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Mirabolic Robinson-Shensted-Knuth correspondence

Roman Travkin

Abstract. The set of orbits of $GL(V)$ in $Fl(V) \times Fl(V) \times V$ is finite, and is parametrized by the set of certain decorated permutations in a work of Magyar, Weyman, Zelevinsky. We describe a Mirabolic RSK correspondence (bijective) between this set of decorated permutations and the set of triples: a pair of standard Young tableaux, and an extra partition. It gives rise to a partition of the set of orbits into combinatorial cells. We prove that the same partition is given by the type of a general conormal vector to an orbit. We conjecture that the same partition is given by the bimodule Kazhdan-Lusztig cells in the bimodule over the Iwahori-Hecke algebra of $GL(V)$ arising from $Fl(V) \times Fl(V) \times V$. We also give conjectural applications to the classification of unipotent mirabolic character sheaves on $GL(V) \times V$.

1. Introduction

1.1.

Let $v \in V$ be a nonzero vector in an N-dimensional vector space over a field k. The stabilizer P_N of v in $\mathrm{GL}_N = \mathrm{GL}(V)$ is called a *mirabolic* subgroup of GL_N. The special properties of the pair $P_N \subset \mathrm{GL}_N$ are among the principal reasons why the representation theory of GL_N is in many respects simpler than that of the other reductive groups over k (see e.g. [2], [10]). One more remarkable feature of the pair $P_N \subset \mathrm{GL}_N$ was discovered by P. Etingof and V. Ginzburg a few years ago. Namely, the quantum Hamiltonian reduction of the differential operators on GL_N with respect to P_N is isomorphic to the spherical trigonometric Cherednik algebra H_N (see e.g. [5]); equivalently, the quantum Hamiltonian reduction of the differential operators on $\mathrm{GL}_N \times V$ with respect to GL_N is isomorphic to H_N. Thus one is led to study the D-modules on $\mathrm{GL}_N \times V$ whose quantum Hamiltonian reduction lies in the category $\mathcal{O}$ for H_N (see [6]). The corresponding perverse sheaves are called *mirabolic character sheaves*; they are close relatives of Lusztig's character sheaves (see e.g. [14]). The present work is a first step towards a classification of mirabolic character sheaves.

1.2.

According to Lusztig's classification of character sheaves, the set of isomorphism classes of unipotent character sheaves on a reductive group G is partitioned into cells, which correspond bijectively to special unipotent classes in G. For $G = \mathrm{GL}_N$, each unipotent class is special, and each cell contains a unique character sheaf; thus the unipotent character sheaves are classified by their (nilpotent) singular supports, so they are numbered by partitions of N.

Finally, recall that the cells in question are the two-sided Kazhdan-Lusztig cells of the finite Hecke algebra $\mathcal{H}_N$. If $\mathrm{Fl}(V)$ stands for the flag variety of $\mathrm{GL}(V)$, then $\mathcal{H}_N$ is the Grothendieck ring of the constructible $\mathrm{GL}(V)$-equivariant mixed Tate complexes on $\mathrm{Fl}(V) \times \mathrm{Fl}(V)$ (multiplication given by convolution). The two-sided cells arise from the two-sided ideals spanned by the subsets of the Kazhdan-Lusztig basis (formed by the classes of Goresky-MacPherson sheaves). This basis is numbered by the symmetric group $\mathfrak{S}_N$, and its partition into two-sided cells is given by the Robinson-Shensted-Knuth algorithm, see [11]. A $\mathrm{GL}(V)$-orbit in $\mathrm{Fl}(V) \times \mathrm{Fl}(V)$ numbered by $w \in \mathfrak{S}_N$ lies in a two-sided cell λ iff a general conormal vector to the orbit is a nilpotent element of type λ, see [18].

1.3.

The starting point of our work is a classification of $\mathrm{GL}(V)$-orbits in $\mathcal{N} \times V$ where $\mathcal{N}$ is the nilpotent cone in $\mathrm{End}(V)$ (it was independently obtained by P. Achar and A. Henderson in [1]). We prove (see section 2.2) that the set of orbits is in a natural bijection with the set $\mathfrak{P}$ of pairs of partitions (ν, θ) such that $|\nu| = N$, and $\nu \supset \theta$, that is $\nu_i \geq \theta_i \geq \nu_{i+1}$ for any $i \geq 1$. Note that $\mathfrak{P}$ arises also in Zelevinsky's classification of restrictions of unipotent irreducible representations of $\mathrm{GL}_N(\mathbb{F}_q)$ to $P_N(\mathbb{F}_q)$ (see [21], Theorem 13.5), and this coincidence is not accidental.

A conormal vector to a $\mathrm{GL}(V)$-orbit in $\mathrm{Fl}(V) \times \mathrm{Fl}(V) \times V$ lies in the variety Z of quadruples (u_1, u_2, v, v^*) where $v \in V$, and $v^* \in V^*$, and u_1, u_2 are nilpotent endomorphisms of V such that $u_1 + u_2 + v \otimes v^* = 0$. The set of orbits of $\mathrm{GL}(V)$ in Z is infinite, and Z is reducible (it has $N+1$ irreducible components of dimension N^2) but we define in 3.2 a collection of closed irreducible subvarieties of Z numbered by the triples $(\nu \supset \theta \subset \nu')$ of partitions such that $|\nu| = |\nu'| = N$. These subvarieties are the images of the closures of the conormal bundles to $\mathrm{GL}(V)$-orbits in $\mathrm{Fl}(V) \times \mathrm{Fl}(V) \times V$; they are mirabolic analogues of the nilpotent orbit closures in $\mathcal{N}$.

The Hecke algebra $\mathcal{H}_N$ acts by the right and left convolution on the Grothendieck group of the constructible $\mathrm{GL}(V)$-equivariant mixed Tate complexes on $\mathrm{Fl}(V) \times \mathrm{Fl}(V) \times V$; we will denote this bimodule by $\mathcal{R}_N$. It comes equipped with a Kazhdan-Lusztig basis numbered by the finite set RB_N of $\mathrm{GL}(V)$-orbits in $\mathrm{Fl}(V) \times \mathrm{Fl}(V) \times V$, described in [16] (see also [15]). Thus we can define a partition of RB_N into *bimodule KL cells.* In this note we define an analogue of the RSK algorithm which is conjectured to be connected with these bimodule cells. Our *mirabolic RSK correspondence* (see subsection 3.5) is a bijection between the set RB_N of colored permutations of $\{1, \ldots, N\}$, and the set of triples $\{(T_1, T_2, \theta)\}$ where T_1 (resp. T_2) is a standard tableau of the shape ν (resp. ν') where $|\nu| = |\nu'| = N$, and θ is another partition such that $\nu \supset \theta \subset \nu'$.

We conjecture that the colored permutations $\tilde{w}, \tilde{w}' \in RB_N$ lie in the same bimodule KL cell iff the output of the mirabolic RSK algorithm on $\tilde{w}, \tilde{w}'$ gives the same partitions: $\nu(\tilde{w}) = \nu(\tilde{w}'),\ \nu'(\tilde{w}) = \nu'(\tilde{w}'),\ \theta(\tilde{w}) = \theta(\tilde{w}')$ (see Theorem 3 for a partial result in this direction). An equivalent form of the conjecture states that the $\mathcal{H}_N$-subquotient bimodules of $\mathcal{R}_N$ supported by the bimodule KL cells are irreducible (cf. Proposition 8). We also define a partition of RB_N into *microlocal two-sided cells* according to the type of a general conormal vector to the corresponding orbit. We prove that the colored permutations $\tilde{w}, \tilde{w}' \in RB_N$ lie in the same microlocal two-sided cell iff the output of the mirabolic RSK algorithm on $\tilde{w}, \tilde{w}'$ gives the same partitions $\nu \supset \theta \subset \nu'$ (see Theorem 2). In subsection 5.8 we describe combinatorially the involution F on RB_N arising from the Fourier-Deligne transform from the category of $\mathrm{GL}(V)$-equivariant sheaves on $\mathrm{Fl}(V) \times \mathrm{Fl}(V) \times V$ to the category of $\mathrm{GL}(V)$-equivariant sheaves on $\mathrm{Fl}(V^*) \times \mathrm{Fl}(V^*) \times V^*$. In subsection 5.9 we give a conjectural application to the classification of unipotent mirabolic character sheaves. In subsection 5.10 we formulate a conjecture on the structure of the asymptotic bimodule over Lusztig's asymptotic ring J for *diagonal* bimodule KL cells: those corresponding to triples $\nu \supset \theta \subset \nu$ (that is, $\nu = \nu'$).

1.4.

Let us emphasize that almost all arguments and constructions in the paper are of elementary combinatorial and linear algebraic nature. For instance, even though the bimodule $\mathcal{R}_N$ over the Hecke algebra $\mathcal{H}_N$ is of geometric origin, it is described explicitly in Propositions 2 and 3. The Kazhdan-Lusztig basis of $\mathcal{R}_N$ is defined by an inductive combinatorial algorithm, similarly to the Kazhdan-Lusztig basis of $\mathcal{H}_N$. Only the description of the W-graph of the $\mathcal{H}_N$-bimodule $\mathcal{R}_N$ in Proposition 9 does rely on geometric considerations.

1.5. Acknowledgments

I am grateful to P. Achar and A. Henderson for sending me [1] prior to its publication, where our Theorem 1 is proved independently (as Proposition 2.3). I am indebted to M. Finkelberg for posing the problem, numerous valuable discussions and help in editing the paper. I am thankful to Independent University of Moscow for education, financial support and various help. I thank P. Etingof for creating the ideal conditions for my work.

2. $\mathrm{GL}(V)$-orbits in $\mathcal{N} \times V$

2.1.

The following Theorem essentially goes back to J. Bernstein, who proved in [2], section 4.2, the finiteness of the set of P_N-orbits in the nilpotent cone of $\mathfrak{gl}_N$. It was independently proved by P. Achar and A. Henderson ([1], Proposition 2.3).

Theorem 1. *Let $\mathcal{N} \subset \mathfrak{gl}(V)$ be the nilpotent cone. There is a one-to-one correspondence between $\mathrm{GL}(V)$-orbits in $\mathcal{N} \times V$ and pairs of partitions (λ, μ) such that $|\lambda| + |\mu| = \sum \lambda_i + \sum \mu_i = N$. Furthermore, if a pair $(u, v) \in \mathcal{N} \times V$ belongs to the orbit corresponding to the pair (λ, μ) then the type of u is equal to $\lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$.*

Proof. Given a pair (λ, μ) such that $|\lambda| + |\mu| = N$, we will construct the pair (u, v) in the following way. Let $\nu = \lambda + \mu$ and u be a nilpotent of type ν. Denote by D_ν the set of boxes of the Young diagram ν, i.e. $D_\nu = \{(i, j) \mid 1 \le j \le \nu_i\}$. Choose a basis $e_{i,j}$ $((i, j) \in D_\nu)$ such that $ue_{i,j} = e_{i,j-1}$ for $2 \le j \le \nu_i$ and $ue_{i,1} = 0$. Let $v = \sum\limits_i e_{i,\lambda_i}$ where we put $e_{i,0} = 0$.

The inverse correspondence is obtained as follows. Let $(u, v) \in \mathcal{N} \times V$. Denote by $Z(u)$ the centralizer of u in the algebra $\mathrm{End}(V)$. Let ν be the type of u and λ be the type of $u|_{Z(u)v}$ and μ be the type of $u|_{V/Z(u)v}$.

Let us prove that these two correspondences are mutually inverse. We will need the following lemma.

Lemma 1. *Let $A \subset \mathrm{End}(V)$ be an associative algebra with identity and $A^\times$ the multiplicative group of A. Suppose the A-module V has finitely many submodules. Then $A^\times$-orbits in V are in one-to-one correspondence with these submodules. Namely, each $A^\times$-orbit has the form $\Omega_S := S \setminus \bigcup\limits_{\text{submodules } S' \subsetneq S} S'$ where S is an $A^\times$-submodule of V.*

Proof. It is clear that the sets Ω_S give us a decomposition of V into a union of locally closed subvarieties. So, we must prove that two points $v, v' \in V$ belong to the same $A^\times$-orbit iff they belong to the same Ω_S, i. e. they generate the same A-submodule $S = Av = Av'$. If v and v' belong to the same $A^\times$-orbit then $v' = av$ for some $a \in A^\times$ and $Av = Aav = Av'$. Conversely, let $v, v' \in \Omega_S$ for some S, so that $Av = Av' = S$. It is easy to see that $A^\times v$ and $A^\times v'$ are constructible dense subsets of S. This implies that $A^\times v \cap A^\times v' \neq \varnothing$ and therefore $A^\times v = A^\times v'$. □

2.1.1. Let us deduce the theorem from the lemma. Fix a partition ν of N. Consider all the $\mathrm{GL}(V)$-orbits in $\mathcal{N} \times V$ consisting of points (u, v) where u has the type ν. These orbits correspond to $\mathrm{GL}(V)_u$-orbits in V where $\mathrm{GL}(V)_u$ is the stabilizer of u in $\mathrm{GL}(V)$. Note that $\mathrm{GL}(V)_u = (Z(u))^\times$. According to the lemma it suffices to prove that V has finitely many $Z(u)$-submodules and find all these submodules. Consider V as a $\mathsf{k}[t]$-module where t acts by u. This module is isomorphic to $\bigoplus\limits_i \mathsf{k}[t]/(t^{\nu_i}\mathsf{k}[t])$. Let $V_i \cong \mathsf{k}[t]/(t^{\nu_i}\mathsf{k}[t]) \subset V$ be the the i-th direct summand of this sum. For each i let $\{e_{i,j}\}_{j=1}^{\nu_i}$ be a basis of V_i such that $ue_{i,j} = e_{i,j-1}$ $(j \geq 2)$ and $ue_{i,1} = 0$. We can write

$$Z(u) = \mathrm{End}_{\mathsf{k}[t]}(V) = \bigoplus_{i,i'} \mathrm{Hom}_{\mathsf{k}[t]}(V_i, V_{i'}) \cong \bigoplus_{i,i'} \mathsf{k}[t]/(t^{\min\{\nu_i,\nu_{i'}\}}\mathsf{k}[t])$$

Let $a_{i,i'}$ be a generator of the $\mathsf{k}[t]$-module $\mathrm{Hom}_{\mathsf{k}[t]}(V_i, V_{i'}) \subset \mathrm{End}_{\mathsf{k}[t]}(V)$ given by

$$a_{i,i'}e_{i_1,j} = \delta_{i,i_1}e_{i',(j-\max\{0,\nu_i-\nu_{i'}\})} \qquad \text{(we put } e_{i,j} = 0 \text{ for } j \leq 0).$$

Now let S be a $Z(u)$-submodule of V. Sinse S is invariant under $a_{i,i'}$ for all i, S has a form $S = \bigoplus_i S_i$ where $S_i \subset V_i$. Further since S invariant under $u \in Z(u)$ all the S_i have the form $u^{\mu_i}V_i$. Put $\lambda_i = \nu_i - \mu_i$. The invariance of S under all $a_{i,i'}$ is equivalent to the fact that λ and μ are partitions, i.e. $\lambda_1 \geq \lambda_2 \geq \dots$ and $\mu_1 \geq \mu_2 \geq \dots$. So we have shown that $Z(u)$-submodules of V are in one-to-one correspondence with pairs of partitions (λ, μ) such that $\lambda + \mu = \nu$. An application of Lemma 1 concludes the proof of the theorem. □

2.2. Comparison with Zelevinsky's parametrization

A. Zelevinsky considers in [21], Theorem 13.5.a) the set $\mathfrak{Z}$ of isomorphism classes of pairs (U, W) where U is an irreducible unipotent complex representation of the finite group $\mathrm{GL}_N(\mathbb{F}_q)$, and W is an irreducible constituent of the restriction of U to the mirabolic subgroup $P_N(\mathbb{F}_q)$. He constructs a natural bijection between $\mathfrak{Z}$ and the set $\mathfrak{P}$ of pairs of partitions (ν, θ) such that $|\nu| = N$, and $\tilde{\nu}_j - 1 \leq \widetilde{\theta}_j \leq \tilde{\nu}_j$ for all j (this is equivalent to $\nu_i \geq \theta_i \geq \nu_{i+1}$ for all i). The following Proposition-Construction establishes a natural bijection between $\mathfrak{P}$, and the set of pairs of partitions (λ, μ) such that $|\lambda| + |\mu| = N$.

Proposition-Construction 1. *Let ν be a partition and $\tilde{\nu}$ the conjugate partition so that $\nu_i \geq j \iff \tilde{\nu}_j \geq i$. There exists a natural one-to-one correspondence between pairs (λ, μ) of partitions such that $\lambda + \mu = \nu$, and partitions θ such that $\tilde{\nu}_j - 1 \leq \widetilde{\theta}_j \leq \tilde{\nu}_j$ for all j (this is equivalent to $\nu_i \geq \theta_i \geq \nu_{i+1}$ for all i). This correspondence is given by*

$$\theta_i = \lambda_{i+1} + \mu_i; \tag{1}$$

$$\begin{cases} \lambda_i = \sum_{k=i}^{\infty}(\nu_k - \theta_k) = \nu_i - \theta_i + \nu_{i+1} - \dots, \\ \mu_i = \sum_{k=i}^{\infty}(\theta_k - \nu_{k+1}) = \theta_i - \nu_{i+1} + \theta_{i+1} - \dots. \end{cases} \tag{2}$$

Proof. It is easy to see that equations (1) and (2) give mutually inverse correspondences. □

We will denote the above correspondence by $(\nu, \theta) = \Upsilon(\lambda, \mu)$, $(\lambda, \mu) = \Xi(\nu, \theta)$.

