IX. MISCELLANEOUS PROBLEMS

A. ELECTRONIC DIFFERENTIAL ANALYZER

Staff: Professor H. Wallman
A. B. Macnee

The work of the past quarter has been divided principally into two parts: (1) engineering and development of the various basic units used in the solution of ordinary differential equations with constant coefficients and (2) preliminary investigations of the solution of some non-linear differential equations.

Linear Differential Equations. Some time was spent in the investigation of the relative merits, in general, of d-c coupled amplifiers versus d-c restored a-c coupled amplifiers. It has been concluded that where the time necessary for d-c restoration (or clamping) is available, this method is the more satisfactory with regards to stability, warm-up time, and economy of tubes and components.

An amplifier of this type having an amplification of 1500 has been designed and tested. Three of these units with suitable feedback connections have been used to solve the equation

$$\frac{d^2y}{dt^2} + k_1 \frac{dy}{dt} \pm k_2 y = 0 \qquad (1)$$

where the k's are fixed numbers. Particular emphasis was placed on the stability of the solution displays on the cathode-ray tube screen; this led to considerable engineering in the initial condition and gate-generators.

The equation

$$\frac{d^2y}{dt^2} + \omega_o^2 y = 0 \qquad (2)$$

has been studied in some detail to determine the restrictions imposed by the finite bandwidth and gain of the amplifiers and integrators. The function of frequency obtained from this equation by the Laplace transform has poles on the imaginary axis in the complex frequency plane. It is particularly sensitive to bandwidth and gain limitations. These limitations introduce respectively negative and positive damping of the exact solution, which is an undamped sine wave. The bandwidth restriction is particularly interesting. If the solution amplitude is to change less than 1 per cent over the solution display time, then

$$\frac{\omega_o^2 T}{2\,\omega_1} \leqq .01 \qquad (3)$$

where T = display time,
ω_o = natural frequency of the solution,
ω_1 = upper half-power frequency of the amplifier.

In particular if the solution period is 1/120 second and the amplifier bandwidth is 400 kc/sec, the maximum permissible value of $f_o = \omega_o/2\pi$ becomes 392 cps. Thus for this case the amplifier bandwidth must be 1000 times the natural frequency of the solution.

Non-Linear Differential Equations. The three units discussed above have been used together with a function generator to solve equations of the form

$$\frac{d^2y}{dt^2} \pm k_1 \frac{dy}{dt} \pm k_2 y = F(y). \tag{4}$$

No particular difficulties have been encountered in solving these equations.

Solutions of two particular equations of this type are shown in Figs. IX-2 and IX-3. Figure IX-2 is a triple exposure photograph of three different solutions of the equation

$$\frac{d^2y}{dt^2} = y^2 \tag{5}$$

with y_o held fixed and $\dot{y}_o$ varied. Equation (5) describes the motion of a particle in a potential field. A typical potential of this type is shown in Fig. IX-1.

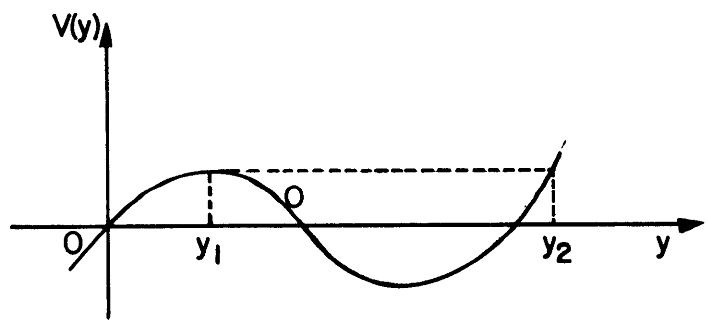

Fig. IX-1. A cubic potential curve.

In Fig. IX-2 for one initial value the particle oscillates stably in the potential well between y_1 and y_2. The other two exposures show cases for which the particle is shot off with sufficient initial kinetic energy to carry it over the potential peak at y_1 after which it drops off rapidly toward minus infinity.

Figure IX-3 displays three solutions of the equation

$$\frac{d^2y}{dt^2} = - \sin y \tag{6}$$

with $\dot{y}_o$ held fixed and y_o varied. This is the exact equation for the motion of a physical pendulum. The manner in which the period of oscillation increases with increasing amplitude can be clearly seen.

