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FACILITY LOCATIONS WITH THE L_1 METRIC IN THE PRESENCE OF BARRIERS TO TRAVEL

by

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ABSTRACT

This paper considers the optimal location of p facilities in the plane, under the assmumption that all travel occurs according to the L_1 (or rectilinear or Manhattan) metric in the presence of impenetrable polygonal barriers to travel. Facility users are distributed over a finite set of demand points, with the weight of each point proportional to its demand intensity. Each demand point is assigned to the closest facility. The objective is to locate facilities so as to minimize average L_1 travel distance to a random demand. It is shown that an optimal set of facility locations can be drawn from a finite set of candidate points, all of which are easy to determine.

Determining the locations of p facilities in a network or space so as to minimize the average distance between facilities and users is commonly called the p-median problem (p=1,2,3,...). Due to its wide applicability, the p-median problem has enjoyed much attention from the operations research and transportation science communities during the last two decades.

A primary concern of p-median research has been to reduce the size of the set of feasible locations one must consider in seeking the optimal solution. When the entire problem is restricted to a given network, with users located only at nodes, Hakimi in 1964 showed that an optimal solution to the p-median problem exists on the nodes. This result reduced the problem of continuous search to a combinatorial one. As a consequence there are now several solution procedures, exact as well as approximate, for the network-constrained p-median problem.

In this paper we examine the p-median problem in two-dimensional Euclidean space having fixed polygonal barriers to travel, under the assumption that all travel occurs according to the L_1 (rectilinear right-angle, or Manhattan) metric. The problem is motivated from urban applications, in which the L_1 metric is often a reasonable approximation to travel behavior and in which lakes, parks, cemeteries, rivers, etc. provide impenetrable barriers to travel. There are other potential applications, such as in printed circuit board design, facilities layout, and routing of power lines. The problem is to locate p facilities in the plane (not in the barriers) so as to minimize average distance between facilities and users (who are assumed to be distributed among a finite number of demand points). The analysis is facilitated by recent results by Larson and Li [8] regarding shortest L_1 paths in the presence of barriers.

In related work, Francis and White [4] showed for the L_1 metric in the absence of barriers that an optimal solution exists at points (x_r, y_q), where

(x_r, y_r) and (x_q, y_q) are user demand points. They present a linear programming formulation for the problem, relying on the fact that the problem can be divided into two independent subproblems, an x-problem and a y-problem.

No result like Hakimi's (or Francis and White's) exists for the L_2 metric.

Our primary result in this paper is the following: an optimal solution to the stated p-median problem (in the plane, in the presence of barriers, with the L_1 metric) exists on a finite set of candidate points. The candidate points can be determined by inspection. Thus, as with the original Hakimi result, the search for an optimum is reduced to a combinatorial one.

I. Problem Formulation

We consider a set of user demand points D and a set of barriers vertices V. The set of <u>fixed nodes</u> is $N = D \cup V$, with node $i \in N$ having coordinates (X_i, Y_i). The set of (x,y) points contained within barrier j is B_j, with $B = \bigcup_j B_j$ representing the set through which travel is forbidden. Feasible facility locations are drawn from $F = R^2 - B$, with $(x(k), y(k))$ denoting the location of facility k and $\overline{\underline{X}} =$ $[(x(1), y(1)),\ldots, (x(k), y(k)),\ldots, (x(p), y(p))]$ being a feasible set of facility locations, $(x(k), y(k)) \in F$ for all $k = 1,\ldots,p$.

We let d_{ij} be the minimal feasible L_1 distance between nodes i and j, and $d(i,\overline{\underline{X}})$ is the distance between a demand point i and the closest of the facilities located in $\overline{\underline{X}}$, (closeness is measured according to minimal length feasible L_1 travel paths). The weight of node i is w_i, where $w_i \geq 0$, $\Sigma_i w_i = 1$.

For a given $\overline{\underline{X}}$, the average travel distance between a random demand point and its closest facility is:

$$\overline{J}\,(\overline{\underline{X}}) = \sum_j w_j\, d(j,\overline{\underline{X}}).$$

Our problem is to find feasible $\overline{\underline{X}}^*$ such that for all feasible $\overline{\underline{X}}$, $\overline{J}(\overline{\underline{X}}^*) \leq \overline{J}\,(\overline{\underline{X}})$. $\overline{\underline{X}}^*$ is the optimum set of p median locations for our problem.

II. Formation of a Grid

Consider the smallest rectangle which encloses all fixed nodes, (i.e., user demand points and barrier vertices) as shown in Figure 1. A grid is then formed by passing lines parallel to the X and Y axes through all nodes, without penetrating a barrier or leaving the rectangle (Figure 2); these lines are called <u>node traversal</u> lines.

