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A. D. MacDonald

Abstract

The confluent hypergeometric function is useful in many problems in theoretical physics, in particular as the solution of the differential equation for the velocity distribution function of electrons in a high frequency gas discharge. This report presents some of the properties of this function together with six-figure tables and charts for the parameter values γ = 0.5, 1.0, 1.5, and 2.0 and a = 0.001, 0.01, 0.05, 0.1, 0.2, 0.25, 0.3, 0.4, 0.5, 0.6, 0.7, 0.75, 0.8, 0.9, and 1.0 as well as tables of the second solution of the differential equation for γ = 1, 2, and 3 at the same values of a. The tables give values for arguments up to 8 by steps of 0.5.

PROPERTIES OF THE CONFLUENT HYPERGEOMETRIC FUNCTION

The confluent hypergeometric functions have proved useful in many branches of physics. They have been used in problems involving both diffusion and sedimentation, for example, in isotope separation and protein molecular weight determinations in the ultracentrifuge. The solution of the equation for the velocity distribution of electrons in high frequency gas discharges may frequently be expressed in terms of these functions. The high frequency breakdown electric field may then be predicted theoretically for gases by the use of such solutions together with kinetic theory.

This report presents some of the properties of the confluent hypergeometric functions together with six-figure tables and charts of the functions for the parameter values γ = 0.5, 1.0, 1.5, and 2.0 and α = 0.001, 0.01, 0.05, 0.1, 0.2, 0.25, 0.3, 0.4, 0.5, 0.6, 0.7, 0.75, 0.8, 0.9, and 1.0 as well as tables of the second solution of the differential equation for γ = 1, 2, and 3 at the same values of α.[1] The tables give values for arguments up to 8 by steps of 0.5. For values of the argument above 8, the asymptotic expansions suffice.

1. Definitions

The confluent hypergeometric function $M(\alpha;\gamma;z)$ is defined as the solution, bounded at the origin of the second order linear homogeneous differential equation

$$z\frac{d^2M}{dz^2} + (\gamma - z)\frac{dM}{dz} - \alpha M = 0 \qquad (1)$$

where γ, α, and z are unrestricted. Equation (1) has a regular singularity at the origin and an irregular singularity at infinity.

If γ is not integral, a second solution of Eq. (1) is given by

$$W(\alpha;\gamma;z) = z^{1-\gamma} M(\alpha - \gamma + 1;\ 2 - \gamma;\ z)\ . \qquad (2)$$

If γ is integral, a second solution is given by

$$W(\alpha;\gamma;z) = M(\alpha;\gamma;z)\left\{\ln z + \psi(1-\alpha) - \psi(\gamma) + C\right\} + \sum_{n=1}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(\gamma)B_n z^n}{\Gamma(\alpha)\Gamma(n+\gamma)n!} +$$

$$+ (-1)^{\gamma}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma)\Gamma(n+\alpha-\gamma+1)\Gamma(\gamma-n-1)(-1)^n}{\Gamma(\alpha)\ n!\ z^{\gamma-n-1}} \qquad (3)$$

[2]

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1. The notation used in this report is that of E. Jahnke and F. Emde, "Tables of Functions", Teubner, Leipzig, 1933.
2. W. J. Archibald, Phil. Mag. (London) 7, 26, 419 (1938).

where $$\psi(a) = \frac{\Gamma'(a)}{\Gamma(a)} \quad ,$$

C is Eulers constant 0.577216 ... ,

and $$B_n = \left(\frac{1}{a} + \frac{1}{a+1} + \dots \frac{1}{a+n-1}\right) - \left(\frac{1}{\gamma} + \frac{1}{\gamma+1} + \dots \frac{1}{\gamma+n-1}\right) - \left(1 + \frac{1}{2} + \dots \frac{1}{n}\right)$$

Extensive tables of the ψ function are available.[1]

2. Series Representation

The following series converges absolutely for all values of z.

$$M(a;\gamma;z) = 1 + \frac{a}{\gamma} z + \frac{a(a+1)}{\gamma(\gamma+1)} \frac{z^2}{2} + \dots = \sum_{n=0}^{\infty} \frac{\Gamma(\gamma)\Gamma(a+n)}{\Gamma(a)\Gamma(\gamma+n)} \frac{z^n}{n!} \tag{4}$$

3. Recurrence Relations

Both M and W satisfy the following recurrence relations

$$\frac{d}{dz} M(a;\gamma;z) = \frac{a}{\gamma} M(a+1;\gamma+1;z) \tag{5}$$

$$aM(a+1;\gamma+1;z) = (a-\gamma)M(a;\gamma+1;z) + \gamma M(a;\gamma;z) \tag{6}$$

$$aM(a+1;\gamma;z) = (z+2a-\gamma)M(a;\gamma;z) + (\gamma-a)M(a-1;\gamma;z) \tag{7}$$

Equation (5) follows directly as the differentiation of Eq. (4), and Eqs. (6) and (7) may be shown directly from the differential equation. A relation between $M(a;\gamma;z)$ and $W(a;\gamma;z)$ may be obtained by the use of the Wronskian of Eq. (1), and the functional relation between the value of a Wronskian at any point in a plane and its value at a given point. Carrying out the required integration results in, for non integral γ ,

$$M(a;\gamma;z)W(a+1;\gamma+1;z) - M(a+1;\gamma+1;z)W(a;\gamma;z) = \frac{\gamma(1-\gamma)}{a} \frac{e^z}{z^\gamma} \ . \tag{8}$$

4. Asymptotic Expansions

$$M(a;\gamma;z) \sim \frac{\Gamma(\gamma)}{\Gamma(a)} e^z z^{a-\gamma} G(1-a;\gamma-a;z) \tag{9}$$

if the real part of z, Re(z) is greater than zero.

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1. H. T. Davis, "Tables of the Higher Mathematical Functions", Vol. I, The Principia Press, Bloomington, Ind., 1933.

$$M(\alpha;\gamma;z) \sim \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}(-z)^{-\alpha}\, G(\alpha;\ \alpha-\gamma-1;\ -z) \qquad (10)$$

if Re(z) is less than zero, where

$$G(\alpha;\gamma;z) = 1 + \frac{\alpha}{1!z} + \frac{\alpha(\alpha+1)\,\gamma\,(\gamma+1)}{2!z^2} + \dots \quad . \qquad (11)$$

For non-integral γ, the asymptotic expansion of $W(\alpha;\gamma;z)$ follows from Eqs. (2), (8), and (9). For integral γ ,

$$W(\alpha;\gamma;z) \sim \pi\ \cot(\pi\alpha)\,\frac{\Gamma(\gamma)}{\Gamma(\alpha)}\, e^z z^{\alpha-\gamma}\ G(1-\alpha;\ \gamma-\alpha;\ z) \qquad (12)$$

$$\mathrm{Re}(z) > 0$$

$$W(\alpha;\gamma;z) \sim i\pi\,\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}(-z)^{-\alpha}\, G(\alpha;\ \alpha-\gamma-1;\ -z) \qquad (13)$$

$$\mathrm{Re}(z) < 0 \ .$$

The asymptotic series (9) to (13) may be developed by expressing the series expansion for $M(\alpha;\gamma;z)$ as a contour integral of the Barnes type, expanding the integrand by the binomial theorem and calculating the residues of each term.

5. Contour Integral Representation

$$M(\alpha;\gamma;z) = \frac{\Gamma(\gamma)}{2\pi i}\int_C \left(1-\frac{z}{t}\right)^{-\alpha} e^t t^{-\gamma}\, dt \qquad (14)$$

where the contour C encircles the origin and the point z in a counter-clockwise direction. There is a cut in the t-plane between t = 0 and t = z.

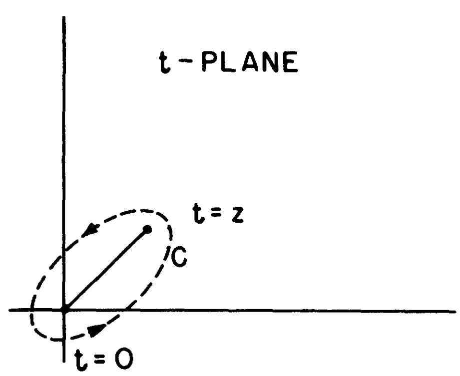


Fig. 1. Contour for integral in Eq. (14)