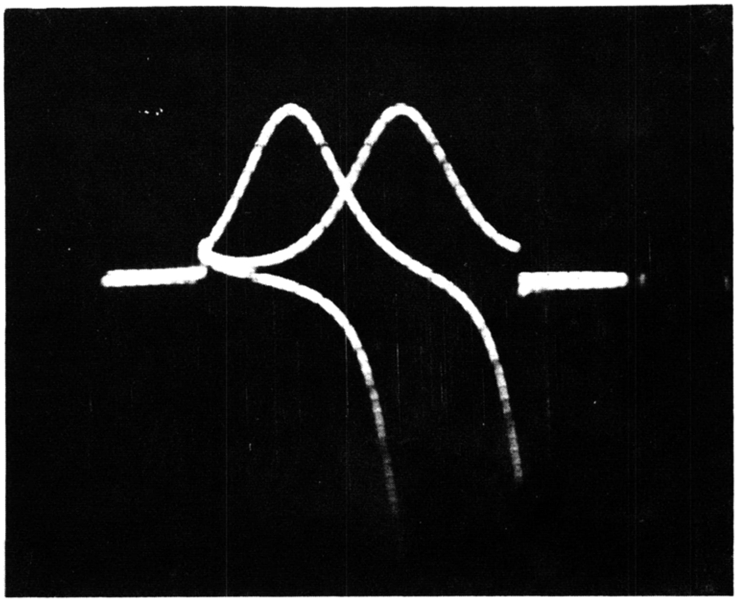

Fig. IX-2. Plot of y versus t for fixed y_o and variable $\dot{y}_o$, where y is the solution of the equation $d^2y/dt^2 = y^2$.

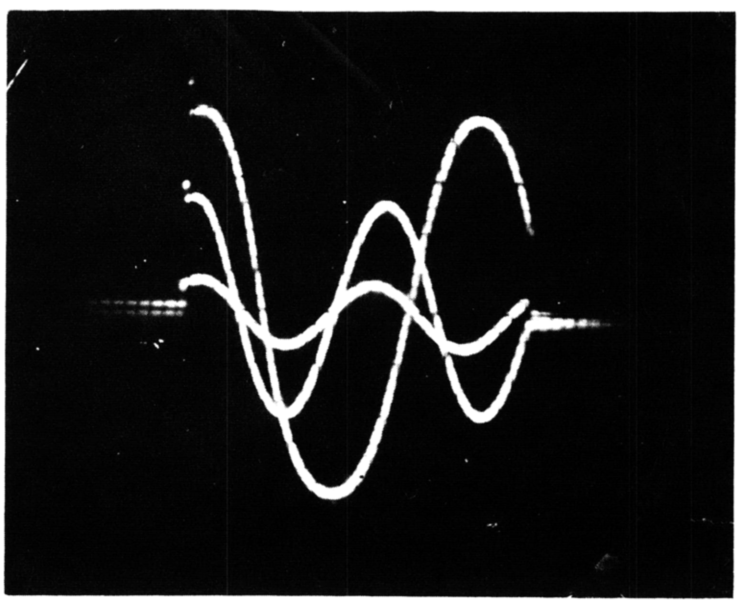

Fig. IX-3. Plot of y versus t for $\dot{y}_o$ fixed, y_o variable, where y is the solution of the equation $d^2y/dt^2 = -\sin y$.

Finally Van der Pol's famous non-linear differential equation

$$\frac{d^2x}{dt^2} - \left[A - 3B\,x^2\right] \frac{dx}{dt} + x = 0 \qquad (7)$$

has been set up and solved[1]. A typical solution of Van der Pol's equation is given in Fig. IX-4.

