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Efficiency of Scalar-Parameterized Mechanisms

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We consider the problem of allocating a fixed amount of an infinitely divisible resource among multiple competing, fully rational users. We study the efficiency guarantees that are possible when we restrict to mechanisms that satisfy certain scalability constraints motivated by large scale communication networks; in particular, we restrict attention to mechanisms where users are restricted to one-dimensional strategy spaces. We first study the efficiency guarantees possible when the mechanism is not allowed to price differentiate. We study the worst-case efficiency loss (ratio of the utility associated with a Nash equilibrium to the maximum possible utility), and show that the proportional allocation mechanism of Kelly (1997) minimizes the efficiency loss when users are price anticipating. We then turn our attention to mechanisms where price differentiation is permitted; using an adaptation of the Vickrey-Clarke-Groves class of mechanisms, we construct a class of mechanisms with one-dimensional strategy spaces where Nash equilibria are fully efficient. These mechanisms are shown to be fully efficient even in general convex environments, under reasonable assumptions. Our results highlight a fundamental insight in mechanism design: when the pricing flexibility available to the mechanism designer is limited, restricting the strategic flexibility of bidders may actually *improve* the efficiency guarantee.

Subject classifications: Game/group decisions: noncooperative, bidding/auctions; Networks/graphs: theory; Utility/preference: theory

1. Introduction

We consider the problem of allocating a fixed amount of an infinitely divisible resource among multiple competing, fully rational users, a problem that has received extensive attention in recent literature on pricing of communication resources (Shenker et al. 1996, Falkner et al. 2000). We investigate a fundamental question: what is the best achievable efficiency when users are fully rational? We study the efficiency guarantees that are possible when we restrict to mechanisms that satisfy certain constraints motivated by large scale communication networks.

Our starting point is a mechanism introduced by Kelly (1997). For the special case of a single resource in fixed supply, his scheme is easily described. Each user chooses a total payment, or *bid*. The resource is then allocated to users in proportion to their bids, and each user pays an amount equal to his bid; equivalently, a market-clearing price is set equal to the sum of the bids divided by the total amount of available resource.

This scheme has two desirable properties: first, the strategy space of the users is “simple”, i.e., one-dimensional; and second, the feedback required from the mechanism to the users is a single price per unit of resource. Both features are central requirements for a mechanism to be practical in the context of large networks. In order to keep the communication from the users to the network simple, it is reasonable to restrict users to one-dimensional bids; we refer to mechanisms that restrict users to such bids as *scalar-parameterized mechanisms*. Further, in communication networks, mechanisms that use “aggregate feedback” are considered scalable (Shenker 1990); this is one motivation for mechanisms that choose only a single price. The Kelly mechanism is designed to incorporate both of these design principles simultaneously (see also Kelly et al. 1998, Srikant 2004).

Kelly (1997) considered a model where each user is endowed with a concave, increasing utility, which is a function of the received resource allocation. He showed that if users do not anticipate the effect of their bid choices on market-clearing prices, then at equilibrium, a fully efficient allocation is achieved—that is, an allocation where the aggregate utility of the users has been maximized. However, this analysis assumed that users are not fully rational, and do not fully anticipate the effects of their bids on prices. By contrast, Johari and Tsitsiklis (2004) considered a game theoretic model of the Kelly mechanism where users are price anticipating, and established that the aggregate utility at a Nash equilibrium is no worse than 75% of the maximal aggregate utility.

In this paper, we investigate the *optimality* of the result of Johari and Tsitsiklis (2004). Our results provide two major insights. First, we show that if we restrict attention to a class of mechanisms where users only submit one-dimensional bids, and the mechanism only chooses a single price, then the Kelly mechanism exhibits the best possible efficiency guarantee. Formally, we consider a

class of market-clearing mechanisms; in these mechanisms each user submits a scalar-parameterized demand function, and a single price is chosen to clear the market. Under reasonable assumptions, we show that the Kelly mechanism minimizes the worst case efficiency loss at a Nash equilibrium when users are price anticipating. This result complements the work of Maheswaran and Basar (2004): they reached similar conclusions for a qualitatively different class of mechanisms involving differentiated prices that are functions of individual bids, as opposed to our single prices that are functions of all bids. See Section 2.1 for details.

Our second major insight is aimed at understanding the importance of the *single price* assumption in the preceding result. To investigate this point, we preserve the assumption that bids are one-dimensional, but remove the restriction that the mechanism only chooses a single price. We ask: how much can we improve the efficiency guarantee if the mechanism is allowed to choose a different price for each user? Using a class of mechanisms inspired by the Vickrey-Clarke-Groves (VCG) approach (Vickrey 1961, Clarke 1971, Groves 1973), we show that if users are restricted to using one-dimensional bids, but the mechanism can price differentiate (i.e., set a different price for each user) then fully efficient Nash equilibria can be achieved—not only in a single resource setting, but also in general convex environments. We call these mechanisms *scalar strategy VCG (SSVCG) mechanisms*.¹ We note that as a special case, SSVCG mechanisms contain an earlier class of mechanisms obtained by Yang and Hajek (2006a) and Maheswaran (2003); we further discuss this relationship in Sections 2.2 and 5.2.

Outline The remainder of the paper is organized as follows. In Section 2, we discuss related work. In Section 3, we give the mathematical framework of our optimal mechanism design problem. We restrict attention to allocation of a single infinitely divisible resource among multiple users. Each user is endowed with a concave, strictly increasing, differentiable utility function, and makes strategic decisions to maximize his utility less payment.

In Section 3.1, we begin by embedding the Kelly mechanism into a broader class of market-clearing mechanisms. We make a simple but crucial observation: a user’s bid θ establishes a relation

¹ A similar result was independently and simultaneously presented by Yang and Hajek (2006b, 2007); see Section 2.2.

$d = \theta/p$ between the quantity d allocated to the user, and the market-clearing price p per unit of the resource. This can be viewed as having the user submit a demand function $D(p, \theta)$, which specifies demand as a function of p . The user is, however, limited to a particular one-parameter family of demand functions, demand functions of the form $D(p, \theta) = \theta/p$; choosing a value for the parameter θ is all the freedom given to the user.

