

## XII. MICROWAVE COMPONENTS

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### A. FERRITES AT MICROWAVE FREQUENCIES

In reference 1, variational principles for the resonant frequencies of a cavity or the cutoff frequency of a waveguide, both completely filled with ferrite, were given. Variational principles for the resonant frequencies of more complicated systems, as well as for the propagation constant of waveguides partially or completely filled with ferrites, have also been obtained and will be briefly discussed here. To illustrate their use and to develop confidence in them, they have been applied to inhomogeneous waveguides the solutions of which are already known. It turns out that the variational principles lead to algebraic expressions which not only obviate the need of solving transcendental equations but also give excellent accuracy even with rather crude trial fields. Furthermore, since the various quantities are explicitly related by algebraic expressions, the latter may be used as design equations.

#### 1. Variational Principles for Resonant Frequencies of Cavities

a. A cavity partially filled with ferrite. Equation 1 of reference 1 can be used in this case provided that  $\epsilon$  is taken within the integral sign.

b. A cavity partially filled with a dielectric. Here we can either use the variational principle described in the preceding paragraph (with  $\mu$  a scalar) or

$$\left[ \omega_n^2 \right] = \frac{\int_V \nabla \times \mathbf{E} \cdot \nabla \times \mathbf{E}^* dv}{\int_V \epsilon \mu \mathbf{E} \cdot \mathbf{E}^* dv} \quad (1)$$

The choice depends on whether it is easier to use  $\mathbf{E}$  or  $\mathbf{H}$  as a trial field. The symbols of Eq. 1 are the same as those used in reference 1.

c. A cavity with inhomogeneous and anisotropic substance. Here we require  $\vec{\epsilon}$ ,  $\vec{\mu}$  to be hermitian. Then we have

$$\left[ \omega_n \right] = -j \frac{\int_V \mathbf{E}^* \cdot \nabla \times \mathbf{H} dv - \int_V \mathbf{H}^* \cdot \nabla \times \mathbf{E} dv}{\int_V \mathbf{E}^* \cdot \vec{\epsilon} \cdot \mathbf{E} dv + \int_V \mathbf{H}^* \cdot \vec{\mu} \cdot \mathbf{H} dv} \quad (2)$$

To utilize Eq. 2, a trial field, say  $\mathbf{E}$ , is assumed.  $\mathbf{H}$  is then found from one of

(XII. MICROWAVE COMPONENTS)

Maxwell's two equations. The rest of the procedure is standard. For the method of deriving Eq. 2 see section 4 below.

The formulas of the preceding paragraphs can be adapted to the cutoff frequencies of waveguides, provided that we replace the resonant frequencies by cutoff frequencies, the volume integrals by surface integrals over the cross section of the guide, and  $E, H$  by their  $z$ -independent parts.

2. Variational Principle for the Propagation Constant  $\gamma$

We have

$$[\gamma] = -j \frac{\int_S H^* \cdot \nabla \times E ds - \int_S E^* \cdot \nabla \times H ds + j\omega \int_S H^* \cdot \vec{\mu} \cdot H ds + j\omega \int_S E^* \cdot \vec{\epsilon} \cdot E ds}{\int_S H^* \cdot a_z \times E ds - \int_S E^* \cdot a_z \times H ds} \quad (3)$$

where  $E, H$  are the  $z$ -independent parts of the electric and magnetic fields,  $E^*, H^*$  are their complex conjugates, and  $a_z$  is the unit vector in the direction of propagation.

To utilize Eq. 3, a trial field, say  $E$ , is assumed;  $H$  is then found from Maxwell's equation

$$\nabla \times E - j\gamma a_z \times E = -j\omega \vec{\mu} \cdot H$$

The rest of the procedure is standard in variational calculation.

3. Application

a. Equation 1 has been applied to the case of the lowest mode of waveguides partially filled with dielectric of permittivity  $\epsilon$ , as shown in Figs. XII-1 and XII-2. A very crude approximation of the field  $E$  gives results that are in excellent agreement with the exact solutions given in reference 2. Moreover, the variational principle enables us to compute cases similar to that of Fig. XII-1, but with the dielectric asymmetrically placed. As a specific result we shall give the expression for the cutoff wavelength  $\lambda_c$  in terms of the geometry of Fig. XII-1.