We call each polygon in the grid which is not a barrier, a "cell." A vertex of cell is called a "corner" to differentiate it from a fixed node (of course, any given point may be both a cell corner and a fixed node). An edge or boundary of a cell is called a "wall."

Some properties of the grid and cells are as follows:

1. Any horizontal or vertical line in the grid passes through a fixed node. Any line segment in the grid which is neither vertical nor horizontal is a part of a barrier edge.
2. A corner has coordinates of the form (X_i,Y_j) or (X_i,y^e) or (x^e,Y_j) where $(X_i y^e)$ and (x^e,Y_j) denote points on some edge e of a barrier.
3. If the coordinates of a corner are of form (X_i,y^e) (or (x^e,Y_j)) we can assume that barrier edge e is neither horizontal nor vertical.
4. For a given vertical (or horizontal) wall of a cell, the two endpoints of the wall cannot be of the form (X_i,y^e), (X_i,y^ℓ) [or of the form (x^e,Y_j), (x^ℓ,Y_j)]. That is, wall endpoints cannot both terminate on barrier edges. (For instance, a

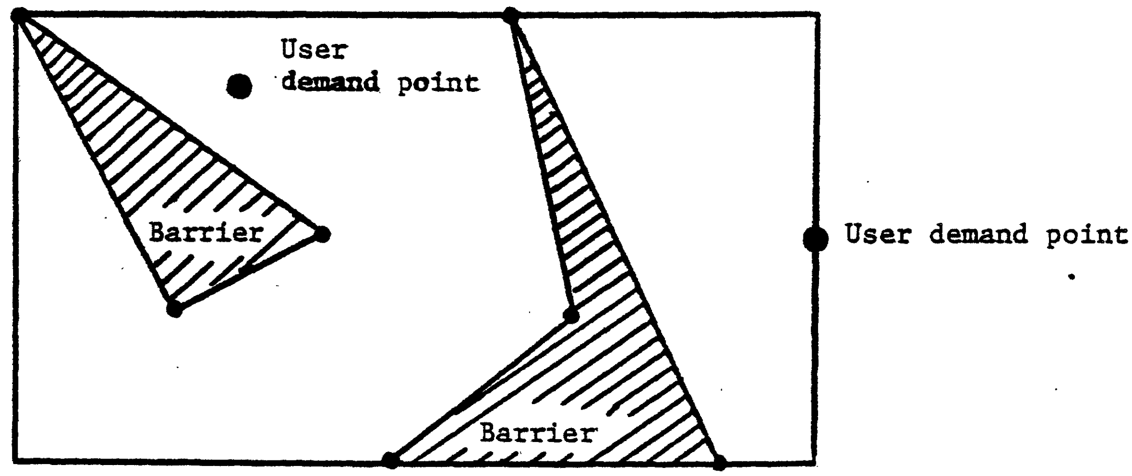

Figure 1: Smallest rectangle enclosing all nodes

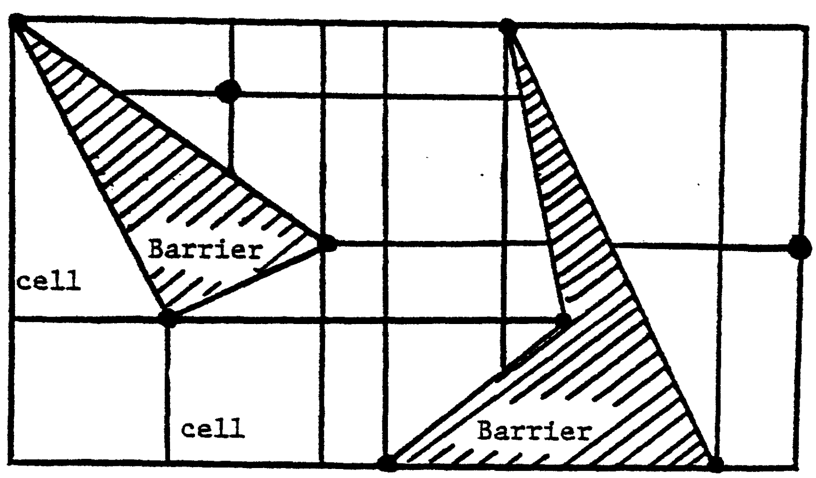

Figure 2: The grid and the cells

vertical wall having one endpoint (X_i,y^e) must pass through fixed node i before intersecting some other barrier edge ℓ, thus terminating the wall prior to intersection with edge ℓ).

