phase portrait (bottom row) in Figure 4-5 illustrate this. A constant competition coefficient of $c = 0.15$ is sufficient ($c > r$) to vanquish a competitor as depicted in (a). Such effort, however, is insufficient when one considers the cost and benefit of the competition. As (b) shows, for the given parameters, the condition will instead be sufficient for the technologies to coexist at the same state! Additional competitive effort has to be exerted to surmount the competition. Increasing α to 0.3 (and satisfying the condition $> r + \Delta$) delivers the final blow.

4.3.2 Increase in the Number of Possible Coexistence Steady States

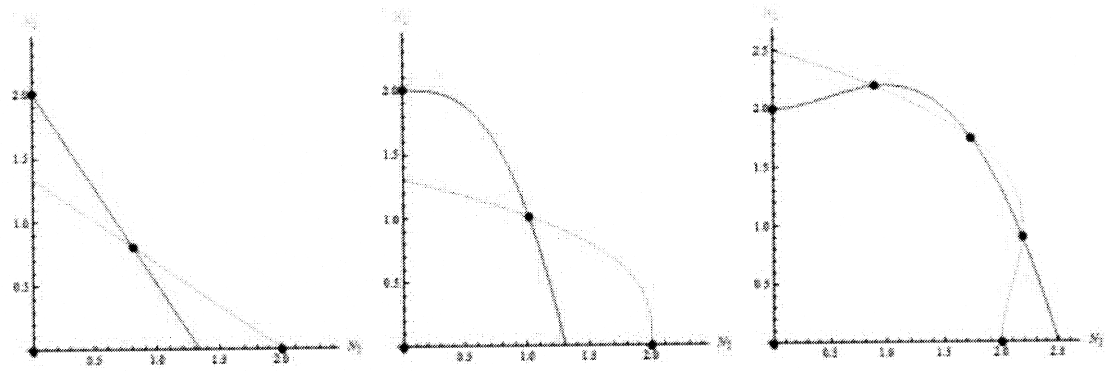
Table 4-3 shows that the constant coefficient has closed-form coexistence steady states. Coexistence equilibrium points for the variable coefficient on the other hand are given as solutions to nonlinear equations for the nullclines. The mathematics becomes more involved and makes it more difficult to discern the detailed dynamics. In any case, valuable insights can be obtained on the general dynamics of coexistence.

Table 4-3. Comparison of Coexistence Steady States

Competition $c_{ij} = constant$		Variable Coefficient $c_{ij} = \dfrac{\alpha_i N_j^2 - \beta_i N_j}{(\gamma_i^2 + N_j^2)K_i}$	
Equilibrium Points	Stable If	Equilibrium Points	Stable If
$\begin{pmatrix} \dfrac{r_2(K_1 r_1 - c_{12}K_2)}{r_1 r_2 - c_{12}c_{21}}, \\ \dfrac{r_1(K_2 r_2 - c_{21}K_1)}{r_1 r_2 - c_{12}c_{21}} \end{pmatrix}$	$c_{12}c_{21} < r_1 r_2$	$r_1 - \dfrac{r_1}{K_1}N_1 - \dfrac{\alpha_1 N_2^2 - \beta_1 N_2}{\gamma_1^2 + N_2^2}N_2 = 0$ $r_2 - \dfrac{r_2}{K_2}N_2 - \dfrac{\alpha_2 N_1^2 - \beta_2 N_1}{\gamma_2^2 + N_1^2}N_1 = 0$	$\dfrac{r_1 r_2}{K_1 K_2} - \left((\partial_2 c_{12})N_2^* + c_{12}\right)\left((\partial_1 c_{21})N_1^* + c_{21}\right) > 0$

As discussed previously, the coexistence equilibrium points are intersection points of the nullclines. For the constant coefficient, the nullclines are lines and their intersection results in only one coexistence steady state. The variable coefficient, on the other hand, has curves as nullclines and has at most nine (9) coexistence steady states[44]. Some of the behaviors, though not exhaustive, are illustrated in Figure 4-6:

[44] Simplifying the equations results in a nonic equation(9th order polynomial), e.g. for N_1: $N_1^9\alpha_2^2(r_1r_2 - \alpha_1\alpha_2) + N_1^8(-K_1r_1r_2\alpha_2^2 + K_2r_2\alpha_2(-2r_1r_2 + 3\alpha_1\alpha_2) - \alpha_2(-3\alpha_1\alpha_2\beta_2 + r_2(\alpha_2\beta_1 + 2r_1\beta_2))) + N_1^7(2K_1K_2r_1r_2^2\alpha_2 + K_2^2r_2^2(r_1r_2 - 3\alpha_1\alpha_2) + 2K_2r_2^2\alpha_2\beta_1 + 2K_1r_1r_2\alpha_2\beta_2 - 3K_2r_2\alpha_1\alpha_2\beta_2 - K_2r_2(-2r_1r_2 + 3\alpha_1\alpha_2)\beta_2 - 3\alpha_1\alpha_2\beta_2^2 + r_1r_2^3\gamma_1^2 + r_2(2\alpha_2\beta_1\beta_2 + r_1\beta_2^2 + r_1\alpha_2^2\gamma_2^2)) + N_1^6(-K_1K_2^2r_1r_2^3 + K_2^3r_2^3\alpha_1 - K_2^2r_2^3\beta_1 - 2K_1K_2r_1r_2^2\beta_2 + 3K_2^2r_2^2\alpha_1\beta_2 - 2K_2r_2^2\beta_1\beta_2 - K_1r_1r_2\beta_2^2 + 3K_2r_2\alpha_1\beta_2^2 + \alpha_1\beta_2^3 - K_1r_1r_2^3\gamma_1^2 - 2K_2r_1r_2^2\alpha_2\gamma_2^2 - K_1r_1r_2\alpha_2^2\gamma_2^2 + K_2r_2\alpha_2(-2r_1r_2 + 3\alpha_1\alpha_2)\gamma_2^2 - r_2(\beta_1\beta_2^2 + \alpha_2(\alpha_2\beta_1 + 2r_1\beta_2)\gamma_2^2)) + N_1^5(K_2^2r_1r_2^3\gamma_2^2 + 4K_1K_2r_1r_2^2\alpha_2\gamma_2^2 + 2K_2^2r_2^2(r_1r_2 - 3\alpha_1\alpha_2)\gamma_2^2 + 4K_2r_2^2\alpha_2\beta_1\gamma_2^2 + 2K_2r_1r_2^2\beta_2\gamma_2^2 + 2K_1r_1r_2\alpha_2\beta_2\gamma_2^2 - 3K_2r_2\alpha_1\alpha_2\beta_2\gamma_2^2 - K_2r_2(-2r_1r_2 + 3\alpha_1\alpha_2)\beta_2\gamma_2^2 + r_2(2\alpha_2\beta_1\beta_2 + r_1(\beta_2^2 + 3r_2^2\gamma_1^2))\gamma_2^2) +$



(a) **Linear: 1 intersection** (b) **Curve: 1 intersection** (c) **Curve: 3 intersections**

Figure 4-6. Sample nullclines and coexistence steady states for variable coefficient

The negative and complex intersection points are not considered in our solution as they do not have physical significance. Only positive and real intersections are counted as viable stationary states. As such, the number of coexistence steady states may be less than the maximum nine solutions that the equation offers. This however, provides more possibilities than the single coexistence state offered by the constant coefficient.

