Investments of Uncertain Cost

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Abstract

I study irreversible investment decisions when projects take time to complete, and are subject to two types of uncertainty over the cost of completion. The first is technical uncertainty, i.e., uncertainty over the amount of time, effort, and materials that will ultimately be required to complete the project, and that is only resolved as the investment takes place. The second is input cost uncertainty, i.e., uncertainty over the prices and quantities of labor and materials that are expected to be required, and which is external to the firm's investment activity. This paper derives simple decision rules that maximize the firm's value, and that are easy to implement. I show how these two types of uncertainty have very different effects on the decision to invest, and how they affect the value of the opportunity to invest.

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1. Introduction.

In most studies of investment under uncertainty, it is the future payoffs from the investment that are uncertain. The same is true for most textbook treatments of project evaluation and capital budgeting, which show, for example, how the CAPM can be used to discount uncertain future cash flows. The emphasis on uncertainty over future payoffs also applies to the growing literature on irreversible investment. Much of that literature studies optimal stopping rules for the timing of sunk costs of known magnitude, in exchange for capital whose value fluctuates stochastically.[1]

At times, however, the cost of an investment is more uncertain than the future value of its payoff. This is often the case for large projects that take considerable time to build. An example is a nuclear power plant, where total construction costs are very hard to predict due to both engineering and regulatory uncertainties. Although the future value of a completed nuclear plant is also uncertain (because the demand for electricity and costs of alternative fuels are uncertain), construction cost uncertainty is much greater, and has often deterred utilities from undertaking new plants. There are many other examples, ranging from large petrochemical complexes, to the development of a new line of jet aircraft, to major urban construction projects. Also, large size is not a requisite for cost uncertainty. Many (if not most) R&D projects involve considerable cost uncertainty; the development of a new drug by a pharmaceutical company is an example.

In addition to their uncertain costs, all of the investments mentioned above are irreversible. Expenditures on nuclear power plants, petrochemical complexes, the development of new drugs, and so on are sunk costs that cannot be recovered should the investment turn out, *ex post*, to have been a bad one. In each case, the investment could turn out to be bad because demand for the product is less than anticipated, or because the cost of the investment turns out to be greater than anticipated. Whatever the reason, the firm cannot "disinvest" and recover the money it spent.[2]

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[1] For an overview of that literature, see Dixit (1992) and Pindyck (1991).

[2] These costs are sunk because they are firm- or industry-specific. A petrochemical plant, for example, can only be used to produce certain chemicals. Although it could be sold to another chemical company, its

This paper studies the implications of cost uncertainty for the decision to undertake an irreversible investment, and for the decision to abandon a project when its cost turns out to be larger than expected. I am concerned with projects that take time to complete, so that two different kinds of uncertainty arise. The first, which I call *technical* uncertainty, relates to the physical difficulty of completing a project: Assuming factor costs are known, how much time, effort, and materials will ultimately be required? An important characteristic of this kind of uncertainty is that it can only be resolved by undertaking and completing the project.[3] One then observes actual costs (and construction time) unfold as the project proceeds. These costs may from time to time turn out to be greater or less than anticipated (as impediments arise or as the work progresses faster than planned), but the total cost of the investment is only known for certain when the project is complete. Another characteristic of this uncertainty is that it is largely diversifiable. It results not from unpredictable changes in input prices, but only from the inability to predict how difficult a project will be, which is likely to be independent of the overall economy.

The second kind of cost uncertainty relates to *input costs.* For example, the prices of labor, land, and materials required for a project are likely to fluctuate over time. Also, unpredictable changes in government regulations can change the required quantities of one or more inputs. For example, new safety regulations may add to labor requirements, or changing environmental regulations may require more capital. Input costs evolve stochastically whether or not the firm is investing, and are more uncertain the farther into the future they are incurred. Hence input cost uncertainty is particularly important for projects that are likely to take a long time to complete, or that are subject to voluntary or involuntary delays. In addition, this kind of uncertainty may be partly nondiversifiable; changes in wage rates,

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cost is mostly sunk, particulary if the industry is competitive. The reason is that the value of the plant will be about the same for all firms in the industry, so there will be little gained from selling it. If the plant turns out to be a "bad" investment for one company, it is likely to be just as bad for other companies.

[3]This is a simplification, in that for some projects cost uncertainty can be reduced by first undertaking additional engineering studies. The investment problem is then more complicated because one has three choices instead of two: start construction now, undertake an engineering study and then begin construction only if the study indicates costs are likely to be low, or abandon the project completely. As I discuss in the concluding section, the model developed in this paper could be extended along this line.

materials costs, etc., are likely to be correlated with overall economic activity.

This paper derives decision rules for irreversible investments subject to both types of cost uncertainty. For simplicity and clarity, I assume that the value of the completed project is known with certainty, but I show how the model can be extended so that this is also stochastic. The decision rules I derive allow for the possibility of abandoning the project midstream, and maximize the value of the firm in a competitive capital market. These rules also have a particularly simple form: Invest as long the expected cost to complete the project is below a critical number. Hence they can be used to evaluate projects, rather than simply characterize investment decisions. In addition, the derivation of the decision rule yields the value of the investment opportunity, i.e., what one would pay for the right to undertake the project. I explore how this value, and the critical expected cost to completion, depend on the level and characteristics of uncertainty, as well as other parameters.

Technical and input cost uncertainty both increase the value of the investment opportunity. The reason is that the payoff function for the investment opportunity is convex in the cost of the investment; letting K be the cost and V be the value of the completed project, the payoff function is $\max[0, V - K]$. Note that the investment opportunity is analogous to a put option, i.e., it gives the holder the right to sell an asset worth an uncertain amount K for a fixed "exercise price" V. As with any option, its value is increased by an increase in the variance of the price of the underlying asset.[4]

However, these two types of uncertainty affect the optimal investment decision in very different ways. Technical uncertainty *raises* the critical expected cost to completion. Hence a project can have an expected cost that makes its conventionally measured NPV negative, but if the variance of the cost is sufficiently high, it can still be economical to begin investing. The reason is that investing reveals information about cost, and thus about the expected net payoff from investing further. It therefore has a shadow value beyond its direct contribution to the completion of the project, which lowers the full expected cost of the investment.[5]

[4]Using put-call parity, we can also think of this as a *call* option with a stochastic exercise price (K) on an asset with a fixed value (V). In my model, the firm has a more complicated compound option; it can spend an uncertain amount of money in return for an option to continue the partially completed project.

[5]It is analogous to the shadow value of production arising from a learning curve, which lowers the full

Also, since information about cost arrives only when investment is taking place, there is no value to waiting.

As an example, a project requires a first phase investment of \$1. Then, with probability .5 the project will be finished, and with probability .5 a second phase costing \$4 will be required. Completion of the project yields a certain payoff of \$2.8. Since the expected cost of the project is \$3, the conventionally measured NPV is negative. But this ignores the value of the option to abandon the project should the second phase be required. The correct NPV is $-1 + (.5)(2.8) = \$0.4$, so one should proceed with at least the first phase.

Input cost uncertainty has the opposite effect — it reduces the critical expected cost. Hence a project could have a conventionally measured NPV that is positive, but it might still be uneconomical to begin investing. The reason is that fluctuations in factor costs occur whether or not investment is taking place, so as with most irreversible investments, there is a value of waiting to see if those costs change before commiting resources. Also, this effect is magnified when stochastic fluctuations in factor costs are correlated with the economy, i.e., in the context of the CAPM, to the extent that the "beta" of cost is high. The reason is that a higher "beta" implies that high cost outcomes, and hence low project values, are more likely to be associated with high stock market returns, so that the investment opportunity is a hedge against nondiversifiable risk. Put another way, a higher "beta" raises the required expected return, and hence discount rate, that must be applied to possible future costs. Since the payoff from completing the project is known, this raises the value of the investment opportunity, as well as the benefit from waiting rather than investing now.

