

XX. ANALOG COMPUTER RESEARCH

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A. INTRODUCTION

The Analog Computer group is engaged in a number of basic investigations in the field of analog and special-purpose computers. The emphasis has been on the underlying concepts and the introduction of new components rather than on the design, construction, or operation of a large computing facility.

We are primarily concerned with analog computers, rather than digital computers, because more basic research seems to be needed in the analog field. There is great need for an increase in the speed and the accuracy of analog equipment.

While some combined analog-digital equipment has been constructed, the emphasis is on special-purpose computers. The underlying philosophy of special-purpose computers is that they are cheap enough and simple enough to be used by the person who has the problem to be solved. They do not interpose the extra link of a large, trained operating staff between the problem and its ultimate solution.

Since a great many of the special-purpose computers that have been built are directly or indirectly related to network theory, our group has maintained an active interest in network theory and has done some work in this field.

At the present time the research falls into three groups: (a) those problems relating to computer systems and the basic philosophy of computers, (b) the design of new computing elements, and (c) applications of modern network theory.

R. E. Scott

B. COMPUTER SYSTEMS

1. Energy and Power in Nonlinear Systems

A basic investigation is being conducted into energy and power relations in nonlinear systems, such as those involved in computers. This investigation begins with a transitional stability study of second-order systems.

In the design of electromechanical multipliers and function generators, various schemes have been employed to extend their dynamic range. Among other things, saturation at high-power level represents a limitation to such efforts. This limitation is similarly encountered in the related field of feedback control systems. The electromechanical device, which is stable in the linear sense, may exhibit relative instability when saturation occurs. Such operation involves the transition between linear and nonlinear modes. A phase-plane study has been made of the transitional stability of two second-order electromechanical systems (Figs. XX-1, XX-2) that are frequently used to form multipliers and function generators.

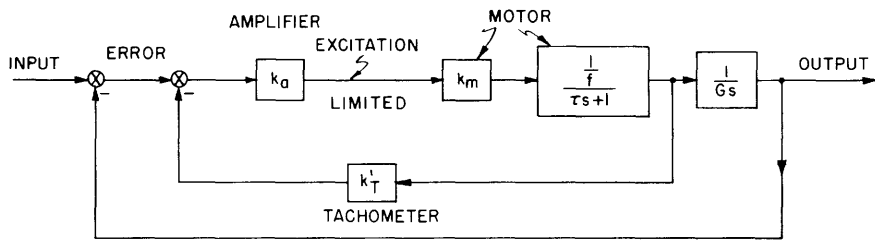


Fig. XX-1

Excitation-limited motor

$$k_T = 1 + \frac{k_a k_m k'_T}{f} = 1 + \text{tachometer loop gain.}$$

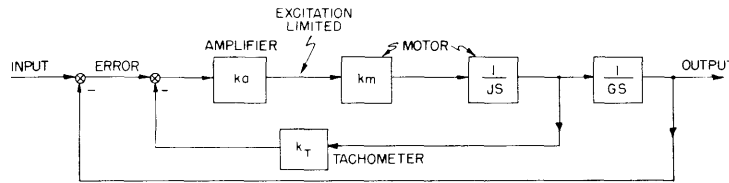


Fig. XX-2

Torque and rate-limited motor

$$\tau = \frac{J}{k_a k_m k_T}$$

$$a_m = \text{maximum acceleration} = \frac{\text{maximum torque}}{\text{inertia}}$$

$$\omega_m = \text{maximum velocity.}$$

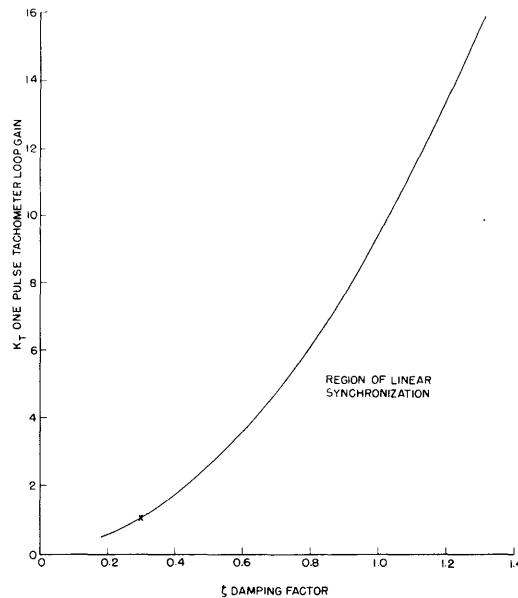


Fig. XX-3

Transitional behavior of the system shown in Fig. XX-1.