The increase in the number of coexistence steady states leads to another behavior not covered in the constant coefficient. Whereas the constant coefficient has a coexistence state independent of the initial conditions, the initial conditions in the variable coefficient model determine to which coexistence state the system would fall into.

$$N_1^4(-3K_1K_2^2r_1r_2^3\gamma_2^2 + 3K_2^3r_2^3\alpha_1\gamma_2^2 - 3K_2^2r_2^3\beta_1\gamma_2^2 - 4K_1K_2r_1r_2^2\beta_2\gamma_2^2 + 6K_2^2r_2^2\alpha_1\beta_2\gamma_2^2 - 4K_2r_2^2\beta_1\beta_2\gamma_2^2 - K_1r_1r_2\beta_2^2\gamma_2^2 + 3K_2r_2\alpha_1\beta_2^2\gamma_2^2 - r_2\beta_1\beta_2^2\gamma_2^2 - 3K_1r_1r_2^3\gamma_1^2\gamma_2^2 - 2K_2r_1r_2^2\alpha_2\gamma_2^4) + N_1^3(2K_2^2r_1r_2^3\gamma_2^4 + 2K_1K_2r_1r_2^2\alpha_2\gamma_2^4 + K_2^2r_2^2(r_1r_2 - 3\alpha_1\alpha_2)\gamma_2^4 + 2K_2r_2^2\alpha_2\beta_1\gamma_2^4 + 2K_2r_1r_2^2\beta_2\gamma_2^4 + 3r_1r_2^3\gamma_1^2\gamma_2^4) + N_1^2(-3K_1K_2^2r_1r_2^3\gamma_2^4 + 3K_2^3r_2^3\alpha_1\gamma_2^4 - 3K_2^2r_2^3\beta_1\gamma_2^4 - 2K_1K_2r_1r_2^2\beta_2\gamma_2^4 + 3K_2^2r_2^2\alpha_1\beta_2\gamma_2^4 - 2K_2r_2^2\beta_1\beta_2\gamma_2^4 - 3K_1r_1r_2^3\gamma_1^2\gamma_2^4) + N_1(K_2^2r_1r_2^3\gamma_2^6 + r_1r_2^3\gamma_1^2\gamma_2^6) - K_1K_2^2r_1r_2^3\gamma_2^6 + K_2^3r_2^3\alpha_1\gamma_2^6 - K_2^2r_2^3\beta_1\gamma_2^6 - K_1r_1r_2^3\gamma_1^2\gamma_2^6 = 0$$

4.3.3 Support for Different Coexistence Modes

The curve nullclines provide another result for the variable coefficient which is not possible for the constant coefficient. A simple summary of possible modes of coexistence can be made for the Lotka-Volterra interaction framework by considering the relation of the coexistence equilibrium points to its carrying capacity. A quadrant can be created from the intersection of the lines perpendicular to the carrying capacities as shown in Figure 4-7:

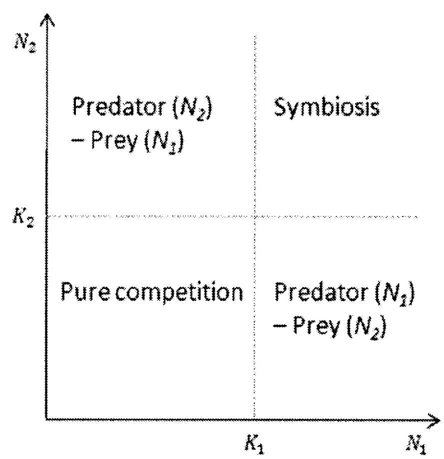
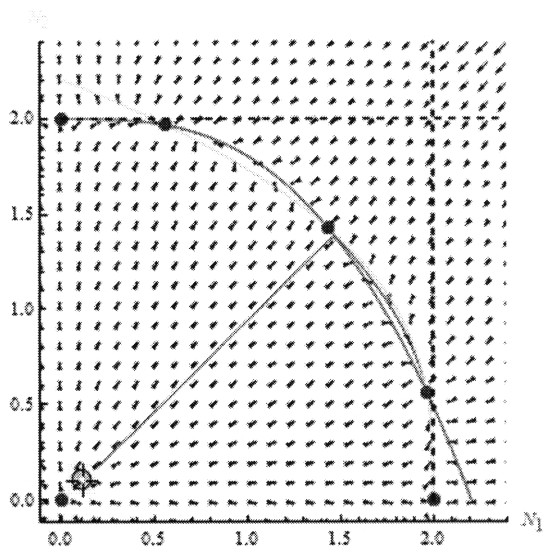


Figure 4-7. Lotka-Volterra interaction framework coexistence steady states for constant coefficient

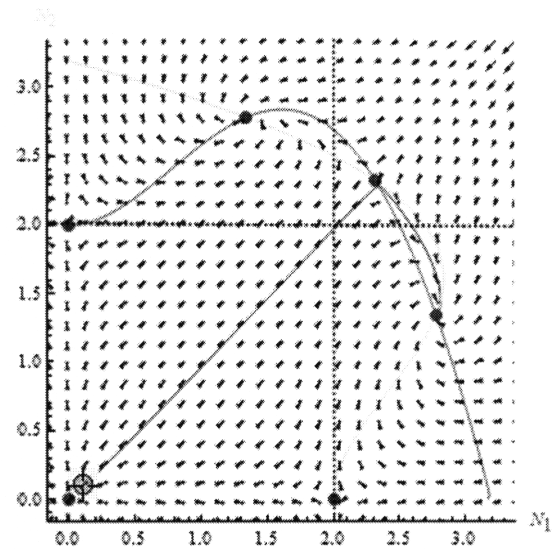
For pure competition, the steady state values would be less than the respective carrying capacities and fall in the lower left quadrant; for symbiosis it would enhance both technologies and hence would have a steady state value higher than its carrying capacity and pushing it to the upper right quadrant; for predator-prey modes, only the predator is enhance at the expense of the prey, and they would be placed at the upper left (predator N_2) or at lower right (predator N_1) depending on who the predator or prey is. Since the constant coefficient can only accommodate one coexistence equilibrium point, the coexistence mode of the interaction is limited to only one quadrant.