For example, suppose an investment can be undertaken now or later. The cost is now \$3, but next period it will either fall to \$2 or rise to \$4, each with probability .5, and then remain at that level. Investing yields a certain payoff of \$3.2, and we will assume the risk-free rate of interest is zero. If we invest now, the project has a conventionally measured NPV of \$0.2. But this ignores the opportunity cost of closing our option to wait for a better outcome (a drop in cost). If we wait until next period, we will only invest if the cost falls

cost of production; see Majd and Pindyck (1989).

to \$2. The NPV if we wait is (.5)(3.2 − 2) = \$0.6, so it is clearly better to wait. Also, the value of being able to wait is 0.6 − 0.2 = \$0.4. Now suppose the "beta" of cost is high, so that the risk-adjusted discount rate is 25 percent per period. Because the payoff from completing the project is certain, this discount rate is only applied to cost. Hence the NPV assuming we wait is now (.5)[3.2 − 2/1.25] = \$0.8. The higher "beta" increases the value of the investment opportunity and also increases the value of waiting, because costs in the future are more heavily discounted, so that the present values of net payoffs are larger.

This paper is related to several earlier studies. The value of information gathering has been explored by Roberts and Weitzman (1981), who developed a model of sequential investment similar to mine in that the project can be stopped in midstream, and the process of investing reduces both the expected cost of completing the project as well the variance of that cost. They derive an optimal stopping rule, and show that it may pay to go ahead with the early stages of an investment even though the NPV of the entire project is negative.[6] Grossman and Shapiro (1986) also study investments for which the total effort required to reach a payoff is unknown. They model the payoff as a Poisson arrival, with a hazard rate specified as a function of the cumulative effort expended. They allow the rate of progress to be a concave function of effort, and focus on the rate of investment, rather than on whether one should proceed or not. My results complement those of these authors, but my model is more general in its treatment of cost uncertainty, and yields relatively simple decision rules.

This paper is related as well to the basic model of irreversible investment by McDonald and Siegel (1986). They consider the payment of a sunk cost I in return for a project worth V, where both V and I evolve as geometric Brownian motions. The optimal investment rule is to wait until V/I reaches a critical value that exceeds 1, because of the opportunity cost of commiting resources rather than waiting for new information about V and/or I. Also, Majd

[6] Weitzman, Newey, and Rabin (1981) use this model to evaluate demonstration plants for synthetic fuel production, and show that learning about costs could justify these early investments. Also, MacKie-Mason (1991) extends the Roberts and Weitzman analysis by accounting for the fact that investors (who pay for the cost of a project) and managers (who decide whether to continue or abandon the project) may have conflicting interests and asymmetric information. He shows that asymmetric learning about cost then leads to inefficient overabandonment of projects. Finally, Zeira (1987) developed a model with adjustment costs in which a firm learns about its payoff function as it accumulates capital.

and Pindyck (1987) study sequential investment when a firm can invest at some maximum rate (so it takes time to complete a project), the project can be abandoned before completion, and the value of the project, received only upon completion, evolves as a geometric Brownian motion. In this paper the firm can also invest at a maximum rate, but it is the cost rather than the value of the completed project that is uncertain.[7]

In the next section, a model of investment is developed that includes both technological and factor price uncertainty, and that is based on the maximization of the firm's market value. In Section 3, numerical solutions are used to show how the value of the investment opportunity and the optimal investment rule depend on the source and amount of uncertainty, as well as other parameters. Section 4 discusses some extensions of the basic model, and Section 5 concludes.

2. The Basic Model.

Consider an investment in a project whose actual cost of completion is a random variable, $\tilde{K}$, and whose expected cost is $K = E(\tilde{K})$. The project takes time to complete; the maximum rate at which the firm can (productively) invest is k. Upon completion, the firm receives an asset (e.g., a factory or new drug) whose value, V, is known with certainty.

If there were no uncertainty over the total cost, valuing the investment opportunity and determining the optimal investment rule would be straightforward. The project will take time $T = K/k$ to complete, so the opportunity to invest is worth:

$$\begin{aligned} F(K) &= \max\left[Ve^{-rK/k} - \int_0^{K/k} ke^{-rt}\,dt,\, 0\right] \\ &= \max\left[(V + k/r)e^{-rK/k} - k/r,\, 0\right] \end{aligned} \tag{1}$$

where r is the (risk-free) rate of interest. Also, the optimal investment rule is to proceed

[7]In related work, Baldwin (1982) analyzes sequential investment decisions when investment opportunities arrive randomly and the firm has limited resources. She values the sequence of opportunities and shows that a simple NPV rule leads to overinvestment, i.e., there is a value to waiting for better opportunities. Likewise, if cost evolves stochastically, it may pay to wait for cost to fall. Also, Myers and Majd (1984) determine the value of a firm's option to abandon a project in return for a scrap value, S, when the value of the project, V, evolves as a geometric Brownian motion (the firm has a put option to sell a project worth V for a price S), and show how this abandonment value affects the decision to invest in the project.

with the project as long as $F(K) > 0$, i.e., as long as K is less than a critical K^*, given by:

$$K^* = (k/r)\log(1 + rV/k).$$

Figure 1 shows $F(K)$ for $V = 10$, $k = 2$, and $r = 0$, .1, and .2. Note that if $r = 0$, $F(K) = V - K$, and $K^* = V$. But if $r > 0$, $F(K) < V - K$, and $K^* < V$. The reason is that the payoff V is received only upon completion at time $T = K/k$, and must be discounted accordingly, whereas the cost of the investment is spread out evenly from $t = 0$ to T. Also note that whatever the value of r, $F(K)$ is a convex function of K, and therefore uncertainty over cost should increase $F(K)$. There is little that can be said at this point, however, about the effect of uncertainty on the optimal investment rule.

Introducing Uncertainty.

I introduce uncertainty over cost by letting the expected cost to go, $K(t)$, follow a controlled diffusion process. Suppose for the moment that $K(t)$ is given by:

$$dK = -Idt + g(I, K)dz, \tag{2}$$

where I is the rate of investment, $z(t)$ is a Wiener process that might or might not be correlated with the economy and the stock market, and $g_I \geq 0$, $g_{II} \leq 0$, and $g_K \geq 0$. Eqn. (2) says that the expected cost to go declines with ongoing investment, but also changes stochastically. Stochastic changes in K might be due to technical uncertainty, in which case $g(0, K) = 0$ and $g_I > 0$, to input cost uncertainty, in which case $g(0, K) > 0$, or to both.[8]

I will again assume that there is a maximum rate of investment k. Let $F(K) = F(K; V, k)$ be the value of the investment opportunity. Then $F(K)$ satisfies:

$$F(K) = \max_{I(t)} E_0 \left[Ve^{-\mu\tilde{T}} - \int_0^{\tilde{T}} I(t)e^{-\mu t}\, dt \right], \tag{3}$$

subject to eqn. (2), $0 \leq I(t) \leq k$, and $K(\tilde{T}) = 0$. Here μ is an appropriate risk-adjusted discount rate, and the time of completion, $\tilde{T}$, is stochastic.

[8]Eqn. (2) is a generalization of Roberts and Weitzman (1981), who also model the expected cost to go as a stochastic process that is controlled by the rate of investment.

For eqn. (2) to make economic sense, more structure is needed. In particular, we would like: (i) $F(K;V,k)$ to be homogeneous of degree one in K, V, and k; (ii) $F_K < 0$, i.e., an increase in the expected cost of an investment should always reduce its value; (iii) the instantaneous variance of dK to be bounded for all finite K and to approach 0 as $K \to 0$; and (iv) if the firm invests at the maximum rate k until the project is complete, $E_0 \int_0^{\tilde{T}} k\,dt = K$, so that K is indeed the expected cost to completion. We can meet these conditions and still allow for reasonably general cost structures by letting $g(I,K) = \beta K(I/K)^{\alpha}$, with $0 \leq \alpha \leq \frac{1}{2}$. That this satisfies conditions (i) and (iii) is obvious. As will become clear later, $0 \leq \alpha \leq \frac{1}{2}$ rather than $0 \leq \alpha < 1$, which also satisfies (i) and (iii), is needed to satisfy (ii). Finally, it is shown in the Appendix that (iv) is also satisfied.