6. Indefinite Integrals

The confluent hypergeometric function has been used in this laboratory most often as a solution of the differential equation for the electron velocity distribution function, and in this application the following integrals have been found useful:

$$\int e^{-z} M(\alpha;\gamma;z)\,dz = \frac{e^{-z}}{\alpha-\gamma+1}\left\{ z\frac{d}{dz}\left[M(\alpha;\gamma;z)\right] + (\gamma-1)M(\alpha;\gamma;z)\right\} \tag{15}$$

$$\int z e^{-z} M(\alpha;\gamma;z)\,dz = \frac{e^{-z}}{\alpha-\gamma+2}\left\{ (z - \frac{\gamma-2}{\alpha-\gamma+1}) \left[(z\frac{d}{dz} M(\alpha;\gamma;z) + (\gamma-1)M(\alpha;\gamma;z)\right] - \right.$$

$$\left. - zM(\alpha;\gamma;z)\right\} . \tag{16}$$

In particular

$$\gamma = 2 \qquad \int z e^{-z} M(\alpha;2;z)\,dz = \frac{e^{-z}}{\alpha} z^2 \frac{d}{dz} M(\alpha;2;z) \tag{17}$$

$$\gamma = 1 \qquad \int z e^{-z} M(\alpha;1;z)\,dz = \frac{e^{-z}}{\alpha+1} z \left\{ (\frac{\alpha z+1}{\alpha})\frac{d}{dz} M(\alpha;1;z) - M(\alpha;1;z)\right\} \tag{18}$$

$$\int z^{\gamma} e^{-z(1-\frac{\alpha}{\gamma})} M(\alpha;\gamma;z)\,dz = \frac{\gamma}{\alpha}\frac{z^{\gamma}}{(1-\frac{\alpha}{\gamma})} e^{-z(1-\frac{\alpha}{\gamma})}\left\{\frac{d}{dz}\left[M(\alpha;\gamma;z)\right] - \frac{\alpha}{\gamma} M(\alpha;\gamma;z)\right\} \tag{19}$$

$$\int z e^{-z(1-\frac{\alpha}{\gamma})} M(\alpha-\tfrac{1}{2};\gamma-1;z)\,dz = \frac{\gamma}{\alpha}\frac{e^{-z(1-\frac{\alpha}{\gamma})}}{(1-\frac{\alpha}{\gamma})}\left\{ z\frac{d}{dz}\left[M(\alpha-\tfrac{1}{2};\gamma-1;z)\right] - \right.$$

$$\left. - M(\alpha-\tfrac{1}{2};\gamma-1;z)\left[\frac{1}{2}+\frac{\alpha}{\gamma}z\right]\right\} . \tag{20}$$

Setting $\gamma = 3/2$, Eqs. (19) and (20) become

$$\int z^{3/2} e^{-z(1-\frac{2}{3}\alpha)} M(\alpha;\tfrac{3}{2};z)\,dz = \frac{3}{2\alpha}\frac{z^{3/2}}{(1-\frac{2}{3}\alpha)} e^{-z(1-\frac{2}{3}\alpha)}\left\{\frac{d}{dz}\left[M(\alpha;\tfrac{3}{2};z)\right] - \tfrac{2}{3}\alpha M(\alpha;\tfrac{3}{2};z)\right\} \tag{21}$$

$$\int z e^{-z(1-\frac{2}{3}\alpha)} M(\alpha-\tfrac{1}{2};\tfrac{1}{2};z)\,dz = \frac{3e^{-z(1-\frac{2}{3}\alpha)}}{2\alpha(1-\frac{2}{3}\alpha)}\left\{ z\frac{d}{dz}\left[M(\alpha-\tfrac{1}{2};\tfrac{1}{2};z)\right] - \right.$$

$$\left. - M(\alpha-\tfrac{1}{2};\tfrac{1}{2};z)\left[\frac{1}{2}+\frac{2}{3}\alpha z\right]\right\} . \tag{22}$$

These integrals are worked out by setting $M(\alpha;\gamma;z) = \frac{S(z)}{f(z)}$, where $f(z)M(\alpha;\gamma;z)$ is the integrand of the required integral. The expression is then differentiated, put into Eq. (1) and the resulting terms integrated by parts. The results may be checked by differentiation and the application of the defining equation in the right hand side of Eqs. (15) - (22).

7. Differential Equations Involving $M(\alpha;\gamma;z)$

Some second order differential equations which have solutions involving the confluent hypergeometric function follow. These are presented in the normal form in order to give as much information as possible in a few equations.[1]

$$\frac{d^2y}{dx^2} - \frac{x^2 + (4\alpha - 2\gamma)x + \gamma^2 - 2\gamma}{4x^2}\, y = 0$$

(23)

$$y = x^{\frac{\gamma}{2}} e^{-\frac{x}{2}} M(\alpha;\gamma;x)$$

$$\frac{d^2y}{dx^2} - \frac{a^2x + b}{4x}\, y = 0$$

(24)

$$y = axe^{-\frac{ax}{2}} M(1 + \frac{b}{4a};\ 2;\ ax)$$

$$\frac{d^2y}{dx^2} - \frac{a^2x^2 + bx + c}{4x^2}\, y = 0$$

(25)

$$y = (ax)^{1/2(1 \pm \sqrt{1 + c})} e^{-\frac{ax}{2}} M(\frac{b}{4a} + \frac{1}{2}(1 \pm \sqrt{1 + c});\ 1 \pm \sqrt{1 + c};\ ax)$$

$$\frac{d^2y}{dx^2} - \frac{a^2x^4 + bx^2 + c}{x^2}\, y = 0$$

(26)

$$y = x^{-1/2}(ax^2)^{1/2(1 \pm \sqrt{\frac{1}{4} + c})} e^{-\frac{ax^2}{2}} M(\frac{b}{4a} + \frac{1}{2}(1 \pm \sqrt{\frac{1}{4} + c});\ 1 \pm \sqrt{\frac{1}{4} + c};\ ax^2)$$

---

1. The transformation of a second order linear homogeneous differential equation to the normal form is simple. If

$$\frac{d^2f}{dx^2} + p(x)\frac{df}{dx} + q(x)f = 0 \quad , \qquad \text{(a)}$$

let

$$f(x) = e^{-1/2\int p\, dx}\, y(x) \quad , \qquad \text{(b)}$$

$$\frac{d^2y}{dx^2} - \frac{a^2x^{2k} + bx^k + c}{(\frac{2}{k})^2 x^2} y = 0$$

$$y = x^{\frac{1}{2}(1 \pm \sqrt{ck^2 + 1})} e^{-\frac{ax^k}{2}} M(\frac{b}{4a} + \frac{1}{2}(1 \pm \sqrt{c + \frac{1}{k^2}}); 1 \pm \sqrt{c + \frac{1}{k^2}}; ax^k) \quad . \tag{27}$$

8. Relations to Other Functions

Several functions useful in theoretical physics may be expressed in terms of the confluent hypergeometric function.

The Whittaker function $M_{km}(z)$ is defined by

$$M_{km}(z) = z^{1 - \frac{\gamma}{2}} e^{-\frac{z}{2}} M(\alpha; \gamma; z) \tag{28}$$

where

$$\alpha = \frac{1}{2} + m - k$$

and

$$\gamma = 2m + 1 \quad .$$

For some physical applications this form is more convenient than $M(\alpha; \gamma; z)$.

Laguerre polynomials $L_n(z)$ result if $\gamma = 1$ and α is a negative integer n; and the associated Laguerre polynomials $L_n^k(z)$ if $\gamma = 1 + k$ and $\alpha = -n + k$, where k is also integral

$$L_n(z) = M(-n; 1; z) \qquad \text{n integral} \tag{29}$$

$$L_n^k(z) = M(k - n; 1 + k; z) \qquad \text{n,k, integral} \quad . \tag{30}$$

---

then (a) becomes

$$\frac{d^2y}{dx^2} + I(x)y = 0 \tag{c}$$

where

$$I(x) = q(x) - \frac{1}{2}\frac{dp(x)}{dx} - \frac{1}{4}\left[p(x)\right]^2 \quad . \tag{d}$$

I is called the invariant of the equation and is the same for any equations which may be transformed into each other by a change of dependent variable.

Hermite polynomials may be expressed by

$$H_n(x) = xM(-\frac{n}{2}; \frac{1}{2}; x^2) \qquad (31)$$

where n is an even integer.

The Bessel function of order n and imaginary argument is given by

$$J_n(ix) = 2x^{1/2}e^{-2x} M(\frac{4n^2 + 9}{8}); 2; 2x) \quad . \qquad (32)$$

9. Further Tables

1. E. Jahnke and F. Emde, "Tables of Functions", p. 275 gives some charts but no tables.
2. Report of the British Association for the Advancement of Science, Section A, Oxford, 1926, gives 5-place tables for $\gamma = \pm \frac{1}{2}, + \frac{3}{2}$, $\alpha = -4(\frac{1}{2})4$*, $z = 0(0.1)1(0.2)3(0.5)8$. Section A, Leeds, 1927, gives 5-place tables for $\gamma = -4(\frac{1}{2})4$, $\alpha = -4(1)4$ and $z = 0(0.02)0.10(0.05)1.0(0.1)2(0.2)3(0.5)8$.
3. R. Gran Olsson, Ingenieur Archiv, 8, 99 (1937), gives 4-place tables for $z = \frac{\lambda \rho^n}{n}$ with $\rho = 0.1$ to 1.0, $n = 2$ or 4, α between -0.675 and 1.65 and γ between 0.5 and 3.0.
4. A. H. Heatley, "Transactions of Royal Society of Canada", 1943, defines a function which is a confluent hypergeometric function in z^2 multiplied by a power of z, and gives a short 5-figure table.

Acknowledgment

The tables of the confluent hypergeometric function have been computed by the Joint Computing Group of the Massachusetts Institute of Technology.

