THE DEPARTURE PROCESS FROM A GI/G/1 QUEUE AND ITS APPLICATIONS TO THE ANALYSIS OF TANDEM QUEUES

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The departure process from a $GI/G/1$ queue and its applications to the analysis of tandem queues

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Abstract

In this paper we characterize exactly the departure process of a GI/G/1 queue and use this characterization to propose a new algorithm for the analysis of single server tandem queues with general distributions. We first establish a close connection of the departure process with the idle time and obtain that in steady state interdeparture times are identically distributed but they are not independent. Using the Hilbert factorization combined with complex analysis methods, we find exactly the Laplace transform of the stationary interval, determine exactly the correlation of the departure times and characterize the counting process of the departure process. We then use these results to propose a new approximate algorithm for the steady-state analysis of tandem queues. We implemented the algorithm and we found that the answers produced by the algorithm are very close to those produced by simulation. We believe that the power of the algorithm rests in the direct exploitation of the characteristics of the departure process. We report several computational results.

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1 Introduction

The study of departure processes in queueing systems is primarily motivated by the need to analyze queueing network models, in which the departure process of one queue is the arrival process of another queue. Probably the most well known result in the theory of departure processes in queueing theory is the result of Burke [2] that the departure process of a M/M/s queue is a Poisson process. It is also well known that if one goes beyond the exponentiallity assumptions, the departure processes become not renewal. There are very few exact results for departure processes if one considers queues with arbitrary distributions. Daley [4] finds the stationary distribution of the departure process for the GI/M/1 and M/G/1 queues and considers structural properties of the departure process for these systems. Whitt [9], [12] approximates the departure process of various systems as renewal processes and uses these approximations for an approximate analysis of a general queueing network.

Queueing network models have recently become the focus of attention in the queueing theory literature primarily because of their applications in manufacturing and communications. Closed form solutions for queueing networks, however, are restricted to Kelly networks (see Kelly [8]) and their special cases. For FCFS models analytical solutions are known only in the case of Poisson arrivals and exponential service times (Jackson networks). Unfortunately, if one goes beyond the exponentiallity assumptions no exact results are known under FCFS. Motivated by the intractability of the models researchers have focused their attention to approximations. Whitt [10], [11] proposes a two moment theory by approximating the processes involved by renewal processes. Harrison and Dai [5] approximate the queueing network model by a Brownian network, which they then solve exactly.

The simplest network one can imagine is a series of queues in tandem. Despite its apparent simplicity, the analysis of tandem queues seems quite challenging. Whitt [12] considers the performance of the QNA to the tandem queue problem and compares it with other approximations. In Whitt [13] these approximations are used for the ordering the queues in series so that the total sojourn time is minimized.

In this paper we characterize exactly the departure process of a GI/G/1 queue and use this characterization to propose a new approximate algorithm for the steady-state analysis of queues in tandem. In section 2 we establish a close connection of the departure process with the idle time, obtain that in steady state interdeparture times are identically distributed but they are not independent and find exactly the Laplace transform of the stationary interval. In section 3 we determine exactly the correlation of the departure times for the GI/G/1 queue and also consider the special cases of the GI/M/1 and M/G/1 queues. In section 4 we characterize the counting process of the departure process. In section 5 we formulate the tandem queue problem, while in section 6 we propose the algorithm for its solution. In the final section we report computational results and compare our results with simulation.

2 The distribution of interdeparture times

In this section we use the Hilbert factorization technique to characterize the properties of the departure process from a GI/G/1 queue. We first briefly review the standard Lindley process for the GI/G/1 queue, and then discuss the interdeparture times.

Consider the Lindley process

$$W_n - I_n = W_{n-1} + X_{n-1} - T_n, \tag{1}$$

where

W_n = the waiting time of the nth customer.

I_n = the idle period prior to the arrival of the nth customer.

T_n = the interarrival time between the $(n-1)$th customer and the nth customer.

X_n = the service time of the nth customer.

The key observation is that the interdeparture time is a sum of two independent random variables, namely the service time and the idle time proceeding the service time. In fact, when the server is busy, the idle time is zero, thus the interdeparture time is equal to the service time. On the other hand, when the server is idle, then the last departure epoch is the beginning of the idle period and the next departure is at the service completion of a customer who arrives during this idle period. These two random variables are independent, since the length of the idle period does not affect the length of the service time of the customer arriving during this idle period. If D_n is the interdeparture time between the $(n-1)$th customer and the nth customer, then

$$D_n = I_n + X_n. \tag{2}$$

We introduce the transforms
$\Phi_n(s) = \mathrm{E}[e^{-sW_n}]$, $\Psi_n(s) = \mathrm{E}[e^{-sI_n}]$, $\alpha_n(s) = \mathrm{E}[e^{-sT_n}]$, $\beta_n(s) = \mathrm{E}[e^{-sX_n}]$, $\Delta_n(s) = \mathrm{E}[e^{-sD_n}]$.
The corresponding transforms in steady-state are denoted by $\Phi(s)$, $\Psi(s)$, $\alpha(s)$, $\beta(s)$, $\Delta(s)$, i.e. we drop n in the previous definitions. Therefore, equations (1) and (2) become in the transform region

$$\begin{aligned} \Phi_n(s) + \Psi_n(-s) - 1 &= \Phi_{n-1}(s)\beta_{n-1}(s)\alpha_n(-s) \\ \Delta_n(s) &= \Psi_n(s)\beta_n(s). \end{aligned}$$

In steady state, by taking n going to infinity, we obtain

$$1 - \Psi(-s) = (1 - \alpha(-s)\beta(s))\Phi(s) \tag{3}$$

$$\Delta(s) = \beta(s)\Psi(s). \tag{4}$$

By differentiating (3) we can find the first two moments of the interdeparture time for the GI/G/1 queue:

$$\begin{aligned} \mathrm{E}[D] &= \mathrm{E}[T], \\ \mathrm{Var}[D] &= \mathrm{Var}[T] + 2\,\mathrm{Var}[X] - 2\,\mathrm{E}[T - X]\,\mathrm{E}[W]. \end{aligned}$$

In the general GI/G/1 case, the solution of the factorization problem can be expressed in terms of Cauchy-Fourier integration as follows:

$$\begin{aligned}\Phi(s) &= \exp\left(\frac{1}{2\pi\sqrt{-1}}\oint_{C_0}\log(\frac{\omega}{\omega-s})\,d\log(1-\alpha(-\omega)\beta(\omega))\right)\\ \Delta(s) &= \beta(s)\left(1-s\,\mathrm{E}[T-X]\exp\left(\frac{1}{2\pi\sqrt{-1}}\oint_{C_1}\log(\frac{\omega}{\omega+s})\,d\log(1-\alpha(-\omega)\beta(\omega))\right)\right),\end{aligned}$$

where the contour C_0 contains the negative half complex plane only and the contour C_1 contains all the positive half complex plane only.

The proof of this requires heavy machinery from complex analysis. Although this a result in closed form, we do not believe that it is numerically useful, since the inversion of such a transform is numerically unstable due to the multiple branch points in the integrand. In order to find numerically useful results, we consider the GI/R/1 and R/G/1 queues, where one of the interarrival or service time distributions have rational Laplace transform. We obtain closed form solutions for these two cases which are computationally very tractable.

2.1 The R/G/1 queue

In this case $\alpha(s)=\frac{\alpha_N(s)}{\alpha_D(s)}$, where $\alpha_D(s)$ is a monic polynomial in s of degree L and $\alpha_N(s)$ is a polynomial of degree less than L. Then

Theorem 1 *For the R/G/1 queue*

$$\begin{aligned}\Phi(s) &= \frac{s\,\mathrm{E}[T-X]\alpha_D(0)}{\alpha_N(-s)\beta(s)-\alpha_D(-s)}\prod_{i=1}^{L-1}\frac{x_i^+-s}{x_i^+},\\ \Delta(s) &= \beta(s)\left(1-s\,\mathrm{E}[T-X]\frac{\alpha_D(0)}{\alpha_D(s)}\prod_{i=1}^{L-1}\frac{x_i^++s}{x_i^+}\right),\end{aligned} \tag{5}$$

where $x=x_1^+,\ldots,x_{L-1}^+,0$ are the roots of the equation

$$\alpha(-x)\beta(x)=1,$$

with $\mathrm{Re}(x_i^+)>0\ (i=1\ldots L-1)$.