We will restrict the analysis to $\alpha = 0$ and $\frac{1}{2}$, which correspond naturally to our two types of cost uncertainty, and which result in simple corner solutions for optimal investment. (As discussed in Section 4, other values of α result in interior solutions where I is varied in response to changes in the variance of dK.) The case of $\alpha = \frac{1}{2}$ corresponds to technical uncertainty; K can change only if the firm is investing, and the instantaneous variance of dK/K increases linearly with I/K. When the firm is investing, the expected change in K over an interval Δt is $-I\Delta t$, but the realized change can be greater or less than this, and K can even increase. As the project proceeds over time, progress will at times be slower and at times faster than expected. The variance of $\tilde{K}$ falls as K falls, but the actual total cost of the project, $\int_0^{\tilde{T}} I\,dt$, is only known when the project is completed.

The case of $\alpha = 0$ corresponds to input cost uncertainty; the instantaneous variance of dK/K is constant and independent of I. Now K will fluctuate even when there is no investment; ongoing changes in the costs of labor and materials will change K irrespective of the firm's progress towards completing the project. And since the project takes time to build, the actual total cost of the project is again only known when the project is complete.

We can allow for both types of uncertainty by combining these two cases in a single equation for the evolution of K:

$$dK = -I dt + \beta(IK)^{1/2} dz + \gamma K dw, \tag{4}$$

where dz and dw are the increments of uncorrelated Wiener processes. We will assume that all risk associated with dz is diversifiable, i.e., dz is uncorrelated with the economy and the stock market. However, dw may be correlated with the market.

Note that eqn. (4) combines uncertainty over the amount of effort required to complete a project, uncertainty over the cost of that effort, and uncertainty over the time the project will take. As we will see, given a maximum rate of expenditure, k, the investment decision only requires estimates of the expected cost to completion, the variance of that cost, and its covariance with the stock market.

The Optimal Investment Rule.

Given that dw in eqn. (4) may be correlated with the market, we cannot use the risk-free rate of interest for the discount rate μ in eqn. (3). We can eliminate μ from the problem, however, if dw is spanned by existing assets in the economy, i.e., if in principle one could replicate movements in dw with some other asset or dynamic portfolio of assets. The investment problem can then be solved using contingent claims methods. If spanning does not hold, we could instead find an optimal investment rule using dynamic programming, but subject to some choice of discount rate μ. Note, however, that without spanning we would have no theory for determining the correct discount rate (other than by making assumptions about the risk preferences of managers or the firm's stockholders).[9]

We will assume that spanning holds. Let x be the price of an asset or dynamic portfolio of assets perfectly correlated with w, so that dx follows:

$$dx = \alpha_x x dt + \sigma_x x dw. \tag{5}$$

The expected return on x will be a risk-adjusted rate, r_x. By the CAPM, $r_x = r + \theta \rho_{xm} \sigma_x$, where θ is the market price of risk,[10] and ρ_{xm} is the instantaneous correlation of x with the market portfolio.

[9]The CAPM, for example, would not apply. Furthermore, the correct discount rate need not be constant. If dw indeed reflects unpredictable changes in the prices of factors such as labor and raw materials, spanning should indeed hold, at least roughly.

[10]That is, $\theta = (r_m - r)/\sigma_m$, where r_m is the expected return on the market, and σ_m is the standard deviation of that return. If we take the New York Stock Exchange Index as the market, $r_m - r \approx .08$ and $\sigma_m \approx .2$, so $\theta \approx .4$.

In the Appendix, standard contingent claims methods are used to show that $F(K)$ must satisfy the following differential equation:

$$\tfrac{1}{2}\beta^2 IKF_{KK} + \tfrac{1}{2}\gamma^2 K^2 F_{KK} - IF_K - \phi\gamma KF_K - I = rF, \tag{6}$$

where $\phi \equiv (r_x - r)/\sigma_x$. Recall that $r_x = r + \theta\rho_{xm}\sigma_x$. Thus $\phi = \theta\rho_{xm}$. Since θ is a economy-wide parameter, the only project-specific parameter needed to determine ϕ is ρ_{xm}, which is equal to the coefficient of correlation between fluctuations in cost and the stock market.

Note that eqn. (6) is the Bellman equation for the stochastic dynamic programming problem given by (3), but with μ replaced by r. Because eqn. (6) is linear in I, the rate of investment that maximizes $F(K)$ is always equal to either 0 or the maximum rate k. Specifically:

$$I = \begin{cases} k & \text{for } \tfrac{1}{2}\beta^2 KF_{KK} - F_K - 1 \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Eqn. (6) therefore has a free boundary at a point K^*, such that $I(t) = k$ when $K \leq K^*$ and $I(t) = 0$ otherwise. The value of K^* must be found as part of the solution for $F(K)$. To determine $F(K)$ and K^*, we solve (6) subject to the following boundary conditions:

$$F(0) = V \tag{8}$$

$$\lim_{K\to\infty} F(K) = 0 \tag{9}$$

$$\tfrac{1}{2}\beta^2 K^* F_{KK}(K^*) - F_K(K^*) - 1 = 0 \tag{10}$$

and $F(K)$ continuous at K^*. Condition (8) says that when the project is complete, the payoff is V. Condition (9) says that when K is very large, the probability is very small that over some finite period of time it will drop enough to begin the project. Finally, condition (10) follows from (7), and is equivalent to the "smooth pasting" condition that $F_K(K)$ be continuous at K^*.

When $I = 0$, eqn. (6) has the following simple analytical solution:

$$F = aK^b \tag{11}$$

where, to satisfy boundary condition (9), b is the negative root of the quadratic equation $\frac{1}{2}\gamma^2 b(b-1) - \phi\gamma b - r = 0$, i.e.,

$$b = \frac{1}{2} + \frac{\phi}{\gamma} - \frac{1}{2\gamma}\sqrt{(\gamma + 2\phi)^2 + 8r} \tag{12}$$

The parameter a is determined from the remaining boundary conditions, together with K^* and the solution for $F(K)$ for $K < K^*$. This must be done numerically, which is relatively easy once eqn. (6) has been appropriately transformed.[11] A family of solutions for $K < K^*$ can be found that satisfy condition (8), but a unique solution, together with the value of a, is determined from (10) and the continuity of $F(K)$ at K^*.

3. Solution Characteristics.

The effects of cost uncertainty are best understood by first examining solutions of eqn. (6) for the case where there is only technical uncertainty, i.e., $\gamma = 0$, and then for the case where there is only input cost uncertainty, i.e., $\beta = 0$. Afterwards we will return to the general case.

Technical Uncertainty.

When only technical uncertainty is present, eqn. (6) reduces to:

$$\tfrac{1}{2}\beta^2 IKF_{KK} - IF_K - I = rF. \tag{13}$$

In this case, K can change only when investment is taking place, so if $K > K^*$ and the firm is not investing, it never will, and $F(K) = 0$. Hence boundary conditions (8) and (10) remain the same, but condition (9) is replaced with $F(K^*) = 0$.

When $r = 0$, eqn. (13) has an analytical solution:

$$F(K) = V - K + \beta^2 \left(\frac{V}{2}\right)^{-2/\beta^2} \left(\frac{K}{\beta^2 + 2}\right)^{(\beta^2+2)/\beta^2}, \tag{14}$$

[11] When $I = k$, eqn. (6) has a first-degree singularity at $K = 0$. To eliminate this, make the substitution $F(K) = f(y)$, where $y = \log K$. Then (6) becomes:

$$f_{yy}(y) - f_y(y) - \frac{2kf_y(y)}{\beta^2 k + \gamma^2 e^y} = \frac{2k + 2rf(y)}{\beta^2 ke^{-y} + \gamma^2},$$

and boundary conditions (8) to (10) are transformed accordingly.

and the critical value of K, K^*, is given by:

$$K^* = (1 + \tfrac{1}{2}\beta^2)V.$$

Eqn. (14) has a simple interpretation. With $r = 0$, $V - K$ would be the value of the investment opportunity were there no possibility of abandoning the project once construction begins. The last term in the equation is the value of the put option, i.e., the option to abandon the project should costs turn out to be much higher than expected. Note that for $\beta > 0$, $K^* > V$, and K^* is increasing in β. The more uncertainty there is, the greater the value of the investment opportunity, and the larger is the maximum expected cost for which beginning to invest is economical.