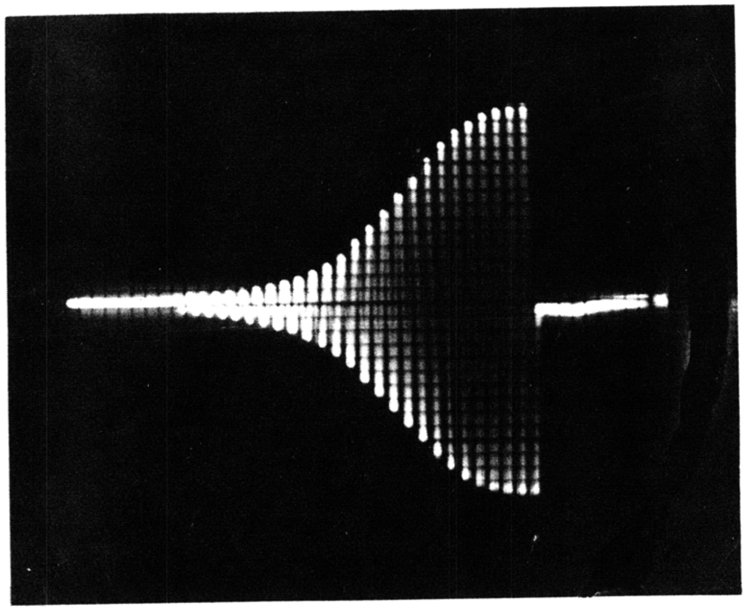

Fig. IX-4. A plot of x versus t for Van der Pol's equation.

Figure IX-5 is another solution of the same equation for the case of A and B both of the order of magnitude unity. Figure IX-6 is a photograph of the build-up to the phase-space limit cycle for this case. A considerable amount of mathematical work has been done on the characteristics of limit cycles of this type[2].

1. This equation is most easily handled by the analyzer if it is first converted by the change of variable $x = \frac{dy}{dt}$ and one integration to

$$\frac{d^2y}{dt^2} - \left[A - B\left(\frac{dy}{dt}\right)^2\right] \frac{dy}{dt} + y = 0.$$

2. N. Minorsky, "Non-linear Mechanics", Edwards Brothers, Ann Arbor, Michigan, pp 113-115, pp 131-133.

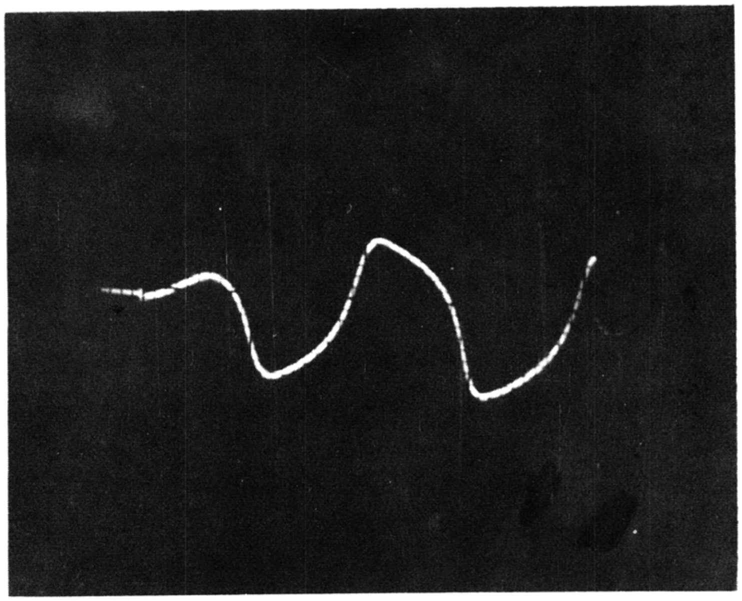

Fig. IX-5. A plot of x versus t for Van der Pol's equation (low Q case)

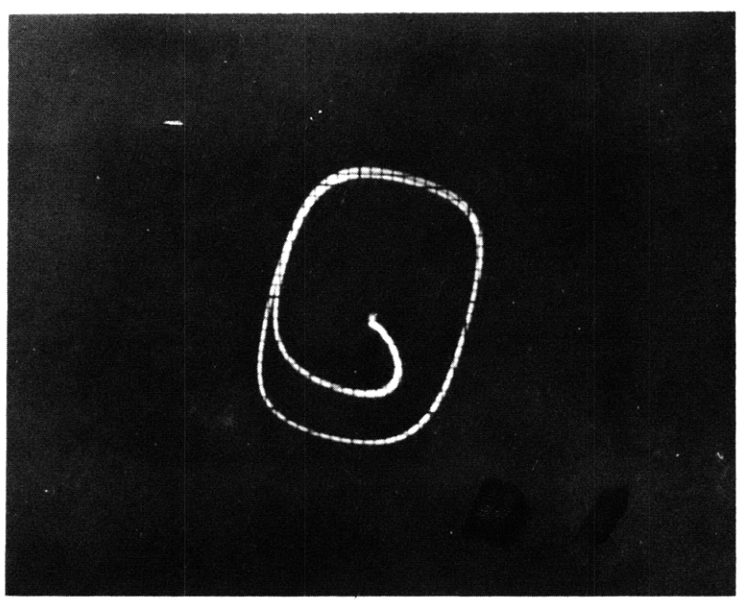

Fig. IX-6 Build-up to limit cycle in the solution of Van der Pol's equation.