We want to show that an optimal solution to the p-median problem exists only at grid corners (X_i,Y_j). This will be proved in several steps. First we will show that a solution on the walls of cells cannot do worse than a solution in the interior of a cell. Next we will show that corners cannot do worse than other points on the walls. Finally, we will show that a facility at a corner (X_i,y^e) can be moved to some corner (X_m,Y_n) without deteriorating the solution.

III. A Network Formulation:

Before proceeding to a network formulation, it is necessary to present a few definitions and results from Larson and Li [8].

i) A stair-case path between (X_i,Y_i) and (X_j,Y_j) is an L_1 path having length $|X_i - X_j| + |Y_i - Y_j|$.

ii) Two points are said to communicate if there is at least one stair-case path between them, i.e, the shortest path between them is not made any longer by the barriers.

iii) Two points are said to communicate simply if they satisfy any of three criteria defined in [8]. One of the three criteria that occurs in our work is that the node traversal lines through the points intersect.

iv) "Path-Push and Amalgamation" is a procedure which operates on any staircase path between nodes i and j to obtain a new equal-length staircase path $i-n_1-n_2 \ldots n_m-j$ where $n_1,\ldots,n_m \in N$ and (i,n_1), $(n_1,n_2),\ldots,(n_m,j)$ are pairs of simply communicating nodes. The new path is called a nodal path.

Consider any feasible solution to the given p-median problem, with facility k located at $(x(k),y(k))$. In computing minimal feasible L_1 distances between facility k and each of the demand points, Larson and Li use "path-push and amalgamation" to show that the original problem in R^2 can be reduced to a network problem. The network associated with facility k is a tree $T(N',A)$, with N' being the set of nodes $N \cup (x(k),y(k))$, A being a set of arcs between simply communicating nodes in N', and $(x(k),y(k))$ being the root node.

Our results rely on a modification to $T(N',A)$. First, we restrict our attention to the subtree $\dot{T} \subseteq T$ containing only demand points that are allocated to facility k, where demand point j is said to be allocated to facility k, if it is closer to facility k than to any other facility. Second, if some demand point in the subtree (say j) does not simply communicate with facility k (i.e., it is not accessible to k via a single link in the subtree), we add its weight w_j to another node q where i) node q simply communicates with facility k and ii) node q lies on a shortest path from node j to facility k. The subtree implied by the Larson/Li results guarantees the existence of such a node q. The geometries of an original problem and the associated modified problem are illustrated in Figures 3, 4.