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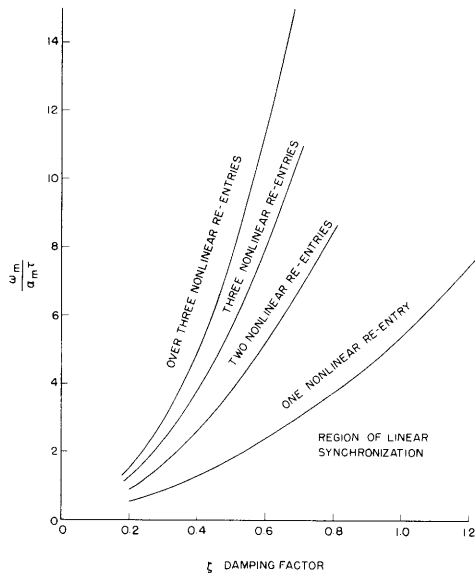


Fig. XX-4

Transitional behavior of the system shown in Fig. XX-2.

The result for the system of Fig. XX-1 is shown in Fig. XX-3. The system is said to be linearly synchronized if the response to a step input of very large amplitude remains linear once it has left the saturated region. The result obtained by W. Hurewicz and N. B. Nichols (Servos with Torque Saturation: Part 1, Rad. Lab. Report 555, May 1944) is indicated by a cross on the curve. The result for the system of Fig. XX-2 is shown in Fig. XX-4. A formula for the approximate number of nonlinear entries when the number n is large, is given by

$$n < \frac{1}{8\xi^2} \frac{\omega_m}{a_m \tau} < n + 1$$

F. F. Lee

2. Investigations and Applications of the Theory of Signal Flow Graphs to Analog Computers

An attempt has been made to indicate to what extent computer component errors (especially of a dynamic nature) affect the form of the equations of the original physical problem being solved. Instead of expressing the resultant computer errors as actual functions of the independent variable, the errors are expressed as modifications of the original differential equations.

Thus far, we have determined the theoretical limit on the number of dynamic components necessary to solve a system of simultaneous linear constant coefficient equations

on a computer. A systematic method of proceeding from the original equations to the final optimum computer connection is in the process of being completed.

Given a set in the form

$$\begin{array}{r} L_{11}x_1 + L_{12}x_2 + \dots + L_{1n}x_n = S_1 \\ \cdot \qquad \qquad \qquad \cdot \\ \cdot \qquad \qquad \qquad \cdot \\ \cdot \qquad \qquad \qquad \cdot \\ L_{n1}x_1 + L_{n2}x_2 + \dots + L_{nn}x_n = S_n \end{array}$$

where L_{pq} is a linear operator (that is, a polynomial in s), and S_q is a known function of the independent variable, a signal flow graph is drawn representing the relations as they are written above, that is, with the unknown functions as sources and the known functions as sinks. The graph then consists of open paths, each path being a single branch. To make this system physically realizable on a computer we must invert at least n paths. The manner in which these paths are to be chosen depends largely upon the characteristics of the computer being used. We shall assume that we are dealing with a computer containing only integrators as dynamic elements. To utilize the inversions to the best advantage, only n paths of inversion will be chosen. These paths intercept all $2n$ nodes, that is, no two paths of inversion will intercept the same node. They are also chosen so that the product of their branch transmissions is as large as possible, to reduce the order of each individual branch operator. If paths of inversion other than those specified above are used, the resultant flow graph will present difficulties in physical realization. However, these difficulties can usually be overcome by performing appropriate loop inversions until the configuration reverts to the one that would have resulted from the correct initial choice of open path inversions.