The variable coefficient, on the other hand, has multiple coexistence equilibrium points. All of the equilibrium points may fall on one of the possible modes or may be distributed to any of the possible modes. An illustration is provided in Figure 4-8.

Figure 4-8 (a) has all the equilibrium points lying in the pure competition quadrant while (b) has equilibrium points in the symbiosis and predator-prey quadrants. One notes however, that the actual coexistence steady state the system would settle into would depend on the initial conditions and the specific parameters. Illustrated in (a) is the competition steady state and (b) predator(N_1) - prey(N_2) steady state (lower right quadrant).



(a) Coexistence equilibrium points in the pure competition quadrant

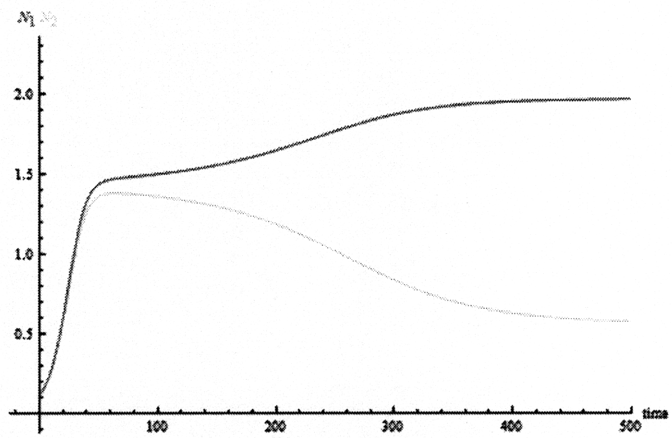


(b) Coexistence equilibrium points in the symbiosis; predator(N_2) - prey (N_1); predator (N_1) - prey(N_2) quadrants

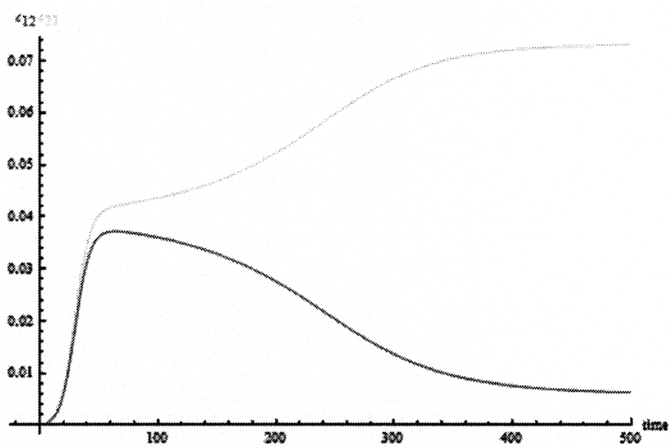
Figure 4-8. Sample of possible modes of coexistence for the variable coefficient

4.3.4 Multi-mode Competition Dynamics

The extent of the competition can be discerned by computing the evolution of the competition coefficients. Consider for instance the time evolution of the dynamics that was depicted in Figure 4-8. The actual time evolution of the competition in Figure 4-8 (a) is shown in Figure 4-9 (a). From the values of $N_1$and N_2 in the time evolution, one can compute the variation of the coefficients. This is shown in Figure 4-9(b). Though the extent of the competition changes as depicted by the changing values of the competition coefficients, the nature of the competition did not. In this example they are locked into a pure competition mode. Since the resulting competition coefficients are less than their respective intrinsic growth rates, both technologies coexist.



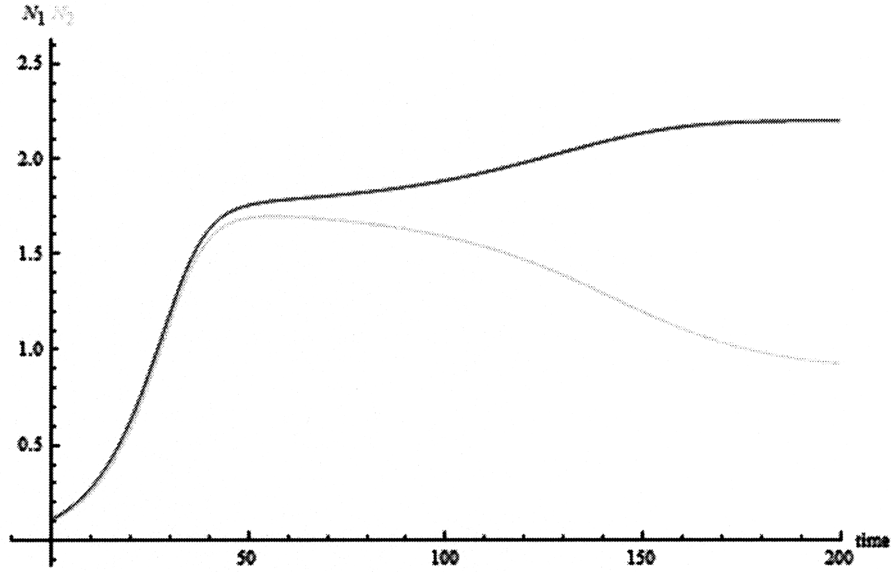
(a) Time evolution



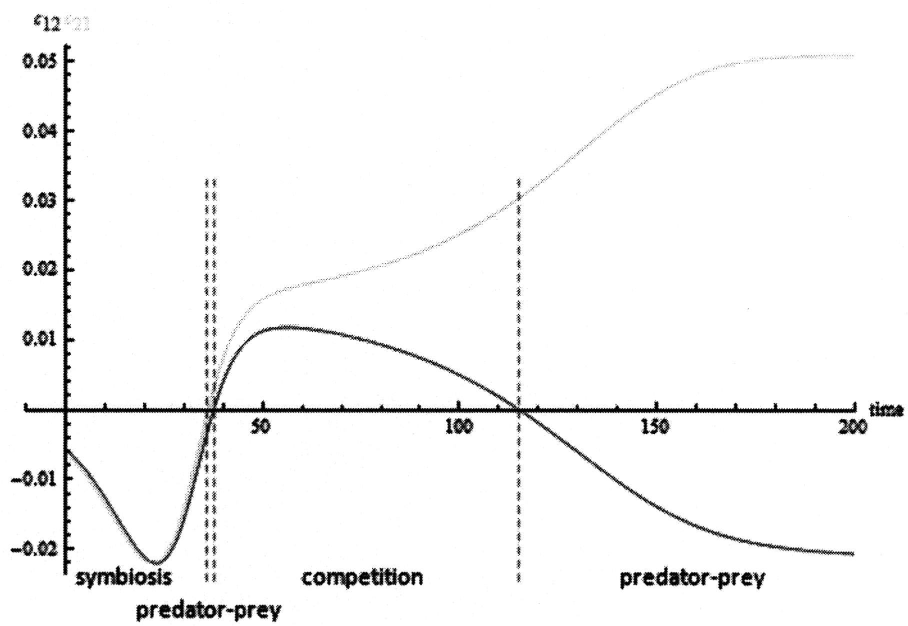
(b) Evolution of the competition coefficients