Corollary 1. *A pair (u,v) lies in an orbit $(\mathcal{N}\times V)_{(\lambda,\mu)}$ such that $(\lambda,\mu)=\Xi(\nu,\theta)$ iff the Jordan type of u is ν, and the Jordan type of $u|_{V/\langle v,uv,u^2v,\dots\rangle}$ is θ.*

Proof is obvious from the construction. □

3. $\mathrm{GL}(V)$-orbits in $\mathrm{Fl}(V)\times\mathrm{Fl}(V)\times V$

3.1.

Let $(F_1,F_2,v)\in\mathrm{Fl}(V)\times\mathrm{Fl}(V)\times V$. Consider the orbit $\mathrm{GL}(V)\cdot(F_1,F_2,v)$. If $v=0$ then this orbit lies in $\mathrm{Fl}(V)\times\mathrm{Fl}(V)\times\{0\}$. Such orbits can be parametrized by permutations of N elements. Otherwise, if $v\neq 0$ the orbit is preimage of an orbit in $\mathrm{Fl}(V)\times\mathrm{Fl}(V)\times\mathbb{P}(V)$. This follows from the fact that if $c\in\mathsf{k}^\times$ then the element (F_1,F_2,cv) can be obtained from (F_1,F_2,v) by the action of the scalar operator $c\cdot\mathrm{id}\in\mathrm{GL}(V)$. Such orbits are in one-to-one correspondence with pairs (w,σ) where $w\in\mathfrak{S}_N$ is a permutation and σ is non-empty, decreasing subsequence of w (see [16]). So $\mathrm{GL}(V)$-orbits in $\mathrm{Fl}(V)\times\mathrm{Fl}(V)\times V$ are indexed by pairs (w,σ) where $w\in\mathfrak{S}_N$ and σ is a decreasing subsequence of w (possibly empty). We will give another proof of this fact in the following lemma.

Lemma 2. *There is a one-to-one correspondence between $\mathrm{GL}(V)$-orbits in $\mathrm{Fl}(V)\times\mathrm{Fl}(V)\times V$ and pairs (w,σ) where $w\in\mathfrak{S}_N$ and $\sigma\subset\{1,2,\dots,N\}$ such that if $i,j\in\sigma$ and $i<j$ then $w(i)>w(j)$. These orbits can be also indexed by pairs (w,β) where $\beta\subset\{1,2,\dots,N\}$ is a subset such that if $i\in\{1,\dots,N\}\setminus\beta$ and $j\in\beta$ then either $i>j$ or $w(i)>w(j)$.*

We denote by RB the set of such pairs (w,β). We think of elements of RB as of words colored in two colors: red and blue. Namely, if $(w,\beta)\in RB$ we consider the word $w(1)\dots w(N)$ and paint $w(i)$ in blue if $i\in\beta$, and we paint it in red if $i\notin\beta$.

Proof. For each $w\in\mathfrak{S}_N$ let Ω_w be the corresponding $\mathrm{GL}(V)$-orbit in $\mathrm{Fl}(V)\times\mathrm{Fl}(V)$. Namely, $(F_1,F_2)\in\Omega_w$ iff there exists a basis $\{e_i\}$ of V such that

$$F_{1,i}=\langle e_1,\dots,e_i\rangle \tag{3}$$

$$F_{2,j}=\langle e_{w(1)},\dots,e_{w(j)}\rangle \tag{4}$$

Consider all the $\mathrm{GL}(V)$-orbits in $\mathrm{Fl}(V)\times\mathrm{Fl}(V)\times V$ consisting of such points (F_1,F_2,v) that $(F_1,F_2)\in\Omega_w$ where w is fixed. Fix a pair $(F_1,F_2)\in\Omega_w$ and let H be its stabilizer in $\mathrm{GL}(V)$. Then these orbits correspond to H-orbits in V. Let $A_k\subset\mathrm{End}(V)$ $(k=1,2)$ be the subalgebra defined by

$$a\in A_k \iff \forall i\quad a(F_{k,i})\subset F_{k,i}.$$

Denote $A=A_1\cap A_2$. Then $H=A^\times$ and we can apply Lemma 1.

Let $\{e_i\}$ be a basis satisfying (3) and (4) and $E_{i,j}$ the operator given by

$$E_{i,j}e_{j'}=\delta_{j,j'}e_i. \tag{5}$$

Then

$$A = \bigoplus_{\substack{i\le i'\\ w(i)\le w(i')}} \mathsf{k}E_{i,i'}.$$

Now it is easy to see that all the A-submodules in V have the form $S(\beta) := \bigoplus_{i\in\beta}\mathsf{k}e_i$ where β satisfies the condition of the lemma. So, applying Lemma 1 proves the second part of the lemma. We will denote by $\Omega_{w,\beta}$ the orbit in $\mathrm{Fl}(V)\times\mathrm{Fl}(V)\times V$ corresponding to (w,β).

For each $(w,\beta)\in RB$ let

$$\sigma = \sigma(\tilde{w}) = \{i\in\beta \mid \forall j\ (j>i)\ \&\ (w(j)>w(i)) \Longrightarrow j\notin\beta\}. \tag{6}$$

It is easy to see that (w,β) and (w,σ) can be reconstructed from each other. So the lemma is proved. □

Note that $\Omega_{w,\beta}$ consists of such triples (F_1,F_2,v) that there exists a basis $\{e_i\}$ satisfying (3), (4) and such that[1]

$$v = \sum_{i\in\beta} e_i. \tag{7}$$

3.2. X, Y, Z **and two-sided microlocal cells**

We denote $\mathrm{Fl}(V)\times\mathrm{Fl}(V)\times V$ by X, and consider the cotangent bundle T^*X. It can be described as the variety of sextuples $(F_1,F_2,v,u_1,u_2,v^*)\in T^*(X)$ where $(F_1,F_2,v)\in X$, u_i $(i=1,2)$ are nilpotent operators on V, u_i preserves F_i and $v^*\in V^*$. The moment map $T^*X\to\mathfrak{gl}(V)^*\cong\mathfrak{gl}(V)$ sends a point $(F_1,F_2,v,u_1,u_2,v^*)\in T^*(X)$ to the sum $u_1+u_2+v\otimes v^*$. The preimage Y of 0 under this map is the union of conormal bundles of $\mathrm{GL}(V)$-orbits in X. So all the irreducible components of Y have the form $Y_{w,\sigma}=\overline{N^*\Omega_{w,\sigma}}$. We determine the type of u_1 for a general point of $N^*\Omega_{w,\sigma}$.

Now consider the projection $\pi\colon Y \to \mathrm{Fl}(V)\times V\times\mathcal{N}\times\mathcal{N}\times V^*$, $(F_1,F_2,v,u_1,u_2,v^*)\longmapsto(F_2,v,u_1,u_2,v^*)$. Let $\tilde{Y}=\pi(Y)$. The preimage of a point $(F_2,v,u_1,u_2,v^*)\in\tilde{Y}$ is isomorphic to the variety $\mathrm{Fl}_{u_1}(V)$ of full flags fixed by u_1. This variety is known to be pure-dimensional and the set of its irreducible components can be identified with the set $\mathrm{St}(\lambda)$ of standard tableaux of the shape λ where λ is the type of u_1. Namely, for each $T\in\mathrm{St}(\lambda)$ the corresponding irreducible component $\mathrm{Fl}_{u_1,T}$ of $\mathrm{Fl}_{u_1}(V)$ is defined as follows. Let $\lambda^{(i)}(T)$ be the shape of the subtableau of T formed by numbers $1,\ldots,i$. Then $\mathrm{Fl}_{u_1,T}$ is the closure of the set $\mathrm{Fl}^T_{u_1}$ of all $F\in\mathrm{Fl}_{u_1}(V)$ such that $u_1|_{F_i}$ has the type $\lambda^{(i)}(T)$.

Let Z be the variety of quadruples (u_1,u_2,v,v^*), where $(u_1,u_2)\in\mathcal{N}$, $v\in V$, $v^*\in V^*$ and $u_1+u_2+v\otimes v^*=0$. Then we have a projection $\pi: Y\to Z$. We say that $\tilde{w},\tilde{w}'\in RB$ belong to the same two-sided microlocal cell if $\pi(Y_{\tilde{w}})=\pi(Y_{\tilde{w}'})$. We denote by $\mathfrak{P}$ the set of pairs of partitions (ν,θ) such that $|\nu|=N$ and $\nu_i\ge\theta_i\ge\nu_{i+1}$ for each $i\ge1$. Further, denote by $\mathbf{T}$ the set of triples of partitions (ν,θ,ν') such that $(\nu,\theta)\in\mathfrak{P}$ and $(\nu',\theta)\in\mathfrak{P}$. For any $\mathbf{t}=(\nu,\theta,\nu')\in\mathbf{T}$ denote by $Z^{\mathbf{t}}$ the

[1] This formula is different from the one in [16]: $v=\sum_{i\in\sigma}e_i$.

set of quadruples $(u_1, u_2, v, v^*) \in Z$ such that the types of u_1, u_2 and $u_1|_{V/k[u_1]v}$ are equal to ν, ν' and θ respectively (it is easy to check that for each quadruple $(u_1, u_2, v, v^*) \in Z$ we have $\mathsf{k}[u_1]v = \mathsf{k}[u_2]v$ and $u_1|_{V/\mathsf{k}[u_1]v} = u_2|_{V/\mathsf{k}[u_1]v}$).

3.3.

We fix $\tilde{w} \in RB$. Let y be a general point of variety $Y_{\tilde{w}} = \overline{N^*(\Omega_{\tilde{w}})}$. We take $\mathbf{t} = \mathbf{t}(\tilde{w}) = (\nu, \theta, \nu') \in \mathbf{T}$ such that $\pi(y) \in Z^{\mathbf{t}}$. We consider the standard Young tableaux $T_1 = T_1(\tilde{w}) \in \mathrm{St}(\nu)$ and $T_2 = T_2(\tilde{w}) \in \mathrm{St}(\nu')$ such that $F_i(y) \in \mathrm{Fl}_{u_i,T_i}$ $(i = 1, 2)$.

Proposition 1. *The map $\tilde{w} \mapsto (\mathbf{t}(\tilde{w}), T_1(\tilde{w}), T_2(\tilde{w}))$ realizes a one-to-one correspondence between RB and the set of triples $(\mathbf{t}, T_1, T_2)$ such that $\mathbf{t} = (\nu, \theta, \nu') \in \mathbf{T}$, $T_1 \in \mathrm{St}(\nu), T_2 \in \mathrm{St}(\nu')$. Moreover $\tilde{w}$ and $\tilde{w}'$ belong to the same two-sided microlocal cell iff $\mathbf{t}(\tilde{w}) = \mathbf{t}(\tilde{w}')$.*

Proof. Denote by $Y^{\mathbf{t},T_1,T_2}$ the set of points $y \in Y$ such that $\pi(y) \in Z^{\mathbf{t}}$ and $F_i(y) \in \mathrm{Fl}_{u_i,T_i}(V)$. These sets are locally closed, disjoint, and Y is their union. We claim that all of them are open subsets of irreducible components of Y. We will use the formula (14) (see page 21) whose proof does not use the proposition we are proving. (See also Remark 1 below.) Note that the number of the sets $Y^{\mathbf{t},T_1,T_2}$ coincides with the number of irreducible components of Y. This follows from the fact that the number of these sets is equal to the rank of the right hand side of the formula (14), and this rank coincides with the cardinality of RB, i.e. with the number of irreducible components of Y. Therefore, if all these sets are irreducible then their closures must be irreducible components of U. In this case we obtain a bijection of required form. Hence it is enough to prove that the sets $Y^{\mathbf{t},T_1,T_2}$ are irreducible. Note that all the fibers of the projection $Y^{\mathbf{t},Y_1,T_2} \to Z^{\mathbf{t}}$ have the form $\mathrm{Fl}_{u_1,T_1(\tilde{w})} \times \mathrm{Fl}_{u_2,T_2(\tilde{w})}$. It means they are irreducible and have the same dimension. So it is enough to prove that $Z^{\mathbf{t}}$ is irreducible.

Let $\mathbf{t} = (\nu, \theta, \nu')$. Let $\mathcal{O}_{\nu,\theta}$ be an orbit in $\mathcal{N} \times V$ corresponding to the pair (ν, θ), i.e. the set of all $(u, v) \in \mathcal{N} \times V$ such that the type of u is equal to ν and the type of $u|_{V/(\mathsf{k}[u]\cdot v)}$ is equal to θ. We have the natural pojection $Z^{\mathbf{t}} \to \mathcal{O}_{\nu,\theta}$. The fiber of this map over a point (u, v) is isomorfic to the set of $v^* \in V^*$ such that $u + v \otimes v^* \in \mathcal{O}_\nu$ where $\mathcal{O}_\nu \subset \mathcal{N}$. One can check that this subset is an open subset of an affine subspace of V^*. So the fibers of this projection are irreducible. Besides, this bundle is homogeneous. Since the orbit $\mathcal{O}_{\nu,\theta}$ is irreducible, we obtain that $Z^{\mathbf{t}}$ is irreducible. □

Remark 1. Instead of using the formula (14), one can directly compute the dimension of the sets $Y^{\mathbf{t},T_1,T_2}$, showing that $\dim Y^{\mathbf{t},T_1,T_2} = \dim Y$, which amounts to proving the equation

$$\dim Z^{\mathbf{t}} = N^2 - n(\nu) - n(\nu')$$

where $\mathbf{t} = (\nu, \theta, \nu')$, and $n(\nu) = \sum_{i \geq 1}(i - 1)\nu_i$.

3.4. Notation

We will call the map $\tilde{w} \mapsto (\mathbf{t}(\tilde{w}), T_1(\tilde{w}), T_2(\tilde{w}))$ constructed in 3.3 *the mirabolic RSK correspondence* and denote it by $\mathrm{RSK}_{\mathrm{mir}}$.

3.5. The description of mirabolic RSK correspondence

We are going to give a combinatorial description of mirabolic RSK correspondence defined in Proposition 1. Let $\tilde{w} = (w, \beta) \in RB$. We will construct step by step a standard Young tableau. Besides we will need a separate row of infinite length (denote it by $r^@$) consisting originally from the symbols "@". We assume that "@" is greater than all the numbers from 1 to N.

We will run next procedure successively for $i = 1, 2, \ldots, N$:

1a. If $i \in \beta$ then insert $w(i)$ into the tableau $T^@$ (originally empty) according to the standard row bumping rule of the RSK algorithm described in [9] (The tableau $T^@$ changes as the next element is inserted).

1b. If $i \notin \beta$ then insert first $w(i)$ into $r^@$ instead of the least element greater than $w(i)$, and then insert the element removed from $r^@$ by replacing into tableau $T^@$ via row bumping algorithm (see [9].)

2. After all the elements $w(1), \ldots, w(N)$ are inserted, we should insert the elements of $r^@$ successively via standard row bumping algorithm.

3a. After that we construct $T_2(\tilde{w})$ from the tableau $T^@$ by throwing out all the symbols "@".

3b. $T_1(\tilde{w})$ is defined as the standard tableau where number "i" stands in the cell that was added into $T^@$ at the i-th step.

3c. Finally, $\mathbf{t}(\tilde{w}) = (\nu, \theta, \nu')$ where $\nu = \mathrm{Sh}(T_1(\tilde{w}))$; $\nu' = \mathrm{Sh}(T_2(\tilde{w}))$; $\theta = (\mathrm{Sh}(T^@_*))_-$ and we have denoted

Sh - the operation of taking the shape of a tableau;

$()_-$ - the operation of removing of the first part of a partition;

$T^@_*$ - the tableau $T^@$ obtained at the last step of the algorithm.

Let us illustrate the above construction by the following example.

