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Working Paper

A Fluid Model of Dynamic Pricing and Inventory Management for Make-to-Stock Manufacturing Systems

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OR 362-02 **August 2002**

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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August, 2002

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Abstract

In this paper, we introduce a fluid model of dynamic pricing and inventory management for make-to-stock manufacturing systems. Instead of considering a traditional model that is based on how price affects demand, we consider a model that relies on how price and level of inventory affect the time a unit of product remains in inventory. Our motivation is based on the observation that in inventory systems, a unit of product incurs a delay before being sold. This delay depends on the unit price of the product, prices of competitors, and the level of inventory of this product. Moreover, delay data is not hard to acquire and is internally controlled and monitored by the manufacturer. It is interesting to notice that this delay is similar to travel times incurred in a transportation network. The model of this paper includes joint pricing, production and inventory decisions in a competitive, capacitated multi-product dynamic environment. In particular, in this paper we (i) introduce a model for dynamic pricing and inventory control that uses delay rather then demand data and establish connections with traditional demand models, (ii) study analytical properties of this model, (iii) establish conditions under which the model has a solution and finally, (iv) establish an algorithm that solves efficiently a discretized version of the model.

Keywords: Dynamic Pricing, Fluid Models.

1 Introduction and Motivation

In recent years, pricing has become very important in a variety of areas including airline revenue management, inventory control and supply chain management. For instance, in the airline industry, revenue management has demonstrated its potential to dramatically improve revenue. Smith *et al.* discuss in [34] how revenue management enabled American Airlines to increase its yearly revenue by nearly 5 %, which led to a $ 1.4 billion profit improvement over a period of three years. Moreover, in recent years, the rapid development of information technology, the Internet and E-commerce has had a very strong influence on the development of pricing.

As a result, pricing theory has been extensively studied by researchers from a variety of fields. These fields include, among others, economics (see for example, [37]), marketing (see for example, [24]), telecommunications (see for example, [20], [21], [31]), revenue management and supply chain management (see for example, [3], [5], [8], [13], [16], [41]). The paper by McGill and van Ryzin [28], and the references therein, provide a thorough review of revenue management and pricing models.

As the nature of pricing is becoming more dynamic and tactical, companies are faced with the challenge of reacting to and taking advantage of these changes. A study by McKinsey and Company on the cost structure of Fortune 1000 companies in the year 2000 shows that pricing is a more powerful lever than variable cost, fixed cost or sales volume improvements. An improvement of 1% in pricing yields an average of 8.6% in operating margin improvement (see Figure 1). Therefore, companies' ability to survive in this very competitive environment depends on the development of efficient pricing models.

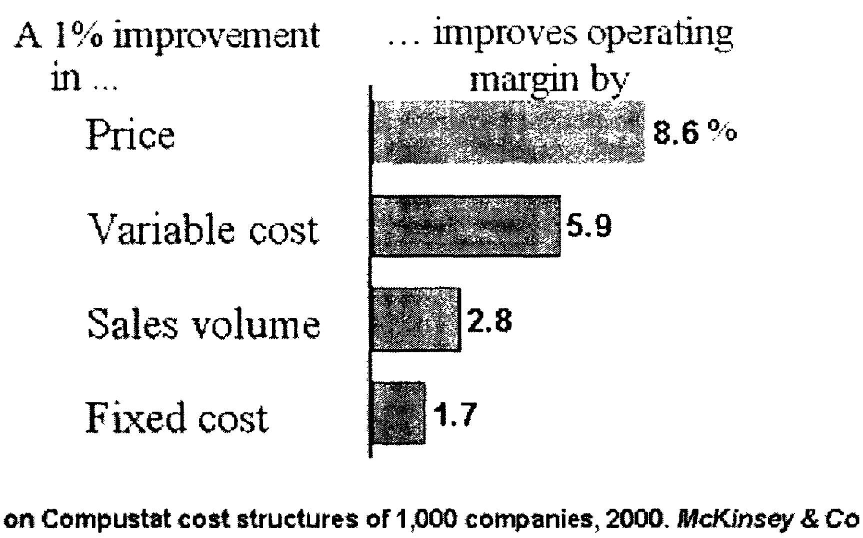

Figure 1: Price as a powerful lever to improve profitability

Make-to-stock manufacturing is the standard for a very large number of industries such as retail (see Ha [17] and Wein [36] for more details on make-to-stock models). Furthermore, a motivation for the use of fluid models is that these models have shown to provide good production and inventory policies in a variety of settings, as illustrated in Avram, Bertsimas and Ricard [2], Bertsimas and Paschalidis

[4], Harrison [18], and Meyn [29]. Nevertheless, these models do not address the pricing aspect of the problem.

In this paper, we consider (i) a multi-product and dynamic environment, (ii) a dynamic production capacity shared amongst all products, (iii) the presence of competition. We address the joint pricing, production and inventory problem, without assuming any fixed relationship between price and inventory. Furthermore, for better numerical tractability, we study the solution of the model assuming a specific price-inventory relationship with parameters that are an output of the model.

Instead of considering a traditional model that assumes an a priori relationship between price and demand with fixed parameters, we consider a model that relies on how price and level of inventory affect the time a unit of product remains in inventory. We refer to this time spent in inventory as delay or sojourn time.

The impetus of considering delay data is motivated from: (1) The widespread recording, by barcode readers, of entrance times and exit times of products in inventory systems, which makes this delay data easily available. (2) The delay data being internal and easily extractable from data warehouses, as opposed to demand data, which is external, and therefore not controlled by the manufacturer. As a result, issues of missing data are not as much present when dealing with delay and inventory data (contrary to demand data). (3) In an environment where price does not vary a lot with time, the estimation of the relationship between price and demand, which is used as an input to the pricing models in the literature, can be quite inaccurate. However, because of the moderate to high variability of inventories with time, the estimation of the relationship between inventory level and sojourn time can be more accurate. A few companies such as *Amazon.com* are currently using sojourn time information to control their inventories and adjust their pricing policies.

The contributions of this paper are the following:
(1) It studies a general continuous-time formulation of the joint dynamic pricing and inventory control problem by establishing insightful analytical properties of this model.
(2) It establishes when the general model has a solution.
(3) It examines the solution of a discretized version of the model by introducing an algorithm for efficiently solving this model and computing pricing policies. As a result, it also discusses how the delay approach we take in this paper directly connects with the traditional demand approach.

The structure of the paper is as follows. In Section 2, we provide the notation and some definitions. In Section 3, we formulate the Dynamic Pricing Model as a continuous-time nonlinear optimization problem. In Section 4, we present a solution algorithm for a discretized version of the model, test it on a small case example, and report on the computational results. In Section 5, we consider the general Dynamic Pricing Model. In particular, we study the analytical properties of its feasible region, and establish, under weak assumptions, the existence of a pricing/production/inventory control policy that maximizes the profit of the company under study over the feasible region. Finally, in Section 6, we provide some conclusions.

2 Notation and Definitions

In this section, we present the notation and some definitions that we use throughout the paper.

2.1 Notation

In this paper, we study a multi-product inventory system that we represent conceptually by a directed network with two nodes O and D, and n links joining these two nodes. Node O represents the arrival of a product to the warehouse and node D represents the delivery of this product to the customer. Each link joining O and D corresponds to a distinct product that the company is selling and is indexed by i, $i \in \{1, ..., n\}$. We assume that the company under study is a Stackelberg leader, and as a result is a price setter. Therefore, competitors' prices are functions of the price of the company under study. These functions can be estimated in practice using regression on the competitors' prices and the prices of the company under study, as illustrated in Subsection 3.2. Below, we describe the inputs and the outputs of the Dynamic Pricing Model. Figure 2 provides a network illustration of the notation introduced below.

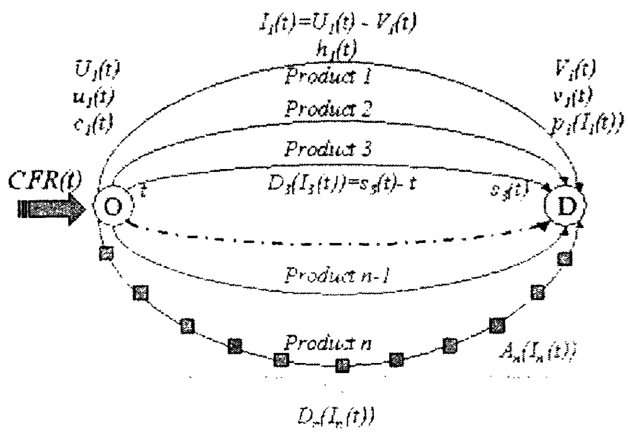

Figure 2: Network representation of the multi-product inventory system

Inputs of the Dynamic Pricing Model

Link variables:

$CFR(t)$	$=$	Shared production capacity rate at time t;
$p_i^c(p_i(.))$	$=$	$(p_{i,j}^c(p_i(.)), j \in \{1, ..., J(i)\})$, vector of price functions of companies competing on product i;

$D_i(I_i) = T_i(I_i, p_i, p_i^c)$	:	product sojourn time function, that is the total time a newly produced unit of product i spends in the inventory system, given an inventory I_i, a unit price $p_i(I_i)$, and a set of competitors' price functions $p_i^c(.)$;
$A_i(I_i)$	:	average product delay function, that is the average time needed to sell a unit of product i (i.e. $A_i(I_i) = \frac{D_i(I_i)}{I_i}$);
B_{1i}	:	a lower bound on the derivative $D_i'(.)$ of the product sojourn time function $D_i(I_i)$;
B_{2i}	:	an upper bound on the derivative $D_i'(.)$ of the product sojourn time function $D_i(I_i)$;
$c_i(t)$	:	production cost of product i at time t;
$h_i(t)$	:	inventory cost of product i at time t.

Time variables:

t	:	index for continuous time;
$[0, T]$	:	production period. After time T, the company under study ceases producing.

Outputs of the Dynamic Pricing Model

Link variables:

$U_i(t)$	:	cumulative production flow of product i during interval $[0, t]$;
$u_i(t)$	:	production flow rate of product i at time t;
$V_i(t)$	:	cumulative sales flow of product i during interval $[0, t]$;
$v_i(t)$	:	sales flow rate of product i at time t;
$I_i(t)$	:	inventory (number of units) of product i at time t;
$p_i(I_i(t))$	:	sales price of one unit of product i given an inventory $I_i(t)$;
$s_i(t)$	:	exit time of a production flow of product type i entering at time t $(s_i(t) = t + D_i(I_i(t)))$.

Time variables:

$[0, T_\infty]$	:	analysis period. It is the interval of time from when the first unit of product is produced to the first instant all products have been sold.

Notice that the control variables are the production flow rates $u_i(.)$ and the unit price functions $p_i(I_i)$.

2.2 Definitions

The following definitions express different types of First In First Out (FIFO) properties. The FIFO property will play a key role in the analysis of our model in Section 5.

Definition 1 *A product verifies the FIFO property if and only if:*

$$\forall (t_1, t_2) \in [0, T]^2, \text{ if } t_1 \leq t_2, \text{ then: } s_i(t_1) \leq s_i(t_2). \tag{1}$$

The above property expresses that a newly produced unit of product cannot be sold before its predecessors. Similarly, a product verifies the Strict FIFO property if and only if the product exit time function is striclty increasing.

Definition 2 *A product verifies the strong FIFO property if and only if:*

$$\exists\, \theta > 0 \text{ such that } \forall (t_1, t_2) \in [0, T]^2, \text{ if } t_1 < t_2, \text{ then: } s_i(t_2) - s_i(t_1) \geq \theta(t_2 - t_1). \tag{2}$$

3 Formulation of the Dynamic Pricing Model

3.1 Modeling Assumptions

Before formulating the model, we describe the setting and the assumptions.

We consider a competitive setting where:
A1) The company under study is a Stackelberg leader (a monopoly is a special case of a Stackelberg leader).
A2) There are multiple products, (these products could also be non-perishable).
A3) The total production capacity rate is bounded by a non-negative capacity flow rate function $CFR(.)$.
A4) There is no substitution between products.
A5) The company under study faces holding costs but no setup costs.
A6) The demand is deterministic.
A7) The unit price $p_i(.)$ is a function of the inventory I_i.

The study of the general dynamic pricing model in Section 5 does not require any assumption on the functional form of the unit price function $p_i(.)$. Instead, the unit price function is an output of the model. However, for the purpose of analyzing a discretized version of the pricing model in Section 4, we assume that the unit price function $p_i(.)$ is linear as a function of the inventory. Notice that Assumption A7 allows us to consider a variety of models for the unit price functions. Examples of such models include linear functions of the type $p_i(I_i) = p_i^{max} - \frac{p_i^{max} - p_i^{min}}{C_i} I_i$ as well as nonlinear functions of the type $p_i(I_i) = \frac{p_i^{max}}{(\frac{p_i^{max}}{p_i^{min}} - 1)\frac{I_i}{C_i} + 1}$, where C_i denotes the inventory capacity, p_i^{max} the maximum allowable price, and p_i^{min} the minimum allowable price. Also, notice that the unit price function $p_i(I_i(t))$ depends on time only through the time-dependence of the inventory $I_i(t)$. Furthermore, the examples we state consider the case where the unit price decreases as inventory increases.

We consider a sojourn time function $D_i(I_i(t)) = T_i(I_i(t), p_i(I_i(t)), p_i^c(p_i(I_i(t))))$ that represents the total time it takes to sell, at time t, a newly produced unit of product i, given a level of inventory $I_i(t)$, a unit price $p_i(I_i(t))$ and a set of competitors' prices $p_i^c(p_i(I_i(t)))$. Notice that the product sojourn time function $D_i(I_i(t))$ resembles the time to traverse a link in a transportation network. For the sake of completeness, in what follows we discuss how sojourn times can be estimated in practice.

3.2 Estimation of Sojourn Times in Practice

A few companies such as *Amazon*.com utilize the sojourn time information to control their inventory levels and adjust their pricing policies. A key motivation for introducing sojourn times in the model of this paper is the availability of sojourn time information in almost every company's data warehouse. Indeed, as a unit of good enters the inventory system, a barcode reader records its entrance time. When this unit is sold, a barcode reader records its exit time. The lag between the entrance time and the exit time is the sojourn time.

Below, we describe how to estimate the sojourn times $D_i(I_i)$ in practice:

- Extract entrance times t_i and exit times $s_i(t_i)$ of units of product i from the data warehouse and record sojourn times $\widehat{D}_i(t_i) = s_i(t_i) - t_i$.

- Record the inventory levels $I_i(t_i)$ and the unit prices $p_i(t_i)$ at entrance times t_i.

- Fit the triplets $(I_i(t_i)$, $p_i(t_i)$, $\widehat{D}_i(t_i)$) into a parametric function $\overline{D}_i(I_i(t_i), p_i(t_i))$.

- Assume a parametric shape for the unit price function $p_i(I_i)$ and plug it in $\overline{D}_i(I_i(t_i), p_i(t_i))$ to derive the sojourn time function $D_i(I_i)$.

Notice that since the vector of competitors' price functions $p_i^c(p_i(.))$ is assumed to be a function of the unit price function $p_i(.)$ of the company under study, it follows that the function $\overline{D}_i(I_i(t_i), p_i(t_i))$ also takes into account the effect of competition.

Finally, notice that the estimation procedure outlined above is easy to implement. The parameters of the sojourn time functions $D_i(I_i)$ can be recalibrated regularly to account for changes in customer behavior and in competitors' pricing policies.

3.3 Model Formulation

In what follows, we discuss a continuous-time analytical model for the dynamic pricing problem. This model takes a fluid dynamics approach by expressing link dynamics, flow conservation, flow propagation and boundary constraints. This formulation extends the Dynamic Network Loading (DNL) model used in the context of the dynamic traffic equilibrium problem (see Friesz *et al.* [15], Wu *et al.* [39], Xu *et al.* [40], and Kachani [19] for more details). Nevertheless, the model in this paper includes the added complexities that it considers a shared production capacity environment, it incorporates the pricing component, and finally, it is placed in the framework of dynamic optimization.