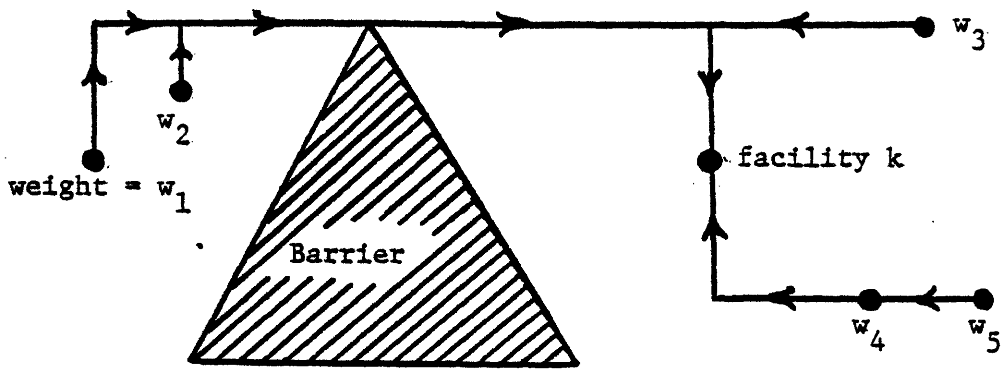

Figure 3: The original demand allocations to facility k

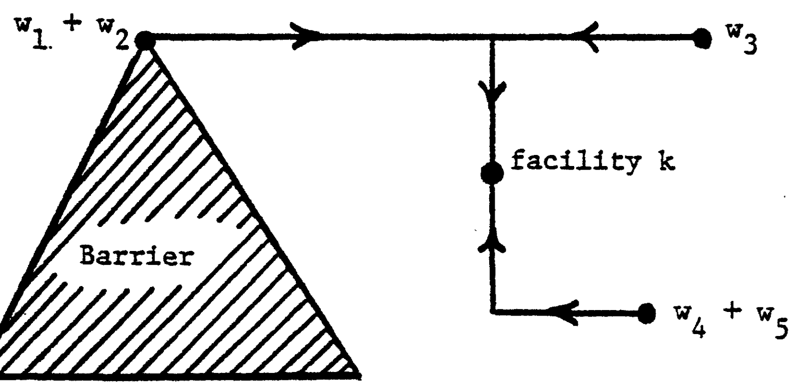

Figure 4: The modified demand allocations to facility k

With the modified problem (Figure 4), all demand to be served by a facility is generated from nodes that simply communicate with the facility. In fact, it can easily be seen that these nodes, called collection points for facility k, simply communicate with all points on the cell containing the facility. So for any set of locations of facilities every demand point in the original problem (Figure 3) has an associated facility-specific collection point, which acts as a representative demand point in the modified problem. The corresponding modified Larson/Li tree contains no nodes not adjacent to the root node at $(x(k), y(k))$.

Suppose $(x^o(k), y^o(k))$ is the position of facility k in a certain feasible solution set $\underline{X}$. The solution obviously consists of optimal assignments (of demand points to facilities) and optimal nodal paths. Now suppose that by moving facility k to $(x^1(k), y^1(k))$ an improvement in the objective function $\bar{J}$ of c units (where $c \geq 0$) is achieved in the corresponding modified problem. Then an improvement of at least c units is achieved in the original problem. This is true because here we are not taking into account the fact that the best allocations and paths may change when the facility is moved to $(x^1(k), y^1(k))$, hence further reducing $\bar{J}$.

IV. Basic Results

In the next several lemmas we will see that an improvement of $c \geq 0$ units can always be achieved in the modified problem, unless $(x(k), y(k))$ is at a corner of a cell.

Lemma 1:

There exists no collection point at (X_q, Y_q) for facility k for which $X_{min} < X_q < X_{max}$ and/or $Y_{min} < Y_q < Y_{max}$ where X_{min}, X_{max}, Y_{min}, Y_{max}

are the respective bounds on x and y in the cell containing $(x(k),y(k))$.

Proof:

Suppose $X_{min} < X_q < X_{max}$ (Figure 5). Then since (X_q,Y_q) communicates with all points in the cell, there must exist a feasible vertical path from (X_q,Y_q) that would divide the cell in two. This is a contradiction to the way the cells are formed. A similar proof applies for $Y_{min} < Y_q < Y_{max}$. ■

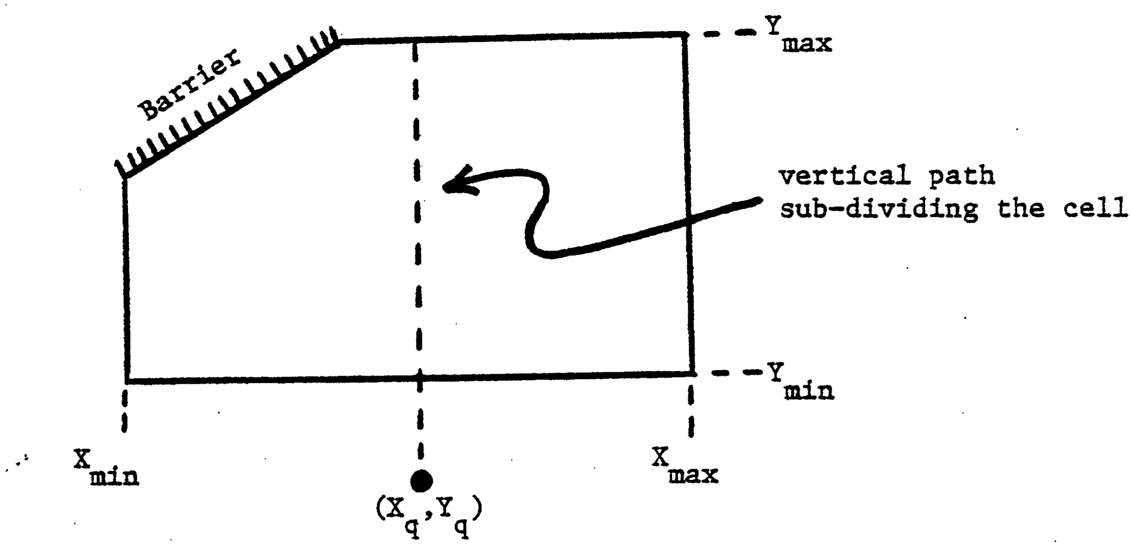

Figure 5: Illustration for Lemma 1

Lemma 2:

Without penalty, all collection points can reach the facility through one of the corners of the cell.