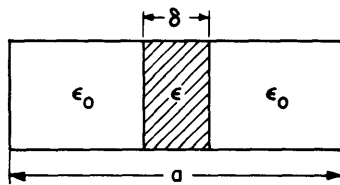


Fig. XII-1

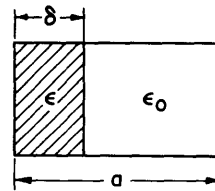


Fig. XII-2

$$\lambda_c = 2a \left[ 1 + \chi_e \left( \frac{\delta}{a} + \frac{1}{\pi} \sin \frac{\pi \delta}{a} \right) \right]^{1/2} \quad (4)$$

b. Next, we shall give an application of Eq. 3 to the case shown in Fig. XII-2.

$$\begin{aligned} \gamma^2 = & -\frac{5}{2} \left( \frac{\pi}{a} \right)^2 + k_o^2 \left[ 1 + \chi_e \frac{\delta}{a} - \frac{\chi_e}{2\pi} \sin \frac{2\pi\delta}{a} \cos^2 \frac{\pi\delta}{a} \right] \\ & + \left\{ \left[ \frac{3}{2} \left( \frac{\pi}{a} \right)^2 - \frac{k_o^2 \chi_e}{2\pi} \sin \frac{2\pi\delta}{a} \sin^2 \frac{\pi\delta}{a} \right]^2 + \frac{16}{9} \left( \frac{k_o^2 \chi_e}{\pi} \right)^2 \sin^6 \frac{\pi\delta}{a} \right\}^{1/2} \end{aligned} \quad (5)$$

where  $k_o^2 = \omega^2 \mu_o \epsilon_o$ . Although Eq. 5 was obtained with a rather crude trial field, it faithfully reproduces all the curves given in Fig. 11.11 of reference 2.

#### 4. Derivation of the Preceding Variational Principles

The variational principles given in reference 1 and that given by Eq. 1 can be derived by following known procedures of which an excellent account may be found in reference 3. The variational principles given by Eqs. 2 and 3 are more complicated, since the functionals  $[\omega]$ ,  $[\gamma]$  depend on more than one function. A brief outline of the derivation of Eq. 3 will, therefore, be given. We start with

$$\nabla \times \mathbf{E} + j\omega \vec{\mu} \cdot \mathbf{H} = j\gamma \mathbf{a}_z \times \mathbf{E} \quad (6)$$

$$\nabla \times \mathbf{H} - j\omega \vec{\epsilon} \cdot \mathbf{E} = j\gamma \mathbf{a}_z \times \mathbf{H} \quad (7)$$

We multiply the first by  $\vec{H}$ , the second by  $\vec{E}$ , integrate over the cross section of the guide, and subtract.  $\vec{E}$ ,  $\vec{H}$  are the adjoints of  $\mathbf{E}$ ,  $\mathbf{H}$ , to be determined presently.

$$\begin{aligned} & \int_S \vec{H} \cdot \nabla \times \mathbf{E} ds - \int_S \vec{E} \cdot \nabla \times \mathbf{H} ds + j\omega \int_S \vec{H} \cdot \vec{\mu} \cdot \mathbf{H} ds + j\omega \int_S \vec{E} \cdot \vec{\epsilon} \cdot \mathbf{E} ds \\ & = j\gamma \left[ \int_S \vec{H} \cdot \mathbf{a}_z \times \mathbf{E} ds - \int_S \vec{E} \cdot \mathbf{a}_z \times \mathbf{H} ds \right] \end{aligned} \quad (8)$$

Equation 8 constitutes a variational principle for  $\gamma$  if we vary  $\vec{E}$ ,  $\vec{H}$ . It remains, however, to relate  $\vec{E}$ ,  $\vec{H}$  to  $\mathbf{E}$ ,  $\mathbf{H}$  in a simple manner. This can be done by suitably transforming Eq. 8 to a variational principle for the determination of  $\vec{E}$ ,  $\vec{H}$ . It then turns out that  $\vec{E}$ ,  $\vec{H}$  must satisfy the complex conjugates of Eqs. 6 and 7 provided that  $\vec{\epsilon}$ ,  $\vec{\mu}$  are hermitian, that is, when the losses are disregarded. Thus  $\vec{E}$ ,  $\vec{H}$  equal  $\mathbf{E}^*$ ,  $\mathbf{H}^*$ .

(XII. MICROWAVE COMPONENTS)

5. Mode-Expansion Solution of Anisotropic-Inhomogeneous Waveguide Problems

In reference 4 a general approach for the solution of waveguide problems with anisotropic and cross-sectionally inhomogeneous media was mentioned, and results of its application were given. A brief exposition of the method follows.

Let  $E(x,y)\exp(-j\gamma z)$  be the electric field,  $H(x,y)\exp(-j\gamma z)$  the magnetic field,  $J_e \exp(-j\gamma z)$  the electric current,  $J_m \exp(-j\gamma z)$  the magnetic current.  $E, H, J_e, J_m$  are three-dimensional vectors depending on the cross-sectional coordinates only. Substituting in Maxwell's equations, we obtain

$$\left. \begin{aligned} \nabla \times E - j\gamma a_z \times E + j\omega\mu_0 H &= J_m \\ \nabla \times H - j\gamma a_z \times H - j\omega\epsilon_0 E &= J_e \end{aligned} \right\} \quad (9)$$

Following the method used for the problem of a cavity, we wish to expand the field in terms of a complete set of modes, of which there are several. One possible choice is a set comprising the usual TE, TM modes completed by a set of irrotational modes. The inclusion of the latter is necessary for the expansion of the irrotational part of the field, as in the case of ferrites, for example. Such a set of modes is perfectly admissible and has the advantage that each mode has physical existence. However, we shall choose another set which, although its individual modes have no physical existence, is simpler and real (the TE, TM set is complex).