Dynamic Pricing Model:

$$\text{Maximize} \quad \sum_{i=1}^{n} \int_0^{T_\infty} [p_i(I_i(t))v_i(t) \ - \ c_i(t)u_i(t) - h_i(t)I_i(t)]dt \tag{3}$$

$$\text{s.t.} \quad \frac{dI_i(t)}{dt} = u_i(t) \ - \ v_i(t), \quad \forall i \in \{1, ..., n\} \tag{4}$$

$$V_i(t) = \int_{\omega \in W} u_i(\omega)d\omega, \quad \forall i \in \{1, ..., n\}, \quad \text{where} \quad W = \{\omega : s_i(\omega) \le t\} \tag{5}$$

$$U_i(0) = 0, \quad V_i(0) = 0, \quad I_i(0) = 0, \quad \forall i \in \{1, ..., n\}$$

$$\sum_{i=1}^{n} u_i(t) \le CFR(t), \tag{6}$$

$$u_i(.) \ge 0, \quad \forall i \in \{1, ..., n\}, \qquad CFR(.) \ge 0. \tag{7}$$

The continuous-time Dynamic Pricing Model (DPM) is maximizing objective function (3) subject to constraints (4)-(7).

Remarks:

- The objective of the company is to maximize its profits. That is, by subtracting production costs and inventory costs from sales.
- The link dynamics equations (4) express the change in inventory at time t as the difference between the production and the sales flow rates.
- The flow propagation equations (5) describe the flow progression over time. Note that a production flow entering link i at time t will be sold at time $s_i(t) = t + D_i(I_i(t))$. Therefore, by time t, the cumulative sales flow of link i should be equal to the integral of all production inflow rates which would have entered link i at some earlier time ω and would have been sold by time t.
- Furthermore, if the product exit time functions $s_i(.)$ are continuous and satisfy the strict First-In-First-Out (FIFO) property, then the flow propagation equations (5) can be equivalently rewritten as

$$V_i(t) = \int_0^{s_i^{-1}(t)} u_i(\omega)d\omega, \quad \forall i \in \{1, ..., n\}. \tag{8}$$

  Notice that $s_i^{-1}(t)$ is the time at which a unit of product i needs to be produced so that it is sold at time t. Furthermore, under the strict FIFO condition, a unit of product i, entering the queue at time t, will be sold only after the units of product i, that entered the queue before it, are all sold. In mathematical terms, this is equivalent to the product exit time functions $s_i(.)$ being strictly increasing. As a result, defining the production time $s_i^{-1}(t)$ makes sense.

  Furthermore, notice $V_i(t)$ represents the cumulative demand. Equation (5) links the demand to the sojourn time. It replaces the demand-price relationship used in traditional models in the literature.

- Constraint (6) assumes that at each time t, the total production flow rate is no more than the total capacity flow rate $CFR(t)$.
- It is not necessary to assume that $I_i(0) = 0$. Instead, we could assume that $I_i(0) = I_{i0} > 0$. However, we consider zero-level inventories at $t = 0$ for simplicity of notation. Hence, in the beginning, we start producing before the demand arrives. As a result, we build inventory, which is characteristic of make-to-stock systems.
- In general, the DPM Model is a continuous-time non-linear optimization problem. The non-linearity of the model comes from the unit price as a function of the inventory, as well as the integral equation (5). In this formulation, the known variables are the product sojourn time functions $D_i(.)$, the production and inventory costs $c_i(.)$ and $h_i(.)$, and the total capacity flow rate function $CFR(.)$. The unknown variables we wish to determine are $u_i(t)$, $v_i(t)$, $U_i(t)$, $V_i(t)$ and $I_i(t)$. Notice that integral equation (5), which connects the production to the sales schedules through the delays incurred in the system due to price and inventory, makes this problem hard to solve.
- Notice that the model is general enough to account for the case where the FIFO property, defined above, is not necessarily verified (notice that Equation (5) does not assume that the FIFO property holds). In Section 5, we investigate when the FIFO property holds. We examine

conditions on the product sojourn time functions $D_i(.)$ and on the production flow rates $u_i(.)$. When the FIFO property holds, the model becomes more tractable.

In the remainder of the paper, we will denote by $F(DPM)$ the feasible region of the DPM Model. In Section 5, we study the analytical properties of this region but also establish conditions under which the model has a solution.

First in Section 4, we examine the solution of a discretized version of the Dynamic Pricing Model by introducing an algorithm for solving the model. In Section 5, we illustrate how our results extend to the general case.

4 Solution Algorithm and Computational Results

In order to study the efficient solution of the Dynamic Pricing Model, in this section, we consider a discretized version of the model. In particular, we introduce and study a relaxation algorithm, illustrate this algorithm on an example, and report some preliminary computational results.

In particular, in the following two subsections, we review the modeling assumptions we impose in order to obtain a discretized version of the model that is computationally tractable. We do not make a direct assumption on the shape of the sojourn time function but rather impose a condition on the demand. As a result, we discuss a connection with traditional demand models.

4.1 A Pricing Model

We consider the case of a linear unit price in terms of inventory and a linear demand arrival rate function in terms of the unit product price. We first need to define the following primitive quantities that are the essential data for our model. Let $\overline{p}_i^{min}$ denote a minimum reference price and $\overline{\lambda}_i^{max}$ denote the corresponding demand arrival rate. Let $\overline{p}_i^{max}$ denote the corresponding reservation price, that is, the minimum price for which there is no demand for product i. These three quantities are input data in the model.

Moreover, in addition to Assumptions A1-A7 that we considered in Section 3, we make the following assumptions, for every product i:

A8) The unit price function $p_i(I_i)$ is linear in terms of the inventory level I_i (see Figure 3). As we discussed in Section 3, we assume that

$$p_i(I_i) = p_i^{max} - \frac{p_i^{max} - p_i^{min}}{C_i} I_i, \tag{9}$$

where C_i denotes the storage capacity, p_i^{min} denotes the minimum allowable price, and p_i^{max} denotes the reservation price. Notice that this function is decreasing in terms of inventory.

A9) The reservation price p_i^{max} is a function of the minimum allowable price p_i^{min}. To illustrate this assumption, we consider the example of two retail stores competing on a product i. If the minimum price of Store 1 is lower than that of Store 2 (that is, $p_{i,1}^{min} < p_{i,2}^{min}$), then Store 1 will be perceived by customers as cheaper. As a result, Store 1 can from time to time take advantage of this perception by having slightly higher prices than Store 2. This observation illustrates that the reservation price of Store 1 is higher than that of Store 2 (that is $p_{i,1}^{max} > p_{i,2}^{max}$). Therefore, p_i^{max} is a decreasing function of p_i^{min}. However, due to customers' sensitivity to high prices, the difference (in absolute value) between the reservation prices of the two stores needs to be smaller than the difference (in absolute values) between their minimum allowable prices. This can be achieved by assuming that the difference between reservation prices is a concave function in terms of the difference between minimum prices. As a result, we consider the following reservation price function that verifies the condition above.

$$p_i^{max}(p_i^{min}) = \overline{p}_i^{max} + sign(\overline{p}_i^{min} - p_i^{min}).|\overline{p}_i^{min} - p_i^{min}|^{\frac{1}{4}}, \tag{10}$$

where $sign(x) = 1$ when $x \geq 0$, and $sign(x) = -1$ when $x < 0$. Note that the exponent term $\frac{1}{4}$ can be replaced by any real $r \in (0,1)$ (since $|\phi| \rightarrow |\phi|^r$ is concave for $r \in (0,1)$).

A10) We assume that the storage capacity for each of the n products of the firm under study is allocated so that the firm is able to satisfy the maximum demand rate λ_i^{max} within a fixed period of time δ that is the same for all products. In mathematical terms $C_i = \lambda_i^{max}.\delta$, $\forall i \in \{1, ..., n\}$. Quantity δ represents the minimum reserve time.

So far, we imposed assumptions on how pricing relates to inventory and the functional form we consider for the pricing in the model. Our goal in this section is to model the delay of a product waiting in inventory. To achieve this, we consider average delay functions $A_i(I_i(t))$ of the hyperbolic form $\frac{\delta}{I_i(t)}$, where δ is the minimum reserve time, and $A_i(I_i(t))$ is defined in Subsection 2.1. This seems to imply that as inventory increases, we price so that the average delay of a product decreases. In what follows, we impose assumptions on the demand arrival rate (as done traditionally in the literature) and demonstrate how these assumptions give rise to these hyperbolic average product delay functions.

4.2 Delay and Demand Models

We assume the following, for every product i:
A11) For every minimum allowable price p_i^{min} corresponds an arrival rate λ_i^{max}. This maximum arrival rate is a hyperbolic function of the form

$$\lambda_i^{max}(p_i^{min}) = \overline{\lambda}_i^{max}.\frac{\overline{p}_i^{min}}{p_i^{min}}. \tag{11}$$

See Allen [1] and Tirole [35] for more details.

A12) A non-homogenous renewal demand arrival process, with rate $\lambda_i(p_i, p_i^{min})$ that is linear as a function of the price p_i (see Figure 4). Similarly to [1], [12], [35], and [38], we assume that

$$\lambda_i(p_i, p_i^{min}) = \lambda_i^{max}(p_i^{min}).\frac{p_i^{max}(p_i^{min}) - p_i}{p_i^{max}(p_i^{min}) - p_i^{min}}. \tag{12}$$

Notice that when the inventory level hits its capacity level (i.e. $I_i = C_i$), then we charge the minimum allowable price (i.e. $p_i(I_i) = p_i^{min}$), and we target the maximum arrival rate (i.e. $\lambda_i(p_i) = \lambda_i^{max}$). On the other hand, when the inventory level is zero (i.e. $I_i = 0$), then we charge the reservation price (i.e. $p_i(I_i) = p_i^{max}$), and we target a zero arrival rate (i.e. $\lambda_i(p_i) = 0$). Figure 5 illustrates Assumptions A9 and A11-A12.

In practice, given a minimum reference price $\overline{p}_i^{min}$, its correponding demand arrival $\overline{\lambda}_i^{max}$ is readily available in a datawarehouse. Furthermore, its corresponding reservation price $\overline{p}_i^{min}$ can be estimated through customers' surveys. Therefore, the parametric functions in Assumptions A8-A12 can be estimated in practice.

In order to provide a connection between demand and delay models, we consider the approximation that Little's law holds for every time t. That is, $I_i(t) = \lambda_i(p_i(I_i(t))).D_i(I_i(t))$. We view $I_i(.)$ as the average length of the queue, $\lambda_i(.)$ as the arrival demand rate, and $D_i(.)$ as the average waiting time in the queue. As a result, this approximation views the sojourn time $D_i(I_i(t))$ as the expected value of $I_i(t)$ interarrivals of the renewal process, that is $\frac{I_i(t)}{\lambda_i(p_i(I_i(t)))}$.

Notice that this approximation looks at the average state of the system. Indeed, the expression $\lambda_i(p_i(I_i(t))) = \frac{I_i(t)}{D_i(I_i(t))}$ describes the average arrival rate for product i. As a result, it is indifferent about how far or close some of the $I_i(t)$ units of product i in the inventory system are from being sold at time t. However, by looking at the system from the perspective of the sojourn time, that is the waiting in the system, the fluid model formulated in Section 3 captures the dynamics of the $I_i(t)$ units of product i in better detail. Furthermore, quantity $v_i(t)$ describes the selling rate of product i exactly and not on average. As a result, the two approaches are different and in general, $\lambda_i(p_i(I_i(t))) \neq v_i(t)$. Nevertheless, in what follows, we will attempt to gain some insight on the relationship between the two approaches. Figure 7 illustrates this relationship in the case of the test example of Subsection 4.6.

In fact, the next lemma shows that the total amount sold in the analysis period $[0, T_\infty]$ is the same as the cumulative demand.

Lemma 1 *For constant product sojourn time functions $D_i(I_i(t)) = \theta_i$, the total cumulative demand is equal to the total cumulative sales:*

$$\int_0^{T_\infty} \lambda_i(p_i(I_i(t)))dt = V_i(T_\infty). \tag{13}$$

Proof:
Since $\lambda_i(p_i(I_i(t))) = \frac{I_i(t)}{D_i(I_i(t))}$, it follows that

$$\begin{aligned}
\int_0^{T_\infty} \lambda_i(p_i(I_i(t)))dt &= \frac{1}{\theta_i}.\int_0^{T_\infty} I_i(t)dt \\
&= \frac{1}{\theta_i}.[\int_0^{T_\infty} U_i(t)dt - \int_{\theta_i}^{T_\infty} V_i(t)dt] \text{ (from equation (4) and the boundary conditions)} \\
&= \frac{1}{\theta_i}.[\int_0^{T_\infty} U_i(t)dt - \int_{\theta_i}^{T_\infty} U_i(t-\theta_i)dt] \text{ (from equation (8))} \\
&= \frac{1}{\theta_i}.[\int_0^{T+\theta_i} U_i(t)dt - \int_0^{T} U_i(t)dt] \text{ (since } T_\infty = s_i(T) = T + \theta_i) \\
&= \frac{1}{\theta_i}.\int_T^{T+\theta_i} U_i(t)dt \\
&= \frac{1}{\theta_i}.\theta_i.U_i(T) \text{ (since production ends at time } T) \\
&= V_i(T_\infty) \text{ (from equation (8))}.
\end{aligned}$$

□

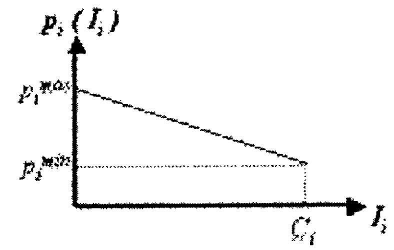

Figure 3: Linear unit price function $p_i(I_i)$

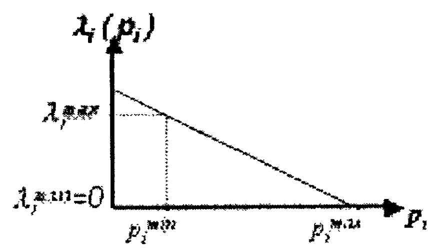

Figure 4: Linear arrival rates $\lambda_i(p_i)$

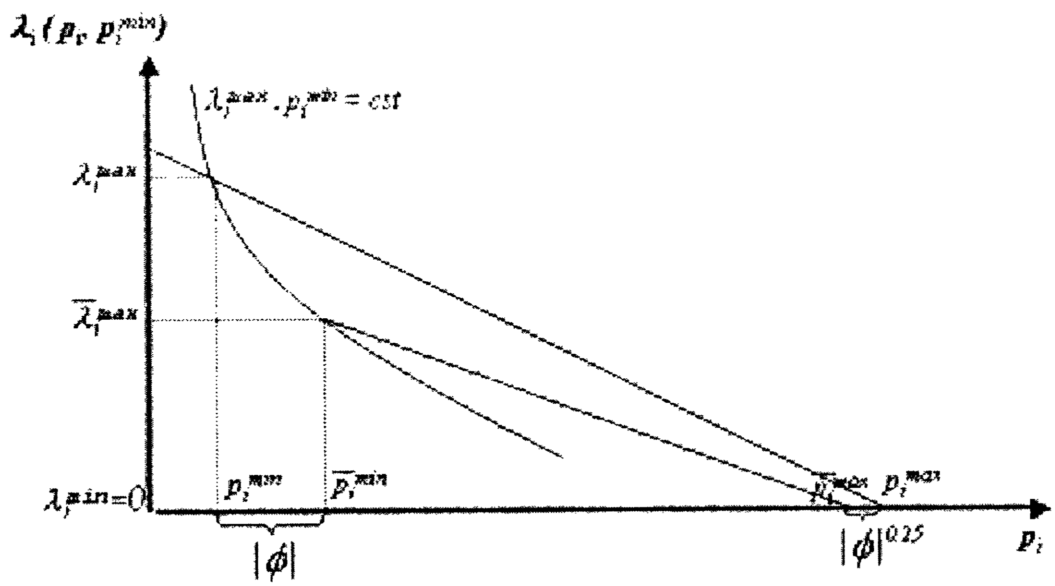

Figure 5: Model for arrival rates, maximum arrival rates, and reservation prices

4.3 Discretized DPM Model

In the remainder of this section, we will consider the discretized version of the Dynamic Pricing Model In particular, this model discretizes the time space by introducing $N = \lfloor \frac{T}{\delta} \rfloor$. We consider $N+1$ intervals of time of length δ and assume that for every discretization interval index $j \in \{0, 1, ..., N\}$ and for every time $t \in [j\delta, (j+1)\delta)$, the following variables in each interval are constant: $CFR(t) = CFR_j$, $u_i(t) = u_{ij}$, $c_i(t) = c_{ij}$, and $h_i(t) = h_{ij}$. The decision variables are the production levels u_{ij} for every product i and for every discretization interval index j, as well as the unit price function parameter p_i^{min}.