The theoretical minimum number of integrators necessary to solve a given set of equations is equal to the order of the determinant of the operational coefficients. This is always either equal to, or less than, the sum of the orders of the highest derivatives of all the dependent variables. Therefore, the next step in the procedure is to determine this minimum and compare it with the number of integrations present in the inverted flow graph. If they are equal, no further advantage in this direction may be obtained by more modifications. If they are not equal, the number of integrators can usually be reduced to the minimum by such manipulations as the introduction of additional nodes, and the like. These further modifications, which do not change the fundamental relations between the variables, may also be necessary in certain cases for reducing the order of some branch transmissions that may still not be realizable on the computer even after the best possible combination of branch inversions. These modifications should also be used to make as many of the transmissions negative as possible, since electronic computer components always have an inherent phase inversion.

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This method has been tried on several examples and seems to have some advantages over the conventional methods of setting up the computer.

Additional investigation is to be performed on modification of the flow-graph configurations to reduce the sensitivity of the solutions to errors in branch transmissions.

T. E. Stern

C. COMPUTER ELEMENTS

1. Square-Law Network

Two networks, shown in Figs. XX-5 and XX-6, were constructed and tested during the past months. Initial tests showed that the network in Fig. XX-5 is superior to that of Fig. XX-6 in both stability and accuracy.

A push-pull stage was then constructed for the second network to enable it to accept both plus and minus signals. At present, the device can be represented by Fig. XX-7. This device uses four straight-line segments to approximate the square-law characteristics. It has about 1 percent static accuracy. With ac signals the accuracy drops to 10.5 percent due to an equal gain in the push-pull stages. Frequency response is unknown at the present time because many potentiometers have been used in the circuit to facilitate experimental adjustments and they decrease the upper frequency range. At present we need a more stable push-pull driving stage that has a maximum output of 3 ma into a 10-kilohm load and a better switching arrangement for the diodes in the network.

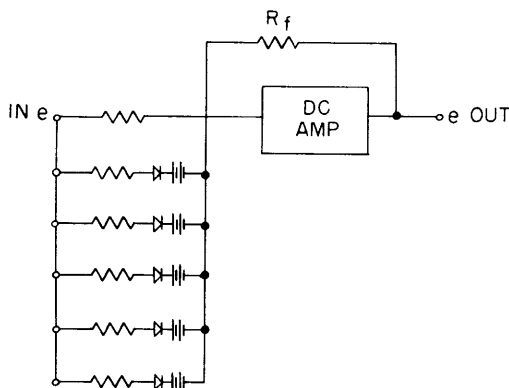


Fig. XX-5

Square-law network using active elements.

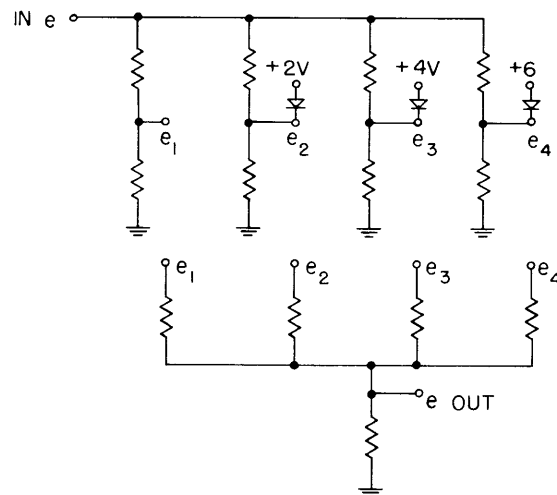


Fig. XX-6

Square-law network using passive elements only.

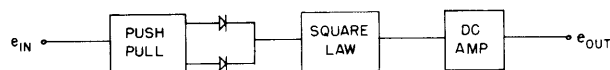


Fig. XX-7

Complete square-law device

 e_{in} = input voltage e_{out} = output voltage k = proportionality constant

$$e_{in}^2 = k e_{out} \quad 12 < k < 35.$$

The principal limitation of square-law devices with germanium diodes is the finite forward and backward resistance. It is safe to say that unless elaborate compensation is used an accuracy of 0.5 percent is the practical upper limit for a square-law device of this nature. Any attempt to go above the limit will sacrifice the simplicity of the network.