* The parenthesis indicates the interval between points.

TABLE I

$$M(\alpha; \frac{1}{2}; z)$$

z \ α	0.001	0.01	0.05	0.1	0.2	0.25	0.3	0.4
0.5	1.00119	1.01194	1.06009	1.12121	1.24661	1.31090	1.37626	1.51027
1.0	1.00289	1.02901	1.14709	1.29939	1.62003	1.78859	1.96279	2.32856
1.5	1.00536	1.05388	1.27525	1.56542	2.19240	2.53016	2.88475	3.64646
2.0	1.00901	1.09069	1.46683	1.96791	3.07856	3.69109	4.34381	5.77622
2.5	1.01447	1.14593	1.75679	2.58359	4.46185	5.52098	6.66544	9.22724
3.0	1.02275	1.22976	2.20028	3.53409	6.63581	8.42141	10.3759	14.8313
3.5	1.03541	1.35822	2.88452	5.01272	10.0714	13.0397	16.3274	23.9477
4.0	1.05495	1.55666	3.94789	7.32760	15.5257	20.4209	25.9018	38.7987
4.5	1.08529	1.86528	5.61060	10.9709	24.2181	32.2551	41.3421	63.0208
5.0	1.13269	2.34797	8.22371	16.7304	38.1153	51.2780	66.2936	102.566
5.5	1.20711	3.10654	12.3483	25.8697	60.3939	81.9241	106.685	167.182
6.0	1.32444	4.30354	18.8825	40.4184	96.1907	131.388	172.166	272.838
6.5	1.51007	6.19896	29.2666	63.6416	153.822	211.354	278.457	445.706
7.0	1.80468	9.20931	45.8139	100.799	246.763	340.810	451.182	728.693
7.5	2.27348	14.0029	72.2444	160.373	396.873	550.643	732.133	1192.16
8.0	3.02120	21.6536	114.548	256.061	639.632	891.119	1189.51	1951.53

TABLE I

$M(\alpha; \frac{1}{2}; z)$

z \ α	0.5	0.6	0.7	0.75	0.8	0.9	1.0
0.5	1.64872	1.79170	1.93929	2.01485	2.09159	2.24869	2.41069
1.0	2.71828	3.13289	3.57336	3.80361	4.04071	4.53595	5.06016
1.5	4.48169	5.39476	6.39020	6.92028	7.47272	8.64722	9.91880
2.0	7.38906	9.19635	11.2129	12.3045	13.4544	15.9372	18.6789
2.5	12.1825	15.5689	19.4269	21.5461	23.7995	28.7328	34.2760
3.0	20.0855	26.2291	33.3600	37.3286	41.5843	51.0165	61.7801
3.5	33.1155	44.0320	56.9179	64.1737	72.0140	89.5838	109.914
4.0	54.5982	73.7236	96.6443	109.688	123.879	156.001	193.640
4.5	90.0171	123.188	163.492	186.649	212.001	269.909	338.545
5.0	148.413	205.515	275.772	316.494	361.329	464.598	588.290
5.5	244.692	342.432	464,078	535.152	613.814	796.395	1017.20
6.0	403.429	569.984	779.473	902.777	1039.91	1360.45	1751.60
6.5	665.142	947.949	1307.14	1520.00	1757.79	2317.32	3005.76
7.0	1096.63	1575.44	2189.08	2555.04	2965.56	3937.50	5142.69
7.5	1808.04	2616.72	3661.88	4288.89	4994.97	6676.33	8776.41
8.0	2980.96	4343.99	6119.53	7190.61	8401.17	11299.4	14944.4

TABLE II

M(α; 1; z)

z \ α	0.001	0.01	0.05	0.1	0.2	0.25	0.3	0.4
0.5	1.00057	1.00571	1.02869	1.05776	1.11703	1.14724	1.17784	1.24021
1.0	1.00132	1.01321	1.06679	1.13540	1.27817	1.35241	1.42859	1.58690
1.5	1.00232	1.02328	1.11839	1.24178	1.50409	1.64328	1.78802	2.09471
2.0	1.00369	1.03705	1.18958	1.39019	1.82607	2.06216	2.31092	2.84820
2.5	1.00558	1.05622	1.28951	1.60066	2.29180	2.67384	3.08162	3.97886
3.0	1.00827	1.08333	1.43200	1.90360	2.97435	3.57807	4.23056	5.69204
3.5	1.01211	1.12226	1.63802	2.34540	3.98620	4.92912	5.96042	8.30961
4.0	1.01769	1.17388	1.93959	2.99716	5.50133	6.96661	8.58729	12.3378
4.5	1.02589	1.26217	2.38583	3.96845	7.78989	10.0641	12.6059	18.5749
5.0	1.03806	1.38594	3.05245	5.42871	11.2729	14.8059	18.7931	28.2840
5.5	1.05628	1.57147	4.05648	7.64094	16.6086	22.1089	28.3718	43.4671
6.0	1.08377	1.85170	5.57965	11.0148	24.8291	33.4155	43.2727	67.3054
6.5	1.12554	2.27781	7.90491	16.1903	37.5574	51.0008	66.5503	104.863
7.0	1.18938	2.92950	11.4742	24.1701	57.3512	78.4616	103.048	164.217
7.5	1.28742	3.93132	16.9795	36.5288	88.2509	121.495	160.459	258.270
8.0	1.43873	5.47833	25.5071	55.7451	136.652	189.144	251.027	407.662

TABLE II

$M(\alpha; 1; z)$

$z \diagdown \alpha$	0.5	0.6	0.7	0.75	0.8	0.9	1.0
0.5	1.30417	1.36974	1.43695	1.47118	1.50584	1.57642	1.64872
1.0	1.75339	1.92832	2.11198	2.20717	2.30466	2.50666	2.71828
1.5	2.42533	2.78110	3.16328	3.36469	3.57319	4.01219	4.48169
2.0	3.44152	4.09470	4.81173	5.19550	5.59683	6.45439	7.38906
2.5	4.99283	6.13341	7.41115	8.10491	8.83720	10.4235	12.18249
3.0	7.38010	9.31770	11.5295	12.7465	14.0421	16.8839	20.0855
3.5	11.0791	14.3181	18.0805	20.1759	22.4244	27.4133	33.1155
4.0	16.8440	22.2065	28.5359	32.1007	35.9535	44.5924	54.5982
4.5	25.8738	34.6994	45.2706	51.2857	57.8302	72.6467	90.0171
5.0	40.0784	54.5508	72.1215	82.2137	93.2612	118.496	148.413
5.5	62.5213	86.1861	115.294	132.159	150.723	193.479	244.692
6.0	98.0333	136.727	184.838	212.935	244.026	316.176	403.429
6.5	154.467	217.647	297.040	343.748	395.682	517.055	665.142
7.0	244.333	347.456	478.319	555.835	642.410	846.076	1096.63
7.5	387.747	556.046	771.567	900.045	1044.14	1385.20	1808.04
8.0	617.064	891.735	1246.47	1459.19	1698.70	2268.89	2980.96

TABLE III

$M(\alpha; \frac{3}{2}; z)$

$z \backslash \alpha$	0.001	0.01	0.05	0.1	0.2	0.25	0.3	0.4
0.5	1.00037	1.00371	1.01860	1.03740	1.07559	1.09498	1.11457	1.15435
1.0	1.00083	1.00832	1.04197	1.08486	1.17344	1.21916	1.26586	1.36222
1.5	1.00141	1.01418	1.07185	1.14616	1.30238	1.38440	1.46911	1.64683
2.0	1.00216	1.02174	1.11076	1.22677	1.47522	1.60799	1.74669	2.04260
2.5	1.00315	1.03168	1.16231	1.33460	1.71071	1.91537	2.13166	2.60095
3.0	1.00447	1.04498	1.23175	1.48119	2.03654	2.34426	2.67326	3.39906
3.5	1.00625	1.06305	1.32681	1.68356	2.49378	2.95088	3.44519	4.55356
4.0	1.00871	1.08798	1.45884	1.96690	3.14379	3.81957	4.55849	6.24152
4.5	1.01215	1.12287	1.64472	2.36876	4.07876	5.07752	6.18124	8.73307
5.0	1.01702	1.17232	1.90964	2.94540	5.43784	6.91746	8.56909	12.4421
5.5	1.02399	1.24321	2.29139	3.78153	7.43214	9.63277	12.1126	18.0052
6.0	1.03408	1.34590	2.84697	5.00536	10.3832	13.6719	17.4106	26.4052
6.5	1.04881	1.49602	3.66268	6.81170	14.7830	19.7229	25.3851	39.1642
7.0	1.07050	1.71729	4.86981	9.49774	21.3863	28.8452	37.4596	58.6465
7.5	1.10268	2.04584	6.66877	13.5185	31.3560	42.6750	55.8392	88.5356
8.0	1.15075	2.53690	9.36657	19.5732	46.4884	63.7471	83.9493	134.584

