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FIELD STRENGTH MEASUREMENTS IN RESONANT CAVITIES

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# MASSACHUSETTS INSTITUTE OF TECHNOLOGY RESEARCH LABORATORY OF ELECTRONICS 

# FIELD STRENGTH MEASUREMENTS IN RESONANT CAVITIES* <br> Leonard C. Maier, Jr. 


#### Abstract

This report gives the theory and examples of a method of measuring the electric and magnetic field strengths in resonant cavities. The fields can be represented as the product of a readily measured constant, a function of the geometrical coordinates and the time factor. The problem of determining this space variation is solved by the application of electromagnetic perturbation theory to relate the shift in the cavity resonant frequency caused by the introduction of a metallic perturbation to the volume integral of the fields removed from the resonator. Three simple solids are examined, a sphere, a prolate spheroid, and an oblate spheroid; the resulting relations between the frequency shift, the shape and size of the perturbation, and the space variation of the fields permit the measurement of the absolute field strength at any point within the resonator. As an example, the method is applied to a section of the linear accelerator and the results are given. With proper use this method can give very accurate measurements of the direction and magnitude of the electric and magnetic fields.


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## FIELD STRENGTH MEASUREMENTS IN RESONANT CAVITIES

## I. INTRODUCTION

The problem of measuring the electric and magnetic field strengths in resonant cavities has become increasingly important with the recent great interest in microwave electronics. At microwave frequencies, ordinary circuit elements become so small that they are physically impractical, and cavity resonators are used in their place. For practical application it is necessary that the mode of oscillation and the values of the fields be known reasonably accurately, at least in some regions within the resonator. They can be calculated analytically for only a few simple geometric shapes which can be readily described by one of the standard coordinate systems (for example, rectangular, cylindrical, spherical, and ellipsoidal cavities). A number of approximate methods of calculation have been developed, but they are often either difficult to apply or rather inaccurate in their results. Therefore it has become increasingly necessary for an accurate method of measuring the electric and magnetic field strengths to be developed.

Prior to the present time, the field strengths in resonant cavities have been measured by extensions from standing wave measurements in wave guides. Electric-field, rod-type probes and magnetic-field, loop-type probes have been inserted into the walls of the resonators, and the electromagnetic energy received has been essentially rectified and presented on a meter. The orientation of the probes for maximum deflection has given the direction of the electric field along the axis of the rod-type probe and the direction of the magnetic field perpendicular to the face of the loop-type probe. While variations of this presentation have been used, they do not overcome the many objections to this method. First and foremost, it is only possible to get relative field strengths, that is, the ratio from one point to another, since the presentation apparatus is exceedingly difficult, if not impossible, to calibrate for absolute measurements. Second, the fields are measured near the region of the walls and only very general arguments based on symmetry and on knowledge of the known fields in a somewhat similarly shaped cavity can be used to give an idea of the field strengths in other regions. Third, the insertion of the probe itself changes the field depending on the size of the probe. Further difficulties are met with the integrating effects of the large size probes necessary when the field strengths are low. With these various difficulties, it is not surprising that more work has not been done on measuring field strengths in resonant cavities.

Recently, there has been developed a method which eliminates many of the objections to the probe-type measurements (1). It follows the general ideas of this report except that it has the objection of giving the field strength only in the immediate neighborhood of the walls of the cavity. In some cases, e.g. the narrow gap of a klystron cavity, this can be a great advantage, for measurements can be made in restricted volumes where the methods of this report are difficult to use.

The method of measuring field strengths in resonant cavities with which this report will be concerned makes use of electromagnetic perturbation theory. This theory relates
the electric and magnetic field strengths with the frequency shift caused by the introduction of a small metallic perturbing volume into the cavity resonator. The magnitude of the frequency shift is proportional to the value of fields at the point of perturbation; by making the relatively simple measurements of frequency shifts, the fields can be computed.

This method overcomes most of the objections previously stated. In the first place, a measurement of absolute field strength can be made once the relation between the shape and size of the perturbing element, the frequency shift, and the fields is known. Second, it is possible to suspend the perturbing element anywhere in the volume of the cavity resonator in such a way that the method of suspension does not appreciably change the field. In the third place, since the shift in resonant frequency can be measured with great accuracy, it is possible to make the perturbing element quite small compared to a wavelength, and thus minimize effects from variations of the field over the dimensions of the measuring device. At the same time, this method retains the advantages of determining the directions of the fields, since the theory will show that certain shapes have preferred directions and that by maximizing the frequency shifts, the directions of the electric and magnetic fields may be determined.

The author has employed this method of field strength measurements on the Linear Accelerator Project at the Research Laboratory of Electronics at the Massachusetts Institute of Technology for the past two years, and feels that it has some value. This report gives a discussion of the theory involved and the methods and results of measurements on a typical cavity whose geometry does not permit a simple analytical solution for the fields.

## II. GENERAL THEORY OF CAVITY FIELDS

Before going further, it would seem advisable to give a brief discussion of the fields in cavity resonators. A cavity resonator is formed by any closed volume, though almost always it is enclosed by high conductivity metal surfaces. To obtain an analytical solution for the fields and the resonant frequency, Maxwell's equation must be solved subject to boundary conditions imposed by the cavity. Combining Maxwell's equations leads to a wave equation, and the fields are derivable from a potential function which satisfies the wave equation. The fields must satisfy the boundary conditions, which, for the generally assumed perfectly conducting wall, are that the tangential electric field and the normal magnetic field are zero on the boundaries. The net result is that such a solution is possible for only a limited number of resonator shapes which are readily defined by standard coordinate systems.

While a cavity whose fields can be calculated is of interest to this report only for checking the theory, many of the concepts needed are included in more general shaped cavities, and it seems reasonable to outline the calculation of the case of a cylindrical cavity. This will also provide the relations needed to verify the theory to be developed. Maxwell's equations may be written in vector form in a cavity with no space charge or
current density, as follows

$$
\begin{array}{ll}
\nabla \times \underline{E}+\frac{\partial \underline{B}}{\partial \mathrm{t}}=0 & \nabla \cdot \underline{D}=0  \tag{1}\\
\nabla \times \underline{H}-\frac{\partial \underline{D}}{\partial \mathrm{t}}=0 & \nabla \cdot \underline{B}=0
\end{array}
$$

In free space, and to a good approximation in air, there are the auxiliary relations

$$
\begin{equation*}
\underline{B}=\mu_{\mathrm{o}} \underline{H} \quad \underline{\mathrm{D}}=\epsilon_{\mathrm{o}} \underline{E} \tag{2}
\end{equation*}
$$

Writing these equations in cylindrical component form and solving them simultaneously gives scaler wave equations for the various components, and it is seen that two general types of solutions satisfy these wave equations. In the transverse magnetic (TM) mode, the magnetic field has no component along the $z$-axis; in the transverse electric (TE) mode, the electric field has no component along the $z$-axis. It is a general result for all cavities of a general cylindrical shape and for a spherical cavity that there exist these two types of modes, but in other cavities, "transverse" may refer to any axis or, in the case of a spherical cavity, to the radius vector. Only the TM modes will be considered, since the cavity used to check the perturbation theory was operated in such a mode, but the solutions for TE modes are similar. Assuming that the fields vary with $e^{j \omega t}$ and substituting the boundary conditions for a cylindrical cavity of length $L$ and radius $R_{o}$, that is,

$$
\begin{aligned}
\mathrm{E}_{\mathbf{z}}=0 & \text { at } \mathbf{r}
\end{aligned}=\mathrm{R}_{\mathrm{o}}, ~ \begin{aligned}
\mathrm{E}_{\mathbf{r}}=0 \quad \text { at } \mathrm{z} & =0 \\
\mathrm{z} & =\mathrm{L},
\end{aligned}
$$

the following are the solutions for the fields.

$$
\begin{align*}
& H_{\theta}=A \cos \left(\frac{1 \pi}{L} z\right) \frac{J_{n+1}\left(\frac{U_{n m}}{R_{o}} r\right)}{U_{n m}} e^{j \omega t}  \tag{3}\\
& E_{r}=-j \frac{\beta}{k} \sqrt{\frac{\mu_{o}}{\epsilon_{o}}} A \sin \left(\frac{1 \pi}{L} z\right) \frac{J_{n+1}\left(\frac{U_{n m}}{R_{o}} r\right)}{\frac{U_{n m}}{R_{o}}} e^{j \omega t}  \tag{4}\\
& E_{z}=\frac{-j}{k} \sqrt{\frac{\mu_{o}}{\epsilon_{o}} A \cos \left(\frac{1 \pi}{L} z\right) J_{n}\left(\frac{U_{n m}}{R_{o}} r\right) e^{j \omega t}} \tag{5}
\end{align*}
$$

where $A$ is an arbitrary constant, $\beta=1 \pi / L, k=2 \pi / \lambda=\omega / c, U_{n m}=m$ mh root of $J_{n}$, $1=0,1,2,3 \ldots, m=1,2,3 \ldots$, and $n=0,1,2,3 \ldots$. This gives, as in all cavities,
an infinite series of possible $\mathrm{TM}_{1 \mathrm{mn}}$ modes depending upon the values of $1, \mathrm{~m}$, and n chosen. For the $\mathrm{TM}_{010}$ mode which was used to check the perturbation theory,

$$
\begin{align*}
& E_{z}=-\frac{j}{k} A \sqrt{\frac{\mu_{o}}{\epsilon_{o}}} J_{o}(k r) e^{j \omega t}  \tag{6}\\
& H_{\theta}=\frac{1}{k} A J_{1}(k r) e^{j \omega t}  \tag{7}\\
& k=\frac{2.405}{R_{o}} \tag{8}
\end{align*}
$$

These solutions represent fields in the cavity multiplied by an arbitrary constant. The value of the constant can also be calculated by making the assumptions that the effects of a coupling hole or probe, and of the finite conductivity of the wall surfaces, have only a second order effect on the geometrical variations of the field. These are valid assumptions with cavities of very high conductivity materials and with reasonably small coupling holes. The unloaded $Q_{0}$ of a cavity may be defined as

$$
\begin{equation*}
Q_{o}=\frac{2 \pi \times \text { Energy stored }}{\text { Energy dissipated per cycle }} \tag{9}
\end{equation*}
$$

The energy dissipated per second is the power $P$ actually put into the cavity, and the energy stored in the resonator is

$$
\begin{equation*}
\frac{1}{2} \epsilon_{o} \int E^{2} d v=\frac{1}{2} \mu_{o} \int H^{2} d v \tag{10}
\end{equation*}
$$

Using these relationships, the value of A can be found in terms of an integral over the cavity volume of the space variation of the fields. For the $\mathrm{TM}_{010}$ mode, whose space variation is known, this integration can be carried out and yields

$$
\begin{equation*}
A=\sqrt{\frac{Q_{o} P k^{2}}{1 / 2 \omega \mu \pi L R_{o}^{2} J_{l}^{2}\left(k R_{o}\right)}} \tag{11}
\end{equation*}
$$

The fields can then be written with the above value of $A$, and since $Q, P$, and $\lambda_{0}$ can all be measured or calculated, the fields are completely determined in this case.

The problem occurs when the fields can not be found analytically, and the constant A contains an unknown integral depending upon the geometrical variations of the fields. Different authors have adopted various concepts to take care of this difficulty. One of the most popular is to carry over the analogies of low-frequency, lumped-constant resonant circuits, and define $Q$ and a shunt impedance, the $Q$ being defined as above, and the shunt impedance as a factor which gives the amount of power which must be supplied to maintain a given voltage across a given path. Others consider the ratio of the shunt impedance to the Q in order to have a constant depending upon the geometry and not
containing the conductivity of the walls. All these have a great disadvantage in that, unlike the $Q$, the shunt impedance has no unique definition and depends completely on the arbitrary path used to define it (2). To avoid this difficulty, cavity fields will be discussed in a method which recognizes that the fields can be represented by a measurable constant multiplying a purely geometrical variation. It has the additional advantage of representing the general solution to a cavity whose fields cannot be found analytically, and of leading directly to the perturbation method of measuring the field strengths.

The following concepts are those developed by J. C. Slater (3). The essential idea behind this theory is that a general solution of the fields in a cavity resonator can be obtained from a summation over certain normal modes of oscillation. These normal modes have orthogonality properties and have certain defined values on the boundaries of the cavity. In terms of such modes, the fields in cavities of any shape may be calculated, and more exact solutions which take into account variations in the fields caused by finite conductivity and coupling devices may be had for the simple shape resonators. This report will not attempt to develop these concepts, but will use them.

As Slater has shown, the $\underline{E}$ and $\underline{H}$ fields of a resonant cavity with no current density and no space charge can be given by the following vector summation over normal modes

$$
\begin{align*}
& \underline{E}=\sum_{a} \underline{E}_{a} \int \underline{E} \cdot \underline{E}_{a} d v  \tag{12}\\
& \underline{H}=\sum_{a} \underline{H}_{a} \int \underline{H} \cdot \underline{H}_{a} d v \tag{13}
\end{align*}
$$

where $\underline{E}_{\mathrm{a}}$ and $\underline{\mathrm{H}}_{\mathrm{a}}$, the normal modes, are time independent solutions of the vector wave equations

$$
\begin{array}{ll}
\nabla \times \nabla \times \underline{E}_{a}=k_{a}^{2} \underline{E}_{a} & \nabla \times \underline{E}_{a}=k_{a} \underline{H}_{a} \\
\nabla \times \nabla \times \underline{H}_{a}=\mathrm{k}_{\mathrm{a}}^{2} \underline{H}_{\mathrm{a}} & \nabla \times \underline{\mathrm{H}}_{\mathrm{a}}=\mathrm{k}_{\mathrm{a}} \underline{\mathrm{E}}_{\mathrm{a}} \tag{15}
\end{array}
$$

with the additional normalization and orthogonality conditions that

$$
\begin{array}{rlrl}
\int \underline{E}_{a} \cdot \underline{E}_{b} d v=\int \underline{H}_{a} \cdot \underline{H}_{b} d v=\delta_{a b} & =0 & a \neq b  \tag{16}\\
& =1 & a=b
\end{array}
$$

Substitution of these relations into Maxwell's equations yields two differential equations for the coefficients in the expansions for $\underline{E}$ and $\underline{H}$,

$$
\begin{equation*}
\epsilon_{o} \mu_{o} \frac{d^{2}}{d t^{2}} \int \underline{E} \cdot \underline{E}_{a} d v+k_{a}^{2} \int \underline{E} \cdot \underline{E}_{a} d v=\mu_{o} \frac{d}{d t} \int_{S^{\prime}}(\underline{n} \times \underline{H}) \cdot \underline{E}_{a} d a-k_{a} \int_{S}(\underline{n} \times \underline{E}) \cdot \underline{H}_{a} d a \tag{17}
\end{equation*}
$$

$$
\epsilon_{0} \mu_{o} \frac{d^{2}}{d t^{2}} \int \underline{H} \cdot \underline{H}_{a} d v+k_{a}^{2} \int \underline{H} \cdot \underline{H}_{a} d v=-k_{a} \int_{S^{\prime}}(\underline{n} \times \underline{H}) \cdot E_{a} d a-\epsilon_{o} \int_{S}(\underline{n} \times \underline{E}) \cdot \underline{H}_{a} d a
$$

where $S$ is the boundary surface on which the tangential component of $E_{a}$ is zero and $S^{\prime}$ the boundary surface on which the tangential component of $\underline{H}_{\mathrm{a}}$ is zero. By accounting for the integrals over these surfaces, the effects of finite conductivity and coupling devices may be computed.