Figure 4-9. Evolution of the competition coefficient. Parameter values: $r_1=r_2=0.1$, $K_1=K_2=2$, $\alpha_1=\alpha_2=1$, $\beta_1=\beta_2=0.5$, $\gamma_1=\gamma_2=6.9$

For different conditions, however, a more dynamic competition ensues where different modes of interaction are accommodated in the evolution of the competition. From the perspective of the competition model, these modes of interaction are discerned based on the sign of the competition coefficients as specified in Table 3-4. The time evolution for the case in Figure 4-8(b) is depicted in Figure 4-10 (a):



(a) Time evolution



(b) Evolution of the competition coefficients and modes of interaction

Figure 4-10. Multi-mode interactions in the evolution of competition. Parameter values: r_1=r_2=0.1, K_1=K_2=2, α_1=α_2=1, β_1=β_2=1.5, γ_1=γ_2=5

Using the recipe of Table 3-4, one can decompose the different modes of interaction in the evolution of the competition. From Figure 4-10 (b), the competition started off as symbiotic where both experienced rapid growth; then for some time, it shifted to predator-prey where N_1 preyed upon N_2 and slowing its growth rate; pure competition then ensued where both growth rates were further slowed (N_1) or reversed (N_2); finally they settled into a predator (N_1)-prey (N_2) mode. Since the predation coefficient is less than the intrinsic growth rate, the prey technology (N_2) coexists with the predator technology (N_1).

The decomposition of the competition to more specific modes of interaction is important in that it allows for a more focused response. Appropriate strategies specific to the prevailing mode of interaction can be planned and implemented accordingly. For example, advertising and image-building strategies would be different when one is the predator or prey in a predator-prey mode[45]. Knowing which role one is in would result in a more targeted and effective response than a blanket strategy.

[45] (Modis, 1997)

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Chapter 5

Conclusion and Recommendations

"Essentially all models are wrong, but some are useful."

- George Box and Norman Draper

In this chapter, we summarize the main results derived in earlier sections. We then conclude our discussion with a synthesis of the results and implications of the model. Lastly, we provide a general outline of steps that can be taken to further develop the model.

5.1 Summary of Results

At the onset, we set-out to construct a model that can accommodate shifting modes of interaction in the evolution of a competition. Essential to the goal is defining the mathematical form of the model that would best explain the stylized facts in the competition between technologies. To go about this, we present the results of the competition model with constant coefficient and the other interaction modes accommodated in the Lotka-Volterra framework. We then define a variable competition coefficient that can integrate the different modes observed. The variable coefficient was based on the cost and benefit the competition would entail on the competitors. We assumed specific mathematical form of the cost and benefit function and then studied the dynamics of the resulting model. A summary of the results is tabulated in Table 5-1:

Table 5-1. Summary of results of constant and variable competition coefficient

	Competition Coefficient (c_{ij}) $\dot{N_1} = r_1 N_1 \left(1 - \frac{N_1}{K_1}\right) - \frac{c_{12}}{K_1} N_1 N_2$ $\dot{N_2} = r_2 N_2 \left(1 - \frac{N_2}{K_2}\right) - \frac{c_{21}}{K_2} N_1 N_2$	
Attributes	**Constant Coefficient** $c_{ij} = constant$	**Variable Coefficient** $c_{ij} = \frac{\alpha_i N_j^2 - \beta_i N_j}{(\gamma_i^2 + N_j^2)}$
Number of possible end states	4	At most 12
Winner-take-all scenarios?	Yes	Yes with modifications on the conditions
Number of coexistence equilibrium points	1	At most 9
Coexistence steady state value	Less than the carrying capacity	Can have values less than or greater than the carrying capacity
Can accommodate predator-prey interaction?	No	Yes
Can accommodate symbiotic interaction?	No	Yes
Can accommodate shifting modes of interaction?	No	Yes

The variable competition coefficient was constructed to be able to reflect some of the observed dynamics in the competition between technologies. Analysis of its dynamics shows that it can cover aspects of competition not possible with constant coefficient model. The model provides a richer set of possibilities with at most 12 possible steady states compared to 4 for the constant coefficient. It can accommodate different modes of interactions in the steady state coexistence between technologies. These modes may be symbiosis or predator-prey aside from pure competition. Not only does it allow different modes of coexistence, it provides mechanism for shifts from one mode to another during the evolution of the competition between technologies.

5.2 Conclusion and Recommendations

The mathematical functions and framework that we have defined to model the dynamics of competition have so far been successful in terms of integrating results from the previous model. Known results like winner-take-all scenarios and coexistence between competitors are accommodated in the model. Appropriate modifications in the conditions are provided as well to cover the variations in the competition.

A unique contribution of the model, however, is that it allows for different coexistence modes. Competition may end up as symbiotic or predatory aside from purely competitive. Not only does it allow for different coexistence modes, but more importantly, it also accommodates shifts from one mode to another in the evolution of the competition. It provides a mechanism where the competition can be broken up to different modes of interaction. This information is crucial in the management of technology where appropriate strategies need to align with prevailing condition.

The model and its results open up the study of the dynamics of competition to closer scrutiny. The analytical form of the model requires an exhaustive and systematic study to understand its implications. With nonic equations (9^{th} order polynomial) to play with, there might be more interesting dynamical outcomes. The conditions for stability for other practical cases need to be worked out to provide appropriate strategies in the management of technology. Generalizations to multiple competitors would add more relevance to the model.

But before the detailed analytical work can proceed in earnest, data on the pertinent variables of the competing technologies need to be collected. This is essential to ascertain that the interaction effects between competing technologies are substantial. Fitting the collected data to the model requires estimating the parameters of the model that would agree with the data. Parameter estimation, as this task is known, is not a trivial endeavor - especially for nonlinear coupled differential equations[46].

Once these tasks are accomplished, the model would be on firmer ground to provide valuable insights and predictions on the dynamic outcomes of competition.