<u>TABLE III</u>

$M(a; \frac{3}{2}; z)$

z \ a	0.5	0.6	0.7	0.75	0.8	0.9	1.0
0.5	1.19496	1.23639	1.27867	1.30013	1.32181	1.36581	1.41069
1.0	1.46265	1.56727	1.67619	1.73231	1.78955	1.90747	2.03008
1.5	1.83603	2.03724	2.25098	2.36271	2.47779	2.71825	2.97293
2.0	2.36445	2.71379	3.09225	3.29293	3.50153	3.94340	4.41972
2.5	3.12228	3.69958	4.33701	4.67964	5.03896	5.81006	6.65520
3.0	4.22221	5.15156	6.19657	6.76558	7.36734	8.67465	10.1300
3.5	5.83596	7.31086	8.99823	9.92826	10.9196	13.0983	15.5592
4.0	8.22631	10.5494	13.2509	14.7569	16.3741	19.9666	24.0800
4.5	11.7973	15.4434	19.7483	22.1740	24.7967	30.6815	37.5051
5.0	17.1722	22.8880	29.7330	33.6282	37.8669	47.4668	58.7290
5.5	25.3164	34.2789	45.1546	51.4005	58.2376	73.8573	92.3820
6.0	37.7301	51.7986	69.0807	79.0905	90.1085	115.483	145.883
6.5	56.7504	78.8684	106.349	122.392	140.141	181.327	231.213
7.0	86.0296	120.865	164.605	190.328	218.923	285.742	367.264
7.5	131.289	186.256	255.955	297.224	343.307	451.696	585.027
8.0	201.510	288.408	399.605	465.867	540.165	715.995	933.960

TABLE IV

M(α; 2; z)

z \ α	0.001	0.01	0.05	0.1	0.2	0.25	0.3	0.4
0.5	1.00027	1.00273	1.01369	1.02751	1.05550	1.06967	1.08397	1.11294
1.0	1.00060	1.00601	1.03025	1.06106	1.12435	1.15685	1.18993	1.25785
1.5	1.00100	1.01000	1.05059	1.10261	1.21111	1.26764	1.32572	1.44671
2.0	1.00149	1.01495	1.07595	1.15490	1.32218	1.41067	1.50252	1.69663
2.5	1.00211	1.02118	1.10809	1.22176	1.46662	1.59822	1.73622	2.03229
3.0	1.00290	1.02915	1.14949	1.30860	1.65736	1.84788	2.04977	2.48953
3.5	1.00392	1.03950	1.20364	1.42314	1.91298	2.18504	2.47638	3.12071
4.0	1.00527	1.05316	1.27558	1.57649	2.26041	2.64661	3.06456	4.00292
4.5	1.00708	1.07145	1.37251	1.78469	2.73886	3.28662	3.88557	5.25020
5.0	1.00953	1.09630	1.50493	2.07108	3.40586	4.18456	5.04464	7.03226
5.5	1.01289	1.13049	1.68812	2.46988	4.34628	5.45813	6.69810	9.60289
6.0	1.01758	1.17811	1.94452	3.03145	5.68592	7.28240	9.07932	13.3435
6.5	1.02416	1.24516	2.30725	3.83038	7.61228	9.91915	12.5383	18.8295
7.0	1.03352	1.34055	2.82543	4.97762	10.4061	13.7614	17.6020	26.9329
7.5	1.04695	1.47747	3.57224	6.63900	14.4892	19.4018	25.0677	38.9800
8.0	1.06636	1.67568	4.65722	9.06341	20.4987	27.7373	36.1450	56.9945

TABLE IV

$M(\alpha; 2; z)$

$z \backslash \alpha$	0.5	0.6	0.7	0.75	0.8	0.9	1.0
0.5	1.14241	1.17238	1.20286	1.21830	1.23386	1.26539	1.29744
1.0	1.32819	1.40100	1.47635	1.51500	1.55430	1.63493	1.71828
1.5	1.57432	1.70881	1.85045	1.92404	1.99953	2.15632	2.32113
2.0	1.90526	2.12919	2.36922	2.49554	2.62620	2.90100	3.19453
2.5	2.35664	2.71117	3.09790	3.30400	3.51894	3.97653	4.47300
3.0	2.98058	3.52712	4.13363	4.46084	4.80484	5.54579	6.36185
3.5	3.85394	4.68461	5.62191	6.13360	6.67570	7.85657	9.17584
4.0	5.09068	6.34434	7.78181	8.57549	9.42253	11.2875	13.3995
4.5	6.86068	8.74766	10.9447	12.1709	13.4887	16.4196	19.7816
5.0	9.41857	12.2588	15.6142	17.5059	19.5523	24.1478	29.4826
5.5	13.1508	17.4299	22.5586	25.4778	28.6551	35.8547	44.3076
6.0	18.6278	25.1013	32.9545	37.4645	42.4019	53.6843	67.0715
6.5	26.7392	36.5564	48.6100	55.5904	63.2735	80.9696	102.176
7.0	38.8235	53.7630	72.3129	83.1393	95.1162	122.906	156.519
7.5	56.9328	79.7471	108.373	125.203	143.909	187.615	240.939
8.0	84.2153	119.176	163.475	189.695	218.969	287.813	372.495

TABLE V

W(α; 1; z)

z \ α	0.001	0.01	0.05	0.1	0.2	0.3	0.4
0.5	0.451439	0.426196	0.309127	0.150598	-0.214888	-0.663856	-1.23141
1.0	1.89194	1.86292	1.72533	1.53174	1.05899	0.439451	-0.386811
1.5	3.29819	3.26973	3.12913	2.91793	2.35375	1.54433	0.387540
2.0	4.95185	4.92945	4.80806	4.60097	3.96186	2.92708	1.32363
2.5	7.07300	7.06474	6.99538	6.82503	6.13735	4.82556	2.59878
3.0	9.93609	9.95435	9.98798	9.90626	9.21692	7.55704	4.44092
3.5	13.9328	13.9970	14.2141	14.3048	13.6979	11.5942	7.18799
4.0	19.6470	19.7877	20.3161	20.7154	20.3389	17.6644	11.3621
4.5	27.9638	28.2294	29.2731	30.2016	30.3121	26.9006	17.7816
5.0	40.2381	40.7052	42.5895	44.4046	45.4407	41.0782	27.7380
5.5	58.5546	59.3450	62.5895	65.8698	68.5728	62.9906	43.2763
6.0	86.1367	87.4439	92.8790	98.5596	104.173	97.0455	67.6449
6.5	127.986	130.119	139.072	148.663	159.260	150.216	106.014
7.0	191.891	195.339	209.934	225.876	244.894	233.5575	166.629
7.5	290.009	295.555	319.193	345.423	378.547	364.631	262.661
8.0	441.374	450.267	488.397	531.272	587.872	571.384	415.179

TABLE V

$W(a;\ 1;\ z)$

z \ a	0.5	0.6	0.7	0.75	0.8	0.9
0.5	-1.97934*	-3.02899	-4.65969	-5.90077	-7.70160	-16.2842
1.0	-1.52411*	-3.17545	-5.80488	-7.83038	-10.7855	-24.9312
1.5	-1.29260	-3.83568	-8.01278	-11.2835	-16.0953	-39.3270
2.0	-1.14446	-5.04445	-11.6571	-16.9231	-24.7392	-62.8442
2.5	-1.03874	-7.03262	-17.5082	-25.9861	-38.6780	-101.156
3.0	-0.958210*	-10.2132	-26.8475	-40.5111	-61.1308	-163.574
3.5	-0.89413*	-15.2625	-41.7527	-63.8073	-97.3352	-265.344
4.0	-0.84156*	-23.2700	-65.5791	-101.235	-155.803	-431.418
4.5	-0.79738*	-35.9852	-103.745	-161.477	-250.371	-702.648
5.0	-0.75958*	-56.2172	-165.005	-258.616	-403.554	-1145.94
5.5	-0.71351*	-88.4821	-263.525	-415.502	-652.002	-1870.92
6.0	-0.69780*	-140.048	-422.239	-669.251	-1055.43	-3057.25
6.5	-0.6721*	-222.628	-678.321	-1080.20	-1711.18	-4999.49
7.0	-0.6485*	-355.114	-1092.05	-1746.47	-2778.03	-8180.73
7.5	-0.6286*	-568.019	-1761.40	-2827.83	-4515.09	-13393.4
8.0	-0.6089*	-910.663	-2845.35	-4584.43	-7345.43	-21937.6

* Because of the nature of the function for these values, the last figure may be in doubt.