Y. C. Ho

2. Delay Lines

The delay lines and their associated equipment built by Dr. C. A. Stutt are being repaired and put into operation for experimental investigations of Dr. M. V. Cerrillo's theory of network synthesis in the time domain.

Y. C. Ho

D. APPLIED NETWORK THEORY

1. Potential Analogs

During the preparation of a technical report from the work on potential analogs the material has been completely revised. It is true that an equation such as Eq. 2 (Quarterly Progress Report, July 15, 1953, p. 101) is quite costly to deal with, and for this reason more examples have not been worked out. In the revised material the main ideas and results remain the same but they are presented in a more precise and satisfactory form.

M. S. Macrakis

2. RC Filter Frequency Transformations

In the techniques of modern network synthesis it is customary to satisfy the requirements for a desired filter on a normalized lowpass prototype and then, through a

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frequency transformation, to obtain the correct type of filter, highpass, or bandpass. It is the purpose of this study to investigate the applicability of frequency transformation in the case of a transfer function of RC character.

a. Lowpass to highpass transformation

Consider a transfer function $Z_{12}(s)$ which satisfies all the RC requirements

$$Z_{12}(s) = C \frac{\prod_{i=1}^n (s^2 + A_i s + B_i)}{\prod_{i=1}^m (s + a_i)} = C \frac{\prod_{i=1}^n (s + \beta_i)(s + \bar{\beta}_i)}{\prod_{i=1}^m (s + a_i)}$$

where $n = 2m$ as the upper limit and a_i is real and positive.

To make the lowpass-to-highpass transformation let $s \rightarrow k/s$. The new transfer function is

$$\begin{aligned} Z_{12}^*(s) &= Z_{12}\left(\frac{k}{s}\right) = \frac{\prod_{i=1}^n (k + \beta_i s)(k + \bar{\beta}_i s)}{\prod_{i=1}^m (k + a_i s)} C, \quad n = 2m \\ &= \frac{s^\rho \prod_{i=1}^n (k + \beta_i s)(k + \bar{\beta}_i s)}{\prod_{i=1}^m (k + a_i s)} C, \quad 2m = n + \rho \end{aligned}$$

or

$$Z_{12}^*(s) = \frac{s^\rho \prod_{i=1}^n (s + k/\beta_i)(s + k/\bar{\beta}_i)}{\prod_{i=1}^m (s + k/a_i)} C \frac{\prod_{i=1}^n |\beta_i|^2}{\prod_{i=1}^m a_i}$$

On examining this equation it is seen that the poles and zeros of the new function are inverted about a circle of radius k . (See Fig. XX-8.) Since the RC character of the transfer function is preserved, that is, all poles are simple and on the real axis, and $n \leq 2m$, the transformation is still applicable.

b. Lowpass to bandpass transformation

For this type of frequency transformation let

$$s \rightarrow k \left(\frac{s}{\omega_0} + \frac{\omega_0}{s} \right)$$

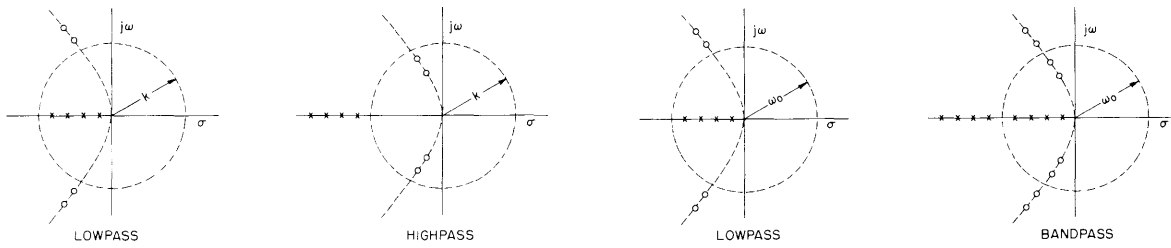


Fig. XX-8

Fig. XX-9

where ω_0 is the center frequency desired, and k is a bandwidth factor. The gain bandwidth product is constant.