Proof

Let $x = x_1^+, \ldots, x_{L-1}^+, 0$ be the roots of the equation

$$\alpha(-x)\beta(x) = 1,$$

with $\mathrm{Re}(x_i^+) > 0\,(i = 1 \ldots L-1)$. The proof of this is easily established by applying Rouche's theorem. Now, (3) can be written as

$$\frac{\Phi(s)}{\frac{s\prod_{i=1}^{L-1}(x_i^+-s)}{\alpha_N(-s)\beta(s)-\alpha_D(-s)}} = -\frac{1-\Psi(-s)}{\frac{s\prod_{i=1}^{L-1}(x_i^+-s)}{\alpha_D(-s)}}. \tag{6}$$

By observing that the expression in the lhs of the equation (6) is analytic for $\mathrm{Re}(s) > 0$ and the expression in the rhs of the the equation (6) is analytic for $\mathrm{Re}(s) < 0$ and using Liouville's theorem we conclude that both expressions should be equal a constant K.

To complete Liouville's theorem, we need that the expressions in both sides of the equation (6) are bounded. But for the lhs, with $\mathrm{Re}(s) > 0$, it is easily seen (since the zeros cancel out) that the denominator is bounded away from 0, and thus for some $\epsilon > 0$,

$$\left|\frac{s\prod_{i=1}^{L-1}(x_i^+ - s)}{\alpha_N(-s)\beta(s) - \alpha_D(-s)}\right| \geq \epsilon.$$

Also $|\Phi(s)| \leq 1$. In an analogous way the denominator of the rhs, with $\mathrm{Re}(s) < 0$, is bounded away from 0, i.e., for some $\epsilon > 0$,

$$\left|\frac{s\prod_{i=1}^{L-1}(x_i^+ - s)}{\alpha_D(-s)}\right| \geq \epsilon,$$

and $|1 - \Psi(-s)| \leq 2$. Thus by applying Liouville 's theorem we conclude that the unique solution to the Hilbert factorization problem is:

$$\Phi(s) = K\frac{s\prod_{i=1}^{L-1}(x_i^+ - s)}{\alpha_N(-s)\beta(s) - \alpha_D(-s)}.$$

By taking the limit as $s \to 0$ and using Hospital's rule we obtain that $K = \frac{E[T-X]\alpha_D(0)}{\prod_{i=1}^{L-1} x_i^+}$, from which (5) follows. □

2.2 The GI/R/1 queue

In this case $\beta(s) = \frac{\beta_N(s)}{\beta_D(s)}$, where $\beta_D(s)$ is a monic polynomial in s of degree M and $\beta_N(s)$ is a polynomial of degree less than M. Then, in a completely analogous way as in the previous section we can prove the following.

Theorem 2 *For the GI/R/1 queue*

$$\begin{aligned} \Phi(s) &= \frac{\beta_D(s)}{\beta_D(0)} \prod_{j=1}^{M} \frac{x_j^-}{x_j^- - s} \\ \Delta(s) &= \beta(s)\left(1 - \frac{\beta_D(-s) - \alpha(s)\beta_N(-s)}{\beta_D(0)} \prod_{j=1}^{M} \frac{x_j^-}{x_j^- + s}\right), \end{aligned} \tag{7}$$

where $x = x_1^-, \ldots, x_M^-$ *are the roots of the equation*

$$\alpha(-x)\beta(x) = 1,$$

with $\mathrm{Re}(x_j^-) < 0\,(j = 1 \ldots M)$.

3 The autocorrelation of the interdeparture times

In the previous section we characterized the distribution of the interdeparture interval. As it is well known the interdeparture times are not independent. In this section we characterize the dependence structure of the interdeparture times by considering the autocorrelation function of the interdeparture times. Given two intervals A, B from a stationary point process the autocorrelation of A, B is defined as

$$\text{Autocorrelation}(A, B) = \frac{E[AB] - E[A]^2}{Var[A]}.$$

We will use the Hilbert factorization technique, which leads us to use the heavy machinery of complex analysis. We first study the intermediate problem of characterizing the the covariance between adjacent interdeparture times in steady state and then study the covariance of arbitrary interdeparture times.

3.1 Adjacent interdeparture times

Throughout this and the next subsection we will use the notation $\{D_\infty, D_{\infty+1}\}$ to denote a pair of two adjacent interdeparture times in steady state. This notation is motivated by the simplicity of the resulting expressions. Our aim in this subsection is to find the joint distribution of $\{D_\infty, D_{\infty+1}\}$.

We first note that since $\{D_n, D_{n+1}\} = \{D_n, I_{n+1} + X_{n+1}\}$ and X_{n+1} is independent from D_n, I_{n+1}, W_{n+1}, it suffices to look at $\{D_n, W_{n+1} - I_{n+1}\}$. The dynamics of the system are described recursively as follows

$$\{D_n, W_{n+1} - I_{n+1}\} = \begin{cases} \{X_n, & W_n + X_n - T_{n+1}\} \quad (W_n > 0) \\ \{X_n + I_n, & X_n - T_{n+1}\} \quad (I_n > 0). \end{cases}$$

By defining[1] $\Phi(s_1, s_2) = \mathrm{E}[e^{-s_1 D_\infty - s_2 W_{\infty+1}}]$,
$\Psi(s_1, s_2) = \mathrm{E}[e^{-s_1 D_\infty - s_2 I_{\infty+1}}]$,
$\Phi(s, 0) = \Psi(s, 0) = \Delta(s)$,
and noting that $\Phi(0, s) = \Phi(s)$, $\Psi(0, s) = \Psi(s)$ and $\Delta(0, s) = \Delta(s)$, we obtain by taking transforms and limits

$$\Phi(s_1, s_2) + \Psi(s_1, -s_2) - \Delta(s_1) = \alpha(-s_2)\beta(s_1 + s_2)(\Phi(s_2) + \Psi(s_1) - 1). \quad (8)$$

In order to solve this factorization problem we use the Laurent expansion of the rhs of (8). Note that all the singularities (λ's) of $\Psi(s_1, -s_2)$ are in the region $\mathrm{Re}(s_2) \geq 0$, i.e. they are equal to the singularities of $\alpha(-s_2)$. Combined with $\Psi(s, 0) = \Delta(s)$, we have that

$$\begin{aligned} \Psi(s_1, -s_2) &= \Delta(s_1) \\ &+ \sum_{\mathrm{Re}(\lambda)>0} \left(\frac{1}{\lambda} + \frac{1}{s_2 - \lambda}\right) \underset{s=\lambda}{\mathrm{Residual}} \left\{ \alpha(-s)\beta(s_1 + s)(\Phi(s) + \frac{\Delta(s_1)}{\beta(s_1)} - 1) \right\} \end{aligned}$$

[1] We follow the notation introduced by Keilson of using the same symbols Φ, Ψ and Δ for both single and double parameter functions. The relation between them is that when the first parameter s_1 of the two parameter functions becomes zero, they reduce to the corresponding single parameter functions.

Expressing this in terms of a Cauchy integral, we obtain that

$$\begin{aligned}\Delta(s_1, s_2) &= \mathrm{E}[e^{-s_1 D_\infty - s_2 D_{\infty+1}}] \\ &= \beta(s_2)\Psi(s_1, s_2) \qquad (9) \\ &= \frac{\beta(s_2)}{2\pi\sqrt{-1}} \oint_{C_2} \alpha(-\omega)\beta(s_1+\omega)(\Phi(\omega) + \frac{\Delta(s_1)}{\beta(s_1)} - 1)(\frac{1}{\omega} - \frac{1}{s_2+\omega})\, d\omega\end{aligned}$$

where the contour C_2 contains 0 and all the singularities of $\alpha(-s)$, but it does not contain $-s_2$ and the singularities of $\Phi(s)$, $\beta(s_1+s)$ for a fixed s_1. Similarly

$$\Phi(s_1, s_2) = \frac{s_2}{2\pi\sqrt{-1}} \oint_{C_4} \frac{\alpha(-\omega)\beta(s_1+\omega)(\Phi(\omega) + \Psi(s_1) - 1)}{\omega(s_2 - \omega)}\, d\omega \qquad (10)$$

where the contour C_4 contains 0, all the singularities of $\Phi(s)$ and $\beta(s_1+s)$ for a fixed s_1, but not the singularities of $\alpha(-s)$ and s_2.