TABLE VI*

W(α; 2; z)

z \ α	0.001	0.01	0.05	0.1	0.2	0.3
0.5	-2.84756	-2.88691	-3.06956	-3.31745	-3.89302	-4.61032
1.0	-0.826838	-0.860261	-1.01596	-1.22869	-1.72682	-2.35223
1.5	0.309981	0.277995	0.127300	-0.082485	-0.587796	-1.24219
2.0	1.25637	1.22589	1.07957	0.869404	0.339596	-0.381060
2.5	2.19777	2.17015	2.03335	1.82680	1.26970	0.441461
3.0	3.23626	3.21386	3.09601	2.90197	2.32212	1.39899
3.5	4.46240	4.44881	4.36451	4.19757	3.60667	2.54611
4.0	5.98155	5.98207	5.95323	5.83603	5.25551	4.02755
4.5	7.93257	7.95505	8.01442	7.98143	7.44757	6.01544
5.0	10.5091	10.5652	10.7616	10.8654	10.4376	8.75589
5.5	13.9883	14.0953	14.5021	14.8225	14.5963	12.6095
6.0	18.7725	18.9563	19.6836	20.3426	20.4708	18.1113
6.5	25.4515	25.7511	26.9652	28.1487	28.8737	26.0617
7.0	34.8962	35.3699	37.3226	39.3153	41.0201	37.6650
7.5	48.3988	49.1353	52.2086	55.4468	58.7361	54.7409
8.0	67.8872	69.0208	73.7968	78.9507	84.7765	80.0513

* This table has been computed by the recurrence formula from tables V and VII, therefore there is a possibility that the last number may be off by ± 1.

TABLE VI

W(α; 2; z)

z \ α	0.4	0.5	0.6	0.7	0.75	0.8	0.9
0.5	-5.53635	-6.79061	-8.61092	-11.5497	-13.8519	-17.2595	-33.9886
1.0	-3.16152	-4.25512	-5.83126	-8.34808	-10.3010	-13.1713	-27.1052
1.5	-2.10960	-3.30286	-5.04257	-7.83570	-10.0042	-13.1876	-28.5827
2.0	-1.37344	-2.78062	-4.87872	-8.29916	-10.9730	-14.9089	-33.9700
2.5	-0.713817	-2.44229	-5.09490	-9.51041	-12.9983	-18.1585	-43.2658
3.0	-0.0180700	-2.20138	-5.66247	-11.5612	-16.2785	-23.3013	-57.6938
3.5	0.801770	-2.01904	-6.64562	-14.7269	-21.2736	-31.0861	-79.4984
4.0	1.84251	-1.87504	-8.18483	-19.4780	-28.7451	-42.7303	-112.264
4.5	3.23110	-1.75769	-10.5130	-26.5545	-39.8812	-60.1275	-161.566
5.0	5.14742	-1.65973	-13.9936	-37.0953	-56.5119	-86.1970	-236.030
5.5	7.85487	-1.57484	-19.1848	-52.8478	-81.4477	-125.433	-349.011
6.0	11.7458	-1.50433	-26.9405	-76.4928	-119.013	-184.769	-521.224
6.5	17.4097	-1.44128	-38.5685	-112.154	-175.879	-274.928	-784.887
7.0	25.7385	-1.38542	-56.0751	-166.190	-262.356	-412.539	-1190.24
7.5	38.0870	-1.33574	-82.5480	-248.446	-394.443	-623.450	-1815.78
8.0	56.5221	-1.29082	-122.750	-374.165	-596.998	-947.952	-2784.47

TABLE VII*

$W(a;\ 3;\ z)$

z\a	0.001	0.01	0.05	0.1	0.2	0.3
0.5	-9.45028	-9.56093	-10.0789	-10.7932	-12.4992	-14.7097
1.0	-3.54739	-3.60145	-3.85360	-4.19907	-5.01404	-6.05166
1.5	-1.68300	-1.72512	-1.92197	-2.19238	-2.83203	-3.64691
2.0	-0.591663	-0.629037	-0.804794	-1.04882	-1.63504	-2.39427
2.5	0.247797	0.213384	0.049784	-0.181573	-0.752615	-1.54374
3.0	1.00348	0.971889	0.819161	0.597061	0.026498	-0.769050
3.5	1.75745	1.72940	1.59010	1.37873	0.803847	-0.045953
4.0	2.56647	2.54338	2.42309	2.22756	1.64962	0.727450
4.5	3.48293	3.46697	3.37463	3.20380	2.62951	1.61682
5.0	4.56562	4.55999	4.50878	4.37634	3.81880	2.69580
5.5	5.88826	5.89753	5.90655	5.83280	5.31382	4.05796
6.0	7.54887	7.57955	7.67629	7.69094	7.24491	5.83007
6.5	9.68179	9.74321	9.96723	10.1137	9.79362	8.18944
7.0	12.4746	12.5801	12.9885	13.3303	13.2170	11.3886
7.5	16.1922	16.3612	17.0366	17.6666	17.8829	15.7908
8.0	21.2120	21.4724	22.5352	23.5900	24.3217	21.9232

---

* The last figure in those values near zero may be in doubt because of the rapid rate of change of the function in this region.

TABLE VII

$W(a; 3; z)$

z \ a	0.4	0.5	0.6	0.7	0.75	0.8	0.9
0.5	-17.6828	-21.8842	-28.2497	-38.9690	-47.6065	-60.6253	-126.177
1.0	-7.42028	-9.31483	-12.1244	-16.7558	-20.4347	-25.9285	-53.2348
1.5	-4.71794	-6.19072	-8.35310	-11.8733	-14.6422	-18.7486	-38.9453
2.0	-3.40247	-4.79827	-6.85123	-10.1849	-12.7967	-16.6563	-35.5145
2.5	-2.54857	-4.00494	-6.17115	-9.70937	-12.4851	-16.5846	-36.5632
3.0	-1.88050	-3.48769	-5.92222	-9.94727	-13.1216	-17.8191	-40.7281
3.5	-1.27858	-3.12059	-5.97664	-10.7772	-14.5938	-20.2630	-47.9994
4.0	-0.671740	-2.84455	-6.30505	-12.2350	-16.9962	-24.1035	-59.0470
4.5	-0.002936	-2.62812	-6.93220	-14.4631	-20.5759	-29.7520	-75.1382
5.0	0.786624	-2.45301	-7.92701	-17.7129	-25.7457	-37.8760	-98.2701
5.5	1.76826	-2.30859	-9.40754	-22.3736	-33.1369	-49.4883	-131.455
6.0	3.03656	-2.18500	-11.5560	-29.0284	-43.6911	-66.0979	-179.187
6.5	4.72289	-2.07949	-14.6452	-38.5414	-58.8040	-89.9440	-248.158
7.0	7.01412	-1.98760	-19.0790	-52.1914	-80.5442	-124.352	-348.355
7.5	10.1798	-1.90670	-25.4551	-71.8746	-111.986	-174.275	-494.721
8.0	14.6125	-1.83475	-34.6591	-100.411	-157.710	-247.113	-709.697