Remarks:

- Notice that we can have a finer discretization by choosing intervals of length $\frac{\delta}{M}$, where M is a positive integer that represents the discretization accuracy. This does not add any complexity in the formulation of the discretized model and in the solution algorithm. Furthermore, the computational burden increases linearly with M. However, for the sake of clarity and brevity, in what follows we choose $M = 1$.

- As discussed in Subsection 2.1, in addition to prices, the control variables also include the production levels.

In what follows, we show that under this piecewise constant discretization, the Dynamic Pricing Model becomes equivalent to solving a quadratic optimization problem.

Proposition 1 *Under Assumptions A8-A12, the solution of the Dynamic Pricing Model is equivalent to solving the following quadratic optimization problem:*

Discretized Quadratic Pricing Model (DQPM):

$$Min_{((u_{ij})_{(i\in\{1,\dots,n\},j\in\{0,\dots,N\})}),((p_i^{min})_{(i\in\{1,\dots,n\})})} \sum_{i=1}^{n}(k_i[\sum_{j=0}^{N-1} u_{ij}u_{ij+1} + \sum_{j=0}^{N} u_{ij}^2] + \sum_{j=0}^{N} g_{ij}u_{ij})$$

$$\sum_{i=1}^{n} u_{ij} \le CFR_j \quad , \quad \forall j \in \{0, 1, ..., N\}$$

$$u_{ij} \ge 0 \quad , \quad \forall i \in \{1, 2, ..., n\}, \ \forall j \in \{0, 1, ..., N\}$$

$$\text{where} \quad g_{ij} = -\delta(p_i^{max} - c_{ij} - \frac{h_{ij} + h_{ij+1}}{2}\delta) \quad , \quad k_i = \frac{\epsilon_i \delta^2}{2}, \text{ and } \epsilon_i = \frac{p_i^{max} - p_i^{min}}{C_i}.$$

Proof:
The capacity constraint above follows directly from its continuous analogue (6) in Section 3. Moreover, the non-negativity constraint also follows from its continuous analogue (7). Next, we establish that the objective function in the DPM Model (see equation (3)) simplifies to the quadratic objective of the DQPM Model formulated above. Notice that we converted the problem to a minimization problem by changing the signs. As a result, in what follows, we will illustrate that the optimal objective value of the original DPM Model is the opposite of the optimal objective value of the DQPM Model.

For $j \in \{0, 1, ..., N\}$, and $t \in [j\delta, (j+1)\delta)$, the previous assumptions together with relations (4)-(7) imply that $I_i(t) = u_{ij}.(t - j\delta) + u_{ij-1}.((j+1)\delta - t)$ (14b), $U_i(t) = u_{ij}.(t - j\delta) + \delta.\sum_{l=0}^{j-1} u_{il}$ and

$v_i(t) = u_{ij-1}$. Furthermore, replacing the unit price as a function of the inventory in equation (3) yields the following objective function:

$$
\begin{aligned}
Obj &= -Min \sum_{i=1}^{n} \int_0^{T_\infty} -p_i(I_i(t))v_i(t) + c_i(t)u_i(t) + h_i(t)I_i(t)dt \\
&= -Min \sum_{i=1}^{n} \int_0^{T_\infty} \epsilon_i I_i(t)v_i(t) - (p_i^{max}v_i(t) - c_i(t)u_i(t) - h_i(t)I_i(t))dt. \qquad (14)
\end{aligned}
$$

Moreover, replacing $v_i(t)$ and $I_i(t)$ from (14b) gives rise to:

$$
\begin{aligned}
\epsilon_i \int_0^{T_\infty} I_i(t)v_i(t)dt &= \epsilon_i \sum_{j=1}^{N+1} \int_{j\delta}^{(j+1)\delta} (j+1)\delta u_{ij-1}^2 - j\delta u_{ij-1}u_{ij} + u_{ij-1}(u_{ij} - u_{ij-1})tdt \\
&= \epsilon_i \delta^2 [\sum_{j=0}^{N-1} (j+2)u_{ij}^2 - \sum_{j=0}^{N-1} (j+1)u_{ij}u_{ij+1} \\
&\quad +0.5 \sum_{j=0}^{N-1} (2j+3)u_{ij}(u_{ij+1} - u_{ij}) + 0.5u_{iN}^2] \\
&= k_i[\sum_{j=0}^{N-1} u_{ij}u_{ij+1} + \sum_{j=0}^{N} u_{ij}^2], \qquad (15)
\end{aligned}
$$

where $k_i = \frac{\epsilon_i \delta^2}{2}$.

Furthermore, $\int_0^{T_\infty} p_i^{max} v_i(t)dt = \delta . p_i^{max} . \sum_{j=0}^{N} u_{ij}$ (16b), and $\int_0^{T_\infty} c_i(t)u_i(t)dt = \delta . \sum_{j=0}^{N} c_{ij}.u_{ij}$ (16c). Replacing $I_i(t)$ from (14b) gives rise to:

$$
\begin{aligned}
\int_0^{T_\infty} h_i(t)I_i(t)dt &= \sum_{j=0}^{N+1} \int_{j\delta}^{(j+1)\delta} h_{ij}[(j+1)\delta u_{ij-1} - j\delta u_{ij} + (u_{ij} - u_{ij-1})t]dt \\
&= \delta^2[0.5h_{i0}u_{i0} + \sum_{j=0}^{N-1} (j+2)h_{ij+1}u_{ij} - \sum_{j=0}^{N-1} (j+1)h_{ij+1}u_{ij+1} \\
&\quad +0.5 \sum_{j=0}^{N-1} (2j+3)h_{ij+1}(u_{ij+1} - u_{ij}) + 0.5h_{iN+1}u_{iN}] \\
&= \frac{\delta^2}{2} \sum_{j=0}^{N} (h_{ij} + h_{ij+1})u_{ij}. \qquad (16)
\end{aligned}
$$

Replacing equations (15), (16b), (16c), and (16) in (14) gives rise to the result of the proposition.

□

Remarks:

- $\frac{g_{ij}}{-\delta}$ represents the profit margin of product i for discretization interval index j.
- The objective function of the DQPM Model is separable by product.
- As shown in the next subsection, the DQPM Model is strictly convex. As a result, the DQPM Model has a unique solution. As a result, we can compute it efficiently.

4.4 An Iterative Relaxation Algorithm

In this subsection, we focus on the efficient solution of the DQPM Model described in Proposition 1. In particular, we propose a solution algorithm that determines optimal production levels for a fixed unit price function (that is, when p_i^{min}, and as a result p_i^{max}, are fixed). In the next subsection, we also will illustrate how to incorporate pricing decisions in the solution algorithm.

The solution algorithm we propose, utilizes ideas from the iterative relaxation scheme of Dafermos ([9]) and Nagurney [30], and the equilibration scheme of Dafermos and Sparrow ([10] and [11]), to the dynamic pricing problem. The key intuition behind this solution method lies with the idea of equilibrating at each time period, the "marginal profits" of the produced products. This idea is extensively used in static traffic equilibrium problems (see Florian and Hearn [14], and Sheffi [33] for more details).

To illustrate this equilibration approach, we need to introduce some additional notation. We define C_{ij} as the opposite of the marginal profit of product i for discretization interval index j. In mathematical terms:

$$C_{ij} = \frac{-\partial Obj}{\partial u_{ij}} \quad = \quad 2k_i u_{ij} + k_i(u_{ij+1} + u_{ij-1}) + g_{ij}, \tag{17}$$

where Obj is defined in Proposition 1.

We define m_{ij} as the opposite of the marginal profit of product i for discretization interval index j at a zero production level. That is, $m_{ij} = k_i(u_{ij+1} + u_{ij-1}) + g_{ij}$. Therefore, $C_{ij} = m_{ij} + 2k_i u_{ij}$. We further introduce the upperscript index k to denote the number of iterations of the algorithm. Hence, at iteration k, our goal is to determine, for every product i, and for every discretization interval index j, the production levels u_{ij}^k. Moreover, we introduce $C_{ij}^k = 2k_i u_{ij}^k + m_{ij}^k$, where $m_{ij}^k = k_i(u_{ij+1}^{k-1} + u_{ij-1}^k) + g_{ij}$. Note that the production levels u_{ij}^k will be computed in increasing order of the discretization interval index j in the algorithm. Hence, in the expression of m_{ij}^k, u_{ij-1} is evaluated at iteration k while u_{ij+1} is evaluated at iteration $k-1$. This approach makes the problem separable at each iteration, which allows us to solve easily.

We introduce an $n \times (N+1)$ matrix with elements $order(i,j)$ that sorts the opposite of the zero-production marginal profits m_{ij}^k, $i \in \{1, 2, ..., n\}$ in a non-decreasing order. That is, for j and k fixed, $m_{order(1,j)j}^k \leq m_{order(2,j)j}^k \leq ... \leq m_{order(n,j)j}^k$.

At every iteration k, the equilibration algorithm computes for every discretization interval index j, an index l_j and a vector α_{ij}^k by equilibrating the corresponding opposite of the marginal profits, that is,

$$\begin{aligned}
C_{order(1,j)j}^k = ... = C_{order(l_j,j)j}^k = \alpha_{order(l_j,j)j}^k \quad &\leq \quad C_{order(l_j+1,j)j}^k, C_{order(l_j+2,j)j}^k, ..., \leq C_{order(n,j)j}^k \\
u_{order(1,j)j}^k > 0, ..., u_{order(l_j,j)j}^k \quad &> \quad 0, \sum_{i=1}^{l_j} u_{order(i,j)j}^k = CFR_j \\
u_{order(l_j+1,j)j}^k = u_{order(l_j+2,j)j}^k \quad &= \quad ... = u_{order(n,j)j}^k = 0.
\end{aligned}$$

Figure 6 provides a network representation of the equilibration algorithm described above. There is an interesting analogy between this equilibration algorithm and static traffic equilibrium (see [14], [33]). Indeed, for a fixed discretization interval index j, we select (i) which products we should produce, and (ii) how much of each of the selected products we should be producing, so that all selected products have equal and minimum opposite marginal profit. In static traffic equilibrium, we select (i) which paths should be used, and (ii) how much traffic should flow on each of the selected paths, so that all selected paths have equal and minimum travel times.

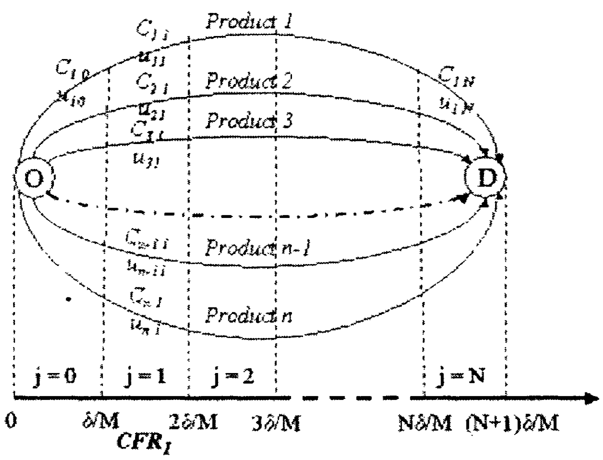

Figure 6: Network representation of the discretized Dynamic Pricing Model

Let ϵ be our tolerance level:

Iterative Relaxation Algorithm:

Step 0: for every time index $j \in \{0, 1, ..., N\}$ and product $i \in \{1, 2, ..., n\}$

$$u_{ij}^{0} = \frac{CFR_j}{n}.$$

Step k: for every time index $j \in \{0, 1, ..., N\}$,

$$\text{Let } \alpha_{order(i,j)j}^{k} = \frac{CFR_j + \sum_{p=1}^{i} \frac{m_{order(p,j)j}^{k}}{2k_{order(p,j)}}}{\sum_{p=1}^{i} \frac{1}{2k_{order(p,j)}}}.$$

If $eq_j = argmin\{i \text{ such that } \alpha_{order(i,j)j}^{k} \leq m_{order(i+1,j)j}^{k}\}$ exists, then set $l_j = eq_j$.
Otherwise, set $l_j = n$.

If $i > l_j$, then $u_{order(i,j)j}^{k} = 0$.
Otherwise, $u_{order(i,j)j}^{k} = \frac{\alpha_{order(l_j,j)j}^{k} - m_{order(i,j)j}^{k}}{2k_{order(i,j)}}$.

Convergence criterion:

If for all $j \in \{0, 1, ..., N\}$ and $i \in \{1, 2, ..., n\}$,
all $u_{ij}^{k} = 0$ satisfy $C_{ij}^{k} \geq \alpha_{order(l_j,j)j}^{k} - \epsilon$, then stop.
Otherwise, set $k = k + 1$ and go to Step k.

Below, we establish that this algorithm converges to the optimal solution of the DQPM Model.

Theorem 1 *The Iterative Relaxation Algorithm converges to the unique optimal solution of the DQPM Model.*

Proof:
The Iterative Relaxation Algorithm is based on considering a separable approximation of C_{ij} (equation (17)) in terms of production levels u_{ij}. For i fixed, let vector $W_i(u) = (2.u_{i0}, 2.u_{i1}, ..., 2.u_{iN})$ and let Z_i be the $(N+1) \times (N+1)$ matrix defined by

$$Z_i = \begin{bmatrix} 0 & 1 & 0 & . & . & . & . & . & 0 \\ 1 & 0 & 1 & 0 & . & . & . & . & 0 \\ 0 & 1 & 0 & 1 & . & . & . & . & . \\ . & 0 & 1 & 0 & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 & 0 & . \\ . & . & . & . & . & 1 & 0 & 1 & 0 \\ . & . & . & . & . & 0 & 1 & 0 & 1 \\ 0 & 0 & . & . & . & . & 0 & 1 & 0 \end{bmatrix}$$

Notice that matrix $Z_i = \nabla_u[(\frac{m_{ij}}{k_i})_{j\in\{0,...,N\}}]$ is the Jacobian matrix of the opposite of the zero-production marginal profits of product i and $W_i(u) = (\frac{C_{ij}-m_{ij}}{k_i})_{j\in\{0,...,N\}}$. We will use the following result from Nagurney [30] to prove that the Iterative Relaxation Algorithm converges.

Lemma 2 *([9], [30]) Assume that there exist a scalar $\gamma > 0$ and a scalar $\lambda \in (0,1)$ such that:*
(F1) For every $j \in \{0, ..., N\}$, $\frac{\partial W_i(u)}{\partial u_{ij}} \geq \gamma$.
(F2) $|||Z_i||| \leq \lambda.\gamma$, (where $|||Z_i|||$ denotes the maximum eigenvalue of matrix Z_i)
Then, the Iterative Relaxation Algorithm converges to an optimal solution.

In order to use the result of thr above lemma, we will show that vector $W_i(u)$ and matrix Z_i verify conditions (F1) and (F2) of Lemma 2 with $\gamma = 2$.

For every $j \in \{0, ..., N\}$, notice that $\frac{\partial W_i(u)}{\partial u_{ij}} = 2$. Hence, condition (F1) is verified with $\gamma = 2$. Since Z_i is a symmetric matrix, it has $N+1$ real eigenvalues (not necessarily distinct). To show that condition (F2) is verified, it suffices to show that every eigenvalue α of matrix Z_i is strictly less than 2.

If α is an eigenvalue of matrix Z_i, there exists a vector $x^\alpha \neq 0$ such that

$$Z_i x^\alpha = \alpha x^\alpha. \tag{18}$$

We will assume that $\alpha \geq 2$ and try to reach a contradiction. If x^α verifies equation (18), then $-x^\alpha$ also verifies it. Hence, without loss of generality, we can assume that $x_0^\alpha \geq 0$. Using an induction over the rows of equation (18), it follows that $0 \leq x_0^\alpha \leq x_1^\alpha \leq ... \leq x_N^\alpha$. Therefore, x^α is a non-negative vector.

Summing up the rows of the vectors in both sides of Equation (18) gives rise to

$$x_0^\alpha + x_N^\alpha + 2.\sum_{j=1}^{N-1} x_j^\alpha = \alpha \sum_{j=0}^{N} x_j^\alpha.$$

Therefore, $(\alpha-1)(x_0^\alpha + x_N^\alpha) + (\alpha-2).\sum_{j=1}^{N-1} x_j^\alpha = 0$. Since $\alpha \geq 2$ and x^α is non-negative, it follows that $x_0^\alpha = x_N^\alpha = 0$. Through an induction argument over the rows of equation (18), it follows that $x_1^\alpha = x_2^\alpha = ... = x_{N-1}^\alpha = 0$. Hence, $x^\alpha = 0$, which contradicts our earlier assumption that $x^\alpha \neq 0$.