The method of application is to solve the vector wave equations in the coordinate system most nearly representing the cavity to obtain the functions $\underline{E}_{a}$ and $\underline{H}_{a}$. The coefficients are found by solving the differential equations with the values $\underline{E}$ and $\underline{H}$ on the surfaces $S$ and $S^{\prime}$. Then, by taking the indicated summations, $\underline{E}$ and $\underline{H}$ are known. In terms of this method of description, the fields of the $\mathrm{TM}_{010}$ cavity oscillating in its lowest mode can be written as follows, after normalization

$$
\begin{align*}
\underline{E}=\underline{E} a \int \underline{E} \cdot \underline{E}_{a} d v=\frac{\underline{K} J_{o}(k r)}{\sqrt{\pi L} R_{o} J_{l}\left(k R_{o}\right)} \int \underline{E} \cdot \underline{E} d v  \tag{19}\\
\underline{H}=\underline{H}_{a} \int \underline{H} \cdot \underline{H}_{a} d v=\frac{\underline{\theta} J_{1}(k r)}{\sqrt{\pi L} R_{o} J_{1}\left(k R_{o}\right)} \int \underline{H} \cdot \underline{H}_{a} d v \tag{20}
\end{align*}
$$

and where the solutions of the differential equations for the coefficients give

$$
\begin{align*}
& \int \underline{H} \cdot \underline{H}_{a} d v=\text { const } e^{j \omega t}  \tag{21}\\
& \int \underline{E} \cdot \underline{E} a d v=-j \sqrt{\frac{\mu_{o}}{\epsilon_{o}}} \int \underline{H} \cdot \underline{H}_{a} d v \tag{22}
\end{align*}
$$

The constant in Eq. 21 for the coefficients can be found in terms of the $Q_{o}$ and input power, by the same method as was used to find Eq. 11

$$
\begin{equation*}
\text { const }=\sqrt{\frac{Q_{0} P \lambda_{a}}{377 \pi}} \tag{23}
\end{equation*}
$$

All quantities in this expression are measurable, and the net result is that the fields can now be expressed in terms of a measurable constant multiplying a term depending com pletely on the geometrical properties of the resonant cavity.

For a general cavity, the approach is exactly the same except that the geometrical factor must somehow be measured. In general terms, the fields will be expressed in terms of $\underline{E}_{o}$ and $\underline{H}_{o}$ where these are time-independent functions of position and are
normalized over the volume of the cavity. These functions correspond to the $\mathrm{E}_{\mathrm{a}}$ and $\underline{H}_{\mathrm{a}}$ above, and are proportional to the true fields. The constant of proportionality can be expressed in terms of purely measurable quantities. The following chapters are then devoted to determining the method and details of measuring $\underline{E}_{o}$ and $\underline{H}_{o}$, and once this is done, the true fields are given by

$$
\begin{align*}
& \underline{E}=-j \sqrt{\frac{\mu_{0}}{\epsilon_{0}} \sqrt{\frac{Q_{0} P \lambda_{o}}{377 \pi}} \underline{E}_{o} e^{j \omega t}}  \tag{24}\\
& \underline{H}=\sqrt{\frac{Q_{0} P \lambda_{0}}{377 \pi}} \underline{H}_{o} e^{j \omega t} \tag{25}
\end{align*}
$$

## III. DERIVATION OF THE PERTURBATION FORMULA

The principal advantage of expressing cavity fields by the methods developed in the previous chapter is that these descriptions of the fields lead to a simple derivation of the perturbation formula. This derivation is given in Ref. 3 but is repeated here for completeness and also in the hope that an understanding of the derivation will aid in comprehension of its use. Consider a cavity in which the functions $\underline{E}_{a}$ and $\underline{H}_{a}$ have already been found. If one of the bounding walls is now pushed into the cavity by a small amount, the final fields, $\underline{E}$ and $\underline{H}$ will be zero in the volume between the initial and final walls. The resulting discontinuity at the perturbed wall of the tangential component of $\underline{H}$ must then be included in the right-hand sides of the differential equations for $\int E \cdot E \cdot d v a n d$ $\int \underline{\mathrm{H}} \cdot \underline{\mathrm{H}}_{\mathrm{a}}$ dv. This surface discontinuity in $\underline{\mathrm{H}}$ will be included as the integral $\int(\underline{\mathrm{n}} \times \underline{\mathrm{H}})$ $\cdot \underline{E}_{a}$ dv over the perturbed wall surface. Since, for a very small perturbation, the perturbed field $\underline{H}$ will be very nearly the original $\underline{H}_{a}$ over the wall surface, the desired integral may be expressed by

$$
\begin{equation*}
\int(\underline{\mathrm{n}} \times \underline{\mathrm{H}}) \cdot \underline{E}_{\mathrm{a}} \mathrm{da} \sim \int\left(\underline{\mathrm{n}} \times \underline{H}_{a}\right) \cdot \underline{E}_{a} d a=\int-\underline{\mathrm{n}} \cdot\left(\underline{\mathrm{E}}_{\mathrm{a}} \times \underline{H}_{a}\right) \mathrm{da} \tag{26}
\end{equation*}
$$

By a vector analysis theorem

$$
\begin{equation*}
\nabla \cdot\left(\underline{E}_{a} \times \underline{H}_{a}\right)=\underline{H}_{a} \cdot\left(\nabla \times \underline{E}_{a}\right)-\underline{E}_{a} \cdot\left(\nabla \times \underline{H}_{a}\right) \tag{27}
\end{equation*}
$$

and from the definitions of $\underline{H}_{a}$ and $E_{a}$ in Eqs. 14 and 15 , this is given by $k_{a}\left(H_{a}^{2}-E_{a}^{2}\right)$. If this expression is now integrated over the small volume removed from the cavity by the perturbation of the wall, then

$$
\begin{equation*}
\int_{\Delta V} \nabla \cdot\left(\underline{E}_{a} \times \underline{H}_{a}\right) d v=k_{a} \int_{\Delta V}\left(H_{a}^{2}-E_{a}^{2}\right) d v \tag{28}
\end{equation*}
$$

Applying the divergence theorem, the left-hand side is transformed into two surface
integrals, one over the original surface and the other over the perturbed surface. Over the original surface, $\underline{n} \cdot\left(\underline{E}_{a} \times \underline{H}_{a}\right)$ can be changed to $\underline{H}_{a} \cdot\left(\underline{n}^{\times} \underline{E}_{a}\right)$, and by the boundary condition defining $\underline{E}_{a}$, this tangential component is zero on the original surface. This leaves the integral over the perturbed surface, and since the normal from the divergence theorem is the opposite of the desired normal, the result is that

$$
\begin{equation*}
\int-\underline{n} \cdot\left(\underline{E}_{a} \times \underline{H}_{a}\right) d a=k_{a} \int_{\Delta V}\left(H_{a}^{2}-E_{a}^{2}\right) d v \tag{29}
\end{equation*}
$$

By the approximation in Eq. 26 this gives the desired $\int(\underline{n} \times \underline{H}) \cdot \underline{E}_{a}$ da over the perturbed volume. If the original field were a constant times $\underline{H}_{a}$, or $\underline{H}_{a} \int \underline{H} \cdot \underline{H}_{a} d v$, the result would have to be multiplied by $\int \underline{H} \cdot \underline{H}_{a}$ dv. Thus

$$
\begin{equation*}
\int(\underline{n} \times \underline{H}) \cdot \underline{E}_{a} d a=k_{a} \int \underline{H} \cdot \underline{H}_{a} d v \int_{\Delta V}\left(H_{a}^{2}-E_{a}^{2}\right) d v \tag{30}
\end{equation*}
$$

Substituting this in Eq. 18 for the coefficients $\int \underline{H}_{j \omega t} \underline{H}_{a} d v$ and assuming that they are proportional to a constant multiplied by $\mathrm{e}^{\mathrm{j} \omega \mathrm{t}}$, then

$$
\begin{equation*}
-\epsilon_{o} \mu_{o} \omega^{2}+k_{a}^{2}=k_{a}^{2} \int_{\Delta V}\left(H_{a}^{2}-E_{a}^{2}\right) d v \tag{31}
\end{equation*}
$$

but

$$
\begin{equation*}
\mathrm{k}_{\mathrm{a}}^{2}=\omega_{\mathrm{a}}^{2} \epsilon_{\mathrm{o}}^{\mu_{o}} \tag{32}
\end{equation*}
$$

and thus

$$
\begin{equation*}
\frac{\omega_{a}^{2}-\omega^{2}}{\omega_{a}^{2}}=\int_{\Delta V}\left(E_{a}^{2}-H_{a}^{2}\right) d v \tag{33}
\end{equation*}
$$

This last result is the electromagnetic perturbation formula, and relates the shift in resonant frequency to the integral $\int\left(\mathrm{E}_{\mathrm{a}}^{2}-\mathrm{H}_{\mathrm{a}}^{2}\right)$ dv over the volume which was removed from the cavity.

There are two warnings which must be remembered in applying this perturbation theory. The first is that it is exact only for infinitesimal perturbations, and cannot be applied for finite distortions. The second is perhaps slightly more subtle, but is nevertheless important. Since the whole derivation, as given above, depends upon the unperturbed fields $\underline{E}_{a}$ and $\underline{H}_{a}$ satisfying the boundary conditions before the perturbation is introduced, the formula will not hold in cases where even an infinitesimal distortion is made from a situation not originally satisfying the boundary conditions. The reasons for making these two distinctions clear will be apparent when, in succeeding chapters, this perturbation theory is applied to various shapes of perturbing volumes.

The foregoing derivation of the perturbation formula for a perturbation of the walls still holds for a perturbing volume placed within the resonator, if the perturbation is
small compared to a wavelength, and if the two warnings given in the preceding paragraph are heeded. The reason for the restriction as to the size of the perturbation is that the derivation given did not take into account the possibility of resonant effects occurring when the perturbing object is of the order of a wavelength. The frequency shifts caused by such a resonance would thoroughly mask the frequency shifts calculated from the application of the perturbation formula. To avoid this difficulty, all the perturbing shapes used will be of dimensions much smaller than a wavelength, and the calculations can be carried out using only the perturbation theory developed in this chapter.

## IV. PERTURBATION EFFECTS OF METALLIC SPHERES

Perhaps the simplest example of the application of the perturbation theory developed in the previous chapter is to the perturbation produced by a metallic sphere in the $E$ and $H$ fields of a cavity resonator. The spherical symmetry and resulting non-directional effects, as well as the magnitude of the perturbations produced, make the metallic sphere an extremely useful tool in measuring field strengths.

The method of applying the perturbation formula is to assume that a perfectly conducting sphere is placed into a constant electro- or magneto-static field, and to solve for the resulting $E$ or $H$ field subject to the boundary conditions on the sphere. This gives fields which satisfy the boundary conditions before the perturbation theory is applied, which, it will be remembered from the preceding chapter, is one of the necessary steps in correctly applying the perturbation formula. The sphere is expanded by an infinitesimal amount, and the E or H fields integrated as indicated by the perturbation formula. This insures that the perturbation formula is correctly applied only to an infinitesimal change in the boundaries, and hence both the warnings given in Chapter III have been heeded. The resulting expression can then be integrated from a sphere of radius zero to the desired size, and the perturbation effect is known for a given sphere in a given electric or magnetic field. Although the perturbation formula will be applied to $E$ and $H$ fields separately, it will hold in a region where both fields are present because of the linear character of Maxwell's equations and the superposition theorem.

The assumptions of constant static fields do not lead to any loss of generality. Since the perturbing sphere will be small compared to a wavelength, the fields found in cavity resonators will be practically constant for distances comparable to the sphere diameter. Static fields may be considered since the perturbation formula is concerned only with an integration over the space variations of the field and is independent of the time variation. The static fields may be considered the equivalents of the geometric field variations discussed in Chapter II.