IX. B. AN AUTOMATIC IMPEDANCE ANALYZER

Staff: Professor H. Wallman
R. E. Scott

Some progress has been made during the last quarter upon the development of an automatic impedance and gain-function analyzer.

Theory. The electrolytic tank used by Linvill and by Hansen and Lundstrom for the analysis of impedance functions was described in the Quarterly Progress Report of January 15, 1948. The method depends upon the mathematical analogy between potential problems and complex variable theory. It is well known that any physically realizable impedance function can be represented as the ratio of two polynomials in the complex frequency λ. This is true for driving-point impedances and for transfer impedances.

$$Z(\lambda) = \frac{(\lambda - \lambda_1)(\lambda - \lambda_3)\cdots\cdots\cdots}{(\lambda - \lambda_2)(\lambda - \lambda_4)\cdots\cdots\cdots}$$

where $Z(\lambda)$ = the impedance function,
λ = the complex frequency variable,
λ_i = zeros of the function for i odd,
λ_i = poles of the function for i even.

$$\log_e Z(\lambda) = \log_e |Z(\lambda)| + i \arg Z(\lambda)$$

$$\log_e |Z(\lambda)| = \log_e |\lambda - \lambda_1| + \log_e |\lambda - \lambda_3| + \cdots\cdots\cdots$$
$$- \log_e |\lambda - \lambda_2| - \log_e |\lambda - \lambda_4| - \cdots\cdots\cdots$$

From potential theory the voltage drop from the origin to any point λ in an infinite conducting plane is

$$V(\lambda) = A \log_e \left| \frac{(\lambda - \lambda_1)(\lambda - \lambda_3)\cdots\cdots\cdots}{(\lambda - \lambda_2)(\lambda - \lambda_4)\cdots\cdots\cdots} \right|$$

where A is a constant depending on the magnitude of the curren entering and leaving the plane, and on the conductivity;
λ_i = point where a current enters the plane for i even
= point where a current leaves the plane for i odd.
(the currents at each point are of equal magnitude)

The similarity of these two equations is at once apparent and it is seen that by measuring the voltage along the imaginary axis, the equivalent impedance function is evaluated at real frequencies. In order to exploit the method to the full it is desirable to have the potential along the imaginary axis displayed on a cathode-ray tube so that the effects of changes in the positions of the poles and zeros may be observed immediately. If such a device could be produced, it would be extremely useful for synthesizing impedances and filters. Arbitrary conditions could be imposed upon the poles and zeros with great ease; for example, they might be confined to the negative real axis. This restriction would limit the network to R, C elements.

Experimental Approach. The first step in producing a practical device is to replace the electrolytic tank by a sheet of uniformly conducting paper. Teledeltos Type H (made by Western Union) has been found satisfactory. It is uniform to better than one per cent and will carry a current of 8 ma without leaving a mark at a sharp electrode. In the actual experiment the current is limited to one-tenth of this value. Currents of this magnitude allow voltages of about twenty volts maximum to be measured along the imaginary axis. In the experiment direct currents are used from current sources with open-circuit voltages of plus and minus 300 volts. In order to prevent arcing when the probes are removed from the paper, it is necessary to provide a limiter diode at each probe to keep the open-circuit voltage below 45 volts.

The voltages along the imaginary axis are picked up on 15 probes and commutated mechanically. The resultant voltage is fed to the X plates of a cathode-ray tube. For an even spacing of the probes and a linear sweep, the voltage observed on the screen is the logarithm of the magnitude of the impedance, plotted against the frequency.

Qualitatively the device was successful. Its small size, however, made it rather inaccurate, and the mechanical commutator was a source of constant trouble. It was decided that it would be desirable to use at least a hundred pick-up probes in the final device and that an electronic commutator might be advisable. For this reason a prototype electronic commutator was constructed.

The Electronic Commutator. An electronic commutator was built which would switch four lines. It consists of a matrix of multivibrators operating at multiple frequencies and a 6AS6 gate tube in each line. In the present circuit two multivibrators are used and their outputs are combined in four ways to produce gate pulses to operate the 6AS6 tubes. To switch n lines, the number of tubes which would be required (exclusive of power supplies) is

$$n + 2 \log_2 n .$$

The circuit operates satisfactorily at frequencies from 1000 to 10,000 cycles per second with an adjacent channel rejection factor of about 50. This would probably be sufficiently good for the application proposed.

