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PIEZOELECTRIC TRANSDUCERS

I. ELECTROMECHANICAL IMPEDANCE MATRIX
II. ELECTRICAL DRIVING POINT IMPEDANCE AND ADMITTANCE

W. ROTH

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I. Electromechanical Impedance Matrix

II. Electrical Driving Point Impedance and Admittance

by

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Part I. Abstract

A piezoelectric transducer operating in the thickness vibration mode is represented as a six terminal network. The mesh equations, electromechanical impedance matrix and equivalent circuit valid for any general conditions of loading and frequency are obtained. All properties of the transducer can thus be determined once the impedance of the loads and the energy sources are specified.

Part II. Abstract

The electrical driving point impedance and admittance of a piezoelectric transducer operating in the thickness vibration mode are derived for all conditions of loading and frequency. Universal curves of these quantities are included for particular cases of importance.

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I. ELECTROMECHANICAL IMPEDANCE MATRIX

1. Introduction

The use of piezoelectric transducers to produce ultrasonic energy for research and commercial purposes has increased rapidly in recent years. As a result, a theoretical analysis of the operation of such elements, which is capable of simplifying the work of the equipment designer, is extremely desirable. The analysis presented here was undertaken with this end in view.

Piezoelectric materials have the property of reacting mechanically to an applied electrical stimulus, and reciprocally, reacting electrically to an applied mechanical stimulus.[1,2] Such bilateral conversion of electrical to mechanical energy, and vice versa, is the function performed by the piezoelectric element in its various applications. Hence, any theoretical conclusions must be in a form equally convenient for use with either electrical or mechanical systems.

Although piezoelectric crystals are useful when operating in many varied modes, the mode principally in use for generation of the higher ultrasonic frequencies is that corresponding to X-cut Quartz - the so-called thickness vibration. This mode is characterized by the colinearity of the mechanical strain and electric field intensity vectors. This is the mode to be considered here.

2. Statement of Problem and Assumptions

Figure 1 illustrates the problem to be considered. The crystal, its physical constants and geometry, and the loading acoustic media are specified, while the relations holding among the four mechanical and two electrical parameters are to be determined. The Rationalized m.k.s. system of units is employed throughout to facilitate the joint use of mechanical and electrical parameters without the introduction of troublesome multiplying factors.[3]

To simplify the problem, the following assumptions are made:

i. The crystal is an infinite slab with two plane, parallel surfaces perpendicular to the direction of propagation of the resulting acoustic disturbance.

ii. The two plane surfaces are electric equipotentials.

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1. W. G. Cady, "Piezoelectricity", McGraw-Hill, New York, 1946. An extremely complete, well-documented book on electromechanical phenomena in crystals, An excellent bibliography is included.

2. W. P. Mason, "Electromechanical Transducers and Wave Filters", Van Nostrand, New York, 1942; p. 195 ff. An equivalent circuit for a piezoelectric transducer is derived and used in several examples. Derivation of the Piezoelectric Equations is given in Appendix C.

3. O. W. Eshbach, "Handbook of Engineering Fundamentals", J. Wiley, New York, 1936. Contains a useful table of conversion factors for various unit systems, p. 1-130 ff, and a discussion of existing systems, p. 3-02 ff.

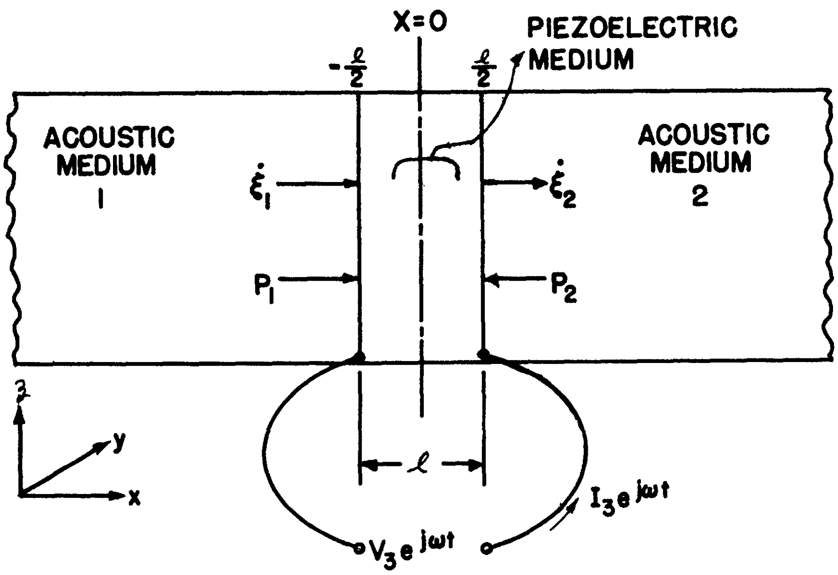

Figure 1. The piezoelectrical medium terminated by two acoustic media and one electrical terminal pair.

iii. The mechanical energy dissipated in the piezoelectric material is negligible.

iv. The electrical or mechanical stresses either applied or produced are not sufficient to cause departure from a linear operating region.

These assumptions are not restrictive when applied to the type of transducer generally used for ultrasonic applications. In accordance with i, the width of the crystal must be very much larger than its thickness. In practice, this ratio is frequently as large as 25 or 50 to 1; hence, this assumption is certainly valid. At very low frequencies, edge effects assume importance since the thickness may be in the order of the width, and a more elaborate theory is required.

Assumption ii restricts the analysis to widths small compared to the electrical wave lengths employed. This is certainly always the case except for operation in the microwave region where the wave lengths become comparable to the crystal dimensions.

Assumption iii is not at all restrictive unless the transducer is operating in a vacuum, in which case, all the mechanical energy is dissipated within the crystal. In most cases of practical importance, the dissipation in the acoustic material and supporting structure greatly exceeds that in the transducer, and no noticeable error is introduced by treating the dissipationless case.

In most cases of importance, iv is not limiting. However, if such a thing as cavitation is encountered when transmitting into liquids, the mechanical loading becomes a function of the intensity and cannot be treated as a constant. In such cases, the present treatment must be considered only as a first order theory which indicates trends and approximate magnitudes rather than quantitative behaviour.

3. Procedure

We shall regard the piezoelectric transducer as a circuit element: either electrical or mechanical depending upon which terminals of the transducer are under examination. By adopting this viewpoint, it becomes possible to characterize the transducer completely by an electromechanical impedance matrix relating electrical and mechanical "currents" to the associated "voltages". More will be said about the definition of these terms in a later section.[4]

Since the mode we are considering has two opposing crystal surfaces from which ultrasonic energy is radiated, to represent this situation we can evidently introduce four mechanical terminals. To affect the crystal electrically, it is necessary to attach electrodes to these faces. This also makes it possible to introduce two electrical terminals. In this manner, the transducer is considered as a six terminal network and we seek the "mesh" equations and the electromechanical impedance matrix which describes the interaction occurring among the variables specified for the respective terminal pairs.

As is the case with pure electrical networks, we must first obtain the equations relating "voltages" and "currents" at the six terminals. From this set of three "mesh" equations, the impedance matrix can be defined and thenceforth used for design purposes. We shall see that all properties of the transducer can be determined from a knowledge of this matrix and the boundary conditions at the terminals.

The impedance matrix is developed in accordance with the following procedure:

i. Maxwell's Equations and the constitutive relations satisfied by the piezoelectric crystal are introduced to obtain the equations satisfied by the electrical parameters.

ii. The Piezoelectric Equations which describe the interrelation between the electrical and mechanical variables for piezoelectric materials are used to develop preliminary mesh equations in conjunction with the results of i.

iii. Newton's First Law is used to derive the wave equation satisfied by the mechanical displacement. After integration, the result is employed to obtain the final mesh equations in the desired form.

iv. The impedance matrix follows directly from the mesh equations by proper definition of the operational process involved.

3.1 <u>Introduction of Maxwell's Equations</u>. In the Rationalized m.k.s. system of units, Maxwell's Equations are[5]

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4. M. F. Gardner and J. L. Barnes, "Transients in Linear Systems", J. Wiley, New York, 1942. Chapter II contains a discussion of the equations of electrical and mechanical systems both alone and in combination.

5. J. A. Stratton, "Electromagnetic Theory", McGraw-Hill, New York, 1941. Chapter I.

$$\nabla \times E + \frac{\partial B}{\partial t} = 0 \tag{1a}$$

$$\nabla \times H - \frac{\partial D}{\partial t} = J \tag{1b}$$

$$\nabla \cdot D = \rho \tag{1c}$$

$$\nabla \cdot B = 0 \tag{1d}$$

The constitutive relations for piezoelectric materials operating in the mode considered here are

$$B = \mu H \tag{1e}$$

$$D_x = \epsilon_x (E_x - d \frac{\partial \xi}{\partial x}) \tag{1f}$$

$$D_y = \epsilon_y E_y \tag{1g}$$

$$D_z = \epsilon_z E_z \tag{1h}$$

where the symbols represent the following quantities:

E = Electric field intensity — volt/meter
B = Flux density — weber/sq. meter
H = Magnetic field intensity — ampere-turn/meter
D = Dielectric displacement — coulomb/sq. meter
J = Current density — ampere/sq. meter
ρ = Charge density — coulomb/cubic meter
μ = Permeability — henry/meter
ϵ = Inductive capacity — farad/meter
d = Piezoelectric constant — volt/meter
ξ = Mechanical displacement — meter

It is to be understood that all variables are functions of both space and time until proved otherwise.

In view of our original assumption of equipotential, parallel, plane surfaces, infinite in extent, it is evident that $E_y = E_z = 0$, and $\partial E_x/\partial y = \partial E_x/\partial z = 0$. From (1g) and (1h), we find that $D_y = D_z = 0$ and by expanding $\nabla \times E$ into its rectangular components, it is clear also that $\nabla \times E = 0$. From the former condition and (1c) we obtain

$$\frac{\partial D_x}{\partial x} = \rho , \tag{2a}$$

while the latter and (1a) result in

$$\frac{\partial B}{\partial t} = 0 . \tag{2b}$$

From this result and (1e), it follows that B and H are independent of time. We know from physical considerations that if a time varying voltage is applied to the electrical terminals, E, D, and J must also vary with t in the same way. We found above that H, and therefore $\nabla \times H$ is independent of t; hence from (1b) we see that $\nabla \times H = 0$. Since both $\nabla \times H$ and $\nabla \cdot H$ are zero, H itself must be a constant vector; and if no external sources of B are present, we are free to impose the conditions that

$$B = H = 0 \quad . \tag{3}$$

The charge density, ρ, appearing in (1c) is real charge density as opposed to polarization charge density, usually denoted by ρ'. As the crystal is a dielectric, ρ is zero except on the surfaces, whereas ρ' is certainly not necessarily zero. Thus, within the crystal we find that

$$\frac{\partial D_x}{\partial x} = 0; \tag{2c}$$

hence D is independent of position - varying only with time. This is an important result and will be used in a later section.

We found above that $\nabla \times E = 0$ and $E_y = E_z = 0$. This permits us to set

$$E_x = -\frac{\partial \varphi}{\varphi x}$$

in the usual manner, where φ is a scalar potential function. Using the definition of voltage difference,

$$V_3 = \int_{\varphi_1}^{\varphi_2} d\varphi \ ,$$

it is immediately evident that

$$V_3 = -\int_{-\ell/2}^{\ell/2} E_x dx \ . \tag{4}$$

This result will be needed in the later development of the mesh equations.

By choosing the time variation of the impressed voltage to be $e^{j\omega t}$ in the usual manner, it is clear that the time derivatives of all the variables are simply the variables multiplied by jω. We found that $\nabla \times H = 0$, hence (1b) becomes

$$J_x = -\frac{\partial D_x}{\partial t} \ .$$

Remembering that $I = J \cdot A$, we finally arrive at the expression for current

$$I_3 = -j\omega AD \ , \tag{5}$$

which is also required for the derivation of the mesh equations. It is no longer necessary to continue the use of subscripts since we found a variation only with t and x — y and z are no longer of concern.

The above detailed discussion of perhaps perfectly obvious facts, has been included for the sake of completeness and rigor. It should be noted that an important result of the above discussion is that no electromagnetic radiation is generated by the transducer since Poynting's vector, E x H, is zero. This is a direct consequence of the assumptions concerning the shape of the crystal and the electrode arrangement, and only applies when these assumed conditions are valid.

3.2. Formulation of the Mesh Equations. The behaviour of the piezoelectric medium for the mode considered here is described by the Piezoelectric Equations[6]

$$- P = a \frac{\partial \xi}{\partial x} + dD \tag{6a}$$

$$E = d \frac{\partial \xi}{\partial x} + bD . \tag{6b}$$

These correspond to the equivalent equations for normal elastic dielectric materials[7,8]

$$- P = \frac{Y(1 - \sigma)}{(1 + \sigma)(1 - 2\sigma)} \frac{\partial \xi}{\partial x} = a' \frac{\partial \xi}{\partial x} \tag{6c}$$

$$E = \frac{1}{\epsilon} D = b' D \tag{6d}$$

where:

- P = Pressure — newton/sq. meter
- a = Stress to strain ratio for infinite slab with $D = 0$ — newton/sq. meter
- ξ = Mechanical displacement — meter
- d = Piezoelectric constant — newton/coulomb or volt/meter
- D = Dielectric displacement — coulomb/sq. meter
- E = Electric field intensity — volt/meter
- b = Electric intensity to dielectric displacement ratio with $\partial \xi / \partial x = 0$ — meter/farad (reciprocal of inductive capacity)
- Y = Young's modulus — newton/sq. meter
- σ = Poisson's ratio — dimensionless.