[46] (Walmag & Delhez, 2005)

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Appendix A

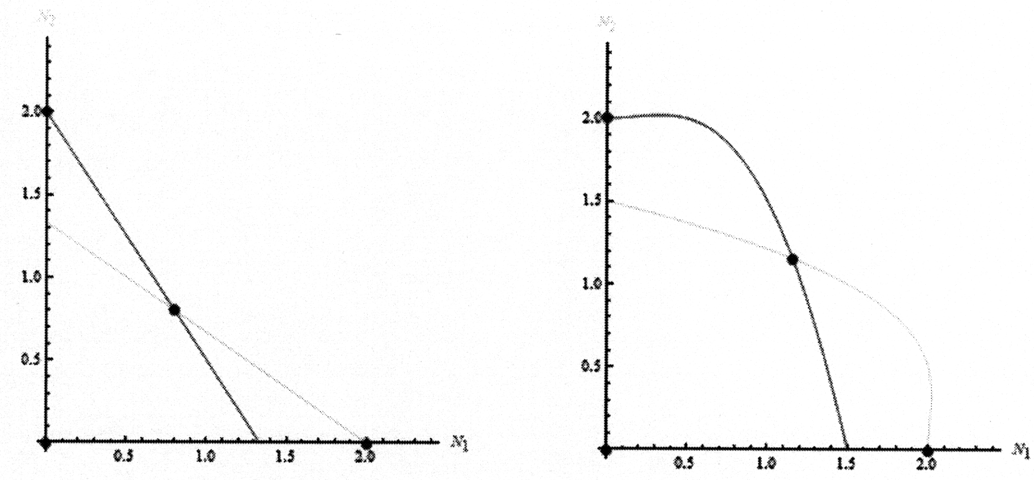
Nonlinear Phase Plane Analysis

We consider in here the phase plane analysis of our models[47]. We can cast the models to the form:

$$\begin{aligned} \frac{dN_1}{dt} &= \dot{N}_1 = N_1 f_1(N_1, N_2) \\ \frac{dN_2}{dt} &= \dot{N}_2 = N_2 f_2(N_1, N_2) \end{aligned} \tag{6.1}$$

The nullclines are the curves where either $f_1(N_1, N_2) = 0$ or $f_2(N_1, N_2) = 0$. They indicate whether the flow is along the N_1-axis or N_2-axis. In general these nullclines are curves but for constant coefficients, these would be lines. Example of nullclines are shown in Figure A-0-1:

[47] We pattern the discussion after (Strogatz, 1994)



(a) Linear nullcline **(b) Curved nullcline**

Figure A-0-1. Nullclines

The blue points shown in the figure above are the points where there are no change in the growth rates of N_1 and N_2 and correspond to a steady state. These equilibrium points are obtained by equating $N_1 = 0$ and $N_2 = 0$. The solutions would be the "trivial" solutions involving the origin $(N_1 = 0, N_2 = 0)$ and the intersection of the nullclines with the axes $(N_1 = 0 \; and \; f_2(N_1, N_2), N_2 = 0 \; and \; f_1(N_1, N_2))$. The "non-trivial" solutions would be the intersection of the nullclines which correspond to the coexistence steady states.

These equilibrium points can be classified according to their stability. To classify them, we compute the Jacobian given below:

$$A = \begin{pmatrix} \dfrac{\partial \dot{N}_1}{\partial N_1} & \dfrac{\partial \dot{N}_1}{\partial N_2} \\ \dfrac{\partial \dot{N}_2}{\partial N_1} & \dfrac{\partial \dot{N}_2}{\partial N_2} \end{pmatrix}_{(N_1^*, N_2^*)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{6.2}$$

We then consider each equilibrium point and compute for its eigenvalues. The eigenvalues are obtained from the characteristic equation:

$$|A - \lambda I| = \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \tag{6.3}$$

The solution is given by:

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} \tag{6.4}$$

where $\tau = Trace(A)$, $\Delta = \det(A)$

The general solution of (6.1) for the specific equilibrium point then becomes

$$\boldsymbol{N}(t) = c_1 e^{\lambda_1 t}\boldsymbol{v}_1 + c_2 e^{\lambda_2 t}\boldsymbol{v}_2 \tag{6.5}$$

Where $\boldsymbol{v}_{1,2}$ are the eigenvectors of A corresponding to λ_1 and λ_2 respectively. The combination of τ and Δ determine the specific value of λ_1 and λ_2. General characteristic of the solution can be obtained based on the sign of the eigenvalue. We are interested in steady state solutions where $\boldsymbol{N}(t)$ does not approach $\pm\infty$ as $t \to \infty$. A sufficient condition would be to have $\mathrm{Re}(\lambda) < 0$. This would happen if $\tau < 0$ and $\Delta > 0$. Equilibrium points that would satisfy these conditions are called stable nodes. The different combinations of τ and Δ and their corresponding classification is given in Figure A-0-2:

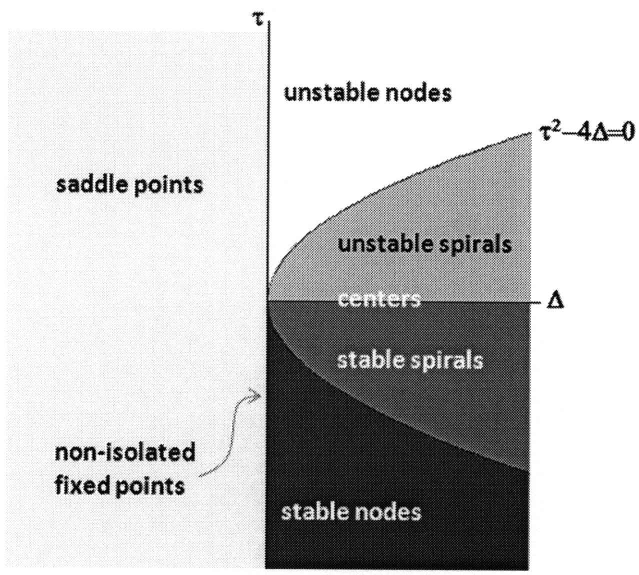


Figure A-0-2. Stability classification of equilibrium points

The solutions can be visualized along the N_1 and N_2 plane where $(N_1(t), N_2(t))$ correspond to a point moving along a curve in the plane. This curve is called the trajectory and the plane filled with the possible trajectories is called the phase portrait.

To assist in the sketching of the trajectories, vector fields are drawn to indicate whether the flow is along the N_1 axis or N_2 axis. The trajectories for the different initial conditions would be tangent to the vector fields. These concepts are illustrated in the Figure A-0-3 below:

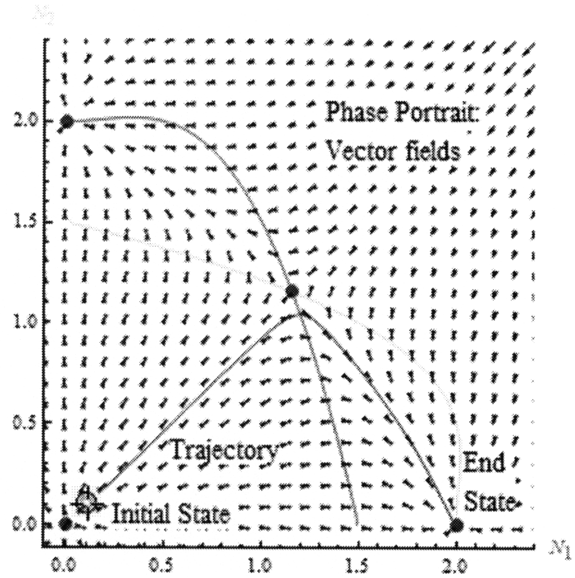


Figure A-0-3. Phase portrait

The set of initial conditions where the trajectory ends in common equilibrium point is called its basin of attraction. The Figure A-0-4 below shows the basin of attraction for the equilibrium point (2,0). Any initial states on this basin of attraction would end up at (2,0). The line demarcating the basin of attraction is called the basin boundary.