Therefore, $\alpha < 2$, which in turn implies that $|||Z_i||| < 2$. Therefore, there exists a scalar $\lambda \in (0,1)$ such that condition (F2) is verified. Lemma 2 implies that the Iterative Relaxation Algorithm converges to an optimal solution.

Furthermore, the quadratic terms of the objective function can be rewritten as

$$\begin{aligned} Q(u) &= \sum_{i=1}^{n} \frac{k_i}{2}[\sum_{j=0}^{N-1}(u_{ij}^2 + 2u_{ij}u_{ij+1} + u_{ij+1}^2) + u_{i0}^2 + u_{iN}^2] \\ &= \sum_{i=1}^{n} \frac{k_i}{2}[\sum_{j=0}^{N-1}(u_{ij} + u_{ij+1})^2 + u_{i0}^2 + u_{iN}^2]. \end{aligned}$$

Notice that $Q(u)$ is a strictly convex quadratic function in terms of the production levels u_{ij}. Hence, the DQPM Model has a unique optimal solution. Therefore, the Iterative Relaxation Algorithm converges to the unique optimal solution.

□

Notice that the iterative relaxation algorithm belongs to a family of linearly converging algorithms (see [10], [30] for more details). It easy to see that by enhancing the algorithm with a binary search, each iteration of the algorithm requires computations of the order of $n.N.log(N)$.

4.5 Determining Optimal Production/Pricing Policies

In this subsection, we show how we can extend the Iterative Relaxation Algorithm to determine both the optimal production levels and the optimal pricing policies.

We introduce the minimum allowable price parameter ϕ defined as $\phi = p_i^{min} - \overline{p}_i^{min}$. Assumptions A7-A11 give rise to the following relations:

$$\begin{aligned} p_i^{min} &= \overline{p}_i^{min} + \phi, \\ p_i^{max} = \overline{p}_i^{max} &- sign(\phi)|\phi|^{\frac{1}{4}}, \\ \lambda_i^{max} = \overline{\lambda}_i^{max} &. \frac{\overline{p}_i^{min}}{\overline{p}_i^{min} + \phi}, \\ p_i(I_i) = \overline{p}_i^{max} - sign(\phi)|\phi|^{\frac{1}{4}} &- \frac{\overline{p}_i^{max} - \overline{p}_i^{min} - sign(\phi)|\phi|^{\frac{1}{4}} - \phi}{\delta.\overline{\lambda}_i^{max}.\frac{\overline{p}_i^{min}}{\overline{p}_i^{min}+\phi}}.I_i. \end{aligned}$$

As a result, in the formulation of the DQPM Model in Proposition 1, the objective function depends on the minimum allowable price parameter ϕ through the parameters k_i and g_{ij} that can be rewritten as:

$$\begin{aligned} k_i = \frac{\delta}{2} &. \frac{\overline{p}_i^{max} - \overline{p}_i^{min} - sign(\phi)|\phi|^{\frac{1}{4}} - \phi}{\overline{\lambda}_i^{max}.\frac{\overline{p}_i^{min}}{\overline{p}_i^{min}+\phi}}, \quad \text{and} \\ g_{ij} = -\delta(\overline{p}_i^{max} - sign(\phi)|\phi|^{\frac{1}{4}} &- c_{ij} - \frac{h_{ij}+h_{ij+1}}{2}\delta). \end{aligned}$$

Hence, for every value of the minimum allowable price parameter ϕ, we can perform the Iterative Relaxation Algorithm (IRA) and obtain an optimal production policy $u(\phi)$ that yields a profit $IRA_{opt}(\phi)$. Therefore, solving the overall DQPM Model is equivalent to maximizing $IRA_{opt}(\phi)$ for $\phi \in (-\overline{p}_i^{min}, \overline{p}_i^{max} - \overline{p}_i^{min})$ such that $p_i^{min}(\phi) \leq p_i^{max}(\phi)$. Notice that this problem is a one-dimensional maximization problem.

As a result, by embedding the Iterative Relaxation Algorithm in a line search procedure for the one-dimensional objective function $IRA_{opt}(\phi)$, we are able to solve the Discretized Dynamic Pricing Model and determine optimal production levels and pricing policies.

4.6 Test Example

In this subsection, we apply the Iterative Relaxation Algorithm in a small test example. We consider 5 products and 10 discretization intervals (i.e. $n = 5$ and $N = 9$). We use as inputs the minimum reference prices $\overline{p}_i^{min}$ and their corresponding reservation prices $\overline{p}_i^{max}$ outlined in Table 1, the shared capacity flow rate vector CFR outlined in Table 2, the production costs c_{ij} and the holding costs h_{ij} provided in Tables 3 and 4 respectively.

$\overline{p}_i^{max}$	$\overline{p}_i^{min}$	
16.25	12.56	*Product 1*
13.25	9.56	*Product 2*
13.25	9.56	*Product 3*
13.25	9.56	*Product 4*
14.85	11.16	*Product 5*

Table 1: Minimum allowable unit reference prices and their corresponding reservation prices

Discretization Interval Index	0	1	2	3	4	5	6	7	8	9
CFR_j	19	21	23	25	27	29	31	33	35	37

Table 2: Shared capacity flow rate per discretization interval

We first assume that the unit price function is fixed (that is, $p_i^{min} = \overline{p}_i^{min}$ and $p_i^{max} = \overline{p}_i^{max}$). We apply the steps of the Iterative Relaxation Algorithm outlined in the previous subsection. For our computations, we used a PC with a Pentium III, 366 MHz, 128 MB RAM, and implemented the algorithm in MATLAB. We chose a tolerance level of $\epsilon = 10^{-9}$ in the convergence criterion. In this example, the algorithm converged in 102 iterations. The running time was 4.2 seconds. Table 5 provides the optimal production levels. The optimal profit associated with these production levels is $\$1,539$.

	0	1	2	3	4	5	6	7	8	9
c_{1j}	8.5938	8.7500	8.8698	8.9709	9.0599	9.1405	9.2144	9.2833	9.3481	9.4093
c_{2j}	5.4000	5.6209	5.7905	5.9333	6.0593	6.1731	6.2778	6.3751	6.4667	6.5533
c_{3j}	4.1698	4.4405	4.6481	4.8231	4.9772	5.1167	5.2449	5.3642	5.4762	5.5823
c_{4j}	4.9209	5.2333	5.4731	5.6751	5.8533	6.0142	6.1622	6.3000	6.4293	6.5517
c_{5j}	7.6599	8.0093	8.2772	8.5033	8.7022	8.8823	9.0478	9.2017	9.3465	9.4833

Table 3: Production costs c_{ij}

	0	1	2	3	4	5	6	7	8	9
h_{1j}	1.6037	1.5135	1.4865	1.4236	1.4107	1.3568	1.3504	1.3013	1.2987	1.2528
h_{2j}	1.4138	1.2862	1.2482	1.1591	1.1409	1.0647	1.0556	0.9861	0.9825	0.9176
h_{3j}	1.2331	1.0770	1.0302	0.9213	0.8989	0.8057	0.7944	0.7095	0.7049	0.6254
h_{4j}	1.0574	0.8770	0.8230	0.6972	0.6714	0.5637	0.5507	0.4526	0.4474	0.3556
h_{5j}	0.8846	0.6830	0.6226	0.4820	0.4532	0.3326	0.3182	0.2085	0.2027	0.1000

Table 4: Holding costs h_{ij}

	0	1	2	3	4	5	6	7	8	9
u_{1j}	0	0	0	0	1.1412	0.0627	3.7224	0.0544	6.2184	0
u_{2j}	0	0.1635	2.7467	0.8509	4.8769	1.4403	6.3578	2.0120	7.7943	2.5721
u_{3j}	13.6804	11.6337	13.5364	14.4020	12.9029	17.1538	11.6776	19.9317	10.4424	22.7265
u_{4j}	5.3196	9.2028	6.7169	9.7471	7.7320	10.3432	8.2041	11.0020	8.6957	11.7015
u_{5j}	0	0	0	0	0.3469	0	1.0381	0	1.8492	0

Table 5: Optimal production levels

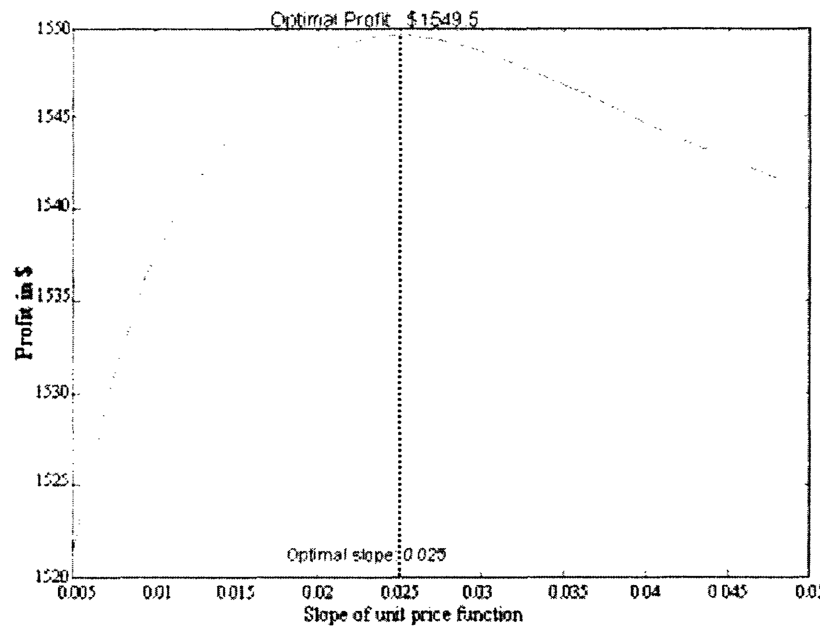

Figure 8: Optimal profit as a function of the slope of the unit price function

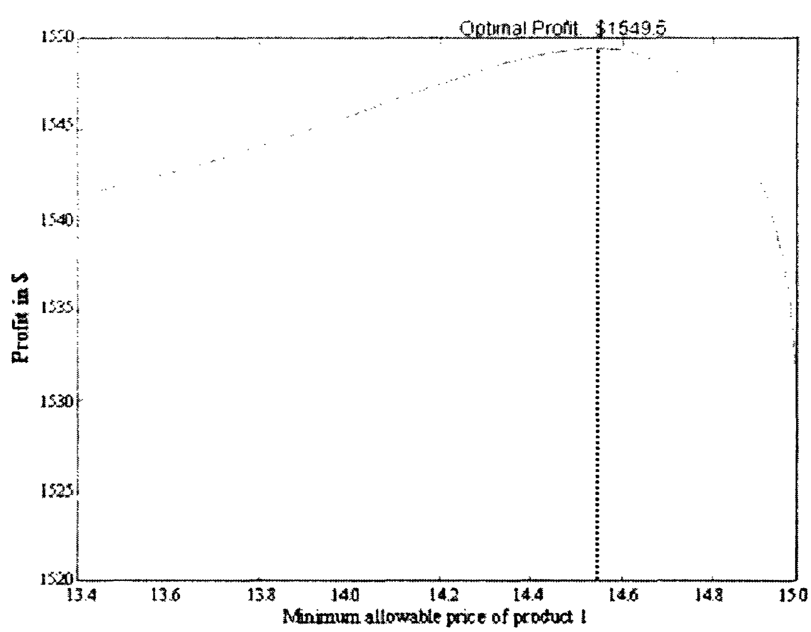

Figure 9: Optimal profit as a function of the minimum allowable price of product 1

Furthermore, Figure 7 illustrates, in this example, the corresponding demand rate $\lambda_3(t)$ and the corresponding sales flow rate $v_3(t)$ for product 3. Note that, as established in Lemma 1, while the profiles of the demand rate and the sales flow rate are different, the two areas under the curves of $\lambda_3(t)$ and $v_3(t)$, depicted in Figure 7, are equal.

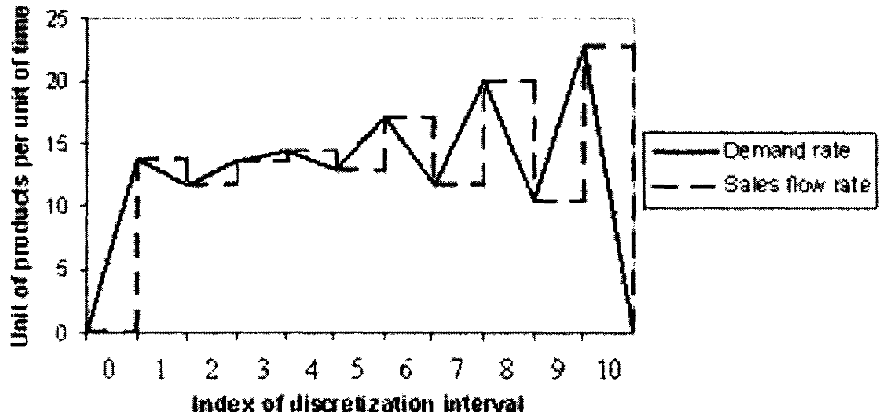

Figure 7: Profile of Demand Rate and Sales Flow Rate of Product 3

Next, we incorporate the pricing aspect in the algorithm as we described in Subsection 4.5. We perform a line search procedure by varying the minimum allowable price parameter ϕ. For every value of ϕ, we run the Iterative Relaxation Algorithm to obtain an optimal objective value $IRA_{opt}(\phi)$.

Figures 8 and 9 show that the profit of the company under study, for this instance of the Discretized Dynamic Pricing Model, is a quasi-concave function in terms of the slope $\frac{p_i^{max}-p_i^{min}}{C_i}$ of the unit price function, and a quasi-concave function of the minimum allowable price of product 1. Notice that the optimal profit is attained for a slope of 0.025 at a value of $\$1,549.5$.

5 The General Dynamic Pricing Fluid Model

In this section, we consider the General Dynamic Pricing Model without considering a discrete approximation, and without imposing the assumptions of Section 4. In particular, we examine key properties of the General Dynamic Pricing Model as formulated in Section 3. In Subsection 5.1, we first study the analytical properties of the feasible region of the Dynamic Pricing Model. Furthermore, in Subsection 5.2, we establish, under weak assumptions, the existence of a pricing/production/inventory control policy that maximizes the profit of the company under study over the feasible region.

5.1 Properties of the Feasible Region of the Dynamic Pricing Model

5.1.1 Unifying Analysis for Non-linear and Linear Sojourn Time Functions

Under sufficient conditions on the production flow rate functions and the sojourn time functions, we prove in this subsection that the feasible region of the Dynamic Pricing Model ($F(DPM)$) is not empty. Furthermore, we provide a unifying analysis of the $F(DPM)$ region for both non-linear and linear sojourn time functions.

In Subsection 5.1.2, we show that if the conditions of Theorem 2 are violated, then the FIFO property is also violated. In this sense, the conditions of Theorem 2 are tight.

In the model presented in Section 3, the production flow rate functions $u_i(.)$ are control variables. In an effort to establish general results, we assume that these functions are Lebesgue integrable. A function is said to be Lebesgue integrable if the set of points where this function is discontinuous is Lebesgue negligible. A set is Lebesgue negligible if its Lebesgue measure is 0.

Definition 3 *A solution is unique (or respectively differentiable) almost everywhere if and only if the set of points where this solution is not unique (or respectively not differentiable) is Lebesgue negligible.*

Later in this paper, we show that the cumulative flow variables are differentiable and the solution to the problem is unique "almost everywhere". We refer to "almost everywhere" by "a.e.".
In what follows, we provide a unifying analysis for both linear and nonlinear sojourn time functions. Corollary 1 shows how linear sojourn time functions can be interpreted as a limit case of nonlinear sojourn time functions and why linear sojourn time functions lead to stronger results.