Consider a constant electrostatic field of magnitude $E_{o}$ directed along the positive $z$-axis. The effect on this field caused by the introduction of an uncharged, perfectly conducting metallic sphere of radius $r_{o}$ is straightforward to work out, and will be taken from the literature (4). The resulting potential function outside the sphere expressed in spherical coordinates is given by

$$
\begin{equation*}
\phi=-E_{o} r \cos \theta+\frac{E_{o} r_{o}^{3}}{r^{2}} \cos \theta \tag{34}
\end{equation*}
$$

The resulting fields are obtained from the gradient of this potential, and are

$$
\begin{align*}
& \mathrm{E}_{\mathrm{r}}=-\frac{\partial \phi}{\partial r}=\mathrm{E}_{\mathrm{o}}\left[1+\frac{2 \mathrm{r}_{\mathrm{o}}^{3}}{\mathrm{r}^{3}}\right] \cos \theta  \tag{35}\\
& \mathrm{E}_{\theta}=-\frac{1}{\mathrm{r}} \frac{\partial \phi}{\partial \theta}=\mathrm{E}_{\mathrm{o}}\left[-1+\frac{\mathrm{r}_{\mathrm{o}}^{3}}{\mathrm{r}^{3}}\right] \sin \theta  \tag{36}\\
& \mathrm{E}_{\psi}=-\frac{1}{\mathrm{r} \sin \theta} \frac{\partial \phi}{\partial \psi}=0 \tag{37}
\end{align*}
$$

The perturbation formula for the case of zero H field is

$$
\begin{equation*}
\frac{\omega_{o}^{2}-\omega^{2}}{\omega_{o}^{2}}=\int_{\Delta V} E_{a}^{2} d v \tag{38}
\end{equation*}
$$

Since

$$
\begin{equation*}
E_{a}^{2}=E_{r}^{2}+E_{\theta}^{2}+E_{\psi}^{2} \tag{39}
\end{equation*}
$$

and

$$
\begin{equation*}
d v=r^{2} \sin \theta d r d \theta d \psi \tag{40}
\end{equation*}
$$

integrating the fields for an infinitesimal expansion of the sphere into the field, from $r_{o}$ to $\mathrm{r}_{\mathrm{o}}+\mathrm{dr}_{\mathrm{o}}$, gives

$$
\begin{align*}
& \quad\left(\int_{r_{o}}^{r_{o}+d r_{o}}\left[1+\frac{2 r_{o}^{3}}{r^{3}}\right]^{2} r^{2} d r \int_{o}^{\pi} \cos ^{2} \theta \sin \theta d \theta\right. \\
& \left.+\int_{r_{o}}^{r_{o}+d r_{o}}\left[1-\frac{r_{o}^{3}}{r^{3}}\right]^{2} r^{2} d r \int_{o}^{\pi} \sin ^{3} \theta d \theta\right) 2 \pi E_{o}^{2}=\int_{V}^{V+d V} E_{a}^{2} d v \tag{41}
\end{align*}
$$

From integral calculus, it is obvious that

$$
\begin{equation*}
\int_{x_{0}}^{x_{0}+d x_{0}} f(x) d x=f\left(x_{0}\right) d x_{0} \tag{42}
\end{equation*}
$$

Applying this to the previous expression, the result for the infinitesimal expansion of the sphere is

$$
\begin{equation*}
\int_{V}^{V+d V} E_{a}^{2} d v=12 \pi E_{o}^{2} r_{o}^{2} d r_{o} \tag{43}
\end{equation*}
$$

For the total perturbation caused by a sphere of radius $r_{o}$, the above expression must be integrated from $r_{o}=0$, to $r_{o}=r_{o}$, and substituting the resulting expression into the perturbation formula gives

$$
\begin{equation*}
\frac{\omega_{o}^{2}-\omega^{2}}{\omega_{o}^{2}}=3 E_{o}^{2} \frac{4 \pi}{3} r_{o}^{3} \tag{44}
\end{equation*}
$$

This, then, gives an expression relating the magnitude $E_{o}$ of the $E$ field to the size and shape of a perturbing volume, and the frequency shift caused by its introduction.

The application of perturbation theory to a metallic sphere in the H field is exactly similar. A potential function for a perfectly conducting metallic sphere in a constant H field of magnitude $\mathrm{H}_{\mathrm{o}}$ along the positive z -axis, is given by

$$
\begin{equation*}
\phi=-H_{o}\left[r \cos \theta+\frac{1}{2} \frac{r_{o}^{3}}{r^{2}} \cos \theta\right] \tag{45}
\end{equation*}
$$

From the gradient of the potential, the components of the field are

$$
\begin{align*}
& H_{r}=H_{o}\left[1-\frac{r_{o}^{3}}{r^{3}}\right] \cos \theta  \tag{46}\\
& H_{\theta}=-H_{o}\left[1+\frac{1}{2} \frac{r_{o}^{3}}{r^{3}}\right] \sin \theta  \tag{47}\\
& H_{\psi}=0 \tag{48}
\end{align*}
$$

and applying the perturbation theory exactly as was done for the $E$ field, the final result for a sphere of radius $r_{o}$ is that

$$
\begin{equation*}
\frac{\omega_{o}^{2}-\omega^{2}}{\omega_{o}^{2}}=-\frac{3}{2} H_{o}^{2} \frac{4 \pi}{3} r_{o}^{3} \tag{49}
\end{equation*}
$$

By applying the superposition principle, the perturbation effect in a region where both fields are present is given by

$$
\begin{equation*}
\frac{\omega_{o}^{2}-\omega^{2}}{\omega_{o}^{2}}=\frac{4 \pi}{3} r_{o}^{3} 3\left(E_{o}^{2}-\frac{1}{2} H_{o}^{2}\right) \tag{50}
\end{equation*}
$$

From this, it is clear that a sphere alone cannot give E and H separately, but only the function ( $\mathrm{E}_{\mathrm{o}}^{2}-\frac{1}{2} \mathrm{H}_{\mathrm{o}}^{2}$ ), and that at least one additional equation is needed. Actually, as it
turns out in a general resonant cavity, there will be five components of the fields which must be determined. This means that to determine $E$ and $H$ completely, except in a region where only E or H alone is present, shapes of perturbing volumes must be examined to obtain other relationships between the various components.

These results have been checked experimentally, as will be shown in a future chapter, but there is also a method of checking them theoretically which has been applied. The results will be mentioned here but the work not carried out, as it does not further the general problem. The method of checking lies in the fact that both spherical cavities and cavities formed by two concentric spheres can be calculated analytically for the fields and frequencies. For the TE modes in a spherical cavity, H is not zero at the center while $E$ is, and the opposite is true for TM modes. By solving for the frequency difference between a spherical cavity and a concentric spherical cavity of the same outer radius, agreement was had with the frequency shift found by using the perturbation formula on the fields in the center of the spherical resonator. These results agreed with those for spheres inserted in constant fields as found in this chapter.

## V. PERTURBATION EFFECTS OF CIRCULAR METALLIC NEEDLES

The next two chapters will be concerned with the perturbation effects of ellipsoids of revolution; the present chapter with ellipses rotated around the major axis to form a circular needle, and the following chapter with ellipses rotated around the minor axis to form a circular disk (5). These two chapters will then give the necessary additional relations for complete determination of the fields.

The problem of the general ellipsoid with all axes different cannot be calculated analytically. This is unfortunate since such a calculation would make possible one series of applications of the perturbation formula, and the results for needles and disks could be found by letting two of the three axes be equal. An additional, though minor, complication results from the fact that the coordinate systems best suited for describing the needles and disks have not been worked out, and there is no readily available potential function for perfectly conducting disk or needle in an electric or magnetic field. Thus, the problem must be worked out from the equations of the ellipsoid, the potentials found for the various orientations of the needle and disk, and then the perturbation formula applied to the resulting fields. This process will be carried out in full for the case of the needle in the present chapter, and merely outlined, with the end results given, in the next chapter on disks.

The process to be followed in finding the potential in the desired coordinate system is the straightforward method of generalized coordinates and is in the literature (6). The coordinates to be used for the circular needle of elliptic cross sections are the points in space determined by the intersections of a family of confocal ellipsoids, $\xi=$ constant, with a family of confocal hyperbolas, $\eta=$ constant, with an angle, $\psi=$ constant, measured around the x -axis. The equations of the coordinate surfaces are given by

$$
\begin{equation*}
\frac{x^{2}}{\xi+a^{2}}+\frac{r^{2}}{\xi+c^{2}}=1, \frac{x^{2}}{a^{2}-\eta}-\frac{r^{2}}{\eta-c^{2}}=1 \tag{51}
\end{equation*}
$$

where

$$
\begin{align*}
& a \geqslant c \\
& \xi>-c^{2} \\
& c^{2}<\eta<a^{2}  \tag{52}\\
& r^{2}=y^{2}+z^{2} \\
& y=r \sin \psi \\
& z=r \cos \psi
\end{align*}
$$

These equations can be solved simultaneously for $\mathrm{x}, \mathrm{y}$ and z , and this yields

$$
\begin{align*}
& x=\left[\frac{\left(\xi+a^{2}\right)\left(a^{2}-\eta\right)}{\left(a^{2}-c^{2}\right)}\right]^{1 / 2}  \tag{53}\\
& y=\left[\frac{\left(\xi+c^{2}\right)\left(\eta-c^{2}\right)}{\left(a^{2}-c^{2}\right)}\right]^{1 / 2} \sin \psi  \tag{54}\\
& z=\left[\frac{\left(\xi+c^{2}\right)\left(\eta-c^{2}\right)}{\left(a^{2}-c^{2}\right)}\right]^{1 / 2} \cos \psi \tag{55}
\end{align*}
$$

The desired coordinates are $\xi, \eta$ and $\psi$, and the metrical coefficients can be calculated from Eqs. 53, 54 and 55 and are

$$
\begin{gather*}
h_{1}=\frac{1}{2}\left[\frac{(\xi+\eta)}{\left(\xi+a^{2}\right)\left(\xi+c^{2}\right)}\right]^{1 / 2} \quad h_{2}=\frac{1}{2}\left[\frac{(\xi+\eta)}{\left(a^{2}-\eta\right)\left(\eta-c^{2}\right)}\right]^{1 / 2} \\
h_{3}=\left[\frac{\left(\xi+c^{2}\right)\left(\eta-c^{2}\right)}{\left(a^{2}-c^{2}\right)}\right]^{1 / 2} \tag{56}
\end{gather*}
$$

The potential function needed will be found by a method given by J. A. Stratton (7). The case of the constant field being directed along the $x$-axis, or, as can be seen from the equation for the ellipsoid, along the major axis of the needle, will be calculated, and from this solution, the potential function for the field along the minor axis can readily be found. For an applied field of magnitude $C_{o}$, along the $x$-axis, the potential of the field is given by

$$
\begin{equation*}
\phi_{0}=-C_{0} x=-C_{0}\left[\frac{\left(\xi+a^{2}\right)\left(\mathrm{a}^{2}-\eta\right)}{\left(\mathrm{a}^{2}-\mathrm{c}^{2}\right)}\right]^{1 / 2} \tag{57}
\end{equation*}
$$

The desired potential function must satisfy Laplace's equation, since there is no free charge; it must reduce to a constant on the surface $\xi=0$, that is, on the surface of the needle; and it must be regular at infinity. The primary potential $\phi_{O}$ is a solution of Laplace's equation, but is not regular at infinity. Let

$$
\begin{equation*}
\phi_{0}=A_{1} F_{1}(\xi) \mathrm{F}_{2}(\eta) \mathrm{F}_{3}(\psi) \tag{58}
\end{equation*}
$$

If the boundary conditions are to be satisfied on the surface $\xi=0$, the potential $\phi_{1}$ of the induced distribution must vary identically with $\eta$ over surfaces of constant $\xi$. Assume then, that

$$
\begin{equation*}
\phi_{1}=A_{2} G_{1}(\xi) F_{2}(\eta) F_{3}(\psi) \tag{59}
\end{equation*}
$$

$\phi_{1}$ must also satisfy Laplace's equation, which can be found from its expression in generalized coordinates (6) with the metrical coefficients given above. It is

$$
\begin{gather*}
\nabla^{2} \phi=0=\sqrt{\xi+a^{2}} \frac{\partial}{\partial \xi}\left[\sqrt{\xi+a^{2}}\left(\xi+c^{2}\right) \frac{\partial \phi}{\partial \xi}\right] \\
+\sqrt{a^{2}-\eta} \frac{\partial}{\partial \eta}\left[\sqrt{a^{2}-\eta}\left(\eta-c^{2}\right) \frac{\partial \phi}{\partial \eta}\right]+\frac{1}{4}\left[\frac{(\xi+\eta)\left(a^{2}-c^{2}\right)}{\left(\xi+c^{2}\right)\left(\eta-c^{2}\right)}\right] \frac{\partial^{2} \phi}{\partial \psi^{2}} \tag{60}
\end{gather*}
$$

Substitution of the expression for $\phi_{1}$ into Laplace's equation, 60, then gives the following differential equation for $G_{1}(\xi)$.

$$
\begin{equation*}
\frac{d^{2} G_{1}}{d \xi^{2}}+\frac{d G_{1}}{d \xi}\left[\frac{1}{2} \frac{1}{\xi+a^{2}}+\frac{1}{\xi+c^{2}}\right]-\frac{G_{1}}{2} \frac{1}{\left(\xi+a^{2}\right)\left(\xi+c^{2}\right)}=0 \tag{62}
\end{equation*}
$$

If one solution of a second-order, linear differential equation is known, in this case, $F_{1}(\xi)$, an independent solution, $G_{1}(\xi)$, can be determined from it by integration (8). That is, if $y_{1}(x)$ is a solution of

$$
\begin{equation*}
\frac{d^{2} y}{d x^{2}}+p(x) \frac{d y}{d x}+q(x) y=0 \tag{63}
\end{equation*}
$$

then

$$
\begin{equation*}
y_{2}(x)=y_{1}(x) \int \frac{e^{-\int p(x) d x}}{\left[y_{1}(x)\right]^{2}} d x \tag{64}
\end{equation*}
$$

Applying this to find $G_{1}(\xi)$ where $F_{1}(\xi)=\sqrt{\xi+a^{2}}$, the result is that

$$
\begin{equation*}
G_{1}(\xi)=\sqrt{\xi+a^{2}} \int \frac{e^{-\frac{1}{2} \int \frac{d \xi}{\xi+a^{2}} e^{-\int \frac{d \xi}{\xi+c^{2}}}}}{\left(\xi+a^{2}\right)} d \xi \tag{65}
\end{equation*}
$$

$$
\begin{equation*}
G_{1}(\xi)=\sqrt{\xi+a^{2}} \int \frac{d \xi}{\left(\xi+a^{2}\right)^{3 / 2}\left(\xi+c^{2}\right)} \tag{66}
\end{equation*}
$$

The limits of integration on $G_{1}(\xi)$ are arbitrary, but to make $\phi_{1}$ vanish at infinity, the limits used are $\xi$ to infinity. Substituting this in the expression for $\phi_{1}$ gives

$$
\begin{equation*}
\phi_{1}=\phi_{0} \frac{A_{2}}{A_{1}} \int_{\xi}^{\infty} \frac{d \xi}{\left(\xi+a^{2}\right)^{3 / 2}\left(\xi+c^{2}\right)} \tag{67}
\end{equation*}
$$

The total potential for the case of the applied field parallel to the major axis is the sum of the primary potential $\phi_{O}$ and the induced potential $\phi_{1}$, and is

$$
\begin{equation*}
\phi_{11}=-C_{0}\left[\frac{\left(\xi+a^{2}\right)\left(a^{2}-\eta\right)}{\left(a^{2}-c^{2}\right)}\right]^{1 / 2}\left[1+A \int_{\xi}^{\infty} \frac{d \xi}{\left(\xi+a^{2}\right)^{3 / 2}\left(\xi+c^{2}\right)}\right] \tag{68}
\end{equation*}
$$

Since nothing in the foregoing derivation restricted it to either an E or an H field, the constant $\mathrm{C}_{\mathrm{o}}$ can be either $\mathrm{E}_{\mathrm{o}}$ or $\mathrm{H}_{\mathrm{o}}$, and A will then be determined from the boundary conditions applying to the field being considered.