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6. Mason: op. cit., p. 202, formulae 6.32. This set is converted to apply to the thickness vibration by changing the y subscripts to x. Displacement is introduced rather than charge from the relation $Q = 1/4\pi$ (Disp.), and the two sets become similar in form but not in units. This conversion is made if $a = y_o/10$, $d = 3 \cdot 10^4 D$ and $b = 36\pi \cdot 10^9/k$. Mason's values for the coefficients for different crystals can be used above if these changes are made.

7. G. Joos, "Theoretical Physics"; Stechert. New York, 1934, p. 165 (with $e_{22} = e_{33} = 0$, $P_{11} = - P$ and $Y = E$).

8. Stratton: op. cit., p. 10 ff.

3.2.1. Mechanical Mesh Equations. From the first Piezoelectric Equation, (6a) and (5), we have

$$- P_1 = a \frac{\partial \xi_1}{\partial x} - \frac{d}{j\omega A} I_3 \qquad (7a)$$

$$- P_2 = a \frac{\partial \xi_2}{\partial x} - \frac{d}{j\omega A} I_3 , \qquad (7b)$$

where the subscript 1 refers to $x = -\ell/2$, and 2 to $x = \ell/2$. In order to combine both mechanical and electrical variables in one equation, it is frequently convenient to consider velocity as the analogue of current, and force the analogue of voltage. It should be noted that this choice is not unique; other analogues are perfectly permissable[9]. We shall follow this procedure here: but since pressure and not force is the parameter of interest, it is convenient to consider velocity analogous to current density, or surface area times velocity analogous to current.

In order to put (7a,b) in proper form, it is necessary to find the relation between $\partial\xi/\partial x$ and $\dot{\xi}$ since the latter is the desired variable. This can be done by integration of the wave equation which expresses ξ as a function of x and t. By application of Newton's First Law to a small volume element of density ρ — kg/m³ — as shown in Fig. 2, we can write

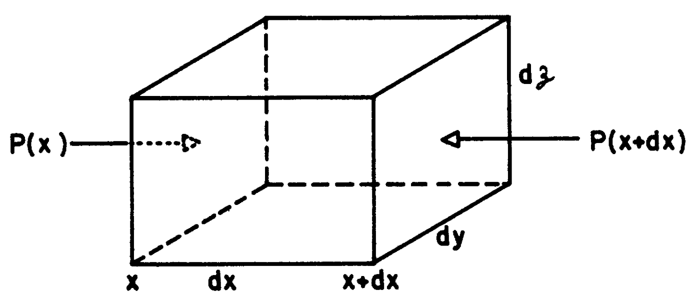

Figure 2. Forces acting on small volume element.

$$\rho dx dy dz \frac{\partial^2 \xi}{\partial t^2} = \left[P(x) - P(x + dx)\right] dy dz .$$

Since

$$P(x + dx) = P(x) + \frac{\partial P(x)}{\partial x} dx + \ldots. ,$$

we find, after neglecting higher order terms,

$$\rho \frac{\partial^2 \xi}{\partial t^2} = - \frac{\partial P}{\partial x} .$$

Upon differentiation of (6a) and introduction in the above, we obtain

$$\rho \frac{\partial^2 \xi}{\partial t^2} = a \frac{\partial^2 \xi}{\partial x^2} + d \frac{\partial D}{\partial x} .$$

9. Gardner and Barnes, op. cit. p. 60 ff.

Recalling from (2c) that D is independent of x, we finally obtain the simple wave equation

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2} , \qquad \text{(8a)}$$

where

$$c = \sqrt{\frac{a}{\rho}} \qquad \text{(8b)}$$

is the velocity of propagation of the elastic wave in the piezoelectric medium.

Assuming a time variation of the form $e^{j\omega t}$ in accordance with an earlier discussion, this integrates simply, and we have

$$\xi = \left[\xi_+ \, e^{-jkx} + \xi_- \, e^{jkx}\right] e^{j\omega t} \qquad \text{(9)}$$

where

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

ω = angular frequency — radian/sec.

λ = wavelength — meter.

This form of solution represents the mechanical displacement as the superposition of two traveling waves; one propagating in the positive x direction, and the other propagating in the negative x direction.

Upon differentiation, we find

$$\frac{\partial \xi_1}{\partial x} = jk\xi_+ \left[-e^{j\theta} + \gamma e^{-j\theta}\right] e^{j\omega t} \qquad \text{(10a)}$$

$$\frac{\partial \xi_2}{\partial x} = jk\xi_+ \left[-e^{-j\theta} + \gamma e^{j\theta}\right] e^{j\omega t} , \qquad \text{(10b)}$$

where

$$\theta = \frac{k\ell}{2}$$

$$\gamma = \frac{\xi_-}{\xi_+} ,$$

and

$$\dot{\xi}_1 = j\omega\xi_+ \left[e^{j\theta} + \gamma e^{-j\theta}\right] e^{j\omega t} \qquad \text{(11a)}$$

$$\dot{\xi}_2 = j\omega\xi_+ \left[e^{-j\theta} + \gamma e^{j\theta}\right] e^{j\omega t} . \qquad \text{(11b)}$$

Solving for ξ_+ in (11a), and substituting this in (10a), we obtain

$$\frac{\partial \xi_1}{\partial x} = \frac{k\dot{\xi}_1}{\omega}\left[\frac{\gamma e^{-j\theta} - e^{j\theta}}{\gamma e^{-j\theta} + e^{j\theta}}\right] . \qquad (12a)$$

In a similar manner, by using (11b) and (10b), we have

$$\frac{\partial \xi_2}{\partial x} = \frac{k\dot{\xi}_2}{\omega}\left[\frac{\gamma e^{j\theta} - e^{-j\theta}}{\gamma e^{j\theta} + e^{-j\theta}}\right] . \qquad (12b)$$

Recalling the fact that

$$\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} ,$$

if we define

$$\gamma = e^{2\psi} , \qquad (13)$$

the above equations take the simple form

$$\frac{\partial \xi_1}{\partial x} = \frac{k\dot{\xi}_1}{\omega} \tanh(\psi - j\theta) \qquad (14a)$$

$$\frac{\partial \xi_2}{\partial x} = \frac{k\dot{\xi}_2}{\omega} \tanh(\psi + j\theta) . \qquad (14b)$$

These two latter equations can now be substituted in (7a,b) to give the proper forms of mesh equations which describe the transducer from the mechanical terminals. By so doing, we obtain finally

$$P_1 = -\frac{\rho c}{A} \tanh(\psi - j\theta)\, A\dot{\xi}_1 + \frac{d}{j\omega A} I_3 \qquad (15a)$$

$$P_2 = -\frac{\rho c}{A} \tanh(\psi + j\theta)\, A\dot{\xi}_2 + \frac{d}{j\omega A} I_3 \qquad (15b)$$

where we have included the relation $\frac{ak}{\omega} = c^2\rho \cdot \frac{\omega}{c} \cdot \frac{1}{\omega} = \rho c$. These equations have the desired form, but an expression for ψ in terms of the mechanical boundary conditions is required in order to make them more readily usable.

If the ratio of pressure to particle velocity is defined as acoustic impedance[10], the boundary conditions at $x = -\ell/2$ and $x = \ell/2$ are specified by stating the acoustic impedances of the loading media. These are determining characteristics of materials, and in general, are both complex and functions of frequency. It should be noted, that the acoustic impedance for a wave traveling in the negative direction in a given medium equals the negative of the impedance for a wave traveling in the positive direction. This is simply seen physically from the fact that at a particular point of reference, the pressure is the same for both waves, but the velocities have opposite signs.

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10. P. M. Morse, "Vibration and Sound", McGraw-Hill, New York, 1936, p. 191 ff.

With this understood, the mechanical boundary conditions are imposed by stating

$$-\mathfrak{z}_1 = \frac{P_1}{\dot{\xi}_1} \qquad (16a)$$

$$\mathfrak{z}_2 = \frac{P_2}{\dot{\xi}_2} \qquad (16b)$$

Employing (15a) and (16a), we find

$$-\mathfrak{z}_1\dot{\xi}_1 = -\rho c \tanh(\psi - j\theta)\dot{\xi}_1 + \frac{d}{j\omega A} I_3 ,$$

and similarly from (15b) and (16b)

$$\mathfrak{z}_2\dot{\xi}_2 = -\rho c \tanh(\psi + j\theta)\dot{\xi}_2 + \frac{d}{j\omega A} I_3 .$$

Reverting to the original exponential form for tanh x and the defining relation for ψ, (13), these become

$$\mathfrak{z}_1\dot{\xi}_1 = \rho c \frac{\gamma e^{-j\theta} - e^{j\theta}}{\gamma e^{-j\theta} + e^{j\theta}} \dot{\xi}_1 - \frac{d}{j\omega A} I_3 \qquad (17a)$$

$$\mathfrak{z}_2\dot{\xi}_2 = -\rho c \frac{\gamma e^{j\theta} - e^{-j\theta}}{\gamma e^{j\theta} + e^{-j\theta}} \dot{\xi}_2 + \frac{d}{j\omega A} I_3 . \qquad (17b)$$

By introducing the expressions for $\dot{\xi}_1$ and $\dot{\xi}_2$ from (11a) and (11b) respectively, the two equations can be manipulated to solve for γ in terms of the boundary impedances and the constants of the crystal. The resulting expression is

$$\gamma = -\left[\frac{(\zeta_1 + 1)e^{j\theta} + (\zeta_2 - 1)e^{-j\theta}}{(\zeta_2 + 1)e^{j\theta} + (\zeta_1 - 1)e^{-j\theta}}\right], \qquad (18)$$

where

$$\zeta_1 = \frac{\mathfrak{z}_1}{\rho c}$$

$$\zeta_2 = \frac{\mathfrak{z}_2}{\rho c} .$$

Finally, from the expression for ψ, (13), we obtain

$$\psi = \frac{1}{2} \ln\left[\left\{\frac{(\zeta_1 + 1)e^{j\theta} + (\zeta_2 - 1)e^{-j\theta}}{(\zeta_2 + 1)e^{j\theta} + (\zeta_1 - 1)e^{-j\theta}}\right\} e^{j\pi}\right] . \qquad (19)$$

Having determined the value of ψ for any general conditions of loading, equations (15a,b) are complete. All the quantities are known with the exception of the variables P, $A\dot{\xi}$, and I. These, however, are just the quantities of interest, so that the remaining mesh equation describing the transducer from the electrical terminals must be derived. This will then give three equations and three unknowns from which the quantities of interest may be obtained.

3.2.2. Electrical Mesh Equations. By integrating the second Piezoelectric Equation, (6b), with respect to x between the limits $-\ell/2$ and $\ell/2$ and then introducing (4), we find

$$-V_3 = d\int_{\ell/2}^{\ell/2} \frac{\partial\xi}{\partial x}\,dx + b\int_{\ell/2}^{\ell/2} D\,dx \,.$$

Recalling that D is independent of x, this becomes

$$-V_3 = d(\xi_2 - \xi_1) + bD\ell \qquad (20)$$

where the subscripts have the same meaning as before.

Since $\dot{\xi} = j\omega\xi$, Eq. (20) assumes the desired form

$$V_3 = \frac{1}{j\omega C_p} A\dot{\xi}_1 - \frac{1}{j\omega C_p} A\dot{\xi}_2 + \frac{1}{j\omega C_E} I_3 \qquad (21)$$

where

$$C_p = \frac{A}{d}$$

$$C_E = \frac{A}{b\ell} \,.$$

C_E is recognized as simply the electrostatic capacity of the parallel plate capacitor formed by the equipotential surfaces of the transducer separated by the dielectric material. C_p is defined here as the piezoelectric capacity of the transducer. This is done simply as a matter of convenience since it is so similar to C_E. This completes the derivation of the three mesh equations. We now proceed with the definition of the impedance matrix and a brief discussion of its use.

3.3. Electromechanical Impedance Matrix. The mesh equations obtained above are repeated here for convenience.

$$P_1 = -\frac{\rho c}{A}\tanh(\psi - j\theta)A\dot{\xi}_1 + \frac{1}{j\omega C_p} I_3 \qquad (15a)$$

$$P_2 = -\frac{\rho c}{A}\tanh(\psi + j\theta)A\dot{\xi}_2 + \frac{1}{j\omega C_p} I_3 \qquad (15b)$$

$$V_3 = \frac{1}{j\omega C_p} A\dot{\xi}_1 - \frac{1}{j\omega C_p} A\dot{\xi}_2 + \frac{1}{j\omega C_E} I_3 \,. \qquad (21)$$

These were obtained for the polarity conditions shown in Fig. 3. Here all arrows point in the positive direction. The lack of symmetry made evident by the signs in (21) can be rectified by reversing the positive direction of $\dot{\xi}_2$. This is shown in Fig. 4 where the symbol v has replaced $\dot{\xi}$ to eliminate confusion. That is, $v_1 = \dot{\xi}_1$, but $v_2 = -\dot{\xi}_2$.