When $r > 0$, eqn. (13) does not have an analytical solution, but can be solved numerically for different values of β. To choose values for β that are reasonable, we need to relate this parameter to the variance of the project's total cost. The Appendix shows that for this case in which $\gamma = 0$, the variance of the cost to completion is given by:

$$\text{Var}(\tilde{K}) = \left(\frac{\beta^2}{2 - \beta^2}\right) K^2. \tag{15}$$

Hence if one standard deviation of a project's cost is 25 percent of the expected cost, β would be 0.343, and if one standard deviation is 50 percent of the expected cost, β would be 0.63. Standard deviations of project cost in the range of 25 to 50 percent are not unusual, so we will use these values for β in the calculations that follow.

Figure 2 shows $F(K)$ as a function of K for $V = 10$, $k = 2$, $r = .05$, and $\beta = 0$, .343, and .63. Observe that $F(K)$ looks like the value of a put option, except that $F(K) = 0$ when K exceeds the "exercise" point K^*. Although $F(K)$ is larger the higher is β, the effect is greatest for larger values of K. Also, the effect of technical uncertainty on the optimal investment rule is moderate; only when $\beta = .63$ does K^* substantially exceed its value for the certainty case. This can also be seen in Figure 3, which shows K^* as a function of β. For K^* to increase by 50 percent (from about 9 to about 13.5) requires a value of β around 1, which in turn implies that the standard deviation of total cost be about 100 percent of the expected cost.

Finally, Figure 4 shows how $F(K)$ depends on the maximum rate of investment, k. (Here, $\beta = .63$.) As in the certainty case, a larger k implies a larger $F(K)$, because the payoff V is expected to be received earlier, and hence is discounted less. Also, when the investment opportunity is worth more, the critical value K^* is larger.

Input Cost Uncertainty.

With only input cost uncertainty, eqn. (6) reduces to:

$$\tfrac{1}{2}\gamma^2 K^2 F_{KK} - IF_K - \phi\gamma K F_K - I = rF. \tag{16}$$

This is again subject to boundary conditions (8) and (9), but condition (10) is replaced with $F_K(K^*) = -1$. Now K can change whether or not investment is taking place, so like a financial put option, $F(K) > 0$ for any finite K.

When $\gamma > 0$, eqn. (16) has no solution when $r = 0$, because then there would be no reason to ever invest. One would always be better off waiting until K fell close to 0 so that the net payoff from investing is larger. It would not matter that substantial time might have to pass for this to happen, because net payoffs would not be discounted.

If $I = 0$, $\tilde{K}$ is lognormally distributed. Then γ can be interpreted as the standard deviation of percentage changes per period (in this case, a year) in K. Determining a value for γ that is reasonable depends on the makeup of cost. For example, a value of .20 would be high for wage rates, but would be low for commodity inputs such as copper, steel, or oil. Figure 5 shows numerical solutions of eqn. (16) for $\gamma = 0$, .2 and .4. (In each case, $V = 10$, $k = 2$, $r = .05$, and $\phi = 0$.) Observe that even when γ is .2, there is a substantial effect on the value of the investment opportunity (particulary when K is large), and on the critical cutoff K^*. When $\gamma = .2$, K^* is about half of what it is when $\gamma = 0$, so that a correct net present value rule would require the payoff from the investment to be about twice as large as the expected cost before the investment is undertaken. This is similar to the kinds of numerical results obtained by McDonald and Siegel (1986) and Majd and Pindyck (1987) for uncertainty over the payoff to an investment, and shows that the effects of input cost uncertainty can also be quantitatively important.

Figure 6 shows the dependence of $F(K)$ and K^* on ϕ, i.e., on the extent to which fluctuations in K are correlated with the economy and the stock market. Recall that $\phi = \theta\rho_{xm} = \theta\rho_{Km}$. A reasonable value for θ, the market price of risk, is 0.4, so we would expect ϕ to be less than this, perhaps on the order of .1 to .3. Figure 6 shows $F(K)$ for $\phi = 0$, .3, and for illustrative purposes, .6. As is clear from this figure, a value of ϕ on the order of .1 will have only a neglible effect on $F(K)$ and K^*. For a value of .3, however, the effect is large, and reduces K^* by around 25 percent compared to $\phi = 0$. Thus input cost uncertainty with a large systematic component can have a substantial impact on the decision to invest.

The General Case.

The value of the investment opportunity and the critical expected cost K^* can be found for any combination of β, γ, and ϕ by numerically solving eqn. (6) and its associated boundary conditions. Since increases in β and γ (or ϕ) have opposite effects on K^*, it is useful to determine the net effect for combinations of these parameters.

Figure 7 and Table 1 show K^* as a function of both β and γ, for $\phi = 0$, $V = 10$, $k = 2$, and $r = .05$. Note that K^* decreases with γ and increases with β, but is much more sensitive to changes in γ. Whatever the value of β, a γ of 0.5 reduces K^* to about a fifth of the value it has when $\gamma = 0$. Also, this drop in K^* would be even larger if there were a systematic component to the input cost uncertainty. Thus for many investments, and particularly for large industrial projects where input costs fluctuate, increasing uncertainty is likely to depress investment. The opposite will be the case only for investments like R&D programs, where technical uncertainty is far more important and β could easily exceed 1.

Figure 8 and Table 2 show $F(K;\beta,\gamma)$ as a function of β and γ for $K = 8.92$, which is the value of K^* when $\beta = \gamma = 0$. Thus the graph shows the "premium" in the value of the investment opportunity that results from the two sources of cost uncertainty. Note that this premium is increasing in both β and γ, but is again more sensitive to γ. Also, if γ is large (say, 0.5), this premium changes very little when β is increased.

Application of the Model.

To use this model for investment decisions, one must arrive at estimates of the β and γ that apply to a project's cost, and, secondarily, an estimate of ϕ or ρ_{Km}. In practice, this

Table 1 — Critical K^* as a Function of β and γ.
(Note: $V = 10, k = 2, r = .05$, and $\phi = 0$.)

β \ γ	0	0.1	0.2	0.3	0.4	0.5
0	8.9257	6.6113	4.9463	3.7524	2.8857	2.2559
0.1	8.9844	6.6504	4.9756	3.7720	2.9016	2.2681
0.2	9.1309	6.7578	5.0537	3.8330	2.9468	2.3032
0.3	9.3750	6.9385	5.1855	3.9307	3.0225	2.3608
0.4	9.7168	7.1875	5.3711	4.0674	3.1274	2.4438
0.5	10.156	7.5098	5.6104	4.2480	3.2617	2.5488
0.6	10.693	7.9053	5.8984	4.4629	3.4277	2.6758
0.7	11.328	8.3691	6.2402	4.7168	3.6230	2.8271
0.8	12.051	8.8965	6.6309	5.0146	3.8477	3.0005
0.9	12.861	9.5020	7.0801	5.3467	4.1016	3.1982
1.0	13.770	10.166	7.5732	5.7178	4.3848	3.4180

requires estimating confidence intervals around projected cost for each source of uncertainty. To break total cost uncertainty down into technical and input cost components, one can utilize the fact that the first is independent of time, whereas the variance of cost due to the second component grows linearly with the time horizon.

For example, a value for γ can be based on an estimate of a one- or two-standard deviation confidence interval for cost T years into the future assuming no investment takes place prior to that time. The estimated T-period standard deviation, $\hat{\sigma}_T$, would come from managers' experience with input costs, or could be derived from an accounting model of cost combined with variance estimates for the evolution of individual factor inputs. Then, $\hat{\gamma} = \hat{\sigma}_T/\sqrt{T}$. For consistency, one would check that estimates of σ_T based on different value of T lead to roughly the same value for $\hat{\gamma}$. Likewise, using eqn. (15) and an initial estimate of expected cost, $K(0)$, a value for β can be based on an estimate of the time-independent standard

Table 2 — $F(K)$ as a Function of β and γ.
(Evaluated at K^* corresponding to $\beta = \gamma = 0$)

β \ γ	0	0.1	0.2	0.3	0.4	0.5
0	0	1.0877	2.1553	3.1588	4.0535	4.8345
0.1	.1384	1.0915	2.1596	3.1599	4.0565	4.8371
0.2	.2026	1.0983	2.1642	3.1670	4.0606	4.8409
0.3	.2428	1.1149	2.1753	3.1747	4.0692	4.8456
0.4	.3924	1.1434	2.1956	3.1878	4.0810	4.8595
0.5	.5199	1.1918	2.2277	3.2146	4.0974	4.8746
0.6	.7499	1.2650	2.2697	3.2440	4.1240	4.8920
0.7	.9067	1.3652	2.3280	3.2837	4.1572	4.9184
0.8	1.1664	1.4942	2.3998	3.3401	4.1978	4.9487
0.9	1.3606	1.6848	2.4939	3.4024	4.2460	4.9884
1.0	1.6034	1.8724	2.5996	3.4764	4.3021	5.0323

deviation of $\tilde{K}$. That standard deviation would summarize managers' confidence intervals for each stage of the project.