We observe that the poles of the new transfer function

$$Z_{12}^{(B)}(s) = Z_{12} \left[k \left(\frac{s}{\omega_0} + \frac{\omega_0}{s} \right) \right]$$

will be solely determined by the transformation of the factor

$$(s + a_i) \rightarrow \frac{k}{\omega_0 s} \left(s^2 + a_i \frac{\omega_0}{k} s + \omega_0^2 \right)$$

However, since the degree of the numerator is smaller than or equal to the degree of the denominator it is sufficient to consider only the factor

$$\left(s^2 + a_i \frac{\omega_0}{k} s + \omega_0^2 \right) = (s + \gamma)(s + \delta)$$

The necessary condition for γ and δ to be positive is

$$\left. \begin{matrix} \gamma \\ \delta \end{matrix} \right\} = \frac{-a_i \omega_0}{2k} \pm \left(\frac{a_i^2 \omega_0^2}{4k^2} - \omega_0^2 \right)^{1/2}$$

$$\frac{a_i^2 \omega_0^2}{4k^2} > \omega_0^2 \quad \frac{a_i}{2k} > 1$$

or

$$k < \frac{a_i}{2}$$

Since k determines the bandwidth, it is clear that there is a low limit in the

$$Q = \frac{\text{bandwidth}}{\omega_0}$$

of the bandpass network. This limit is determined by the pole a_i which is closest to the

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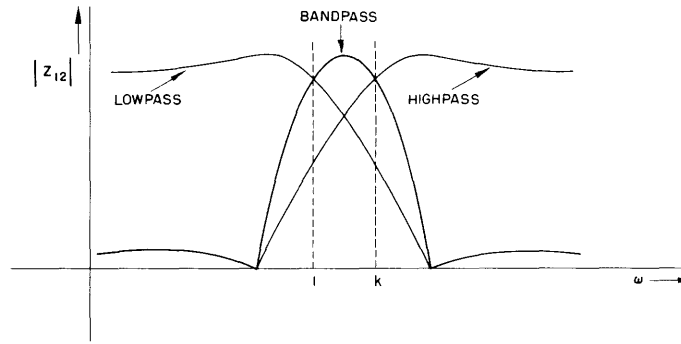


Fig. XX-10

origin in the lowpass structure. This constitutes the only restriction on the transformation from a lowpass RC structure to another bandpass. It is obvious that for

$$Z_{12}(0) = Z_{12}^{(B)} \left[k \left(\frac{s}{\omega} + \frac{\omega_0}{s} \right) \right]$$

$s = \pm j\omega_0$, therefore ω_0 is the center frequency and for

$$Z_{12}(\pm j) = Z_{12}^{(B)} \left[k \left(\frac{s}{\omega_0} + \frac{\omega_0}{s} \right) \right]$$

$$s = \pm j \left\{ \frac{\omega_0}{2k} \pm \left[\omega_0^2 - \left(\frac{\omega_0}{2k} \right)^2 \right]^{1/2} \right\}$$

Hence the bandwidth ΔW is

$$\Delta W = \frac{\omega_0}{k}$$

The minimum Q of the RC network therefore becomes

$$Q = \frac{\Delta W}{\omega_0} = \frac{1}{k} < \frac{a_{\min}}{2}$$

If the roots of the equation are examined it is seen

$$\gamma \delta = \left(\frac{a_i \omega_0}{2k} \right)^2 - \left(\frac{a_i \omega_0}{2k} \right)^2 + \omega_0^2$$

or

$$\gamma \cdot \delta = \omega_0^2$$

Therefore γ_i and δ_i constitute inverse pairs about the periphery of a circle from the

origin with radius ω_0 . (See Fig. XX-9.) From Fig. XX-9 the following relation

$$Z_{12}^{(B)}(s) = Z_{12}(s) Z_{12}^*(s)$$

is obvious for the case of $\omega_0 = k = 1$. This suggests an alternative procedure of superimposing the magnitude characteristic of a lowpass and a properly derived highpass filter as in Fig. XX-10.