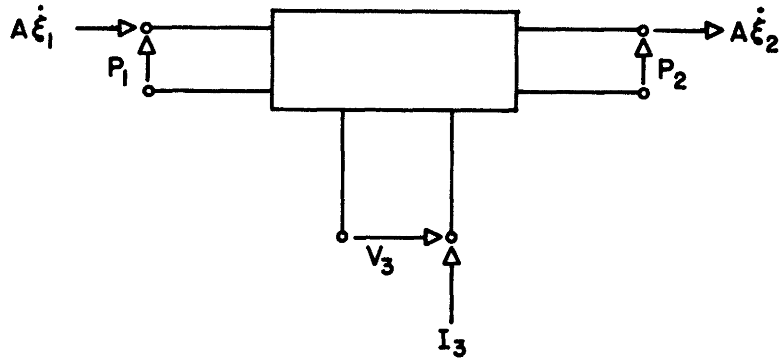

Figure 3. Six terminal network with original positive polarities.

We see by inspection of Fig. 4 that the six terminal network is now completely symmetric: all currents are taken positive when feeding into the network at the particular terminals shown. The revised mesh equations now become

$$P_1 = \frac{\rho c}{A} \tanh\left(-\psi + j\frac{\omega}{c}\frac{\ell}{2}\right) Av_1 + \frac{1}{j\omega C_p} I_3 \qquad (22a)$$

$$P_2 = \frac{\rho c}{A} \tanh\left(\psi + j\frac{\omega}{c}\frac{\ell}{2}\right) Av_2 + \frac{1}{j\omega C_p} I_3 \qquad (22b)$$

$$V_3 = \frac{1}{j\omega C_p} Av_1 + \frac{1}{j\omega C_p} Av_2 + \frac{1}{j\omega C_E} I_3 \quad . \qquad (22c)$$

These can be written in matrix form if the usual rules of matrix multiplication are

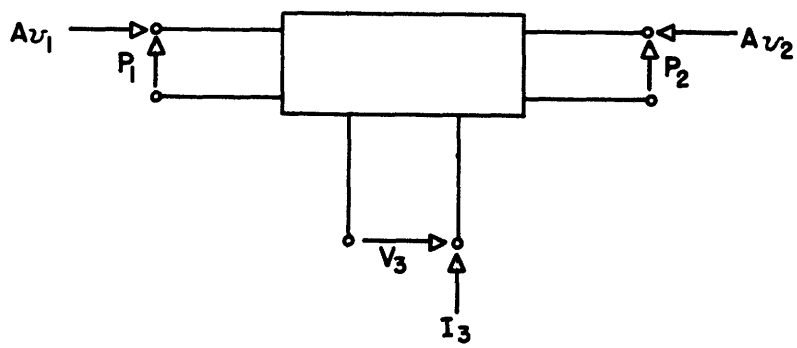

Figure 4. Six terminal network with revised positive polarities.

followed[11]. From (22a,b,c) we can immediately write the matrix equation

11. E. A. Guillemin, "Communication Networks," Vol. II, J. Wiley, 1935, Chap. IV. The Application of matrix algebra to network analysis and the concept of an impedance matrix is discussed in considerable detail.

$$\begin{bmatrix} P_1 \\ P_2 \\ V_3 \end{bmatrix} = [Z] \times \begin{bmatrix} Av_1 \\ Av_2 \\ I_3 \end{bmatrix} \tag{23}$$

where the electromechanical impedance matrix $[Z]$ becomes

$$[Z] = \begin{bmatrix} \frac{\rho c}{A}\tanh(-\psi + j\frac{\omega}{c}\frac{\ell}{2}) & 0 & \frac{1}{j\omega C_p} \\ 0 & \frac{\rho c}{A}\tanh(\psi + j\frac{\omega}{c}\frac{\ell}{2}) & \frac{1}{j\omega C_p} \\ \frac{1}{j\omega C_p} & \frac{1}{j\omega C_p} & \frac{1}{j\omega C_E} \end{bmatrix} \tag{24}$$

This matrix is seen to possess some very special properties. Since $Z_{13} = Z_{31}$, and $Z_{23} = Z_{32}$, the transducer obeys the Reciprocity Theorem if the proper precautions are taken as to the equality of the internal impedances of any power generating and indicating apparatus introduced[12]. This, of course, must always be the case when applying reciprocity tests.

We also see that $Z_{13} = Z_{23}$ and $Z_{31} = Z_{32}$. This implies that the mechanical terminal pairs are equivalent so far as the electrical terminals are concerned. This is, of course, obvious on physical grounds alone. If it were not so clearly substantiated by the theory, we would be correct in regarding the latter with suspicion — to put it mildly!

The third point of interest is that $Z_{12} = Z_{21} = 0$. This states that the velocity of one crystal face has no effect on the pressure at the other face. Only the current flowing into the electrical terminals and the velocity of the surface in question affects this pressure.

Finally, we recognize that the only impedance elements comprising the matrix which are not only functions of frequency, but also of acoustic loading once the crystal type and geometry are specified, are the two mechanical self-impedances, Z_{11} and Z_{22}. These are functions of loading since ψ is determined by the acoustic media at the two surfaces. If it were not for this complicating factor, the piezoelectric transducer would be a simpler element with which to work both theoretically and experimentally, since the remaining self- and mutual-impedances assume simple forms.

In the particular case of symmetric loading, we see from (19) that $\psi = j\frac{\pi}{2}$. Since $\tanh j(\frac{\omega}{c}\frac{\ell}{2} \pm \frac{\pi}{2}) = -j\cot\frac{\omega\ell}{c2}$, we obtain the impedance matrix

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12. Guillemin: loc. cit. Vol. 1, p. 152. This is also an excellent reference for those unfamiliar with methods of network analysis.

$$[Z_s] = \begin{bmatrix} \frac{\rho c}{jA}\cot(\frac{\omega}{c}\frac{\ell}{2}) & 0 & \frac{1}{j\omega C_p} \\ 0 & \frac{\rho c}{jA}\cot(\frac{\omega}{c}\frac{\ell}{2}) & \frac{1}{j\omega C_p} \\ \frac{1}{j\omega C_p} & \frac{1}{j\omega C_p} & \frac{1}{j\omega C_E} \end{bmatrix} \qquad (25)$$

which is only a function of frequency once the crystal is specified. Thus, the design of systems using symmetrical acoustic loading is much simpler than the corresponding asymmetrical system.

By inspection of the general Z matrix, an equivalent circuit for the transducer which is valid for any operating conditions can immediately be determined. This is shown in Fig. 5.

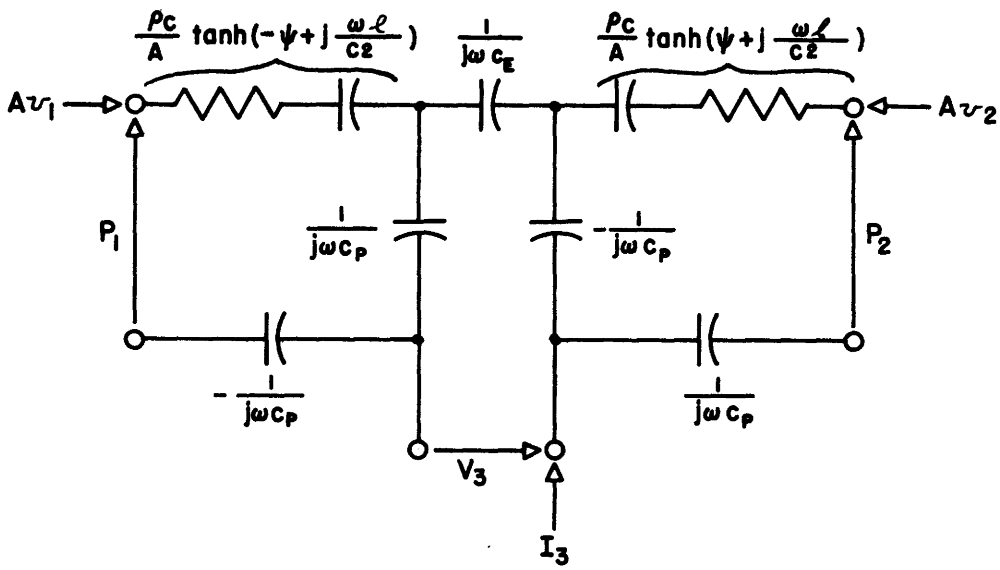

Figure 5. Six terminal equivalent circuit valid for any general terminating condition.

4. Conclusions

The electromechanical impedance matrix, mesh equations or equivalent circuit obtained above can be used to determine the operating characteristics of the transducer for any general conditions of driving or loading, regardless of the terminals in question. The procedure is identical to that used in connection with pure electrical networks whose impedance matrix is specified.

Once the boundary conditions satisfied by the respective terminal pairs are stated in terms of the actual values of loading impedances or energy sources, sufficient information is introduced to permit the solution of the set of linear equations for the unknown parameters. Due to the simple form of our final equations, these solutions can be obtained by those familiar with any of the simple techniques for solving a set of three linear equations. Of course, more sophisticated methods can also be applied in which the matrix is used directly without use of the mesh equations. The methods for solution of such a set of linear equations are numerous and varied, and hence, are beyond the scope of this paper. It is intended that the forms given be suitable for solution by all these methods: the choice is left entirely to the reader.

As an example of the use of the above theory for the solution of a design problem, the properties of a piezoelectric transducer driven electrically and loaded acoustically in any arbitrary manner are investigated in Part II. General formulae of driving point impedance and acoustic power output are developed, and curves of these quantities for particular values of loading are plotted as functions of frequency.

II. ELECTRICAL DRIVING POINT IMPEDANCE AND ADMITTANCE

1. Introduction

In Part I of this paper, we saw that a piezoelectric transducer can be represented by a six terminal network. The mesh equations, electromechanical impedance matrix and equivalent circuit for a crystal operating in the thickness vibration mode were derived. Here, we shall apply these results to determine the electrical driving point impedance and admittance for such a transducer when loaded by any two arbitrary acoustic media. This problem is of fundamental importance to the designer who is faced with the task of developing electrical apparatus capable of properly exciting the piezoelectric crystal when loaded by specified media.

2. Symmetrical Acoustic Loading

In order to illustrate the method we shall use later for the general problem without becoming overburdened with algebraic manipulation, we shall consider first the case of symmetrical acoustic loading. By use of Eq. (16) and the further relations introduced above, $\dot{\xi}_1 = v_1$ and $\dot{\xi}_2 = - v_2$, we specify the boundary conditions existing at the mechanical terminals in the general case by letting

$$\mathfrak{Z}_1 = -\frac{P_1}{v_1} \qquad (26a)$$

and

$$\mathfrak{Z}_2 = -\frac{P_2}{v_2} . \qquad (26b)$$

Of course, for the symmetrical case, $\mathfrak{Z}_1 = \mathfrak{Z}_2$, so we can omit subscripts and merely write

$$\mathfrak{Z} = -\frac{P}{v} . \qquad (27)$$

This now applies to either of the two mechanical terminal pairs.

By inspection of the impedance matrix for symmetrical loading given by (25) plus the additional conditions introduced by (27), the equivalent circuit presented in Fig. 5 becomes that shown in Fig. 6.

The method by which the minus sign in (27) is introduced in this circuit deserves particular comment. The pressure, P, has been considered a positive quantity in all the preceding work when the stress applied to a crystal face produces compression with the electrical terminals open-circuited, i.e., $I_3 = 0$. Hence, P denotes a pressure "rise" completely analogous to a voltage "rise" introduced when discussing a source of electromotive force.

In this sense, the arrows at the terminals of the circuit of Fig. 5 indicate

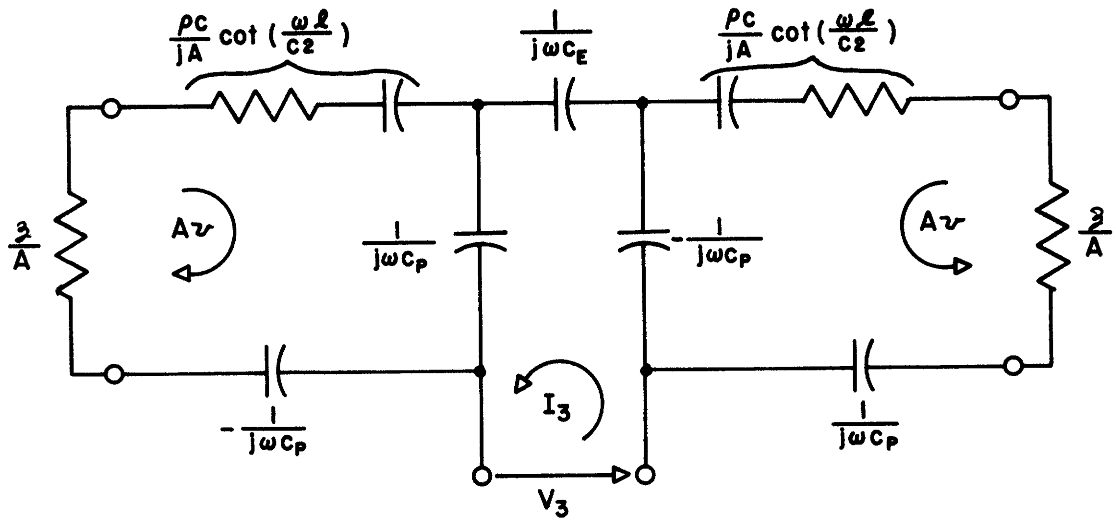

Figure 6. Two terminal equivalent circuit including two symmetric mechanical terminations.

directions of pressure and voltage rises. If we choose to keep both the same positive directions for current flow in the respective meshes as above and positive values for the terminating impedances, (27) shows that P is negative. Thus, the directions of pressure rises at the two mechanical terminal pairs must be opposite to those indicated in Fig. 5; or the pressure "drops" must be in the arrow directions. By following the velocity mesh currents in Fig. 6, it will be seen that this is indeed the case.