3.6. An example

Let $N = 10$, $w = 7, 2, 5, 1, 6, 9, 3, 8, 10, 4$; $\beta = \{1, 2, 3, 4, 7\}$. The tableau $T^@$ and the row $r^@$ obtained at the i-th algorithm step will be denoted as $T^@_i$ and $r^@_i$ respectively. So:

1. $T_1^{@} = \begin{array}{l} 7 \end{array}$ $\qquad r_1^{@} = \ @\ \ @\ \ \ldots$

2. $T_2^{@} = \begin{array}{l} 2 \\ 7 \end{array}$ $\qquad r_2^{@} = \ @\ \ @\ \ \ldots$

3. $T_3^{@} = \begin{array}{ll} 2 & 5 \\ 7 & \end{array}$ $\qquad r_3^{@} = \ @\ \ @\ \ \ldots$

4. $T_4^{@} = \begin{array}{ll} 1 & 5 \\ 2 & \\ 7 & \end{array}$ $\qquad r_4^{@} = \ @\ \ @\ \ \ldots$

5. $T_5^{@} = \begin{array}{lll} 1 & 5 & @ \\ 2 & & \\ 7 & & \end{array}$ $\qquad r_5^{@} = \ 6\ \ @\ \ @\ \ \ldots$

6. $T_6^{@} = \begin{array}{llll} 1 & 5 & @ & @ \\ 2 & & & \\ 7 & & & \end{array}$ $\qquad r_6^{@} = \ 6\ \ 9\ \ @\ \ @\ \ \ldots$

7. $T_7^{@} = \begin{array}{llll} 1 & 3 & @ & @ \\ 2 & 5 & & \\ 7 & & & \end{array}$ $\qquad r_7^{@} = \ 6\ \ 9\ \ @\ \ @\ \ \ldots$

8. $T_8^{@} = \begin{array}{llll} 1 & 3 & 9 & @ \\ 2 & 5 & @ & \\ 7 & & & \end{array}$ $\qquad r_8^{@} = \ 6\ \ 8\ \ @\ \ @\ \ \ldots$

9. $T_9^{@} = \begin{array}{lllll} 1 & 3 & 9 & @ & @ \\ 2 & 5 & @ & & \\ 7 & & & & \end{array}$ $\qquad r_9^{@} = \ 6\ \ 8\ \ 10\ \ @\ \ @\ \ \ldots$

10. $T_{10}^{@} = \begin{array}{lllll} 1 & 3 & 6 & @ & @ \\ 2 & 5 & 9 & & \\ 7 & @ & & & \end{array}$ $\qquad r_{10}^{@} = \ 4\ \ 8\ \ 10\ \ @\ \ @\ \ \ldots$

11. $$\begin{array}{lllll} 1 & 3 & 4 & @ & @ \\ 2 & 5 & 6 & & \\ 7 & 9 & & & \\ @ & & & & \end{array} \quad \begin{array}{lllll} 1 & 3 & 4 & 8 & @ \\ 2 & 5 & 6 & @ & \\ 7 & 9 & & & \\ @ & & & & \end{array} \quad \begin{array}{lllll} 1 & 3 & 4 & 8 & 10 \\ 2 & 5 & 6 & @ & @ \\ 7 & 9 & & & \\ @ & & & & \end{array} \quad \begin{array}{lllllll} 1 & 3 & 4 & 8 & 10 & @ & @\ldots \\ 2 & 5 & 6 & @ & @ & & \\ 7 & 9 & & & & & \\ @ & & & & & & \end{array} = \boxed{T_*^{@}}$$

12. $T_1(\tilde{w}) = \begin{array}{lllll} 1 & 3 & 5 & 6 & 9 \\ 2 & 7 & 8 & & \\ 4 & 10 & & & \end{array}$ $\qquad T_2(\tilde{w}) = \begin{array}{lllll} 1 & 3 & 4 & 8 & 10 \\ 2 & 5 & 6 & & \\ 7 & 9 & & & \end{array}$

As result we have $\nu = \mathrm{Sh}(T_1(\tilde{w})) = (5,3,2)$; $\nu' = \mathrm{Sh}(T_2(\tilde{w})) = (5,3,2)$; $\theta = (\mathrm{Sh}(T^{@}_*))_- = (\infty,5,2,1)_- = (5,2,1)$. Note that coincidence of ν and ν' is purely accidental.

Theorem 2. *For any* $\tilde{w} \in RB$ *the triple* $(\mathbf{t}(\tilde{w}), T_1(\tilde{w}), T_2(\tilde{w}))$ *obtained by the mirabolic algorithm described in* 3.5 *coincides with the triple* $(\mathbf{t}(\tilde{w}), T_1, T_2)$ *defined in* 3.3.

Proof. Consider colored permutation $\tilde{w}_+ \in RB_{3N}$ defined by the formulas

$$\tilde{w}_+ = (w_+, \beta_+)$$
$$w_+(i) = \begin{cases} i+2N & \text{if} \quad i \leq N \\ w(i-N)+N & \text{if} \quad N < i \leq 2N \\ i-2N & \text{if} \quad i > 2N \end{cases}$$
$$\beta_+ = \{i+N | i \in \beta\}$$

Consider a general point $x \in Y_{\tilde{w}_+}$, $x = (F_1, F_2, u_1, u_2, v, v^*)$. Denote by S the annihilator of $\mathsf{k}[u_1^*] \cdot v^*$. We are going to describe the relative position of flags $F_1 \cap S$ and $F_2 \cap S$.

3.7. The relative position of flags $F_1 \cap S$ and $F_2 \cap S$

Define 2 sequences of subsets $\{\gamma_m\}$ and $\{\delta_m\}$ $(m \geq 1)$ inductively as follows:

1. $\gamma_1 = \{1, \ldots, 3N\} \setminus \beta_+$.
2. δ_m consists of all $i \in \gamma_m$ such that there exists no $j \in \gamma_m$ satisfying both inequalities $j < i$ and $w_+(j) < w_+(i)$.
3. $\gamma_{m+1} = \gamma_m \setminus \delta_m$.

It is easy to check that $\delta_m \neq \varnothing$ iff $1 \leq m \leq N$, moreover, the minimal element of δ_m is equal to m and the maximal one is equal to $m+2N$. Define a permutation $w_1' : \{N+1, \ldots, 3N\} \to \{N+1, \ldots, 3N\}$ as follows:

$$w_1'(i) = \begin{cases} w_+(i) & \text{if } i \in \beta_+ \\ w_+(j), \text{ where } j = \max\{l \in \delta_m | l < i\} & \text{if } i \in \delta_m \end{cases}$$

Lemma 3. *The flags* $F_1 \cap S$ *and* $F_2 \cap S$ *are in relative position* w_1.

Proof. Choose a basis $e_1, \ldots, e_{3N}$ of V_+ such that $F_{1,i} = \langle e_1, \ldots, e_i \rangle$; $F_{2,j} = \langle e_{w_+(1)}, \ldots, e_{w_+(j)} \rangle$. Denote by $\{e_i^*\}$ the dual basis. Then by sufficiently general choice of the point x and the basis $\{e_i\}$ we will have $(u_*)^m v^* = \sum_{i \in \gamma_{m-1}} a_{m,i} e_i^*$ where the coefficients $a_{m,i} \neq 0$. Note that the space S is the intersection of kernels of functionals $(u^*)^m v^*$, where $0 \leq m \leq N-1$. Hence it is transversal to the spaces $F_{1,N}$ and $F_{2,N}$. Therefore i-dimensional subspaces of the flags $F_1 \cap S$ and $F_2 \cap S$ have a form $F_{1,i+N} \cap S$ and $F_{2,i+N} \cap S$.

Denote by $r_{i,j}(w_1')$ the number of all i' such that $i' \leq i$ and $w_1'(i') \leq j$. Then to prove the lemma we have to show that $\dim F_{1,i} \cap F_{2,j} \cap S = r_{i,j}(w_1')$ for any $i, j \in \{N+1, \ldots, 3N\}$. Define $r_{i,j}(w_+)$ in the same way. Then $\dim F_{1,i} \cap F_{2,j} =$

$r_{i,j}(w_+)$. Denote by $R_{i,j}$ the set of all $i' \leq i$ such that $w_+(i') \leq j$. Then $F_{1,i} \cap F_{2,j}$ has a basis $\{e_{i'}\}$ where $i' \in R_{i,j}$.

Note that if $m \leq m'$ and $\delta_m \cap R_{i,j} \neq \varnothing$ then $\delta_{m'} \cap R_{i,j} \neq \varnothing$, so we can find $k_{i,j} \geq 0$ such that $\delta_m \cap R_{i,j} \neq \varnothing$ iff $m \leq k_{i,j}$. Then $(u^*)^m v^*|_{F_{1,i} \cap F_{2,j}} \neq 0$ iff $m \leq k_{i,j}$. Moreover, for $m = 1, \ldots, k_{i,j}$ these functionals are linearly independent. By this reason the space $F_{1,i} \cap F_{2,j} \cap S$ being the intersection of kernels of these functionals has dimension $\dim F_{1,i} \bigcap F_{2,j} \bigcap S = \dim F_{1,i} \bigcap F_{2,j} - k_{i,j} = r_{i,j}(w_+) - k_{i,j}$.

It remains to prove that $r_{i,j}(w_+) - k_{i,j} = r_{i,j}(w'_1)$. We have the following equalities:

$$R_{i,j}(w_+) = (R_{i,j}(w_+) \cap \beta_+) \bigcup (\bigcup_{m=1}^{N} R_{i,j}(w_+) \cap \delta_m);$$

$$R_{i,j}(w'_1) = (R_{i,j}(w'_1) \cap \beta_+) \bigcup (\bigcup_{m=1}^{N} R_{i,j}(w'_1) \cap \delta_m).$$

From the definition of w'_1 we obtain $R_{i,j}(w'_1) \cap \beta = R_{i,j}(w_+) \cap \beta$. Besides,

in the case $m > k_{i,j}$ we have $R_{i,j}(w'_1) \cap \delta_m = R_{i,j}(w_+) \cap \delta_m = \varnothing$;

in the case $m \leq k_{i,j}$ we have $R_{i,j}(w'_1) \cap \delta_m = R_{i,j}(w_+) \cap \delta_m \setminus \{i_m\}$, where i_m is the minimal element of $R_{i,j}(w_+) \cap \delta_m$.

This implies that the set $R_{i,j}(w'_1)$ can be obtained from the set $R_{i,j}(w_+)$ by removing the elements $i_1, \ldots, i_{k_{i,j}}$, whence we get the required equality: $r_{i,j}(w'_1) = r_{i,j}(w_+) - k_{i,j}$. □

3.8.

Let, as before, $x = (F_1, F_2, u_1, u_2, v, v^*)$ be a general point of variety $Y_{\tilde{w}_+}$; $S = (\mathsf{k}[u_1^*]v^*)^\perp$. Let $u = u_1|_S = -u_2|_S$, and let T'_1 and T'_2 be the standard tableaux such that $F_1 \cap S \subset \mathrm{Fl}_{u,T'_1}$; $F_2 \cap S \subset \mathrm{Fl}_{u,T'_2}$.

Lemma 4. *One can make the flags $F_1 \cap S$ and $F_2 \cap S$ to be any points of varieties Fl_{u,T'_1} and Fl_{u,T'_2} by an appropriate choice of a point x.*

Proof. Consider any $F_1^S \in \mathrm{Fl}_{u,T'_1}$ and $F_2^S \in \mathrm{Fl}_{u,T'_2}$ and let the flags F'_1, F'_2 be defined (for $k = 1, 2$) as follows: $\begin{cases} F'_{k,i} = F_{k,i} & \text{if } i \leq N \\ F'_{k,i} = F^S_{k,i-N} + F_{k,N} & \text{if } i > N \end{cases}$

Then $x' = (F'_1, F'_2, u_1, u_2, v, v^*) \in Y$. Note that the correspondence $(F'_1, F'_2) \mapsto x'$ defines a map $f : \mathrm{Fl}_{u,T'_1} \times \mathrm{Fl}_{u,T'_2} \to Y$.

Since $\mathrm{Fl}_{u,T'_1} \times \mathrm{Fl}_{u,T'_2}$ is irreducible, the image of f belongs to one irreducible component of Y. As this image contains the point x, it lies in $Y_{\tilde{w}_+}$. Finally, as F_1^S and F_2^S are arbitrary points of Fl_{u,T'_1} and Fl_{u,T'_2}, replacing x by x' proves the lemma. □

3.9.

According to Lemma 3, the relative position of $F_1 \cap S$ and $F_2 \cap S$ is given by the permutation w_1, so using the result of Spaltenstein ([18]), we see that the pair T_1', T_2' corresponds to w_1 by the classical RSK correspondence.

Now note that the spaces $F_{1,2N}$ and $F_{2,2N}$ are invariant with respect to both operators u_1 and u_2. Let $V = F_{1,2N} \cap F_{2,2N}$. Then a sixtuple $(F_1 \cap V, F_2 \cap V, u_1|_V, u_2|_V, v, v^*|_V)$ is a general point of variety $Y_{\tilde{w}}$. Let $(\mathbf{t}(\tilde{w}), T_1(\tilde{w}), T_2(\tilde{w}))$ be the triple defined in 3.3 and $\mathbf{t}(\tilde{w}) = (\nu, \theta, \nu')$. Then $F_1 \cap V \in \mathrm{Fl}_{u_1|_V, T_1(\tilde{w})}$, $F_2 \cap V \in \mathrm{Fl}_{u_2|_V, T_1(\tilde{w})}$ and θ is the type of the nilpotent $u_1|_{V/\mathsf{k}[u_1]v}$.

Lemma 5. *The tableaux $T_1(\tilde{w})$ and $T_2(\tilde{w})$ are obtained of the tableaux T_1' and T_2' by removing the numbers $N+1, \ldots, 2N$.*

Proof. By the reason of symmetry, it suffices to prove the lemma for $T_1(\tilde{w})$ which we will denote by T for short. Recall that T_1' is defined by the condition $F_1 \cap S \subset \mathrm{Fl}_{u_1|_S, T_1'}$. So denoting by $\tilde{T}_1$ the tableau obtained from T_1' by removing the numbers greater than N, we have

$$F_1 \cap S \cap F_{1,2N} \in \mathrm{Fl}_{u_1|_{S\cap F_{1,2N}}, \tilde{T}_1}.$$

Note that the spaces V and $S \cap F_{1,2N}$ are both complementary to $F_{1,N}$ inside $F_{1,2N}$. Therefore they can be identified with $F_{1,2N}/F_{1,N}$. Under this identification the operators $u_1|_V$ and $u_1|_{S\cap F_{1,2N}}$ go to the same operator $u_1|_{F_{1,2N}/F_{1,N}}$. Similarly, the flags $F_1 \cap V$ and $F_1 \cap S \cap F_{1,2N}$ go to the same flag $(F_1 \cap F_{1,2N})/F_{1,N}$. From this we obtain that

$$(F_1 \cap F_{1,2N})/F_{1,N} \in \mathrm{Fl}_{u_1|_{F_{1,2N}/F_{1,N}}, T_1} \qquad \text{and} \qquad (F_1 \cap F_{1,2N})/F_{1,N} \in \mathrm{Fl}_{u_1|_{F_{1,2N}/F_{1,N}}, \tilde{T}_1}.$$

Now F_1 being a general point of a certain component of the variety Fl_{u_1} we obtain that $T_1 = \tilde{T}_1$. □

Lemma 6.

$$\theta = (\mathrm{Sh}(T_1'))_- = (\mathrm{Sh}(T_2'))_-.$$

Proof. By definition of the tableaux T_1' and T_2', their shape coincides with the type of nilpotent $u = u_1|_S = u_2|_S$. On the other hand, θ is the type of $u_1|_{V/\mathsf{k}[u_1]v}$. Define $L := \mathsf{k}[u]v$ and consider the space $D = (F_{1,N} + F_{2,N}) \cap S + L$. It is invariant under u, moreover $u|_D$ has only one Jordan block (it can be checked directly). Besides, $D \cap V = L$ and $D + V = S$. Therefore $u|_{V/L}$ has the same type as $u|_{S/D}$. Define $d := \dim D$. Then from the equalities $S = D + V$, $\dim V = N$, $\dim S = 2N$, it follows $d \geq N$. So d is the least power of u vanishing on S. Hence, the type of $u|_{S/D}$ is obtained from the type of $u|_S$ by removing the maximal part of the partition. □

3.10. The completion of proof of Theorem 2

Let $(v^c, \theta^c, (\nu')^c, T_1^c, T_2^c)$ be the result of application to $\tilde{w}$ of the algorithm described in 3.5. We have to prove that this quintuple coincides with $(v(\tilde{w}), \theta(\tilde{w}), \nu'(\tilde{w}), T_1(\tilde{w}), T_2(\tilde{w}))$.

Note that the result of application of algorithm 3.5 will not change if instead of infinite row of symbols "@" we will take finite sequence $N+1, \ldots, 2N$. Then $i+N \in \delta_m$ iff at the i-th step of the algorithm $w(i)$ is being inserted into the m-th position of $r^{@}$. In this case $w_1(i)$ is the number inserted into $T^{@}$ at the i-th step of the algorithm. Hence, if we apply to w_1 the classical RSK algorithm and after that throw out from the tableaux $T_1(w_1)$ and $T_2(w_1)$ all the numbers greater than N then we obtain the same pair of tableaux as the pair T_1^c and T_2^c obtained by the algorithm 3.5.

Moreover, the partition θ^c has the form $\theta^c = (\mathrm{Sh}(T_1(w_1)))_-$. We have proved above that

$$T_1(w_1) = T_1'; \quad T_2(w_1) = T_2'; \quad \theta = (\mathrm{Sh}(T_1'))_-.$$

In view of Lemma 5 we obtain

$$T_1^c = T_1(\tilde{w}); \quad T_2^c = T_2'(\tilde{w}) \quad \text{and} \quad \theta^c = \theta(\tilde{w}).$$

The proof of Theorem 2 is completed. □

4. Hecke algebra and mirabolic bimodule

4.1.

Let Σ be a finite set and E be a vector space over $\mathbb{C}$ with basis $\{e_\alpha\}_{\alpha \in \Sigma}$. Then the algebra $\mathrm{End}(E)$ of all linear operators on E can be described as the algebra of $\mathbb{C}$-valued functions on $\Sigma \times \Sigma$ with the multiplication given by convolution:

$$(f * g)(\alpha, \beta) = \sum_{\gamma \in \Sigma} f(\alpha, \gamma) g(\gamma, \beta)$$

If a finite group G acts on Σ then it also acts on E and $\mathrm{End}(E)$. Denote by $H = \mathrm{End}_G(E) \subset \mathrm{End}(E)$ the algebra of G-invariants in $\mathrm{End}(E)$. It consists of all functions on $\Sigma \times \Sigma$ that are constant on each G-orbit.