As an alternative to the electronic commutator which has been described on the preceding page, a cyclophon tube might be used. These tubes have 25 separate plates and can be used to commutate 25 lines. Such a tube has been received from Federal Telecommunications Laboratories and it is being tested for our applications at the present time.

IX. C. PROTON-VELOCITY METER

Staff: L. D. Smullin
P. Lally

Work has begun on a microwave proton-velocity meter to be used on the one-million volt Van de Graaff generator now under construction in the Laboratory for Nuclear Science and Engineering.

The proposed method uses the H_2^+ beam that is separated from the proton beam magnetically, as the beam in a "klystron amplifier". That is, the beam is passed through a bunching cavity, driven at 3000 Mc/sec by an external stabilized oscillator and amplifier. The velocity-modulated beam then flows through a drift tube about 2 meters long to a catcher cavity, where it induces 3000-Mc/sec oscillations. The phase of this induced current is compared with that of the exciting current in a conventional circuit. Changes in beam velocity will be observed as changes in phase. It is intended to use the phase information to control an electron current flowing up the Van de Graaff machine to regulate the voltage. Response time of 10 to 100 μsec, and regulation to one part in 10^4 are hoped for with this system. Figure IX-7 shows a block diagram of the complete system.

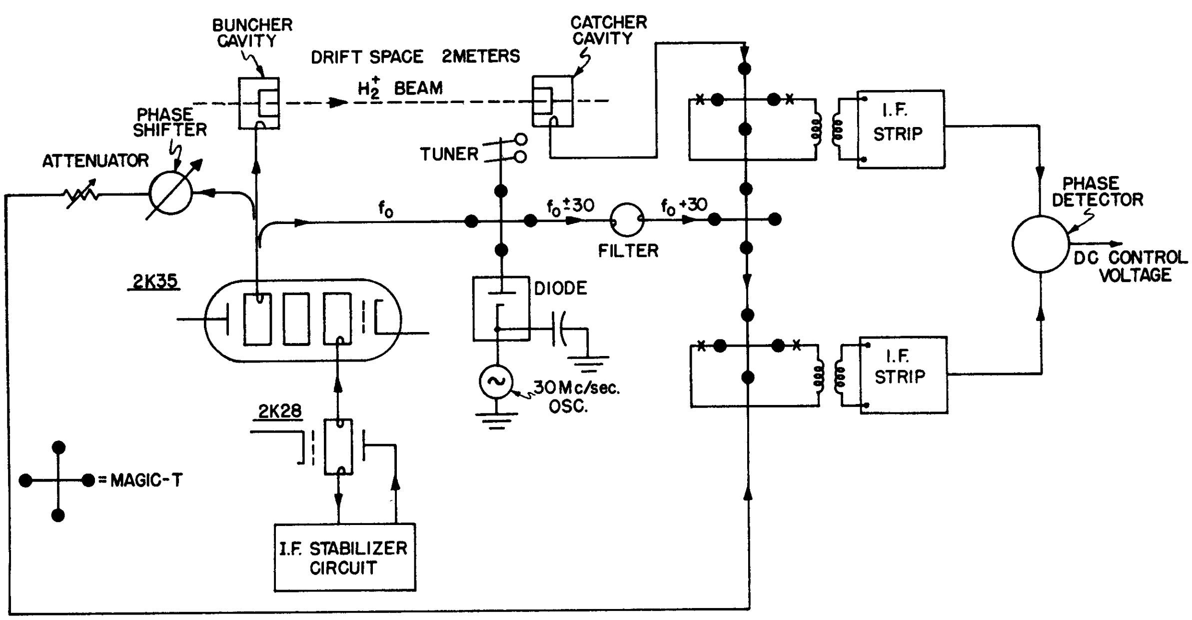

Fig. IX-7. Block diagram of proton-velocity meter.

The stabilized klystron and amplifier are well under way and work is beginning on the "synchronous local oscillator" and on power supplies for the i-f strips in the phase detector circuit.

With about 30 watts of driving power at the bunching, it is hoped to get an output power of about 3×10^{-8} watts/$(\mu a)^2$ of beam current. This should make it possible to measure phase to within about ± 5 degrees with wide-band receivers (2-5 Mc/sec wide). If slower response times are permissible, greater receiver sensitivity and accuracy can be obtained.