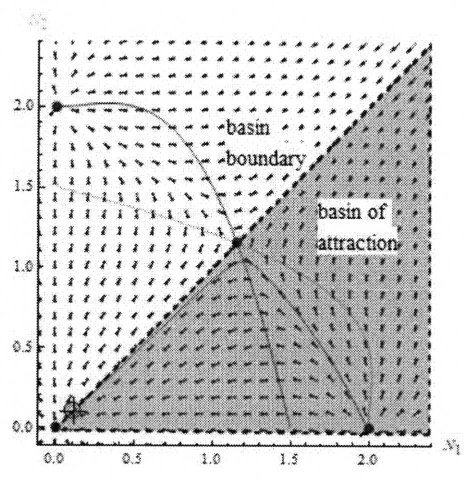


Figure A-0-4. Basin of attraction

Appendix B

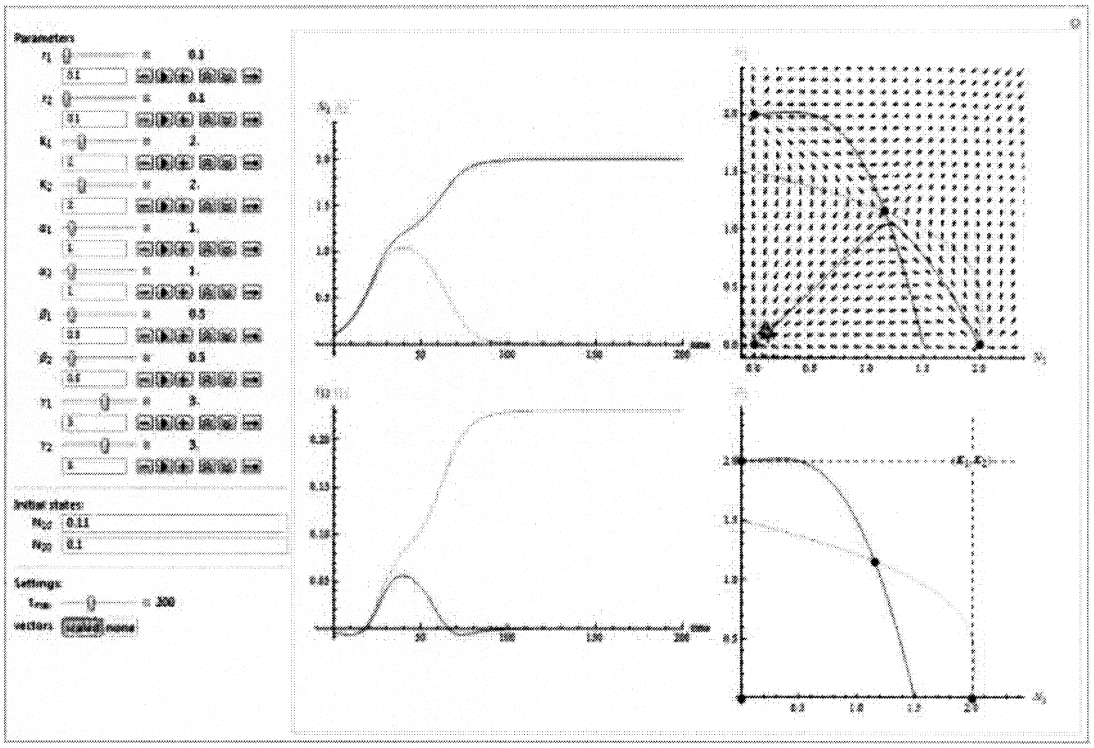
Code Listing

Computer programs written in Mathematica 6 were created to facilitate simulation of the model.

```
Manipulate[
 Module[
  {eqn1, eqn2, statpts, sol, n1statptsmax, n2statptsmax, n1maxdata, n2maxdata, n1max, n2max, n1n2max,
   coeff1, coeff2, plot0, plot1, plot2, gr1, gr2, startv, pl2list, pl3list, colorn1 = RGBColor[1, 0, 0],
   colorn2 = RGBColor[0, 1, 0], plpts = 50, vecplts = 25, padding = {{30, 30}, {20, 20}}},

  (*stationary points*)
  eqn1 = r1 n1[t] - r1/k1 n1[t]^2 - ((a1 n2[t]^2 - b1 n2[t])/(c1^2 + n2[t]^2)) (n1[t] n2[t])/k1;
  eqn2 = r2 n2[t] - r2/k2 n2[t]^2 - ((a2 n1[t]^2 - b2 n1[t])/(c2^2 + n1[t]^2)) (n1[t] n2[t])/k2;
  statpts = Solve[eqn1 == 0 && Abs[n1[t]] == n1[t] && eqn2 == 0 && Abs[n2[t]] == n2[t], {n1[t], n2[t]}];

  (* numerical solution to coupled nonlinear eqns *)
  sol = Quiet[NDSolve[
     {n1'[t] == eqn1,
      n2'[t] == eqn2,
      n1[0] == n10,
      n2[0] == n20},
     {n1, n2}, {t, 0, tmax}]];
```

```
(* axes settings *)
n1statptsmax = 1.2 Max[n1[t] /. statpts];
n2statptsmax = 1.2 Max[n2[t] /. statpts];
n1maxdata = 1.2 Max[Table[Quiet[Extract[Evaluate[n1[t] /. sol], 1]], {t, 0, tmax, tmax/(10 plpts)}]];
n2maxdata = 1.2 Max[Table[Quiet[Extract[Evaluate[n2[t] /. sol], 1]], {t, 0, tmax, tmax/(10 plpts)}]];
n1max = Max[n10, n1statptsmax, n1maxdata]; n2max = Max[n20, n2statptsmax, n2maxdata]; n1n2max = Max[n1maxdata, n2maxdata];

(* plot of c12[t] and c21[t] *)
coeff1 = Evaluate[((a1 n2[t]^2 - b1 n2[t])/(c1^2 + n2[t]^2)) /. sol];
coeff2 = Evaluate[((a2 n1[t]^2 - b2 n1[t])/(c2^2 + n1[t]^2)) /. sol];
plot0 = Plot[{coeff1, coeff2}, {t, 0, tmax},
   PlotRange -> {{-0.05 tmax, tmax}, Automatic(*{-0.05 n1n2max,n1n2max}*)}, AxesOrigin -> {0, 0},
   (*{-0.05 tmax,-0.05 n1n2max},*)
   PlotStyle -> {colorn1, colorn2}, AxesLabel -> {"time", Style["c12", colorn1] Style[ "c21", colorn2]},
   ImageSize -> {400, 250}, ImagePadding -> padding];

(* plot of n1[t] and n2[t] *)
plot1 = Plot[{Evaluate[n1[t] /. sol], Evaluate[n2[t] /. sol]}, {t, 0, tmax},
   PlotRange -> {{-0.05 tmax, tmax}, {-0.05 n1n2max, n1n2max}}, AxesOrigin -> {0, 0}(*{-0.05 tmax,-0.05 n1n2max}*),
   PlotStyle -> {colorn1, colorn2}, AxesLabel -> {"time", Style["N1", colorn1] Style[ "N2", colorn2]},
   ImageSize -> {400, 250}, ImagePadding -> padding];

(* phase trajectory of n1[t] and n2[t] *)
plot2 = ParametricPlot[{Extract[Evaluate[n1[t] /. sol], 1], Extract[Evaluate[n2[t] /. sol], 1]}, {t, 0, tmax},
   PlotRange -> {{-0.05 n1max, n1max}, {-0.05 n2max, n2max}}, AxesOrigin -> {-0.05 n1max, -0.05 n2max},
   AxesLabel -> {Style["N1", colorn1] , Style["N2", colorn2]}];
startv = Graphics[Locator[Dynamic[{n10, n20}], Background -> Yellow, LocatorRegion -> Automatic],
   PlotRange -> {{-0.05 n1max, n1max}, {-0.05 n2max, n2max}}];

(* nullclines of n1[t] and n2[t] *)
gr1 = ContourPlot[r1 x - r1/k1 x^2 == ((a1 y^2 - b1 y)/(c1^2 + y^2)) (x y)/k1, {x, 0, n1max}, {y, 0, n2max}, ContourStyle -> colorn2,
   PlotPoints -> plpts, Frame -> False, Axes -> True, AxesLabel -> {Style["N1", colorn1] , Style["N2", colorn2]}];
gr2 = ContourPlot[r2 y - r2/k2 y^2 == ((a2 x^2 - b2 x)/(c2^2 + x^2)) (x y)/k2, {x, 0, n1max}, {y, 0, n2max}, ContourStyle -> colorn1,