Theorem 2 *If the pair $(D_i(.), u_i(.))$ satisfies the following conditions:*
(B1) The product sojourn time function $D_i(.)$ is continuously differentiable, and there exist two non-negative constants B_{1i} and B_{2i} such that for every iventory level I_i, $0 \leq B_{1i} \leq D_i'(I_i) < B_{2i}$.
(B2) The production flow rate function $u_i(.)$ is Lebesgue integrable, non-negative and does not exceed $\frac{1}{B_{2i}-B_{1i}}$ (that is, the inverse of the maximum variation of the sojourn time of product i in terms of its inventory).
Then, the feasible region $F(DPM)$ has the following properties:
(1) $F(DPM)$ is well-defined, (that is, the product inventory $I_i(.)$, the sales flow rate $v_i(.)$, and the cumulative variables can be uniquely (a.e.) determined by the product sojourn time function $D_i(.)$ and the production rate $u_i(.)$ on the analysis period $[0, T_\infty]$).
(2) The Strong FIFO property holds.

Proof: See Appendix 7.3.

Remarks:

- Conditions (B1)-(B2) are the minimal conditions to ensure that the FIFO property is verified. Intuitively, this is true since $B_{2i} - B_{1i}$ represents the maximum variation of the sojourn time in terms of inventory. During a time interval Δt of inventory decrease, the variation of inventory $I_i(t) - I_i(t + \Delta t)$ is bounded by the quantity $\frac{1}{B_{2i}-B_{1i}}.\Delta t$. Therefore, the variation of sojourn time $D_i(I_i(t)) - D_i(I_i(t + \Delta t))$ is bounded by $(B_{2i} - B_{1i}).\frac{1}{B_{2i}-B_{1i}}.\Delta t$. As a result, this variation is bounded by Δt. Hence, $s_i(t) = t + D_i(I_i(t)) \leq t + \Delta t + D_i(I_i(t + \Delta t)) = s_i(t + \Delta t)$, which is the FIFO property. Moreover, during a time interval Δt of inventory increase, since the sojourn time functions are non-decreasing, $D_i(I_i(t)) \leq D_i(I_i(t + \Delta t))$. Therefore, $s_i(t) \leq s_i(t + \Delta t)$, which is the FIFO property.

- Notice that if the product sojourn time function $D_i(.)$ is linear, then conditions (B1)-(B2) of Theorem 2 simplify significantly. Indeed, in this case, $D_i'(.) = cst = B_{1i}$. Moreover, for any arbitrarily small positive scalar ϵ, by introducing $B_{2i} = B_{1i} + \epsilon$, Condition (B1) of Theorem 2 is verified. Furthermore, since Condition (B2) can be rewritten as $u_i(t) \leq \frac{1}{\epsilon}$, which allows us to consider production rates that are arbitrarily large. Therefore, the following corollary follows.

- This result suggests that the maximum variation of the sojourn time of a product in terms of its inventory connects with the production rate of the product. In particular, when the maximum variation of the sojourn time of a product in terms of its inventory is large (or small), then the production rate of the product should be small (or large) in order to sell the units of this product using FIFO.

Corollary 1 *If the pair $(D_i(.), u_i(.))$ satisfies the following conditions:*
(C1) The product sojourn time function $D_i(.)$ is linear and non-negative.
(C2) The production flow rate function $u_i(.)$ is Lebesgue integrable and non-negative.
Then, conditions (B1)-(B2) of Theorem 2 also hold.

In summary, the results of this subsection establish that by constraining the production capacity with the maximum variation of sojourn time with inventory:

- The effect of the variation of inventory with time can be limited, so that the FIFO property holds.

- When the FIFO property holds, the feasible region $F(DPM)$ is non-empty, and we can uniquely determine the sales flow rate and inventory in terms of the production flow rate.

5.1.2 Tightness of the Conditions of Theorem 2

In this subsection, we illustrate using a counter-example that conditions (B1)-(B2) in Theorem 2 are tight.

Theorem 3 *For any arbitrarily small positive scalar δ, there exist a product sojourn time function $D_i(.)$ and a production flow rate function $u_i(.)$ that verify the following conditions*
(D1) $D_i(.)$ is continuously differentiable and nondecreasing;
(D2) $u_i(.)$ is non-negative, Lebesgue integrable and bounded from above by M_i;
(D3) $\frac{1}{M_i} < Max\{D_i'(I_i), I_i \in R\}$ - $Min\{D_i'(I_i), I_i \in R\} \leq \frac{1}{M_i} + \delta$,
violating the FIFO property.

Proof: To show this, we will construct a production flow rate function $u_i(.)$ and a product sojourn time function $D_i(.)$ such that $(u_i(.), D_i(.))$ verify conditions (D1)-(D3) of Theorem 3, violating the FIFO property.

Let δ and M_i be any positive scalars, B_{1i} and β be any non-negative scalars, and ϵ and α be any two positive scalars such that: $\epsilon < \alpha$. Let ω be a positive scalar such that $\omega \in (\alpha, 2\alpha - \epsilon)$.

We first construct the product sojourn time function $D_i(.)$. We define $D_i(.)$ on three contiguous intervals: $[0, (\alpha-\epsilon)M_i]$, $((\alpha-\epsilon)M_i, \alpha M_i)$ and $[\alpha M_i, +\infty)$. On the first and third intervals, $D_i(.)$ is affine with a slope on its first affine piece less than the slope on its second affine piece. On the second interval, $D_i(.)$ is an exponential, nondecreasing and continuously differentiable function. Let I_{i1}, y_{i1}, I_{i2}, y_{i2}, γ_{i1} and γ_{i2} be given by:

$$\begin{array}{lll} I_{i1} = (\alpha-\epsilon)M_i & \text{and,} & y_{i1} = D_i(I_{i1}) = \alpha + B_{1i}(\alpha-\epsilon)M_i, \\ I_{i2} = \alpha M_i & \text{and,} & y_{i2} = D_i(I_{i2}) = \beta + (B_{1i} + \frac{1}{M_i} + \delta)\alpha M_i, \\ \gamma_{i1} = \frac{B_{1i}+\frac{1}{M_i}+\delta}{y_{i2}} - \frac{2}{I_{i2}-I_{i1}} & \text{and,} & \gamma_{i2} = \frac{B_{1i}}{y_{i1}} - \frac{2}{I_{i1}-I_{i2}}. \end{array}$$

Consider the following product sojourn time function $D_i(.)$:

$$D_i(I_i) = \begin{cases} \alpha + B_{1i}I_i & \text{, if} \quad I_i \in [0, (\alpha-\epsilon)M_i] \\ y_{i2}(\frac{I_i - I_{i1}}{I_{i2}-I_{i1}})^2 e^{\gamma_{i1}(I_i - I_{i2})} + y_{i1}(\frac{I_i - I_{i2}}{I_{i1}-I_{i2}})^2 e^{\gamma_{i2}(I_i - I_{i1})} & \text{, if} \quad I_i \in ((\alpha-\epsilon)M_i, \alpha M_i) \\ \beta + (B_{1i} + \frac{1}{M_i} + \delta)I_i & \text{, if} \quad I_i \in [\alpha M_i, +\infty). \end{cases}$$

Notice that $D_i(.)$ is continuously differentiable and nondecreasing on $[0, +\infty)$.

Consider the production flow rate function $u_i(.)$ given by:

$$u_i(t) = \begin{cases} M_i & \text{, if} \quad t \in [0, \omega), \\ 0 & \text{, if} \quad t \in [\omega, +\infty). \end{cases}$$

Notice that functions $u_i(.)$ and $D_i(.)$, as defined above, verify conditions (D1)-(D3) of Theorem 3.

In what follows, we solve constraints (4)-(7) of the $F(DPM)$ on intervals $[0, \alpha-\epsilon)$ and $[\alpha, \omega]$. That is, we express the variables $U_i(.)$, $v_i(.)$, $V_i(.)$, $I_i(.)$, and $s_i(.)$ in terms of the data above. Then, we show that the FIFO property is violated at $t = \omega$.

Notice that for $t \in [0, \alpha-\epsilon)$, $u_i(t) = M_i$. Hence, $U_i(t) = M_i t$. Furthermore, since $\alpha - \epsilon \leq \alpha = D_i(0) = t_1$, it follows that $V_i(t) = 0$. Thus, $I_i(t) = U_i(t) - V_i(t) = M_i t$.

For $t \in [\alpha, \omega]$, there exists $z \in [0, \alpha-\epsilon)$ such that $s_i(z) = t$. Hence $z + D_i(I_i(z)) = t$. Thus, $z + \alpha + B_{1i}M_i z = t$. It follows that $z = s_i^{-1}(t) = \frac{t-\alpha}{1+B_{1i}M_i}$. Using equation (8) of the DPM formulation, that describes the relationship between the cumulative sales and the production flow rate, we obtain

$$V_i(t) \;=\; \int_0^{s_i^{-1}(t)} u_i(w)dw = \int_0^{\frac{t-\alpha}{1+B_{1i}M_i}} M_i dw = \frac{(t-\alpha)}{1+B_{1i}M_i}M_i.$$

Hence, $v_i(t) = \frac{M_i}{1+B_{1i}M_i}$. Therefore, we obtain $I_i(t) = M_i t - \frac{t-\alpha}{1+B_{1i}M_i}M_i = \frac{\alpha M_i + B_{1i}M_i^2 t}{1+B_{1i}M_i}$. Since $t \geq \alpha$, it follows that $I_i(t) \geq \alpha M_i$. Hence, by definition of $D_i(.)$, it follows that $D_i'(I_i(t)) = (B_{1i} + \frac{1}{M_i} + \delta)$.

Next, we show that $s_i'(\omega) < 0$. Indeed, since $s_i'(t) = 1 + D_i'(I_i(t))(u_i(t) - v_i(t))$, it follows that

$$\begin{aligned} s_i'(\omega) &= 1 + D_i'(I_i(\omega))(u_i(\omega) - v_i(\omega)) \\ &= 1 + (B_{1i} + \frac{1}{M_i} + \delta)(0 - \frac{M_i}{1+B_{1i}M_i}) = -\delta\frac{M_i}{1+B_{1i}M_i} < 0. \end{aligned}$$

This implies that the exit time function $s_i(.)$ is strictly decreasing at $t = \omega$. Hence, the FIFO property is violated for $t = \omega$.

□

5.1.3 Properties of the Feasible Region

In this subsection, we present some properties of the $F(DPM)$. These properties are not only useful in understanding the structure of the model, but will also be useful in Subsection 5.2 in order to prove the existence of a solution to the Dynamic Pricing Model.

Let $D = (D_1, ..., D_n)$, $p = (p_1, ..., p_n)$, and $u = (u_1, ..., u_n)$ denote respectively a vector of product sojourn time functions, a vector of unit price functions, and a vector of production flow rate functions. $(D(.), p(.), u(.))$ is feasible if each component $(D_i, p_i, u_i(.))$ verifies conditions (B1)-(B2) of Theorem 2 as well as capacity equation (7). In this case, using Theorem 2, the product inventory functions $I_i(.)$, the sales flow rates $v_i(.)$, and the cumulative variables can be uniquely determined through the product sojourn time functions $D_i(.)$, the unit price functions $p_i(.)$, and the production rates $u_i(.)$ on the analysis period $[0, T_\infty]$.

Proposition 2 *Assume that for every product i, the unit price function p_i is bounded by a scalar p_i^{max}. Then, the feasible region $F(DPM)$ is non-empty and bounded.*

Proof:
First, notice that $(D(.), p(.), 0)$ lies in the feasible region $F(DPM)$.

Further, let CFR denote the minimum total capacity, i.e. $CFR = min_{t \in [0,T]}(CFR(t))$. We assume, without loss of generality, that $CFR > 0$, and show that we can construct a feasible solution $(D(.), p(.), u(.))$ with $u(.) \neq 0$. Given a vector of product sojourn time functions $D(.)$, and a vector of unit price functions $p(.)$, let M denote the scalar $M = min(CFR, \frac{1}{max(\frac{1}{CFR}, (B_{2i} - B_{1i})_{\{i:B_{2i} - B_{1i} > 0\}})})$. Let $(\alpha_1, ..., \alpha_n)$ denote a finite sequence of non-negative scalars such that $\sum_{i=1}^{n} \alpha_i = 1$. For every $i \in \{1, ..., n\}$, and for every $t \in [0, T]$, let $u_i(t) = \alpha_i M$. It follows that vector $(D(.), p(.), u(.))$ as well as every vector $(D(.), p(.), \overline{u}(.))$, with $0 \leq \overline{u}(.) \leq u(.)$, are feasible.

Moreover, from the proof of Theorem 2, it follows that the flow rate functions $u_i(.)$ and v_i are bounded by CFR, the cumulative flow rate functions $U_i(.)$, $V_i(.)$ and $I_i(.)$ are bounded by $CFR.T_\infty$, and the sojourn time functions $D_i(.)$ and the exit time functions $s_i(.)$ are bounded by T_∞. Furthermore, by assumption, the unit price functions $p_i(.)$ are bounded. Therefore, the $F(DPM)$ region is bounded.

□

Proposition 3 *If vectors $(D(.), p(.), u(.))$ and $(D(.), q(.), w(.))$ are feasible, then, for every $\lambda \in [0, 1]$, vector $(D(.), \lambda p(.) + (1 - \lambda)q(.), \lambda u(.) + (1 - \lambda)w(.))$ is also feasible. In this sense, the feasible region $F(DPM)$ is convex.*

Proof:
We assume that $(D(.), p(.), u(.))$ and $(D(.), q(.), w(.))$ are feasible. For any $\lambda \in [0, 1]$, it is easy to see that $(D(.), \lambda p(.) + (1 - \lambda)q(.), \lambda u(.) + (1 - \lambda)w(.))$ verifies conditions (B1)-(B2) of Theorem 2 as well as capacity equation (6).

□

Proposition 4 *If a sequence $(p^j(.))_{j \in \mathbb{N}}$ of vectors of unit price functions converges to $(p(.))$, and a sequence $(u^j(.))_{j \in \mathbb{N}}$ of vectors of production flow rates converges to $(u(.))$, and, if for every j, vector $(D(.), p^j(.), u^j(.))$ is feasible, then, the limit $(D(.), p(.), u(.))$ is also feasible. In this sense, the $F(DPM)$ region is closed.*

Proof:
Let us assume that for all $j \in \aleph$, vectors $(D(.), p^j(.), u^j(.))$ are feasible. Then, it is easy to see that the limit $(D(.), p(.), u(.))$ verifies conditions (B1)-(B2) of Theorem 2 as well as capacity equation (6).

□

5.2 Existence of an Optimal Production/Inventory Control Policy

In this subsection, we establish one of the fundamental results of this paper. That is, we illustrate that under weak assumptions, the DPM Model possesses an optimal solution.

Theorem 4 *Assume that the following conditions hold:*
(E1) The price inventory functions $p_i(I_i)$ are continuously differentiable and bounded from above by scalars p_i^{max}.
(E2) The product sojourn time functions $D_i(.)$ are continuously differentiable, and there exist two non-negative constants B_{1i} and B_{2i} such that for every inventory level I_i, $0 \leq B_{1i} \leq D_i'(I_i) < B_{2i}$.
(E3) The shared capacity flow rate function $CFR(.)$ is Lebesgue integrable, non-negative and bounded from above by $\min_{1,\ldots,n} \frac{1}{B_{2i}-B_{1i}}$.
Then, the Dynamic Pricing Model has an optimal solution.

In order to establish the above result, we first formulate the DPM Model as a variational inequality problem.

5.2.1 A Variational Inequality Formulation for the Dynamic Pricing Model

The DPM Model introduced in Section 3 can be summarized as the problem of finding a vector $e^*(.) = (u_i^*(.), v_i^*(.), I_i^*(.), p_i^*(.))_{i\in\{1,\ldots,n\}} \in F(DPM)$ that maximizes the objective function:

$$\begin{aligned} F(e(.)) &= \sum_{i=1}^{n} \int_0^{T_\infty} f_i(e_i(t))dt \\ &= \sum_{i=1}^{n} \int_0^{T_\infty} p_i(I_i(t))v_i(t) - c_i(t)u_i(t) - h_i(t)I_i(t)dt. \end{aligned}$$

If $G_i(t, e_i(t))$ denotes the gradient $\nabla f_i(e_i(t))$ of $f_i(e_i(t))$, then notice that

$$G_i(t, e_i(t)) = (-c_i(t), p_i(I_i(t)), p_i'(I_i(t)v_i(t) - h_i(t), v_i(t)).$$

Therefore, the DPM Model is equivalent to solving the following variational inequality problem: Find a vector $e^* \in F(DPM)$ satisfying

$$\begin{aligned} &\textstyle\sum_{i=1}^{n} \int_0^{T_\infty} (-c_i(t), p_i(I_i^*(t)), p_i'(I_i^*(t)v_i^*(t) - h_i(t), v_i(t)). \\ &(u_i(t) - u_i^*(t), v_i(t) - v_i^*(t), I_i(t) - I_i^*(t), p_i(I_i(t)) - p_i^*(I_i^*(t)))^T dt \leq 0, \end{aligned} \tag{19}$$

for all vectors $e(.) \in F(DPM)$.
Variational inequality (19) can be written in compact form as: Find a vector $e^* \in F(DPM)$, such that for all vectors $e(.) \in F(DPM)$,

$$< G(e^*), e^* - e >\leq 0, \tag{20}$$

where $< x, y >$ denotes the scalar product $\sum_{i=1}^{n} \int_0^{T_\infty} x_i(t).y_i(t)dt$ of two vectors x and y.
In what follows, we will refer to $G(.)$ as the Dynamic Pricing Map.