While the Kelly mechanism assumes a specific form for $D(p, \theta)$, an entire family of market-clearing mechanisms results if we consider alternative forms for $D(p, \theta)$. This added generality retains the simplicity of the Kelly mechanism: each user selects from a one-dimensional strategy space, and a single price is charged to all users. Formally, we restrict attention to *smooth market-clearing mechanisms* defined by a differentiable parameterized demand function $D(p, \theta)$.² In these mechanisms, each user r chooses a strategy $\theta_r \geq 0$. A price $p^* = p_D(\boldsymbol{\theta})$ is then chosen to *clear the market*, i.e., so that $\sum_r D(p^*, \theta_r) = C$, where C is the supply, the amount of available resource. Each user r then receives an allocation $D(p^*, \theta_r)$, and pays $p^* D(p^*, \theta_r)$. A smooth market-clearing mechanism is thus entirely specified by the parametric family of demand functions $D(p, \theta)$. We emphasize that although the class of demand functions is constrained in this way, no such constraints are imposed on the true utility functions of the users (other than concavity and monotonicity).

In Section 3.2, we recapitulate existing results regarding the Kelly mechanism. We note that the Kelly mechanism yields full efficiency when users are price taking, and guarantees an *efficiency* of 75%: i.e., the aggregate utility at a Nash equilibrium is no less than 75% of the maximal aggregate utility when users are price anticipating, and this bound is tight.

With this background, our objective is to maximize the market's efficiency. In particular, we wish to determine whether the 75% guarantee of the Kelly mechanism can be improved. In Section 4, we consider a class \mathcal{D} of smooth market-clearing mechanisms where: (1) users' payoffs are concave when users are *price anticipating*, i.e., when they fully anticipate the functional form of the market-clearing price $p_D(\boldsymbol{\theta})$; and (2) for each p , the range of $D(p, \theta)$, as θ varies over the nonnegative real

² Some notion of smoothness is required to ensure that dimension is a reasonable measure of strategic flexibility; for example, in the absence of a differentiability requirement, an arbitrary amount of information can be communicated in a single scalar, e.g., using a space-filling curve.

numbers, is the entire interval $[0, \infty)$. We explain why these assumptions are reasonable, and give examples of mechanisms in \mathcal{D} beyond the Kelly mechanism.

We show in Theorem 1 that in this class of mechanisms, the Kelly mechanism minimizes the worst case efficiency loss when users are price anticipating, and is the unique mechanism with this property. In the process, we also give a structural description of all mechanisms in this class: any mechanism satisfying the given assumptions must be of the form $D(p, \theta) = \theta/B(p)$, where B is concave, strictly increasing, and invertible on $(0, \infty)$. Notably, this structural characterization implies any such mechanism is a *proportional allocation mechanism*: the allocation is always made in proportion to the bids, as is true for the Kelly mechanism. Thus mechanisms in \mathcal{D} differ only in the prices they choose for any given vector of users' bids.

For any smooth market-clearing mechanism, users have one-dimensional strategy spaces, and the mechanism only chooses a single price. Because of these constraints, even the highest performance mechanism—the Kelly mechanism—suffers a positive efficiency loss. In Section 5, we consider the implications of removing the “single price” constraint, while retaining the constraint that strategy spaces must be one-dimensional. We show in Section 5.2 that if we consider mechanisms with scalar strategy spaces, and allow the mechanism to choose a different price for each user, then in fact full efficiency is achievable at Nash equilibrium; this is shown via a class of mechanisms called SSVCG mechanisms (see above); we show that as a special case of this mechanism, one can recover the mechanism and main conclusions of Yang and Hajek (2006a) and Maheswaran (2003). The result involves adapting the well-known class of Vickrey-Clarke-Groves (VCG) mechanisms to the case of scalar strategy spaces. For SSVCG mechanisms, we are also able to consider far more general resource allocation environments. In particular, we show that for a very general class of convex resource allocation problems, if the utility of a user depends only on a scalar function of the vector of resources allocated to that user, then SSVCG mechanisms ensure that full efficiency is achievable at Nash equilibrium. We show that an efficient Nash equilibrium always exists, and also give reasonable sufficient conditions under which all Nash equilibria are efficient. As an illustration,

we apply our SSVCG framework to a standard multicommodity flow problem. We conclude with a discussion of extensions and open problems in Section 6.

2. Related Work

In this section we review threads of the existing literature related to our work. One body of recent work concerns the characterization of efficiency loss in scalar-parameterized mechanisms; we compare this work with the results of Section 4. A second thread of the literature studies the design of scalar-parameterized mechanisms that achieve full efficiency; we compare this work with the results of Section 5.

2.1. Efficiency Loss of Scalar-Parameterized Mechanisms

In Section 4, we show that in the class \mathcal{D} of smooth market-clearing mechanisms, the Kelly mechanism uniquely minimizes the worst case efficiency loss when users are price anticipating. The efficiency loss of the Kelly mechanism was first studied by Johari and Tsitsiklis (2004); subsequently, several others have studied efficiency loss of related resource allocation mechanisms, including Chen and Zhang (2005), Johari et al. (2005), Johari and Tsitsiklis (2006), Maheswaran (2003), Maheswaran and Basar (2004), Moulin (2007), Sanghavi and Hajek (2004) and Yang and Hajek (2006a). More generally, several works in the recent literature aim at quantifying efficiency losses in games, e.g., for routing (Czumaj and Voecking 2002, Koutsoupias and Papadimitriou 1999, Mavronicolas and Spirakis 2001), traffic networks (Correa et al. 2004, Roughgarden and Tardos 2002), network design problems (Anshelevich et al. 2003, Fabrikant et al. 2003), and supply chain management (Roels and Perakis 2007); for a more comprehensive discussion on work in this area, the reader is referred to the survey text of Nisan et al. (2007).

The most closely related result to ours is presented by Maheswaran and Basar (2004). They consider mechanisms where each user r chooses a bid w_r , and the allocation is made proportional to each user's bid. However, rather than assuming that every user pays w_r as in the Kelly mechanism, Maheswaran and Basar consider a class of mechanisms where the user pays $c(w_r)$, where c is a

convex function.³ They show that in this class of mechanisms, the Kelly mechanism (i.e., a linear function c) achieves the minimal worst case efficiency loss when users are price anticipating.