   PlotPoints -> plpts, AxesLabel -> {Style["N1", colorn1] , Style["N2", colorn2]}];
(*combination plots*)
pl2list = {gr1, gr2, Graphics[{PointSize[Large], Blue, Point[{n1[t], n2[t]} /. statpts]}],
   Graphics[{Dashed, Line[{{k1, 0}, {k1, n2max}}], Line[{{0, k2}, {n1max, k2}}],
     Text["(K1,K2)", {k1, k2}, Background -> White]}]};
pl3list = {plot2, gr1, gr2, startv, Graphics[{PointSize[Large], Blue, Point[{n1[t], n2[t]} /. statpts]}]};
If[vec ≠ "none", AppendTo[pl3list,
   VectorFieldPlots`VectorFieldPlot[{r1 x - r1/k1 x^2 - ((a1 y^2 - b1 y)/(c1^2 + y^2)) (x y)/k1, r2 y - r2/k2 y^2 - ((a2 x^2 - b2 x)/(c2^2 + x^2)) (x y)/k2 },
    {x, -0.05 n1max, n1max}, {y, -0.05 n2max, n2max}, PlotPoints -> vecplts]]];

Grid[{{plot1, Show[pl3list, AspectRatio -> 1, ImageSize -> {300, 300}]},
  {plot0, Show[pl2list, AspectRatio -> 1, ImageSize -> {300, 300}]}}, ItemSize -> {{30, 25}, {30, 25}},
 Alignment -> {{Right, Left}, {Bottom, Top}}]
],
```

```
"Parameters",
 {{r1, r1init, "r₁"}, r1min, r1max, deltar1, ImageSize → Tiny, Appearance → "Labeled"},
 {{r2, r2init, "r₂"}, r2min, r2max, deltar2, ImageSize → Tiny, Appearance → "Labeled"},
 {{k1, k1init, "K₁"}, k1min, k1max, deltak1, ImageSize → Tiny, Appearance → "Labeled"},
 {{k2, k2init, "K₂"}, k2min, k2max, deltak2, ImageSize → Tiny, Appearance → "Labeled"},
 {{a1, a1init, "α₁"}, a1min, a1max, deltaa1, ImageSize → Tiny, Appearance → "Labeled"},
 {{a2, a2init, "α₂"}, a2min, a2max, deltaa2, ImageSize → Tiny, Appearance → "Labeled"},
 {{b1, b1init, "β₁"}, b1min, b1max, deltab1, ImageSize → Tiny, Appearance → "Labeled"},
 {{b2, b2init, "β₂"}, b2min, b2max, deltab2, ImageSize → Tiny, Appearance → "Labeled"},
 {{c1, c1init, "γ₁"}, c1min, c1max, deltac1, ImageSize → Tiny, Appearance → "Labeled"},
 {{c2, c2init, "γ₂"}, c2min, c2max, deltac2, ImageSize → Tiny, Appearance → "Labeled"},
 Delimiter,
 "Initial states:",
 {{n10, 0.11, "N₁₀"}, ControlType → InputField, ImageSize → Tiny},
 {{n20, 0.1, "N₂₀"}, ControlType → InputField, ImageSize → Tiny},
 Delimiter,
 "Settings:",
 {{tmax, tmaxinit, "tₘₐₓ"}, tmaxmin, tmaxmax, deltatmax, ImageSize → Tiny, Appearance → "Labeled"},
 {{vec, "scaled", "vectors"}, {"scaled", "none"}, ImageSize → Tiny},

 Initialization :→ (
   r1init = 0.1; r1min = 0.01; r1max = 5.; deltar1 = 0.005;
   r2init = 0.1; r2min = 0.01; r2max = 5.; deltar2 = 0.005;
   k1init = 2.; k1min = 1.; k1max = 5.; deltak1 = 0.05;
   k2init = 2.; k2min = 1.; k2max = 5.; deltak2 = 0.05;
   a1init = 1.; a1min = 0.01; a1max = 10.; deltaa1 = 0.01;
   a2init = 1.; a2min = 0.01; a2max = 10.; deltaa2 = 0.01;
   b1init = 0.5; b1min = 0.01; b1max = 5.; deltab1 = 0.01;
   b2init = 0.5; b2min = 0.01; b2max = 5.; deltab2 = 0.01;
   c1init = 3.; c1min = 0.01; c1max = 5.; deltac1 = 0.01;
   c2init = 3.; c2min = 0.01; c2max = 5.; deltac2 = 0.01;
   tmaxinit = 200; tmaxmin = 5; tmaxmax = 500; deltatmax = 10;
   Get["VectorFieldPlots`"]), SynchronousUpdating → False,
 ControlPlacement → Left, AutorunSequencing → {3, 5, 6, 8}, TrackedSymbols → Manipulate]
```



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References

Abernathy, W. J., & Utterback, J. M. (1978). Patterns of Industrial Innovation. Technology Review , 80 (7), 40-47.