N. DeClaris

3. Duality of Ideal Transformers

Given an electrical circuit containing R's, L's, C's, and sources, there is a technique that enables one to interchange voltage with current and junction points with meshes by merely topological considerations. The following theorem extends the application of this technique to circuits containing ideal transformers. The significance of the theorem is obvious, since one has to recognize the fact that ideal transformers are very frequently used in circuit analysis.

Theorem: The dual of an n-winding ideal transformer with N_k turns in each winding consists of n two-winding ideal transformers of $1:N_k$ turns ratio each, and with the secondary windings connected in series; the secondary voltages are taken in the same clockwise (or counterclockwise) direction.

Proof: Consider an ideal transformer with n-windings. (See Fig. XX-11.) Each of the windings has N_k number of turns. Let the voltage polarities and current flow directions be assumed to be those shown in Fig. XX-11. Since the system is lossless, the instantaneous power is zero.

$$\sum_{k=0}^n V_k I_k = 0 \quad (1)$$

Furthermore, by definition

$$\frac{V_1}{N_1} = \frac{V_2}{N_2} = \frac{V_k}{N_k} = \text{constant} \quad (2)$$

Divide Eq. 1 by Eq. 2.

$$\frac{N_k}{V_k} \sum_{k=0}^n V_k I_k = 0$$

Since

$$\frac{N_k}{V_k} = \text{constant}$$

$$\sum_{k=0}^n N_k I_k = 0 \quad (3)$$

It is obvious that Eqs. 2 and 3 describe the system completely.

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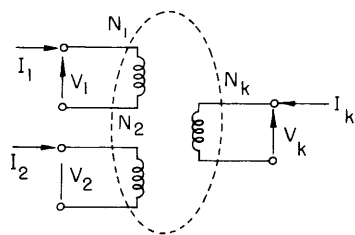


Fig. XX-11

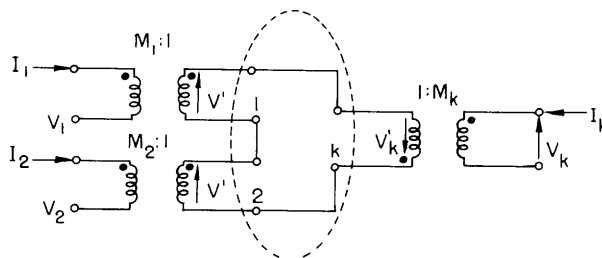


Fig. XX-12

In order to obtain this dual set of equations, it is sufficient to interchange I_k and V_k . Thus we have

$$\frac{I_1}{N_1} = \frac{I_2}{N_2} = \frac{I_k}{N_k} = \text{constant} \quad (4)$$

$$\sum_{k=0}^n N_k V_k = 0 \quad (5)$$

Obviously the power equation has not been altered

$$\sum_{k=0}^n V_k I_k = 0$$

Therefore the system defined by Eqs. 4 and 5 constitutes the dual of the circuit shown in Fig. XX-12. In order to recognize the system more easily, let

$$\frac{1}{N_k} = M_k$$

Equations 4 and 5 are rewritten as

$$M_1 I_1 = M_2 I_2 = M_k I_k = \text{constant} \quad (6)$$

$$\sum_{k=0}^n \frac{V_k}{M_k} = 0 \quad (7)$$

The circuit of Fig. XX-12 corresponds to Eqs. 6 and 7, as can be easily verified.

For a general orientation on this topic the reader is referred to E. A. Guillemin, *Communication Networks*, Vol. II, chap. V, John Wiley and Sons, Inc., New York, 1935 and E. C. Cherry, "The Duality Between Interlinked Electric and Magnetic Circuits and the Formation of Transformer Equivalent Circuits," *Proc. Phys. Soc. (London)* B 62, 101, 1949.

N. DeClaris