4. Extensions.

This section shows how the model can be extended to account for uncertainty over the future value of the completed project, and to allow for more general processes for $K(t)$.

Uncertainty over the Value of the Completed Project.

As before, we will let the evolution of K be given by eqn. (4), but we will also assume that V evolves stochastically:

$$dV = \alpha_v V dt + \sigma_v V dz_v, \tag{17}$$

where the Wiener process dz_v is assumed to be uncorrelated with dz or dw. Thus future values of V are lognormally distributed, and since the project takes time to complete, the payoff is necessarily uncertain. For simplicity, we will assume that there is no systematic component to any of the Wiener processes. Then we can find the optimal investment rule using dynamic programming, discounting with the risk-free rate of interest.

The value of the investment opportunity is again given by eqn. (3), but with V now stochastic, and hence replaced by $V(\tilde{T})$. The Bellman equation for this problem is:

$$rF = \max_{I(t)} \left\{ -I(t) - IF_K + \tfrac{1}{2}\beta^2 IKF_{KK} + \tfrac{1}{2}\gamma^2 K^2 F_{KK} + \alpha_v V F_V + \tfrac{1}{2}\sigma_v^2 V^2 F_{VV} \right\} \tag{18}$$

This is linear in I, and equation (7) again applies. The optimal rule is to invest whenever $K \le K^*(V)$. Note that eqn. (18) is an elliptic partial differential equation with a free boundary along the line $K^*(V)$. The solution must satisfy the following boundary conditions:

$$F(0, V) = V \tag{19}$$

$$\lim_{V \to 0} F(K, V) = 0 \tag{20}$$

$$\lim_{K \to \infty} F(K, V) = 0 \tag{21}$$

$$\tfrac{1}{2}\beta^2 K^* F_{KK}(K^*, V) - F_K(K^*, V) - 1 = 0 \tag{22}$$

and $F(K, V)$ and $F_K(K, V)$ continuous at $K^*(V)$. Condition (20) reflects the fact that 0 is an absorbing barrier for V; the other conditions have the same interpretation as before.

When $K > K^*(V)$, so that $I = 0$, eqn. (18) has the following analytical solution:

$$F(K, V) = m(K/V)^{\omega}, \tag{23}$$

where

$$\omega = \left(\frac{1}{2} + \frac{\alpha_v - \sigma_v^2}{\gamma^2 + \sigma_v^2} \right) \left(1 - \sqrt{1 + \frac{2r(\gamma^2 + \sigma_v^2)}{(\gamma^2 + 2\alpha_v - \sigma_v^2)^2}} \right) \tag{24}$$

To find $F(K, V)$ for $K < K^*(V)$, use the continuity of $F(K, V)$ and $F_K(K, V)$ at K^* to eliminate m:

$$F(K^*, V) = (K^*/\omega) F_K(K^*, V) \tag{25}$$

Eqn. (18) together with conditions (19) and (25) must then be solved numerically using a finite difference method. The optimal boundary, $K^*(V)$, is solved for simultaneously along with $F(K, V)$.

Generalizing the Process for $K(t)$.

We have imposed restrictions on the process for $K(t)$ that resulted in a particularly simple investment rule, and that allowed us to clearly differentiate between two types of cost uncertainty. Specifically, we let $K(t)$ follow:

$$dK = -Idt + \beta K(I/K)^{\alpha}dz, \tag{26}$$

and restricted α to be 0 or $\frac{1}{2}$. Here we briefly examine the optimal investment rule for $0 < \alpha < \frac{1}{2}$. For simplicity, we will again assume that dz is uncorrelated with the economy or the stock market. We will also assume that the payoff V is fixed and certain.

The Bellman equation is now:

$$rF = \max_{I(t)} \left\{ -I(t) - IF_K + \tfrac{1}{2}\beta^2 I^{2\alpha} K^{2(1-\alpha)} F_{KK} \right\} \tag{27}$$

Maximizing with respect to I gives the optimal investment rule in terms of $F(K)$:

$$I^*(K) = \left[\frac{\alpha\beta^2 K^{2(1-\alpha)} F_{KK}}{1 + F_K} \right]^{1/(1-2\alpha)} \tag{28}$$

Substituting $I^*(K)$ into eqn. (27) yields the following nonlinear differential equation for $F(K)$:

$$rF = 1 + F_K - (\alpha\beta^2 K^{2-2\alpha} F_{KK})^{1/(1-2\alpha)}(1 + F_K)^{-2\alpha/(1-2\alpha)} \tag{29}$$

To obtain $F(K)$, eqn. (29) must be solved (numerically) subject to boundary conditions (8) and (9).

Eqn. (29) has solutions for which $-1 < F_K \leq 0$ and $F_{KK} > 0$.[12] Note from eqn. (28) that $I \to 0$ as $K \to 0$, so for small K, I falls as the net payoff $V - K$ rises. This is the opposite of Grossman and Shapiro's (1986) finding that I will rise as the net payoff rises when there are decreasing returns to effort. In my model there are constant returns to effort, and I falls because the variance of $\tilde{K}$ falls as K falls, so that the shadow value of learning falls.

[12] At $K = 0$, F_K must be greater than -1 as long as construction takes finite time and the discount rate is positive. Likewise, F_{KK} must remain finite as $K \to 0$.

5. Conclusions.

The model developed in this paper, as well as such predecessors as Roberts and Weitzman (1981) and Grossman and Shapiro (1986), belong to a broad class of optimal search problems analyzed by Weitzman (1979). In what he characterized as a "Pandora's box" problem, one must decide how many investment opportunities with uncertain outcomes should be undertaken, and in what order. In this paper, each dollar spent towards completion of a project is of as an investment opportunity, and the uncertain outcome is the amount of progress that results. The model developed here is more general in that expected outcomes can evolve stochastically even when no investment is taking place (input cost uncertainty), but more restrictive in that the order in which dollars are spent is predetermined.

One advantage of this model is that it leads to a simple investment rule that is relatively easy to apply in practice. Also, the restrictions that have been imposed on the process for $K(t)$ allowed us to clearly differentiate between two types of cost uncertainty. As we have seen in the previous section, some of the restrictive assumptions in the model can be relaxed (e.g., that V is non-stochastic), but at the cost of considerable computational complexity. Other restrictions can be relaxed as well. For example, we can relax the restriction that technical uncertainty is the same for each phase of the project (i.e., the uncertainty over the first third of a project's anticipated cost is the same as for the last third) by making β in eqn. (13) a function of K. As long as $\beta(K)$ is a smooth monotonic function, it is reasonably straightforward to obtain numerical solutions for $F(K)$.

The sources and amounts of cost uncertainty will vary greatly across different projects. However, based on the ranges of parameter values that would apply to the bulk of large capital investments, factor cost uncertainty is likely to be more important than technical uncertainty in terms of its effect on the investment rule and the value of the investment opportunity. The opposite may be the case for some R&D projects. And although we found that K^* is not very sensitive to β, this was based on the assumption, discussed above, that the uncertainty is the same across all phases of the project. Increases in K^* may be much larger if much of a project's uncertainty gets resolved during its early phases.

Appendix

A. Mean and Variance of $\tilde{K}$.

Here I show that if $K(t)$ follows a controlled diffusion of the form:

$$dK = -kdt + \beta K(k/K)^{\alpha} dz, \qquad (A-1)$$

then $K(t)$ is indeed the expected cost to completion. Let:

$$M(K) = E_t\left[\int_t^{\tilde{T}} k\, d\tau | K(t)\right], \qquad (A-2)$$

where $\tilde{T}$ is the first passage time for $K = 0$. We will show that $M(K) = K$.