From (9), we can compute the covariance:

$$\begin{aligned}\mathrm{Cov}(D_\infty, D_{\infty+1}) &= \lim_{s_1,s_2\to 0} \frac{\partial^2}{\partial s_1 \partial s_2}\Delta(s_1, s_2) - \mathrm{E}[D]^2 \\ &= \mathrm{E}[T]\,\mathrm{E}[X-T] \qquad (11) \\ &\quad + \frac{1}{2\pi\sqrt{-1}} \oint_{C_3} \frac{1}{\omega^2}\alpha(-\omega)(\mathrm{E}[X-T]\beta(\omega) + \dot{\beta}(\omega)\Phi(\omega))\, d\omega\end{aligned}$$

where the contour C_3 contains all the singularities of $\alpha(-s)$ but it does not contain the singularities of $\Phi(s)$, $\beta(s)$ and 0.

For the R/G/1 queue with

$$\begin{aligned}\alpha(s) &= \frac{\alpha_N(s)}{\alpha_D(s)} \\ \alpha_D(s) &= \prod_i (s+\lambda_i)^{L_i},\end{aligned}$$

(11) becomes

$$\begin{aligned}\mathrm{Cov}(D_\infty, D_{\infty+1}) &= \mathrm{E}[T]\,\mathrm{E}[X-T] \\ &+ \sum_i L_i \lim_{s\to\lambda_i} \frac{d^{L_i-1}}{ds^{L_i-1}} \frac{\alpha_N(-s)\left(\mathrm{E}[X-T]\beta(s) + \Phi(s)\frac{d\beta(s)}{ds}\right)}{s^2 \frac{d^{L_i}}{ds^{L_i}}\alpha_D(-s)}.\end{aligned}$$

We evaluated this expression numerically, using the symbolic software package Mathematica [14]. The evaluation requires several differentiations, especially when the transform of the interarrival distribution has poles of high multiplicity like the Erlang distribution. We also note that we verified using Mathematica that indeed $\text{Cov}(D_\infty, D_{\infty+1}) = 0$ for the M/M/1 case.

3.2 Pairs of arbitrary interdeparture times

We are now ready to find the generating function for the autocorrelation function of the departure process in steady state. We note first that for $k \geq 2, X_{n+k-1}$ and T_{n+k} are independent from $D_n, I_{n+k-1}, W_{n+k-1}$. The dynamics are can be written as follows:

$$\{D_n, W_{n+k} - I_{n+k}\} = \begin{cases} \{D_n, \quad W_{n+k-1} + X_{n+k-1} - T_{n+k}\} & (W_{n+k-1} > 0) \\ \{D_n, \quad X_{n+k-1} - T_{n+k}\} & (I_{n+k-1} > 0) \end{cases} \tag{12}$$

We define $\Phi_c(z, s_1, s_2) = \sum_{k=1}^{\infty} z^k \, \mathrm{E}[e^{-s_1 D_\infty - s_2 W_{\infty+k}}]$,
$\Psi_c(z, s_1, s_2) = \sum_{k=1}^{\infty} z^k \, \mathrm{E}[e^{-s_1 D_\infty - s_2 I_{\infty+k}}]$,
$\Phi_c(z, s, 0) = \Psi_c(z, s, 0) = \frac{z}{1-z}\Delta(s)$,
$\Delta_c(z, s_1, s_2) = \sum_{k=1}^{\infty} z^k \, \mathrm{E}[e^{-s_1 D_\infty - s_2 D_{\infty+k}}] = \beta(s_2)\Psi_c(z, s_1, s_2)$.
By taking transforms of (12) and summing we obtain

$$\Phi_c(z, s_1, s_2) - z\Phi(s_1, s_2) + \Psi_c(z, s_1, -s_2) - z\Psi(s_1, -s_2) - \frac{z^2\Delta(s_1)}{1-z} = z\alpha(-s_2)\beta(s_2)\Phi_c(z, s_1, s_2) \tag{13}$$

where $\Phi(s_1, s_2), \Psi(s_1, s_2)$ were calculated in the previous subsection (see (9), (10)).

We will solve (13) for the $R/R/1$ case. We will use the Green's function method, a technique most commonly used for solving differential equations, originated by Keilson [7]. The difficulty of this problem is the presence of the functions $\Phi(s_1, s_2)$ and $\Psi(s_1, s_2)$ in the equation. Without them equation (13) is a simple factorization problem, called the homogeneous problem. A solution for this homogeneous problem is called the Green function, or in ODE terminology it is also known as the general solution.

The first step in our calculation is the determination of the Green function, which is obtained by solving

$$(1 - z\alpha(-s_2)\beta(s_2))\tilde{\Phi}_c(z, s_1, s_2) = \frac{z^2\Delta(s_1)}{1-z} - \tilde{\Psi}_c(z, s_1, -s_2).$$

Using the usual factorization arguments (see the proof of theorem 1) we obtain the Green functions as follows:

$$\begin{aligned} \tilde{\Phi}_c(z, s_1, s_2) &= -\frac{\beta_D(s_2)}{\prod_{j=1}^{M}(s_2 - x_j^-(z))}C \\ \tilde{\Psi}_c(z, s_1, -s_2) &= \frac{z^2\Delta(s_1)}{1-z} + \frac{\prod_{i=1}^{L}(x_i^+(z) - s_2)}{\alpha_D(-s_2)}C \end{aligned}$$

for some constant C, where $x_1^+(z), \ldots, x_L^+(z), x_1^-(z), \ldots, x_M^-(z)$ are the roots of the equation

$$z\alpha(-x)\beta(x) = 1$$

with the property that $\Re x_i^+(z) > 0$, $(i = 1, \ldots, L)$ and $\mathrm{Re}(x_j^-(z)) < 0$, $(j = 1, \ldots, M)$. The latter is established by applying Rouche theorem.

In the second step of the calculation we substitute C by a function $s_2\chi_c(z, s_1, s_2)$, so that it will compensate for the missing terms $\Phi(s_1, s_2), \Psi(s_1, s_2)$ in the homogeneous equation. Guided by the solution to the homogeneous problem we assume that

$$\Phi_c(z, s_1, s_2) = \frac{z\Phi(s_1, s_2)}{1 - z\alpha(-s_2)\beta(s_2)} - \frac{s_2\beta_D(s_2)}{\prod_{j=1}^{M}(s_2 - x_j^-(z))}\chi_c(z, s_1, s_2) \quad (14)$$

$$\Psi_c(z, s_1, -s_2) = z\Psi(s_1, -s_2) + \frac{z^2\Delta(s_1)}{1-z} + \frac{s_2\prod_{i=1}^{L}(x_i^+(z) - s_2)}{\alpha_D(-s_2)}\chi_c(z, s_1, s_2) \quad (15)$$

Our goal is to determine $\chi_c(z, s_1, s_2)$ so that all the conditions are satisfied. Since $\Phi_c(z, s_1, s_2)$ is regular in $\mathrm{Re}(s_2) > 0$, we must have

$$\chi_c(z, s_1, s_2) = \sum_{i=1}^{L} \frac{C_i(z, s_1)}{s_2 - x_i^+(z)}.$$

To find $C_i(z, s_1)$, we must set the residual of (14) at $s_2 = x_i^+(z)$ equal to zero, and thus

$$C_i(z, s_1) = -\frac{z\alpha_D(-x_i^+(z))\Phi(s_1, x_i^+(z))}{x_i^+(z)\prod_{\substack{k=1 \\ k\neq i}}^{L}(x_k^+(z) - x_i^+(z))}.$$