In the case of the primary field being directed along the $y$ - or $z$-axis, it is simple to sketch the derivation of the potential to find the changes in the final function $\phi_{\perp} . F_{1}(\xi)$ will in this case be $\sqrt{\xi+c^{2}}$ and the resulting integration for $G_{1}(\xi)$ will give

$$
\begin{equation*}
G_{1}(\xi)=\sqrt{\xi+c^{2}} \int_{\xi}^{\infty} \frac{d \xi}{\left(\xi+c^{2}\right)^{2} \sqrt{\xi+a^{2}}} \tag{69}
\end{equation*}
$$

The final potential, in this case, of the primary field perpendicular to the major axis of the needle, will be

$$
\begin{equation*}
\phi_{\perp}=-C_{0}\left[\frac{\left(\xi+c^{2}\right)\left(\eta-c^{2}\right)}{\left(a^{2}-c^{2}\right)}\right]^{1 / 2} \sin \psi\left[1+A \int_{\xi}^{\infty} \frac{d \xi}{\left(\xi+c^{2}\right)^{2} \sqrt{\xi+a^{2}}}\right] \tag{70}
\end{equation*}
$$

in contrast to the expression for $\phi_{11}$ given in Eq. 68.
There are obviously four cases which must be calculated, two for $E_{0}$, parallel and perpendicular to the major axis a of the needle, and two for $H_{0}$. The procedure to be used is similar to that followed for spheres. The fields will be found when the needle is placed in a previously constant field, and the perturbation formula applied to an infinitesimal expansion of the needle into these fields. The resulting expression will then be integrated from a needle of zero size to one of semi-major and semi-minor axes a and c. The cases of $\mathrm{E}_{\mathrm{o}}$ and $\mathrm{H}_{\mathrm{o}}$ parallel to the major axis will be calculated in detail to illustrate
the method used, while the calculations for the other two cases will be merely outlined and the results given.

Consider a needle of semi-major axis a and semi-minor axis c placed in a constant electric field of magnitude $E_{0}$. The potential in the presence of the needle, as found in Eq. 68 will be given by

$$
\begin{equation*}
\phi_{11}=-\mathrm{E}_{\mathrm{o}}\left[\frac{\left(\xi+\mathrm{a}^{2}\right)\left(\mathrm{a}^{2}-\eta\right)}{\left(\mathrm{a}^{2}-\mathrm{c}^{2}\right)}\right]^{1 / 2}\left[1+\mathrm{A}_{\mathrm{E}} \mathrm{~V}\right] \tag{71}
\end{equation*}
$$

where

$$
\begin{gather*}
V=\int_{\xi}^{\infty} \frac{d \xi}{\left(\xi+a^{2}\right)^{3 / 2}\left(\xi+c^{2}\right)}  \tag{72}\\
=\frac{-1}{\left(a^{2}-c^{2}\right)^{3 / 2}}\left[\frac{2 \sqrt{a^{2}-c^{2}}}{\sqrt{\xi+a^{2}}}+\ln \frac{\sqrt{\xi+a^{2}}-\sqrt{a^{2}-c^{2}}}{\sqrt{\xi+a^{2}}+\sqrt{a^{2}-c^{2}}}\right] \tag{73}
\end{gather*}
$$

The fields, $E_{\xi}, E_{\eta}$, and $E_{\psi}$ are found from the gradient of this potential, and the value of $A$ from the boundary conditions that the tangential components of $E, E_{\eta}{ }^{\text {and }} \mathrm{E}_{\psi}$, are zero on the surface of the needle, $\xi=0$.

$$
\begin{equation*}
\mathrm{E}_{\eta}=-\frac{1}{\mathrm{~h}_{2}} \frac{\partial \phi_{11}}{\partial \eta}=\mathrm{E}_{\mathrm{o}}\left[\frac{\left(\xi+\mathrm{a}^{2}\right)\left(\eta-\mathrm{c}^{2}\right)}{(\xi+\eta)\left(\mathrm{a}^{2}-\mathrm{c}^{2}\right)}\right]^{1 / 2}\left[1+\mathrm{A}_{\mathrm{E}} \mathrm{~V}\right] \tag{74}
\end{equation*}
$$

If $E_{\eta}=0$, at $\xi=0$ for all values of $\eta$ and $\psi$, then

$$
\begin{equation*}
A_{E}=\frac{-1}{\nabla_{o}} \tag{75}
\end{equation*}
$$

where

$$
\begin{equation*}
V_{0}=\frac{-1}{\left(a^{2}-c^{2}\right)^{3 / 2}}\left[\frac{2 \sqrt{a^{2}-c^{2}}}{a}+\ln \frac{a-\sqrt{a^{2}-c^{2}}}{a+\sqrt{a^{2}-c^{2}}}\right] \tag{76}
\end{equation*}
$$

which makes

$$
\begin{align*}
& \mathrm{E}_{\eta}=\mathrm{E}_{\mathrm{o}}\left[\frac{\left(\xi+\mathrm{a}^{2}\right)\left(\eta-c^{2}\right)}{(\xi+\eta)\left(\mathrm{a}^{2}-\mathrm{c}^{2}\right)}\right]^{1 / 2}\left[1-\frac{\mathrm{V}}{\mathrm{~V}_{o}}\right]  \tag{77}\\
& \mathrm{E}_{\xi}=\mathrm{E}_{o}\left[\frac{\left(\xi+\mathrm{c}^{2}\right)\left(\mathrm{a}^{2}-\eta\right)}{(\xi+\eta)\left(a^{2}-c^{2}\right)}\right]^{1 / 2}\left[1-\frac{V}{V_{o}}+\frac{2}{\left.V_{o} \sqrt{\xi+a^{2}\left(\xi+c^{2}\right.}\right)}\right]  \tag{78}\\
& E_{\psi}=0 \tag{79}
\end{align*}
$$

The next step is to apply the perturbation formula to an expansion of the needle, and integrate these fields over the expansion. The second warning in applying the perturbation formula must be thoroughly considered here. The perturbation formula holds only for an expansion of the perturbing volume to a similar volume. In the case of the sphere, this was automatically accounted for, since any two spheres are similar. Here, the needle can be expanded in many ways. The surface, $\xi=0$, represents the surface of the needle, and it might then be thought that an expansion from $\xi=0$ to $d \xi$, or from one ellipsoid to a confocal ellipsoid infinitesimally larger would be wanted. But the perturbation formula would not hold in this case because the needle would not be expanding all dimensions equally into the field, and the resulting confocal, slightly larger ellipsoid would not be similar to the original. What is then needed is the equation of a circular needle of elliptic cross section similar to the original, but infinitesimally larger. The equation of the basic needle is

$$
\begin{equation*}
\frac{x^{2}}{a^{2}}+\frac{r^{2}}{c^{2}}=1 \tag{80}
\end{equation*}
$$

and the equation for a similar needle is

$$
\begin{equation*}
\frac{x^{2}}{a^{2}}+\frac{r^{2}}{c^{2}}=a^{2} \quad \text { where } \quad a^{2}=\frac{a^{\prime^{2}}}{a^{2}}=\frac{c^{\prime^{2}}}{c^{2}} \tag{81}
\end{equation*}
$$

Putting the values of $\mathrm{x}^{2}$ and $\mathrm{r}^{2}$ found in Eqs. 53, 54 and 55 into the expression for the similar needle, and solving for the value of $\xi_{o}$ denoting the surface of the similar needle

$$
\begin{equation*}
\xi_{0}=\frac{a^{2} c^{2}\left(a^{2}-1\right)}{\eta} \tag{82}
\end{equation*}
$$

When the similar needle is only infinitesimally greater than the basic one

$$
\begin{equation*}
a=\frac{a^{\prime}}{a}=\frac{a+d a}{a}=\frac{c^{\prime}}{c}=\frac{c+d c}{c}=1+d a \tag{83}
\end{equation*}
$$

and

$$
\begin{equation*}
d \xi_{o}=\frac{2 \mathrm{a}^{2} \mathrm{c}^{2} \mathrm{da}}{\eta} \tag{84}
\end{equation*}
$$

This value will then be the upper limit of integration for the application of the perturbation formula to the expansion of the basic ellipsoid. The volume increment is given by

$$
\begin{align*}
\mathrm{dv} & =\mathrm{h}_{1} \mathrm{~h}_{2} \mathrm{~h}_{3} \mathrm{~d} \xi \mathrm{~d} \eta \mathrm{~d} \psi  \tag{85}\\
& =\frac{1}{2} \frac{(\xi+\eta) \mathrm{d} \xi \mathrm{~d} \eta \mathrm{~d} \psi}{\sqrt{\xi+\mathrm{a}^{2}} \sqrt{\mathrm{a}^{2}-\eta} \sqrt{\mathrm{a}^{2}-\mathrm{c}^{2}}} \tag{86}
\end{align*}
$$

With this information, the fields $E_{\xi}, E_{\eta}$ and $E_{\psi}$ found in $E q s .77,78$ and 79 can now be
integrated over the infinitesimal expansion of the needle. This results in

$$
\begin{equation*}
\int_{V}^{V+d V} E^{2} d v=\frac{1}{2} \int_{0}^{2 \pi} d \psi \int_{c^{2}}^{a^{2}} d \eta \int_{0}^{d \xi} o d \xi\left(E_{\xi}^{2}+E_{\eta}^{2}+E_{\psi}^{2}\right)\left[\frac{(\xi+\eta)}{\sqrt{\xi+a^{2}} \sqrt{a^{2}-\eta} \sqrt{a^{2}-c^{2}}}\right] \tag{87}
\end{equation*}
$$

However, $E_{\psi}=0$, and $E_{\eta}$ and $E_{\xi}$ both contain terms in $\left[1-V / V_{o}\right]$, which, integrated between the limits on $\xi$, are equal to zero. Carrying out the remaining integrations over $\xi$ gives

$$
\begin{equation*}
\int_{V}^{V+d V} E^{2} d v=\frac{\pi E_{o}^{2}}{\left(a^{2}-c^{2}\right)^{3 / 2}} \frac{c^{2}}{a} \frac{4}{V_{o}^{2} a^{2} c^{4}} \int_{c^{2}}^{a^{2}} \sqrt{a^{2}-\eta} d \eta d \xi \tag{88}
\end{equation*}
$$

Substituting the value of $d \xi_{o}$ given in Eq. 84, and performing the integration over $\eta$ gives as a result

$$
\begin{equation*}
\int_{V}^{V+d V} E^{2} d v=\frac{-8 \pi E_{o}^{2}}{\left(a^{2}-c^{2}\right)^{3 / 2}} \frac{d a}{V_{o}^{2}}\left[\frac{2 \sqrt{a^{2}-c^{2}}}{a}+\ln \frac{a-\sqrt{a^{2}-c^{2}}}{a+\sqrt{a^{2}-c^{2}}}\right] \tag{89}
\end{equation*}
$$

and substituting in the value of $\mathrm{V}_{\mathrm{o}}$ from Eq. 76 gives

$$
\begin{equation*}
\int_{V}^{V+d V} E^{2} d v=\frac{8 \pi E_{o}^{2}\left(a^{2}-c^{2}\right)^{3 / 2} d a}{\left[\ln \frac{a+\sqrt{a^{2}-c^{2}}}{a-\sqrt{a^{2}-c^{2}}}-\frac{2 \sqrt{a^{2}-c^{2}}}{a}\right]} \tag{90}
\end{equation*}
$$

This is the result of the integration over the infinitesimal expansion of the volume of the needle. To find the total perturbation caused by the introduction of the needle, the above expression must be integrated from $a=c=0$ to a and $c$. For similar needles, $\beta=c / a$ is a constant and $d a=d a / a=d c / c$. Substituting and integrating, this will give the total perturbation caused by the introduction of the needle into the field

$$
\begin{equation*}
\frac{\omega_{0}^{2}-\omega^{2}}{\omega_{0}^{2}}=\frac{8 \pi E_{0}^{2}\left(1-\beta^{2}\right)^{3 / 2} \int_{0}^{a} a^{2} d a}{\left[\ln \frac{1+\sqrt{1-\beta^{2}}}{1-\sqrt{1-\beta^{2}}}-2 \sqrt{1-\beta^{2}}\right]}=\frac{2 E_{0}^{2}\left(1-\beta^{2}\right)^{3 / 2} \frac{4 \pi}{3} a^{3}}{\left[\ln \frac{1+\sqrt{1-\beta^{2}}}{1-\sqrt{1-\beta^{2}}}-2 \sqrt{1-\beta^{2}}\right]} \tag{91}
\end{equation*}
$$

This is an exact expression for the perturbation effect produced by the introduction of a needle of semi-major axis a and semi-minor axis $c$ with the applied field parallel to a. It holds for all values of $\beta$, and in particular, as $\beta$ approaches unity, a approaches $c$,
the needle approaches a sphere, and the expression reduces to

$$
\begin{equation*}
\frac{\omega_{o}^{2}-\omega^{2}}{\omega_{o}^{2}}=3 E_{o}^{2} \frac{4 \pi}{3} a^{3} \tag{93}
\end{equation*}
$$

which is the expression found in Eq. 44 for the perturbation of a metallic sphere.
Figure 1 contains the above relation normalized to the perturbation effect of a sphere of radius $r_{o}=a$, and plotted as $F_{1}(\beta)$ against $\beta$ from zero to one. Figure 2 contains the same function on an expanded scale for small $\beta$.