We make use of the fact that the functional $M(K)$ must satisfy the Kolmogorov backward equation corresponding to (A – 1):

$$\tfrac{1}{2}\beta^2 k^{2\alpha} K^{2-2\alpha} M_{KK} - kM_K + k = 0, \qquad (A-3)$$

subject to the boundary conditions (i) $M(0) = 0$ and (ii) $M(\infty) = \infty$. (See Karlin and Taylor (1981), Chapter 15.) Clearly $M(K) = K$ is a solution of (A – 3) and the associated boundary conditions. Now consider a more general solution of the form $M(K) = K + h(K)$, where $h(K)$ is an arbitrary function of K. By direct integration,

$$h_K(K) = C \exp\left[\frac{2K^{2\alpha-1}}{(2\alpha-1)\beta^2 k^{2\alpha}}\right]. \qquad (A-4)$$

But since $\lim_{K\to\infty} h_K(K) = C$, the constant C must equal zero to satisfy boundary condition (ii). Hence $M(K) = K$.

For the case of $\alpha = \frac{1}{2}$ (technical uncertainty), we can also find the variance of the cost to go, i.e.,

$$\text{Var}(K) = E_t\left[\int_t^{\tilde{T}} k\, d\tau | K\right]^2 - K^2(t). \qquad (A-5)$$

Let $G(K) = E_t\left[\int_t^{\tilde{T}} k\, d\tau | K\right]^2$. Then $G(K)$ must satisfy the following Kolmogorov equation:

$$\tfrac{1}{2}\beta^2 kKG_{KK} - kG_K + 2kK = 0, \qquad (A-6)$$

subject to the boundary conditions $G(0) = 0$ and $G(\infty) = \infty$. (See Karlin and Taylor (1981), page 203.) The solution to (A – 6) is $G(K) = 2K^2/(2-\beta^2)$, so the variance is:

$$\text{Var}(K) = \left(\frac{\beta^2}{2-\beta^2}\right) K^2. \tag{$A-7$}$$

B. Derivation of Equation (6).

Given a replicating asset or portfolio whose price x follows eqn. (5), we can value the firm's investment opportunity as a contingent claim. First, denote $\delta \equiv r_x - \alpha_x$. Now consider the following portfolio: hold the investment opportunity, worth $F(K)$, and sell short n units of the asset with price x. The value of this portfolio is then $\Phi = F(K) - nx$, and the instantaneous change in this value is $d\Phi = dF - ndx$. Since the expected rate of growth of x is $\alpha_x < r_x$, the short position will require a payment stream over time at the rate $n(r_x - \alpha_x)x = n\delta x$. Also, insofar as investment is taking place, holding the investment opportunity implies a payment stream $I(t)$. Thus over an interval dt, the total return on the portfolio is $dF - ndx - n\delta x dt - I(t)dt$.

Next, using Ito's Lemma, write dF as:

$$\begin{aligned} dF &= F_K dK + \tfrac{1}{2}F_{KK}(dK)^2 \\ &= -IF_K dt + \beta(IK)^{1/2}F_K dz + \gamma K F_K dw + \tfrac{1}{2}\beta^2 IKF_{KK}dt + \tfrac{1}{2}\gamma^2K^2F_{KK}dt \end{aligned}$$

Substituting (5) for dx, the total return on the portfolio over an interval dt is therefore:

$$\begin{aligned} -IF_K dt &+ \beta(IK)^{1/2}F_K dz + \gamma K F_K dw + \tfrac{1}{2}\beta^2 IKF_{KK}dt + \tfrac{1}{2}\gamma^2K^2F_{KK}dt \\ &- n\alpha_x x dt - n\sigma_x x dw - n\delta x dt - I dt. \end{aligned}$$

By setting $n = \gamma K F_K/\sigma_x x$, we can eliminate the terms in dw, and thereby remove nondiversifiable risk from the portfolio. With n chosen this way, the only risk the portfolio carries is diversifiable, and hence the expected rate of return on the portfolio must be the risk-free rate, r. Using this value of n and equating the expected portfolio return to $r(F - nx)dt$ yields equation (6) for $F(K)$.

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FIGURE 1 - NO COST UNCERTAINTY
(V = 10, k = 2)

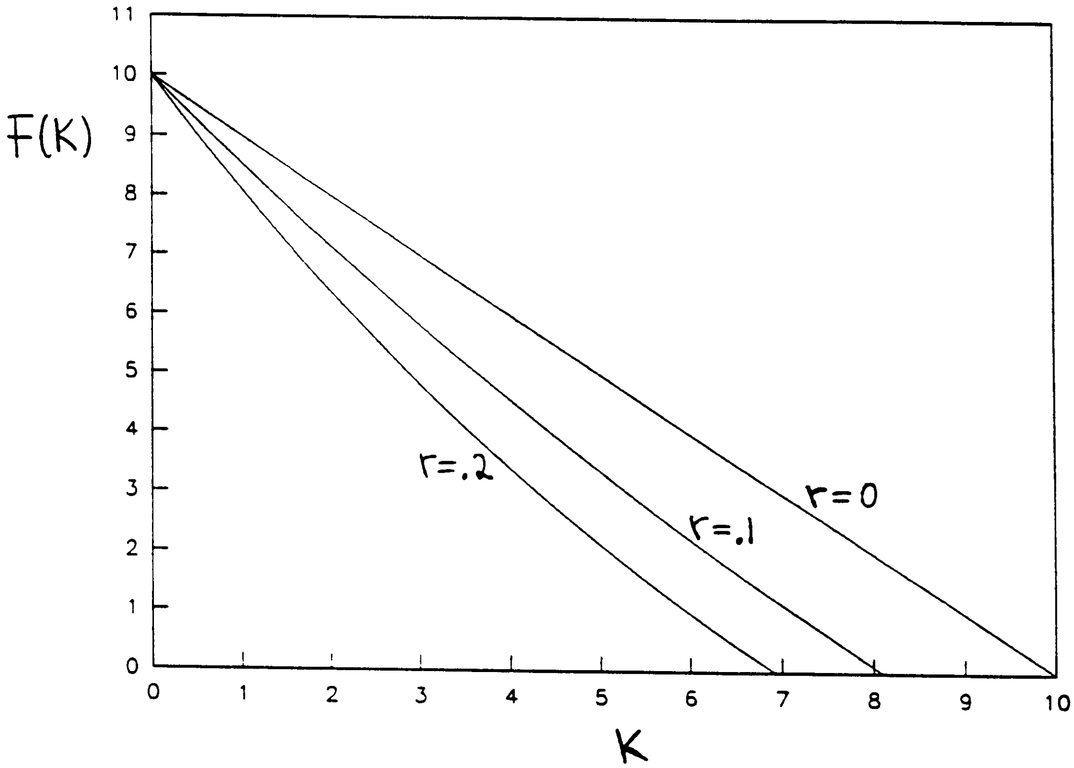

FIGURE 2 - TECHNICAL UNCERTAINTY
(V = 10, k = 2, r = .05, γ = φ = 0)