We note that in general, in Maheswaran and Basar’s model, different users are charged *different* per unit prices for the resource they obtain (i.e., pricing is *nonuniform*), unless c is linear—exactly the Kelly mechanism. By contrast, in *every* mechanism in the class of mechanisms we consider, all users are charged the same per unit price (i.e., the market-clearing price), where the price is a function of the vector of all bids, which depends on the specific mechanism chosen. Thus our results prove optimality of the Kelly mechanism in a distinct regime from Maheswaran and Basar.

We also note that our work is substantially different in another way from Maheswaran and Basar, because we do not postulate *a priori* that the mechanism must allocate the resource in proportion to users’ bids; instead, this emerges as a *consequence* of our rather natural assumptions on the mechanisms in \mathcal{D} . Several other works on efficiency of resource allocation mechanisms, including Maheswaran and Basar (2004) and Yang and Hajek (2006a), assume *a priori* that allocations are made in proportion to users’ bids. In this sense, our result lends a rigorous foundation to the intuition that proportional allocation mechanisms—those that allocate the resource in proportion to users’ bids—yield a natural approach to resource allocation among competing users.

We conclude this section by noting that because the class \mathcal{D} induces constraints on the communication both *to* and *from* the mechanism, a strictly positive efficiency loss is inevitable. Our approach in Section 4 can be viewed as an optimization problem that aims at minimizing efficiency loss subject to communication constraints.

2.2. Fully Efficient Scalar-Parameterized Mechanisms

In Section 5, we relax the assumption that all users must pay the same per-unit price for the resource. We construct a class of scalar strategy VCG mechanisms that ensure that Nash equilibria are fully efficient. The literature in both economics and more recently in operations research has

³ We note that the authors’ assumption that c is convex is largely to ensure existence of a Nash equilibrium; in our context, a similar role is played by the second condition in Definition 4.

a significant body of work related to this problem. The seminal results of Vickrey (1961), Clarke (1971), and Groves (1973) established that if users can submit entire utility functions, and the mechanism can choose individualized prices (i.e., one price per user), then full efficiency can be achieved as a dominant strategy equilibrium. In our setting, however, because of the restriction to one-dimensional strategy spaces, it is not possible to achieve full efficiency in dominant strategies.

Reichelstein and Reiter (1988) presented a key general result in the theory of mechanism design with restricted strategy spaces. Their paper calculates the minimal strategy space dimension that allows for fully efficient Nash equilibria for a general class of economic models known as *exchange economies*. Their results apply to general convex environments, such as those studied in Section 5.3; however, their mechanisms require a strategy space per user of dimension $(J - 1) + J/(R(R - 1))$, where J is the number of resources being allocated (e.g., the number of links in a network), and are “asymmetric,” in the sense that different users are treated differently, even if they all choose the same strategies. We are able to use a strategy space smaller than their bound in part because each user’s utility depends only on a scalar function of the allocation to that user, and in part because we assume quasilinear preferences.

More recently, for the special case of multicommodity flow, Semret (1999) presented a class of mechanisms similar to the VCG class, but where each user has a two-dimensional strategy space; the author demonstrates existence of an approximately efficient Nash equilibrium, but does not establish conditions under which all equilibria are efficient. Maheswaran and Basar (2004) (as well as the earlier work, Maheswaran 2003) and Yang and Hajek (2006a) independently suggested a mechanism for allocation of a single infinitely divisible resource in fixed supply. They demonstrated that Nash equilibria of this mechanism are efficient. Remarkably, as we will see in Section 5, their mechanism can be recovered as a special case of the SSVCG mechanisms we develop.

We note here that simultaneously and independently of our work described in Section 5, Yang and Hajek (2007) proposed a class of VCG-like mechanisms with scalar strategy spaces; these results were first presented by Yang and Hajek (2006b). In some respects their work is more specialized than ours: they restrict attention to network resource allocation settings, and consider a less general

class of mechanisms. However, their paper also considers various extensions not discussed here: “soft” capacity constraints, specified through penalty functions; revenue of SSVCG mechanisms; and dynamic algorithms that converge to efficient Nash equilibria.

Subsequent to our work (Johari and Tsitsiklis 2005) and the work of Yang and Hajek (2007), several papers have presented related constructions of mechanisms that use limited communication yet achieve fully efficient Nash equilibria (Dimakis et al. 2006, Stoenescu and Ledyard 2006, Moulin 2006). Building on the work of Semret (1999) discussed above, Dimakis et al. (2006) establish that a VCG-like mechanism where agents submit a pair of strategies (requested price and quantity) can achieve fully efficient equilibrium for a related resource allocation game. Stoenescu and Ledyard (2006) consider the problem of resource allocation by building on the notion of minimal message spaces addressed in earlier literature on mechanism design, and propose a class of efficient mechanisms with restricted strategy spaces. Finally, Moulin (2006) considers the design of efficient scalar-parameterized mechanisms where the net monetary transfer from users is minimized.

Several other papers have considered the consequences of restricted communication in a discrete setting where complexity is measured in bits, rather than dimension. Blumrosen et al. (2007) consider a setting where each bidder in an auction for a single indivisible item can only transmit a fixed number of bits to the auctioneer. Fadel and Segal (2007) use a communication complexity-theoretic approach to study the number of bits necessary to achieve efficiency under a variety of notions of equilibrium. Both papers treat substantively different environments from those studied here, and of course the methods of analysis are substantially different as well.

3. Preliminaries

The basic model we study is similar to the one studied by Kelly (1997) and Johari and Tsitsiklis (2004). We consider a collection of R users bidding for a share of a finite, infinitely divisible resource of capacity C . We begin by describing the users. Each user r is endowed with a *utility function* $U_r(\cdot)$, that determines the monetary value of any resource allocation to user r . We let $\mathbf{U} = (U_1, \dots, U_R)$ denote the vector of utility functions. We make the following assumption, *which will remain in force throughout the paper*, unless explicitly mentioned otherwise.

ASSUMPTION 1. For each r , and over the domain $d_r \geq 0$, the utility function $U_r(d_r)$ is nonnegative, concave, strictly increasing, and continuous; and over the domain $d_r > 0$, $U_r(d_r)$ is continuously differentiable. Furthermore, the right directional derivative at 0, denoted $U_r'(0)$, is finite. We let \mathcal{U} denote the set of all utility functions satisfying these conditions.

Note that although we make rather strong differentiability assumptions, these are not essential to the argument; however, they ease the technical presentation. We call a triple (C, R, \mathbf{U}) , where $C > 0$, $R > 1$, and $\mathbf{U} \in \mathcal{U}^R$, a *utility system*; our goal is to design a resource allocation mechanism which has high efficiency for all utility systems.