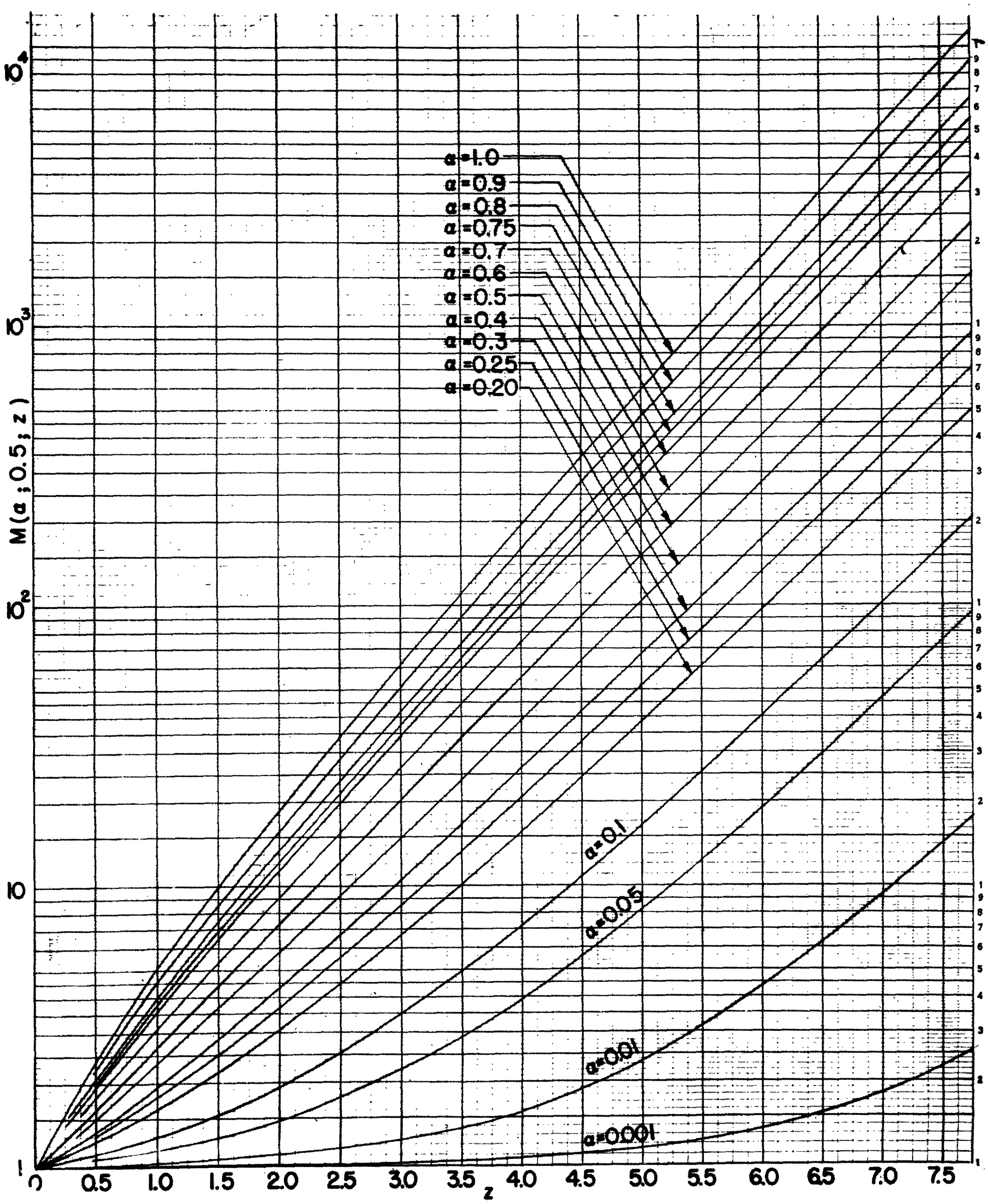
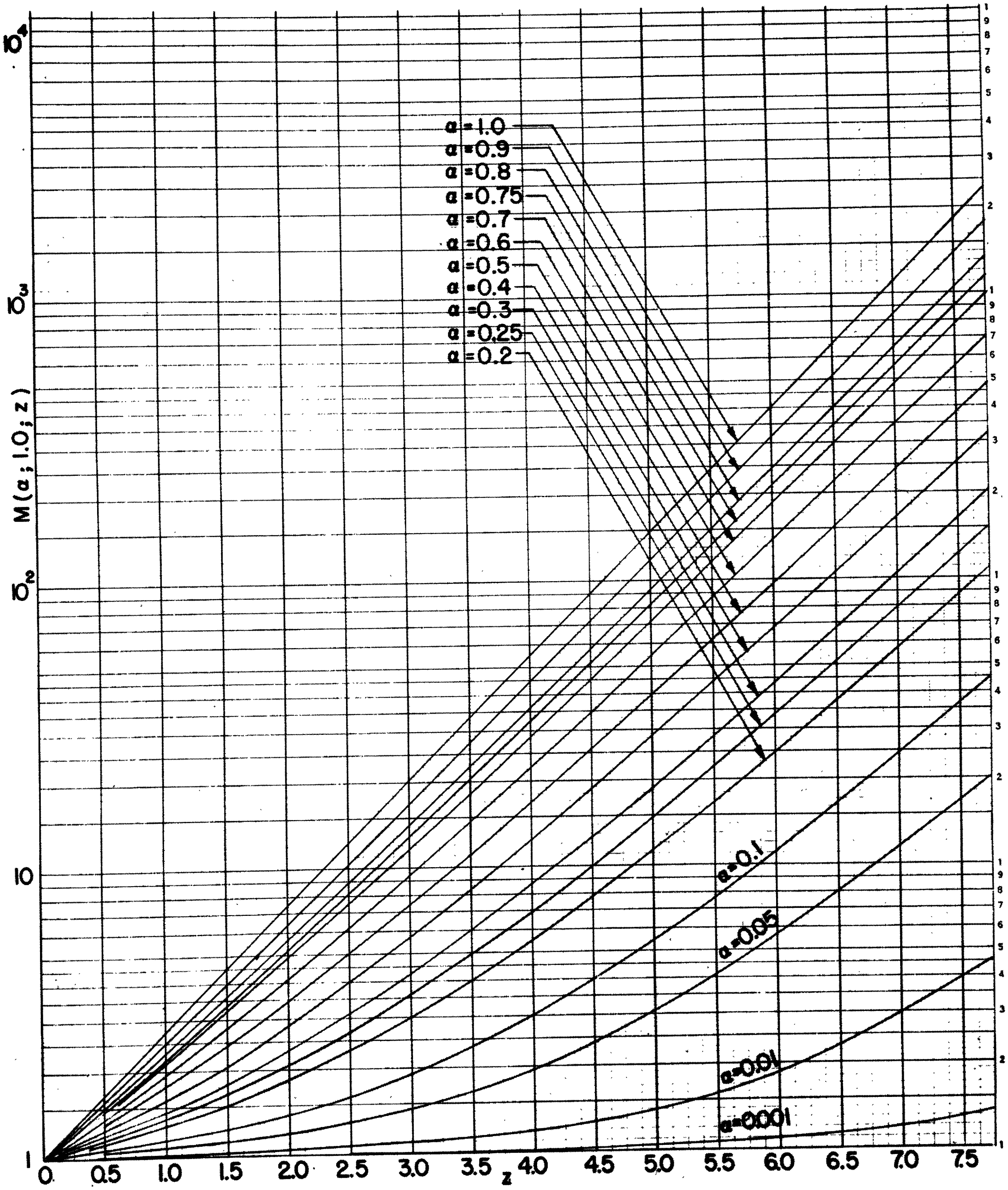
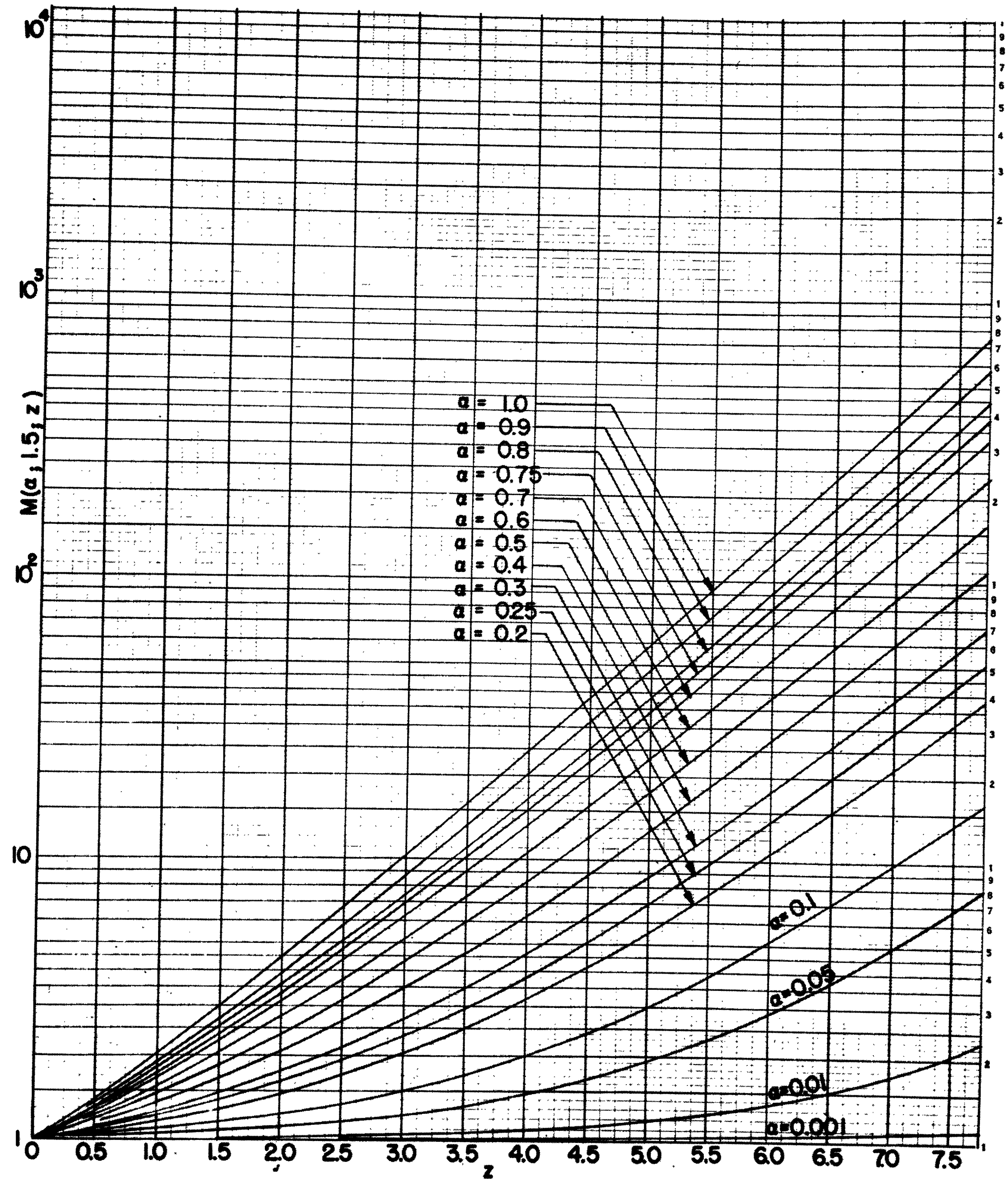

M(a ; 0.5 ; z)
10⁴
10³
10²
10
1
a=1.0
a=0.9
a=0.8
a=0.75
a=0.7
a=0.6
a=0.5
a=0.4
a=0.3
a=0.25
a=0.20
a=0.1
a=0.05
a=0.01
a=0.001
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
z
4.0
4.5
5.0
5.5
6.0
6.5
7.0
7.5