Proof:

From Lemma 1, we know that the collection point at (X_q,Y_q) is such that $X_q \notin (X_{min},X_{max})$ and $Y_q \notin (Y_{min},Y_{max})$.

Suppose the staircase path from (X_q,Y_q) to facility k does not enter the cell at a corner. Then by the "Path Push and Amalgamation Process" the path can be altered without penalty so as to enter at a corner, for otherwise there would be a barrier vertex (X_i,Y_i) with $X_i \in (X_{min},X_{max})$ or $Y_i \in (Y_{min},Y_{max})$ in which case the cell should have been subdivided - a contradiction. ■

Lemma 3:

Let $(x(k),y(k))$ be in the interior of a cell. The solution cannot be worsened by moving $(x(k),y(k))$ to a boundary point of the cell.

Proof:

Let w_1 be the weight of all collection points (X_j,Y_j) with $Y_j \le Y_{min}$ and w_2 of collection points (X_j,Y_j) with $Y_j \ge Y_{max}$. Let y_1 and y_2 be the minimum and maximum value that y can achieve within the cell at $x = x(k)$ (Figure 6). The y-distance component of the objective function is:

$$D_Y = w_1(y(k)-Y_{min}) + w_2(Y_{max} - y(k)) + \text{a constant term} = (w_1-w_2)\,y(k)$$
$$+ \text{terms independent of } y(k).$$

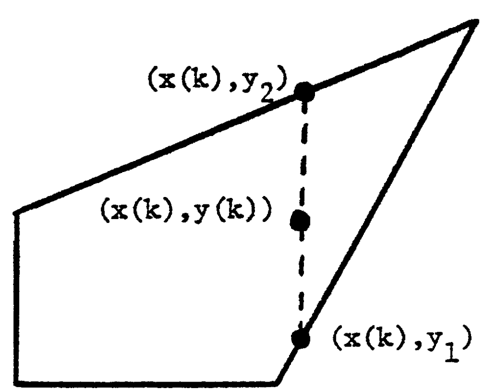

Figure 6: Diagram for Lemma 3

Obviously D_y is minimized at an extreme value of $y(k)$, i.e. Y_1 or Y_2 depending on whether $w_1 - w_2 \geq 0$ or ≤ 0. ■

Lemma 4:

The solution cannot be worsened by moving a facility from a cell wall (excluding corners) to a corner.

Proof:

If the wall containing the facility is vertical or horizontal, the proof is identical to the proof of Lemma 3.

Now suppose that the boundary containing facility k has a slope $s \neq 0$ or $s \neq \infty$, extending from (x_1, y_1) to (x_2, y_2). We prove the result for the case $0 \leq s < \infty$ when the barrier is immediately to the right of facility k. (other cases are proved similarly). We can write the objective function as:

$$D_x + D_y = w_1 \, [(x(k) - x_{min}) + (y(k) - y_{min})] + w_2 \, [(x(k) - x_{min}) + (y_{max} - y(k))] + w_3 \, [(x_{max} - x(k)) + (y_{max} - y(k))] + \text{constant},$$

where w_1, w_2, w_3 correspond to weights associated with collection points that are, respectively, southwest, northwest, and northeast of the cell containing the facility.* Simplifying and using the fact that on the boundary $y(k) = \alpha + sx(k)$ for some constant α, we have:

$$D_x + D_y = \beta + x(k) \, [(1+s)w_1 + (1-s)w_2 - (1+s)w_3],$$

or a linear function of $x(k)$. Thus $D_x + D_y$ is minimized at an extreme value of $x(k)$, either X_1 or X_2, corresponding to a corner, (X_1, Y_1) or (X_2, Y_2). ■

Combining the results of Lemmas 3 and 4, we have thus far proved that $(x(k), y(k))$ must be at some corner of a cell.

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*Here we have associated the positive x direction with East and the positive y direction with North.

We would now like to exclude all those corners which are of the form (x^e, Y_j) or (X_i, y^e).

Lemma 5:

Moving $(x(k), y(k))$ from (x^e, Y_j) or (X_i, y^e) to some adjacent or "nearly adjacent" corner with coordinates (X_m, Y_n) cannot worsen the solution. ("Nearly adjacent" is defined in the proof).

Proof:

Let the facility be at (x^e, Y_j). We assume the previous orientation of the barrier with edge e (Figure 7), having slope s. Since (x^e, Y_j) is not a vertex of edge e (otherwise it would be of form (X_i, Y_i)), it must be at a corner shared by exactly two cells, say cells 1 and 2.

Let (x_1, y_1) and (x_2, y_2) be the other corners of cells 1 and 2 on edge e. Also, let (X_i, Y_j) be the other endpoint of the boundary partitioning the two cells.