With this clearly in mind, it is evident that the complete solution of the symmetrical transducer problem can be obtained from an analysis of the relatively simple three mesh circuit of Fig. 6. Using the method of determinants and Cramer's rule[13], we find

$$I_3 = \frac{\begin{vmatrix} \frac{\rho c}{jA}\cot\theta + \frac{\mathcal{Z}}{A} & 0 & 0 \\ 0 & \frac{\rho c}{jA}\cot\theta + \frac{\mathcal{Z}}{A} & 0 \\ \frac{1}{j\omega C_p} & \frac{1}{j\omega C_p} & V_3 \end{vmatrix}}{\begin{vmatrix} \frac{\rho c}{jA}\cot\theta + \frac{\mathcal{Z}}{A} & 0 & \frac{1}{j\omega C_p} \\ 0 & \frac{\rho c}{jA}\cot\theta + \frac{\mathcal{Z}}{A} & \frac{1}{j\omega C_p} \\ \frac{1}{j\omega C_p} & \frac{1}{j\omega C_p} & \frac{1}{j\omega C_E} \end{vmatrix}}, \quad (28)$$

13. E. A. Guillemin, "Communication Networks", Vol. I, Wiley, New York, 1935, p. 147 ff.

where $\Theta = \frac{\omega}{c}\frac{\ell}{2}$ as before.

Since the electrical driving point impedance of this symmetrical case is $Z_s = V_3/I_3$ ohms, we have from (28)

$$Z_s = \frac{\left[\frac{\rho c}{jA}\cot\Theta + \frac{\mathfrak{Z}}{A}\right]\left[\frac{1}{j\omega C_E}\left(\frac{\rho c}{jA}\cot\Theta + \frac{\mathfrak{Z}}{A}\right) + \frac{1}{\omega^2 C_p^2}\right] - \frac{1}{j\omega C_p}\left[\frac{1}{j\omega C_p}\left(\frac{\rho c}{jA}\cot\Theta + \frac{\mathfrak{Z}}{A}\right)\right]}{\left[\frac{\rho c}{jA}\cot\Theta + \frac{\mathfrak{Z}}{A}\right]^2}$$

or

$$Z_s = \frac{1}{j\omega C_E} + \frac{A}{\omega^2 C_p{}^2 \rho c}\left[\frac{2}{\zeta - j\cot\Theta}\right] \qquad (29)$$

where

$$\zeta = \frac{\mathfrak{Z}}{\rho c}.$$

Although ζ is dimensionless, we shall refer to it as the normalized load impedance.

By comparison of the Piezoelectric Equations, (6), with those for a two-mesh electrical network, further simplification can be achieved. For a two-mesh electrical network, the mesh equations can be written in the form[14]

$$V_1 = Z_{11} I_1 + Z_{12} I_2$$

$$V_2 = Z_{21} I_1 + Z_{22} I_2 \quad .$$

In the analysis of such systems, it is usually convenient to define the dimensionless quantity, K, known as the "coupling coefficient", by the relation[15]

$$K^2 = \frac{Z_{12} Z_{21}}{Z_{11} Z_{22}} \quad .$$

Equations (6) have the same form as those above, and here, too, it is convenient to define a dimensionless coupling coefficient. In this instance, we obtain

$$K^2 = \frac{d^2}{ab} \quad . \qquad (30)$$

If we recall from (21) that $C_p = \frac{A}{d}$ and $C_E = \frac{A}{b\ell}$, and from (8b), that $C = \sqrt{\frac{a}{\rho}}$, we find

$$K^2 = \frac{A}{C_p^2 \rho c} \times \frac{C_E 2\Theta}{\omega} \quad . \qquad (31)$$

Substitution of this result in (29) gives finally

14. Guillemin: loc. cit., Vol. II, p. 135.

15. A lower case letter is usually used, but to avoid confusion with k appearing in (9), the upper case will be used throughout.

$$Z_s = \frac{1}{j\omega C_E}\left[1 - \frac{K^2/\theta}{\cot\theta + j\zeta}\right] . \qquad (32)$$

This is the driving point impedance for any conditions of frequency and symmetrical loading consistent with our original assumptions. Although we used the network of Fig. 6 to find first I_3 and then Z_s, it should be realized that the remaining mesh currents and then the power dissipated by the real parts of the loading impedances can similarly be obtained in a straightforward manner. In this way, it is a simple matter to determine the acoustic power output in each load as a function of frequency once the electrical driving source is specified. We shall see later that this is also the case for the general problem of unsymmetrical loading.

3. General Acoustic Loading

In this section, we shall determine the driving point impedance and admittance for the general transducer problem following the procedure used in the simpler case of symmetrical loading. The boundary conditions are given by (26 a,b), and after introducing these in the circuit of Fig. 5, we obtain that shown in Fig. 7.

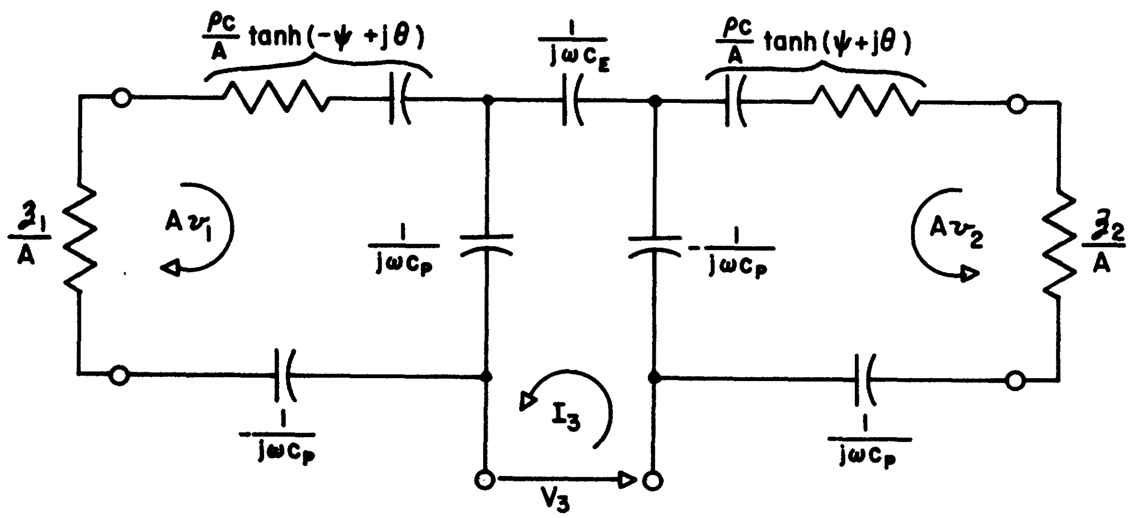

Figure 7. Two terminal equivalent circuit including two asymmetric mechanical terminations.

Writing the equation for I_3 in determinant form as before, we have

$$
I_3 = \frac{\begin{vmatrix} \frac{\rho c}{A}\tanh(-\psi + j\theta) + \frac{\mathfrak{Z}_1}{A} & 0 & 0 \\ 0 & \frac{\rho c}{A}\tanh(\psi + j\theta) + \frac{\mathfrak{Z}_2}{A} & 0 \\ \frac{1}{j\omega C_p} & \frac{1}{j\omega C_p} & V_3 \end{vmatrix}}{\begin{vmatrix} \frac{\rho c}{A}\tanh(-\psi + j\theta) + \frac{\mathfrak{Z}_1}{A} & 0 & \frac{1}{j\omega C_p} \\ 0 & \frac{\rho c}{A}\tanh(\psi + j\theta) + \frac{\mathfrak{Z}_2}{A} & \frac{1}{j\omega C_p} \\ \frac{1}{j\omega C_p} & \frac{1}{j\omega C_p} & \frac{1}{j\omega C_E} \end{vmatrix}} .
$$

Solving this determinantal equation for the ratio V_3/I_3 gives the result

$$
Z = \frac{1}{j\omega C_E}\left[1 - \frac{K^2}{j2\theta} S\right] , \tag{33a}
$$

where

$$
S = \left[\frac{1}{\tanh(\psi + j\theta) + \zeta_2} + \frac{1}{\tanh(-\psi + j\theta) + \zeta_1}\right] . \tag{33b}
$$

For applications in which the crystal and acoustic loads are specified, the solution is now complete since all the parameters entering in the above expression for Z are known. However, we wish here to investigate the behaviour of this impedance as we vary the parameters; so, unfortunately, a considerable amount of algebra is necessary in order to convert (33) to a more useful form. Our goal is to obtain expressions for the real and imaginary parts of Z in terms of the system parameters: crystal constants, normalized load impedances and frequency.

Recalling the results of (12) and (14), we have

$$
\tanh(\psi + j\theta) = \frac{\gamma e^{j\theta} - e^{-j\theta}}{\gamma e^{j\theta} + e^{-j\theta}} \tag{34a}
$$

$$
\tanh(-\psi + j\theta) = -\frac{\gamma e^{-j\theta} - e^{j\theta}}{\gamma e^{-j\theta} + e^{j\theta}} , \tag{34b}
$$

while from (18),

$$
\gamma = -\left[\frac{(\zeta_1 + 1)e^{j\theta} + (\zeta_2 - 1)e^{-j\theta}}{(\zeta_2 + 1)e^{j\theta} + (\zeta_1 - 1)e^{-j\theta}}\right] . \tag{35}
$$

Substitution of (35) in (34a) and (34b), respectively, leads to

$$\tanh(\Psi + j\Theta) = \frac{\zeta_2 + \zeta_1 \cos 2\Theta + j\sin 2\Theta}{\cos 2\Theta - 1 + j\zeta_1 \sin 2\Theta} \tag{36a}$$

$$\tanh(-\Psi + j\Theta) = \frac{\zeta_1 + \zeta_2 \cos 2\Theta + j\sin 2\Theta}{\cos 2\Theta - 1 + j\zeta_2 \sin 2\Theta} \; . \tag{36b}$$

Using these results for substitution in (33b), we have

$$S = \frac{2(\cos 2\Theta - 1) + j(\zeta_1 + \zeta_2)\sin 2\Theta}{(\zeta_1 + \zeta_2)\cos 2\Theta + j(1 + \zeta_1\zeta_2)\sin 2\Theta} \; . \tag{37}$$

Rationalization of this expression to obtain real and imaginary parts results in

$$S = S_1 + jS_2 \tag{38a}$$

where

$$S_1 = \frac{\left[\zeta_1 + \zeta_2\right]\left[2(1 - \sec 2\Theta) + (1 + \zeta_1\zeta_2)\tan^2 2\Theta\right]}{\left[\zeta_1 + \zeta_2\right]^2 + \left[1 + \zeta_1\zeta_2\right]^2 \tan^2 2\Theta} \tag{38b}$$

and

$$S_2 = \frac{\left[\tan 2\Theta\right]\left[(\zeta_1 + \zeta_2)^2 + 2(1 + \zeta_1\zeta_2)(\sec 2\Theta - 1)\right]}{\left[\zeta_1 + \zeta_2\right]^2 + \left[1 + \zeta_1\zeta_2\right]^2 \tan^2 2\Theta} \; . \tag{38c}$$

If we now consider the driving point impedance as comprising a resistance R and reactance X in series, from (33a) and (38), we find

$$Z = R + jX \tag{39a}$$

where

$$R = \frac{K^2 S_1}{\omega c_E 2\Theta} \; , \tag{39b}$$

and

$$X = \frac{K^2}{\omega c_E 2\Theta}\left[S_2 - \frac{2\Theta}{K^2}\right] \; . \tag{39c}$$