Fig. 2. $M(a; 0.5; z)$



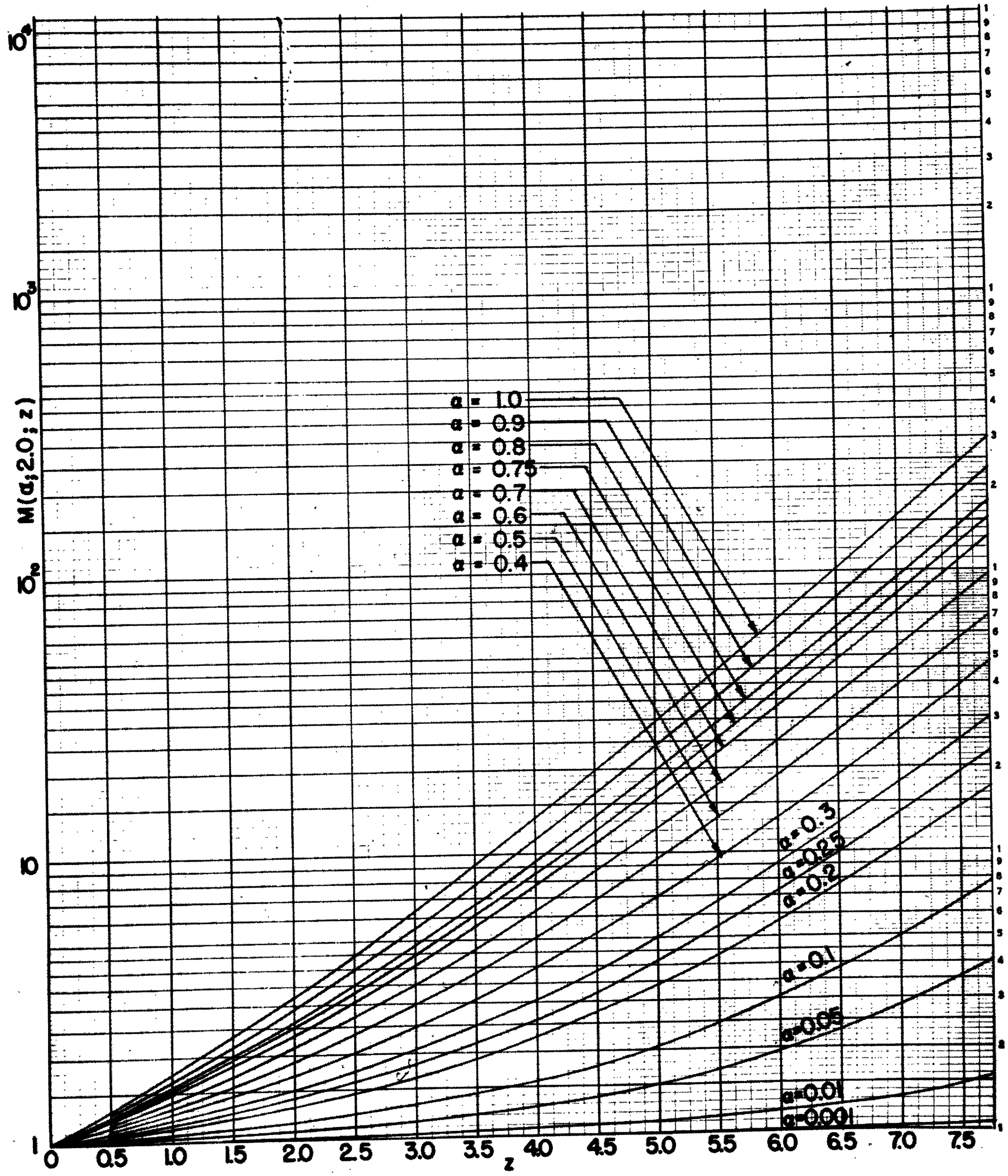
10^4
10^3
10^2
10
1
M(a; 1.0; z)
a=1.0
a=0.9
a=0.8
a=0.75
a=0.7
a=0.6
a=0.5
a=0.4
a=0.3
a=0.25
a=0.2
a=0.1
a=0.05
a=0.01
a=0.001
0.
0.5
1.0
1.5
2.0
2.5
3.0
3.5
z
4.0
4.5
5.0
5.5
6.0
6.5
7.0
7.5

Fig. 3. M(α; 1.0; z)



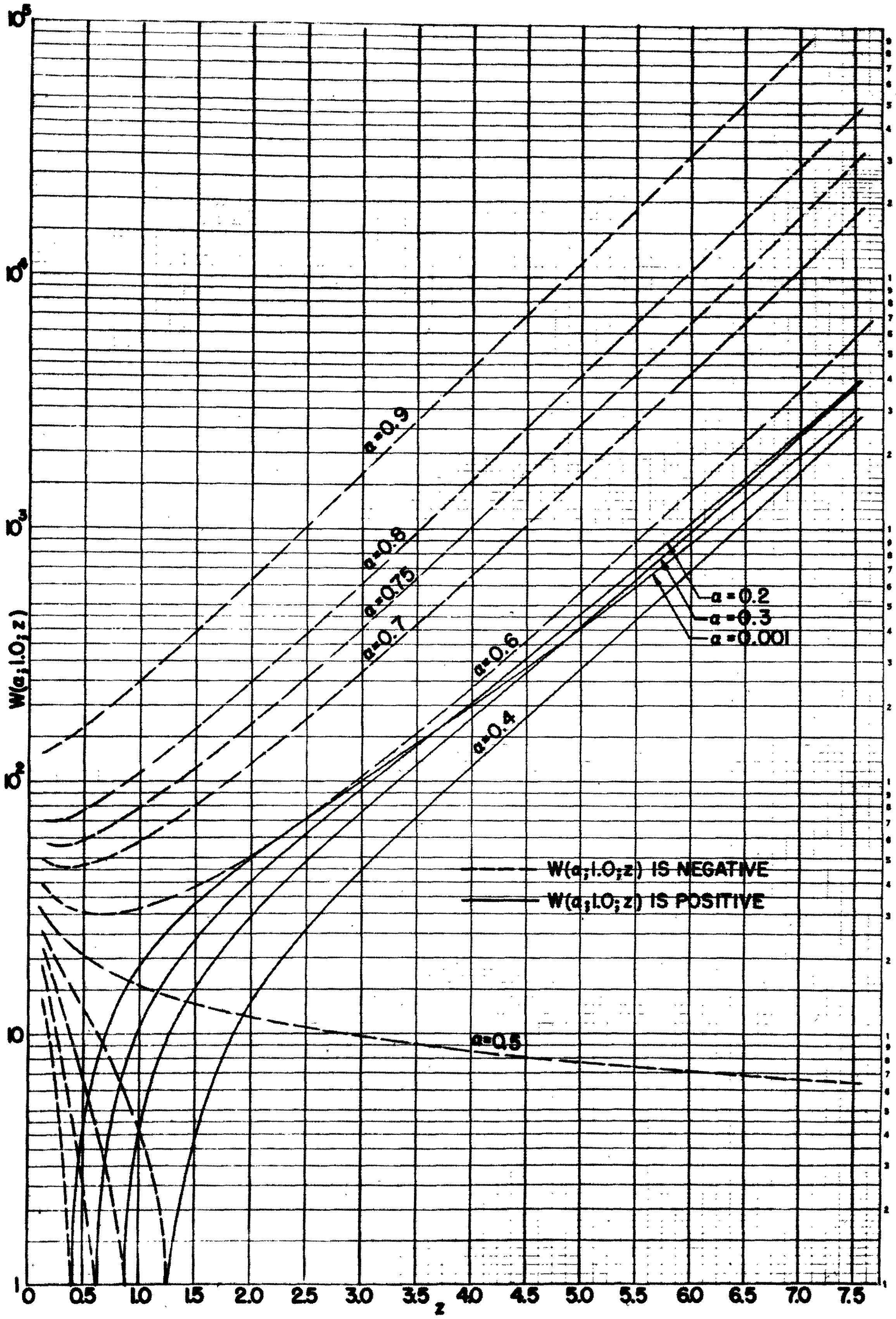
M(a ; 1.5 ; z)
10⁴
10³
10²
10
1
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
5.5
6.0
6.5
7.0
7.5
z
a = 1.0
a = 0.9
a = 0.8
a = 0.75
a = 0.7
a = 0.6
a = 0.5
a = 0.4
a = 0.3
a = 0.25
a = 0.2
a=0.1
a=0.05
a=0.01
a=0.001

Fig. 4. M(a; 1.5; z)