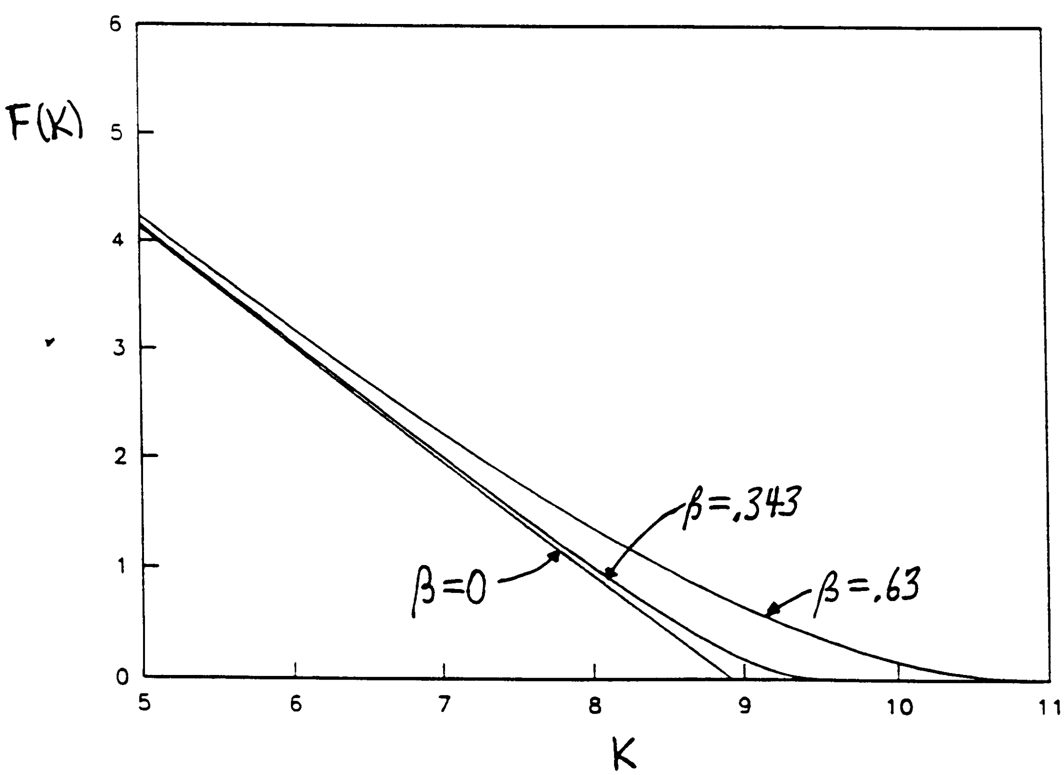

FIGURE 3 - DEPENDENCE OF K* ON β
($V = 10$, $k = 2$, $r = .05$, $\gamma = \phi = 0$)

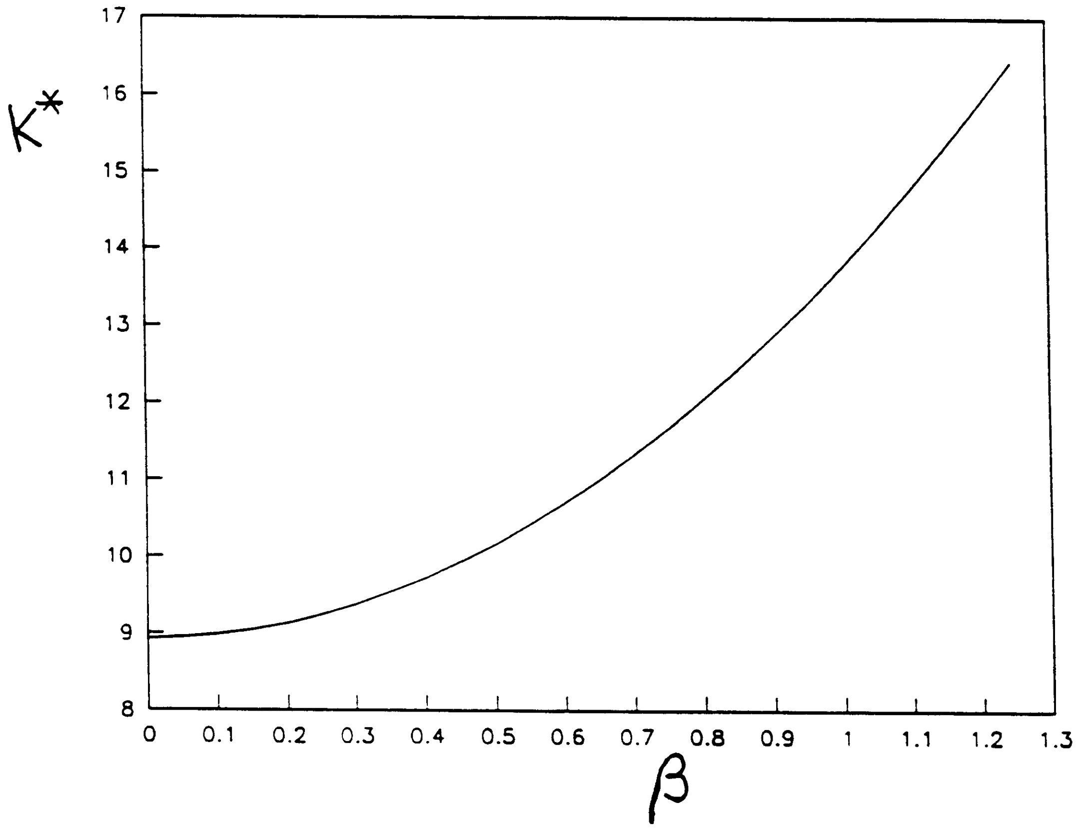

FIGURE 4 - CHANGES IN MAXIMUM RATE OF INVESTMENT
($V = 10$, $r = .05$, $\beta = .63$, $\gamma = \phi = 0$)

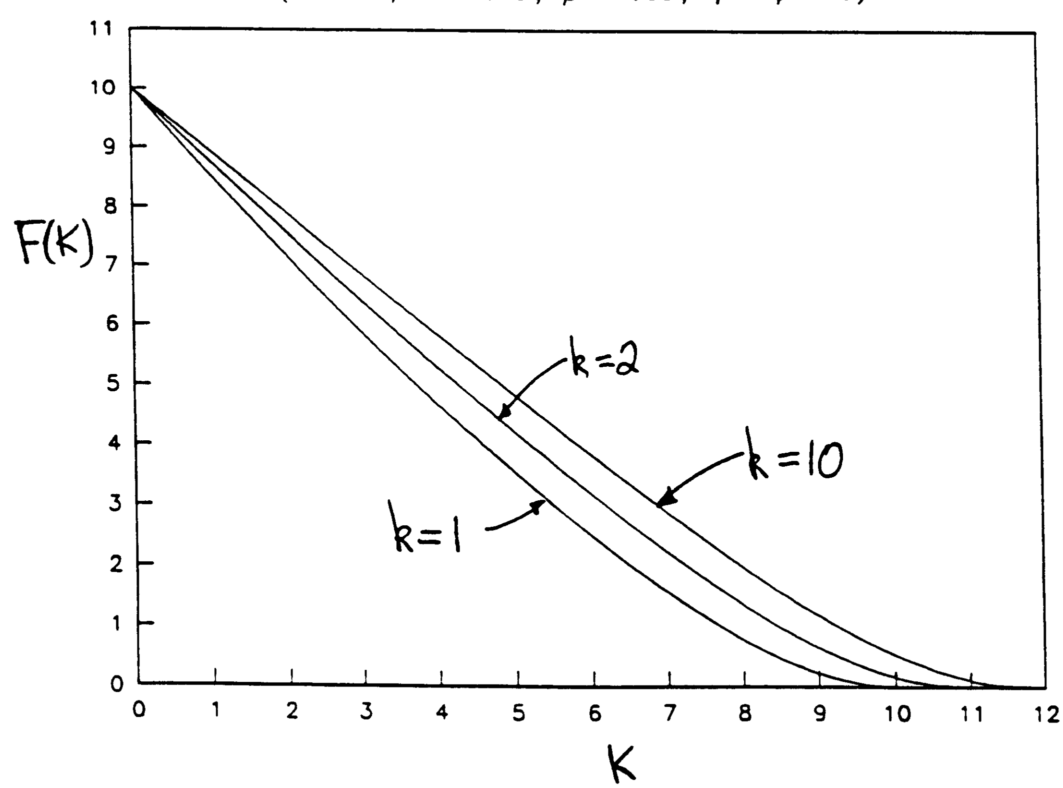

FIGURE 5 - INPUT COST UNCERTAINTY
(V = 10, k = 2, r = .05, β = 0, ϕ = 0)