5.2.2 Properties of the Dynamic Pricing Map

In this subsection, we establish some properties of the Dynamic Pricing Map $G(.)$. These properties will be useful in establishing that the General Dynamic Pricing Model has a solution. We first introduce a definition from functional analysis (for more details, see Kirillov [22], Kolmogorov and Fomin [23], and Rudin [32]).

Definition 4 *(Weak Continuity):*
(i) A sequence $(u_n)_{n\in\aleph}$ in a normed space is said to converge weakly to u, if, for every bounded linear map $LM(.)$, $(LM(u_n))_{n\in\aleph}$ converges to $LM(u)$.
(ii) A map MP from a normed space to another is said to be weakly continuous if, for every sequence of functions $(u_n)_{n\in\aleph}$ weakly converging to u, the sequence $(||MP(u_n) - MP(u)||)_{n\in\aleph}$ converges to 0.

We establish the weak continuity of the Dynamic Pricing Map.

Theorem 5 *If the price inventory functions $p_i(I_i)$ are continuously differentiable and bounded from above by scalars p_i^{max}, then conditions (B1)-(B2) of Theorem 2 imply that the Dynamic Pricing Map $G(.)$ is weakly continuous.*

Proof: See Appendix 7.4.

We now define the notion of pseudo-monotonicity introduced by Brezis [6] and show that the Dynamic Pricing Map $G(.)$ is pseudo-monotone.

Definition 5 *(Pseudo-monotonicity) A bounded map MP is pseudo-monotone over X if, whenever a sequence $(u^k)_{k\in\aleph} \in X^{\aleph}$ weakly converging to u satisfies limsup $< MP(u^k), u^k - x > \leq 0$, $\forall x \in X$, it also satisfies liminf $< MP(u^k), u^k - x > \geq < MP(u), u - x >$, $\forall x \in X$.*

Lemma 3 *The Dynamic Pricing Map $G(.)$ is pseudo-monotone over the $F(DPM)$ region.*

Proof: Notice that $G(.)$ is weakly continuous on the $F(DPM)$ region, and from Proposition 1, the $F(DPM)$ region is bounded. Therefore, $G(.)$ is a bounded map. Let $diam(F(DPM))$ denote the diameter of the $F(DPM)$ region and let $(e_k)_{k\in\aleph}$ denote a sequence of elements of the $F(DPM)$ region weakly converging to e. Then, for $y \in F(DPM)$,

$$\begin{aligned} < G(e_k) - G(e), e_k - y > &\leq ||G(e_k) - G(e)||.||e_k - y|| \\ &\leq diam(F(DPM)).||G(e_k) - G(e)||. \end{aligned}$$

Since $G(.)$ is weakly continuous on the $F(DPM)$, it follows that the sequence $(||G(e_k) - G(e)||)_{k\in\aleph}$ converges to 0. Hence $lim_{k\to\infty} < G(e_k) - G(e), e_k - y >= 0$. It follows that:

$$\begin{aligned} lim_{k\to\infty} < G(e_k), e_k - y > &= lim_{k\to\infty} < G(e), e_k - y > \\ &= < G(e), e - y > . \end{aligned}$$

Hence, the Dynamic Pricing Map $G(.)$ is pseudo-monotone over the $F(DPM)$ region.

□

5.2.3 Existence of An Optimal Solution for the Dynamic Pricing Model

We are now ready to prove Theorem 4 which establishes the existence of solution for the Dynamic Pricing Model.

Proof of Theorem 4: Under conditions (E1)-(E3), Theorem 5 holds, that is, the Dynamic Pricing Map $G(.)$ is weakly continuous. Using Lemma 3, it follows that the Dynamic Pricing Map $G(.)$ is pseudo-monotone over the $F(DPM)$ region. From Propositions 2-4, the $F(DPM)$ region is non-empty, bounded, closed and convex. Using Lemma 4 with $K = F(DPM)$, $A(.) = G(.)$ and $z = 0$, and the variational inequality formulation (20), it follows that the Dynamic Pricing Model has an optimal solution.

Lemma 4 *(Brezis [6], [7])*
Let K be an non-empty, bounded, convex and closed set. Let $A(.)$ denote a map from K to L that is pseudo-monotone map over K. Then, for every vector $z \in L$, there exists a vector $e^* \in K$ such that $< A(e^*), e - e^* > \geq < z, e - e^* >$ is verified for every vector $e \in K$.

For more details on the above lemma, see [6], [7], [25], [26], and [27].

□

6 Conclusions

In this paper, we studied a continuous-time fluid dynamics model for dynamic pricing and inventory management. We formulated the model as a continuous-time non-linear optimization problem. The key characteristic of this model is that it incorporates the delay of price and level of inventory in affecting demand. By considering the case of a hyperbolic average delay function for product i, $A_i(I_i(t)) = \frac{\delta}{I_i(t)}$ (this corresponds to linear demand arrival rates λ_i in terms of the unit price), linear unit price functions $p_i(I_i(t))$ in terms of the inventory, and finally, piecewise constant production cost, inventory cost and shared production capacity functions ($c_i(t)$, $h_i(t)$, and $CFR(t)$ respectively) we were able to reformulate the model as a quadratic optimization problem. This allowed us to propose a solution algorithm that solves this version of model in this case. We tested this algorithm on a small example and reported on the computational results. We generalized our results to the general DPM Model. In particular, we provided a unifying analysis for both linear and non-linear product delay functions. Under sufficient conditions on the production flow rate functions and the product sojourn time functions, we established that the feasible region of the Dynamic Pricing Model ($F(DPM)$) is non-empty, and that the FIFO property holds. We showed that in the case of linear product sojourn time functions, the assumptions we imposed for non-linear product sojourn time functions simplify significantly. We provided a generic counterexample illustrating that the assumptions we imposed, to ensure that FIFO holds, are the tightest possible. We established key properties of the feasible region, such as boundedness, closedness and convexity. Furthermore, for this general DPM Model, we established under weak assumptions the existence of an optimal pricing/production/inventory control policy that maximizes the profit of the firm over the feasible region.

In summary, some of the insights obtained from the analysis of this paper are the following:

- The fluid model we considered describes the selling rate of a unit of product through its sojourn time in the system. Our motivation is based on the belief that delay (sojourn time) data is easy to acquire but also due to the fact that it is internally controlled by the manufacturer. This approach allowed us to describe the system in greater detail by accounting explicitly how each unit of product waits in inventory before being sold.

- The model studied in this paper connects and is consistent with traditional demand models when the demand-price relationship is linear.

- The discretized version of the model is a Quadratic Programming Model. Its special structure allowed us to devise a solution algorithm that determines efficiently the optimal pricing/production/inventory control policy in a capacitated environment. This policy is dynamic and is based on the equilibration of the marginal profits of the products.

- Furthermore, we generalized our approach without considering a time discretization. We established key properties of the general Dynamic Pricing Model that allowed us to establish that the general model also has a solution.

- For example, the general continuous time Dynamic Pricing Model has a solution when the shared capacity (which is an upper bound on the total production rate) is small (or large) and the maximum variation of the delay in terms of the inventory is large (or small).

We hope that the results of this research will lay the foundations for the use of the delay of price and level of inventory in affecting demand in supply chain and inventory management systems.

7 Appendix

7.1 Lemmas

A *diffeomorphism* is a continuously differentiable function that has a continuously differentiable inverse. The following lemma gives sufficient conditions for a function to be a diffeomorphism. This result will be used to establish, under certain assumptions, that the Dynamic Pricing Model leads to product exit time functions that are diffeormorphisms.

Lemma 5 *Let $g(.)$ be a continuously differentiable function on $[0, T]$. If for every scalar $x \in [0, T]$ $g'(x) \neq 0$, then $g(.)$ is invertible on $[0, T]$, its inverse function $g^{-1}(.)$ is continuously differentiable on $[min(g(0), g(T)), max(g(0), g(T))]$ and, $g^{-1\prime}(x) = \frac{1}{g'(g^{-1}(x))}$.*

Proof: Since $g(.)$ is a continuously differentiable function, then $g'(.)$ is continuous. Since for every $x \in [0, T]$, $g'(x) \neq 0$, then $g'(.)$ has a constant sign. Hence, $g(.)$ is either strictly increasing or strictly decreasing. Since every strictly monotone function is invertible, it follows that $g(.)$ is invertible. Let $g^{-1}(.)$ denote the inverse function of $g(.)$. Then, $g(g^{-1}(x)) = x$. If we differentiate both sides of the above equality, we obtain: $g^{-1\prime}(x)g'(g^{-1}(x)) = 1$. Since $g'(x) \neq 0$ on [0,T], $g'(g^{-1}(x)) \neq 0$. It follows that $g^{-1\prime}(x) = \frac{1}{g'(g^{-1}(x))}$.

□

Remark: In the proof of Theorems 1 and 2, we use Lemma 5 where $g(.)$ is replaced with the product exit time function $s_i(.)$.

Lemma 6 *Let $f(.)$ be a continuous and strictly increasing function on interval $[a,b]$. For $x \in [f(a), f(b)]$, the set $W_x = \{w \in [a,b] | f(w) \leq x\}$ is the interval $[a, f^{-1}(x)]$.*

Proof: The proof follows easily.

Lemma 7 *If $g_i(.)$ is a continuously differentiable function over a compact set $[0,Y]$, then, there exists a scalar $\overline{B}_{2i}$ such that: $\overline{B}_{2i} = Max\{g_i'(x), x \in [0,Y]\}$.*

Proof:
Since $g_i(.)$ is continuously differentiable, $g_i'(.)$ is continuous over the compact set [0,Y]. Therefore, $g_i'(.)$ attains its maximum.

□

Remark: In the proof of Theorems 1 and 2, we use Lemma 7 where $g_i(.)$ is replaced with the product sojourn time function $D_i(.)$ and x is replaced with the inventory level I_i.

7.2 A Preliminary Result

The following theorem is needed in the proof of Theorem 2 of Subsection 5.1.1.

Theorem 6 *If the pair $(D_i(.), u_i(.))$ satisfies the following conditions:*
(A1) The product sojourn time function $D_i(.)$ is continuously differentiable, and there exist two non-negative scalars B_{1i} and B_{2i} such that for every inventory level I_i, $0 \leq B_{1i} \leq D_i'(I_i) < B_{2i}$.
(A2) The production flow rate function $u_i(.)$ is Lebesgue integrable, non-negative and bounded from above by $\overline{M}_i = \frac{M_i}{1+B_{1i}M_i}$ on $[0,T]$, where M_i is a positive scalar.
(A3) $M_i \leq \frac{1}{B_{2i}-B_{1i}}$.
Then, the feasible region $F(DPM)$ has the following properties:
(1) $F(DPM)$ is well defined (that is, the product inventory $I_i(.)$, the sales flow rate $v_i(.)$, and the cumulative variables can be uniquely determined by the product sojourn time function $D_i(.)$ and the production rate $u_i(.)$ on the analysis period $[0, T_\infty]$).
(2) The Strong FIFO property holds.

Before providing the proof of Theorem 6, we establish additional preliminary results. Condition (A3) of Theorem 6 can either hold as an equality or as a strict inequality. The following lemma shows that the proof of Theorem 6 can be reduced to an easier proof where condition (iii) can be replaced by $B_{2i} - B_{1i} = \frac{1}{M_i}$.

Lemma 8 *In Theorem 6, one can assume that $B_{2i} - B_{1i} = \frac{1}{M_i}$.*

Proof: If $B_{2i} - B_{1i} < \frac{1}{M_i}$, let $B_{3i} = B_{1i} + \frac{1}{M_i}$. For every scalar I_i, it follows that $0 \leq B_{1i} \leq D_i'(I_i) < B_{3a}$, since $B_{2i} < B_{3i}$.

□

Consider the following sequence of time instants defined by: $t_0 = 0$, $t_1 = s_i(t_0)$ and $t_{j+1} = s_i(t_j)$. We prove the results of Theorem 6 by induction over the index j of interval $[t_j, t_{j+1})$. We first establish that the induction proof is valid.

Lemma 9 *For every non-negative integer j, $t_{j+1} - t_j \geq D_i(0) > 0$. Furthermore, there exists an integer n, such that $T \in [t_n, t_{n+1})$.*

Proof: For a given non-negative integer j, $t_{j+1} = s_i(t_j) = t_j + D_i(I_i(t_j))$. Therefore, $t_{j+1} - t_j = D_i(I_i(t_j))$. Since $D_i(.)$ is a nondecreasing function and since for every t, $I_i(t) \geq 0$, it follows that $D_i(I_i(t_j)) \geq D_i(0)$. Since by assumption $D_i(0) > 0$, it follows that $t_{j+1} - t_j \geq D_i(0) > 0$.

If $n_0 = \lceil \frac{T}{D_i(0)} \rceil$, it follows that $t_{n_0} \geq T$. Hence, $\text{Max}\{j | t_j \leq T\}$ exists. Let $n = \text{Max}\{j | t_j \leq T\}$. It follows that $n \leq n_0$ and $T \in [t_n, t_{n+1})$.

□

Let Y be defined by $Y = \int_0^T u_i(w)dw$. Y represents the total number of units of product i that are produced. Since the product delay function $D_i(.)$ is continuously differentiable and bounded, using Lemma 7, there exists $\overline{B}_{2i}$ such that $\overline{B}_{2i} = Max\{g_i'(x), x \in R\}$. Below is a series of three lemmas that we need in the induction proof of Theorem 6. The following lemma shows that there exists a constant θ that will serve to construct a lower bound on the product exit time function $s_i(.)$.

Lemma 10 *For every $\widehat{B}_{2i} \in [\overline{B}_{2i}, B_{2i})$ and for every $t \in [0,T]$, it follows that $\theta + \widehat{B}_{2i}u_i(t) \in (0,1]$, where $\theta = \frac{1+(B_{1i}-\widehat{B}_{2i})M_i}{1+B_{1i}M_i}$.*

Proof: Since for every $I_i \in [0, Y]$, $D_i'(I_i) < B_{2i}$, and since $\overline{B}_{2i} = Max\{D_i'(I_i), I_i \in [0,Y]\}$, it follows that $\overline{B}_{2i} < B_{2i}$. Let $\widehat{B}_{2i} \in [\overline{B}_{2i}, B_{2i})$. From condition (A2) of Theorem 6, $0 \leq u_i(t) \leq \frac{M_i}{1+B_{1i}M_i}$. Therefore,

$$1 \leq 1 + \widehat{B}_{2i}u_i(t) \leq 1 + \frac{\widehat{B}_{2i}M_i}{1+B_{1i}M_i}.$$

By subtracting $\frac{\widehat{B}_{2i}M_i}{1+B_{1i}M_i}$ from each side of these inequalities, it follows that:

$$1 - \frac{\widehat{B}_{2i}M_i}{1+B_{1i}M_i} \leq 1 - \frac{\widehat{B}_{2i}M_i}{1+B_{1i}M_i} + \widehat{B}_{2i}u_i(t) \leq 1.$$

Using Lemma 8, we can assume that $B_{2i} = B_{1i} + \frac{1}{M_i}$. Since $\widehat{B}_{2i} < B_{2i}$, it follows that $\widehat{B}_{2i} < B_{1i} + \frac{1}{M_i}$. Hence, $\frac{\widehat{B}_{2i}M_i}{1+B_{1i}M_i} < 1$. Thus, $1 - \frac{\widehat{B}_{2i}M_i}{1+B_{1i}M_i} > 0$. Therefore,

$$0 < 1 - \frac{\widehat{B}_{2i}M_i}{1+B_{1i}M_i} \leq \frac{1+(B_{1i}-\widehat{B}_{2i})M_i}{1+B_{1i}M_i} + \widehat{B}_{2i}u_i(t) \leq 1.$$

It follows that $0 < \theta + \widehat{B}_{2i}u_i(t) \leq 1$.