Now let $\mathsf{k} = \mathbb{F}_q$ be a finite field of q elements. Let V, X, Y be as in the previous section. Let Σ be the set of k-points of $\mathrm{Fl}(V)$ and $G = GL(V)$. Then the algebra H from the previous paragraph is called Hecke algebra. It has a basis consisting of characteristic functions of orbits. Denote by T_w the characteristic function of Ω_w considered as an element of H. Now consider the vector space R of G-invariant $\mathbb{C}$-valued functions on $X(\mathsf{k})$ where $X = \mathrm{Fl}(V) \times \mathrm{Fl}(V) \times V$. It has a natural structure of H-bimodule. Namely, if $f \in H, g \in R$ then

$$(f * g)(F_1, F_2, v) = \sum_{F \in [\mathrm{Fl}(V)](\mathsf{k})} f(F_1, F) g(F, F_2, v),$$

$$(g * f)(F_1, F_2, v) = \sum_{F \in [\mathrm{Fl}(V)](\mathsf{k})} g(F_1, F, v) f(F, F_2).$$

If $\tilde{w} \in RB$, let $T_{\tilde{w}} \in R$ denote the characteristic function of the corresponding orbit $\Omega_{\tilde{w}} \subset X$. Note that the involutions $(F_1, F_2) \leftrightarrow (F_2, F_1)$ and $(F_1, F_2, v) \leftrightarrow (F_2, F_1, v)$ induce anti-automorphisms of the algebra H and the bimodule R. These anti-automorphisms send T_w to $T_{w^{-1}}$ and $T_{\tilde{w}}$ to $T_{\tilde{w}^{-1}}$ where $\tilde{w}^{-1} = (w^{-1}, w(\beta))$ for $\tilde{w} = (w, \beta)$.

4.2. Explicit formulas for the action of H in R

We are now going to compute the H-action on R in the basis $\{T_{\tilde{w}}\}$. It is known that the algebra H is generated by the elements T_{s_i} where $s_i = (i, i+1)$ is the elementary transposition. So, it suffices to compute $T_{s_i} T_{\tilde{w}}$ and $T_{\tilde{w}} T_{s_i}$. We will compute only $T_{\tilde{w}} T_{s_i}$, since the other product can be obtained by applying the above anti-automorphisms.

Proposition 2. *Let $\tilde{w} = (w, \beta) \in RB$ and let $s = s_i \in \mathfrak{S}_N$, $i \in \{1, \ldots, N-1\}$.*

Denote $\tilde{w}s = (ws, s(\beta))$ and $\tilde{w}' = (w, \beta \bigtriangleup \{i+1\})$. Let $\sigma = \sigma(\tilde{w})$ and $\sigma' = \sigma(\tilde{w}s)$ be given by (6)*. Then*

$$T_{\tilde{w}} T_s = \begin{cases} T_{\tilde{w}s} & \text{if } ws > w \text{ and } i+1 \notin \sigma', \\ T_{\tilde{w}s} + T_{(\tilde{w}s)'} & \text{if } ws > w \text{ and } i+1 \in \sigma', \\ T_{\tilde{w}'} + T_{\tilde{w}'s} & \text{if } ws < w \text{ and } \beta \cap \iota = \{i\}, \\ (q-1) T_{\tilde{w}} + q T_{\tilde{w}s} & \text{if } ws < w \text{ and } i \notin \sigma, \\ (q-2) T_{\tilde{w}} + (q-1)(T_{\tilde{w}'} + T_{\tilde{w}s}) & \text{if } ws < w \text{ and } \iota \subset \sigma \end{cases} \tag{8}$$

where $\iota = \{i, i+1\}$.

4.3. Tate sheaves

It is well-known that H is the specialization under $\mathbf{q} \mapsto q$ of a $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$-albebra $\mathbf{H}$. The formulas (8) being polynomial in q, we may (and will) view R as the specialization under $\mathbf{q} \mapsto q$ of a $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$-bimodule $\mathbf{R}$ over the $\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$-algebra $\mathbf{H}$. We consider a new variable $\mathbf{v}$, $\mathbf{v}^2 = \mathbf{q}$, and extend the scalars to $\mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}]$: $\mathcal{H} := \mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}] \otimes_{\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]} \mathbf{H}$; $\mathcal{R} := \mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}] \otimes_{\mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]} \mathbf{R}$.

Recall the basis $\{H_w := (-\mathbf{v})^{-\ell(w)} T_w\}$ of $\mathcal{H}$ (see e.g. [17]), and the Kazhdan-Lusztig basis $\{\underline{\tilde{H}}_w\}$ (*loc. cit.*); in particular, for a simple transposition s, $\underline{\tilde{H}}_s = H_s - \mathbf{v}^{-1}$. For $\tilde{w} \in RB$, we denote by $\ell(\tilde{w})$ the difference $\dim(\Omega_{\tilde{w}}) - \mathbf{n}$, where $\mathbf{n} := \frac{N(N-1)}{2} = \dim(\mathrm{Fl}(V))$. We introduce a new basis $\{H_{\tilde{w}} := (-\mathbf{v})^{-\ell(\tilde{w})} T_{\tilde{w}}\}$ of $\mathcal{R}$. In this basis the right action of the Hecke algebra generators $\underline{\tilde{H}}_s$ takes the form:

Proposition 3. *Let $\tilde{w} = (w, \beta) \in RB$ and let $s = s_i \in \mathfrak{S}_N$, $i \in \{1, \ldots, N-1\}$. Denote $\tilde{w}s = (ws, s(\beta))$ and $\tilde{w}' = (w, \beta \bigtriangleup \{i+1\})$. Let $\sigma = \sigma(\tilde{w})$ and $\sigma' = \sigma(\tilde{w}s)$*

be given by (6). *Then*

$$H_{\tilde w}\underline{\tilde H}_s = \begin{cases} H_{\tilde w s} - \mathbf{v}^{-1}H_{\tilde w} & \text{if } ws > w \text{ and } i+1 \notin \sigma', \\ H_{\tilde w s} - \mathbf{v}^{-1}H_{(\tilde w s)'} - \mathbf{v}^{-1}H_{\tilde w} & \text{if } ws > w \text{ and } i+1 \in \sigma', \\ H_{\tilde w'} - \mathbf{v}^{-1}H_{\tilde w} - \mathbf{v}^{-1}H_{\tilde w' s} & \text{if } ws < w \text{ and } \beta \cap \iota = \{i\}, \\ H_{\tilde w s} - \mathbf{v}H_{\tilde w} & \text{if } ws < w \text{ and } i \notin \sigma, \\ (\mathbf{v}^{-1} - \mathbf{v})H_{\tilde w} + (1-\mathbf{v}^{-2})(H_{\tilde w'} + H_{\tilde w s}) & \text{if } ws < w \text{ and } \iota \subset \sigma \end{cases} \tag{9}$$

where $\iota = \{i, i+1\}$.

It is well known that $\mathcal{H}$ is the Grothendieck ring (with respect to convolution) of the derived constructible G-equivariant category of Tate Weil $\overline{\mathbb{Q}}_l$-sheaves on $\mathrm{Fl}(V) \times \mathrm{Fl}(V)$, and multiplication by $\mathbf{v}$ corresponds to the twist by $\overline{\mathbb{Q}}_l(-\frac{1}{2})$ (so that $\mathbf{v}$ has weight 1), see e.g. [4]. In particular, H_w is the class of the shriek extension of $\underline{\overline{\mathbb{Q}}_l}[\ell(w)+\mathbf{n}](\frac{\ell(w)+\mathbf{n}}{2})$ from the corresponding orbit, and $\underline{\tilde H}_w$ is the selfdual class of the Goresky-MacPherson extension of $\underline{\overline{\mathbb{Q}}_l}[\ell(w)+\mathbf{n}](\frac{\ell(x)+\mathbf{n}}{2})$ from this orbit. Similarly, we will prove that $\mathcal{R}$ is the Grothendieck group of the derived constructible G-equivariant category of Tate Weil $\overline{\mathbb{Q}}_l$-sheaves on X, and $\mathcal{H}$-bimodule structure is given by convolution. In particular, $H_{\tilde w}$ is the class of the star extension of $\underline{\overline{\mathbb{Q}}_l}[\ell(\tilde w)+\mathbf{n}](\frac{\ell(\tilde w)+\mathbf{n}}{2})$ from the orbit $\Omega_{\tilde w} \subset X$. We will denote by $j_{!*}\underline{\overline{\mathbb{Q}}_l}[\ell(\tilde w)+\mathbf{n}](\frac{\ell(\tilde w)+\mathbf{n}}{2})$ the selfdual Goresky-MacPherson extension of $\underline{\overline{\mathbb{Q}}_l}[\ell(\tilde w)+\mathbf{n}](\frac{\ell(\tilde w)+\mathbf{n}}{2})$ from $\Omega_{\tilde w} \subset X$, and we will denote by $\underline{\tilde H}_{\tilde w}$ its class in the Grothendieck group.

Recall that a G-equivariant constructible Weil complex F on X is called *Tate* if any cohomology sheaf of its restriction $i^*_{\tilde w}F$ and corestriction $i^!_{\tilde w}F$ to any orbit $\Omega_{\tilde w}$ admits a filtration with successive quotients of the form $\underline{\overline{\mathbb{Q}}_l}(m)$, $m \in \frac{1}{2}\mathbb{Z}$. If for any $\tilde w \in RB$ the sheaf $j_{!*}\underline{\overline{\mathbb{Q}}_l}[\ell(\tilde w)+\mathbf{n}](\frac{\ell(\tilde w)+\mathbf{n}}{2})$ is Tate, then the shriek extension $j_!\underline{\overline{\mathbb{Q}}_l}[\ell(\tilde w)+\mathbf{n}](\frac{\ell(\tilde w)+\mathbf{n}}{2})$ is Tate as well (see Remark between Lemmas 4.4.5 and 4.4.6 of [4]). Note also that the G-equivariant geometric fundamental group of any orbit $\Omega_{\tilde w}$ is trivial. Hence the classes $H_{\tilde w} = [j_!\underline{\overline{\mathbb{Q}}_l}[\ell(\tilde w)+\mathbf{n}](\frac{\ell(\tilde w)+\mathbf{n}}{2})]$ do form a $\mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}]$-basis of the Grothendieck group of G-equivariant Tate sheaves on X, and this Grothendieck group is isomorphic to $\mathcal{R}$.

In order to prove the Tate property of $j_{!*}\underline{\overline{\mathbb{Q}}_l}[\ell(\tilde w)+\mathbf{n}](\frac{\ell(\tilde w)+\mathbf{n}}{2})$, we need to study certain analogues of Demazure resolutions of the orbit closures $\overline{\Omega}_{\tilde w}$.

4.4. Demazure type resolutions

We consider the elements $\tilde w_i = (w, \beta_i) \in RB$ such that $w = \mathrm{id}$ (the identity permutation), and $\beta_i = \{1, \ldots, i\}$, where $i = 0, \ldots, N$. We set $\underline{\tilde H}_{\tilde w_i} := \sum_{0 \le j \le i} (-\mathbf{v})^{j-i} H_{\tilde w_j}$. This is the class of the selfdual (geometrically constant) IC sheaf on the closure of the orbit $\Omega_{\tilde w_i}$.

We fix $k \quad (0 \le k \le N)$, and a pair of sequences $i_1, \ldots, i_r$ and $j_1, \ldots, j_s$ of integers between 1 and $N-1$. Let $S = S^{j_1,\ldots,j_s;k}_{i_1,\ldots,i_r}$ be a variety of collections of flags and vectors $(F_0, \ldots, F_r, F'_0, \ldots, F'_s, v)$ such that:

1. $(F_r, F'_0, v) \in \overline{\Omega}_{\tilde{w}_k}$;
2. $(F_{p-1}, F_p) \in \overline{\Omega}_{s_{i_p}}$ for any $p \in \{1, \dots, r\}$;
3. $(F'_{q-1}, F'_q) \in \overline{\Omega}_{s_{j_q}}$ for any $q \in \{1, \dots, s\}$.

In other words,

$$S = \overline{\Omega}_{s_{i_1}} \times_{\mathrm{Fl}(V)} \dots \times_{\mathrm{Fl}(V)} \overline{\Omega}_{s_{i_r}} \times_{\mathrm{Fl}(V)} \overline{\Omega}_{\tilde{w}_k} \times_{\mathrm{Fl}(V)} \overline{\Omega}_{s_{j_1}} \times_{\mathrm{Fl}(V)} \dots \times_{\mathrm{Fl}(V)} \overline{\Omega}_{s_{j_s}}$$

Consider a map $\phi = \phi^{j_1,\dots,j_s}_{i_1,\dots,i_r} : S^{j_1,\dots,j_s;k}_{i_1,\dots,i_r} \to X$ which takes $F_0, \dots, F_r, F'_0, \dots, F'_s$ to (F_0, F'_s, v).

Proposition 4. *For any $\tilde{w} \in RB$ there exist $i_1, \dots, i_r; j_1, \dots, j_s$ and k such that:*

a) $\phi(S) = \overline{\Omega}_{\tilde{w}}$, moreover ϕ is an isomorphism over $\Omega_{\tilde{w}}$.

b) The sheaf $\phi_*(\underline{\mathbb{Q}}_l)$ is Tate.

Proof. a) We proceed by induction in $\ell(\tilde{w}) = \dim \Omega_{\tilde{w}} - \mathbf{n}$. Assume the proposition 4 is true for any $\tilde{w}'$ such that $\ell(\tilde{w}') < \ell(\tilde{w})$.

Let $\tilde{w} = (w, \beta)$. If $w = \mathrm{id}$ then $\tilde{w} = \tilde{w}_k$ for some k. We choose $i_1, \dots, i_r$ and $j_1, \dots, j_s$ to be the empty sequences. Then the map ϕ is an embedding $\Omega_{\tilde{w}} \hookrightarrow X$, and the proposition is true. Otherwise ($w \neq \mathrm{id}$) it is easy to show that either $ws_i < w$ and $\tilde{w}s_i = (ws_i, s_i(\beta)) \in RB$, or $s_i w < w$ and $(s_i w, \beta) \in RB$. Without loss of generality we can restrict ourselves to the first case (the second one is obtained replacing $\tilde{w}$ by $\tilde{w}^{-1} = (w^{-1}, w(\beta))$).

Let $S' = S^{j_1,\dots,j_s;k}_{i,i_1,\dots,i_r}, \phi' = \phi^{j_1,\dots,j_s}_{i,i_1,\dots,i_r}$. Then

$$S' = \overline{\Omega}_{s_i} \times_{\mathrm{Fl}(V)} S. \tag{10}$$

By the induction hypothesis, S contains an open dense set mapping isomorphically by ϕ onto $\Omega_{\tilde{w}s_i}$. It follows that a map $\Omega_s \times_{\mathrm{Fl}(V)} \Omega_{\tilde{w}s_i} \to X$ has an image lying in $\overline{\Omega}_{\tilde{w}}$, moreover, this map is an isomorphism over $\Omega_{\tilde{w}}$. According to (10), S' contains an open dense subset isomorphic to $\Omega_{s_i} \times_{\mathrm{Fl}(V)} \Omega_{\tilde{w}s_i}$, hence $\phi'(S') = \overline{\Omega}_{\tilde{w}}$, and $\phi^{-1}(\Omega_{\tilde{w}})$ is isomorphic to $\Omega_{\tilde{w}}$.

b) We will prove by induction that any fiber of ϕ is paved by the pieces isomorphic to $\mathbb{A}^k \times \mathbb{G}_m^n$. Moreover, the union of pieces constructed at each step is a closed subvariety of the fiber.

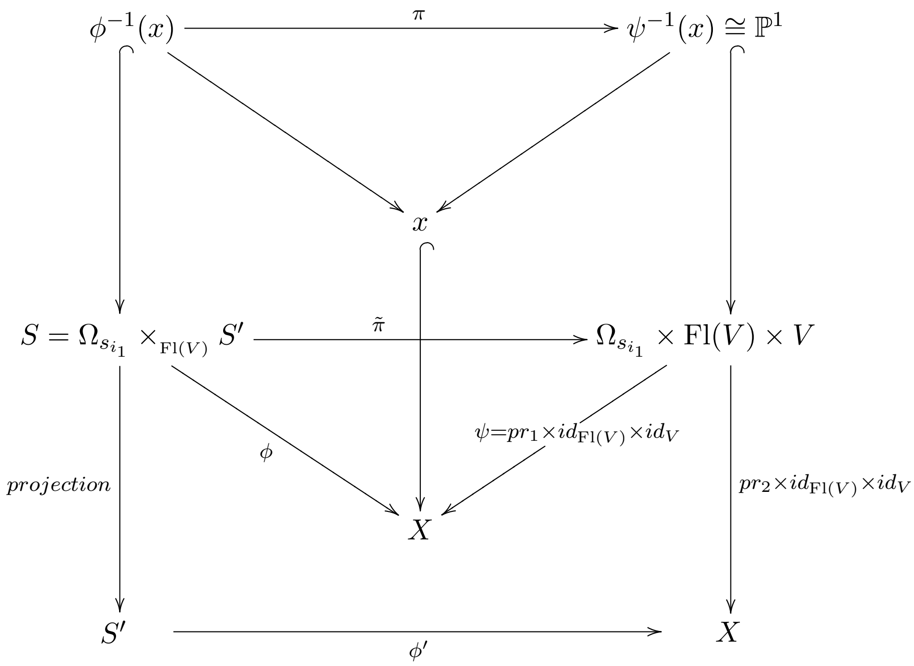
If $r = s = 0$ then any nonempty fiber is just a point, and the statement is obvious. Otherwise without loss of generality we can assume $r > 0$.