Fig. 1


Fig. 2

The same method is followed exactly to find the perturbation effect produced by a needle with its major axis along the applied $H$ field The gradient of the same $\phi_{1 l}$ is taken to obtain the fields, but in this case $A_{H}$ is determined by the boundary condition on $H$, that is, that the component of $H$ perpendicular to the surface of the needle, $H_{\xi}$, is equal to zero on the surface, $\xi=0$. Performing these operations gives

$$
\begin{align*}
& A_{H}=\frac{a c^{2}}{2-a c^{2} v_{o}}  \tag{94}\\
& H_{\xi}=H_{o}\left[\frac{\left(\xi+c^{2}\right)\left(a^{2}-\eta\right)}{(\xi+\eta)\left(a^{2}-c^{2}\right)}\right]^{1 / 2}\left[1+A_{H} V-\frac{2 A_{H}}{a c^{2}}\right] \tag{95}
\end{align*}
$$

$$
\begin{align*}
& H_{\eta}=H_{o}\left[\frac{\left(\xi+c^{2}\right)\left(\eta-c^{2}\right)}{(\xi+\eta)\left(a^{2}-c^{2}\right)}\right]^{1 / 2}\left[1+A_{H} \mathrm{~V}\right]  \tag{96}\\
& H_{\psi}=0 \tag{97}
\end{align*}
$$

Integrating the fields over the volume between the basic needle and the infinitesimally greater similar needle, and making the substitutions for $A_{H}$ and $V_{o}$, results in

$$
\begin{equation*}
\int_{V}^{V+d V} H^{2} d v=\frac{-8 \pi H_{o}^{2}\left(a^{2}-c^{2}\right)^{3 / 2} d a}{\left[\frac{2 a \sqrt{a^{2}-c^{2}}}{c^{2}}-\ln \frac{a+\sqrt{a^{2}-c^{2}}}{a-\sqrt{a^{2}-c^{2}}}\right]} \tag{98}
\end{equation*}
$$

If, as before, $\beta=c / a$ and $d a=d a / a$, the final result for the total perturbation produced by the introduction of a needle of semi-major axis a and semi-minor axis c into a magnetic field directed along a is

$$
\begin{equation*}
\frac{\omega_{o}^{2}-\omega^{2}}{\omega_{o}^{2}}=-\frac{H_{o}^{2}\left(1-\beta^{2}\right)^{3 / 2} \frac{4 \pi}{3} a^{3}}{\left[\frac{\sqrt{1-\beta^{2}}}{\beta^{2}}-\frac{1}{2} \ln \frac{1+\sqrt{1-\beta^{2}}}{1-\sqrt{1-\beta^{2}}}\right]} \tag{99}
\end{equation*}
$$

This function is plotted as $F_{2}(\beta)$ in Figs. 1 and 2, after being normalized to the perturbation effect of a sphere in a magnetic field, and the figures show the agreement with the sphere when $\beta=1$.

The two cases of the applied fields perpendicular to the major axis, that is, applied along c, can be calculated with the same procedure. For this case, $\phi_{\perp}$ must be used, and, as found in Eq. 70 it can be written as

$$
\begin{equation*}
\phi_{\perp}=-C_{0}\left[\frac{\left(\xi+c^{2}\right)\left(\eta-c^{2}\right)}{\left(a^{2}-c^{2}\right)}\right]^{1 / 2} \sin \psi[1+A U] \tag{100}
\end{equation*}
$$

where

$$
\begin{equation*}
U=\int_{\xi}^{\infty} \frac{d \xi}{\left(\xi+c^{2}\right)^{2} \sqrt{\xi+a^{2}}}=\frac{1}{\left(a^{2}-c^{2}\right)^{3 / 2}}\left[\frac{\sqrt{\xi+a^{2}} \sqrt{a^{2}-c^{2}}}{\left(\xi+c^{2}\right)}-\frac{1}{2} \ln \frac{\sqrt{\xi+a^{2}}+\sqrt{a^{2}-c^{2}}}{\sqrt{\xi+a^{2}}-\sqrt{a^{2}-c^{2}}}\right] \tag{101}
\end{equation*}
$$

and

$$
\begin{equation*}
U_{o}=\frac{1}{\left(a^{2}-c^{2}\right)^{3 / 2}}\left[\frac{a \sqrt{a^{2}-c^{2}}}{c^{2}}-\frac{1}{2} \ln \frac{a+\sqrt{a^{2}-c^{2}}}{a-\sqrt{a^{2}-c^{2}}}\right] \tag{102}
\end{equation*}
$$

For the applied E field, $\mathrm{A}_{\mathrm{E}}$ is determined from the boundary condition that $\mathrm{E}_{\psi}=\mathrm{E}_{\boldsymbol{\eta}}=0$
at $\xi=0$, or that

$$
\begin{equation*}
A_{E}=\frac{-1}{U_{0}} \tag{103}
\end{equation*}
$$

and the gradient of $\phi_{\perp}$ then gives

$$
\begin{align*}
& E_{\xi}=E_{o}\left[\frac{\left(\xi+a^{2}\right)\left(\eta-c^{2}\right)}{(\xi+\eta)\left(a^{2}-c^{2}\right)}\right]^{1 / 2} \sin \psi\left[1-\frac{U}{U_{0}}-\frac{2}{U_{0}\left(\xi+c^{2}\right) \sqrt{\xi+a^{2}}}\right]  \tag{104}\\
& E_{\eta}=E_{o}\left[\frac{\left(\xi+a^{2}\right)\left(a^{2}-\eta\right)}{(\xi+\eta)\left(a^{2}-c^{2}\right)}\right]^{1 / 2} \sin \psi\left[1-\frac{U}{U_{0}}\right]  \tag{105}\\
& E_{\psi}=E_{0} \cos \psi\left[1-\frac{U}{U_{0}}\right] \tag{106}
\end{align*}
$$

Integrating the fields between the two similar needles and substituting in the values of $U_{o}$ and $\beta$ gives

$$
\begin{equation*}
\int_{V}^{V+d V} E^{2} d v=\frac{8 \pi E_{o}^{2}\left(1-\beta^{2}\right)^{3 / 2} a^{2} d a}{\left[\frac{\sqrt{1-\beta^{2}}}{\beta^{2}}-\frac{1}{2} \ln \frac{1+\sqrt{1-\beta^{2}}}{1-\sqrt{1-\beta^{2}}}\right]} \tag{107}
\end{equation*}
$$

and integrating from zero to $a$, gives the final result that

$$
\begin{equation*}
\frac{\omega_{o}^{2}-\omega^{2}}{\omega_{0}^{2}}=\frac{2 E_{o}^{2}\left(1-\beta^{2}\right)^{3 / 2} \frac{4 \pi}{3} a^{3}}{\left[\frac{\sqrt{1-\beta^{2}}}{\beta^{2}}-\frac{1}{2} \ln \frac{1+\sqrt{1-\beta^{2}}}{1-\sqrt{1-\beta^{2}}}\right]} \tag{108}
\end{equation*}
$$

which, normalized to a sphere in the electric field is plotted as $\mathrm{F}_{3}(\beta)$ on Figs. 1 and 2.
The result for the applied H field perpendicular to the major axis a of the needle is given by

$$
\begin{equation*}
\frac{\omega_{o}^{2}-\omega^{2}}{\omega_{o}^{2}}=-\frac{2 H_{o}^{2}\left(1-\beta^{2}\right)^{3 / 2} \frac{4 \pi}{3} a^{3}}{\left[\frac{\left(1-2 \beta^{2}\right)}{\beta^{2}} \sqrt{1-\beta^{2}}+\frac{1}{2} \ln \frac{1+\sqrt{1-\beta^{2}}}{1-\sqrt{1-\beta^{2}}}\right]} \tag{109}
\end{equation*}
$$

This result is also plotted in Figs. 1 and 2.
In conclusion, the effects of the four possible orientations of a needle are listed below as well as the functions which are plotted in Figs. 1 and 2.
$E_{o}$ parallel to a

$$
\begin{align*}
& \frac{\omega_{0}^{2}-\omega^{2}}{\omega_{0}^{2}}=\frac{E_{0}^{2}\left(1-\beta^{2}\right)^{3 / 2} \frac{4 \pi}{3} a^{3}}{\left[\frac{1}{2} \ln \frac{1+\sqrt{1-\beta^{2}}}{1-\sqrt{1-\beta^{2}}}-\sqrt{1-\beta^{2}}\right]}  \tag{110}\\
& F_{1}(\beta)=\frac{\frac{1}{3}\left(1-\beta^{2}\right)^{3 / 2}}{\left[\frac{1}{2} \ln \frac{1+\sqrt{1-\beta^{2}}}{1-\sqrt{1-\beta^{2}}}-\sqrt{1-\beta^{2}}\right]} \tag{111}
\end{align*}
$$

$E_{o}$ perpendicular to a

$$
\begin{align*}
& \frac{\omega_{o}^{2}-\omega^{2}}{\omega_{o}^{2}}=\frac{2 E_{o}^{2}\left(1-\beta^{2}\right)^{3 / 2} \frac{4 \pi}{3} a^{3}}{\left[\frac{\sqrt{1-\beta^{2}}}{\beta^{2}}-\frac{1}{2} \ln \frac{1+\sqrt{1-\beta^{2}}}{1-\sqrt{1-\beta^{2}}}\right]}  \tag{112}\\
& F_{2}(\beta)=\frac{\frac{2}{3}\left(1-\beta^{2}\right)^{3 / 2}}{\left[\frac{\sqrt{1-\beta^{2}}}{\beta^{2}}-\frac{1}{2} \ln \frac{1+\sqrt{1-\beta^{2}}}{1-\sqrt{1-\beta^{2}}}\right]} \tag{113}
\end{align*}
$$

$\mathrm{H}_{\mathrm{o}}$ parallel to a

$$
\begin{align*}
& \frac{\omega_{0}^{2}-\omega^{2}}{\omega_{0}^{2}}=\frac{-H_{o}^{2}\left(1-\beta^{2}\right)^{3 / 2} \frac{4 \pi}{3} a^{3}}{\left[\frac{\sqrt{1-\beta^{2}}}{\beta^{2}}-\frac{1}{2} \ln \frac{1+\sqrt{1-\beta^{2}}}{1-\sqrt{1-\beta^{2}}}\right]}  \tag{114}\\
& F_{3}(\beta)=\frac{\frac{2}{3}\left(1-\beta^{2}\right)^{3 / 2}}{\left[\frac{\sqrt{1-\beta^{2}}}{\beta^{2}}-\frac{1}{2} \ln \frac{1+\sqrt{1-\beta^{2}}}{1-\sqrt{1-\beta^{2}}}\right]} \tag{115}
\end{align*}
$$

$\mathrm{H}_{\mathrm{o}}$ perpendicular to a

$$
\begin{align*}
& \frac{\omega_{0}^{2}-\omega^{2}}{\omega_{0}^{2}}=-\frac{2 \mathrm{H}_{\mathrm{o}}^{2}\left(1-\beta^{2}\right)^{3 / 2} \frac{4 \pi}{3} \mathrm{a}^{3}}{\left[\frac{1-2 \beta^{2}}{\beta^{2}} \sqrt{1-\beta^{2}}+\frac{1}{2} \ln \frac{1+\sqrt{1-\beta^{2}}}{1-\sqrt{1-\beta^{2}}}\right]}  \tag{116}\\
& F_{4}(\beta)=\frac{\frac{4}{3}\left(1-\beta^{2}\right)^{3 / 2}}{\left[\frac{1-2 \beta^{2}}{\beta^{2}} \sqrt{1-\beta^{2}}+\frac{1}{2} \ln \frac{1+\sqrt{1-\beta^{2}}}{1-\sqrt{1-\beta^{2}}}\right]} \tag{117}
\end{align*}
$$

The $F(\beta)$ functions are the ratios of the frequency shift caused by a needle of semi-major axis a and semi-minor axis $\beta$ a to the frequency shift of a sphere of radius a placed in the same field. The method of applying these results is straightforward. Knowing the dimensions of the needle and its orientation with respect to the field, the calculations of the perturbation effect for a sphere of radius equal to the a of the needle multiplied by the value of $F(\beta)$ for the orientation gives the desired relation between the shape of the perturbation, its size, and the field strength.

To be more specific, consider a cavity oscillating in a $\mathrm{TM}_{011}$ mode. At any general point there will be three components of the fields: $H_{o}$ perpendicular to the axis, $E_{o r}$ perpendicular to the axis, and $\mathrm{E}_{\mathrm{oz}}$ parallel to the axis of the cavity. If three frequency shifts are now measured, the first caused by a ball of radius a; the second, by a needle of semi-major axis a directed perpendicular to the axis of the cavity and of semi-minor axis $\beta$ a; and the third, by the same needle with major axis parallel to the axis of the cavity; calling these frequency shifts $\Delta f_{1}, \Delta f_{2}$, and $\Delta f_{3}$ and realizing that for small frequency shifts

$$
\begin{equation*}
\frac{\omega_{\mathrm{o}}^{2}-\omega^{2}}{\omega_{\mathrm{o}}^{2}} \sim \frac{2 \Delta \mathrm{f}}{\mathrm{f}_{\mathrm{o}}} \tag{118}
\end{equation*}
$$

the following equations can be written for the field components

$$
\begin{gather*}
E_{o r}^{2}+E_{o z}^{2}-\frac{1}{2} H_{o}^{2}=\frac{2 \Delta f_{1}}{f_{o} \frac{4 \pi}{3} a^{3}}  \tag{119}\\
F_{1}(\beta) E_{o r}^{2}+F_{2}(\beta) E_{o z}^{2}-\frac{1}{2} F_{4}(\beta) H_{o}^{2}=\frac{2 \Delta f_{2}}{f_{o} \frac{4 \pi}{3} a^{3}}  \tag{120}\\
F_{2}(\beta) E_{o r}^{2}+F_{1}(\beta) E_{o z}^{2}-\frac{1}{2} F_{4}(\beta) H_{o}^{2}=\frac{2 \Delta f_{3}}{f_{o} \frac{4 \pi}{3} a^{3}} \tag{121}
\end{gather*}
$$

Thus there are three equations and three unknowns, and an exact solution for the fields
is possible by the simultaneous solutions of the above equations. This procedure is extremely tedious to carry out and very good approximate solutions can be found by proper choice of $\beta$. This will be taken up in detail in the last two chapters. Obviously for cavities oscillating in a $\mathrm{TE}_{011}$ mode, there will be three equations relating the two components of $\mathrm{H}_{\mathrm{o}}$ with the one component of $\mathrm{E}_{\mathrm{o}}$.

## VI. PERTURBATION EFFECTS OF CIRCULAR METALLIC DISKS

This chapter will outline the derivation and give the final results of the perturbation effects of circular metallic disks, formed by revolving an ellipse around its minor axis. The procedure to be followed in finding the potential and in applying the perturbation formula is exactly the same as in the previous chapter and any steps omitted here may be found by reference to it.