We assume that utility is measured in monetary units; thus, if user r receives a rate allocation d_r , but must pay w_r , he receives a net payoff given by:⁴

$$U_r(d_r) - w_r.$$

Given any vector of utility functions $\mathbf{U} \in \mathcal{U}^R$, our goal is to maximize aggregate utility, as defined in the following problem:

SYSTEM(C, R, \mathbf{U}):

$$\text{maximize } \sum_{r=1}^R U_r(d_r) \tag{1}$$

$$\text{subject to } \sum_{r=1}^R d_r \leq C; \tag{2}$$

$$\mathbf{d} \geq 0. \tag{3}$$

We will say that \mathbf{d} *solves* *SYSTEM*(C, R, \mathbf{U}) if \mathbf{d} is an optimal solution to (1)-(3).

3.1. Smooth Market-Clearing Mechanisms

In general, the utility system (C, R, \mathbf{U}) is unknown to the mechanism designer, so a mechanism must be designed to elicit information from the users. The following definition captures the types of market mechanisms that we study. Let $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$.

⁴ We follow the terminology used in the communications literature. The economics literature frequently uses “valuation” and “utility” instead of our “utility” and “payoff”, respectively.

DEFINITION 1. A differentiable function $D : (0, \infty) \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is said to define a *smooth market-clearing mechanism* if for all $C > 0$, for all $R > 1$, and for all nonzero $\theta \in (\mathbb{R}^+)^R$, there exists a unique solution $p > 0$ to the following equation:

$$\sum_{r=1}^R D(p, \theta_r) = C. \quad (4)$$

We let $p_D(\theta)$ denote this solution.⁵

We can interpret a smooth market-clearing mechanism in terms of a market-clearing process. Each user r chooses a parameter θ_r , which specifies a demand function $p \mapsto D(p, \theta_r)$, and determines user r 's demand for the resource as a function of the price p of the resource. The mechanism then chooses a single price $p = p_D(\theta)$ so that aggregate demand equals the available capacity (cf. (4)). Each consumer r then receives a resource allocation of $D(p, \theta_r)$, and pays $pD(p, \theta_r)$. Note that (1) each user is restricted to a *one-dimensional* strategy space; and (2) the mechanism only uses a *single price* to allocate the resource; i.e., the mechanism does not price discriminate.

REMARK 1. While our definition implicitly restricts the strategy θ_r of each user to \mathbb{R}^+ , the subsequent analysis can be adapted to hold even if the strategy space of each user is allowed to be $[c, \infty)$, where $c \in \mathbb{R}$.

REMARK 2. The market-clearing price is undefined if $\theta = \mathbf{0}$. As we will see below, when we formulate a game between consumers for a given function D , we will assume that the payoff to all users is $-\infty$ if the composite strategy vector is $\theta = \mathbf{0}$. Note that this is slightly different from the definition in Johari and Tsitsiklis (2004), where the payoff to a user with utility function U and strategy $\theta = 0$ is $U(0)$. We will discuss this distinction further later; we simply note for the moment that it does not affect our results.

We will study the performance of smooth market-clearing mechanisms through their equilibria. Two notions of equilibrium are relevant to our analysis: *competitive equilibrium* and *Nash equilibrium*. In a competitive equilibrium, users act as *price takers*; that is, they do not anticipate the

⁵ Note that we suppress the dependence of this solution on C ; where necessary we will emphasize this dependence.

effect of their actions on the market-clearing price. Conversely, at a Nash equilibrium, users are completely rational and anticipate the functional dependence of the market-clearing price $p_D(\boldsymbol{\theta})$ on their own strategic decisions. Nash equilibrium is a *complete information* solution concept: implicitly, we assume that players' payoff functions are common knowledge. The formal definitions of competitive equilibrium and Nash equilibrium are as follows.

DEFINITION 2. Given a utility system (C, R, \mathbf{U}) and a smooth market-clearing mechanism D , we say that a nonzero vector $\boldsymbol{\theta} \in (\mathbb{R}^+)^R$ is a *competitive equilibrium* if, for $\mu = p_D(\boldsymbol{\theta})$, there holds for all r :

$$\theta_r \in \arg \max_{\bar{\theta}_r \geq 0} [U_r(D(\mu, \bar{\theta}_r)) - \mu D(\mu, \bar{\theta}_r)]. \quad (5)$$

DEFINITION 3. Given a utility system (C, R, \mathbf{U}) and a smooth market-clearing mechanism D , we say that a nonzero vector $\boldsymbol{\theta} \in (\mathbb{R}^+)^R$ is a *Nash equilibrium* if there holds for all r :

$$\theta_r \in \arg \max_{\bar{\theta}_r \geq 0} Q_r(\bar{\theta}_r; \boldsymbol{\theta}_{-r}). \quad (6)$$

where

$$Q_r(\theta_r; \boldsymbol{\theta}_{-r}) = \begin{cases} U_r(D(p_D(\boldsymbol{\theta}), \theta_r)) - p_D(\boldsymbol{\theta}) D(p_D(\boldsymbol{\theta}), \theta_r), & \text{if } \boldsymbol{\theta} \neq \mathbf{0}; \\ -\infty, & \text{if } \boldsymbol{\theta} = \mathbf{0}. \end{cases} \quad (7)$$

Notice that the payoff Q_r is $-\infty$ if the composite strategy vector is $\boldsymbol{\theta} = \mathbf{0}$, since in this case no market-clearing price exists.

One prominent example of a smooth market-clearing mechanism is the *Kelly mechanism*, studied in the context of communication networks by Kelly (1997); we discuss this mechanism in the next section.