Let $S = S^{j_1,\dots,j_s;k}_{i_1,i_2,\dots,i_r}$ and $S' = S^{j_1,\dots,j_s;k}_{i_2,\dots,i_r}$. We have a commutative diagram:

$$\begin{array}{ccc} S = \Omega_{s_{i_1}} \times_{\mathrm{Fl}(V)} S' & \xrightarrow{\tilde{\pi}} & \Omega_{s_{i_1}} \times \mathrm{Fl}(V) \times V \\ & \searrow^{\phi} \quad \swarrow_{\psi = pr_1 \times id_{\mathrm{Fl}(V)} \times id_V} & \\ & X & \end{array} \tag{11}$$

where $\tilde{\pi}(F_0, \dots, F_r, F'_1, \dots, F'_s) = ((F_0, F_1), F'_s, v)$.

It is easy to see that the fibers of the map ψ are isomorphic to $\mathbb{P}^1$. For each point $x \in X$ we obtain the corresponding map $\pi : \phi^{-1}(x) \to \psi^{-1}(x) \cong \mathbb{P}^1$. We have the following commutative diagram, whose middle part coincides with diagram 11:



(12)

All the 4 squares in this diagram are Cartesian.

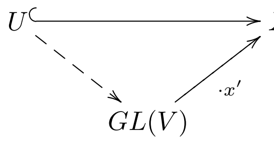
Denote by $\varkappa : \psi^{-1}(x) \to X$ the composition of maps from the commutative diagram 12. This map is an embedding.

For each point $y \in \psi^{-1}(x)$ we have $\pi^{-1}(y) \cong (\phi')^{-1}(\varkappa(y))$. By the induction hypothesis, all the fibers of ϕ' can be decomposed into pieces of required form. If $x = (F, F', v)$ then the image L of the map $\varkappa$ consists of triples (F, F'', v) such that $(F', F) \subset \overline{\Omega}_{s_i}$. For $x' \in X$ the fiber $(\phi')^{-1}(x')$ depends only on the orbit $\Omega_{\tilde{u}}$ which contains x'. The line L can intersect 2 or 3 such orbits; one intersection is open in L, and any other intersection is just one point.

Let $U = L \bigcap \Omega_{\tilde{u}}$ be the intersection open in L. Since all the fibers of π admit a required decomposition, it is enough to construct the decomposition of the set $\pi^{-1}(U') \cong \phi{-}1(U)$ where $U' = \varkappa{-}1(U) \cong U$. This follows from the fact that the bundle $(\phi')^{-1}(U) \to U$ is trivial.

Indeed, in this case for any $x' \in U$ we have $\phi'^{-1}(U) \cong U \times (\phi')^{-1}(x')$, because $\phi'^{-1}(x')$ admits the required decomposition, and U is isomorphic to either $\mathbb{A}^1$ or $\mathbb{G}_m$.

It remains to prove the triviality of the bundle $(\phi')^{-1}(U) \to U$. Note that the bundle $S' \to X$ is $GL(V)$-equivariant. Choose a point $x' \in U$ and consider a map $GL(V) \to X$, given by $g \to g \cdot x'$. Then the induced bundle $S \times_X GL(V) \to GL(V)$ is trivial, so it is enough to prove that there exists the dotted arrow in the diagram


(13)

This can be checked directly.

So the proof of proposition 4 is finished.

□

Corollary 2. *Bimodule $\mathcal{R}$ is generated by the elements $e_i = T_{\tilde{w}_i}$.*

Proof. We prove by induction on $\ell(\tilde{w})$ that $T_{\tilde{w}} \in \sum_i \mathcal{H}e_i\mathcal{H}$.

Choose $i_1, \ldots, i_r, j_1 \ldots, j_s$ and k as in Proposition 4. Then

$$T_{j_s} \cdot \ldots \cdot T_{j_1} \cdot T_{\tilde{w}_k} \cdot T_{i_r} \cdot \ldots \cdot T_{i_1} = T_{\tilde{w}} + \sum_{\tilde{u} < \tilde{w}} a_{\tilde{u}} T_{\tilde{u}}$$

where $a_{\tilde{u}} \in \mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}]$, and $\tilde{u} < \tilde{w}$ means $\Omega_{\tilde{u}} \subset \overline{\Omega_{\tilde{w}}}$. The left hand side of this equality belongs to $\mathcal{H}e_k\mathcal{H}$ (recall that $e_k = T_{\tilde{w}_k}$). Besides, by the induction hypothesis, $T_{\tilde{u}} \in \sum_i \mathcal{H}e_i\mathcal{H}$ for each $\tilde{u} < \tilde{w}$. Hence, $\sum_{\tilde{u}} a_{\tilde{u}} T_{\tilde{u}} \in \sum_i \mathcal{H}e_i\mathcal{H}$.

From this we can conclude that $T_{\tilde{u}} \in \sum_i \mathcal{H}e_i\mathcal{H}$. □

Corollary 3. *For any $\tilde{w} \in RB$, the sheaf $j_{!*}\underline{\overline{\mathbb{Q}}_l}[\ell(\tilde{w}) + \mathbf{n}](\frac{\ell(\tilde{w})+\mathbf{n}}{2})$ is Tate.*

Proof. Follows from Proposition 4.b) by the Decomposition Theorem. □

Corollary 4. *The Grothendieck group of the derived constructible G-equivariant category of Tate Weil $\overline{\mathbb{Q}}_l$-sheaves on X is isomorphic to $\mathcal{R}$ as an $\mathcal{H}$-bimodule with respect to convolution.*

4.5. Duality and the Kazhdan-Lusztig basis of $\mathcal{R}$

Recall the involution (denoted by $h \mapsto \overline{h}$) of $\mathcal{H}$ which takes $\mathbf{v}$ to $\mathbf{v}^{-1}$ and $\overline{\underline{\tilde{H}}_w}$ to $\underline{\tilde{H}}_w$. It is induced by the Grothendieck-Verdier duality on $\mathrm{Fl}(V) \times \mathrm{Fl}(V)$. We are going to describe the involution on $\mathcal{R}$ induced by the Grothendieck-Verdier duality on X.

Recall the elements $\tilde{w}_i = (w, \beta_i) \in RB$ such that $w = \mathrm{id}$ (the identity permutation), and $\beta_i = \{1, \ldots, i\}$, where $i = 0, \ldots, N$. We set $\underline{\tilde{H}}_{\tilde{w}_i} := \sum_{0 \le j \le i} (-\mathbf{v})^{j-i} H_{\tilde{w}_j}$. This is the class of the selfdual (geometrically constant) IC sheaf on the closure of the orbit $\Omega_{\tilde{w}_i}$.

Proposition 5. *a) There exists a unique involution $r \mapsto \overline{r}$ on $\mathcal{R}$ such that $\overline{\underline{\tilde{H}}_{\tilde{w}_i}} = \underline{\tilde{H}}_{\tilde{w}_i}$ for any $i = 0, \ldots, N$, and $\overline{hr} = \overline{h}\overline{r}$, and $\overline{rh} = \overline{r}\overline{h}$ for any $h \in \mathcal{H}$ and $r \in \mathcal{R}$.*

b) The involution in a) is induced by the Grothendieck-Verdier duality on X.

Proof. The uniqueness in a) follows since $\mathcal{R}$ is generated as an $\mathcal{H}$-bimodule by the set $\{\underline{\tilde{H}}_{\tilde{w}_i},\ i = 0, \ldots, N\}$, according to Corollary 2. Now the Grothendieck-Verdier duality on X clearly induces the involution on $\mathcal{R}$ satisfying a); whence the existence and b). □

Now let $\tilde{w}_1 < \tilde{w}_2$ stand for the adjacency Bruhat order on RB described combinatorially in [15], section 1.2.

Proposition 6. *a) For each $\tilde{w} \in RB$ there exists a unique element $\underline{\tilde{H}}_{\tilde{w}} \in \mathcal{R}$ such that $\overline{\underline{\tilde{H}}_{\tilde{w}}} = \underline{\tilde{H}}_{\tilde{w}}$, and $\underline{\tilde{H}}_{\tilde{w}} \in H_{\tilde{w}} + \sum_{\tilde{y}<\tilde{w}} \mathbf{v}^{-1}\mathbb{Z}[\mathbf{v}^{-1}]H_{\tilde{y}}$.*

b) For each $\tilde{w} \in RB$ the element $\underline{\tilde{H}}_{\tilde{w}}$ is the class of the selfdual G-equivariant IC-sheaf with support $\bar{\Omega}_{\tilde{w}}$. In particular, for $\tilde{w} = \tilde{w}_i$, the element $\underline{\tilde{H}}_{\tilde{w}_i}$ is consistent with the notation introduced before Proposition 5.

Proof. a) is a particular case of [13], Lemma 24.2.1.

b) We already know that $H_{\tilde{w}}$ is the class of $j_!\underline{\overline{\mathbb{Q}}_l}[\ell(\tilde{w}) + \mathbf{n}](\frac{\ell(\tilde{w})+\mathbf{n}}{2})$, and $j_{!*}\underline{\overline{\mathbb{Q}}_l}[\ell(\tilde{w}) + \mathbf{n}](\frac{\ell(\tilde{w})+\mathbf{n}}{2})$ is Tate. Now b) follows from the Beilinson-Bernstein-Deligne-Gabber purity theorem by the standard argument (see e.g. [3], section 6). □

4.6. Pointwise purity

We are now going to show that the sheaves $j_{!*}\underline{\overline{\mathbb{Q}}_l}[\ell(\tilde{w}') + \mathbf{n}](\frac{\ell(\tilde{w}')+\mathbf{n}}{2})$ are pointwise pure. We choose a point $F_0 \in \mathrm{Fl}(V)$ and denote by $B \in \mathrm{GL}(V)$ the corresponding Borel subgroup. We consider the preimage of F_0 under the second projection $X' := X_{F_0} = pr_2^{-1}(F_0) \subset X$, $X_{F_0} = \mathrm{Fl}(V) \times \{F_0\} \times V \cong \mathrm{Fl}(V) \times V$. Fix $\tilde{w} \in RB$, and let $\Omega' = \Omega_{\tilde{w},F_0} = \Omega_{\tilde{w}} \cap X_{F_0}$ be the corresponding B-orbit in X_{F_0}. Choose a point $x \in X_{F_0}$.

Lemma 7. *There exists a one-parameter subgroup $\chi :\ \mathbb{G}_m \to B$ preserving x, whose eigenvalues in the normal space $N_{x,X'}\Omega'$ are all positive.*

Proof. Let $\tilde{w} = (w, \beta)$, $x = (F_0, F_1, v)$. Choose a basis $e_1, \ldots, e_N$ of V such that $F_{0,i} = \langle e_1, \ldots, e_i\rangle$, $F_{1,i} = \langle e_{w(1)}, \ldots, e_{w(i)}\rangle$, $v = \sum_{i\in\sigma} e_i$ where σ is defined by (6). Define sets $\beta_r, \gamma_r, \sigma_r, \delta_i \subset \{1, \ldots, N\}$, $i = 1, 2, \ldots$ inductively as follows:

- $\beta_1 = \beta$, $\gamma_1 = \{1, \ldots, N\} \setminus \beta$;
- $\sigma_r = \{i \in \beta_r \mid \nexists j \in \beta_r\ (j > i \text{ and } w(j) > w(i))\}$;
  $\delta_r = \{i \in \gamma_r \mid \nexists j \in \gamma_r\ (j < i \text{ and } w(j) < w(i))\}$;
- $\beta_{r+1} = \beta_r \setminus \sigma_r$, $\gamma_{r+1} = \gamma_r \setminus \delta_r$.

Let $(k_1, \ldots, k_N)$ be the integer sequence given by $k_i = 1-r$ if $i \in \sigma_r$, and $k_i = r$ if $i \in \delta_r$. Then it is not hard to verify that the homomorphism $t \mapsto \mathrm{diag}(t^{k_1}, \ldots, t^{k_N})$ satisfies the required condition. □

Proposition 7. *The intersection cohomology sheaf $j_{!*}\underline{\overline{\mathbb{Q}}_l}[\ell(\tilde{w}') + \mathbf{n}](\frac{\ell(\tilde{w}')+\mathbf{n}}{2})$ of an orbit $\Omega_{\tilde{w}'} \subset X$ is pointwise pure of weight zero.*

Proof. We denote the locally closed embedding $\Omega_{\tilde{w}',F_0} \hookrightarrow X'$ by a. The statement of the proposition is equivalent to pointwise purity of $a_{!*}\underline{\overline{\mathbb{Q}}_l}$. We choose a point $x \in \Omega' = \Omega_{tw,F_0} \subset \overline{\Omega}_{\tilde{w}',F_0}$. Due to B-equivariance, it suffices to prove the purity of the stalk of $a_{!*}\underline{\overline{\mathbb{Q}}_l}$ at x. We claim that there is a locally closed subvariety $Y' \subset X'$ such that a) $Y' \cap \Omega' = x$; b) Y' is smooth at x; c) the tangent space $T_xX' = T_x\Omega' \oplus T_xY'$; d) Y' is stable under the action of the one-parametric subgroup $\chi(\mathbb{G}_m)$; moreover, Y' is contracted to x by this action. The existence of Y' with required properties follows from Sumihiro equivariant embedding theorem [20].

We denote the locally closed embedding $\Omega_{\tilde{w}',F_0} \cap Y' \hookrightarrow Y'$ by a'. The intersection cohomology sheaf $a'_{!*}\underline{\overline{\mathbb{Q}}_l}$ on Y' is $\chi(\mathbb{G}_m)$-equivariant, and hence its stalk at x is pure, see e.g. [19]. The morphism $b: B \times Y' \to X'$, $(g,y) \mapsto g(y)$ is smooth at the point (e,x) (where e is the neutral element of B) due to the condition c) above. It follows that $b^*a_{!*}\underline{\overline{\mathbb{Q}}_l}|_{(e,x)} = pr_2^* a'_{!*}\underline{\overline{\mathbb{Q}}_l}|_{(e,x)}$ where $pr_2: B \times Y' \to Y'$ is the second projection. We conclude that the stalk $a_{!*}\underline{\overline{\mathbb{Q}}_l}|_x$ is pure. The proposition is proved. □

Corollary 5. *We define the Kazhdan-Lusztig polynomials* $P_{\tilde{w},\tilde{y}}$ *by* $\underline{\tilde{H}}_{\tilde{w}} = \sum_{\tilde{y} \leq \tilde{w}} P_{\tilde{w},\tilde{y}} H_{\tilde{y}}$. *Then all the coefficients of* $P_{\tilde{w},\tilde{y}}$ *are nonnegative integers. The coefficient of* $\mathbf{v}^{-k}$ *in* $P_{\tilde{w},\tilde{y}}$ *vanishes if* $k \not\equiv \ell(\tilde{w}) - \ell(\tilde{y}) \pmod 2$. □

4.7. The structure of the $\mathcal{H}$-module $\mathcal{R}$

It is known that the algebra $\mathcal{H} \otimes_{\mathbb{Z}[\mathbf{v},\mathbf{v}^{-1}]} \mathbb{Q}(\mathbf{v})$ is isomorphic to the group algebra of symmetric group $\mathbb{Q}(\mathbf{v})[\mathfrak{S}_N]$. Hence, the isomorphism classes of irreducible modules over $\mathcal{H} \otimes_{\mathbb{Z}[\mathbf{v},\mathbf{v}^{-1}]} \mathbb{Q}(\mathbf{v})$ are indexed by the set of partitions of N. We denote by V_ν the irreducible module corresponding to a partition ν.

Proposition 8. $\mathcal{H} \otimes_{\mathbb{Z}[\mathbf{v},\mathbf{v}^{-1}]} \mathbb{Q}(\mathbf{v})$*-bimodule* $\mathcal{R} \otimes_{\mathbb{Z}[\mathbf{v},\mathbf{v}^{-1}]} \mathbb{Q}(\mathbf{v})$ *has the following decomposition into irreducible bimodules:*

$$\mathcal{R} \otimes_{\mathbb{Z}[\mathbf{v},\mathbf{v}^{-1}]} \mathbb{Q}(\mathbf{v}) = \bigoplus_{(\tilde{\nu},\tilde{\theta},\tilde{\nu}') \in \mathbf{T}} V_\nu^* \otimes_{\mathbb{Q}(\mathbf{v})} V_{\nu'} \tag{14}$$

where the sum is taken over all the triples of partitions ν, θ, ν' *such that* $|\nu| = |\nu'| = N$ *and for any* $i \geq 1$ *we have* $\nu_i \geq \theta_i \geq \nu_i - 1$; $\nu'_i \geq \theta_i \geq \nu'_i - 1$.