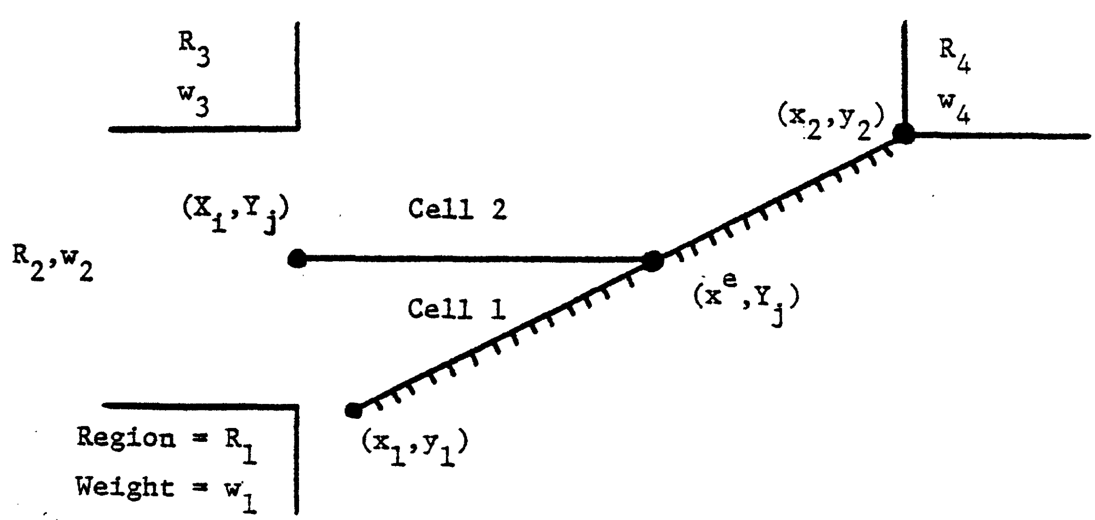


Figure 7: Diagram for Lemma 5

Assuming collection points at (X_q,Y_q), we identify four collection point regions as follows:

R_1 (weight w_1): $X_q \leq \text{Min}\ (X_i,x_1)$, $Y_q \leq y_1$

R_2 (weight w_2): $X_q \leq \text{Min}\ (X_i,x_1)$, $Y_q = Y_j$

R_3 (weight w_3): $X_q \leq \text{Min}\ (X_i,x_1)$, $Y_q \geq y_2$

R_4 (weight w_4): $X_q \geq x_2, Y_q \geq y_2$.

Suppose we move the facility to $(x(k),y(k))$, defining $\Delta x \equiv x(k) - x^e$, $\Delta y \equiv y(k) - Y_j$. For the three possible linear movements along cell walls away from (x^e,Y_j), we can write:

$$\begin{aligned}(D_x + D_y)_1 &= c + \Delta x\ (w_1[1+s] + w_2\ [1+s] + w_3\ [1-s] + w_4\ [-1-s]) \\ &= c + \beta_1 \Delta x \\ (D_x + D_y)_2 &= c + \Delta x\ (-w_1 - w_2 - w_3 + w_4) \\ &= c + \beta_2 \Delta x \\ (D_x + D_y)_3 &= c + \Delta x\ (w_1\ [-1-s] + w_2\ [-1+s] + w_3\ [-1+s] + w_4\ [1+s]) \\ &= c + \beta_3 \Delta x,\end{aligned}$$

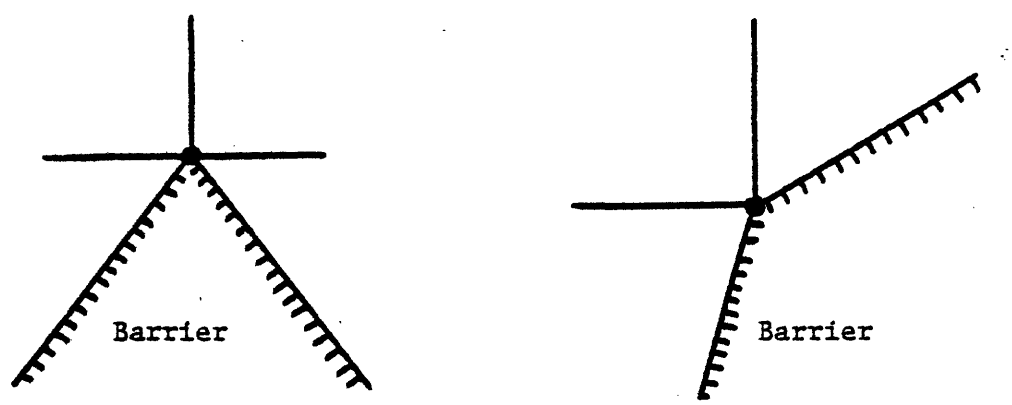
corresponding to movement toward R_4, R_2 and R_1, respectively; the constant c is the average distance to (x^e,Y_j). If $\beta_2 > 0$, $w_4 > w_1+w_2+w_3$, implying $\beta_1 < 0$. Hence, at least one of the coefficients of Δx, β_1, β_2 or β_3, must be non-positive, implying improvement or at least nondeterioration in the objective functions at one of the three corners adjacent to (x^e,Y_j). If (x_1,y_1) or (x_2,y_2) is preferred and is not a barrier vertex, we repeat the argument until we have reached a point (a "nearly adjacent" corner) of the form (X_i,Y_j) or a barrier vertex. ■