Hence

$$
\begin{aligned}
\chi_c(z,s_1,s_2) &= \sum_{i=1}^{L} \frac{z\alpha_D(-x_i^+(z))\Phi(s_1,x_i^+(z))}{x_i^+(z)(x_i^+(z)-s_2)\prod_{\substack{k=1\\k\neq i}}^{L}(x_k^+(z)-x_i^+(z))} \\
&= \frac{z}{2\pi\sqrt{-1}}\oint_{C_5}\frac{\alpha_D(-\omega)}{\omega(s_2-\omega)}\Phi(s_1,\omega)\frac{d\omega}{\prod_{i=1}^{L}(x_i^+(z)-\omega)} \\
&= \frac{z\alpha_D(-s_2)(\Delta(s_1)-\Psi(s_1,-s_2))}{s_2\prod_{i=1}^{L}(x_i^+(z)-s_2)} \\
&\quad + \frac{z}{2\pi\sqrt{-1}}\oint_{C_6}\frac{\alpha_N(-\omega)\beta(s_1+\omega)(\Phi(\omega)+\Psi(s_1)-1)}{\omega(\omega-s_2)\prod_{i=1}^{L}(x_i^+(z)-\omega)}\,d\omega
\end{aligned}
$$

where the contour C_6 contains $0, s_2$, all the singularities of $\Phi(s)$ and $\beta(s_1+s)$ for a fixed s_1 but it does not include $x_i^+(z)\,(i=1,\ldots,L)$. The contour C_5 includes $x_i^+(z)(i=1,\ldots,L)$ only. Note that the last step of the above expressions is attained by substituting (8), (10) and using the complimentary contour integration. The interested reader can check the algebra:

$$
\begin{aligned}
\chi_c(z,s_1,s_2) &= \frac{1}{2\pi\sqrt{-1}}\oint_{C_5}\frac{z\alpha_D(-x)}{x(s_2-x)}\Phi(s_1,x)\frac{dx}{\prod_{i=1}^{L}(x_i^+(z)-x)} \\
&= \left(\frac{1}{2\pi\sqrt{-1}}\right)^2\oint_{C_4}\oint_{C_5}\frac{z\alpha_D(-x)\alpha(-\omega)\beta(s_1+\omega)(\Phi(\omega)+\Psi(s_1)-1)}{(s_2-x)\omega(x-\omega)\prod_{i=1}^{L}(x_i^+(z)-x)}\,dx\,d\omega \\
&= \frac{1}{2\pi\sqrt{-1}}\oint_{C_4}\frac{z\alpha_N(-\omega)\beta(s_1+\omega)(\Phi(\omega)+\Psi(s_1)-1)}{\omega(\omega-s_2)\prod_{i=1}^{L}(x_i^+(z)-\omega)}\,d\omega \\
&\quad + \frac{1}{2\pi\sqrt{-1}}\oint_{C_4}\frac{z\alpha_D(-s_2)\alpha(-\omega)\beta(s_1+\omega)(\Phi(\omega)+\Psi(s_1)-1)}{\omega(s_2-\omega)\prod_{i=1}^{L}(x_i^+(z)-s_2)}\,d\omega \\
&= \frac{1}{2\pi\sqrt{-1}}\oint_{C_4}\frac{z\alpha_N(-\omega)\beta(s_1+\omega)(\Phi(\omega)+\Psi(s_1)-1)}{\omega(\omega-s_2)\prod_{i=1}^{L}(x_i^+(z)-\omega)}\,d\omega \\
&\quad + \frac{z\alpha_D(-s_2)\Phi(s_1,s_2)}{s_2\prod_{i=1}^{L}(x_i^+(z)-s_2)} \\
&= \frac{1}{2\pi\sqrt{-1}}\oint_{C_4}\frac{z\alpha_N(-\omega)\beta(s_1+\omega)(\Phi(\omega)+\Psi(s_1)-1)}{\omega(\omega-s_2)\prod_{i=1}^{L}(x_i^+(z)-\omega)}\,d\omega \\
&\quad + \frac{z\alpha_D(-s_2)\{\Delta(s_1)-\Psi(s_1,-s_2)+\alpha(-s_2)\beta(s_1+s_2)(\Phi(s_2)+\Psi(s_1)-1)\}}{s_2\prod_{i=1}^{L}(x_i^+(z)-s_2)} \\
&= \frac{z\alpha_D(-s_2)(\Delta(s_1)-\Psi(s_1,-s_2))}{s_2\prod_{i=1}^{L}(x_i^+(z)-s_2)} \\
&\quad + \frac{z}{2\pi\sqrt{-1}}\oint_{C_6}\frac{\alpha_N(-\omega)\beta(s_1+\omega)(\Phi(\omega)+\Psi(s_1)-1)}{\omega(\omega-s_2)\prod_{i=1}^{L}(x_i^+(z)-\omega)}\,d\omega.
\end{aligned}
$$

From the definition of $\Delta_c(z, s_1, s_2)$ and (15), we get[2]

$$\Delta_c(z, s_1, s_2) = \frac{z}{1-z}\Delta(s_1)\beta(s_2) + \frac{1}{2\pi\sqrt{-1}} \oint_{C_7} \frac{z s_2 \beta(s_2)\beta(s_1+\omega)\alpha_N(-\omega)}{\omega(s_2+\omega)\alpha_D(s_2)} \prod_{i=1}^{L} \left(\frac{x_i^+(z)+s_2}{x_i^+(z)-\omega}\right) (\Phi(\omega) + \frac{\Delta(s_1)}{\beta(s_1)} - 1)\, d\omega$$

where the contour C_7 contains $x_i^+(z)$ $(i = 1, \ldots, L)$, but it does not contain 0, $-s_2$ and the singularities of $\Phi(s)$, $\beta(s_1 + s)$.

The generating function $R(z)$ for the autocorrelation function of the departure process is thus given:[3]

$$\begin{aligned} R(z) &= \sum_{k=1}^{\infty} z^k \frac{\mathrm{E}[D_\infty D_{\infty+k}] - \mathrm{E}[D]^2}{\mathrm{Var}[D]} \\ &= \frac{1}{\mathrm{Var}[D]} \lim_{s_1,s_2\to 0} \frac{\partial^2}{\partial s_1 \partial s_2}\Delta_c(z, s_1, s_2) - \frac{\mathrm{E}[D]^2 z}{\mathrm{Var}[D](1-z)} \\ &= \frac{\mathrm{E}[T]\,\mathrm{E}[X-T]\, z}{\mathrm{Var}[D](1-z)} \\ &+ \frac{1}{2\pi\sqrt{-1}} \oint_{C_8} \frac{\alpha_D(-\omega)}{\omega^2 \alpha_D(0)} \prod_{i=1}^{L} \left(\frac{x_i^+(z)}{x_i^+(z)-\omega}\right) \left(\frac{\mathrm{E}[X-T]}{\mathrm{Var}[D]} + \frac{\Phi(\omega)}{\mathrm{Var}[D]} \frac{d\log(\beta(\omega))}{d\omega}\right) d\omega \end{aligned} \tag{16}$$

---

[2]For the general $GI/G/1$,

$$\begin{aligned} \Delta_c(z, s_1, s_2) &= \frac{z}{1-z}\Delta(s_1)\beta(s_2) + \\ &\frac{1}{2\pi\sqrt{-1}} \oint_{C_9} (\Phi(\omega) + \frac{\Delta(s_1)}{\beta(s_1)} - 1) \frac{z s_2 \beta(s_2)\beta(s_1+\omega)\alpha(-\omega)}{\omega(s_2+\omega)} \\ &\exp\left(\frac{1}{2\pi\sqrt{-1}} \oint_{C_9} \log(\frac{x+s_2}{x-\omega})\, d\log(1 - z\alpha(-x)\beta(x))\right) d\omega, \end{aligned}$$

[3]For the general GI/G/1

$$\begin{aligned} R(z) &= \frac{\mathrm{E}[T]\,\mathrm{E}[X-T]\, z}{\mathrm{Var}[D](1-z)} \\ &+ \frac{1}{2\pi\sqrt{-1}} \oint_{C_9} \left(\frac{\mathrm{E}[X-T]}{\omega^2\,\mathrm{Var}[D]} + \frac{\Phi(\omega)}{\omega^2\,\mathrm{Var}[D]} \frac{d\log(\beta(\omega))}{d\omega}\right) \\ &\exp\left(\frac{1}{2\pi\sqrt{-1}} \oint_{C_9} \log(\frac{x}{x-\omega})\, d\log(1 - z\alpha(-x)\beta(x))\right) d\omega \end{aligned}$$

where the contour C_9 contains the positive half complex plane only.

where the contour C_8 contains $x_i^+(z)$ $(i = 1, \ldots, L)$, but it does not contain 0 and the singularities of $\Phi(s)$ and $\beta(s)$.

Again for the $M/M/1$ we symbolically verified using Mathematica that the result is indeed correct. The final form is relatively simple in terms of the roots $x_i^+(z)$, but in order to find the autocorrelation function numerically, we need to differentiate several times with respect to z, which is a nontrivial task. To obtain numerical answers we experimented using the fast Fourier transform algorithm. As an accuracy check, we used the results from the previous subsection for adjacent interdeparture times.

In order to acquire some insight about the behavior of the departure process we next consider two simpler cases.