Although these are the expressions for which we have been looking, they can be improved insofar as their applicability to a large range of design problems is concerned. We shall see later that a crystal which is symmetrically vacuum loaded, $\zeta_1 = \zeta_2 = 0$, has its first "resonance" for $\Theta = \frac{\pi}{2}$. Since $\Theta = \frac{\omega}{c}\frac{\ell}{2} = \frac{2\pi}{\lambda}\frac{\ell}{2}$, if we define the quantity λ_o from the condition that $\frac{2\pi}{\lambda_o}\frac{\ell}{2} = \frac{\pi}{2}$, we have

$$\lambda_o = 2\ell \; . \tag{40a}$$

In other words, for the frequency at which this first resonance occurs, we arbitrarily let the thickness of the crystal, ℓ, equal $\lambda_o/2$. Since the velocity of propagation within the crystal, c, is a constant of the piezoelectric material, we are able to define the quantity, ω_o, by the equation

$$\omega_o = \frac{2\pi c}{\lambda_o} . \qquad (40b)$$

Finally, by defining the dimensionless parameter, δ, by

$$\delta = \frac{\omega}{\omega_o} = \frac{\lambda_o}{\lambda} , \qquad (40c)$$

we obtain from above

$$\theta = \delta\frac{\pi}{2} . \qquad (40d)$$

Upon introducing these relations in the expressions for the impedance, (40), we obtain the final results:

$$Z = R + jX \qquad (41a)$$

where

$$R = \frac{K^2}{\delta^2 \frac{\pi}{2}\omega_o C_E}\left\{\frac{\left[\zeta_1 + \zeta_2\right]\left[1 - \sec\delta\pi + \frac{1}{2}(1 + \zeta_1\zeta_2)\tan^2\delta\pi\right]}{\left[\zeta_1 + \zeta_2\right]^2 + \left[1 + \zeta_1\zeta_2\right]^2\tan^2\delta\pi}\right\} \qquad (41b)$$

and

$$X = \frac{K^2}{\delta^2 \frac{\pi}{2}\omega_o C_E}\left\{\frac{\tan\delta\pi\left[\frac{1}{2}(\zeta_1 + \zeta_2)^2 + (1 + \zeta_1\zeta_2)(\sec\delta\pi - 1)\right]}{\left[\zeta_1 + \zeta_2\right]^2 + \left[1 + \zeta_1\zeta_2\right]^2\tan^2\delta\pi} - \frac{\delta\frac{\pi}{2}}{K^2}\right\} \qquad (41c)$$

The driving point admittance is obtained very easily from these results. Since the admittance, Y, is the sum of a conductance G and susceptance B in parallel, we have

$$Y = G + jB . \qquad (42a)$$

By definition, $Y = \frac{1}{Z}$; thus, from (41a) and (42a) it follows immediately that

$$G = \frac{R}{R^2 + X^2} \qquad (42b)$$

and

$$B = -\frac{X}{R^2 + X^2} . \qquad (42c)$$

Equations (41) and (42) are the general relations valid for all conditions of loading and frequency. The crystal constants, K, ω_0 and C_E, are only dependent upon the type of piezoelectric material used and its dimensions; whereas, the normalized load impedances, ζ_1 and ζ_2, are determined by the choice of crystal and the loading media. It must be remembered that in general ζ is complex, although in the majority of ultrasonic applications, ζ can be considered real.

The two representations of the transducer driving its acoustic loads as electrical circuit elements are shown in Fig. 8. The series form is useful when the source of electrical energy has a high internal impedance, while the shunt form is useful for a source having a low internal impedance. These two limiting cases are usually approximated by constant current and constant voltage sources, respectively.

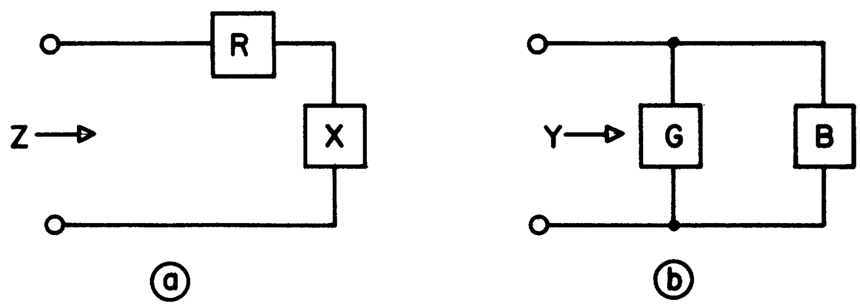

Figure 8. a. Two terminal series equivalent circuit.
b. Two terminal shunt equivalent circuit.

4. Particular Cases of Acoustic Loading

In order to increase the utility of the foregoing analysis, in this section, curves of the four quantities R, X, G, and B are presented as functions of frequency for a variety of loading conditions. Some of these conditions have been chosen since they frequently arise in actual ultrasonic applications, and hence are of practical importance, while others were chosen to indicate the behaviour of the transducer when loaded by certain idealized media.

The latter, although not of practical use in a quantitative sense, are important in that they increase our understanding of the operation of the transducer. Frequently, such information indicates what can and cannot be expected from a particular choice of conditions, and hence is of use in separating reasonable from unreasonable applications.

In an attempt to make these results applicable to all cases in which the thickness vibration mode is employed, the common factor, $K^2/\frac{\pi}{2}\omega_0 C_E$, is not included in the calculated values. This factor, which is wholly dependent upon the crystal used, is introduced after the values are taken from the curves; thus, many of the curves are applicable to any crystal chosen. For the same reason, the values of the circuit elements are plotted against the dimensionless parameter δ, rather than frequency, thus permitting use of the curves for any frequency region. More will be said concerning the universality of each set of curves in the discussion to follow.

4.1. <u>Resistance versus Frequency</u>. (cf. Figs. 9-13) If the factor $2K^2/\pi\omega_0 c_E$ is removed from the right side of (41b), we obtain the dimensionless expression

$$R\,\frac{\pi\omega_0 C_E}{2K^2} = \frac{1}{\delta^2}\left\{\frac{\left[\zeta_1 + \zeta_2\right]\left[1 - \sec\delta\pi + \frac{1}{2}(1 + \zeta_1\zeta_2)\tan^2\delta\pi\right]}{\left[\zeta_1 + \zeta_2\right]^2 + \left[1 + \zeta_1\zeta_2\right]^2\tan^2\delta\pi}\right\}. \qquad (43)$$

It is this expression which is plotted as a function of δ for various assumed values of ζ_1 and ζ_2. Although the ordinates and abscissae of the resulting curves are dimensionless, it is convenient to refer to them as resistance and frequency, respectively.

By inspection of (43), several points of interest are evident. First, we see that the term on the right is entirely free from all explicit crystal constants -- although they are present implicitly in the values of ζ corresponding to various loading media.[16] Hence, the curves of resistance versus frequency are truly universal if applied for loads having the ζ's which are specified.

Secondly, we notice that the value of the bracketed term is periodic, having a period along the frequency axis of 2. As a result, in order to have complete information concerning its behaviour for any value of δ, we need only compute values in the interval $0 \leq \delta \leq 2$.

Finally, we see that the quantity $1/\delta^2$ is a modulating function for the periodic term. The complete behavior of the resistance for any frequency can thus be obtained if its values for any one period are given. In accordance with this, all the resistance curves but one are plotted for the frequency interval of most general interest -- $0 \leq \delta \leq 2$. The one exception has been plotted for values of δ up to 9.5 merely as an illustrative example to show the variation of resistance as the crystal is driven at frequencies higher than its fundamental.

From Fig. 8a, it is evident that the total acoustic power output, if we assume a constant current generator, is proportional to R. For this limiting case, we can obtain the bandwidth of the transducer by measuring the width of the curves between half-amplitude points. This will vary somewhat according to the harmonic considered due to the $1/\delta^2$ factor.

A measure of the bandwidth is important for pulsed operation of the transducer since it is a determining factor of the minimum pulse width that can properly be transmitted. The bandwidth for any actual electrical source is readily obtained from these plots of R and Fig. 8a by first finding the current as a function of frequency, and then obtaining the product I^2R, again as a function of frequency. The width between half-power points of the resulting function is the desired bandwidth figure.

It should be noticed, that numerical values of ζ are given on the curves in addition to the corresponding media: the latter are enclosed in parentheses. These

16. A. B. Wood, "A Textbook of Sound", Bell, London, 1944, pp. 562-63. All values of acoustic impedance used here have been taken from this source.

latter captions only apply when the transducer is Quartz. They have been included for added convenience since Quartz is generally used for high frequency work.

At this point, it would be desirable to discuss in detail the many interesting features of the curves and their relations to one another; however, in the interests of brevity, this must grudgingly be omitted. An attempt has been made to include enough material on the curves, themselves, to make them self-explanatory after a careful perusal. A summary of the most important features is included in the Conclusions.

4.2. Reactance versus Frequency. (cf. Figs. 14-18) If we convert (41c) to dimensionless form, we obtain

$$X\frac{\pi\omega_o C_E}{2K^2} = \frac{1}{\delta^2}\left\{\frac{\tan\delta\pi\left[\frac{1}{2}(\zeta_1+\zeta_2)^2+(1+\zeta_1\zeta_2)(\sec\delta\pi-1)\right]}{\left[\zeta_1+\zeta_2\right]^2+\left[1+\zeta_1\zeta_2\right]^2\tan^2\delta\pi}\right\} - \frac{\pi}{2\delta K^2} . \qquad (44)$$

The first term on the right side of this equation is similar to that discussed above in that it is independent of the crystal used, it has a period of 2, and it includes the $1/\delta^2$ modulating function. Here, however, we have an additional term wich does depend upon the choice of crystal since it includes the coupling coefficient, K.

Referring to the reactance curves, we see that the latter term is a hyperbola around which the variations produced by the periodic term occur. These variations from the true hyperbola can be applied universally even though the hyperbola, itself, will have different values due to the presence of K. These curves have been computed taking $K = 0.10$ which is the value for Quartz; for other crystals, it is a simple matter to superimpose the periodic variations shown here upon the proper hyperbola.

The variations of the reactance curves for $\zeta_2 = 2.83 \times 10^{-5}$ (Air) are extremely sharp in all the cases plotted except that for which $\zeta_1 = 1.00$ (Quartz), but do not become infinite. Although values of the reactance functions are given for values of δ very close to the peaks, these are not to be taken as the maxima since the calculations were made for increments in δ of .001. If smaller increments had been used, larger values for the maxima would have been found. Differentiation to obtain the exact values of δ corresponding to these maxima lead to complicated transcendental equations. The time needed for solution of these equations could not be justified in view of the purpose for which the curves are intended; thus, the values are given merely to indicate the order of magnitudes of the peaks of the reactance functions.

4.3. Conductance versus Frequency. (cf. Figs. 19-22) By use of Eq. (42b) and the dimensionless resistance and reactance expressions, (43) and (44) respectively, it is clear that we have all the information necessary for plotting curves of $G\pi\omega_o C_E/2K^2$ against δ. The question of universality becomes complicated in this case since both R and X appear. Although clear-cut conclusions cannot be made, several general observations may be of value to the designer.

Inspection of the R and X curves shows that X^2 is very much larger than R^2 throughout most of the frequency range, exclusive of the resonance regions, for the majority of loading conditions. We, therefore, obtain the expression

$$G \cong \frac{R}{X^2} ,$$

which is valid when δ is removed from the resonances. Here, X follows a true hyperbola which was shown above to be dependent upon the crystal chosen. In these regions then, the conductance curves are not universal as to magnitudes, although the contours can be applied quite generally.

For the regions of resonance, the above approximation no longer holds because we can no longer neglect R^2. Under these conditions, the variations in the X hyperbolae and the values of R become the determining factors. Since both of these quantities were shown to be independent of the piezoelectric material, the conductance curves near resonance are more universally applicable than for other frequency domains. The latter statement becomes more nearly true for lighter loads — smaller values of ζ — since, in these cases, the values for the variations of X completely overshadow those for the parent hyperbolae.

If more exact values of conductance are needed for a transducer other than Quartz, they can be obtained in a straightforward manner from the universal resistance and the corrected reactance curves. For most design problems, however, this procedure will not be necessary because the curves presented here will be found sufficiently accurate.

From Fig. 8b, we recognize that the total acoustic power output, assuming a constant voltage source, is proportional to G. Thus, as discussed above in connection with the curves of R and a constant current source, we can obtain useful information concerning the bandwidth of the transducer when driving various loading media. By considering the curves of resistance and conductance as acoustic power output for the two electrical sources mentioned, we can appreciate the extreme importance of the internal source impedance and can estimate the behaviour to expect with sources other than these two limiting cases.

4.4. Susceptance versus Frequency. (cf. Figs. 23-26) From (42c), it follows that for regions other than resonance where we can neglect R^2, we have

$$B \cong -\frac{1}{X} .$$

Near resonance R^2 must again be included, and the final result is that the B curves are straight lines with variations appearing at the resonances. The remarks concerning universality of the susceptance curves are entirely similar to those given above in the discussion of the conductance; hence, we shall not discuss this further here.

5. Conclusions

The impedance and admittance curves plotted as functions of frequency present us with much information concerning the operating characteristics of piezoelectric transducers under varied conditions of loading and frequency. Although most of the details have been left to the actual curves, a listing of the important trends should be of considerable value in assisting us to comprehend the behaviour of such elements. For detailed verifications of the items listed, reference to the curves and equations is required.

i. Bandwidth increases when the load impedance increases. In most cases this increase continues for load impedances greater than that of the crystal.

ii. Loading of both crystal surfaces leads to greater bandwidths than are obtained by keeping one surface unloaded.

iii. Maximum power output at resonance for a given driving source varies inversely to transducer bandwidth: power output decreases for an increase in bandwidth, and vice versa.

iv. Operation at very low frequencies requires heavy loading of both surfaces and a source with a high internal impedance.

v. Loading by materials having impedances greater than that of the transducer leads to doubly-peaked resonance curves analogous to those obtained with overcoupled electrical circuits.

vi. With a high internal impedance source, the resonance frequencies decrease with an increase in loading; but with a low internal impedance source, the resonance frequencies remain very nearly constant.

vii. With a high internal impedance source, the power output varies inversely to the square of the order of the harmonic of the fundamental frequency. With a low internal impedance source and light loading, this is also roughly the case; but with heavy loading, the power output is very nearly independent of the order of the harmonic.