□

Lemma 11 *For every $t \in [t_0, t_1)$, the set $W = \{w | s_i(w) \leq t\}$ is empty and hence $V_i(t) = 0$.*

Proof: For every $t \in [t_0, t_1)$ and for every $w \in [0, t]$,

$$s_i(w) = w + D_i(I_i(w)) \geq \;\; 0 + D_i(0) \geq t_1 > t.$$

Hence, the set W defined by $W = \{\omega : s_i(\omega) \leq t\}$ is empty for $t \in [t_0, t_1)$. From equation (5), it follows that $V_i(t) = 0$, for $t \in [t_0, t_1)$.

□

The following lemma is needed in the induction step of the proof of Theorem 6.

Lemma 12 *For* $t \in [t_{j+1}, t_{j+2})$,

$$\frac{u_i(s_i^{-1}(t))}{\widehat{B}_{2i}u_i(s_i^{-1}(t)) + \theta} \leq \frac{M_i}{1 + B_{1i}M_i}, \quad (where\ \theta = \tfrac{1+(B_{1i}-\widehat{B}_{2i})M_i}{1+B_{1i}M_i}). \tag{21}$$

Proof: Let $t \in [t_{j+1}, t_{j+2})$. From condition (A2) of Theorem 6, $u_i(s_i^{-1}(t)) \leq \overline{M}_i = \frac{M_i}{1+B_{1i}M_i}$. By multiplying each side of the inequality by $B_{2i} - \widehat{B}_{2i}$, we obtain that

$$\begin{aligned}
(B_{2i} - \widehat{B}_{2i})u_i(s_i^{-1}(t)) &\leq \frac{(B_{2i} - \widehat{B}_{2i})M_i}{1 + B_{1i}M_i} \\
&\leq \frac{1 + (B_{1i} - \widehat{B}_{2i})M_i}{1 + B_{1i}M_i}, \quad (\text{ from Lemma 8 }) \\
&\leq \theta, \quad (\text{by definition of } \theta).
\end{aligned}$$

Since $\theta > 0$ and $\widehat{B}_{2i}u_i(s_i^{-1}(t)) \geq 0$, it follows that: $\frac{B_{2i}u_i(s_i^{-1}(t))}{\widehat{B}_{2i}u_i(s_i^{-1}(t))+\theta} \geq 1$. Hence, from Lemma 8, we obtain that $\frac{u_i(s_i^{-1}(t))}{\widehat{B}_{2i}u_i(s_i^{-1}(t))+\theta} \geq \frac{1}{B_{2i}} \geq \frac{M_i}{1+B_{1i}M_i}$.

□

We are now ready to provide an induction proof that establishes the results of Theorem 6.

Proof of Theorem 6:
The induction proof is over the index j of interval $[t_j, t_{j+1})$. The induction hypothesis for interval $[t_j, t_{j+1})$ is that the following properties hold:
(i) $s_i(.)$ *is differentiable (a.e.) and continuous over* $[t_j, t_{j+1})$, *and* $s_i'(t) \geq \widehat{B}_{2i}u_i(t) + \theta > 0$;
(ii) $v_i(.)$ *is differentiable (a.e.) over* $[t_j, ti + 1)$;
(iii) For every $t \in [t_j, t_{j+1})$, $v_i(t) \leq M_i'$; *and*
(iv) the $F(DPM)$ *has a solution on* $[0, t_{j+1})$ *and this solution is unique (a.e.).*

We first examine the base case on interval $[t_0, t_1)$. We then assume that the induction hypothesis holds for $[t_j, t_{j+1})$ and prove that it holds for $[t_{j+1}, t_{j+2})$. The proof of the base case is an easy application of Lemma 11.

We assume that the production flow rate functions $u_i(.)$ are given. Hence, they are unique (a.e.). Since the integral operator is unique, the integral $U_i(.)$ is also unique (a.e.). In order to prove the uniqueness (a.e.) of a solution to the $F(DPM)$ on each interval of the induction, it remains to prove that $V_i(.)$ and $s_i(.)$ are unique (a.e.) on these intervals. Then by uniqueness of the differentiation operator, $v_i(.)$ is unique (a.e.). Furthermore, using equation (4) and the initial conditions, it follows that $I_i(.) = U_i(.) - V_i(.)$. Hence, $I_i(.)$ is unique (a.e.).

Base Case: Time interval $[t_0, t_1)$.
For $t \in [t_0, t_1)$,

$$\begin{aligned}
I_i(t) &= U_i(t) - V_i(t) \\
&= U_i(t) - 0 \qquad (\ V_i(t) = 0\ ,\ \text{from Lemma 11}\ ) \\
&= \int_0^t u_i(w)dw.
\end{aligned}$$

Hence, $I_i(.)$ is differentiable (a.e.) and continuous on $[t_0, t_1)$. Moreover, since the integral operator is unique and $U_i(.)$ is unique (a.e.), it follows that $I_i(.)$ is unique (a.e.). Furthermore, the function $s_i(t) = t + D_i(I_i(t))$ is continuous, since $I_i(.)$ and $D_i(.)$ are continuous. The exit time function $s_i(.)$ is differentiable and unique (a.e.) on $[t_0, t_1)$, since $I_i(.)$ is differentiable and unique (a.e.) and $D_i(.)$ is differentiable and unique. By differentiating each term in the expression of $s_i(t)$, we obtain, for $t \in [t_0, t_1)$:

$$\begin{aligned} s_i'(t) &= 1 + D_i'(I_i(t))\frac{dI_i(t)}{dt} \\ &= 1 + u_i(t)D_i'(I_i(t)). \end{aligned}$$

Since $u_i(t) \geq 0$ and $D_i'(I_i(t)) \geq 0$, it follows that $s_i'(t) \geq 1$. Using Lemma 10, we obtain: $s_i'(t) \geq \theta + \widehat{B}_{2i}u_i(t) > 0$. Therefore, the product exit time function $s_i(.)$ is differentiable (a.e.) and strictly increasing on $[t_0, t_1)$. Furthermore, from Lemma 11, $V_i(.) = 0$. Thus, $V_i(.)$ is both differentiable and unique (a.e.) over $[t_0, t_1)$ and for $t \in [t_0, t_1)$, $v_i(.)$ is unique (a.e.) and $v_i(t) = 0 \leq M_i'$. Hence, the $F(DPM)$ has a solution on interval $[0, t_1)$ and this solution is unique (a.e.).

Induction Step: Time interval $[t_{j+1}, t_{j+2})$.
From the induction hypothesis, we know that the product exit time function $s_i(.)$ is differentiable and unique (a.e.), continuous and strictly increasing on $[t_j, t_{j+1})$. From Lemma 5, it follows that $s_i^{-1}(.)$ is differentiable and unique (a.e.), and continuous over $[s_i(t_j), s_i(t_{j+1})) = [t_{j+1}, t_{j+2})$. Using Equation (8) in the model formulation, it follows that:

$$\forall t \in [t_{j+1}, t_{j+2}), \qquad Y_i(t) = \int_0^{s_i^{-1}(t)} u_i(w)dw.$$

Since $s_i^{-1}(.)$ is differentiable and unique (a.e.) on $[t_{j+1}, t_{j+2})$, $V_i(.)$ is differentiable and unique (a.e.). By differentiating $V_i(.)$, we obtain: $v_i(t) = (s_i^{-1})'(t)u_i(s_i^{-1}(t))$. Using Lemma 5, $s_i^{-1\prime}(t) = \frac{1}{s_i'(s_i^{-1}(t))}$. Therefore,

$$v_i(t) = \frac{u_i(s_i^{-1}(t))}{s_i'(s_i^{-1}(t))}. \tag{22}$$

Furthermore, from the induction hypothesis, $s_i'(s_i^{-1}(t)) \geq \theta + \widehat{B}_{2i} > 0$. Hence, $0 < \frac{1}{s_i'(s_i^{-1}(t))} \leq \frac{1}{\widehat{B}_{2i}u_i(s_i^{-1}(t))+\theta}$. Since for every $t \in [t_{j+1}, t_{j+2})$, $0 \leq u_i(s_i^{-1}(t)) \leq M_i'$, it follows that:

$$\forall t \in [t_{j+1}, t_{j+2}), \qquad v_i(t) \leq \frac{u_i(s_i^{-1}(t))}{\widehat{B}_{2i}u_i(s_i^{-1}(t)) + \theta}. \tag{23}$$

Using Lemma 12, it follows that $v_i(.)$ is unique (a.e.) and $v_i(t) \leq M_i'$. Therefore, both the production and sales flow rate functions $u_i(.)$ and $v_i(.)$ are bounded from above by M_i'. This shows that if an upper bound is verified at the entrance of a link, it is also maintained at its exit.

If $t_{j+1} \geq T$, the induction ends and the proof is complete. Otherwise, $t_{j+1} < T$, and

$$I_i(t) = \int_0^t u_i(w)dw - \int_0^t v_i(w)dw.$$

Hence, $I_i(.)$ is differentiable (a.e.) and continuous on $[t_{j+1}, t_{j+2})$. Moreover, since the integral operator is unique and both $u_i(.)$ and $v_i(.)$ are unique (a.e.), it follows that $I_i(.)$ is unique (a.e.). Furthermore, the function $s_i(t) = t + D_i(I_i(t))$ is continuous, since both $I_i(.)$ and $D_i(.)$ are continuous. The exit time function $s_i(.)$ is differentiable and unique (a.e.) on $[t_{j+1}, t_{j+2})$, since $I_i(.)$ is differentiable and unique (a.e.) and $D_i(.)$ is differentiable and unique. By differentiating each term in $s_i(t)$, we obtain:

$$s_i'(t) = 1 + D_i'(I_i(t))\frac{dI_i(t)}{dt} = 1 + (u_i(t) - v_i(t))D_i'(I_i(t)), \quad \text{(from equation (4))}.$$

We discuss two cases: $u_i(t) - v_i(t) \geq 0$ and $u_i(t) - v_i(t) < 0$. First, we consider the case $u_i(t) - v_i(t) \geq 0$. Since $D_i'(I_i(t)) \geq B_{1i} \geq 0$, from Lemma 10, it follows that:

$$s_i'(t) \geq 1 \geq \widehat{B}_{2i}u_i(t) + \theta > 0.$$

Now, consider the case $u_i(t) - v_i(t) < 0$. Since $D_i'(I_i(t)) \leq \overline{B}_{2i} \leq \widehat{B}_{2i}$, it follows that:

$$s_i'(t) \geq 1 + \widehat{B}_{2i}(u_i(t) - v_i(t)).$$

Since $v_i(t) \leq M' = \frac{M_i}{1+B_{1i}M_i}$, we obtain:

$$\begin{aligned} s_i'(t) &\geq 1 + \widehat{B}_{2i}u_i(t) - \frac{\widehat{B}_{2i}}{1 + B_{1i}M_i}M \\ s_i'(t) &\geq \widehat{B}_{2i}u_i(t) + \frac{1 + (B_{1i} - \widehat{B}_{2i})M_i}{1 + B_{1i}M_i} \\ s_i'(t) &\geq \widehat{B}_{2i}u_i(t) + \theta > 0. \end{aligned}$$

Hence, we have showed that properties (i)-(iii) of the induction hypothesis hold on interval $[t_{j+1}, t_{j+2})$. Furthermore, we have proved that the $F(DPM)$ has a solution on interval $[0, t_{j+2})$ and that this solution is unique (a.e.).

□

Next, we show that the induction terminates after a finite number steps. This means that a construction algorithm, based on the induction proof of Theorem 6, will determine a feasible point of the $F(DPM)$ region in a finite number of steps.

Lemma 13 *The induction terminates after a finite number of steps, i.e. T_∞ is finite.*

Proof: Let $n_0 = Max\{n \in N, t_n \leq T\}$. From Lemma 9, n_0 exists and $T \in [t_{n_0}, t_{n_0+1})$. The induction proof, at all steps $i \leq n_0$, ensures that $s_i(.)$ is continuous and strictly increasing over $[0, t_{n_0+1})$. Hence, $Max\{s_i(t), t \in [0,T]\} = s_i(T)$ exists and is finite. Since $T_\infty = Max\{s_i(t), t \in [0,T]\}$, it follows that T_∞ is finite. Hence the induction terminates after a finite number of steps.

□

7.3 Proof of Theorem 2

If $B_{1i} = 0$, then $M_i' = M_i$. In this case, both Theorem 6 and Theorem 2 have the same conditions and provide the same result of existence and uniqueness (a.e.) of a solution to the $F(DPM)$. Next, we only consider the case where $B_{1i} > 0$.

Since Theorem 6 and 2 have in common the first and the third conditions, using Lemma 8, one can assume $B_{2i} = B_{1i} + \frac{1}{M_i}$ in the proof of Theorem 2. In the proof to follow, we will assume that: $B_{1i} > 0$ and $B_{2i} = B_{1i} + \frac{1}{M_i}$.

Consider now the following sequence of time instants defined by: $t_0 = 0$, $t_1 = s_i(t_0)$ and $t_{j+1} = s_i(t_j)$. We prove the results of Theorem 2 by induction over the index j of interval $[t_j, t_{j+1})$. Let Y be the defined by $Y = \int_0^T u_i(w)dw$. Below, we provide two preliminary results that we use in the proof of Theorem 2.

Lemma 14 *There exists* $\widehat{B}_{2i} \in [\overline{B}_{2i}, B_{2i})$ *such that* $\theta_j = \frac{1+(B_{1i}-\widehat{B}_{2i})M}{1+B_{1i}M+\sum_{k=2}^{i}(\widehat{B}_{2i}M)^k} \in (0,1)$.

Proof: From condition (B1) of Theorem 2, $\forall I_i \in [0, Y]$, $D_i'(I_i) < B_{2i}$. Using Lemma 7, $\overline{B}_{2i} < B_{2i}$.

Let $\widehat{B}_{2i} = Max(\frac{\overline{B}_{2i}+B_{2i}}{2}, \frac{\frac{1}{M_i}+B_{2i}}{2})$. From Lemma 8, we can assume that $B_{2i} = B_{1i} + \frac{1}{M_i}$. Since $B_{1i} > 0$, it follows that $B_{2i} > \frac{1}{M_i}$. Hence, $\widehat{B}_{2i} \in [\overline{B}_{2i}, B_{2i})$.

Since $\widehat{B}_{2i} < B_{2i}$, it follows that $\widehat{B}_{2i} - B_{1i} < B_{2i} - B_{1i}$. Using Condition (B3) of Theorem 2, we obtain that $\widehat{B}_{2i} - B_{1i} < \frac{1}{M_i}$. Thus, $1 + (B_{1i} - \widehat{B}_{2i})M > 0$. Since B_{1i}, $\widehat{B}_{2i}$ and M are positive, it follows that the denominator of θ_j is greater than 1, and hence $\theta_j > 0$. Furthermore,

$$1 + (B_{1i} - \widehat{B}_{2i})M < 1 + B_{1i}M < 1 + B_{1i}M + \sum_{k=2}^{i}(\widehat{B}_{2i}M)^k.$$

Since $\theta_j = \frac{1+(B_{1i}-\widehat{B}_{2i})M}{1+B_{1i}M+\sum_{k=2}^{i}(\widehat{B}_{2i}M)^k}$, it follows that $\theta_j < 1$.

□

Lemma 15 is essential for the induction step in the proof of Theorem 2.

Lemma 15 *For any interval index* j, *and for every* $t \in [t_{j+1}, t_{j+2})$,

$$\frac{u_i(s_i^{-1}(t))}{\widehat{B}_{2i}u_i(s_i^{-1}(t)) + \theta_j} \leq \alpha_{i+1}, \quad (24)$$

where $\alpha_j = \frac{M\sum_{k=0}^{i-1}(\widehat{B}_{2i}M)^k}{1+B_{1i}M+\sum_{k=2}^{i}(\widehat{B}_{2i}M)^k}$.