Proof. Choose a finitely generated $\mathbb{Z}[\mathbf{v},\mathbf{v}^{-1}]$-algebra $\mathcal{A} \subset \mathbb{Q}(\mathbf{v})$ such that $\mathcal{H} \otimes_{\mathbb{Z}[\mathbf{v},\mathbf{v}^{-1}]} \mathcal{A}$ is isomorphic to a direct sum of matrix algebras over $\mathcal{A}$, so that V_ν is defined over $\mathcal{A}$. Then it suffices to prove that this isomorphism holds after the specialization $\cdot \otimes_{\mathcal{A}} \mathbb{C}$ which takes $\mathbf{v} \mapsto \sqrt{q}$ where q is a prime power such that $\mathcal{A} \not\ni (\mathbf{v}^2 - q)^{-1}$. In this case the left hand side of formula (14) can be interpreted as

$$\mathrm{End}_{P(\mathsf{k})}(E) \bigoplus \mathcal{H} \otimes_{\mathbb{Z}[\mathbf{v},\mathbf{v}^{-1}]} \mathbb{C}$$

where $P(\mathsf{k}) \subset \mathrm{GL}_N$ is the stabilizer of $v \neq 0$, and E is the vector space introduced in subsection 4.1. According to [21], Theorem 13.5.a), the irreducible components of $P(\mathsf{k})$-module E are indexed by partitions θ with $|\theta| \leq N$. We denote by W_θ the irreducible representation of $P(\mathsf{k})$ indexed by θ. Denote by U_ν the irreducible unipotent representation of $G(\mathsf{k})$ indexed by ν. Then the restriction of U_ν to $P(\mathsf{k})$ is a direct sum of the representations W_θ (with multiplicity one) for all θ such that $\nu_i \geq \theta_i \geq \nu_i - 1$ for any i and $\theta \neq \nu$.

As a representation of the group $G(\mathsf{k})$, E admits a decomposition as follows:

$$E = \bigoplus_\nu U_\nu \otimes V_\nu$$

(Here $G(\mathsf{k})$ acts on V_ν trivially). Therefore, as a representation of $P(\mathsf{k})$, E can be written in the form

$$E = \bigoplus_{\substack{\nu_i \geq \theta_i \geq \nu_i - 1 \\ \nu \neq \theta}} W_\theta \otimes V_\nu$$

It follows that

$$\mathrm{End}_{P(k)}(E) = \bigoplus_{(\tilde{\nu},\tilde{\theta},\tilde{\nu}') \in \mathbf{T}}{}' \; V_\nu^* \otimes V_\nu'$$

where the sum $\bigoplus'$ is the same as in formula (14), but the case $\nu = \theta = \nu'$ is excluded. Besides, we have $\mathcal{H} \otimes_{\mathbb{Z}[\mathbf{v},\mathbf{v}^{-1}]} \mathbb{C} = \bigoplus_\nu V_\nu^* \otimes V_\nu$. Adding these equalities, we obtain the required result.

□

5. Bimodule KL cells

5.1.

Consider all possible subbimodules of bimodule $\mathcal{R}$ spanned by subsets of basis $\underline{\tilde{H}}_{\tilde{w}}$. We say that two coloured permutation $\tilde{w}$ and $\tilde{w}'$ belong to the same Kazhdan-Lusztig bimodule cell if for each such subbimodule $\mathcal{M}$ we have $\underline{\tilde{H}}_{\tilde{w}} \in \mathcal{M} \iff \underline{\tilde{H}'}_{\tilde{w}} \in \mathcal{M}$. If instead of subbimodules we consider left or right submodules then we obtain the definition of left and right Kazhdan-Lusztig cells.

For $N = 3$, a big part (for β nonempty) of RB is depicted in [15] 1.3 with the help of Latin alphabet. It is a union of 13 two-sided KL cells: $\{\max\}$, $\{z,u\}$, $\{y,o,p,h\}$, $\{x,t\}$, $\{v,w,m,n\}$, $\{s,i,k,b\}$, $\{r,g\}$, $\{q,f\}$, $\{l,c\}$, $\{j,d\}$, $\{e\}$, $\{a\}$, $\{\min\}$. We take this opportunity to add two order relations missing in *loc. cit.*: $c \lessdot l$, $r \lessdot v$.

Conjecture 1. *The bimodule KL cells coincide with the two-sided microlocal cells.*

We are only able to prove an inclusion in one direction, see Theorem 3 below. First we have to formulate and prove a few lemmas.

5.2.

Consider the projection $\pi_i'\colon Y \to \mathrm{Fl}^{(i)}(V) \times \mathrm{Fl}(V) \times V \times \mathcal{N} \times \mathcal{N} \times V^*$ where $\mathrm{Fl}^{(i)}(V)$ is the variety of flags consisting of subspaces of V which have dimensions $0, \ldots, i-1, i+2, \ldots, N$. This projection sends a point $(F_1, F_2, v, u_1, u_2, v^*)$ to $(\tilde{F}_1, F_2, v, u_1, u_2, v^*)$ where $\tilde{F}_1$ is obtained from F_1 by deleting the subspaces of dimensions $i, i+1$. Let $Y_i = \pi_i'(Y)$.

Besides, for each $i \in \{1, \dots, N-1\}$ consider the set $\Phi_i \subset RB$ defined as follows: $(w, \beta) \in \Phi_i$ iff for a general $(F_1, F_2, v, u_1, u_2, v^*) \in Y_{w,\beta}$ we have $u_1|_{F_{1,i+1}/F_{1,i-1}} = 0$. Denote by $s_i = (i, i+1)$ the elementary transposition.

Lemma 8. *a) Let $(w, \beta) \in RB, i \in \{1, \dots, N-1\}$. Then $(w, \beta) \in \Phi_i$ iff $w(i) > w(i+1)$ and $\beta \cap \{i, i+1\} \neq \{i\}$.*

b) Let (w, β) and (w', β') be distinct colored permutations. Then $\pi'_i(Y_{w,\beta}) = \pi'_i(Y_{w',\beta'})$ iff $w(j) = w'(j)$ when $j \in \{1, \dots, N\} \setminus \{i, i+1, i+2\}$, $\beta \triangle \beta' := (\beta \setminus \beta') \cup (\beta' \setminus \beta) \subset \{i, i+1, i+2\}$, and one of the following conditions is satisfied up to interchanging (w, β) and (w', β'):

1. $w(i) < w(i+2) < w(i+1)$,
   $\beta \cap \{i, i+1, i+2\} \in \{\varnothing, \{i\}, \{i, i+2\}, \{i, i+1, i+2\}\}$,
   $w' = ws_i,\ \beta' = s_i(\beta)$;
2. $w(i+2) < w(i) < w(i+1)$,
   $\beta \cap \{i, i+1, i+2\} \in \{\varnothing, \{i+2\}, \{i, i+2\}, \{i, i+1, i+2\}\}$,
   $w' = ws_{i+1},\ \beta' = s_{i+1}(\beta)$;
3. $w(i+2) < w(i+1) < w(i),\ \beta \cap \{i, i+1, i+2\} = \{i, i+2\}$,
   $w' = ws_{i+1},\ \beta' = s_{i+1}(\beta)$;
4. $w(i+2) < w(i) < w(i+1),\ \beta \cap \{i, i+1, i+2\} = \{i\}$,
   $w' = ws_i,\ \beta' = s_i(\beta)$;
5. $w(i+2) < w(i+1) < w(i),\ \beta \cap \{i, i+1, i+2\} = \{i\}$,
   $w' = w,\ \beta' = \beta \cup \{i+1\}$.

Proof. a) Let $\{e_i\}$ be a basis of V and $x = (F_1, F_2, v) \in \Omega_{w,\beta}$ be the element given by (3), (4), (7). We must find the conormal space $N^*_x\Omega_{w,\beta} \subset T^*_x X$. It is isomorphic to the space of triples (u_1, u_2, v^*) such that u_k is a nilpotent preserving F_k and $u_1 + u_2 + v \otimes v^* = 0$. Let

$$u_k = \sum_{i,j=1}^{N} (u_k)_{i,j} E_{i,j}, \quad v^* = \sum_{i=1}^{N} c_i e_i^* \tag{15}$$

where $E_{i,j}$ is defined by (5) and e_i^* is the basis dual to e_i. Then the last relation is equivalent to the fact that the following conditions are satisfied:

$$\begin{cases} c_i = 0 & \text{for } i \in \beta; \\ (u_1)_{i,j} = -(u_2)_{i,j} & \text{for } i, j \in \beta \text{ or } i, j \notin \beta; \\ (u_k)_{i,j} = 0 & \text{for } i \notin \beta,\ j \in \beta; \\ (u_1)_{i,j} + (u_2)_{i,j} = -c_j & \text{for } i \in \beta,\ j \notin \beta,\ i < j,\ w(i) < w(j); \\ (u_1)_{i,j} = -c_j,\ (u_2)_{i,j} = 0 & \text{for } i \in \beta,\ j \notin \beta,\ w(i) > w(j); \\ (u_1)_{i,j} = 0,\ (u_2)_{i,j} = -c_j & \text{for } i \in \beta,\ j \notin \beta,\ i > j. \end{cases}$$

If we substitute $j = i+1$, we obtain the statement a) of the lemma.

b) Note that the fiber $(\pi'_i)^{-1}(\tilde{y})$ over an arbitrary point $\tilde{y} = (\tilde{F}_1, F_2, v, u_1, u_2, v^*) \in Y_i$ is isomorphic to the variety of full flags in the 3-dimensional space

$F_{1,i+2}/F_{1,i-1}$ fixed by $u_{1,i} = u_1|_{\tilde{F}_{1,i+2}/\tilde{F}_{1,i-1}}$. The structure of this variety depends on the type of $u_{1,i}$. There are three possibilities for this type: (3), (2, 1) and (1, 1, 1). Denote $W = \pi_i'(Y_{w,\beta})$, $W' = \pi_i'(Y_{w',\beta'})$. Since π_i' is proper, W and W' are closed. Suppose $W = W'$. This means that for a general point $\tilde{y} \in W$ the fiber $(\pi_i')^{-1}(\tilde{y})$ is reducible, so the type of $u_{1,i}$ equals (2, 1). Such fiber has a form of a union of two intersecting projective lines:

$$(\pi_i')^{-1}(\tilde{y}) = l_1(\tilde{y}) \cup l_2(\tilde{y});$$
$$l_1(\tilde{y}) = \{(F_1, F_2, v, u_1, u_2, v^*) \in (\pi_i')^{-1}(\tilde{y}) \mid F_{1,i}/F_{1,i-1} = \operatorname{im} u_{1,i}\};$$
$$l_2(\tilde{y}) = \{(F_1, F_2, v, u_1, u_2, v^*) \in (\pi_i')^{-1}(\tilde{y}) \mid F_{1,i+1}/F_{1,i-1} = \ker u_{1,i}\}.$$

Let $U \subset W$ be the set of all $\tilde{y} \in W$ such that the type of $u_{1,i}$ equals (2, 1). It is an open dense subset in W. Consider the sets $U_k = \bigcup_{\tilde{y} \in U} l_k(\tilde{y})$. The set $U_1 \cup U_2 = (\pi_i')^{-1}(U)$ is an open subset of $(\pi_i')^{-1}(W)$. Since $U_1 \cup U_2 \subset Y_{w,\beta} \cup Y_{w',\beta'}$ and U_k are irreducible, we must have either $Y_{w,\beta} = \overline{U_1}$, $Y_{w',\beta'} = \overline{U_2}$ or $Y_{w,\beta} = \overline{U_2}$, $Y_{w',\beta'} = \overline{U_1}$. Without loss of generality, we can assume that we have the first case. Then it is easy to see that $(w, \beta) \in \Phi_{i+1} \setminus \Phi_i$, $(w', \beta') \in \Phi_i \setminus \Phi_{i+1}$.

Conversely, if $(w, \beta) \in \Phi_i \setminus \Phi_{i+1}$ (resp. $(w, \beta) \in \Phi_{i+1} \setminus \Phi_i$) then the type of $u_{1,i}$ for a general $\tilde{y} \in W$ is (2, 1). Therefore, we have $Y_{w,\beta} = \overline{U_1}$ (resp. $Y_{w,\beta} = \overline{U_2}$). So, there exists exactly one $(w', \beta') \in \Phi_{i+1} \setminus \Phi_i$ (resp. $(w, \beta) \in \Phi_i \setminus \Phi_{i+1}$) such that $\pi_i'(Y_{w,\beta}) = \pi_i'(Y_{w',\beta'})$.

It is clear that the condition $(w, \beta) \in \Phi_{i+1} \setminus \Phi_i$ is equivalent to the fact that the first two parts of one of the conditions of the lemma are satisfied. So, we must prove that if (w', β') is given by the last two equations of this condition then we have $W = W'$. Choose a general $y \in Y_{w,\beta}$ and let $\tilde{y} = \pi_i'(y)$. Since $(w, \beta) \in \Phi_{i+1} \setminus \Phi_i$, we have $y \in l_1(\tilde{y})$. We must prove that $l_2(\tilde{y}) \subset Y_{w',\beta'}$. Let $y = (F_1, F_2, v, u_1, u_2, v^*)$ be given by (3), (4), (7) and (15) for some basis $\{e_i\}$.

First suppose the condition 1 of the lemma is satisfied. Then we have $a := (u_1)_{i,i+1} \neq 0$, $b := (u_1)_{i,i+2} \neq 0$, $(u_1)_{i+1,i+2} = 0$. So,

$$\ker u_{1,i} = \langle e_i, be_{i+1} - ae_{i+2} \rangle \bmod F_{1,i-1}.$$

Consider the space $F'_{1,i+1}$ such that $F'_{1,i+1}/F_{1,i-1} = \ker u_{1,i}$. Then

$$F'_{1,i+1} = g \cdot F_{1,i+1} \text{ where } g = \mathrm{id} - (a/b)E_{i+2,i+1} \in \mathrm{GL}(V).$$

A general point $y_1 \in l_2(\tilde{y})$ has a form $y_1 = (F_1'', F_2, v, u_1, u_2, v^*)$ where $F''_{1,i+1} = F'_{1,i+1}$ and $F''_{1,j} = F_{1,j}$ for $j \neq i, i+1$. If $F_1'' \neq g \cdot F_1$ then $F_1'' = ghs_i \cdot F_1$ where $h = \mathrm{id} + cE_{i,i+1}$ for some $c \in \mathsf{k}$, and $s_i \in \mathrm{GL}(V)$ is given by $s_i e_j = e_{s_i(j)}$.

Denote $g' = ghs_i$. Let $y_1' = (g')^{-1} \cdot y_1$. Then

$$y_1' = \big(F_1,\ (g')^{-1} \cdot F_2,\ (g')^{-1} \cdot v,\ (\mathrm{Ad}(g')^{-1}) \cdot u_1,\ (\mathrm{Ad}(g')^{-1}) \cdot u_2,\ (g')^* v^*\big)$$

Note that g and h preserve F_2, so we have

$$(g')^{-1} \cdot F_2 = s_i h^{-1} g^{-1} \cdot F_2 = s_i \cdot F_2$$

Therefore $(g')^{-1}F_{2,j} = \langle e_{w'(1)}, \ldots, e_{w'(j)} \rangle$ for all j. Further,

$$(g')^{-1}v = \sum_{j\in\beta} d_j s_i e_j = \sum_{j\in\beta'} d_{s_i(j)} e_j$$

where $d_j \in \mathsf{k}$ and $d_j \neq 0$ for general a, b, c. This implies that $y_1' \in N^*\Omega_{w',\beta'} \subset Y_{w',\beta'}$. Hence $y_1 = g' \cdot y_1' \in Y_{w',\beta'}$. Thus any general point $y_1 \in l_2(\tilde{y})$ lies in $Y_{w',\beta'}$. Therefore $l_2(\tilde{y}) \subset Y_{w',\beta'}$. QED.

Other cases can be considered in a similar way. □

5.3.

For each $(w,\beta) \in RB$ there exists at most one $(w',\beta') \in RB$ such that conditions of the above lemma are satisfied. We will denote it by $(w',\beta') = K_i(w,\beta)$.

Lemma 9. *Let $W = \pi(Y_{w,\beta})$ be the image of an irreducible component of Y. Choose an open dense subset $U \subset W$ such that the type λ of the nilpotent u_1 is the same for all points of U. Consider the set $C_W = \{(w',\beta') \in RB \mid \pi(Y_{w',\beta'}) = W\}$. There exists a natural bijection $\tau_W \colon C_W \to \mathrm{St}(\lambda)$ such that for each $p_0 \in U$ and $(w',\beta') \in C_W$ we have $Y_{w',\beta'} \cap \pi^{-1}(p_0) = \mathrm{Fl}_{u_1,\tau_W(w',\beta')} \times \{p_0\}$.*

Proof. For each $T \in \mathrm{St}(\lambda)$ consider the set

$$U_T = \bigcup_{p\in U} \left(\mathrm{Fl}_{u_1(p),T} \times \{p\}\right) \subset \pi^{-1}(U) \subset Y. \tag{16}$$

We have $\bigcup_{T\in\mathrm{St}(\lambda)} U_T = \pi^{-1}(U)$. The sets U_T are irreducible components of $\pi^{-1}(U)$. Note that the equation $\pi(Y_{w',\beta'}) = W$ is equivalent to the fact that $Y_{w',\beta'}$ dominates W (we use that π is proper). In this case $Y_{w',\beta'}$ must coincide with $\overline{U_T}$ for some T. In particular, $Y_{w,\beta} = \overline{U_{T_0}}$ for some $T_0 \in \mathrm{St}(\lambda)$. Since $\dim U_T = \dim U_{T_0} = \dim Y_{w,\beta} = \dim Y$ for each $T \in \mathrm{St}(\lambda)$, each $\overline{U_T}$ is an irreducible component of Y such that $\pi(\overline{U_T}) = W$. So, we have a one-to-one correspondence between the sets C_W and $\mathrm{St}(\lambda)$. Obviously, this correspondence can be described as in the statement of the lemma. □

5.4.