The coordinate system is defined by the following equations

$$
\begin{equation*}
\frac{r^{2}}{\xi+a^{2}}+\frac{z^{2}}{\xi+c^{2}}=1 \quad \frac{r^{2}}{a^{2}-\eta}-\frac{z^{2}}{\eta-c^{2}}=1 \tag{122}
\end{equation*}
$$

from which

$$
\begin{align*}
& x=\left[\frac{\left(\xi+a^{2}\right)\left(a^{2}-\eta\right)}{\left(a^{2}-c^{2}\right)}\right]^{1 / 2} \sin \psi  \tag{123}\\
& y=\left[\frac{\left(\xi+a^{2}\right)\left(a^{2}-\eta\right)}{\left(a^{2}-c^{2}\right)}\right]^{1 / 2} \cos \psi  \tag{124}\\
& z=\left[\frac{\left(\xi+c^{2}\right)\left(\eta-c^{2}\right)}{\left(a^{2}-c^{2}\right)}\right]^{1 / 2} \tag{125}
\end{align*}
$$

and

$$
\begin{align*}
& h_{1}=\frac{1}{2}\left[\frac{(\xi+\eta)}{\left(\xi+a^{2}\right)\left(\xi+c^{2}\right)}\right]^{1 / 2}  \tag{126}\\
& h_{2}=\frac{1}{2}\left[\frac{(\xi+\eta)}{\left(a^{2}-\eta\right)\left(\eta-c^{2}\right)}\right]^{1 / 2}  \tag{127}\\
& h_{3}=\left[\frac{\left(\xi+a^{2}\right)\left(a^{2}-\eta\right)}{\left(a^{2}-c^{2}\right)}\right]^{1 / 2} \tag{128}
\end{align*}
$$

By the procedure used previously, the potentials for the applied field parallel to the semi-major axis a is given by

$$
\begin{equation*}
\phi_{11}=-C_{0}\left[\frac{\left(\xi+a^{2}\right)\left(a^{2}-\eta\right)}{\left(a^{2}-c^{2}\right)}\right]^{1 / 2} \sin \psi[1+A v] \tag{129}
\end{equation*}
$$

where

$$
\begin{align*}
V & =\int_{\xi}^{\infty} \frac{d \xi}{\left(\xi+a^{2}\right)^{2} \sqrt{\xi+c^{2}}}  \tag{130}\\
& =\frac{-1}{\left(a^{2}-c^{2}\right)^{3 / 2}}\left[\frac{\sqrt{\xi+c^{2}} \sqrt{a^{2}-c^{2}}}{\left(\xi+a^{2}\right)}+\tan ^{-1} \frac{\sqrt{\xi+c^{2}}}{\sqrt{a^{2}-c^{2}}}-\frac{\pi}{2}\right] \tag{131}
\end{align*}
$$

and

$$
\begin{equation*}
V_{0}=\frac{-1}{\left(a^{2}-c^{2}\right)^{3 / 2}}\left[\frac{c}{a^{2}} \sqrt{a^{2}-c^{2}}+\tan ^{-1} \frac{c}{\sqrt{a^{2}-c^{2}}}-\frac{\pi}{2}\right] \tag{132}
\end{equation*}
$$

The potential for the applied field perpendicular to the major axis of the disk is similarly given by

$$
\begin{equation*}
\phi_{\perp}=-C_{O}\left[\frac{\left(\xi+c^{2}\right)\left(\eta-c^{2}\right)}{\left(a^{2}-c^{2}\right)}\right]^{1 / 2}[1+A U] \tag{133}
\end{equation*}
$$

where

$$
\begin{align*}
U & =\int_{\xi}^{\infty} \frac{d \xi}{\left(\xi+a^{2}\right)\left(\xi+c^{2}\right)^{3 / 2}}  \tag{134}\\
& =\frac{2}{\left(a^{2}-c^{2}\right)^{3 / 2}}\left[\frac{\sqrt{a^{2}-c^{2}}}{\sqrt{\xi+c^{2}}}+\tan ^{-1} \frac{\sqrt{\xi+c^{2}}}{\sqrt{a^{2}-c^{2}}}-\frac{\pi}{2}\right] \tag{135}
\end{align*}
$$

and

$$
\begin{equation*}
\mathrm{U}_{\mathrm{o}}=\frac{2}{\left(\mathrm{a}^{2}-\mathrm{c}^{2}\right)^{3 / 2}}\left[\frac{\sqrt{\mathrm{a}^{2}-c^{2}}}{\mathrm{c}}+\tan ^{-1} \frac{c}{\sqrt{a^{2}-c^{2}}}-\frac{\pi}{2}\right] \tag{136}
\end{equation*}
$$

Here, as in the previous chapter, the A's in each case are determined by the boundary conditions that tangential $E$ and normal $H$ must be zero on the surface of the disk, $\xi=0$.

For $E_{o}$ applied parallel to the semi-major axis a, the gradient of $\phi_{11}$ determines the fields, and the only component of $E$ which does not go to zero when the integration is performed between zero and $d \xi{ }_{o}$ is $E_{\xi}$

$$
\begin{equation*}
A_{E}=\frac{-1}{V_{0}} \tag{137}
\end{equation*}
$$

$$
\begin{equation*}
E_{\xi}=\left[\frac{\left(\xi+c^{2}\right)\left(a^{2}-\eta\right)}{(\xi+\eta)\left(a^{2}-c^{2}\right)}\right]^{1 / 2} \sin \psi\left[1-\frac{V}{V_{0}}+\frac{2}{V_{o}\left(\xi+a^{2}\right) \sqrt{\xi+c^{2}}}\right] \tag{138}
\end{equation*}
$$

Integrating the square of this field between the two similar ellipsoids, and substituting $\beta=c / a$ and $d \xi$

$$
\begin{equation*}
\int_{V}^{V+d V} E^{2} d v=\frac{8 \pi E_{o}^{2}\left(1-\beta^{2}\right)^{3 / 2} a^{2} d a}{\left[\frac{\pi}{2}-\tan ^{-1} \frac{\beta}{\sqrt{1-\beta^{2}}}-\beta \sqrt{1-\beta^{2}}\right]} \tag{139}
\end{equation*}
$$

For the total perturbation effect of the disk, the above expression, integrated from zero to a, gives

$$
\begin{equation*}
\frac{\omega_{o}^{2}-\omega^{2}}{\omega_{o}^{2}}=\frac{2 E_{o}^{2}\left(1-\beta^{2}\right)^{3 / 2} \frac{4 \pi}{3} a^{3}}{\left[\frac{\pi}{2}-\tan ^{-1} \frac{\beta}{\sqrt{1-\beta^{2}}}-\beta \sqrt{1-\beta^{2}}\right]} \tag{140}
\end{equation*}
$$

Normalization of this to the effect of a sphere of radius a in the same field yields

$$
\begin{equation*}
F_{5}(\beta)=\frac{\frac{2}{3}\left(1-\beta^{2}\right)^{3 / 2}}{\left[\frac{\pi}{2}-\tan ^{-1} \frac{\beta}{\sqrt{1-\beta^{2}}}-\beta \sqrt{1-\beta^{2}}\right]} \tag{141}
\end{equation*}
$$

Similar procedures for the other three field orientations with similar normalizations to spheres in the same field give
$\mathrm{E}_{\mathrm{o}}$ perpendicular to a

$$
\begin{align*}
& \frac{\omega_{0}^{2}-\omega^{2}}{\omega_{0}^{2}}=\frac{E_{o}^{2}\left(1-\beta^{2}\right)^{3 / 2} \frac{4 \pi}{3} a^{3}}{\left[\frac{\sqrt{1-\beta^{2}}}{\beta}+\tan ^{-1} \frac{\beta}{\sqrt{1-\beta^{2}}}-\frac{\pi}{2}\right]}  \tag{142}\\
& F_{6}(\beta)=\frac{\frac{1}{3}\left(1-\beta^{2}\right)^{3 / 2}}{\left[\frac{\sqrt{1-\beta^{2}}}{\beta}+\tan ^{-1} \frac{\beta}{\sqrt{1-\beta^{2}}}-\frac{\pi}{2}\right]} \tag{143}
\end{align*}
$$

$\mathrm{H}_{\mathrm{o}}$ perpendicular to a

$$
\begin{align*}
& \frac{\omega_{o}^{2}-\omega^{2}}{\omega_{0}^{2}}=\frac{-H_{o}^{2}\left(1-\beta^{2}\right)^{3 / 2} \frac{4 \pi}{3} a^{3}}{\left[\frac{\pi}{2}-\tan ^{-1} \frac{\beta}{\sqrt{1-\beta^{2}}}-\beta \sqrt{1-\beta^{2}}\right]}  \tag{144}\\
& F_{7}(\beta)=\frac{\frac{2}{3}\left(1-\beta^{2}\right)^{3 / 2}}{\left[\frac{\pi}{2}-\tan ^{-1} \frac{\beta}{\sqrt{1-\beta^{2}}}-\beta \sqrt{1-\beta^{2}}\right]} \tag{145}
\end{align*}
$$

$\mathrm{H}_{\mathrm{o}}$ parallel to a


Fig. 3

$$
\begin{equation*}
\frac{\omega_{0}^{2}-\omega^{2}}{\omega_{0}^{2}}=\frac{-2 \mathrm{H}_{\mathrm{o}}^{2}\left(1-\beta^{2}\right)^{3 / 2} \frac{4 \pi}{3} \mathrm{a}^{3}}{\left[\tan ^{-1} \frac{\beta}{\sqrt{1-\beta^{2}}}+\frac{\left(2-\beta^{2}\right)}{\beta} \sqrt{1-\beta^{2}}-\frac{\pi}{2}\right]} \tag{146}
\end{equation*}
$$

$$
\begin{equation*}
F_{8}(\beta)=\frac{\frac{4}{3}\left(1-\beta^{2}\right)^{3 / 2}}{\left[\tan ^{-1} \frac{\beta}{\sqrt{1-\beta^{2}}}+\frac{\left(2-\beta^{2}\right)}{\beta} \sqrt{1-\beta^{2}}-\frac{\pi}{2}\right]} \tag{147}
\end{equation*}
$$

The four functions of $\beta$ given above are plotted in Fig. 3. There are several features of interest which can be readily seen. In the first place, for disks oriented with their faces, or semi-major axes, parallel to $\mathrm{E}_{\mathrm{O}}$ and perpendicular to $\mathrm{H}_{\mathrm{O}}$, the field function measured is ( $\mathrm{E}_{\mathrm{o}}^{2}-\frac{1}{2} \mathrm{H}_{\mathrm{o}}^{2}$ ), which is the same as for a sphere. In this case, however, the disk does give the direction of the fields. In the limit, as $\beta$ approches zero, there is no perturbation by a disk with its face perpendicular to $\mathrm{E}_{\mathrm{o}}$ and parallel to $\mathrm{H}_{\mathrm{o}}$, but there is an effect for the opposite orientations in each field. This is certainly logical since the boundary conditions demand that there be no tangential E and no normal H on the surface of a perfect conductor. As all four functions approach unity, they give the same frequency shift as a sphere, since a and c are then equal. A comparison of Figs. 1 and 3 gives the essential differences between the usefulness of needles and disks in measuring the fields. $F_{1}(\beta)$ for low values of $\beta$ is
much greater than the other three functions, and this permits a single field component to give the major perturbation effect. For disks there are two functions, $F_{5}(\beta)$ and $F_{7}(\beta)$, which are greater than the others. This results in the fact that the re are only certain field arrangements which permit the disk to measure a single component. For example, for a cavity oscillating in a TM mode with the $H_{o}$ field only in the $\theta$ direction, an infinitely thin disk with its face parallel to $H_{o}$ and parallel to the r-axis will measure only $\mathrm{E}_{\mathrm{or}}$, and with its face parallel to $\mathrm{H}_{\mathrm{o}}$ and parallel to the z -axis will measure only $\mathrm{E}_{\mathrm{oz}}$, but there is no orientation which will measure $\mathrm{H}_{\mathrm{o}}$ alone. In the general TM mode, with both $\mathrm{H}_{\mathrm{o}}$ and $\mathrm{H}_{\mathrm{or}}$, and in the TE modes, there are no orientations which will measure a single component.

## VII. EXPERIMENTAL VERIFICATION OF PERTURBATION EFFECTS

The relations developed in the preceding chapters between the size and shape of the perturbing object, the frequency shift, and the field strength have been experimentally verified. The method of measuring the frequency shifts and the results of these experiments will be discussed in the present chapter.

The ease with which small frequency shifts can be measured in a high $Q$ cavity resonator is one of the major justifications for developing this perturbation method of determining field strengths. At first glance the problem of measuring a change in frequency of a tenth to a hundredth of a megacycle from a resonant frequency of three thousand megacycles might seem formidable, but it need not be so. The method employed here utilizes the stabilized microwave oscillator circuit developed by R. V. Pound (9). This circuit stabilizes a McNally klystron to a resonant cavity by means of MagicTee discriminator feedback loops to the klystron reflector plate. The klystron is thus electronically tuned to the resonant frequency of the cavity, and, when the circuit is properly designed, it can follow considerable shifts in the cavity frequency. The degree of stabilization obtained depends upon the $Q$ of the cavity used, as well as the gain and stability of the feedback loops. To measure the frequency shifts produced by the perturbing volume, two Pound oscillators were used, and a beat signal obtained in the lowfrequency band. One oscillator was stabilized with the cavity to be measured, and the other with a high $Q$ tunable wavemeter, which gave control over the beat frequency obtained with no perturbation. This beat frequency was fed into an ordinary AM radio receiver, and by introducing the perturbing volumes into the cavity whose fields were to be measured, the changes in resonant frequency were measured with the receiver. The stabilizing cavities and circuits were reasonably well insulated, both mechanically and thermally, and the beat signal obtained was found to vary less than a kilocycle over a five- to ten-minute period. By using a well calibrated receiver, with an expanded frequency dial, it was relatively simple to measure frequency shifts of one kilocycle in a beat frequency of approximately five megacycles. This permitted the use of quite small (compared to a wavelength) perturbing volumes and eliminated much of the difficulty of the fields' varying rapidly over the perturbation.