Summarizing, we have now proved that in the modified problem, each facility can be moved to a point (X_i,Y_j) without deteriorating the solution. But this automatically implies the same result in the original problem.

Hence we conclude that an optimal solution to the stated p-median problem is at (X_i, Y_j) where points X_i and Y_j are such that a horizontal line through (X_i, Y_i) and a vertical line through (X_j, Y_j) intersect each other before intersecting a barrier or leaving the rectangle.

V. A Further Result

The potential sites for facilities can be reduced even further. It can be shown that there is no need to include node traversal lines from certain barrier vertices. In particular, the lines need not be included unless they can be extended on both sides of the vertex. (See Figure 8)



(a) Vertical node traversal line is superfluous

(b) Both node traversal lines are superfluous

Figure 8: Illustration of Superfluous Node Traversal Lines

Lemma 6:

A node traversal line which can be extended in only one direction from a barrier is superfluous in the sense that it can be removed without increasing the minimum feasible value of the objective function. Any two cells partitioned by a superfluous node traversal line can be treated as a single cell.

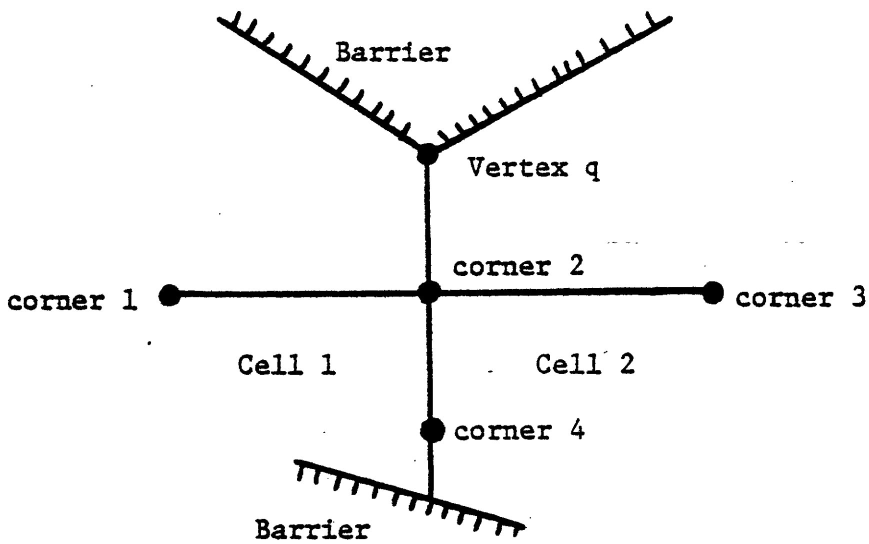


Figure 9: Diagram for Lemma 6

Proof:

Let cells 1 and 2 be partitioned by a superfluous node traversal line emerging from vertex q (Figure 9). Let corner 2 be one of the two common corners. Let corners 1 and 3 be the adjacent corners in the two cells. We show that any path entering cell 1 or cell 2 through corner 2, can enter through corners 1 or 3 without penalty.

For any collection point (X_j,Y_j), we must have $X_j \leq X_{min}^{(1)}$ (where $X_{min}^{(1)}$ is the minimal value of x in cell 1), or $X_j = x(k)$, or $X_j \geq X_{max}^{(2)}$ (where $X_{max}^{(2)}$ is the maximal value of x in cell 2).

If $X_j \leq X_{min}^{(1)}$ or $X_j \geq X_{max}^{(2)}$, then the staircase path from the collection point to the facility (which is in either of the two cells) may enter the cells at corners 1 or 3, respectively.