Let $\pi_i \colon \mathrm{Fl}(V) \to \mathrm{Fl}^{(i)}(V)$ be the natural projection. For each $i \in \{1, \ldots, N-1\}$ consider the set $\Phi_i' \subset \mathrm{St}(\lambda)$ defined as follows: $T \in \Phi_i'$ iff for a general $F \in \mathrm{Fl}_{u_1,T}$ we have $u_1|_{F_{i+1}/F_{i-1}} = 0$.

Lemma 10. *a) Let $T \in \mathrm{St}(\lambda)$. Then $T \in \Phi_i' \iff r_i(T) < r_{i+1}(T) \iff c_i(T) \geq c_{i+1}(T)$ where $r_i(T)$ (resp. $c_i(T)$) stands for the number of row (resp. column) in T containing i.*

b) Let $T, T' \in \mathrm{St}(\lambda)$ and $T \neq T'$. Then $\pi_i(\mathrm{Fl}_{u_1,T}) = \pi_i(\mathrm{Fl}_{u_1,T'})$ iff one of the following conditions is satisfied up to interchanging T and T' :

1.[2] *$r_{i+2}(T) \leq r_i(T) < r_{i+1}(T)$ and T' is obtained from T by interchanging $i+1$ and $i+2$.*

[2] the condition $r_{i+2}(T) \leq r_i(T) < r_{i+1}(T)$ is equivalent to $c_{i+2}(T) > c_i(T) \geq c_{i+1}(T)$.

2.[3] *$r_i(T) < r_{i+2}(T) \leq r_{i+1}(T)$ and T' is obtained from T by interchanging i and $i+1$.*

Proof. a) This statement is equivalent to Lemma 5.11 in [18].

b) Arguments similar to those used in the proof of Lemma 8 b) show that we can define an involution $K'_i\colon \Phi'_i \triangle \Phi'_{i+1} \to \Phi'_i \triangle \Phi'_{i+1}$ such that $K'_i(\Phi'_i \setminus \Phi'_{i+1}) = \Phi'_{i+1} \setminus \Phi'_i$ and such that a pair of tableaux $T \neq T' \in \mathrm{St}(\lambda)$ satisfies $\pi_i(\mathrm{Fl}_{u_1,T}) = \pi_i(\mathrm{Fl}_{u_1,T'})$ iff $T \in \Phi'_i \triangle \Phi'_{i+1}$ and $T' = K'_i(T)$. Thus we must prove that this involution can be described by the conditions 1, 2 of the lemma.

Suppose $T \in \Phi'_i \setminus \Phi'_{i+1}$ and $T' = K'_i(T) \in \Phi'_{i+1} \setminus \Phi'_i$. Then the first part of one of the conditions 1, 2 must be satisfied, and we must prove that T' is given by the second part. The equation $\pi_i(\mathrm{Fl}_{u_1,T}) = \pi_i(\mathrm{Fl}_{u_1,T'})$ implies that for $j \in \{0,\dots,N\} \setminus \{i, i+1\}$ and for general $F \in \mathrm{Fl}_{u_1,T}$ and $F' \in \mathrm{Fl}_{u_1,T'}$ the types of $u_1|_{F_j}$ and $u_1|_{F'_j}$ are the same. This means that T and T' can differ only in the position of $i,\ i+1,\ i+2$.

Moreover, choose a general point $\tilde{F} \in \pi_i(\mathrm{Fl}_{u_1,T})$. Let $l_k(\tilde{F})$ $(k = 1, 2)$ be two irreducible components of $\pi_i^{-1}(\tilde{F})$ defined similarly to the proof of Lemma 8 b), and let F', F'' be general points of $l_1(\tilde{F}), l_2(\tilde{F})$ respectively. Then F' (resp. F'') is a general point of $\mathrm{Fl}_{u_1,T'}$ (resp. $\mathrm{Fl}_{u_1,T}$). In particular, $F' \in \mathrm{Fl}^{T'}_{u_1}$, $F'' \in \mathrm{Fl}^{T}_{u_1}$. Let $l_1(\tilde{F}) \cap l_2(\tilde{F}) = \{F_1\}$, and let T_1 be the tableau such that $F_1 \in \mathrm{Fl}^{T_1}_{u_1}$. Then we obtain that T (resp. T') can differ from T_1 only in the position of i and $i+1$ (resp. $i+1$ and $i+2$).

If T satisfies the first part of the condition 2 of the lemma, the last condition and the conditions $T \in \Phi'_i \setminus \Phi'_{i+1}$ and $T' \in \Phi'_{i+1} \setminus \Phi'_i$ imply the desired statement.

If the first part of the condition 1 is satisfied, there is another a priori possible case:

$$\begin{cases} r_i(T_1) < r_{i+1}(T_1) < r_{i+2}(T_1) \\ c_i(T_1) > c_{i+1}(T_1) > c_{i+2}(T_1) \\ T \text{ is obtained from } T_1 \text{ by interchanging } i \text{ and } i+1 \\ T' \text{ is obtained from } T_1 \text{ by interchanging } i+1 \text{ and } i+2 \end{cases}$$

In this case consider the tableau T'' obtained from T by interchanging $i+1$ and $i+2$. If we apply the above argument to the pair $K'_i(T''), T''$ instead of T, T', we will obtain that $K'_i(T'') = T$, contradicting $K'_i(T') = T$. So, the lemma is proved. □

5.5.

For each $T \in \mathrm{St}(\lambda)$ there exists at most one tableau T' satisfying the conditions of Lemma 10. Denote it by $T' = K'_i(T)$.

Lemma 11. *Let W be the image of an irreducible component of Y under the map π. Then for each $i \in \{1,\dots,N-1\}$ we have $\tau_W(\Phi_i \cap C_W) \subset \Phi'_i$ and for each $i \in \{1,\dots,N-2\}$ we have $\tau_W \circ K_i = K'_i \circ \tau_W$.*

[3] the condition $r_i(T) < r_{i+2}(T) \leq r_{i+1}(T)$ is equivalent to $c_i(T) \geq c_{i+2}(T) > c_{i+1}(T)$.

Proof. In the proof of Lemma 9 we have shown that for each $(w, \beta) \in C_W$ we have $Y_{w,\beta} = \overline{U_T}$ where $T = \tau_W(w, \beta) \in \mathrm{St}(\lambda)$. Now the first inclusion follows immediately from the definition of Φ_i and Φ'_i. Let us prove the second equation. Suppose $(w', \beta') = K_i(w, \beta)$. Let $\tilde{\pi}_i \colon Y_i \to \tilde{Y}$ be the projection satisfying $\tilde{\pi}_i \circ \pi'_i = \pi$. Then

$$W = \pi(Y_{w,\beta}) = \tilde{\pi}_i(\pi'_i(Y_{w,\beta})) = \tilde{\pi}_i(\pi'_i(Y_{w',\beta'})) = \pi(Y_{w',\beta'})$$

So, $(w', \beta') \in C_W$. Denote $T = \tau_W(w, \beta)$, $T' = \tau_W(w', \beta')$. Then for each $p_0 \in U$ we have

$$\pi_i(\mathrm{Fl}_{u_1(p_0),T}) = \pi_i(Y_{w,\beta} \cap \pi^{-1}(p_0)) = \pi'_i(Y_{w,\beta}) \cap \tilde{\pi}_i^{-1}(p_0)$$

and the same for T'. The equation $(w', \beta') = K_i(w, \beta)$ means that $\pi'_i(Y_{w,\beta}) = \pi'_i(Y_{w',\beta'})$ and $(w, \beta) \neq (w', \beta')$. Taking the intersection with $\pi^{-1}(p_0)$, we get $\pi_i(\mathrm{Fl}_{u_1,T}) = \pi_i(\mathrm{Fl}_{u_1,T'})$ and $T \neq T'$ (this inequality follows from the fact that τ_W is bijective).

So, we get $T' = K'_i(T)$. □

5.6.

We write down the action of Hecke algebra generators on bimodule $\mathcal{R}$ in the Kazhdan-Lusztig basis $\underline{\tilde{H}}_{\tilde{w}}$. For $i \in \{1, \ldots, N-1\}$, recall the subset $\Phi_i \subset RB$ introduced in 5.2.

Let $\tilde{w}, \tilde{w}' \in RB$ and $\tilde{w}' < \tilde{w}$. Consider the restriction $IC(\overline{\Omega}_{\tilde{w}})|_{\Omega_{\tilde{w}'}}$ of the IC-sheaf of $\overline{\Omega}_{\tilde{w}}$ to $\Omega_{\tilde{w}'}$. It is a constant Tate complex on $\Omega_{\tilde{w}'}$ concentrated in cohomological degrees less than $-\mathbf{n} - \ell(\tilde{w}')$. We denote by $\mu(\tilde{w}', \tilde{w}) = \mu(\tilde{w}, \tilde{w}')$ the dimention $\dim H^{-\mathbf{n}-\ell(\tilde{w}')-1}(IC(\overline{\Omega}_{\tilde{w}})_x)$ where $x \in \Omega_{\tilde{w}'}$.

Proposition 9. *For any $\tilde{w} \in RB$ and $i \in 1, \ldots, N-1$ we have*

$$\underline{\tilde{H}}_{\tilde{w}} \cdot \underline{\tilde{H}}_{s_i} = \begin{cases} -(\mathbf{v}^{-1} + \mathbf{v})\underline{\tilde{H}}_{\tilde{w}}, & \text{if} \quad \tilde{w} \in \Phi_i \\ \underline{\tilde{H}}_{\tilde{w}*s_i} + \sum\limits_{\substack{\tilde{w}' < \tilde{w} \\ \tilde{w}' \in \Phi_i}} \mu(\tilde{w}', \tilde{w})\underline{\tilde{H}}'_{\tilde{w}} & \text{if} \quad \tilde{w} \notin \Phi_i \end{cases}$$

Proof. By definition, $\underline{\tilde{H}}_{\tilde{w}}$ is the class of the IC-sheaf of the orbit closure $\overline{\Omega}_{\tilde{w}}$, $\underline{\tilde{H}}_{\tilde{w}} = [IC(\overline{\Omega}_{\tilde{w}})]$. Therefore $\underline{\tilde{H}}_{\tilde{w}} \cdot \underline{\tilde{H}}_{s_i}$ is the class of the direct image of the IC-sheaf under the map $\psi \colon S = \overline{\Omega}_{s_i} \times_{\mathrm{Fl}(V)} \overline{\Omega}_{\tilde{w}} \to X$. If $\tilde{w} \in \Phi_i$ then the image of this map coincides with $\overline{\Omega}_{\tilde{w}}$ and all its fibers are isomorphic to $\mathbb{P}^1$, hence we obtain the required formula. If $\tilde{w} \notin \Phi_i$ then $\mathrm{im}(\psi) = \overline{\Omega}_{\tilde{w}*s}$ $(s = s_i)$ and all the fibers of ψ are isomorphic either to $\mathbb{P}^1$ or to a point. We claim that the direct image of the IC-sheaf under the map ψ is perverse. Indeed, pick an orbit $\Omega_{\tilde{w}'}$ inside $\overline{\Omega}_{\tilde{w}*s}$. We need to show that $\psi_*(IC(S))|_{\Omega_{\tilde{w}'}}$ is concentrated in degrees $\leq -\mathbf{n} - \ell(\tilde{w}')$. Let $x \in \Omega_{\tilde{w}'}$ and $Q = \psi^{-1}(x)$. Then we have $\psi_*(IC(S))_x = R\Gamma(Q, IC(S)|_Q)$. If $\tilde{w}' = \tilde{w} * s$ then Q is a point and $R\Gamma(Q, IC(S)|_Q) = \overline{\mathbb{Q}}_l[\mathbf{n} + \ell(\tilde{w}')]$. If Q is a point but $\tilde{w}' \neq \tilde{w} * s$ then the properties of IC-sheaf imply that $H^m(IC(S)|_Q) = 0$ for $m \geq -\mathbf{n} - \ell(\tilde{w}')$.

Otherwise, if $Q \cong \mathbb{P}^1$ (this happens if and only if $\tilde{w}' * s < \tilde{w}$), let $U = Q \cap \phi^{-1}(\Omega_{\tilde{w}'*s})$ where $\phi\colon S \to \overline{\Omega}_{\tilde{w}}$ is the projection to the second factor. Then U is open and dense in Q, and $IC(S)|_U$ is constant. The complex $IC(S)|_U$ is concentrated in degrees $\le -\mathbf{n} - \ell(\tilde{w}') - 2$, and $IC(S)|_{Q\setminus U}$ is concentrated in degrees $\le -\mathbf{n} - \ell(\tilde{w}') - 1$. From this we obtain that $H^m(IC(S)|_Q) = 0$ for $m > -\mathbf{n} - \ell(\tilde{w}')$ and $\dim H^{-\mathbf{n}-\ell(\tilde{w}')}(IC(S)|_Q) = \dim H^{-\mathbf{n}-\ell(\tilde{w}')-2}(IC(S)_{x'})$ where $x' \in U$. Note that if we identify U with $\phi(U)$, we have $IC(S)|_U = IC(\overline{\Omega}_{\tilde{w}})|_{\phi(U)}[1]$. Besides $\phi(U) \subset \Omega_{\tilde{w}'*s}$. If $\tilde{w}' \notin \Phi_i$ then $\tilde{w}' * s > \tilde{w}'$, and therefore $\dim H^{-\mathbf{n}-\ell(\tilde{w}')-2}(IC(S)_{x'}) = \dim H^{-\mathbf{n}-\ell(\tilde{w}'*s)}(IC(\overline{\Omega}_{\tilde{w}})_{\phi(x')}) = 0$. If $\tilde{w}' \in \Phi_i$ then $\tilde{w}' * s = \tilde{w}'$, and therefore $\dim H^{-\mathbf{n}-\ell(\tilde{w}')-2}(IC(S)_{x'}) = \dim H^{-\mathbf{n}-\ell(\tilde{w}*s)-1}(IC(\overline{\Omega}_{\tilde{w}})_{\phi(x')}) = \mu(\tilde{w}' * s, \tilde{w}) = \mu(\tilde{w}', \tilde{w})$. So we get

$$\dim H^m(\psi_*(IC(S))_x) = \dim H^m(IC(S)|_Q) = 0 \quad \text{if } m > -\mathbf{n} - \ell(\tilde{w}');$$

$$\dim H^{-\mathbf{n}-\ell(\tilde{w}')}(\psi_*(IC(S))_x) = \dim H^{-\mathbf{n}-\ell(\tilde{w}')}(IC(S)|_Q) = \begin{cases} 1 & \text{if } \tilde{w}' = \tilde{w} * s; \\ \mu(\tilde{w}', \tilde{w}) & \text{if } \tilde{w}' < \tilde{w} \text{ and } \tilde{w} \in \Phi_i; \\ 0 & \text{otherwise.} \end{cases}$$

Now, taking in account that $\psi_*(IC(S))$ is selfdual, we obtain the desired decomposition. □

Remark 2. Note that Proposition 9 implies that the bimodule $\mathcal{R}$ arises from a certain $\mathfrak{S}_N \times \mathfrak{S}_N^0$-graph Γ_{mir} in the terminology of [11], where $\mathfrak{S}_N^0 \cong \mathfrak{S}_N$ is the opposed group to $\mathfrak{S}_N$, i. e. $\mathfrak{S}_N^0 = \{g^0,\ g \in \mathfrak{S}_N\}$ with multiplication given by $g^0 h^0 = (hg)^0$. The set of vertices of Γ_{mir} is RB; the labels $I_{\tilde{w}}$ are defined by $I_{\tilde{w}} = \{\, s_i^0 \mid \tilde{w} \in \Phi_i \,\} \cup \{\, s_i \mid \tilde{w}^{-1} \in \Phi_i \,\}$; the edges are $\{\tilde{w}, \tilde{w}'\}$ such that $\tilde{w}' < \tilde{w}$, $\mu(\tilde{w}', \tilde{w}) \ne 0$ and $I_{\tilde{w}} \ne I_{\tilde{w}'}$; finally, the multiplicities are $\mu(\tilde{w}, \tilde{w}')$.

5.7. One-sided microlocal cells

Let $W = \pi(Y_{\tilde{w}}) \quad (\tilde{w} \in RB)$ be the image of an irreducible component of Y. We define the *right microlocal cell* corresponding to W as the set C_W described in Lemma 9. We define a *left microlocal cell* as the image of a right microlocal cell under the involution $\tilde{w} \mapsto \tilde{w}^{-1}$. In terms of bijection $\mathrm{RSK}_{\mathrm{mir}}$ introduced in 3.4, two-sided microlocal cells are given by condition $\mathbf{t}(\tilde{w}) = const$. The left microlocal cells are given by conditions $\mathbf{t}(\tilde{w}) = const$ and $T_1(\tilde{w}) = const$, and the right microlocal cells are given by conditions $\mathbf{t}(\tilde{w}) = const$ and $T_2(\tilde{w}) = const$. Each two-sided microlocal cell is a union of left microlocal cells and of right microlocal cells as well; moreover, each left microlocal cell and right microlocal cell inside the same two-sided microlocal cell intersect exactly in one element. Two-sided microlocal cells are the minimal subsets which are unions of both left and right microlocal cells.