3.2. The Kelly Mechanism

In this section, we define the *Kelly mechanism* and survey some of its basic properties. In the Kelly mechanism, each user r submits a payment (also called a *bid*) of w_r to the resource manager. Given the vector $\mathbf{w} = (w_1, \dots, w_r)$, the resource manager chooses an allocation $\mathbf{d} = (d_1, \dots, d_r)$. The

mechanism does not price discriminate; thus each user is charged the same price $\mu > 0$, leading to $d_r = w_r/\mu$. If the entire capacity C is allocated, then we expect the price μ to satisfy:

$$\sum_r \frac{w_r}{\mu} = C.$$

The preceding equality can only be satisfied if $\sum_r w_r > 0$, in which case we have:

$$\mu = \frac{\sum_r w_r}{C}. \quad (8)$$

In other words, if the manager chooses to allocate the entire resource, and does not price discriminate between users, then for every nonzero \mathbf{w} there is a *unique* price $\mu > 0$ which must be chosen by the network, given by the previous equation. The allocation $d_r(\mathbf{w})$ to user r is thus:

$$d_r(\mathbf{w}) = \begin{cases} \frac{w_r}{\sum_s w_s} C, & \text{if } w_r > 0; \\ 0, & \text{if } w_r = 0, \end{cases} \quad (9)$$

and the payoff to user r (when $\mathbf{w} \neq \mathbf{0}$) is $U_r(d_r(\mathbf{w})) - w_r$.

The Kelly mechanism is a smooth market-clearing mechanism. To see this interpretation, note that when a user chooses a total payment w_r , it is as if the user has chosen a demand function $D(p, w_r) = w_r/p$ for $p > 0$. The resource manager then chooses a price μ so that $\sum_r D(\mu, w_r) = C$, i.e., so that the aggregate demand equals the supply C . For the specific form of demand functions we consider here, this leads to the expression for μ given in (8). User r then receives an allocation given by $D(\mu, w_r)$, and makes a payment $\mu D(\mu, w_r) = w_r$.

The following remarks summarize the key insights about the Kelly mechanism.

REMARK 3. Notice that the Kelly mechanism allocates the resource *in proportion to the bids of the users* (from (9)). We refer to mechanisms of this form as *proportional allocation mechanisms*; they will play a key role in our analysis in Section 4.

REMARK 4. When users act as price takers, then given a price $\mu > 0$, each user r chooses $w_r \geq 0$ to maximize $U_r(w_r/\mu) - w_r$. For the Kelly mechanism, *a competitive equilibrium always exists, and the resulting allocation is fully efficient* (Kelly 1997); this result is a special case of the first fundamental theorem of welfare economics (Mas-Colell et al. 1995, Chapter 16).

REMARK 5. When users act as price anticipators, they maximize their payoff with the knowledge that the price will be set according to (8). Johari and Tsitsiklis analyze this setting through the Nash equilibria of the resulting game; they show that for any utility system, *at the unique Nash equilibrium of this game, the aggregate utility at the resulting allocation achieves at least 75% of the maximum possible aggregate utility.*

In the next section, we consider a broader class of mechanisms that includes the Kelly mechanism as a special case, and use this setting to investigate the optimality of the Kelly mechanism.

4. Optimality of the Kelly Mechanism

In this section we ask the following type of fundamental question: is a particular mechanism “desirable” among a class of mechanisms satisfying certain “reasonable” properties? Defining desirability is the simpler of the two tasks: we consider a mechanism to be desirable if it minimizes worst case efficiency loss when users are price anticipating. Importantly, we ask for this efficiency property *independent* of the utility functions of the market participants, and their number. That is, the mechanisms we seek are those that perform well under broad assumptions on the nature of the preferences of market participants. We will show that under a specific set of assumptions, the Kelly mechanism in fact minimizes the worst case efficiency loss when users are price anticipating.

We now frame the specific class \mathcal{D} of market mechanisms we consider in this section, defined as follows.

DEFINITION 4. The class \mathcal{D} consists of all functions $D(p, \theta)$ such that the following conditions are satisfied:

1. D defines a smooth market-clearing mechanism (cf. Definition 1).
2. For all $C > 0$, and for all $U_r \in \mathcal{U}$, a user’s payoff is *concave if he is price anticipating*; that is, for all R , and for all $\theta_{-r} \in (\mathbb{R}^+)^R$, the function:

$$U_r(D(p_D(\theta), \theta_r)) - p_D(\theta)D(p_D(\theta), \theta_r)$$

is concave in $\theta_r > 0$ if $\theta_{-r} = \mathbf{0}$, and concave in $\theta_r \geq 0$ if $\theta_{-r} \neq \mathbf{0}$.

3. For all $p > 0$, and for all $d \geq 0$, there exists a $\theta \geq 0$ such that $D(p, \theta) = d$.

We pause here to briefly discuss the conditions in the previous definition. The second one allows us to characterize Nash equilibria in terms of only first order conditions. To justify this condition, we note that some assumption of quasiconcavity is generally used to guarantee existence of pure strategy Nash equilibria (Fudenberg and Tirole 1991). The third condition ensures that given a price p and desired allocation $d \in [0, C]$, each user can make a choice of θ to guarantee precisely the allocation d . This is an “expressiveness” condition on the mechanism, that ensures all possible demands can be chosen at any market-clearing price. The following example gives a family of mechanisms that lie in \mathcal{D} .

EXAMPLE 1. Suppose that $D(p, \theta) = \theta p^{-1/\alpha}$, where $\alpha \geq 1$. It is easy to check that this class of mechanisms satisfies $D \in \mathcal{D}$ for all choices of α ; when $\alpha = 1$, we recover the Kelly mechanism. The market-clearing condition yields that $p_D(\boldsymbol{\theta}) = (\sum_r \theta_r / C)^{1/\alpha}$. Note that as a result, the *allocation* to user r at a nonzero vector $\boldsymbol{\theta}$ is:

$$D(p_D(\boldsymbol{\theta}), \theta_r) = \frac{\theta_r}{\sum_s \theta_s} C.$$

In other words, regardless of the value of α , the market clearing allocations are chosen proportional to the bids. This remarkable fact is a special case of a more general result we establish below: *all mechanisms in \mathcal{D} are proportional allocation mechanisms*; they differ only in the market-clearing price that is chosen.

Our interest is in the worst-case ratio of aggregate utility at any Nash equilibrium to the optimal value of *SYSTEM*. Formally, for $D \in \mathcal{D}$ we define a constant $\rho(D)$ as follows:

$$\rho(D) = \inf \left\{ \frac{\sum_{r=1}^R U_r(D(p_D(\boldsymbol{\theta}), \theta_r))}{\sum_{r=1}^R U_r(d_r^S)} \mid C > 0, R > 1, \mathbf{U} \in \mathcal{U}^R, \right. \\ \left. \mathbf{d}^S \text{ solves } \text{SYSTEM}(C, R, \mathbf{U}), \text{ and } \boldsymbol{\theta} \text{ is a Nash equilibrium} \right\}.$$

Note that since all $U \in \mathcal{U}$ are strictly increasing and nonnegative, the aggregate utility $\sum_{r=1}^R U_r(d_r^S)$ is positive for any utility system (C, R, \mathbf{U}) with $C > 0$, and any optimal solution \mathbf{d}^S to *SYSTEM*.