3.3 The departure process of a M/G/1 queue

Let $\lambda = \frac{1}{\mathrm{E}[T]}$ be the arrival rate, $\mu = \frac{1}{\mathrm{E}[X]}$ be the service rate, $\rho = \frac{\lambda}{\mu}$ be the traffic intensity. Then the interdeparture time transform pdf is:

$$\Delta(s) = \rho \frac{s+\mu}{s+\lambda} \beta(s).$$

If $r_n = \text{Autocorrelation}(D_\infty, D_{\infty+n}) = \frac{\mathrm{E}[D_\infty D_{\infty+n}] - \mathrm{E}[D]^2}{\mathrm{Var}[D]}$, $R(z) = \sum_{n=1}^{\infty} z^n r_n$ and $y(z) = \beta(x_1^+(z))$, we evaluate the complex integral (16) using the Pollaczek-Khintchine formula for $\Phi(s)$:

$$R(z) = \frac{1-\rho}{1-(1-\mathrm{C}_{\mathrm{X}}^2)\rho^2} \left(\frac{1}{1-zy(z)} - 1 + \frac{1}{1-z} \left(\frac{1}{y(z)} - \frac{1}{y(z) + z\frac{d}{dz}y(z)} - z \right) \right)$$

where $\mathrm{C}_{\mathrm{X}}^2$ is the square coefficient of variation of the service time distribution and

$$\begin{aligned} y(z) &= \beta(\lambda(1 - zy(z))) \\ &= \sum_{n=0}^{\infty} \frac{(-\lambda z)^n}{(n+1)!} \frac{d^n}{d\lambda^n} \beta(\lambda)^{n+1}. \end{aligned}$$

The last expression is obtained by applying the Lagrange implicit function formula. In particular, the autocorrelation for adjacent departures is

$$r_1 = r_{-1} = \frac{1-\rho}{1-(1-\mathrm{C}_{\mathrm{X}}^2)\rho^2} \left(\beta(\lambda) - 1 - \lambda \frac{d}{d\lambda} \log(\beta(\lambda)) \right)$$

Using Mathematica we computed for the M/D/1 the various autocorrelations.

$$\begin{aligned}
r_0 &= 1 \\
r_1 = r_{-1} &= \frac{1}{1+\rho}e^{-\rho} - \frac{1-\rho}{1+\rho} \\
r_2 = r_{-2} &= e^{-2\rho} - \frac{1-\rho}{1+\rho} \\
r_3 = r_{-3} &= \frac{2+4\rho+3\rho^2}{2(1+\rho)}e^{-3\rho} - \frac{1-\rho}{1+\rho} \\
r_4 = r_{-4} &= \frac{3+9\rho+12\rho^2+8\rho^3}{3(1+\rho)}e^{-4\rho} - \frac{1-\rho}{1+\rho} \\
r_5 = r_{-5} &= \frac{24+96\rho+180\rho^2+200\rho^3+125\rho^4}{24(1+\rho)}e^{-5\rho} - \frac{1-\rho}{1+\rho} \\
r_n = r_{-n} &= \sum_{k=0}^{n-1}\left(1-\frac{k}{n}\right)\frac{(n\rho)^k}{k!(1+\rho)}e^{-n\rho} - \frac{1-\rho}{1+\rho}.
\end{aligned}$$

A proof of the last expression is as follows:

$$R(z) = \frac{1}{1+\rho}\left(\frac{zy(z)}{1-zy(z)} + \frac{1}{1-z}\left(\frac{z\frac{d}{dz}y(z)}{y(z)\left(y(z)+z\frac{d}{dz}y(z)\right)} - z\right)\right).$$

Since

$$y(z) = e^{-\rho(1-zy(z))} \Rightarrow \frac{d}{dz}y(z) = \rho y(z)\left(y(z)+z\frac{d}{dz}y(z)\right).$$

Hence

$$\frac{z\frac{d}{dz}y(z)}{y(z)\left(y(z)+z\frac{d}{dz}y(z)\right)} = z\rho.$$

Now let

$$\begin{aligned}
\xi(z) &= \frac{zy(z)}{1-zy(z)} \\
&= z(1+\xi(z))e^{-\frac{\rho}{1+\xi(z)}} \\
&= \sum_{n=1}^{\infty}\frac{z^n}{n!}\lim_{\xi\to 0}\frac{d^{n-1}}{d\xi^{n-1}}\left((1+\xi)e^{-\frac{\rho}{1+\xi}}\right)^n \\
&= \sum_{n=1}^{\infty}\frac{z^n}{n}e^{-n\rho}\frac{1}{2\pi\sqrt{-1}}\oint_{C_{10}}\left(\frac{1+\xi}{\xi}\right)^n e^{\frac{\xi}{1+\xi}n\rho}\,d\xi \\
&= \sum_{n=1}^{\infty}z^n\sum_{k=0}^{\infty}\frac{(n\rho)^k}{k!}e^{-n\rho}\frac{1}{2\pi\sqrt{-1}}\oint_{C_{10}}\frac{1}{n}\left(\frac{1+\xi}{\xi}\right)^{n-k}d\xi
\end{aligned}$$

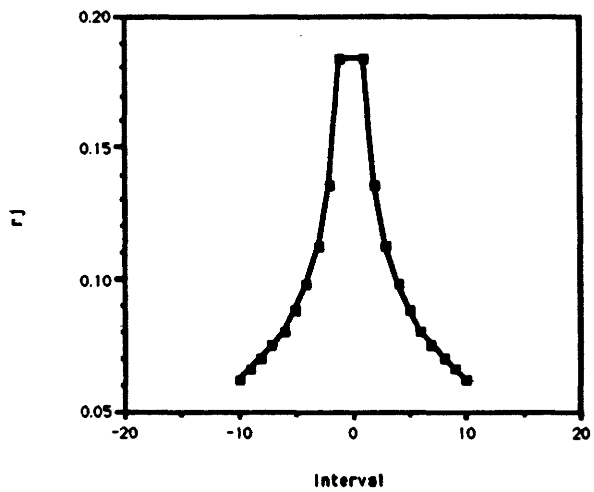

Figure 1: Autocorellations of the the departure process in a M/D/1 queue as $\rho \to 1$

$$= \sum_{n=1}^{\infty} z^n \sum_{k=0}^{n-1} \frac{(n\rho)^k}{k!} e^{-n\rho} \left(1 - \frac{k}{n}\right)$$

where the infinitesimal contour C_{10} contains 0 only. By expanding $R(z)$, we have the expression.

Clearly when $\rho \to 0$ (the light traffic limit) the autocorrelations vanish. Surprisingly, however, when the traffic intensity $\rho \to 1$ (the heavy traffic limit), the autocorrelations do not vanish. In figure 1 we present the autocorellation structure of the departure process of the M/D/1 queue as $\rho \to 1$.

3.4 The departure process of a GI/M/1 queue

Similar to the M/G/1 case, let

$$\begin{aligned} y(z) &= \alpha(-x_1^-(z)) \\ &= \alpha(\mu(1 - zy(z))) \\ &= \sum_{n=0}^{\infty} \frac{(-\mu z)^n}{(n+1)!} \frac{d^n}{d\mu^n} \alpha(\mu)^{n+1}. \end{aligned}$$

We obtain

$$\begin{aligned}
\Phi(s) &= \frac{(1-y(1))(s+\mu)}{s+\mu(1-y(1))} \\
\Delta(s) &= \frac{\mu}{s+\mu}\frac{y(1)s-\mu(1-y(1))\alpha(s)}{s-\mu(1-y(1))} \\
R(z) &= \frac{\rho(1-y(1))(y(1)-\rho)}{y(1)\{C_v^2(T)(1-y(1))-2\rho(y(1)-\rho)\}}\frac{zy(z)}{1-zy(z)}.
\end{aligned}$$

The last expression is obtained from the evaluation of the following alternative formula with $M=1$;

$$R(z)=\frac{\mathrm{E}[T]\,\mathrm{E}[X-T]\,z}{\mathrm{Var}[D](1-z)}$$
$$-\frac{1}{2\pi\sqrt{-1}}\oint_{C_{11}}\frac{\beta_D(0)(1-z)}{\omega^2(\beta_D(\omega)-z\beta_N(\omega)\alpha(-\omega))}\prod_{j=1}^{M}\left(\frac{x_j^-(z)-\omega}{x_j^-(z)}\right)\left(\frac{\mathrm{E}[X-T]}{\mathrm{Var}[D]}+\frac{\Phi(\omega)}{\mathrm{Var}[D]}\frac{d\log(\beta(\omega))}{d\omega}\right)d\omega$$

where the contour C_{11} contains 0, all the singularities of $\Phi(s)$ and $\beta(s)$.