For checking the perturbation theory developed previously, the $\mathrm{TM}_{010}$ cavity described in Chapter II was used as one of the stabilizing cavities, and the frequency shifts measured for various shapes and sizes of perturbations. From Chapter II, Eq. 20,

$$
\begin{gather*}
\mathrm{E}_{\mathrm{a}}^{2}=\frac{\mathrm{J}_{\mathrm{o}}^{2}(\mathrm{kr})}{\pi \mathrm{L} R_{o}^{2} \mathrm{~J}_{1}^{2}\left(k R_{o}\right)} \quad \mathrm{H}_{\mathrm{a}}^{2}=\frac{\mathrm{J}_{1}^{2}(\mathrm{kr})}{\pi \mathrm{L} R_{o}^{2} J_{1}^{2}\left(k R_{o}\right)}  \tag{148}\\
\mathrm{k}=\frac{2 \pi}{\lambda_{o}}=\frac{2.405}{R_{o}}
\end{gather*}
$$

For the cavity used,

$$
\begin{align*}
\mathrm{L} & =3.000 \mathrm{in} .  \tag{151}\\
\mathrm{R}_{\mathrm{o}} & =1.612 \mathrm{in} .  \tag{152}\\
\mathrm{f}_{\mathrm{o}} & =2798.4 \mathrm{Mc}  \tag{153}\\
\mathrm{E}_{\mathrm{a}}^{2} & =0.1510 \mathrm{~J}_{\mathrm{o}}^{2}(\mathrm{kr}) \mathrm{in} .^{-3}  \tag{154}\\
\mathrm{H}_{\mathrm{a}}^{2} & =0.1510 \mathrm{~J}_{1}^{2}(\mathrm{kr}) \mathrm{in} .^{-3} \tag{155}
\end{align*}
$$

For the following measurements, the perturbing volume was glued with Ambroid cement to ordinary thin silk thread. The frequency shift caused by the thread could not be measured; this indicates the very slight effect it had on the fields. For a sphere in this cylindrical cavity

$$
\begin{equation*}
\mathrm{E}_{\mathrm{a}}^{2}-\frac{1}{2} \mathrm{H}_{\mathrm{a}}^{2}=\frac{\omega_{\mathrm{o}}^{2}-\omega^{2}}{4 \pi \mathrm{r}_{\mathrm{o}}^{2} \omega_{\mathrm{o}}^{2}}=\frac{2 \Delta \mathrm{f}}{4 \pi \mathrm{r}_{\mathrm{o}}^{2} \mathrm{f}_{\mathrm{o}}} \tag{156}
\end{equation*}
$$

Table I-A gives the results for three different spheres first in the $E$ field alone on the axis of the cavity, and then at the maximum of $\left(E_{a}^{2}-\frac{1}{2} H_{a}^{2}\right)$, with the percentage error of each measurement. From the low percentage error in the $E$ field measurements, it is seen that the perturbation theory checks to within a few percent, both the magnitude of the effect, and the cubic variation with sphere radius. For the measurements at the maximum of $\left(E_{a}^{2}-\frac{1}{2} H_{a}^{2}\right)$, the percentage error is somewhat greater and can be accounted for by the difficulty in getting the sphere directly at the maximum and by the perturbation of the fields caused by the hole needed to introduce the spheres.

Table I-B gives the results for the measurements made on disks. Four disks were used, three of very thin metal to check the variation with semi-major axis a and one of $\beta=0.455$ to give some idea of the correlation of theory with variations of $\beta$. Because of the difficulty of aligning the disk in a mixed $E$ and $H$ field, and because disks are not

Table I-A Spheres

| $\mathrm{r}_{0}(\mathrm{in}$. | $\Delta \mathrm{f}(\mathrm{Mc})$ | $\begin{gathered} \mathrm{E}_{\mathrm{a}}^{2}\left(\text { in. }{ }^{-3}\right. \text { ) } \\ \text { (meas.) (calc.) } \end{gathered}$ |  | Percentage of Error | $\Delta \mathrm{f}(\mathrm{Mc})$ | $\begin{aligned} & E_{a}^{2}-\frac{1}{2} H \\ & \text { (meas.) } \end{aligned}$ | $\begin{aligned} & \left(\text { in. }{ }^{-3}\right) \\ & (\text { calc. }) \end{aligned}$ | Percentage of Error |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.125 | 5.09 | 0.149 | 0.1510 | 1.3 | 0.693 | 0.0203 | 0.0214 | 5.1 |
| 0.09375 | 2.14 | 0.148 | 0.1510 | 2.0 | 0.298 | 0.0206 | 0.0214 | 3.3 |
| 0.0625 | 0.645 | 0.150 | 0.1510 | 0.6 | 0.090 | 0.0210 | 0.0214 | 1.9 |

Table I-B Disks

| $a($ in. $)$ | $\beta$ | $\Delta f(\mathrm{Mc})$ | $F_{5}(\beta)$ | $\mathrm{E}_{\mathrm{a}}^{2}$ (in. $^{-3}$ ) <br> (calc.) |  | Percentage <br> of Error |
| :--- | :--- | :--- | :--- | :--- | :---: | :---: |
| 0.122 | 0 | 1.91 | 0.424 | 0.143 | 0.1510 | 5.3 |
| 0.094 | 0 | 0.92 | 0.424 | 0.149 | 0.1510 | 1.3 |
| 0.066 | 0 | 0.330 | 0.424 | 0.154 | 0.1510 | 2.0 |
| 0.121 | 0.48 | 2.52 | 0.690 | 0.118 | 0.1510 | 22.6 |

Table I-C Needles

| a (in.) | $\beta$ | $\Delta f(\mathrm{Mc})$ | $\mathrm{F}_{1}(\beta)$ | $\underset{\text { (meas.) }}{\mathrm{E}_{\mathrm{a}}^{2} \text { (in. }}$ | $\begin{aligned} & -3 \text { ) } \\ & \text { (calc.) } \end{aligned}$ | Percentage of Error |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.212 | 0.052 | 4.30 | 0.125 | 0.206 | 0.1510 | 29.2 |  |
| 0.191 | 0.101 | 3.54 | 0.165 | 0.176 | 0.1510 | 16.9 |  |
| 0.136 | 0.46 | 2.28 | 0.438 | 0.118 | 0.1510 | 22.6 |  |
|  |  |  | $F_{3}(\beta)$ |  |  |  |  |
| 0.212 | 0.052 | 0.059 | 0.00182 | 0.194 | 0.1510 | 28.4 |  |
| 0.136 | 0.46 | 1.19 | 0.168 | 0.160 | 0.1510 | 6.0 |  |
|  |  |  |  | $\mathrm{F}_{4}(\beta)$ | $\begin{aligned} & \mathrm{F}_{2} \mathrm{E}_{\mathrm{a}}^{2}-\frac{1}{2} \mathrm{~F}_{4} \mathrm{H}_{\text {a }}^{2} \text { (in. }^{-3} \text { ) } \\ & \text { (mas. } \text { (alc.) } \end{aligned}$ |  | Percentage of Error |
| 0.212 | 0.052 | 0.0125 | 0.00182 | 0.00358 | 0.000490 | 0.000481 | 1.4 |
| 0.136 | 0.46 | 0.241 | 0.168 | 0.243 | 0.0359 | 0.0368 | 2.5 |

as useful in measuring fields, the checking was done only for the E field along the axis. The expression for the perturbation produced by a disk with its semi-major axis directed along the E field is

$$
\begin{equation*}
\frac{\omega_{0}^{2}-\omega^{2}}{\omega_{0}^{2}}=\frac{4 \pi a^{3} F_{5}(\beta)}{\omega_{0}^{2}} E_{a}^{2} \tag{157}
\end{equation*}
$$

from which

$$
\begin{equation*}
E_{a}^{2}=\frac{2 \Delta f}{4 \pi \mathrm{a}^{3} \mathrm{~F}_{5}(\beta) \mathrm{f}_{\mathrm{o}}} \tag{158}
\end{equation*}
$$

The percentage errors for the thin disks are only slightly larger than for spheres and can easily be accounted for in the difficulty of correctly measuring a. The odd sizes of these thin disks result from the fact that they were punched from thin stock and the clearance between the inner and outer surfaces of the punch left burrs on the disks. The results for the $\beta=0.48$ disk, however, has a large percentage error, and the reason for this is probably the very poor tolerances held in machining. As will be seen in the case of needles, the perturbation theory seems to be quite critical as to the crosssection shape of the disks and needles. This $\beta=0.48$ disk was examined under a microscope and its cross section clearly was not elliptic.

Table I-C gives the results for needles placed in the $\mathrm{E}_{\mathrm{a}}$ field at the center of the cavity and at the value of the maximum of $\left[F_{2}(\beta) E_{a}^{2}-\frac{1}{2} F_{4}(\beta) H_{a}^{2}\right]$. The needles were first aligned with their semi-major axis along the axis of the cavity and hence along the $\mathrm{E}_{\mathrm{a}}$ field. In this case,

$$
\begin{equation*}
E_{a}^{2}=\frac{2 \Delta f}{4 \pi a^{3} F_{1}(\beta) f_{o}} \tag{159}
\end{equation*}
$$

In the second experiment, the needles were oriented with the semi-major axis perpendicular to the axis of the cavity, but still in the center of the cavity, and in this case

$$
\begin{equation*}
E_{a}^{2}=\frac{2 \Delta f}{4 \pi a^{3} F_{3}(\beta) f_{o}} \tag{160}
\end{equation*}
$$

In the third case, they were still oriented as in the second, but were displaced from the center to the maximum negative values of $\Delta f$. Then

$$
\begin{equation*}
F_{3}(\beta) E_{a}^{2}-\frac{1}{2} F_{4}(\beta) H_{a}^{2}=\frac{2 \Delta f}{4 \pi a^{3} f_{o}} \tag{161}
\end{equation*}
$$

From the percentage errors plotted in Table I-C, it is seen that there seems to be rather poor agreement with the theory, but, as in the case of disks, the reason is that the needles were not truly ellipsoids of revolution. Examination under a microscope showed
that the $\beta=0.101$ needle was the closest to an ellipsoid, the $\beta=0.46$ next, and the $\beta=$ 0.052 the farthest, which agrees with the changes in percentage error. The experimental error also shows that the effects of the non-elliptic cross section are least important when the fields are at right angles to the a axis and most important when they are parallel. Because of the machining difficulties, it was decided that it would not be worthwhile to attempt making needles of better cross section, though it should not be impossible to do so.

## VIII. FIELD STRENGTHS OF AN UNKNOWN CAVITY

The application of the perturbation method of field strength measurements has been made to a cavity whose fields cannot be found analytically. The unknown cavity used was a typical section of the linear accelerator under construction at the Massachusetts Institute of Technology. The choice of the cavity was dictated by the fact that any considerations of the output energy of the accelerator must be prefaced by a knowledge of at least the axial distribution and magnitude of the electric field. These fields have been computed (10) by the metho ds mentioned earlier for general cavities, but an experimental verification of these computations was necessary. In short, the accelerator cavity can be considered a typical unknown cavity, while at the same time the results of the measurements are of immediate use.

In measuring any unknown resonant cavity some thought should be devoted to the general characteristics dictated by the shape of the cavity and the field distribution of the most nearly similar simple cavity. In the present case the accelerator cavity is an iris-loaded cylinder. The irises are thin metallic circular rings placed periodically along the axis of the cavity transverse to it. It would seem reasonable to expect this cavity to have both TM and TE modes, with field distributions similar to those of a cylindrical cavity in the region between irises, and to be thoroughly perturbed from these distributions near the iris. Furthermore, the periodicity of the irises should give a periodicity of the field distributions. The accelerator operates in the TM $\pi$ mode ( $180^{\circ}$ phase shift between each iris) and this is the oscillation which will be examined. Since it is periodic with the irises, and since it is a TM mode and hence symmetrical with the $z$-axis, if the fields are measured from a point midway between the irises to one iris, and from the center to the walls, these conditions of symmetry will give the fields over the whole cavity.

For a completely unknown cavity certain preliminary measurements would have to be made to determine these features, and these measurements will be described for completeness. One simple method of identifying modes in a cavity is to feed a frequencymodulated signal into the cavity through a Magic-Tee. The rectified output from one arm of the Magic-Tee can be put on a cathode-ray oscilloscope whose sweep is adjusted to the modulation frequency. This essentially gives an impedance vs. frequency plot, and the resonance curve of the cavity is presented. By having a calibrated wave meter in parallel, the frequencies of the resonances can be measured. The TM and TE modes can
be identified by perturbation methods as follows. For cavities with an axis of cylindrical symmetry, a long thin metallic wire along this axis will violently change the resonant frequency of a TM mode and barely affect a TE mode. The reason, of course, is that


Fig. 4
the effect of the wire resembles the perturbation effect of a needle, and, as found in Chapter V, has a large effect on a transverse field. In a TM mode there is an electric field parallel to the axis, but there is none in a TE mode. The periodicity effects can
 also be measured by moving a sphere along the axis and noting the variations in frequency. For cavities without cylindrical symmetry, mode identification can still be made in this manner by the proper choice of the point of insertion for the perturbing wire.

Returning to the problem of measuring the linear accelerator cavity, once the desired mode is identified, the cavity is used to stabilize one of the two Pound oscillators as described in Chapter VII. Since the oscillator will of course lock on any resonance, care must be taken that it locks on the desired one. For the TM mode under consideration, this can be checked
by moving a sphere along the axis. Since there is only an $E$ field present, for the $z$ mode there should be a zero of this field, or no frequency shift, at the position of each iris, and the maximum points between irises should be of equal magnitude. Figure 4 shows the results of such a measurement, with the positions of the irises and end walls
plotted to the same scale. The variations in maximum amplitude can be explained by the coupling hole in the center section, and the lowering of the $Q$ by the end-walls terminating the two half sections. To examine the symmetry along a diameter, a sphere was moved completely across the cavity along a diameter midway between the irises, and the results are given in Fig. 5. As can be seen, the only lack of symmetry comes near one wall and is caused by the hole necessary for introducing the sphere. These two figures prove the statement made previously that, because of the symmetrical properties of the TM mode, the fields need be measured only in a plane bounded by a radius midway between irises, a radius through the iris, the center line, and the wall.

For this cavity, three fields must be determined, $\mathrm{E}_{\mathrm{oz}}, \mathrm{E}_{\mathrm{or}}$, and $\mathrm{H}_{\mathrm{o}}$, where the subscript o refers to geometrical variations of the fields as mentioned in Chapter II. They are, of course, the components of the fields measured by the perturbation effects developed previously. To measure these three fields, three equations relating them are needed, and if, as shown in Chapter V, a sphere and a needle in two orientations at right angles are used, the necessary equations may be found. These relations are

$$
\begin{gather*}
E_{o r}^{2}+E_{o z}^{2}-\frac{1}{2} H_{o}^{2}=\frac{2 \Delta f_{1}}{4 \pi r_{o}^{3} f_{o}}  \tag{162}\\
F_{1}(\beta) E_{o r}^{2}+F_{2}(\beta) E_{o z}^{2}-\frac{1}{2} F_{4}(\beta) H_{o}^{2}=\frac{2 \Delta f_{2}}{4 \pi a^{3} f_{o}}  \tag{163}\\
F_{2}(\beta) E_{o r}^{2}+F_{1}(\beta) E_{o z}^{2}-\frac{1}{2} F_{4}(\beta) H_{o}^{2}=\frac{2 \Delta f_{3}}{4 \pi a^{3} f_{o}} \tag{164}
\end{gather*}
$$

For experimental convenience, the values of $r_{o}$ and a were chosen to keep the maximum values of $\Delta f$ on one expanded frequency dial of the receiver used. The needle chosen was that of $\beta=0.101$ used for checking theory in Chapter VII, since it gave the most accurate results with its a axis parallel to the field and since its $\beta$ was small enough to measure essentially one component of the field. A series of measurements was made to get an accurate correction factor to account for the effect of the imperfect elliptic cross section. These measurements gave that the E fields, measured parallel to a, were 17 percent larger than the theory predicts, while the E and H perpendicular to a, agree with the theory to within experimental error. Increasing $F_{1}(\beta)$ by 17 percent, and finding $\mathrm{F}_{2}(\beta)$ and $\mathrm{F}_{4}(\beta)$ from Figs. 1 and 2 , with $\mathrm{r}_{\mathrm{o}}=0.094$ inches, $\mathrm{a}=0.191$ inches, and $\mathrm{f}_{\mathrm{o}}=$ 2810.0 megacycles, the three equations for the fields are

$$
\begin{align*}
& \mathrm{E}_{\mathrm{or}}^{2}+\quad \mathrm{E}_{\mathrm{oz}}^{2}-0.5000 \mathrm{H}_{\mathrm{o}}^{2}=0.0687 \Delta \mathrm{f}_{1}  \tag{165}\\
& \mathrm{E}_{\mathrm{or}}^{2}+0.0356 \mathrm{E}_{\mathrm{oz}}^{2}-0.0347 \mathrm{H}_{\mathrm{o}}^{2}=0.0420 \Delta \mathrm{f}_{2} \tag{166}
\end{align*}
$$

$$
\begin{equation*}
0.0356 \mathrm{E}_{\mathrm{or}}^{2}+\quad \mathrm{E}_{\mathrm{oz}}^{2}-0.0347 \mathrm{H}_{\mathrm{o}}^{2}=0.0420 \Delta \mathrm{f}_{3} \tag{167}
\end{equation*}
$$

Once the measurements have been made to find $\Delta f_{1}, \Delta f_{2}$, and $\Delta f_{3}$ at each point in space, these three equations can be solved simultaneously to give the three components of the fields. This formal approach is tedious to apply, and in some cases, where the fields are rapidly varying, gives results which are less accurate than those obtained by a more practical approximate solution.