IX. D. ELECTRONIC POTENTIAL MAPPING

Staff: Dr. S. Goldman
C. K. Chien
W. E. Vivian

The new scanning pickup tube using electrostatic focus and deflection, which was specially constructed for small spot size, was tested and found to have a very small spot size as expected. There was considerably less interaction between pickup grids than was found with the earlier magnetic-type tube, and it appeared that this tube would be more suitable for use in every way. Unfortunately there was an internal open lead to one of the pickup grids, so that the tube will have to be repaired before it is satisfactory for displaying potential maps.

An alternative switching system to the cathode-ray tube types has also been developed using gate circuits. This has been tried and appears to be superior to the use of an electron-beam switching tube.

During the past quarter, actual displays of the potential distribution on the surface of the left front of the chest have been made. These tests showed clearly which areas of the chest were active during the different parts of the heart-beat cycle and showed marked differences in the heart activity of a subject after exercise and the same subject during prolonged relaxation.

A detailed account of the circuits and equipment and of the technique and results will be given in the next progress report.

E. PHYSICAL LIMITATIONS OF R-F RADIATING SYSTEMS

Staff: Professor L. J. Chu
Dr. M. V. Cerrillo
Dr. M. Loewenthal

A report on the results obtained thus far is in preparation.

IX. F. MATHEMATICAL PROBLEMS

Staff: R. M. Redheffer

Further work has been done on the problem of separating Laplace's equation. The case in which the solution has the form $R(u,v,w)X(u)Y(v)Z(w)$ has been solved completely. It appears that there are 26 different co-ordinate systems, though most of them are merely inversions of the well-known cases in which $R=1$. Investigation of the case in which the solutions have the form $S(u,v)R(u,v,w)Z(w)$ has been nearly completed. Results will be reported in the near future.

G. THE SCATTERING OF ELECTROMAGNETIC WAVES. MATRIX METHODS

Staff: Professor H. Mueller
N. G. Parke

Prior Work in Field. At the inception of this research there were two algebraic methods which had been developed to attack the general problem of scattering of radiation by matter. One of the algebras, due to Jones, had as its elements the electric vectors. The other algebra, due to Mueller, had as its elements the Stokes' vectors, (I,M,C,S) where I,M,C,S are four intensitie measured with the aid of a polarizer and wave plate. There also existed Wiener's generalized harmonic analysis which provides the general technique for handling the statistical aspects of the radiation.

Generalized Optical Algebra. We have succeeded in constructing, on the basis of modern algebraic techniques, a general wave algebra in which Jones' algebra and Wiener's generalized harmonic analysis play an essential role and which leads to a generalization of Mueller's algebra which is capable of handling, from a uniform standpoint, the problem of scattering from crystals, liquids, and electrolytes. We have a statistical description of the Stokes' vector which gives a new picture of line broadening and a new insight into the phenomenon of partial polarization.

The algebra uses the tensor notation with the added generalization that a continous variable, here time, is used as an integration index. This leads to a generalization of the summation convention to include the "convolution" and Wiener's "correlation" processes. To simplify the manipulation, we write

$$f(t-\underline{t_o})g(\underline{t_o}) \quad \text{for} \quad \int_{-\infty}^{\infty} f(t-t_o)g(t_o)dt_o \qquad \text{(convolution)},$$

and

$$f(t+\underline{\underline{t_o}})g^*(\underline{\underline{t_o}}) \quad \text{for} \quad \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t+t_o)g^*(t_o)dt_o \qquad \text{(correlation)}.$$

In this notation, the repetition of t_o underlined once means "convolution" and the repetition of t_o underlined twice means "correlation". The summation convention on subscripts and superscripts is the familiar one. The net result is a smooth-working algebra into which one can readily translate the physical problem and which then allows rapid and meaningful manipulation to obtain the desired solution. This technique was to generalize Mueller's matrices and arrive at the general law for their addition

$$M = \sum N_{\alpha\beta} M^{\alpha\beta}$$

where the form of $N_{\alpha\beta}$ depends upon the statistical interrelation of the individual scattering centers, $M_{\alpha\alpha}$ is the Mueller matrix of scattering center α, and $M_{\alpha\beta}$ is the "relative" Mueller matrix for scattering centers α and β.

We believe that the applicability of this algebra is broader than the original problem field it was designed to solve. It appears that it can be extended to handle microwave problems involving radiation defined by random functions. The basic elements of the algebra would, in this case, be the characteristic sets of functions for the guides and cavities involved.