For D/M/1 system, contrary to M/D/1, we found that when the traffic intensity $\rho\to 1$ (the heavy traffic limit) the autocorrelations vanish, but when $\rho\to 0$ (the light traffic limit) the autocorrelation between adjacent departures does not vanish.

$$\begin{aligned}
\lim_{\rho\to 0}R(z) &= -\frac{z}{2} \\
\lim_{\rho\to 1}R(z) &= 0
\end{aligned}$$

To see this, we first compute

$$\begin{aligned}
R(z) &= -\frac{zy(z)}{2y(1)}\frac{1-y(1)}{1-zy(z)} \\
y(z) &= e^{-\frac{1}{\rho}+\frac{zy(z)}{\rho}} \\
\mathrm{E}[W] &= \frac{1}{\mu}\frac{y(1)}{1-y(1)}
\end{aligned}$$

Since $\lim_{\rho\to 1}\mathrm{E}[W]=\infty$, we have $\lim_{\rho\to 1}y(1)=1$, which immediately gives

$$\lim_{\rho\to 1}R(z)=0.$$

Using the Lagrange implicit function formula, we obtain

$$R(z)=-\frac{z\sum_{n=0}^{\infty}\frac{(n+1)^{n-1}}{n!}\left(\frac{z}{\rho}e^{-\frac{1}{\rho}}\right)^n}{2\sum_{n=0}^{\infty}\frac{(n+1)^{n-1}}{n!}\left(\frac{1}{\rho}e^{-\frac{1}{\rho}}\right)^n}\,\frac{1-e^{-\frac{1}{\rho}}\sum_{n=0}^{\infty}\frac{(n+1)^{n-1}}{n!}\left(\frac{1}{\rho}e^{-\frac{1}{\rho}}\right)^n}{1-e^{-\frac{1}{\rho}}\sum_{n=0}^{\infty}\frac{(n+1)^{n-1}}{n!}\left(\frac{z}{\rho}e^{-\frac{1}{\rho}}\right)^n}.$$

Since $\lim_{\rho\to 0}\frac{1}{\rho}e^{-\frac{1}{\rho}} = 0$, and $\lim_{\rho\to 0} e^{-\frac{1}{\rho}} = 0$, we have

$$\lim_{\rho\to 0} R(z) = -\frac{z}{2}.$$

The results for both M/D/1 and D/M/1 can be explained intuitively if we observe that when the interdeparture time approaches exponentiallity, i.e. in the light traffic limit for M/D/1 or in the heavy traffic limit for D/M/1, the correlations disappear, and the departure process approaches a Poisson process.

4 On the departure point process

Bertsimas and Nakazato [1] have found a distributional Little's law for queueing systems under FCFS. If the input process to an arbitrary queueing system is a stationary point process, not necessarily a renewal process, and $N_a^*(t)$ is the number of arrivals up to time t, given that the first interarrival time is distributed as the forward recurrence time of the interarrival stationary distribution, then under FCFS the z-transform $G_L(z) = E[z^L]$ of the queue length distribution and the waiting time cdf $F_W(t)$ in steady-state are related as follows:

$$G_L(z) = \int_0^\infty K(t,z)dF_W(t),$$

where $K(z,t) = \sum_{n=0}^{\infty} z^n Pr\{N_a^*(t) = n\}$ is the equilibrium counting process of the arrival process.

Motivated by this distributional Little's law we will study in this section the equilibrium counting process for the departure process. The reason we are interested in the equilibrium counting process for the departure process is that in the next section we will study queues in tandem, where the departure process of one queue is the input process for the next queue. Since our analysis of tandem queues only considers waiting times under FCFS, using our distributional law we will be able to compute queue lengths provided we have a method to calculate the equilibrium counting process for the departure process.

In order to find the equilibrium counting process of the departure process we will first need to compute the joint distribution of $\{\sum_{r=0}^{k-1} D_{\infty+r}, D_{\infty+k}\}$. The analysis in this section is almost identical to the case of the autocorrelation function in the previous section, but slightly more complicated.

We first write the equations of the dynamics:

$$\sum_{r=0}^{k-1} D_{n+r} = \begin{cases} \sum_{r=0}^{k-2} D_{n+r} + X_{n+k-1} + I_{n+k-1} & (W_{n+k-1} > 0) \\ \sum_{r=0}^{k-2} D_{n+r} + X_{n+k-1} & (I_{n+k-1} > 0) \end{cases}$$

$$W_{n+k} - I_{n+k} = \begin{cases} W_{n+k-1} + X_{n+k-1} - T_{n+k} & (W_{n+k-1} > 0) \\ X_{n+k-1} - T_{n+k} & (I_{n+k-1} > 0) \end{cases}$$

We introduce the generating functions in the transform region

$\Phi(z, s_1, s_2) = \sum_{k=0}^{\infty} z^k \, \mathrm{E}[e^{-s_1 \sum_{r=0}^{k-1} D_{\infty+r} - s_2 W_{\infty+k}}]$.

$\Psi(z, s_1, s_2) = \sum_{k=0}^{\infty} z^k \, \mathrm{E}[e^{-s_1 \sum_{r=0}^{k-1} D_{\infty+r} - s_2 I_{\infty+k}}]$.

$\Delta(z, s_1, s_2) = \sum_{k=0}^{\infty} z^k \, \mathrm{E}[e^{-s_1 \sum_{r=0}^{k-1} D_{\infty+r} - s_2 D_{\infty+k}}] = \beta(s_2)\Psi(z, s_1, s_2)$.

$\Delta(z, s) = \Delta(z, s, 0) = \Phi(z, s, 0) = \Psi(z, s, 0) = 1 + z\beta(s)\Psi(z, s, s)$.

In terms of generating functions the equations for the dynamics become

$$\begin{aligned} &\Phi(z, s_1, s_2) - \Phi(s_2) + \Psi(z, s_1, -s_2) - \Psi(-s_2) - \Delta(z, s_1) + 1 \\ &= z\alpha(-s_2)\beta(s_1 + s_2)(\Phi(z, s_1, s_2) + \Psi(z, s_1, s_1) - \Delta(z, s_1)), \end{aligned} \tag{17}$$

where $\Phi(s), \Psi(s)$ were defined in section 2 as the transforms of the waiting time and idle time distribution respectively.

We will solve (17) for the R/R/1 case using again the Green's function method. An additional trick we need is that the Hilbert factorization for the homogeneous part of (17) must be performed between $\Phi(z, s_1, s_2) + \Psi(z, s_1, s_1) - \Delta(z, s_1)$ and $\Psi(z, s_1, -s_2) - \Psi(z, s_1, s_1) - \Psi(-s_2) + \Psi(s_1)$, because of the presence of the additional unknown terms $\Psi(z, s_1, s_1), \Delta(z, s_1)$. The derivation then proceeds along the lines of the previous section, so we omit the details.

Let $x_i^+(z, s)$, $x_i^-(z, s)$ be the roots of the equation

$$z\alpha(-x)\beta(s + x) = 1,$$

such that $\mathrm{Re}(x_i^+(z,s)) > 0$, $(i = 1, \ldots, L)$ and $\mathrm{Re}(x_j^-(z,s)) < 0$, $(j = 1, \ldots, M)$.

Then using similar arguments as in the previous section we find[4]

$$\begin{aligned}
\Delta(z,s) &= 1+ \\
\frac{1}{2\pi\sqrt{-1}} \oint_{C_{12}} & \frac{zs\beta(s)\alpha_D(-\omega)}{\omega(s+\omega)(1-z\beta(s))\alpha_D(0)} \prod_{i=1}^{L} \left(\frac{x_i^+(z,s)}{x_i^+(z,s)-\omega}\right) (\Phi(\omega) + \frac{\Delta(s)}{\beta(s)} - 1)\, d\omega
\end{aligned}$$

$$\begin{aligned}
\Delta(z,s_1,s_2) &= \Delta(s_2) + \\
\frac{1}{2\pi\sqrt{-1}} \oint_{C_{13}} & \left(\frac{s_1\beta(s_2)\alpha_D(-\omega)}{\omega(s_1+\omega)(1-z\beta(s_1))\alpha_D(0)} \prod_{i=1}^{L} \left(\frac{x_i^+(z,s_1)}{x_i^+(z,s_1)-\omega}\right)\right. \\
&+ \left.\frac{(s_2-s_1)\beta(s_2)\alpha_D(-\omega)}{(s_1+\omega)(s_2+\omega)\alpha_D(s_2)} \prod_{i=1}^{L} \left(\frac{x_i^+(z,s_1)+s_2}{x_i^+(z,s_1)-\omega}\right)\right) (\Phi(\omega) + \frac{\Delta(s_1)}{\beta(s_1)} - 1)\, d\omega
\end{aligned}$$

where both contours C_{12} and C_{13} contain 0 and $x_i^+(z,s_1)$ $(i = 1, \ldots, L)$, but they do not contain any singularity of $\Phi(s)$. In addition, contour C_{12} excludes $-s$, and C_{13} excludes s_1 and $-s_2$.