It is well known (5) that complete sets for vector fields can be generated from solutions of

$$(\nabla^2 + \alpha_n^2)\psi_n = 0, \quad \psi_n = 0 \text{ on boundary}$$

$$(\nabla^2 + \beta_n^2)\phi_n = 0, \quad \frac{\partial\phi_n}{\partial n} = 0 \text{ on boundary}$$

as a result of operating on  $\psi_n, \phi_n$  by  $\nabla, \nabla \times a_z, \nabla \times \nabla \times a_z$ .

Application of this procedure yields the set

$$E_n^a = \frac{\nabla\psi_n \times a_z}{\alpha_n}, \quad E_n^b = a_z \phi_n, \quad E_n^c = \frac{1}{\beta_n} \nabla\phi_n \quad (10)$$

for the expansion of fields satisfying the electric type of boundary conditions, and

$$H_n^a = a_z \psi_n, \quad H_n^b = \frac{\nabla\phi_n \times a_z}{\beta_n}, \quad H_n^c = \frac{1}{\alpha_n} \nabla\psi_n \quad (11)$$

for the expansion of fields of the magnetic type. These modes satisfy a number of interesting relations, orthogonality being the most important. Moreover, they are normalized, provided that  $\phi_n, \psi_n$  are normalized. We next expand  $E, J_e$  in terms of

$E_n^a, E_n^b, E_n^c; H, J_m$  in terms of  $H_n^a, H_n^b, H_n^c$ , substitute in Eq. 9, and equate coefficients of identical modes. There results the system of equations

$$\left. \begin{aligned} \alpha_n e_n^a + j\omega\mu_0 h_n^a &= \int J_m \cdot H_n^a ds \\ -j\gamma e_n^a + j\omega\mu_0 h_n^c &= \int J_m \cdot H_n^c ds \\ -j\omega\epsilon_0 e_n^a + \alpha_n h_n^a + j\gamma h_n^c &= \int J_e \cdot E_n^a ds \end{aligned} \right\} \quad (12)$$

$$\left. \begin{aligned} \beta_n e_n^b + j\gamma e_n^c + j\omega\mu_0 h_n^b &= \int J_m \cdot H_n^b ds \\ -j\omega\epsilon_0 e_n^b + \beta_n h_n^b &= \int J_e \cdot E_n^b ds \\ -j\omega\epsilon_0 e_n^c - j\gamma h_n^b &= \int J_e \cdot E_n^c ds \end{aligned} \right\} \quad (13)$$

$e_n^p$  ( $p = a, b, c$ ) are the coefficients of  $E_n^p$  in the expansion of  $E$ ,  $h_n^p$  have an analogous meaning. The integrals are over the cross section of the waveguide. They represent the coupling between various modes.

The grouping of Eq. 12 and Eq. 13 is intentional. When there are no electric or magnetic currents Eq. 12 yields the TE modes, Eq. 13 the TM modes, of an empty guide. For ferrites in waveguides we have  $J_e = j\omega\epsilon_0 \chi_e E$  and  $J_m = -j\omega\mu_0 \overline{\chi}_m \cdot H$ . In this case Eqs. 12 and 13 reduce, after rearrangement, to a homogeneous set of equations. The eigenvalues are the values of the propagation constant  $\gamma$  which render the determinant of the system zero.

Equations 12 and 13 are suitable for approximate calculations. Frequently we know, on physical grounds, the small number of modes that are predominant in the unknown field. We therefore assume the solution to be given just by these. Equations 12 and 13 serve, then, to determine the relative strength of each constituent mode.

## 6. Application

The results given in reference 4 are based on the application of the preceding method. The two results given in sections 3a and 3b of the present report, and obtained by the application of the variational method, have also been obtained as a direct application of Eqs. 12 and 13. This is to be expected, since the trial field in the variational method

(XII. MICROWAVE COMPONENTS)

was assumed to contain the same modes as the field used in Eqs. 12 and 13, and since it is known that orthogonality of a set insures finality of its coefficients of expansion (6).

7. Integral Equation Method of Solving Anisotropic-Inhomogeneous Waveguide Problems

The essence of this method is the formulation of the problem in the form of an integral equation by means of suitable Green's functions.