Acknowledgments

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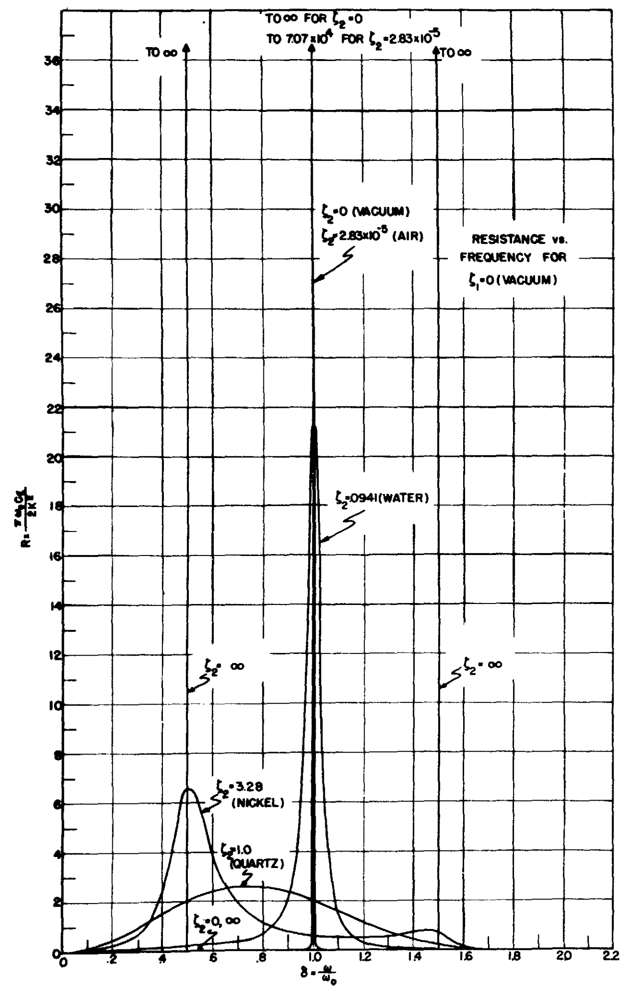

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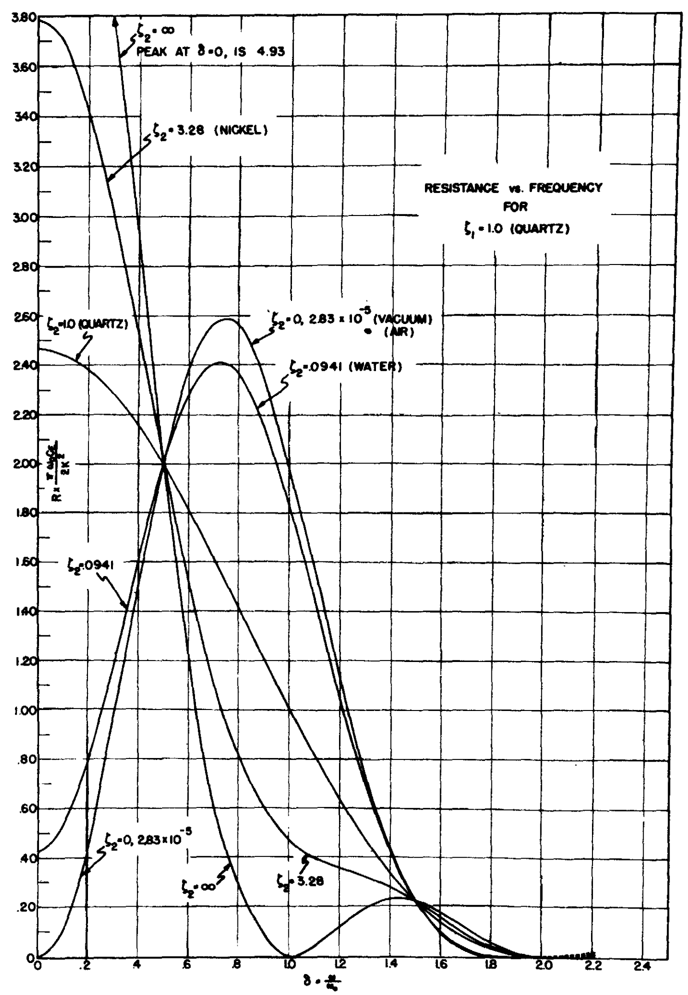

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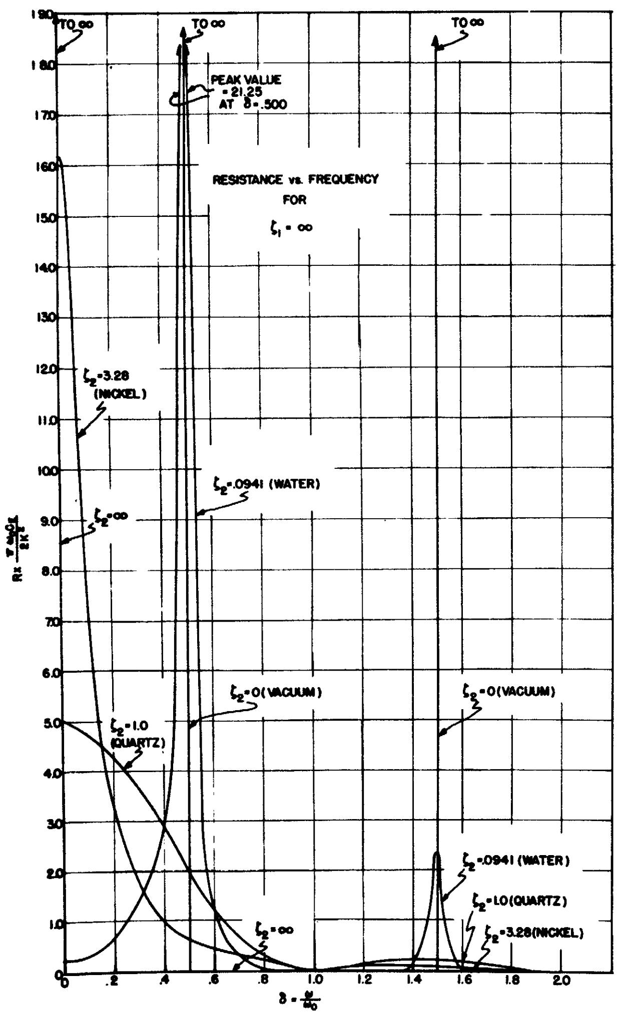

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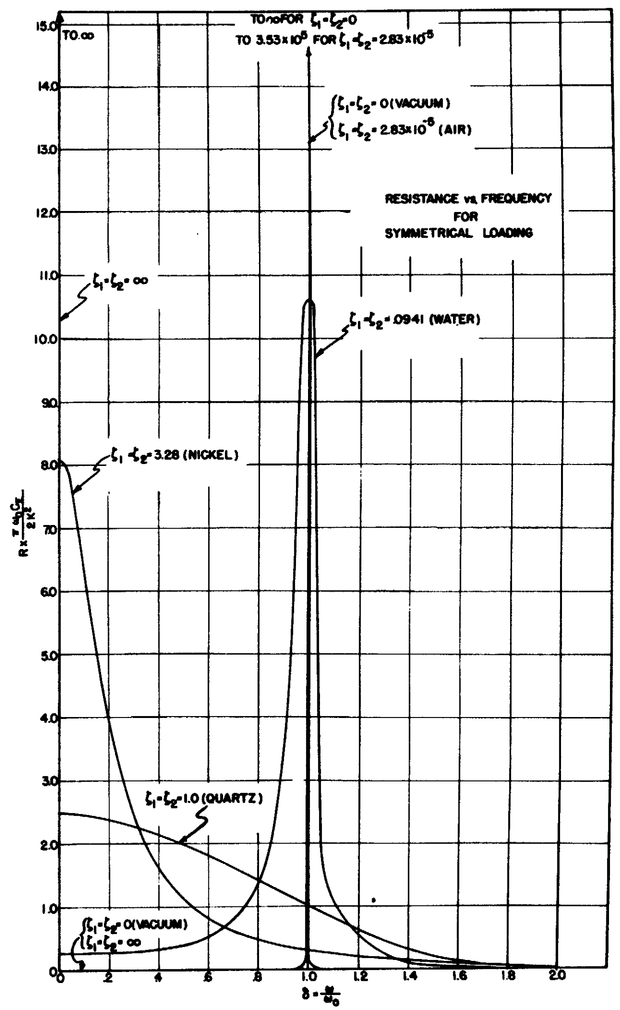

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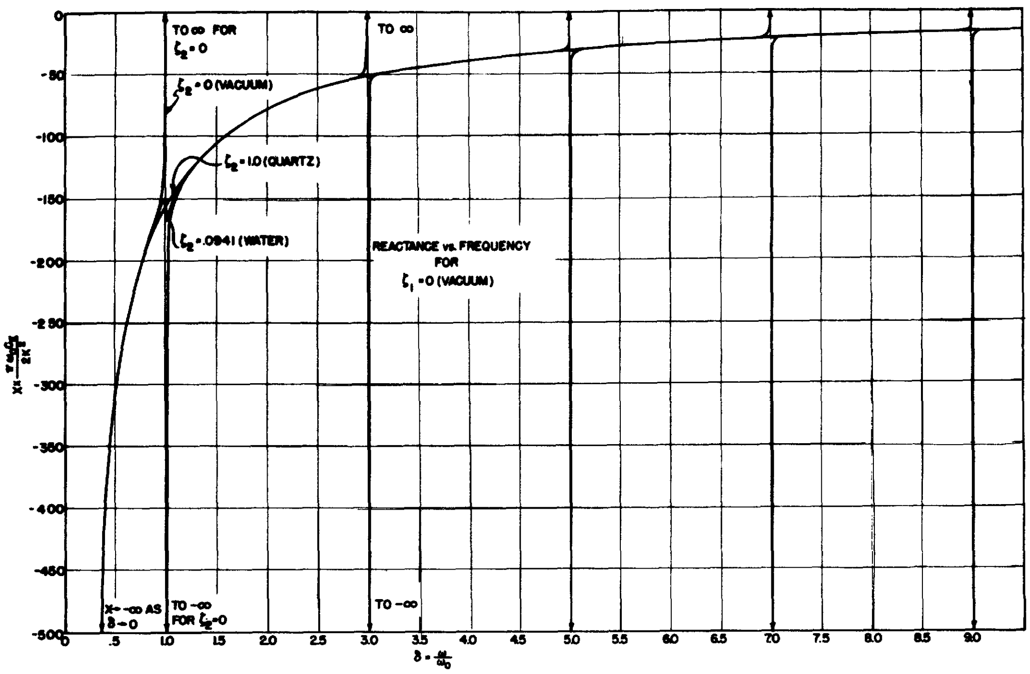

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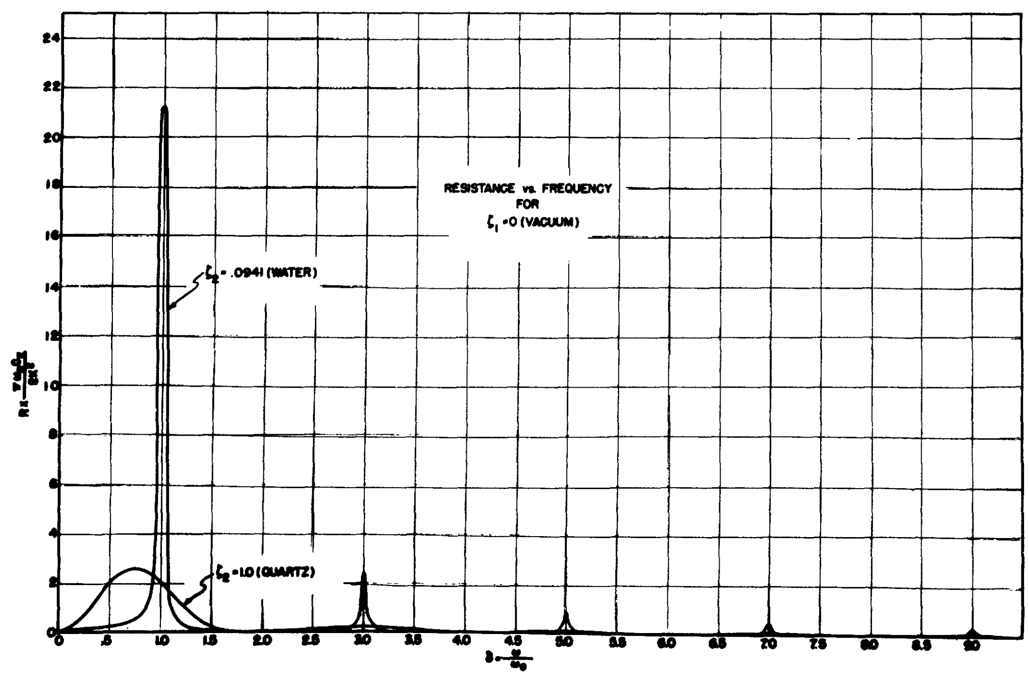