Abrams, P. A. (1980). Are Competition Coefficients Constant? Inductive Versus Deductive Approaches. The American Naturalist , 730-735.

Agarwal, R., Sarkar, M. B., & Echambadi, R. (2002). The Conditionign Effect of Time on Firm Survival: An Industry Life Cycle Approach. Academy of Management Journal , 45 (5), 971-994.

Bhargava, S. C. (1989). Generalized Lotka-Volterra Equations and the Mechanism for Technological Substitution. Technological Forecasting and Social Change , 35, 319-326.

Brandenburger, A. M., & Nalebuff, B. J. (1996). Co-opetition. Broadway.

Christensen, C. M. (1997). Innovator's Dilemma: When New Technologies Cause Great Firms to Fail. Boston, MA: Harvard Business School Press.

Farrell, C. J. (1993). A Theory of Technological Progress. Technological Forecasting and Social Change , 161–178.

Frank, R. H., & Cook, P. J. (1995). The Winner-Take-All Society. New York: Free Press.

Gilbert, C. (2003). The Disruption Opportunity. MITSloan Management Review , 27–32.

Hayashi, T. (2005). Facilities Sharing and Network Interaction: A Lotka-Volterra Approach. Stanford: SJC Discussion Paper: DP-2005-003-E.

Hernandez, M.-J. (1998). Dynamics of Transitions between Population Interactions: A Nonlinear Interaction alpha-Function Defined. Proceedings of the Royal Society B: Biological Sciences , 1433-1440.

Hernandez, M.-J., & Barradas, I. (2003). Variation in the Outcome of Population Interactions: Bifurcations and Catastrophes. Journal of Mathematical Biology , 46, 571-594.

Holling, C. S. (1966). The Functional Response of Invertebrate Predators to Prey Density. Memoirs of the Entomological Society of Canada , 48, 1-86.

Iyer, B., Lee, C.-H., & Venkatraman, N. (2006). Managing in a "Small World Ecosystem": Lessons from the Software Sector. California Management Review , 48 (3), 28-47.

Lenox, M. L., Rockart, S. F., & Lewin, A. Y. (2007). Interdependency, Competition, and Industry Dynamics. Management Science , 53 (4), 599-615.

Lotka, A. J. (1920). Undamped Oscillations Derived from the Law of Mass Action. J. Amer. Chem. Soc. , 42, 1595-1599.

Malhotra, D., Ku, G., & Murnighan, J. K. (2008). When Winning is Everything. Harvard Business Review , 78-86.

Malthus, T. R. (1798). An Essay on the Principle of Population as it Effects the Future Improvement of Mankind. London: J. Johnson.

Maurer, S. M., & Huberman, B. A. (2003). Competitive Dynamics of Web Sites. Journal of Economic Dynamics and Control , 2195-2206.

May, R. M. (1973). Stability and Complexity in Model Ecosystems. Princeton: Princeton University Press.

Modis, T. (1997). Genetic Re-Engineering of Corporations. Technological Forecasting and Social Change , 56, 107-118.

Morris, S. A., & Pratt, D. (2003). Analysis of the Lotka-Volterra Competition Equations as a Technological Substitution Model. Technological Forecasting and Social Change , 70, 103-133.

Murray, J. D. (1989). Mathematical Biology (3rd ed., Vol. 1). Berlin: Springer.

Odum, E. P. (1953). Fundamentals of Ecology. Philadelphia-London: W.B. Saunders.

Pistorius, C. W., & Utterback, J. M. (1996). A Lotka-Volterra Model for Multi-mode Technological Interaction: Modeling Competition, Symbiosis, and Predator-Prey Modes. In R. Mason, L. Lefebure, & T. Khalil (Ed.), Proceeding of the Fifth International Conference on Management of Technology (pp. 61-70). Miami: Elsevier Advance Technology.

Pistorius, C. W., & Utterback, J. M. (1997). Multi-mode Interaction among Technologies. Research Policy , 26, 67-84.

Pistorius, C. W., & Utterback, J. M. (1995). The Death Knell of Mature Technologies. Technology Forecasting and Social Change , 50, 133-151.

Porter, A. L., Roper, A. T., Mason, T. W., Rossini, F. A., Banks, J., & Wiederholt, B. J. (1991). Forecasting and Management of Technology. John Wiley & Sons.

Rothaermel, F. T. (2000). Technological Discontinuities and the Nature of Competition. Technology Analysis and Strategic Management , 12 (2), 149-160.

Samuelson, P. A. (1971). Generalized Predator-Prey Oscillation in Ecological Economic Equilibrium. Proceeding National Academy Sciences, 68, pp. 980-983.

Solomon, S., Richmond, P., Biham, O., & Malcai, O. (2004). Co-evolutionist Stochastic Dynamics: Emergence of Power Laws. In P. Bourgine, & J.-P. Nadal, Cognitive Economics: An Interdisciplinary Approach (pp. 169-179). Berlin: Springer .

Strogatz, S. H. (1994). Nonlinear Dynamics and Chaos: with Applications in Physics, Biology, Chemistry, and Engineering. Reading, MA: Addison-Wesley Publishing.

Tisdell, C. A. (2004). Economic Competition and Evolution: Are there Lessons from Ecology? Contemporary Economic Policy , 22 (2), 179-193.

Utterback, J. M. (1994). Mastering the Dynamics of Innovation. Harvard Business School Press.

Utterback, J. M., & Acee, H. J. (2005). Disruptive Technologies: An Expanded View. International Journal of Innovation Management , 9 (1), 1–17.

Utterback, J. M., & Suarez, F. F. (1993). Innovation, Competition, and Industry Structure. Research Policy , 1-21.

Verhulst, P. F. (1838). Notice sur Ia loi que la population suit dans son accroissement. Correspondences Mathematigues et Physiques , 10, 113-121.

Volterra, V. (1931). Lecons sur le theorie mathematique de la lutte pour la vie. Paris: Gauthiers-Villars.

Walmag, J. M., & Delhez, E. J. (2005). A Trust-region Method Applied to Parameter Identification of a Simple Prey–predator Model. Applied Mathematical Modelling , 29, 289–307.

Weil, H. B., & Utterback, J. M. (2005). The Dynamics of Innovative Industries. Proceedings of the International System Dynamics Conference. Boston: Cambridge-MIT-Institute (CCI).