From Eq. 9 we have the inhomogeneous equation

$$\nabla^2 \mathbf{H} + (k_0^2 - \gamma^2) \mathbf{H} = \mathbf{P}(\mathbf{J}_e, \mathbf{J}_m) \quad (14)$$

where

$$\begin{aligned} \mathbf{P} = & -j\omega\epsilon_0 \mathbf{J}_m - \nabla \times \mathbf{J}_e + j\gamma a_z \times \mathbf{J}_e - \frac{j}{\omega\mu_0} \nabla(\nabla \cdot \mathbf{J}_m) - \frac{\gamma}{\omega\mu_0} a_z \nabla \cdot \mathbf{J}_m \\ & - \frac{\gamma}{\omega\mu_0} \nabla(a_z \cdot \mathbf{J}_m) + j \frac{\gamma^2}{\omega\mu_0} a_z (a_z \cdot \mathbf{J}_m) \end{aligned}$$

A similar equation can be written for the E-field. Next we define a magnetic Green's dyadic  $\mathbf{G}_h$  by the following relation where  $\delta$  is the delta function depending on the transverse coordinates, and  $\mathbf{I}$  is the idem factor:

$$\nabla^2 \mathbf{G}_h + (k_0^2 - \gamma^2) \mathbf{G}_h = \mathbf{I} \delta \quad (15)$$

Physically,  $\mathbf{G}_h$  is the magnetic field caused by a filamentary distribution of current with  $\exp(-j\gamma z)$  dependence and unit amplitude in an otherwise empty waveguide. Proper combination of Eq. 14 and Eq. 15 yields

$$\mathbf{H} = \int \mathbf{G}_h \cdot \mathbf{P} \, ds \quad (16)$$

where the integration is over the cross section of the guide. A similar equation is obtained for the E-field. In the usual case where  $\mathbf{J}_e, \mathbf{J}_m$  are given in terms of E, H, Eq. 16 and its companion for the E-field constitute a pair of integral equations and the usual tricks for approximate solutions can be applied. In particular, they are very suitable for perturbation calculations, provided that the Green's dyadics are expandable in a set of modes. It can easily be shown that Eqs. 10 and 11 are just the sets we need, and that  $\mathbf{G}_h$  is given by

$$\mathbf{G}_h = \sum_n \left[ \frac{\mathbf{H}_n^a \mathbf{H}_n^a}{\gamma^2 - k_0^2 + a_n^2} + \frac{\mathbf{H}_n^b \mathbf{H}_n^b}{\gamma^2 - k_0^2 + \beta_n^2} + \frac{\mathbf{H}_n^c \mathbf{H}_n^c}{\gamma^2 - k_0^2 + \beta_n^2} \right]$$

## (XII. MICROWAVE COMPONENTS)

with a similar expression for the electric Green's dyadic. This method has been worked out in detail. In addition to giving an integral equation formulation of the problem it completely contains the results of the mode-expansion method. In fact, Eqs. 12 and 13 are directly obtainable by this method. Note that the Green's dyadics defined here are fundamentally different from those usually defined in waveguides.

### 8. Remarks

All three methods, the variational, the mode-expansion, and the integral-equation methods, have been successfully applied to the following problems, in addition to those already mentioned: rectangular waveguide with a dielectric layer perpendicular to the electric field, ferrite slab in a rectangular waveguide with transverse steady magnetic field, ferrite rod concentric with circular guide and with longitudinal steady magnetic field, eccentric rod in circular guide with transverse steady magnetic field. The results are very promising and will be published soon.

After having completed the development of the mode-expansion method we have found that a somewhat similar investigation was made by Schelkunoff (7), who treated the problem from the coupled-transmission-lines point of view.

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3. P. M. Morse and H. Feshbach, Methods of theoretical physics, Part II, chap. 9-4, (McGraw-Hill Book Company, Inc., New York, 1953).
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### B. STRIP TRANSMISSION SYSTEM

A Fourier integral analysis of the strip transmission system that was described earlier (1) yielded results that compared favorably with experiment. However, due to the complexity of the integral equations derived, the field pattern and the dependence of

## (XII. MICROWAVE COMPONENTS)

the propagation constant upon the various parameters of the system are obscured and difficult to determine, making them impractical from an engineering standpoint.

It has been observed that for a given system the phase velocity as a function of frequency is essentially constant over an extended range of frequencies; the variation is less than 2 percent over a frequency range extending from 2 kMc/sec to 10 kMc/sec. The constancy of the phase velocity would seem to indicate that a homogeneous plane wave assumption is a good approximation to the dominant mode within this frequency range. However, since in the strip system the medium is only regionally homogeneous, the dominant mode is actually a TE-TM mode, and thus the validity of this assumption requires justification. A theoretical and experimental study along these lines is being made.

M. Schetzen

### References

1. Quarterly Progress Report, Research Laboratory of Electronics, M.I.T., April 15, 1953, p. 86.