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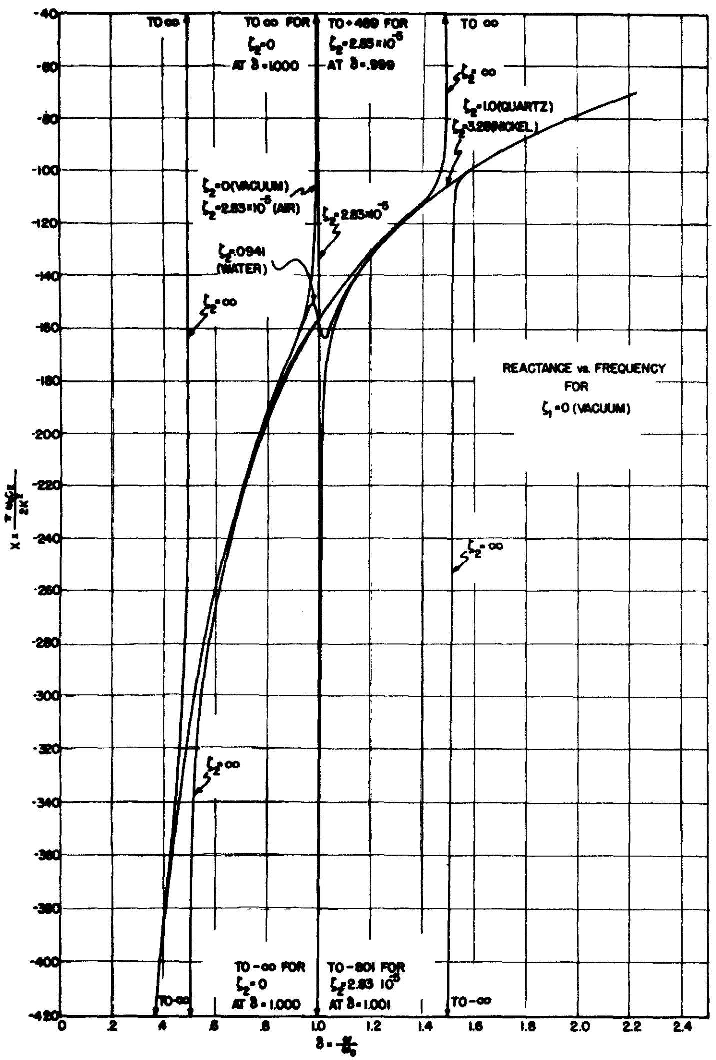

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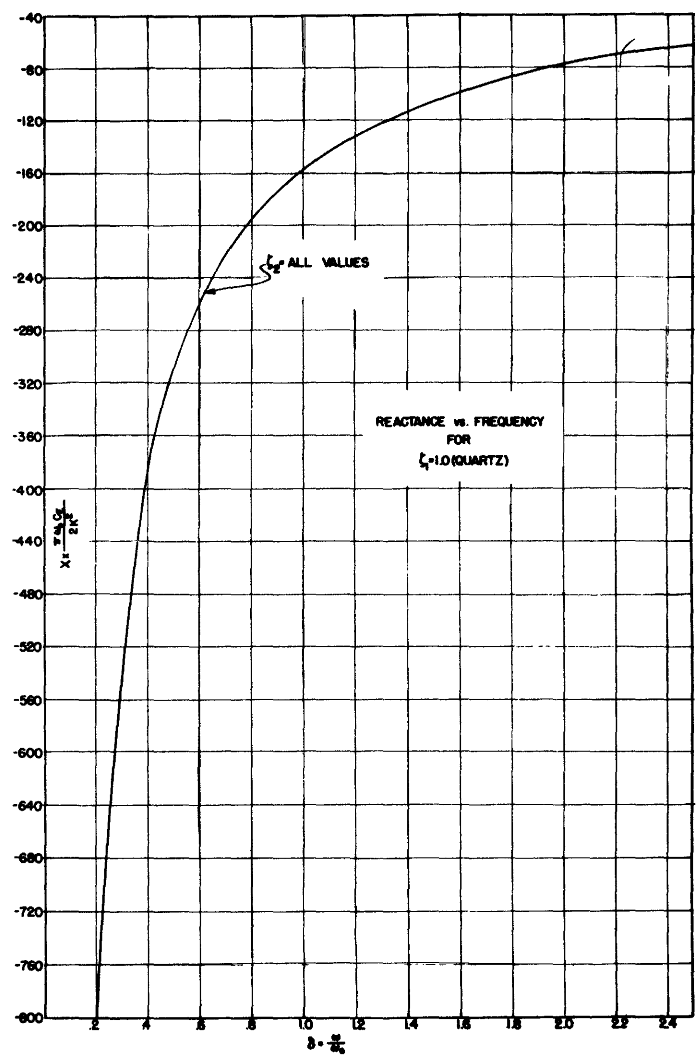

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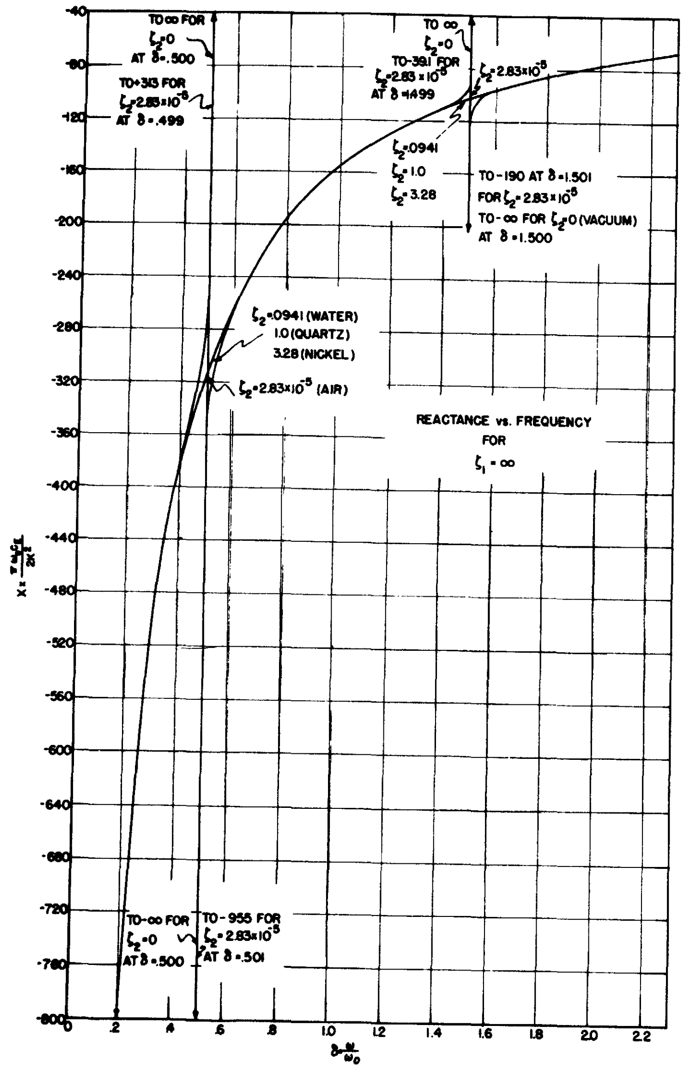

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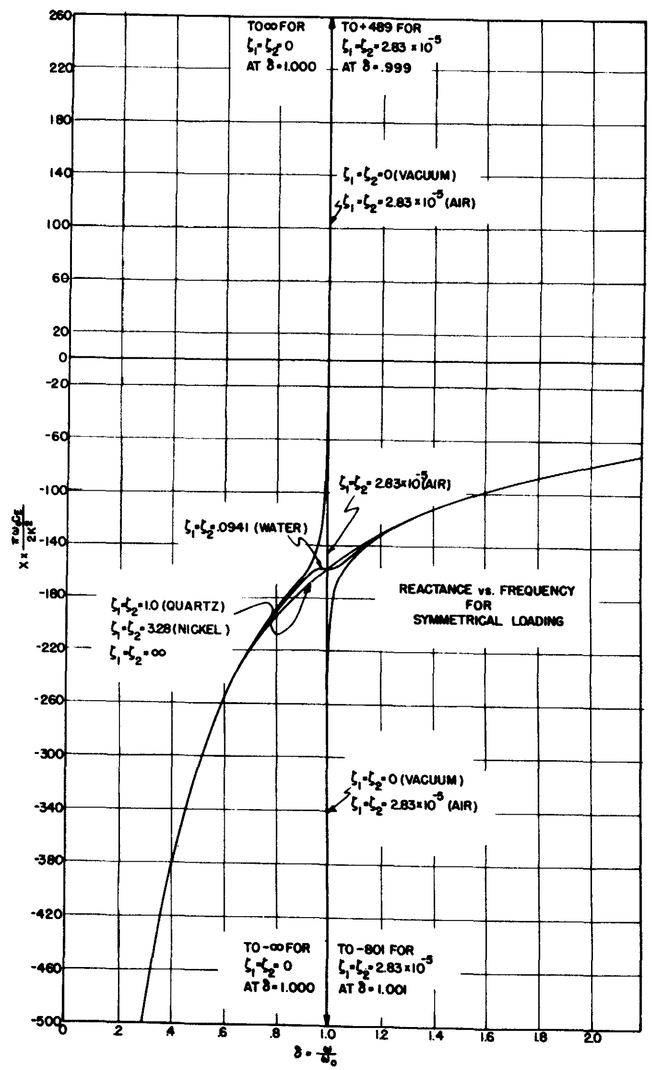

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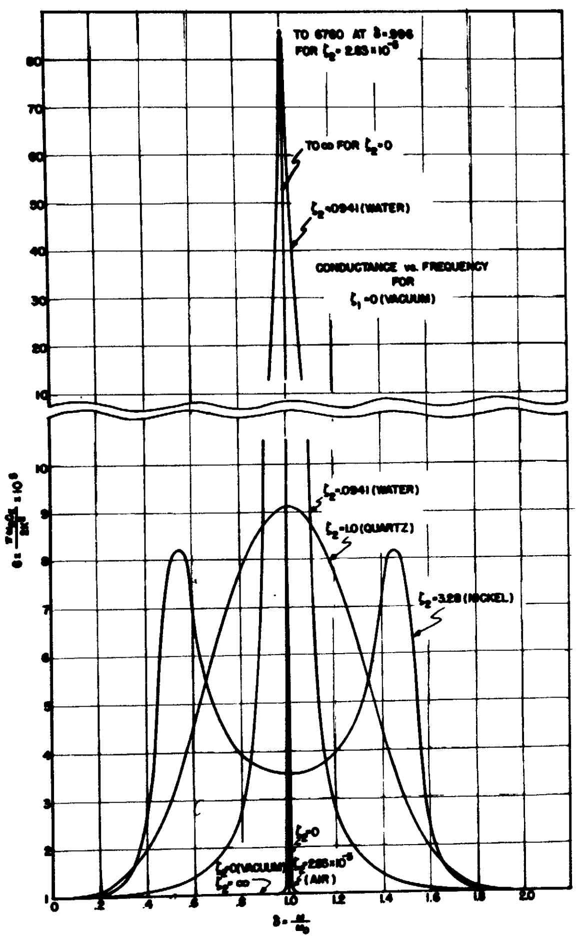

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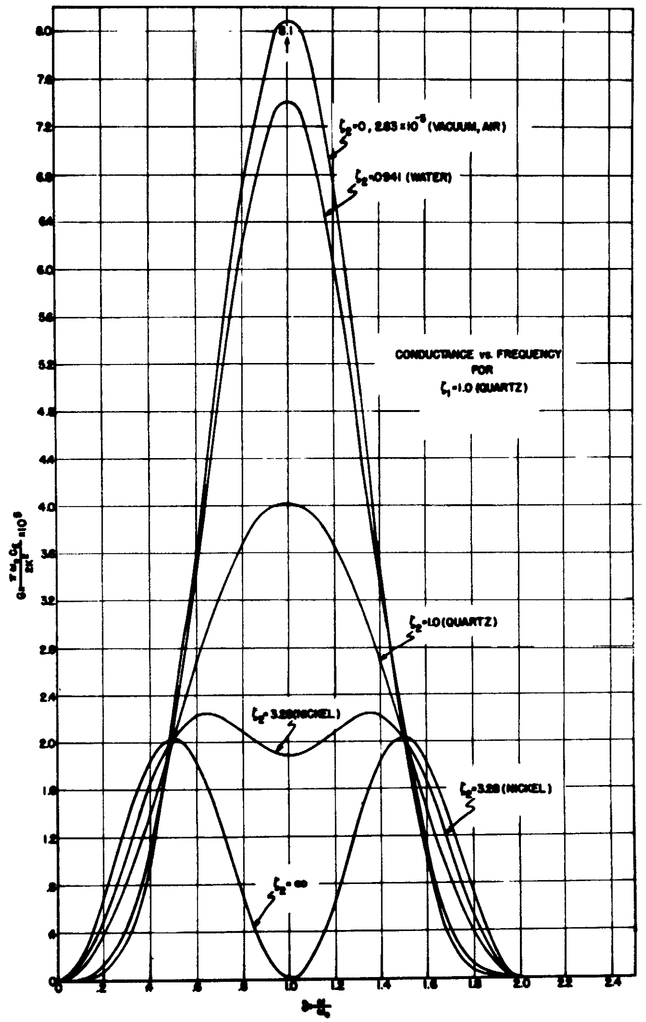