From an examination of the relative magnitudes of the coefficients in the last two equations found from the needle perturbation measurements, it is evident that, in a region where the three components of the fields are equal, the equations can be considered to represent a sing le component, $\mathrm{E}_{\mathrm{or}}$ or $\mathrm{E}_{\mathrm{oz}}$. Furthermore, from a knowledge of the space variations of the TM fields in a cylindrical cavity, it is clear that $\mathrm{E}_{\mathrm{oz}}$ should be large on the axis midway between the irises and decrease toward the walls and toward the iris. $E_{\text {or }}$ should be small midway between irises on the center and increase near the walls and iris, though the boundary conditions demand that it be zero on the iris surfaces. $H_{o}$ should be small in the center and a maximum near the walls, and decrease toward the irises. The first relationship can be found by eliminating $E_{\text {or }}$ in two needle equations.

$$
\begin{equation*}
\mathrm{E}_{\mathrm{oz}}^{2}=\frac{0.0420\left(\Delta \mathrm{f}_{3}-0.0356 \Delta \mathrm{f}_{1}\right)+0.0335 \mathrm{H}_{\mathrm{o}}^{2}}{(1-0.0013)} \tag{168}
\end{equation*}
$$

A second relationship can be found by eliminating $H_{o}$ from the sphere and first needle equations.

$$
\begin{equation*}
\mathrm{E}_{\mathrm{or}}^{2}=\frac{0.0420 \Delta \mathrm{f}_{2}-0.00474 \Delta \mathrm{f}_{1}+0.0338 \mathrm{E}_{\mathrm{oz}}^{2}}{0.9306} \tag{169}
\end{equation*}
$$

Figures 6 through 12 are $\Delta f_{1}, \Delta f_{2}$, and $\Delta f_{3}$ plotted against the radial position for different values of $z$ starting at the iris and going to the center of the cavity. Referring to these figures, it is seen that $\Delta f_{3}$ is always greater than $\Delta f_{1}$, except in the region close to the walls, and from the general variation of $\mathrm{E}_{\mathrm{OZ}}$ and $\mathrm{H}_{\mathrm{O}}$, mentioned earlier, $\mathrm{E}_{\mathrm{OZ}}$ is greater than $H_{o}$ except near the walls. Thus, to an error varying from zero at the center to a few percent near the walls, $\mathrm{E}_{\mathrm{OZ}}^{2}$ can be written approximately as

$$
\begin{equation*}
\mathrm{E}_{\mathrm{oz}}^{2}=0.0421 \Delta \mathrm{f}_{3} \tag{170}
\end{equation*}
$$

$E_{o z}^{2}$ is plotted in Figs. 13 through 19. Knowing $E_{o z}$ and the values of $\Delta f_{2}$ and $\Delta f_{1}, E_{o r}^{2}$ can then be found and this is also plotted in Figs. 13 through 19. From the plots of $E_{o z}^{2}$ and $E_{o r}^{2}$, the value of $H_{o}^{2}$ can be found from the perturbation equation of the sphere. This is plotted on the same figures.

The general features of the fields found in Figs. 13 through 19 can now be pointed out. Figure 19 represents a radius midway between the irises; on this figure $E_{\text {or }}$ is zero, and




it increases as the iris is approached until in Fig. 13, which is 0.11 cm from the center of the cavity, it is the only component of the fields which can be measured. In Fig. 13, the maximum of $\mathrm{E}_{\text {or }}$ is at $\mathrm{r}=3.10 \mathrm{~cm}$ and it decreases rapidly beyond this point. Since the diameter of the hole in the iris is 3.18 cm , this appears to be correct, since the boundary conditions on $\mathrm{E}_{\mathrm{or}}$ demand that it be zero over the iris surfaces. At $\mathrm{r}=0, \mathrm{E}_{\mathrm{Oz}}$ is a maximum in Fig. 19 and decreases as $\sin \pi z / L$ as the iris is approached. For a cylindrical cavity, $E_{o z}$ would fall off $J_{o}(r)$, but the irises make it increase as they are approached, because of the boundary conditions. This increase of $\mathrm{E}_{\mathrm{Oz}}$ with r can be seen, as well as the fact that $\mathrm{E}_{\mathrm{oz}}$ must be zero on the walls to satisfy the boundary conditions. $H_{o}$, it will be noticed, is plotted only on the radii near the center of the cavity. The reason for this is that it is found by the subtraction of $E_{o}^{2}$ from the curve for $\Delta f_{1}$. In the regions near the irises, $\mathrm{E}_{\mathrm{O}}^{2}$ is very rapidly varying, and small errors become magnified during the subtraction of terms of the same order of magnitude, thus introducing large errors in $\mathrm{H}_{\mathrm{o}}^{2}$. Another way of looking at this is that there is no perturbing shape which will measure essentially $H_{o}$ alone, as will the needles for the components of $E_{0}$. That $H_{o}$ cannot be determined accurately in a region where $E_{o}$ is varying rapidly seems to be a fundamental disadvantage of this perturbation method. The last feature that should be pointed out is that $H_{o}$ is less on Fig. 19 than on 18, while it should vary as $\cos \pi z / L$. This can be accounted for by the perturbation effect of the hole through which the sphere and needles were introduced.

The symmetry of the fields can be used to determine the distribution in the rest of

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the volume between the irises, and from the variations along the axis given in Fig. 4, the fields are known in the other sections. This really means that the geometric variations of the field, $\mathrm{E}_{\mathrm{o}}$ and $\mathrm{H}_{\mathrm{o}}$, are determined. For the absolute field strengths, the results of Chapter II give that the actual $E$ field is the value of $E_{o}$ multiplied by the time variation and a constant determined by the $Q, \lambda_{0}$, and the input power, all of which can be readily measured. The value of the constant and the expressions for the fields in volts or amperes per unit length are given in Eqs. 23 and 25. This results in

$$
\begin{align*}
& E=\sqrt{\frac{377 Q_{O} P \lambda_{o}}{\pi}} E_{O} \sin \omega_{O} t  \tag{171}\\
& H=\sqrt{\frac{Q_{O} P \lambda_{o}}{377 \pi}} H_{o} \cos \omega_{o} t \tag{172}
\end{align*}
$$

where $\mathrm{E}_{\mathrm{o}}$ and $\mathrm{H}_{\mathrm{o}}$, expressed in the proper units of length, are given in Figs. 13 to 19. These values of the fields have been checked with the results of the calculations given in Reference 10, and agreement between the two is well within experimental errors.

One interesting application of the perturbation method of measuring field strengths has been used extensively for relative values of the axial $E$ field in the accelerator cavities. The accelerator is a series of seven groups of sixteen sections identical with that measured in this chapter, each group being separated by an RF attenuator. For many reasons, it was necessary to make sure that there was even excitation along the axis of each group and in the process of arranging this, many measurements had to be taken as various adjustments were made. Since using the best method described in Chapter VII would be tedious and since only the relative magnitudes of the fields were of interest, a simple solution was found. The accelerator cavity was used to stabilize a klystron
oscillator as before, but, in this case, the oscillator output was fed into a Magic-Tee discriminator, a d-c amplifier, and ultimately to a recording milliameter. A long thread was used to suspend a sphere into the cavity and one end of this thread was attached to the tape of the recorder. As the recorder tape travelled it moved the sphere in the cavity, and changed the cavity frequency, which essentially fed a frequency-modulated signal into the discriminator. This signal was rectified into d-c voltages, which, when amplified, were recorded on the tape. Since the sphere was kept along the axis of the cavity and since $E_{o z}$ is the only field component on the axis, the output of the recorder was a plot of the relative size of $E_{o z}$ with position along the cavity axis. Thus a group of sixteen sections could be measured in about a minute, depending upon the maximum speed of the recording milliameter. This proved to be an extremely useful feature of the perturbation field strength measurements, and gave extremely reproducible results.

## IX. CONCLUSION

The purpose of this chapter will be to correlate the information on the effects of the various perturbing volumes found previously in the light of their application to general field strength measurements. One specific example has been worked out in some detail in the preceding chapter, and the general procedure to be followed for any cavity would be the same. For a given resonant mode, the first things to be investigated would certainly be the resonant frequency and any general characteristics of the mode. In spherical and cylindrical cavities (not only circular cylinders, but also cylinders in the general sense) the mode characteristics would be the TE or TM mode types described earlier. For other cavities, the characteristics would be such information as the presense of symmetry planes and the positions of the maxima. The method of finding these mentioned in the previous chapter can be extended to any resonator, and gives the necessary information with a minimum of equipment and effort. Once the mode type and symmetry planes are determined, some thought must be given to the shape and size of the perturbations best suited for measuring the particular fields.
'The choice of optimum perturbing volumes depends upon a number of factors, chiefly the fields to be measured, the equipment available for frequency measurements, and the ease with which a given shape can be suspended and moved throughout the volume. The problem of the fields is the most complicated and will be considered first. For the general cavity there will be six components of the fields, three of E and three of H , corresponding to the projections of the fields along the coordinate axes being used. To measure these components, six independent equations relating them must be found and solved simultaneously. The most convenient perturbing volumes for obtaining these relationships would probably be a needle and a disk of low $\beta$, each oriented successively along the three desired coordinate axes to give the six equations. Since the se perturbing shapes accentuate certain field components, they would lend themselves best to approximate solutions. For preliminary measurements to establish the general variations of the
fields and for measurements in regions where only one field is present, the sphere might be the most useful perturbing volume because of its non-directional characteristics. In cylindrical cavities, there will be only five components of the fields because of the TE and TM mode types, and for these measurements one of the orientations of the disk should be omitted. The details of these measurements and of the possible approximate solutions of the simultaneous equations would generally follow the procedure given in Chapter VIII, but without a specific field to discuss, it is not worthwhile to pursue the subject further. With the shapes of the perturbations decided upon, the size would be determined by the maximum and minimum conveniently measured frequency shifts.

Very little work has been done on the method of suspending the perturbing shape in the cavity. The simple and expedient method of cementing it to a piece of fine thread has been satisfactory for the measurements made thus far. However, for the problems which may arise in measuring more complex fields, more elaborate methods of suspension might be necessary. One obvious approach would be a fine dielectric rod, thin enough not to change the fields materially by its introduction, but strong enough to allow manipulation of the perturbation. With such a rod and a thin needle, the electric lines could be traced out in space by maximizing the frequency shift.

Once the direction of the electric field has been established and a rigid method of suspending the perturbation has been found, it is possible to measure the total fields without measuring the components separately. This results from the fact that $E$ and $H$ are usually at right angles (this is by no means a general condition, since even for a simple shaped cavity with walls of appreciable resistance, E and H are no longer at right angles). However, for the case when they are, the following procedure will measure both the direction and magnitude of the total fields. After having determined the direction of the E field with a needle of low $\beta$ as described above, a thin disk of as low $\beta$ as possible can be placed with its face parallel to the direction of $E$ and rotated around this direction as an axis. From the results of Chapter VI, it is clear that when the rotation is such that the resonant frequency is a minimum, the disk is aligned with its face parallel to both $E$ and $H$. With this orientation, the frequency shift from the resonant frequency with no perturbation present can be used in Eq. 140 to calculate the magnitude of $E$ alone, since there is no perturbation of H by a thin disk with its face along H . As the disk is rotated around the $E$ direction, the resonant frequency will increase to a maximum when the disk is $90^{\circ}$ from its former position, or with its face parallel to $E$ and perpendicular to $H$. This change in frequency is caused by the $H$ perturbation and is the value to be used in Eq. 144 to compute the magnitude of H . The direction of H is along the normal to the face of the disk at this orientation. This method may be used in any cavity where the total fields are to be found and it can be a check on the methods of finding components previously discussed.

Another line of approach which might be useful in complex fields would be the perturbations caused by non-metallic shapes. These perturbations can be worked out from considerations similar to those developed in this report, and might well be of some value.

For the sake of completeness, it might be worthwhile to evaluate the perturbation method of measuring field strengths in terms of the objections to other methods as pointed out in the introduction. In the first place, this method gives a value of the absolute field strength in terms of the measurable $Q_{o}$ and input power, or the value of the geometrical variation of the fields. Secondly, the fields can be measured in any part of the resonator volume, and not merely in the region of the wall surfaces. The method of insertion of the measuring device has a negligible effect, and the perturbation can be made very small compared to a wavelength, which avoids appreciable variations of the fields over its length. From the results of the measurements in Chapters VII and VIII, the experimental errors are only a few percent, and without too much effort, could be reduced still further if greater accuracy were needed. The principal disadvantage lies in the fact that the $H$ field must be found from the $E$ fields, and can be in considerable error in regions where $E$ is rapidly varying. Everything considered, however, this perturbation method of measuring the fields appears to be of value.

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[^0]:    * This report is essentially the same as a thesis of the same title submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Physics at the Massachusetts Institute of Technology.