Now recall that $\mathbf{t}(\tilde{w}) = (\nu, \theta, \nu') =: (\nu(\tilde{w}), \theta(\tilde{w}), \nu'(\tilde{w}))$.

Theorem 3. *a) Each left (right, two-sided) microlocal cell is contained in a left (resp. right, bimodule) Kazhdan-Lusztig cell.*

b) Conversely, for $\tilde{w}_1, \tilde{w}_2$ in the same left (right, bimodule) Kazhdan-Lusztig cell, we have $\nu(\tilde{w}_1) = \nu(\tilde{w}_2)$ (resp. $\nu'(\tilde{w}_1) = \nu'(\tilde{w}_2)$, resp. $\nu(\tilde{w}_1) = \nu(\tilde{w}_2)$ and $\nu'(\tilde{w}_1) = \nu'(\tilde{w}_2)$).

Proof. It suffices to prove the theorem for one-sided cells and, by the reason of symmetry, only for right-handed ones. Let us formulate the following auxiliary proposition.

Proposition 10. *Two elements $\tilde{w}, \tilde{w}'$ lie in the same right microlocal cell iff there is a sequence $\tilde{w} = \tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_m = \tilde{w}'$ such that for each $j = 1, \ldots, m-1$ there is $i \in \{1, \ldots, N-2\}$ such that $\tilde{w}_{j+1} = K_i(\tilde{w}_j)$ (see 5.3).*

Proof. It is easy to see from the definition of operations K_i that if $\tilde{w}' = K_i(\tilde{w})$ then $\tilde{w}$ and $\tilde{w}'$ lie in the same right microlocal cell. This implies the "only if" direction. Conversely, let $\tilde{w}$ and $\tilde{w}'$ lie in a microlocal cell corresponding to W. Consider the bijection $\tau_W : C_W \to \mathrm{St}(\lambda)$ of Lemma 9. In view of Lemma 11 it suffices to prove that any 2 standard Young tableaux of the same shape can be obtained from each other by a successive application of operations K_i'.

It can be checked directly. □

It is easy to check that if $\tilde{w}' = K_i(\tilde{w})$ then, up to permutation of $\tilde{w}$ and $\tilde{w}'$, we have $\tilde{w} \lessdot \tilde{w}'$, moreover

$$\begin{cases} \tilde{w}' = \tilde{w} * s_i \\ \tilde{w} \in \Phi_{i+1} \\ \tilde{w}' \notin \Phi_{i+1} \end{cases} \quad \text{or} \quad \begin{cases} \tilde{w}' = \tilde{w} * s_{i+1} \\ \tilde{w} \in \Phi_i \\ \tilde{w}' \notin \Phi_i \end{cases}$$

Observe that if $\tilde{w} < \tilde{w}'$ and $\ell(\tilde{w}') = \ell(\tilde{w}) + 1$ then $\mu(\tilde{w}, \tilde{w}') = 1$, so, taking in account Proposition 9, it follows that if $\tilde{w}' = K_i(\tilde{w})$ then $\tilde{w}'$ and $\tilde{w}$ lie in the same right Kazhdan-Lusztig cell. Therefore, in view of Proposition 10, each microlocal cell lies in a Kazhdan-Lusztig cell. So the proof of Theorem 3.a) is finished.

For the proof of b), we can realize the $\mathcal{H}$-bimodule $\mathcal{R}$ in the Grothendieck group of G-equivariant Hodge D-modules on X. Then we have the functor of singular support from the category of G-equivariant Hodge D-modules on X to the category of G-equivariant coherent sheaves on T^*X supported on Y. Similarly, we have the functor of singular support from the category of G-equivariant Hodge D-modules on $\mathrm{Fl}(V) \times \mathrm{Fl}(V)$ to the category of G-equivariant coherent sheaves on the Steinberg variety of G. These functors are compatible with the convolution action. Thus if $IC_{\tilde{w}_1}$ is a direct summand of the convolution of $IC_{\tilde{w}_2}$ with IC_w, and $({}^1u_1,\ {}^1u_2,\ {}^1v,\ {}^1v^*)$ (resp. $({}^2u_1,\ {}^2u_2,\ {}^2v,\ {}^2v^*)$) is a general element in the conormal bundle to $\Omega_{\tilde{w}_1}$ (resp. $\Omega_{\tilde{w}_2}$), then ${}^1u_1$ must lie in the closure of G-orbit of ${}^2u_1$ (and similarly, ${}^1u_2$ must lie in the closure of G-orbit of ${}^2u_2$). The proof of b) is completed. □

5.8. Fourier duality

In this subsection we will write $X(V), Y(V), \Omega_{\tilde{w}}(V), \ldots$ instead of $X, Y, \Omega_{\tilde{w}}, \ldots$ to emphasize the dependence on V. All the statements in this subsection are straightforward and left to the reader as an exercise.

Note that we have a canonical isomorphism $Y(V) \cong Y(V^*)$, $(F_1, F_2, v, u_1, u_2, v^*) \mapsto (F_1^*, F_2^*, v^*, u_1^*, u_2^*, v)$. Therefore we get a bijection between the sets of their irreducible components, which gives rise to an involution F on RB.

Proposition 11. *For any $\tilde{w} = (w, \beta) \in RB$ we have $\mathrm{F}(\tilde{w}) = (w_0 w w_0, \{1, \ldots, N\} \setminus w_0(\beta))$ where $w_0 \in \mathfrak{S}_N$ is the longest element, i. e. $w_0(i) = N + 1 - i$.* □

Further, we have an isomorphism $\psi\colon Z(V) \xrightarrow{\sim} Z(V^*)$. It carries images of irreducible components of $Y(V)$ to images of irreducible components of $Y(V^*)$, therefore $\psi(Z_{\mathbf{t}}(V)) = Z_{\mathbf{t}^*}(V^*)$ for some $\mathbf{t}^* \in \mathbf{T}$.

Proposition 12. *If $\mathbf{t} = (\nu, \theta, \nu') \in \mathbf{T}$ then $\mathbf{t}^* = (\nu, \theta^*, \nu')$ where $\theta_i^* = \min\{\nu_i, \nu_i'\} + \max\{\nu_{i+1}, \nu_{i+1}'\} - \theta_i$.* □

Proposition 13. *If $\mathrm{RSK}_{\mathrm{mir}}(\tilde{w}) = (\mathbf{t}, T_1, T_2)$ then $\mathrm{RSK}_{\mathrm{mir}}(\mathrm{F}(\tilde{w})) = (\mathbf{t}^*, T_1^*, T_2^*)$ where $\mathbf{t}^*$ is the same as in Proposition 12, and T_1^*, T_2^* are conjugate tableaux to T_1, T_2 (see [9] for the definition). Besides, the partition $\theta^*(\tilde{w}) = \theta(\mathrm{F}(\tilde{w}))$ is the shape of the tableau $T_N^{@}$ from 3.5 with all @'s removed.* □

Corollary 6. *The involution F on RB carries left, right, and two-sided microlocal cells to left, right, and two-sided microlocal cells, respectively.* □

Now consider the Fourier-Deligne transform FD from the derived constructible G-equivariant category of $\overline{\mathbb{Q}}_l$-sheaves on $X(V) = \mathrm{Fl}(V) \times \mathrm{Fl}(V) \times V$ to the derived constructible G-equivariant category of $\overline{\mathbb{Q}}_l$-sheaves on $X(V^*) = \mathrm{Fl}(V^*) \times \mathrm{Fl}(V^*) \times V^* \cong \mathrm{Fl}(V) \times \mathrm{Fl}(V) \times V^*$. It gives rise to an involution F on $\mathcal{R}$ which is compatible with the automorphism of the algebra $\mathcal{H}$ induced by conjugation with w_0 on the Coxeter group $\mathfrak{S}_N$. It carries G-equivariant IC-sheaves on $X(V)$ to G-equivariant IC-sheaves on $X(V^*)$. Therefore we obtain the following

Proposition 14. *For any $\tilde{w} \in RB$ we have $\mathrm{F}(\underline{\tilde{H}}_{\tilde{w}}) = \underline{\tilde{H}}_{\mathrm{F}(\tilde{w})}$.* □

Corollary 7. *The involution F on RB carries left, right, and bimodule Kazhdan-Lusztig cells to left, right, and bimodule Kazhdan-Lusztig cells, respectively.* □

5.9. Relation to mirabolic character sheaves

Recall the definition of unipotent mirabolic character sheaves on $\mathrm{GL}(V) \times V$, cf. [7] 4.1 and [8] 5.2. We consider the following diagram of $\mathrm{GL}(V)$-varieties and $\mathrm{GL}(V)$-equivariant maps:

$$\mathrm{GL}(V) \times V \xleftarrow{pr} \mathrm{GL}(V) \times \mathrm{Fl}(V) \times V \xrightarrow{f} \mathrm{Fl}(V) \times \mathrm{Fl}(V) \times V.$$

In this diagram, the map pr is given by $pr(g,x,v) := (g,v)$, while the map f is given by $f(g,x,v) := (gx,x,gv)$. The group $\mathrm{GL}(V)$ acts diagonally on all the product spaces in this diagram, and acts on itself by conjugation.

The functor CH from the constructible derived category of l-adic sheaves on $\mathrm{Fl}(V)\times\mathrm{Fl}(V)\times V$ to the constructible derived category of l-adic sheaves on $\mathrm{GL}(V)\times V$ is defined as $\mathsf{CH} := pr_* f^![-\dim\mathrm{Fl}(V)]$. Now let $\mathcal{F}$ be a $\mathrm{GL}(V)$-equivariant perverse sheaf on $\mathrm{Fl}(V)\times\mathrm{Fl}(V)\times V$. The irreducible perverse constituents of $\mathsf{CH}\mathcal{F}$ are called unipotent mirabolic character sheaves on $\mathrm{GL}(V)\times V$. Clearly, this definition is a direct analogue of Lusztig's definition of character sheaves.

Recall the following examples of unipotent mirabolic character sheaves (see [6] 5.4). For $M\le N$ let $\tilde{\mathfrak{X}}_{N,M}$ be a smooth variety of triples $(g,F_\bullet,v)$ where $g\in\mathrm{GL}(V)$, and $F_\bullet\in\mathrm{Fl}(V)$ is a complete flag preserved by g, and $v\in F_M$. We have a proper morphism $\pi_{N,M}:\ \tilde{\mathfrak{X}}_{N,M}\to\mathrm{GL}(V)\times V$ (forgetting $F_\bullet$) with the image $\mathfrak{X}_{N,M}\subset\mathrm{GL}(V)\times V$ formed by all the pairs (g,v) such that $\dim\langle v,gv,g^2v,\ldots\rangle\le N-M$. According to Corollary 5.4.2 of *loc. cit.*, we have

$$(\pi_{N,M})_*IC(\tilde{\mathfrak{X}}_{N,M})\simeq\bigoplus_{|\mu|=M}^{|\lambda|=N-M}L_\mu\otimes L_\lambda\otimes\mathcal{F}_{\lambda,\mu}$$

for certain unipotent mirabolic character sheaves $\mathcal{F}_{\lambda,\mu}$ supported on $\mathfrak{X}_{N,M}$, and L_λ, resp. L_μ, is an irreducible representation of $\mathfrak{S}_{N-M}$, resp. $\mathfrak{S}_M$.

We conjecture the following formula for the class of $\mathsf{CH}\underline{\tilde{H}}_{\tilde{w}}$ in the K-group of unipotent mirabolic Weil sheaves.

Conjecture 2. $\mathsf{CH}\underline{\tilde{H}}_{\tilde{w}}=\sum_{|\lambda|+|\mu|=N}f_{\lambda,\mu}(\underline{\tilde{H}}_{\tilde{w}})[\mathcal{F}_{\lambda,\mu}]$ *where* $f_{\lambda,\mu}$ *is a functional* $\mathcal{R}\to\mathbb{Z}[\mathbf{v},\mathbf{v}^{-1}]$ *such that* $f_{\lambda,\mu}(hr)=f_{\lambda,\mu}(rh)$ *for any* $r\in\mathcal{R},\ h\in\mathcal{H}$. *Moreover, in the decomposition* (14) *of Proposition* 8, $f_{\lambda,\mu}$ *vanishes on all the summands except for* $V_\nu^*\otimes V_\nu$ *corresponding to* $(\tilde{\nu},\tilde{\theta},\tilde{\nu})\in\mathbf{T}$ *where* $\Upsilon(\lambda,\mu)=(\nu,\theta)$.

5.10. Asymptotic bimodule

For a partition ν of N, let $c_\nu\subset\mathfrak{S}_N$ be the corresponding two-sided KL cell. Let $a(c_\nu)=a_\nu=N^2-N-n_\nu:=\frac{N^2-N}{2}-\sum_{i\ge1}(i-1)\nu_i$ be its a-function. For multiplication in $\mathcal{H}$ we have $\underline{\tilde{H}}_w\cdot\underline{\tilde{H}}_y=\sum_{z\in\mathfrak{S}_N}m_{w,y,z}\underline{\tilde{H}}_z$, for $m_{w,y,z}\in\mathbb{Z}[\mathbf{v},\mathbf{v}^{-1}]$. If $w,y,z\in c_\nu$ then, according to Lusztig, the degree of $m_{w,y,z}$ is less than or equal to a_ν. Let $\gamma_{w,y,z}\in\mathbb{Z}$ be the coefficient of $\mathbf{v}^{a_\nu}$ in $m_{w,y,z}$. Lusztig's asymptotic ring J_ν is defined as a ring with a basis $\{t_w,\ w\in c_\nu\}$ and multiplication $t_w\cdot t_y=\sum_{z\in c_\nu}\gamma_{w,y,z}t_z$. By the classical RSK algorithm, c_ν is in bijection with the set of pairs of standard tableaux $\{(T_1,T_2)\}$ of shape ν. According to [12] 3.16.b), the ring J_ν with basis $\{t_w\}$ is isomorphic to the matrix ring $\mathrm{Mat}_{\mathrm{St}(\nu)}$ with the basis of elementary matrices $\{e_{T_1,T_2}\}$, so that t_w goes to e_{T_1,T_2} where (T_1,T_2) are constructed from w by the classical RSK algorithm.

Now for a pair of partitions $\nu\supset\theta$ we consider the corresponding bimodule KL cell $c_{\nu\supset\theta\subset\nu}\subset RB$. For $\tilde{w}\in RB$, and $y\in\mathfrak{S}_N$ we have $\underline{\tilde{H}}_{\tilde{w}}\cdot\underline{\tilde{H}}_y=\sum_{\tilde{z}\in RB}m_{\tilde{w},y,\tilde{z}}\underline{\tilde{H}}_{\tilde{z}}$, and $\underline{\tilde{H}}_y \cdot \underline{\tilde{H}}_{\tilde{w}} = \sum_{\tilde{z}\in RB} m_{y,\tilde{w},\tilde{z}}\underline{\tilde{H}}_{\tilde{z}}$. We conjecture that for $\tilde{w},\tilde{z} \in c_{\nu\supset\theta\subset\nu}$, and $y \in c_\nu$, the degrees of $m_{\tilde{w},y,\tilde{z}}$ and $m_{y,\tilde{w},\tilde{z}}$ are less than or equal to a_ν. We denote by $\gamma_{y,\tilde{w},\tilde{z}}$ the coefficient of $\mathbf{v}^{a_\nu}$ in $m_{y,\tilde{w},\tilde{z}}$, and we denote by $\gamma_{\tilde{w},y,\tilde{z}}$ the coefficient of $\mathbf{v}^{a_\nu}$ in $m_{\tilde{w},y,\tilde{z}}$. We define the asymptotic bimodule $J_{\nu\supset\theta\subset\nu}$ over J_ν as a bimodule with a basis $\{t_{\tilde{w}},\ \tilde{w} \in c_{\nu\supset\theta\subset\nu}\}$ and the action $t_{\tilde{w}} \cdot t_y = \sum_{\tilde{z}\in c_{\nu\supset\theta\subset\nu}} \gamma_{\tilde{w},y,\tilde{z}} t_{\tilde{z}}$, and $t_y \cdot t_{\tilde{w}} = \sum_{\tilde{z}\in c_{\nu\supset\theta\subset\nu}} \gamma_{y,\tilde{w},\tilde{z}} t_{\tilde{z}}$.

Conjecture 3. *The based bimodule* $J_{\nu\supset\theta\subset\nu}, \{t_{\tilde{w}},\ \tilde{w} \in c_{\nu\supset\theta\subset\nu}\}$ *is isomorphic to the based regular bimodule* $\mathrm{Mat}_{\mathrm{St}(\nu)}, \{e_{T_1,T_2}\}$*, so that* $t_{\tilde{w}}$ *goes to* e_{T_1,T_2} *where* (T_1,T_2) *are constructed from* $\tilde{w}$ *by the mirabolic RSK algorithm.*

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