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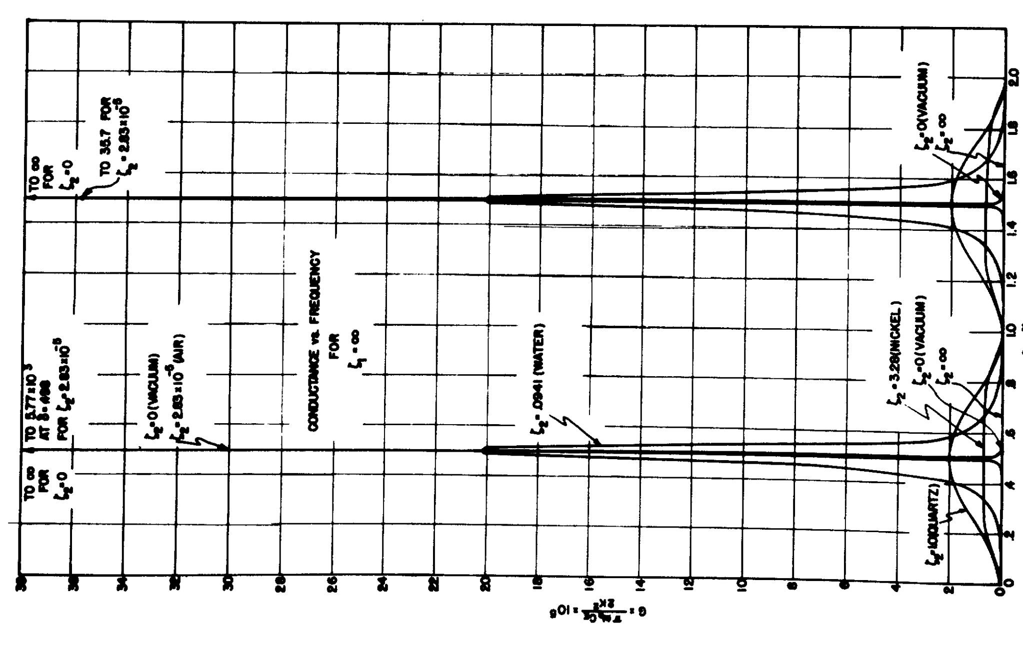

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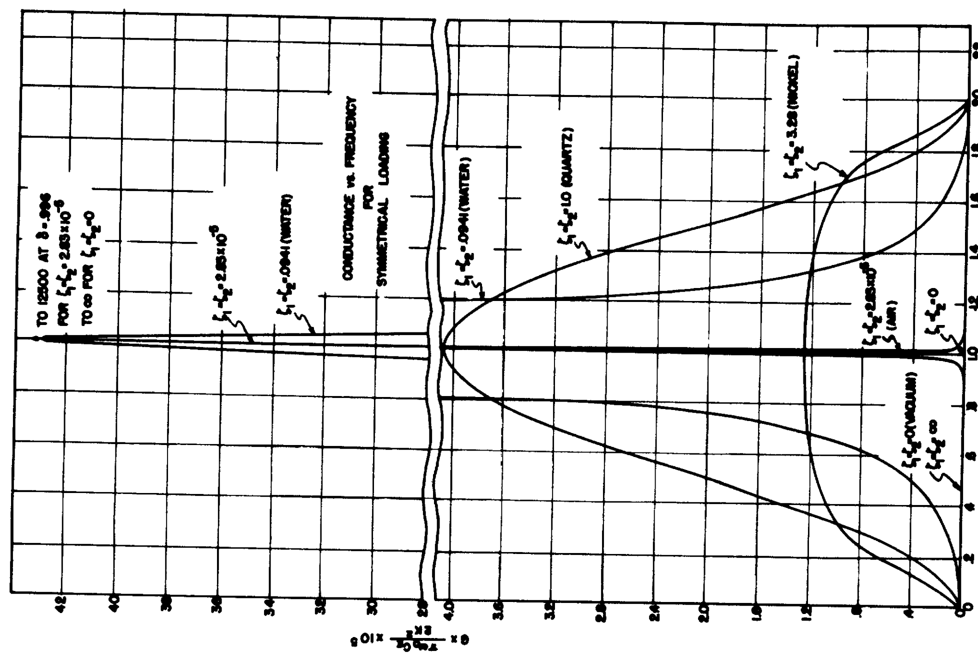

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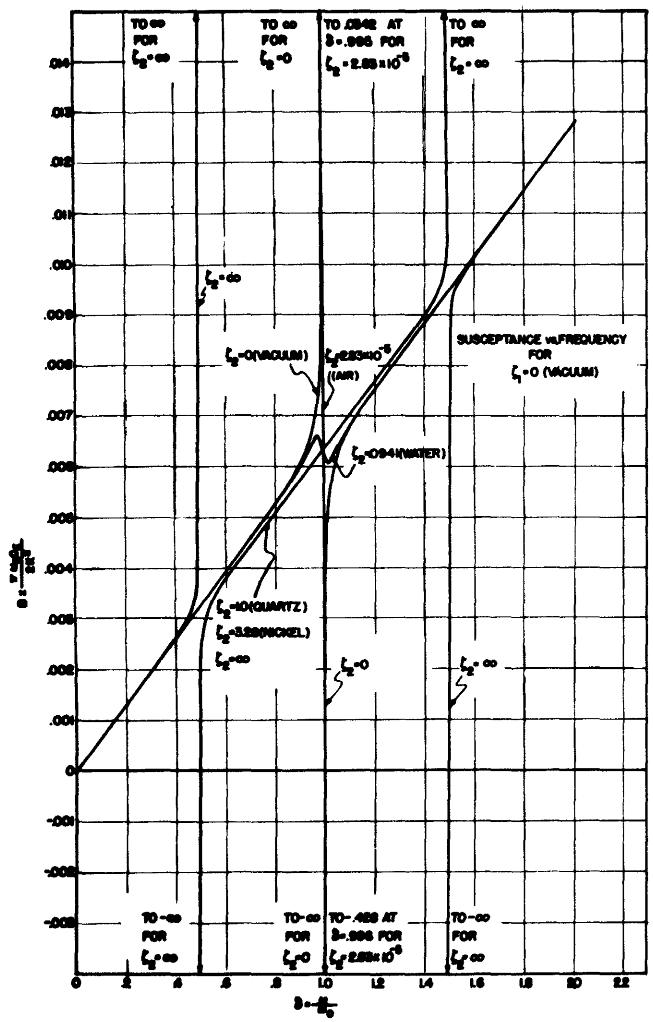

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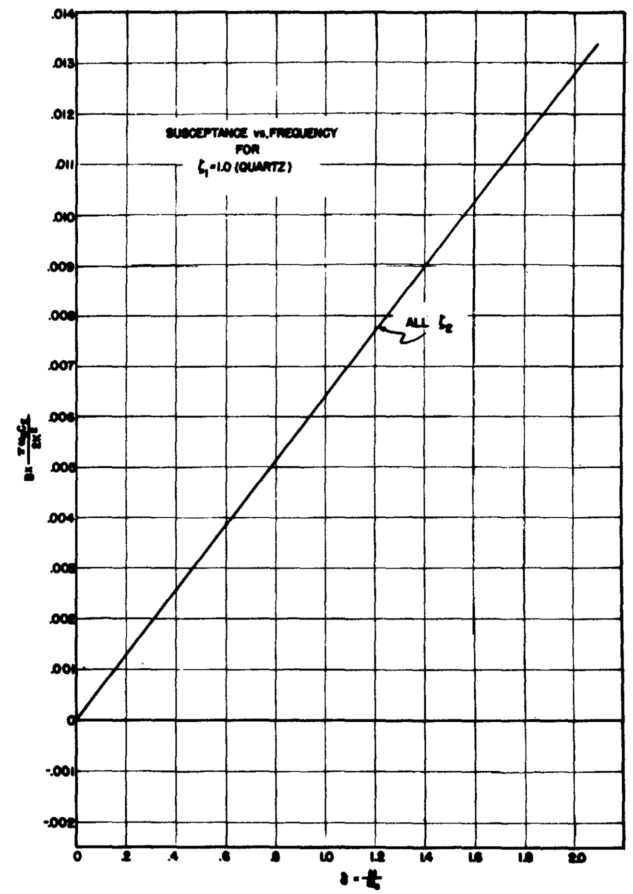

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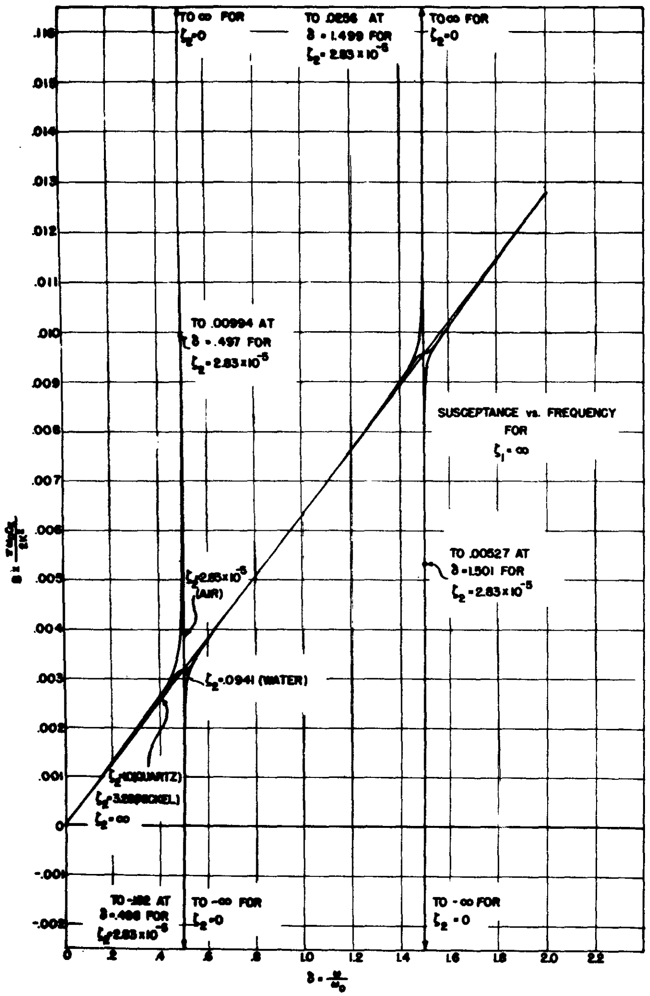

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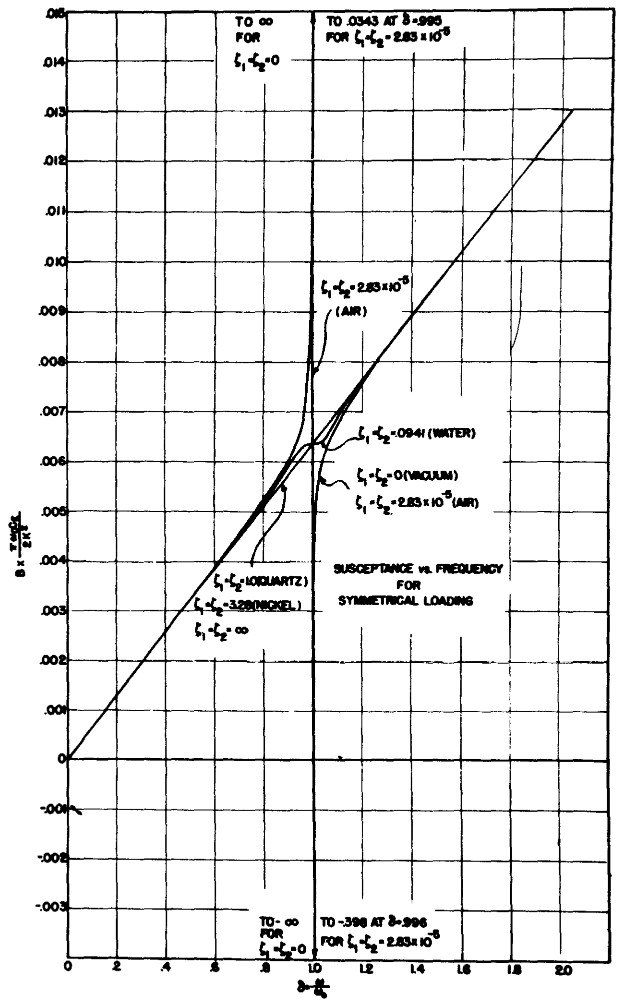

Figure 26.