However, Nash equilibria may not exist for some utility systems (C, R, \mathbf{U}) ; in this case we set $\rho(D) = 0$.

Our main result in this section is the following theorem.

THEOREM 1. *Given $D \in \mathcal{D}$:*

1. *For any utility system (C, R, \mathbf{U}) , there exists a competitive equilibrium $\boldsymbol{\theta}$. Furthermore, for any such $\boldsymbol{\theta}$, the resulting allocation \mathbf{d} given by $d_r = D(p_D(\boldsymbol{\theta}), \theta_r)$ solves $\text{SYSTEM}(C, R, \mathbf{U})$.*
2. *There exists a concave, strictly increasing, differentiable, and invertible function $B: (0, \infty) \rightarrow (0, \infty)$ such that for all $p > 0$ and $\theta \geq 0$:*

$$D(p, \theta) = \frac{\theta}{B(p)}.$$

3. *$\rho(D) \leq 3/4$, and this bound is met with equality if and only if $D(p, \theta) = \Delta\theta/p$ for some $\Delta > 0$.*

Before continuing to the proof, we make the following key remark.

REMARK 6. Parts 1 and 2 of the theorem provide a *characterization* of the types of mechanisms allowed by the constraints that define \mathcal{D} . In particular, notice that for nonzero $\boldsymbol{\theta}$, the market-clearing condition (4) yields

$$C = \sum_{r=1}^R D(p_D(\boldsymbol{\theta}), \theta_r) = \frac{\sum_{r=1}^R \theta_r}{B(p_D(\boldsymbol{\theta}))}, \quad (10)$$

which implies that

$$D(p_D(\boldsymbol{\theta}), \theta_r) = \frac{\theta_r}{\sum_{s=1}^R \theta_s} C; \quad (11)$$

in other words, every mechanism in \mathcal{D} is a proportional allocation mechanism. As a result, we conclude that for a given vector $\boldsymbol{\theta}$, when the market clears, mechanisms in \mathcal{D} differ from the Kelly mechanism only in the market-clearing price—the allocation is the same. Part 3 of the theorem then characterizes the efficiency loss of mechanisms of this form.

We emphasize that the theorem here is distinguished in part from related work because the allocation rule (11) was not assumed in advance. Rather, the result here starts from a set of simple assumptions on the structure of mechanisms to be considered (the definition of the class \mathcal{D}), and uses them to *prove* that any mechanism in the class must lead to the allocation in (11).

Proof. Throughout the proof we fix a particular mechanism $D \in \mathcal{D}$.

Step 1: For all $R > 1$ and $\boldsymbol{\theta}_{-r} \in (\mathbb{R}^+)^{R-1}$, the functions $D(p_D(\boldsymbol{\theta}), \theta_r)$ and $-p_D(\boldsymbol{\theta})D(p_D(\boldsymbol{\theta}), \theta_r)$ are concave in $\theta_r > 0$ if $\boldsymbol{\theta}_{-r} = \mathbf{0}$, and concave in $\theta_r \geq 0$ if $\boldsymbol{\theta}_{-r} \neq \mathbf{0}$. This conclusion follows by considering linear utility functions with very large and very small slope, respectively. Let $U_r(d_r) = \alpha d_r$, with $\alpha > 0$. If we apply Condition 2 of Definition 4 with $\alpha \rightarrow \infty$, then it follows that $D(p_D(\boldsymbol{\theta}), \theta_r)$ is concave in θ_r for nonzero $\boldsymbol{\theta}$. Similarly, if we let $\alpha \rightarrow 0$, it follows that $-p_D(\boldsymbol{\theta})D(p_D(\boldsymbol{\theta}), \theta_r)$ is concave in θ_r for nonzero $\boldsymbol{\theta}$.

Step 2: A user's payoff is concave if he is price taking. In other words, we will show that for all $U \in \mathcal{U}$ and for all $p > 0$, $U(D(p, \theta)) - pD(p, \theta)$ is concave in θ . The key idea is to use a limiting regime where both the number of users and the capacity grow large, so that users that are price anticipating effectively become price taking. The details are deferred to the e-companion.

Step 3: There exists a positive function B such that $D(p, \theta) = \theta/B(p)$ for $p > 0$ and $\theta \geq 0$. By Step 2, a user's payoff is concave when he is price taking. By choosing $U(d) = \alpha d$ where $\alpha > p$ (resp., $\alpha < p$) it follows that $D(p, \theta)$ must be concave (resp., convex) in θ for a given $p > 0$. Thus for fixed $p > 0$, $D(p, \theta)$ is an affine function of θ . Condition 3 in Definition 4 then implies that the constant term must be zero, while the coefficient of the linear term is positive; thus $D(p, \theta) = \theta/B(p)$ for some positive function $B(p)$.

Before continuing, we note that the previous step already implies the fact that *any mechanism* $D \in \mathcal{D}$ must be a *proportional allocation mechanism*. This follows from Remark 6.

Step 4: For all utility systems (C, R, \mathbf{U}) , there exists a competitive equilibrium, and it is fully efficient. This step relies critically on Condition 3 in Definition 4: given a price μ , a user can first determine his optimal choice of quantity, and then choose a parameter θ to express this choice.

Formally, fix a utility system (C, R, \mathbf{U}) . Let \mathbf{d} be an optimal solution to $SYSTEM(C, R, \mathbf{U})$, and let μ be the corresponding Lagrange multiplier, i.e., μ satisfies the following optimality conditions:

$$U'_r(d_r) = \mu, \quad \text{if } d_r > 0; \quad (12)$$

$$U'_r(0) \leq \mu, \quad \text{if } d_r = 0. \quad (13)$$

For each r , choose $\theta_r = B(\mu)d_r$. Clearly $\mu = p_D(\boldsymbol{\theta})$; furthermore, the optimality conditions (12)-(13) ensure that (5) holds for all r . Thus $\boldsymbol{\theta}$ is a competitive equilibrium.