If $X_j = x(k)$ then, we need to retrace the optimal path from the original demand point to the facility. Suppose the path is (Demand point) $-n_1 \ldots$

$-n_{m-1}$ $-n_m$ -facility, where n_m = vertex q. Clearly, the x coordinate of node n_{m-1}, $X_{n_{m-1}}$ is such that $X_{n_{m-1}} \leq X^{(1)}_{min}$ or $X_{n_{m-1}} \geq X^{(2)}_{max}$. Again by the path push amalgamation argument, we can alter the path without penalty, causing it to enter at corner 1 or corner 3 rather than corner 2. Hence corner 2 is unnecessary. Similarly the other corner (corner 4) common between the two cells can be shown to be unnecessary, implying that cells 1 and 2 may be treated as a single cell. ■

VI. Solving the P-Median Problem

Once we have reduced the stated p-median problem to a discrete search problem, we can use any of the existing algorithms available for solving the p-median on a network.

An attempt to solve the given p-median problem consists of the following two steps.

1. Setting up the problem, which requires
   i) Identifying the candidate points
   ii) Finding the distance matrix for the network whose node set is $N \cup$ {candidate points not in N}.
2. Solving the problem, whose solution set is combinatorially large.

The complexity of step 1 is determined by the complexity of finding the distance matrix which is $0(n^4)$. The complexity of step 2 obviously depends on the algorithm one chooses.

Due to the problem's considerable size, one of the future steps is to find those algorithms which best exploit its structure. Even if one has as few as 30 fixed nodes, there may be as many as 900 potential sites for the facilities. Exact procedures therefore, are likely to become unwieldy and one might be forced to rely on efficient heuristics.

VII. A Simple Example

Consider the following example of two barriers and three demand points (Figure 10).

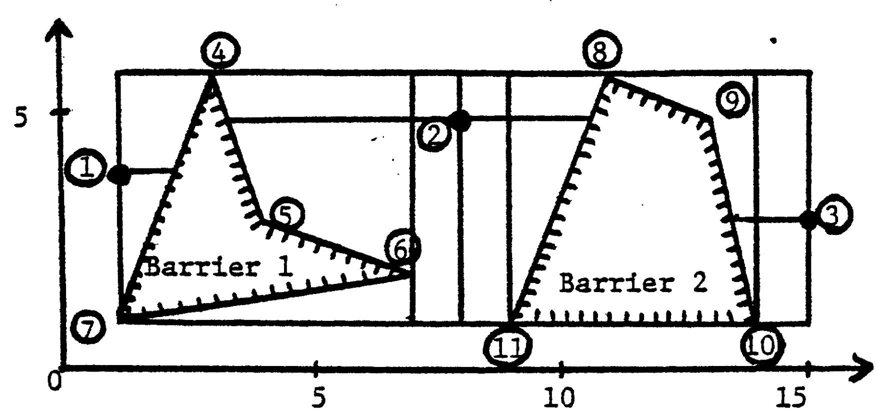


Figure 10: Illustration of the Example

The demand points are nodes 1, 2 and 3 and carry weights 0.3, 0.4 and 0.3, respectively. The barrier vertices are nodes 4 through 11. Table I shows the coordinates (X_i,Y_i) for $i = 1, 2, \ldots 11$.

Node	Coordinates
1	(1,4)
2	(8,5)
3	(15,3)
4	(3,6)
5	(4,3)
6	(7,2)
7	(1,1)
8	(11,6)
9	(13,5)
10	(14,1)
11	(9,1)

Table I: Coordinates (X_i,Y_i)

After forming the grid as explained in section II and taking into account the result in Section V, we identify 23 candidate points (for location of p facilities) as:

(1,6), (3,6), (7,6), (8,6), (9,6), (11,6), (14,6), (15,6), (7,5), (8,5), (9,5), (13,5), (1,4), (4,3), (14,3), (15,3), (7,2), (1,1), (7,1), (8,1), (9,1), (14,1) and (15,1)

(Note that with the result of Section V, there would have been 10 more candidate points created by 5 node traversal lines through nodes 5, 6 and 9).

The solution to 1, 2 and 3 medians (obtained by observation) is shown in Table II.

$\underline{P}$	$\underline{\bar{X}}^*$	$\bar{J}(\underline{\bar{X}}^*)$
1	(8,6)	$\cdot 3(9)+\cdot 4(1)+\cdot 3(10) = 6\cdot 1$
2	(8,5),(15,3)	$\cdot 3(10)+\cdot 4(0)+\cdot 3(0) = 3$
3	(1,4),(8,5),(15,3)	0

Table II: Solutions To The Simple Example

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