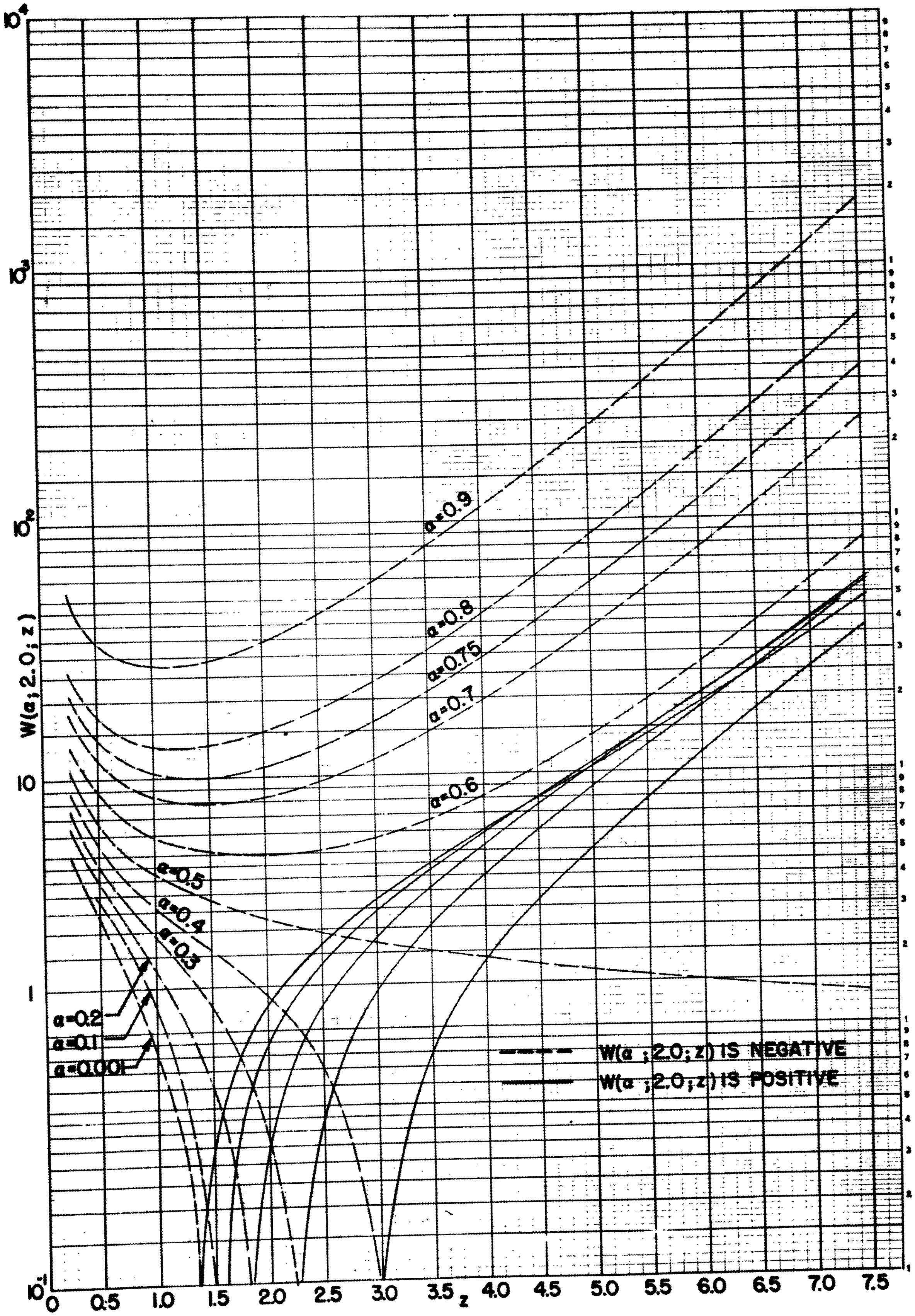
M(a;2.0;z)
z
a = 1.0
a = 0.9
a = 0.8
a = 0.75
a = 0.7
a = 0.6
a = 0.5
a = 0.4
a=0.3
a=0.25
a=0.2
a=0.1
a=0.05
a=0.01
a=0.001

Fig. 5. M(a; 2.0; z)



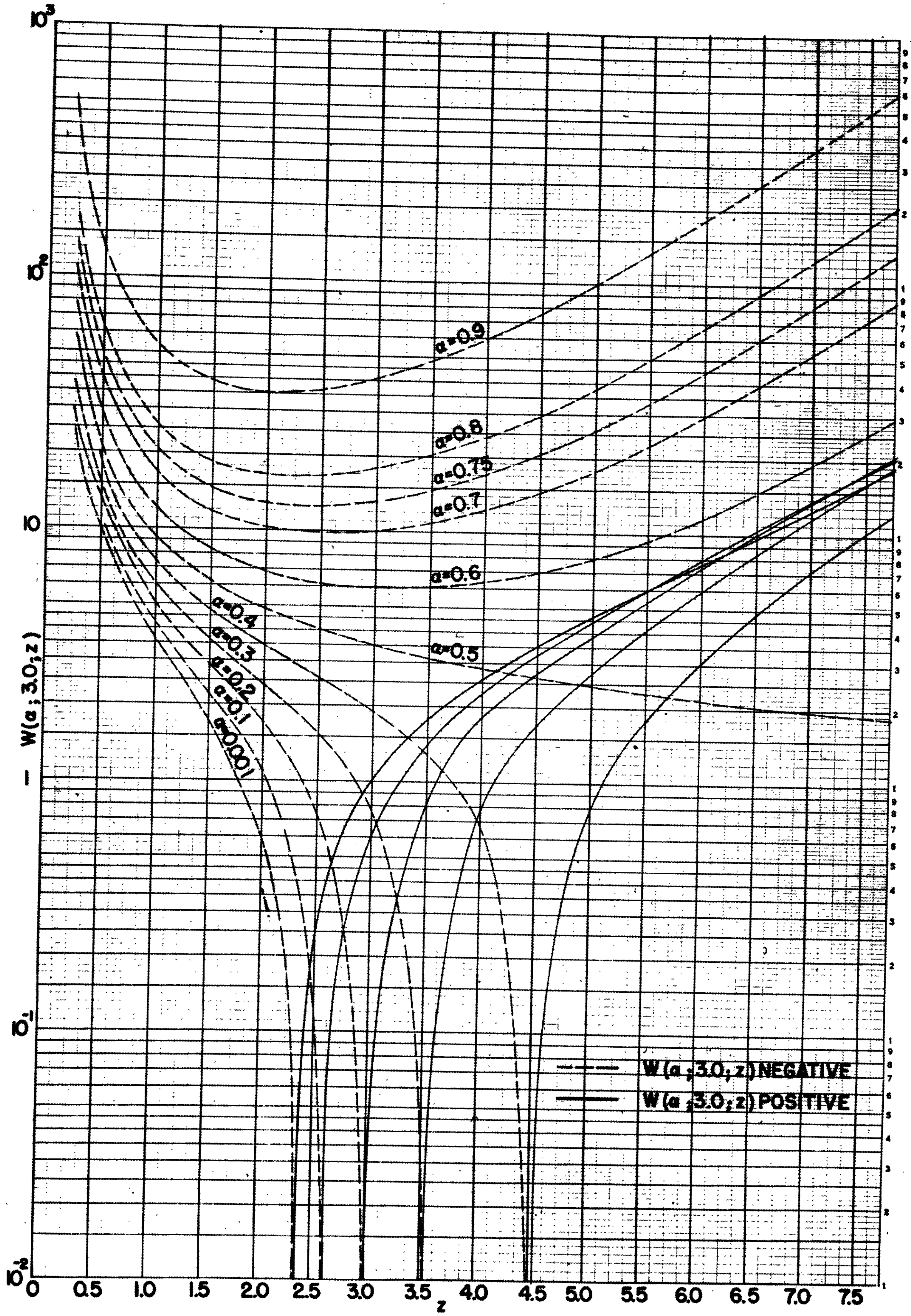
10⁵
10⁴
10³
10²
10
1
W(a; 1.0; z)
a=0.9
a=0.8
a=0.75
a=0.7
a=0.6
a=0.4
a=0.5
a=0.2
a=0.3
a=0.001
W(a; 1.0; z) IS NEGATIVE
W(a; 1.0; z) IS POSITIVE
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
5.5
6.0
6.5
7.0
7.5
z

Fig. 6. W(α; 1.0; z)



10^4
10^3
10^2
10
1
10^{-1}
W(α; 2.0; z)
α=0.9
α=0.8
α=0.75
α=0.7
α=0.6
α=0.5
α=0.4
α=0.3
α=0.2
α=0.1
α=0.001
W(α ; 2.0; z) IS NEGATIVE
W(α ; 2.0; z) IS POSITIVE
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
z
4.0
4.5
5.0
5.5
6.0
6.5
7.0
7.5

Fig. 7. W(α; 2.0; z)



10^3
10^2
10
1
10^{-1}
10^{-2}
W(a; 3.0; z)
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
z
4.0
4.5
5.0
5.5
6.0
6.5
7.0
7.5
a=0.9
a=0.8
a=0.75
a=0.7
a=0.6
a=0.5
a=0.4
a=0.3
a=0.2
a=0.1
a=0.001
W(a ;3.0; z) NEGATIVE
W(a ;3.0; z) POSITIVE

Fig. 8. W(α; 3.0; z)