Proof: By replacing α_{i+1} with its value given above, inequality (24) is equivalent to:

$$\frac{u_i(s_i^{-1}(t))}{\widehat{B}_{2i}u_i(s_i^{-1}(t)) + \theta_j} \leq \frac{M\sum_{k=0}^{i}(\widehat{B}_{2i}M)^i}{1 + B_{1i}M + \sum_{k=2}^{i+1}(\widehat{B}_{2i}M)^i}.$$

It follows that $u_i(s_i^{-1}(t))(1+B_{1i}M+\sum_{k=2}^{i+1}(\widehat{B}_{2i}M)^k) \leq (u_i(s_i^{-1}(t))\widehat{B}_{2i}+\theta_j)M\sum_{k=0}^{i}(\widehat{B}_{2i}M)^k$. Through algebraic manipulations of the above expression, inequality (24) can be equivalently rewritten as

$$u_i(s_i^{-1}(t))(1+B_{1i}M+\sum_{k=2}^{i+1}(\widehat{B}_{2i}M)^k) \leq u_i(s_i^{-1}(t))\sum_{k=1}^{i+1}(\widehat{B}_{2i}M)^k+\theta_j M\sum_{k=0}^{i}(\widehat{B}_{2i}M)^k.$$

Hence, $u_i(s_i^{-1}(t))(1+B_{1i}M) \leq u_i(s_i^{-1}(t))\widehat{B}_{2i}M+\theta_j M\sum_{k=0}^{i}(\widehat{B}_{2i}M)^k$. Thus, it follows that:

$$u_i(s_i^{-1}(t))(1+(B_{1i}-\widehat{B}_{2i})M)-\theta_j M\sum_{k=0}^{i}(\widehat{B}_{2i}M)^k \leq 0.$$

Using Lemma 8, $1+(B_{1i}-\widehat{B}_{2i})M = (B_{2i}-\widehat{B}_{2i})M$. Since $\widehat{B}_{2i} \leq B_{2i} = B_{1i}+\frac{1}{M_i}$, it follows that: $\frac{\sum_{k=0}^{i}(\widehat{B}_{2i}M)^k}{1+B_{1i}M+\sum_{k=2}^{i+1}(\widehat{B}_{2i}M)^k} \leq 1$. Hence, we obtain:

$$u_i(s_i^{-1}(t))(B_{2i}-\widehat{B}_{2i})M-(1+(B_{1i}-\widehat{B}_{2i})M)M \leq 0.$$

Thus, $u_i(s_i^{-1}(t))(B_{2i}-\widehat{B}_{2i})-(B_{2i}-\widehat{B}_{2i})M \leq 0$. By dividing each term of the inequality with the positive scalar $B_{2i}-\widehat{B}_{2i}$, it follows that $u_i(s_i^{-1}(t)) \leq M$. Using Condition (B3) of Theorem 2, we verify the inequality.

□

We are now ready to provide an induction proof that establishes Theorem 2.

Proof of Theorem 2:
Recall that the induction proof is over the index j of interval $[t_j, t_{j+1})$. The induction hypothesis for interval $[t_j, t_{j+1})$ is that the following properties hold:
(i) $s_i(.)$ *is differentiable (a.e.) and continuous over* $[t_j, t_{j+1})$, *and* $s_i'(t) \geq \widehat{B}_{2i}u_i(t)+\theta_j$;
(ii) $V_i(.)$ *is differentiable (a.e.) over* $[t_j, t_{j+1})$;
(iii) $\forall t \in [t_j, t_{j+1})$, $v_i(t) \leq \alpha_j$; *and*
(iv) the $F(DPM)$ *has a solution on* $[0, t_{j+1})$ *and this solution is unique (a.e.).*

Base Case: Time interval $[t_0, t_1)$.

From Lemma 11, for every $t \in [t_0, t_1)$, $V_i(t) = 0$. The proof of this Base Case is similar to the proof of the first Base Case of Theorem 6. As a result, we do not provide it here.

Induction Step: Time interval $[t_{j+1}, t_{j+2})$.

From the induction hypothesis, we know that the link exit time function $s_i(.)$ is differentiable and unique (a.e.), continuous, and strictly increasing on $[t_j, t_{j+1})$. From Lemma 5, it follows that $s_i^{-1}(.)$ is differentiable and unique (a.e.), and continuous over $[s_i(t_j), s_i(t_{j+1})) = [t_{j+1}, t_{j+2})$. Using equation 8 in the model formulation, it follows that

$$\forall t \in [t_{j+1}, t_{j+2}), \qquad V_i(t) = \int_0^{s_i^{-1}(t)} u_i(w)dw.$$

Since $s_i^{-1}(.)$ is differentiable and unique (a.e.) on $[t_{j+1}, t_{j+2})$, $V_i(.)$ is differentiable and unique (a.e.). By differentiating $V_i(.)$, we obtain: $v_i(t) = (s_i^{-1})'(t)u_i(s_i^{-1}(t))$. Using Lemma 5, it follows that:

$$v_i(t) = \frac{u_i(s_i^{-1}(t))}{s_i'(s_i^{-1}(t))}.$$

Furthermore, from the induction hypothesis, $s_i'(s_i^{-1}(t)) \geq \theta_j + \widehat{B}_{2i}u_i(s_i^{-1}(t)) > 0$. Hence, $0 < \frac{1}{s_i'(s_i^{-1}(t))} \leq \frac{1}{\widehat{B}_{2i}u_i(s_i^{-1}(t))+\theta_j}$. Note that for every $t \in [t_{j+1}, t_{j+2})$, $0 \leq u_i(s_i^{-1}(t)) \leq M$. Thus,

$$\forall t \in [t_{j+1}, t_{j+2}), \qquad v_i(t) \leq \frac{u_i(s_i^{-1}(t))}{\widehat{B}_{2i}u_i(s_i^{-1}(t)) + \theta_j}. \tag{25}$$

Using Lemma 15, it follows that $v_i(.)$ is unique (a.e.) and $v_i(t) \leq \alpha_{i+1}$.

If $t_{j+1} \geq T$, the induction ends and the proof is complete. Otherwise, $t_{j+1} < T$, and

$$I_i(t) = \int_0^t u_i(w)dw - \int_0^t v_i(w)dw.$$

Hence, $I_i(.)$ is differentiable (a.e.) and continuous on $[t_{j+1}, t_{j+2})$. Moreover, since the integral operator is unique and both $u_i(.)$ and $v_i(.)$ are unique (a.e.), it follows that $I_i(.)$ is unique (a.e.). Furthermore, the function $s_i(t) = t + D_i(I_i(t))$ is continuous, since both $I_i(.)$ and $D_i(.)$ are continuous. The exit time function $s_i(.)$ is differentiable and unique (a.e.) on $[t_{j+1}, t_{j+2})$, since $I_i(.)$ is differentiable and unique (a.e.), and $D_i(.)$ is differentiable and unique. By differentiating each term in $s_i(t)$, we obtain:

$$\begin{aligned} s_i'(t) &= 1 + D_i'(I_i(t))\frac{dI_i(t)}{dt} \\ &= 1 + (u_i(t) - v_i(t))D_i'(I_i(t)). \end{aligned}$$

We discuss two cases: $u_i(t) - v_i(t) \geq 0$ and $u_i(t) - v_i(t) < 0$. First, consider the case $u_i(t) - v_i(t) \geq 0$. Since $D_i'(I_i(t)) \geq B_{1i} \geq 0$, it follows that $s_i'(t) \geq 1 > \theta_j$. Now, consider the case $u_i(t) - v_i(t) < 0$. Since $D_i'(I_i(t)) \leq \overline{B}_{2i} \leq \widehat{B}_{2i}$, it follows that: $s_i'(t) \geq 1 + \widehat{B}_{2i}(u_i(t) - v_i(t))$. Since $v_i(t) \leq \alpha_{i+1}$, we obtain:

$$\begin{aligned} s_i'(t) &\geq 1 + \widehat{B}_{2i}u_i(t) - \widehat{B}_{2i}\frac{M\sum_{k=0}^{i}(\widehat{B}_{2i}M)^k}{1 + B_{1i}M + \sum_{k=2}^{i+1}(\widehat{B}_{2i}M)^k}, \\ s_i'(t) &\geq \widehat{B}_{2i}u_i(t) + \frac{1 + (B_{1i} - \widehat{B}_{2i})M}{1 + B_{1i}M + \sum_{k=2}^{i+1}(\widehat{B}_{2i}M)^k}, \\ s_i'(t) &\geq \widehat{B}_{2i}u_i(t) + \theta_{i+1} > 0. \end{aligned}$$

Hence, we have showed that properties (i)-(iii) of the induction hypothesis hold on interval $[t_{j+1}, t_{j+2})$. Furthermore, we have proved that the $F(DPM)$ has a solution on interval $[0, t_{j+2})$ and that this solution is unique (a.e.). The proof of Theorem 2 is now complete.

□

From Lemma 13, T_∞ is finite and the induction terminates after a finite number of steps. This means that a construction algorithm, based on the induction proof of Theorem 2, will determine a feasible point of the $F(DPM)$ region in a finite number of steps.

7.4 Proof of Weak Continuity of the Dynamic Pricing Map

The proposition below summarizes some results from functional analysis that are useful to prove Theorem 5 (for more details, see Kirillov [22], and Kolmogorov and Fomin [23]).

Proposition 5 *[22], [23]*
(i) If f and g are two weakly continuous maps, then the maps $f+g$, $f.g$ and $f(g)$ are weakly continuous.
(ii) If f is a weakly continuous map on the set of real numbers and has a constant sign, then the map $\frac{1}{f}$ is weakly continuous.
(iii) The integral operator from the space of bounded functions on $L^1([0,T_\infty])$ to $L^2([0,T_\infty])$ defined as $u(.) \mapsto \int_0^t u(w)dw$ is weakly continuous.

Proof of Theorem 5: Property (iii) of Proposition 5 implies that $u_i(.) \mapsto U_i(.)$ is weakly continuous. We will prove by induction over the time intervals $[t_j, t_{j+1})$ (defined in the proof of Theorem 2), that the maps $u_i(.) \mapsto V_i(.)$, $u_i(.) \mapsto I_i(.)$, $u_i(.) \mapsto s_i(.)$, $u_i(.) \mapsto v_i(.)$, $u_i(.) \mapsto s_i^{-1}(.)$ and $u_i(.) \mapsto (s_i^{-1})'(.)$ are weakly continuous.

We first need to establish a preliminary result.

Lemma 16 *Under conditions (B1)-(B3) of Theorem 2, if the product exit time operator $u_i \mapsto s_i(.)$ is weakly continuous on the interval $[t_j, t_{j+1})$, then its inverse operator $u_i \mapsto s_i^{-1}(.)$ is weakly continuous on the interval $[t_{j+1}, t_{j+2})$.*

Proof: We assume that the product exit time operator $u_i \mapsto s_i(.)$ is weakly continuous on the interval $[t_j, t_{j+1})$.

From the proof of Theorem 2 in Subsection 7.3, we know that for every $t \in [t_j, t_{j+1})$, $s_i'(t) \leq \theta_j$, where $\theta_j \in (0,1)$ as defined in Lemma 14. Hence, $s_i^{-1}(.)$ is Lipschitz continuous on $[t_{j+1}, t_{j+2})$ with parameter $\frac{1}{\theta_j}$.

Furthermore, for every $t \in [t_j, t_{j+1})$,

$$\begin{aligned} s_i'(t) = 1 + D_i'(I_i(t))(u_i(t) - v_i(t)) &\leq 1 + D_i'(I_i(t))u_i(t), \\ &\leq 1 + B_{2i}M_i. \end{aligned}$$

Hence, $s_i(.)$ is Lipschitz continuous on $[t_j, t_{j+1})$ with parameter $1 + B_{2i}M_i$.

Let $(u_i^k(.))_{k\in\mathbb{N}}$ denote a weakly converging sequence of product flow rate functions to $u_i(.)$. Let $s_i^k(.)$ denote the product exit time function corresponding to $u_i^k(.)$.

Furthermore,

$$\begin{aligned} \int_{t_{j+1}}^{t_{j+2}} |(s_i^k)^{-1}(w) - s_i^{-1}(w)|^2 dw &= \int_{s_i(t_j)}^{s_i(t_{j+1})} |(s_i^k)^{-1}(w) - s_i^{-1}(w)|^2 dw \\ &= \int_{t_j}^{t_{j+1}} |(s_i^k)^{-1}(s_i(w)) - s_i^{-1}(s_i(w))|^2 s_i'(w) dw \\ &= \int_{t_j}^{t_{j+1}} |(s_i^k)^{-1}(s_i(w)) - w|^2 s_i'(w) dw \\ &= \int_{t_j}^{t_{j+1}} |(s_i^k)^{-1}(s_i(w)) - (s_i^k)^{-1}(s_i^k(w))|^2 s_i'(w) dw \\ &\leq \frac{1 + B_{2i}M_i}{\theta_j^2} \int_{t_j}^{t_{j+1}} |s_i^k(w) - s_i(w)|^2 dw. \end{aligned}$$

Since $u_i \mapsto s_i(.)$ is weakly continuous on the interval $[t_j, t_{j+1})$, it follows that $u_i \mapsto s_i^{-1}(.)$ is weakly continuous on the interval $[t_{j+1}, t_{j+2})$.

□

Induction Proof:
Base Case: Time interval $[t_0, t_1)$.
On $[t_0, t_1)$, $v_i(t) = V_i(t) = 0$ and $I_i(t) = U_i(t)$. Hence, the maps $u_i(.) \mapsto V_i(.)$, $u_i(.) \mapsto I_i(.)$, and $u_i(.) \mapsto v_i(.)$ are weakly continuous. Furthermore, since $s_i(t) = t + D_i(I_i(t))$, and $D_i(.)$ are continuous functions (and therefore weakly continuous), using property (i) of Proposition 5, it follows that the map $u_i(.) \mapsto s_i(.)$ is weakly continuous. Using Lemma 16 and property (i) of Proposition 5, it follows that the map $u_i(.) \mapsto s_i^{-1}(.)$ is also weakly continuous on $[t_1, t_2)$.

Moreover $(s_i^{-1})'(t) = \frac{1}{s_i'(s_i^{-1}(t))} = \frac{1}{1+D_i'(I_i(s_i^{-1}(t)))(u_i(s_i^{-1}(t))-v_i(s_i^{-1}(t))}$. Using properties (i) and (ii) of Proposition 5, we obtain that $u_i(.) \mapsto (s_i^{-1})'(.)$ is also weakly continuous on $[t_1, t_2)$.

Induction Step: Time interval $[t_{j+1}, t_{j+2})$. From the induction hypothesis, we know that the maps $u_i(.) \mapsto s_i^{-1}(.)$ and $u_i(.) \mapsto (s_i^{-1})'(.)$ are weakly continuous on $[t_{j+1}, t_{j+2})$. Since $v_i(w) = u_i(s_i^{-1}(w)).(s_i^{-1})'(w)$, using property (i) of Proposition 5, it follows that the map $u_i(.) \mapsto v_i(.)$ is weakly continuous. Property (iii) of Proposition 5 implies that $u_i(.) \mapsto V_i(.)$ is weakly continuous. Since $I_i(.) = U_i(.) - V_i(.)$ and $s_i(t) = t + D_i(I_i(t))$, property (i) of Proposition 5 implies that $u_i(.) \mapsto I_i(.)$ and $u_i(.) \mapsto s_i(.)$ are weakly continuous.

Using Lemma 16, it follows that $u_i(.) \mapsto s_i^{-1}(.)$ is also weakly continuous on interval $[t_{j+2}, t_{j+3})$. Moreover, since $(s_i^{-1})'(t) = \frac{1}{1+D_i'(I_i(s_i^{-1}(t)))(u_i(s_i^{-1}(t))-v_i(s_i^{-1}(t))}$, using properties (i) and (ii) of Proposition 5, we obtain that $u_i(.) \mapsto (s_i^{-1})'(.)$ is also weakly continuous on $[t_{j+2}, t_{j+3})$. The induction proof is now complete. Since the price inventory function $p_i(I_i)$ is continuously differentiable, both $p_i(I_i)$ and $p_i'(I_i)$ are continuous (and hence weakly continuous). Property (i) of Proposition 5 implies that the Dynamic Pricing Map $G(.)$, defined in Subsection 5.2.1, is weakly continuous.

□

Acknowledgments

Preparation of this paper was supported, in part, by the PECASE Award DMI-9984339 from the National Science Foundation, the Charles Reed Faculty Initiative Fund, the New England University of Transportation Research Grant and the Singapore MIT Alliance Program.

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