Finally, we check that every competitive equilibrium yields an allocation that is an optimal solution to $SYSTEM(C, R, \mathbf{U})$. By Condition 3 in Definition 4, note that for any price $\mu > 0$, we have:

$$\max_{d_r \geq 0} (U_r(d_r) - \mu d_r) = \max_{\theta_r \geq 0} (U_r(D(\mu, \theta_r)) - \mu D(\mu, \theta_r)).$$

Thus if $\boldsymbol{\theta}$ is a competitive equilibrium, then the optimality conditions (12)-(13) must hold with $\mu = p_D(\boldsymbol{\theta})$ and $d_r = D(\mu, \theta_r)$. We conclude that \mathbf{d} is an optimal solution to $SYSTEM(C, R, \mathbf{U})$. This establishes the first claim of the theorem.

Step 5: B is an invertible, differentiable, strictly increasing, and concave function on $(0, \infty)$.

The technical details are deferred to the e-companion. In the sequel, we let $\Phi : (0, \infty) \rightarrow (0, \infty)$ be the (necessarily differentiable) inverse of B .

Step 6: Let (C, R, \mathbf{U}) be a utility system. A vector $\boldsymbol{\theta} \geq 0$ is a Nash equilibrium if and only if at least two components of $\boldsymbol{\theta}$ are nonzero, and there exists a nonzero vector $\mathbf{d} \geq 0$ and a scalar $\mu > 0$ such that $\theta_r = \mu d_r$ for all r , $\sum_{r=1}^R d_r = C$, and the following conditions hold:

$$U'_r(d_r) \left(1 - \frac{d_r}{C}\right) = \Phi(\mu) \left(1 - \frac{d_r}{C}\right) + \mu \Phi'(\mu) \left(\frac{d_r}{C}\right), \quad \text{if } d_r > 0; \quad (14)$$

$$U'_r(0) \leq \Phi(\mu), \quad \text{if } d_r = 0. \quad (15)$$

In this case $d_r = D(p_D(\boldsymbol{\theta}), \theta_r)$, $\mu = \sum_{r=1}^R \theta_r / C$, and $\Phi(\mu) = p_D(\boldsymbol{\theta})$. The proof is developed using the first order necessary and sufficient conditions for a vector $\boldsymbol{\theta}$ to be a Nash equilibrium; details

are deferred to the e-companion.

Step 7: Let (C, R, \mathbf{U}) be a utility system. Then there exists a unique Nash equilibrium. Our approach is to demonstrate existence of a Nash equilibrium by finding a solution $\mu > 0$ and $\mathbf{d} \geq 0$ to (14)-(15), such that $\sum_{r=1}^R d_r = C$. The technical details are deferred to the e-companion.

Step 8: If the function Φ is not linear, there exists a utility system (C, R, \mathbf{U}) such that at the unique Nash equilibrium θ , the aggregate utility is strictly less than $3/4$ of the maximal aggregate utility. Consider a utility system with the following properties. Let $C = 1$. Fix $\mu > 0$, and let $U_1(d_1) = Ad_1$, where $A > \Phi(\mu)$. We will choose the remaining utilities so that we obtain a solution to the Nash conditions (14)-(15) with market-clearing price $\Phi(\mu)$.

We start by calculating d_1 by assuming it is nonzero, and applying (14):

$$d_1 = \frac{(A - \Phi(\mu))C}{A - \Phi(\mu) + \mu\Phi'(\mu)}. \quad (16)$$

We then choose users $2, \dots, R$ to have identical linear utility functions, with slopes less than A . As we will see, this will be possible if R is large enough. Formally, let $d = (C - d_1)/(R - 1)$, and (cf. (14)) define:

$$\alpha = \frac{\Phi(\mu)C + (\mu\Phi'(\mu) - \Phi(\mu))d}{C - d}. \quad (17)$$

Let $U_r(d_r) = \alpha d_r$ for $r = 2, \dots, R$. Note that as R increases to infinity, d tends to zero and α converges to $\Phi(\mu)$, which is less than A . This guarantees that d_1 must be nonzero at any Nash equilibrium, which then implies that d_1 is indeed given by (16). In turn, letting $d_r = d$ for $r = 2, \dots, R$, this implies that (d_1, \dots, d_R) and μ are a valid solution to (14)-(15), when users have utility functions U_1, \dots, U_R .

Now consider the limiting ratio of Nash aggregate utility to maximal aggregate utility, as $R \rightarrow \infty$, in which case, $d \rightarrow 0$, and $\alpha \rightarrow \Phi(\mu)$. Regardless of R , a solution to $SYSTEM(C, R, \mathbf{U})$ is to allocate

the entire resource to user 1, so the maximal aggregate utility is AC . Thus the limiting ratio of Nash aggregate utility to maximal aggregate utility becomes:

$$\frac{Ad_1 + \alpha(C - d_1)}{AC} = \frac{(A - \Phi(\mu))}{A - \Phi(\mu) + \mu\Phi'(\mu)} + \left(1 - \frac{(A - \Phi(\mu))}{A - \Phi(\mu) + \mu\Phi'(\mu)}\right) \left(\frac{\Phi(\mu)}{A}\right). \quad (18)$$

We will show that for a suitable choice of A and μ , with $\mu > 0$ and $A > \Phi(\mu)$, this value is less than $3/4$.

Given that the nonnegative and convex function Φ has been assumed to be nonlinear, it is straightforward to check that there exists some $\mu > 0$ for which $\psi = \mu\Phi'(\mu)/\Phi(\mu) > 1$. At this value of μ , the right-hand side of (18) can be rewritten in the form

$$\frac{(1-x)^2}{1+(\psi-1)x} + x, \quad (19)$$

where $x = \Phi(\mu)/A$. By choosing $x = 1/2$, i.e., $A = 2\Phi(\mu)$, we see that that the ratio is indeed strictly smaller than $3/4$, establishing the claim.