We should note, however, that inverting the above expressions is a nontrivial matter. As an accuracy check of the algebra, we have tested using Mathematica that the formulas for the M/M/1 queue are indeed correct.

---

[4]For the general GI/G/1 queue,

$$\begin{aligned}
\Delta(z,s) &= 1+ \\
\frac{1}{2\pi\sqrt{-1}} \oint_{C_{14}} & (\Phi(\omega) + \frac{\Delta(s)}{\beta(s)} - 1) \frac{zs\beta(s)}{\omega(s+\omega)(1-z\beta(s))} \\
& \exp\left(\frac{1}{2\pi\sqrt{-1}} \oint_{C_{15}} \log(\frac{x}{x-\omega})\, d\log(1 - z\alpha(-x)\beta(s+x))\right) d\omega
\end{aligned}$$

$$\begin{aligned}
\Delta(z,s_1,s_2) &= \Delta(s_2) + \\
\frac{1}{2\pi\sqrt{-1}} \oint_{C_{14}} & (\Phi(\omega) + \frac{\Delta(s_1)}{\beta(s_1)} - 1) \\
& \left(\frac{s_1\beta(s_2)}{\omega(s_1+\omega)(1-z\beta(s_1))} \exp\left(\frac{1}{2\pi\sqrt{-1}} \oint_{C_{15}} \log(\frac{x}{x-\omega})\, d\log(1 - z\alpha(-x)\beta(s_1+x))\right)\right. \\
&+ \left.\frac{(s_2-s_1)\beta(s_2)}{(s_1+\omega)(s_2+\omega)} \exp\left(\frac{1}{2\pi\sqrt{-1}} \oint_{C_{15}} \log(\frac{x+s_2}{x-\omega})\, d\log(1 - z\alpha(-x)\beta(s_1+x))\right)\right) d\omega
\end{aligned}$$

where the contour C_{14} contains only the nonnegative half complex plane, and the contour C_{15} contains the positive half complex plane.

We are now ready to compute the backward equilibrium counting process of the departure process

$$K(t,z) = \Pr[D^*_\infty > t] + \sum_{n=1}^{\infty} z^n \left(\Pr[D^*_{\infty+n-1} + \sum_{r=0}^{n-2} D_{\infty+r} \le t] - \Pr[D^*_{\infty+n} + \sum_{r=0}^{n-1} D_{\infty+r} \le t] \right),$$

where $D^*_n =$ is the time elapsed from the $(n-1)$th departure until a random epoch, which occurs before the nth departure.

$K(t,z)$ can be found provided the inverse of $\Delta(z, s_1, s_2)$ is given. If we define $\varphi_n(t_1,t_2)$ as the inverse transform of

$$\Delta(z,s_1,s_2) = \sum_{n=0}^{\infty} \int_0^\infty \int_0^\infty z^n e^{-s_1 t_1 - s_2 t_2} \varphi_n(t_1,t_2)\, dt_1\, dt_2$$

then the generating function for the backward equilibrium counting process is

$$K(t,z) = 1 - (1-z) \times \sum_{n=1}^{\infty} z^n \int_0^t \left(\frac{\left(\int_0^\tau \int_0^\infty \varphi_n(t_1,t_2)\, dt_2\, dt_1\right)^2}{\int_0^\tau \int_0^\infty t_2 \varphi_n(t_1,t_2)\, dt_2\, dt_1} - \int_0^\tau \frac{\varphi_n(\tau_0, \tau-\tau_0) \int_0^{\tau_0} \int_0^\infty \varphi_n(t_1,t_2)\, dt_2\, dt_1}{\int_0^{\tau_0} \int_0^\infty t_2 \varphi_n(t_1,t_2)\, dt_2\, dt_1}\, d\tau_0 \right) d\tau,$$

since

$$\Pr[\sum_{r=0}^{k-1} D_{\infty+r} \le \tau_1, D^*_{\infty+k} \le \tau_2] = \frac{\int_0^{\tau_1} \int_0^\infty \varphi_k(t_1,t_2)\, dt_2\, dt_1}{\int_0^{\tau_1} \int_0^\infty t_2 \varphi_k(t_1,t_2)\, dt_2\, dt_1} \int_0^{\tau_1} \int_0^{\tau_2} \int_\tau^\infty \varphi_k(t_1,t_2)\, dt_2\, d\tau\, dt_1.$$

Having characterized the departure process of a GI/G/1 queue we apply in the next section our results to the analysis of tandem queues.

5 Formulation of the tandem queue problem

Consider a queueing system that consists of Q single server queues in tandem. Our goal is to characterize the waiting time distribution in each of the queues. Using our distributional Little's law and our characterization of the equilibrium counting process of the departure process we will be able to compute the queue length distribution as well. Arrivals come to the system only through the first queue with a general renewal process, while the service time at each queue follows a general distribution. In order to model the system we define

T_n = the interarrival time between the $(n-1)$th customer and the nth customer at the first queue.

$X_{i,n}$ = the service time of the nth customer at the ith queue.

$W_{i,n}$ = the waiting time of the nth customer in the ith queue.

$I_{i,n}$ = the idle period of the ith server prior to the arrival of the nth customer to the ith queue.

$D_{i,n}$ = the interdeparture time between the $(n-1)$th customer and the nth customer from the ith queue.

The corresponding random variables in steady state are denoted by T, X_i, W_i, I_i and D_i, i.e. dropping n in the previous definitions. Let L_i denote the time average queue length at the ith queue. Let us also define the steady state transforms:

$\alpha(s) = \mathrm{E}[e^{-sT}]$,

$\beta_i(s) = \mathrm{E}[e^{-sX_i}]$,

$\Phi_i(s) = \mathrm{E}[e^{-sW_i}]$,

$G_i(z) = \mathrm{E}[z^{L_i}]$,

$\Psi_i(s) = \mathrm{E}[e^{-sI_i}]$,

$\Delta_i(s) = \mathrm{E}[e^{-sD_i}]$.

The analysis problem for the tandem queue problem can now be stated as follows: Given the transforms $\alpha(s)$, $\beta_i(s)$, such that $\rho_i = \frac{\mathrm{E}[X_i]}{\mathrm{E}[T]} < 1$, find the transforms $\Phi_i(s)$, $G_i(z)$, $\Psi_i(s)$ and $\Delta_i(s)$.

5.1 System Dynamics

The dynamics of the system can be described as follows:

$$\begin{cases} W_{i,n} - I_{i,n} = W_{i,n-1} + X_{i,n-1} - D_{i-1,n} & (i = 1 \ldots Q) \\ D_{i,n} = I_{i,n} + X_{i,n} & (1 \leq i \leq Q) \\ D_{0,n} = T_n & \end{cases}$$

Observe that $X_{i,n-1}$ is independent from both $W_{i,n-1}$ and $D_{i-1,n}$. The random variables $W_{i,n-1}$ and $D_{i-1,n}$, however, for finite n are not independent. It is not clear what happens in the limit as n goes to infinity. It is this dependency structure that complicates the problem considerably. We were unable to solve the problem exactly. In the next section, however, we make the simplifying assumption that in steady-state W_i and D_{i-1} are indeed independent. As a result, with the simplifying assumption, we are able to solve the problem exactly.

6 The approximation

In this section we propose an algorithm based on the following independence assumption:

Assumption A

The random variables W_i and D_{i-1} are independent.

For a series of $M/M/1$ queues this assumption is accurate, since the input process to each queue is Poisson, i.e. all the D_i's are exponential independent random variables. In the general case, if the departure process were a renewal process, the assumption would also be accurate. As a result, our method can also be seen as a renewal approximation, which approximates the interdeparture interval exactly, however. Moreover, unlike Whitt [12], we use information about the entire interdeparture distribution rather than its first two moments. In the next section we report numerical results, which are very close to simulation experiments.