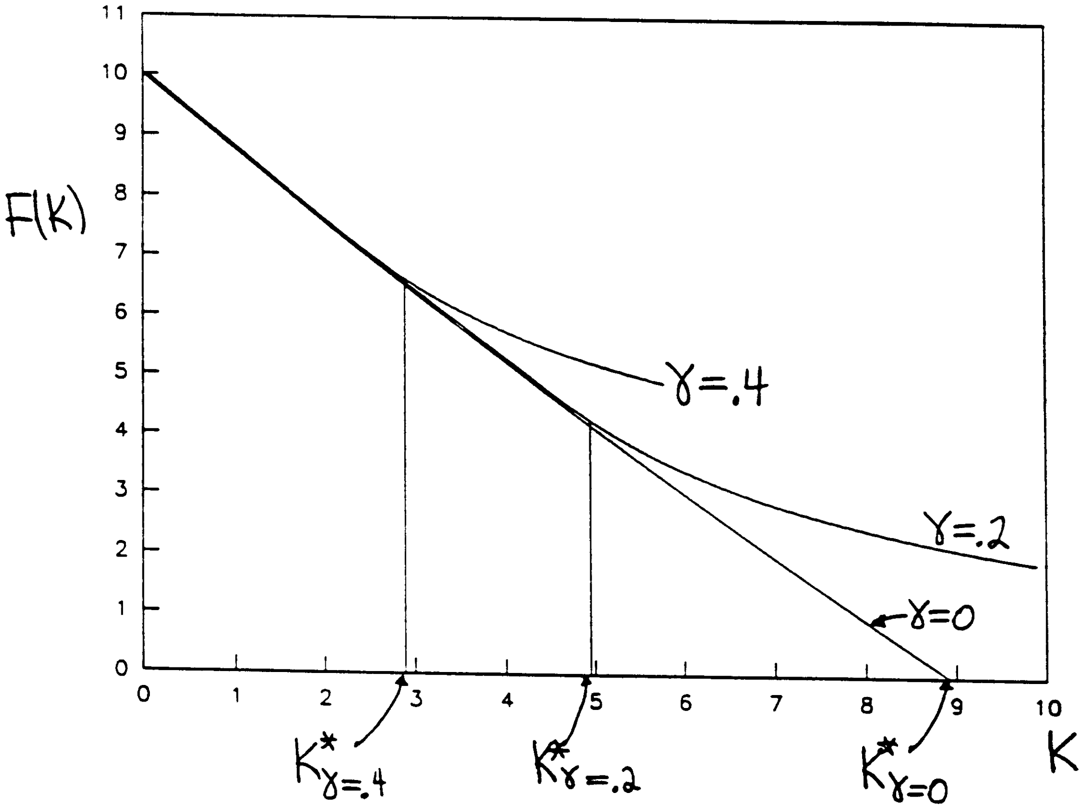

FIGURE 6 - INPUT COST UNCERTAINTY WITH SYSTEMATIC RISK
(V = 10, k = 2, r = .05, β = 0, γ = .2)

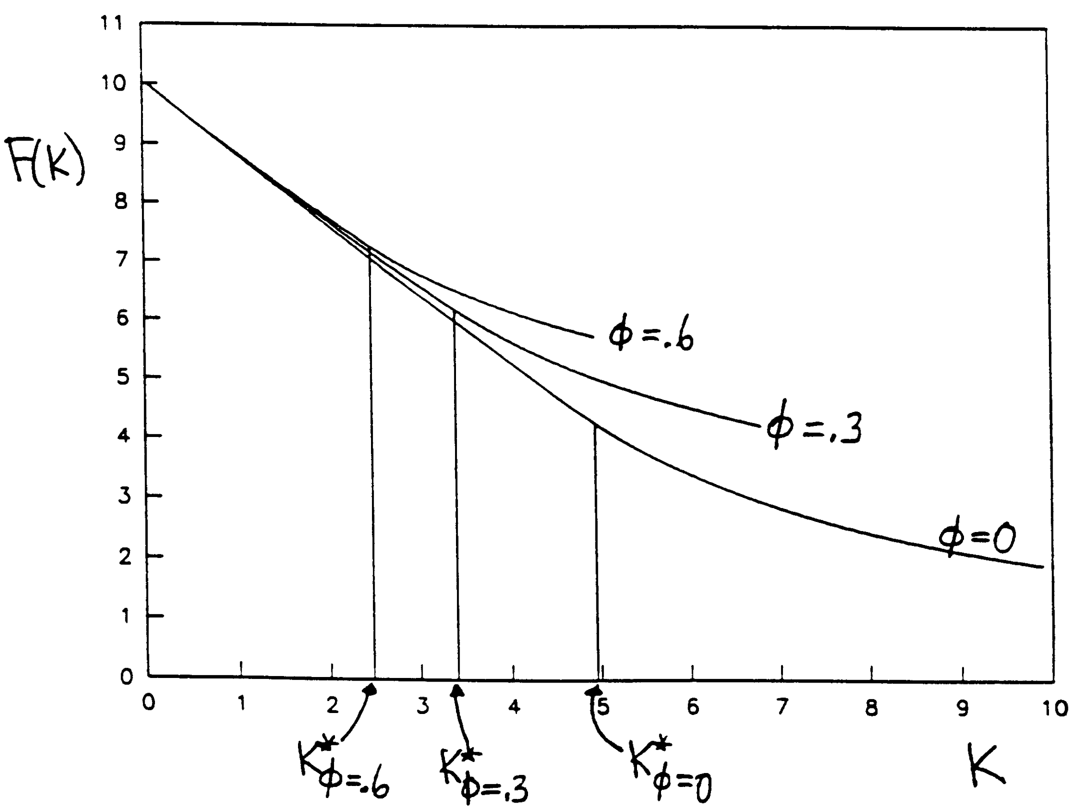

FIGURE 7 - CRITICAL K^* AS FUNCTION OF β AND γ

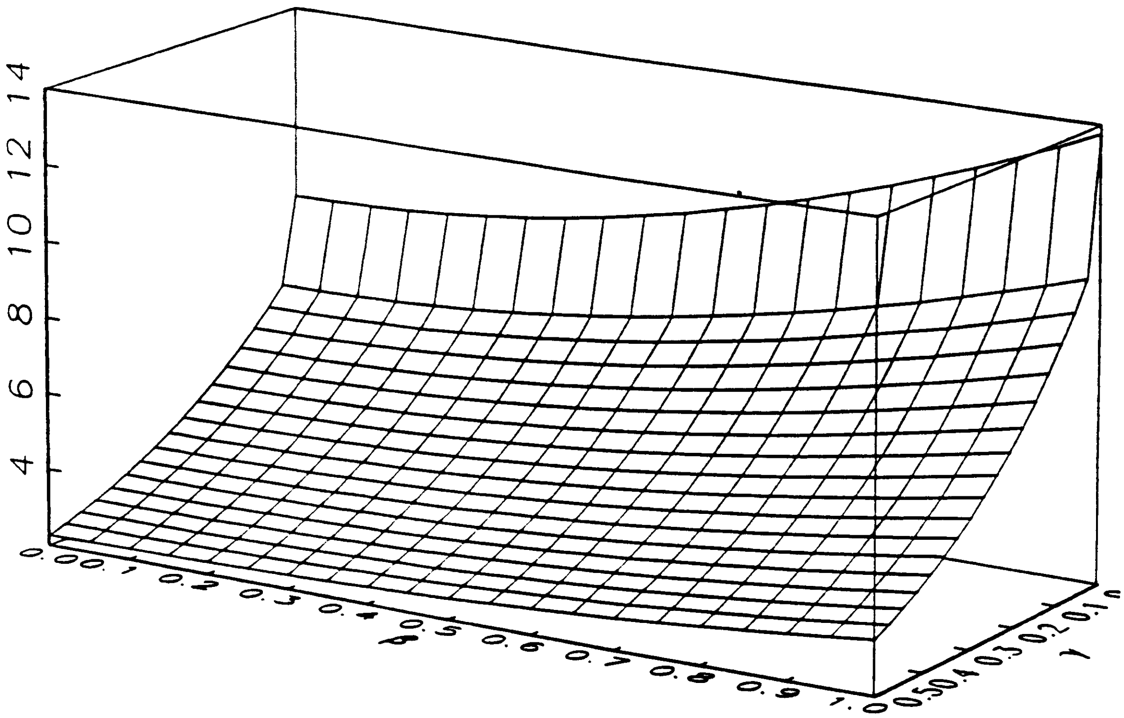

FIGURE 8 - F(K) AS FUNCTION OF β AND γ
EVALUATED AT K^* CORRESPONDING TO $\beta = \gamma = 0$