Although this is not essential for the method, for tractability purposes we further restrict our attention to a series of $R/R/1$ queues, i.e. we assume that

- $\alpha(s) = \frac{\alpha_N(s)}{\alpha_D(s)}$, where $\alpha_D(s), \alpha_N(s)$ is a monic polynomial of degree d_0 and a polynomial of degree $\leq d_0$,
- $\beta_i(s) = \frac{\beta_{Ni}(s)}{\beta_{Di}(s)}$ where $\beta_{Di}(s), \beta_{Ni}(s)$ is a monic polynomial of degree M_i and a polynomial of degree $\leq M_i$.

Assumption A forces D_i to have rational Laplace transform, i.e. $\Delta_i(s) = \mathrm{E}[e^{-sD_i}] = \frac{\Delta_{Ni}(s)}{\Delta_{Di}(s)}$, with $\Delta_{Di}(s), \Delta_{Ni}(s)$ a monic polynomial of degree d_i and a polynomial of degree $\leq d_i$. The polynomials $\Delta_{Ni}(s), \Delta_{Di}(s)$ will be determined in the algorithm. For notational purposes we will denote $\Delta_0(s) = \alpha(s)$.

We now take transforms in the equation of the basic dynamics, take limits and use assumption A. We thus obtain

$$\begin{aligned} (1 - \Delta_{i-1}(-s)\beta_i(s))\Phi_i(s) &= 1 - \Psi_i(-s) \\ \Delta_i(s) &= \beta_i(s)\Psi_i(s) \\ \Delta_0(s) &= \alpha(s). \end{aligned}$$

Using the usual Hilbert factorization arguments we can solve these equations inductively from $i = 1$ to $i = Q$ as follows:

Tandem queue Algorithm

For $i = 1, \ldots, Q$

1. Solve the equation

$$\Delta_{i-1}(-x_i)\beta_i(x_i) = 1,$$

and let $x_{i,1}^+, \ldots, x_{i,d_{i-1}-1}^+, 0, x_{i,1}^-, \ldots, x_{i,M_i}^-$ denote the $d_{i-1} + M_i$ roots.

2. Solutions for the i queue:

$$\begin{aligned} \Phi_i(s) &= \frac{\beta_{Di}(s)}{\beta_{Di}(0)} \prod_{j=1}^{M_i} \frac{x_{i,j}^-}{x_{i,j}^- - s} \\ G_i(z) &= 1 + \lambda \sum_{j=1}^{M_i} \frac{(1-z)(1-\Delta_{i-1}(x_{i,j}^-))}{x_{i,j}^-(1 - z\Delta_{i-1}(x_{i,j}^-))} \prod_{\substack{k=1 \\ k \neq j}}^{M_i} \frac{x_{i,k}^-}{x_{i,k}^- - x_{i,j}^-}, \end{aligned}$$

where the last formula is from the distributional Little's law.

3. Compute the interdeparture transform pdf:

$$\Delta_i(s) = \beta_i(s)\left(1 - s\frac{1-\rho_i}{\lambda}\frac{\Delta_{D_{i-1}}(0)}{\Delta_{D_{i-1}}(s)}\prod_{j=1}^{d_{i-1}-1}\frac{x_{j,i}^{+}+s}{x_{j,i}^{+}}\right).$$

The bottleneck part in the above algorithm is step 1. Clearly $d_i = M_i + d_{i-1}$ and thus $d_i = d_0 + \sum_{k=1}^{i} M_k$, where d_0 is the order of the interarrival time pdf. In order to solve for the ith queue we need to solve a polynomial equation with d_i roots. As a result, we need to find in total $\sum_{i=1}^{Q} d_i = d_0Q + \sum_{r=1}^{Q}(Q+1-r)M_r$ roots.

In the next section we examine the performance of the algorithm compared to simulation results.

7 The implementation and performance of the algorithm

We programmed the algorithm using Mathematica. The main reason for choosing Mathematica is the availability of built-in very accurate root finding techniques for polynomials, and symbolic manipulation operations. The algorithm takes as input the transforms of the interarrival distribution at the first queue and the transforms of the service time transforms. Alternatively, one can give as input the first moments and the coefficient of variation of these random variables. The algorithm then automatically fits distributions with rational Laplace transforms. The algorithm computes the expectation and the coefficient of variation of the waiting time, the queue length and the departure process. It also computes the autocorrelations for the departure process, which is an indication if our approximation is accurate, as well as the waiting time and interdeparture distribution by inverting numerically the Laplace transforms. For the numerical inversion of the Laplace transforms we used the algorithm by Hosono [6].

We ran several examples. As a general rule we found that the correlations of the departure process were very low, in the neighborhood of 0.01. This means that we expect that the departure processes are almost renewal and thus assumption A is accurate. In the cases where simulation results were available from the literature we found that the algorithm was quite accurate. In comparison with the QNA package of Whitt [10] we found that our algorithm gave answers close to those obtained by QNA. We present below three examples.

Example 1 (Whitt [11])

There are two queues in tandem. The characteristics of each queue are:

$E[T] = 1, C_T^2 = 1, E[X_1] = 0.6, C_{X_1}^2 = 2, E[X_2] = 0.8, C_{X_2}^2 = 4,$

i.e. $\rho_1 = 0.6$ and $\rho_2 = 0.8$.

Running our algorithm we found for the first queue

$E[W_1] = 1.35, C_{W_1}^2 = 2.54526, E[L_1] = 1.95, C_{L_1}^2 = 1.92209,$

$E[D_1] = 1, C_{D_1}^2 = 1.36,$

and for the second queue

$E[W_2] = 8.69681, C_{W_2}^2 = 1.57799, E[L_2] = 9.49681, C_{L_2}^2 = 1.49431,$

$E[D_2] = 1, C_{D_2}^2 = 3.00128.$

The simulation results of Whitt [11] were $E[W_1] = 1.360, E[W_2] = 8.131$. In order to understand the behavior of the algorithm we calculated the correlation structure of the departure process. We found that $r_0 = 1$, $r_1 = -0.0118$, $r_2 = -0.0129$, $r_3 = -0.0118$, i.e. the input process to the second queue is almost renewal.

Example 2 (Whitt [11])

There are also two queues in tandem. The characteristics of each queue are:

$E[T] = 1, C_T^2 = 1, E[X_1] = 0.3, C_{X_1}^2 = 0.5, E[X_2] = 0.2, C_{X_2}^2 = 0.5,$

i.e. $\rho_1 = 0.3$ and $\rho_2 = 0.2$.

Running our algorithm we found for the first queue

$E[W_1] = 0.0964286, C_{W_1}^2 = 2.54526, E[L_1] = 0.396429, C_{L_1}^2 = 3.11346,$

$E[D_1] = 1, C_{D_1}^2 = 0.955,$

and for the second queue

$E[W_2] = 0.0264537, C_{W_2}^2 = 10.746, E[L_2] = 0.226454, C_{L_2}^2 = 1.49431,$

$E[D_2] = 1, C^2_{D_2} = 0.952674.$

The simulation results of Whitt [11] were $E[W_1] = 0.096, E[W_2] = 0.026$. In this case the correlation structure of the departure process is as follows:

$r_0 = 1,\ r_1 = 0.0124,\ r_2 = 0.0055,\ r_3 = 0.0026.$

In both examples the results of the algorithm are close to the ones obtained by simulation. An interesting remark can be made about the relative accuracy of the method in the two examples. In the second example, where the results of the simulation and our method are identical, the correlations are very close to 0, while in example 1 the correlations are higher and therefore the results of the simulation and our method differ slightly.

Example 3

As it is quite difficult to find reliable simulation results in the literature we present a random example so that researchers can compare these results with simulation results and with their methods. This example consists of four queues in tandem, with $E[T] = 1, C^2_T = 0.75$ and $E[X_1] = 0.5, C^2_{X_1} = 0.5, \rho_1 = 0.5,$

$E[X_2] = 0.6, C^2_{X_2} = 0.6, \rho_2 = 0.6,$

$E[X_3] = 0.7, C^2_{X_3} = 0.7, \rho_3 = 0.7,$

$E[X_4] = 0.8, C^2_{X_4} = 0.8, \rho_4 = 0.8.$

We found that $E[W_1] = 0.292606, E[W_2] = 0.53196, E[W_3] = 1.09019, E[W_4] = 2.42284.$

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