The proof of the theorem is completed by observing that when Φ is linear, we have $D(p, \theta) = \Delta\theta/p$ for some $\Delta > 0$, recovering the Kelly mechanism. Thus the worst-case efficiency ratio is known to be exactly equal to $3/4$ (Johari and Tsitsiklis 2004). \square

REMARK 7. In fact, it is possible to prove something much stronger than the previous theorem: for any mechanism $D \in \mathcal{D}$, we can explicitly characterize the exact worst case efficiency loss when users are price anticipating. We have shown that all mechanisms D satisfying the conditions of the theorem must be of the form $D(p, \theta) = \theta/B(p)$. Recall from the proof that $\Phi(\mu)$ is the differentiable inverse of B . If we define

$$\psi^* = \max_{\mu > 0} \left[\frac{\mu\Phi'(\mu)}{\Phi(\mu)} \right] = \left(\inf_{p > 0} \left[\frac{pB'(p)}{B(p)} \right] \right)^{-1}, \quad (20)$$

we can then show that $\rho(D) = G(\psi^*)$, where the function G is given by

$$G(\psi) = \begin{cases} \frac{3}{4}, & \text{if } \psi = 1; \\ \frac{2\psi^2 - 3\psi\sqrt{\psi} + \sqrt{\psi}}{(\psi - 1)^2\sqrt{\psi}}, & \text{if } \psi > 1. \end{cases} \quad (21)$$

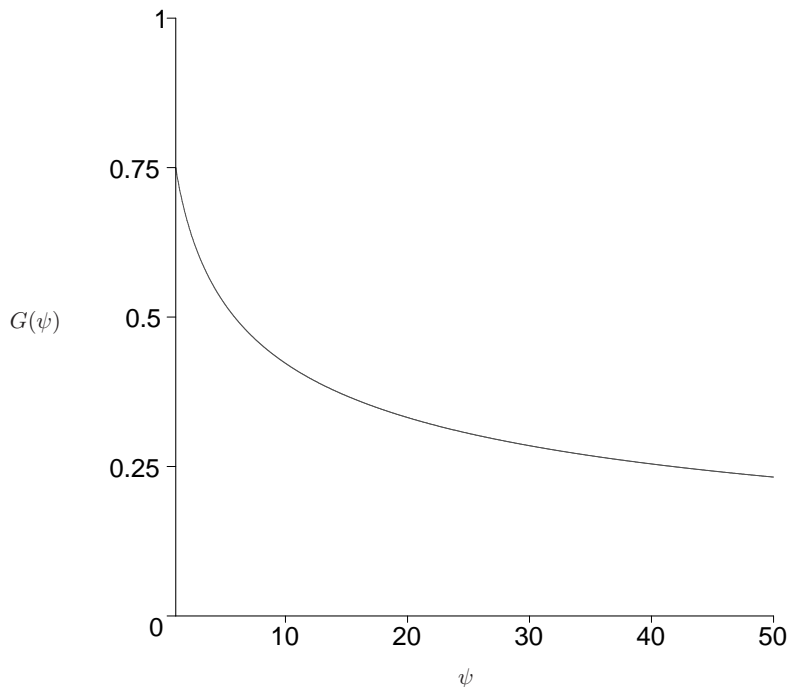


Figure 1 The function $G(\psi)$ defined in (21). Note that $G(\psi)$ is strictly decreasing, with $G(1) = 3/4$.

Details are available in Chapter 5 of Johari (2004); the function G is plotted in Figure 1. The proof involves (i) showing that examples constructed as in Step 8 of the proof of Theorem 1, in fact yield the *worst case* efficiency loss when users are price anticipating, and (ii) picking the worst possible values of ψ and x . For any ψ , the value $G(\psi)$ is just the minimum of the expression (19) over $x \in (0, 1)$. Furthermore, G is strictly decreasing for $\psi \geq 1$, which means that the worst possible value of the expression (19) is obtained by employing the largest possible value of ψ . Note that the quantity $pB'(p)/B(p)$ is the *elasticity* of $B(p)$ (Varian 1992), a common measure of the degree of “nonlinearity” of a function. Thus $1/\psi^*$ is the minimal elasticity of $B(p)$ over all $p > 0$. To summarize, the worst case efficiency loss of such a mechanism is governed by the degree of nonlinearity of $B(p)$, as measured through the quantity ψ^* defined in (20).

REMARK 8. We note one potentially undesirable feature of the family of market-clearing mechanisms considered: the payoff to user r is defined as $-\infty$ when the composite strategy vector is $\theta = 0$ (cf. (7)). This definition is required because when the composite strategy vector is $\theta = 0$, a market-clearing price may not exist. One possible remedy is to restrict attention instead to mech-

anisms where $D(p, \theta) = 0$ if $\theta = 0$, for all $p \geq 0$; in this case we can *define* $p_D(\theta) = 0$ if $\theta = 0$, and let the payoff to user r be $U_r(0)$ if $\theta_r = 0$. This condition amounts to a “normalization” on the market-clearing mechanism. Furthermore, this modification now exactly captures the mechanism of Johari and Tsitsiklis (2004), where $Q_r(0; \mathbf{w}_{-r}) = U_r(0)$ for all $\mathbf{w}_{-r} \geq 0$. It is straightforward to show that this modification does not alter the conclusion of Theorem 1. The argument involves a modification of the class of mechanisms \mathcal{D} , which includes the requirement $D(p, 0) = 0$.

REMARK 9. We note a key desirable feature of the mechanisms considered in Theorem 1, particularly in the context of communication networks. In general, even though the strategy space of the users is one-dimensional, the market-clearing price $p_D(\theta)$ may have a complex dependence on the vector θ . However, under the conditions of Theorem 1, the market-clearing price is only a function of $\sum_s \theta_s$, so that the market-clearing process does not require identification of individual users interacting with the mechanism, or even the number of users. This is a desirable scaling property for market mechanisms to be deployed in large scale networks (Shenker 1990).

REMARK 10. As discussed in Section 2.1, Maheswaran and Basar (2004) also considered a class of mechanisms that allocate the resource according to the bids made by users. However, for every mechanism they consider, the payment made by a user depends *only* on the bid of that user, whereas in our setting this condition holds only for the Kelly mechanism. Thus even though we have shown that we can restrict attention to proportional allocation mechanisms in parts 1 and 2 of the theorem, the class of proportional allocation mechanisms we consider is distinct from the class considered by Maheswaran and Basar, and thus our proof of optimality of the Kelly mechanism is distinct as well.

5. The Vickrey-Clarke-Groves (VCG) Approach

The mechanisms we considered in the last section had several restrictions placed on them; chief among these are that (1) users are restricted to using “simple” strategy spaces; and (2) the mechanism uses only a single price to clear the market. One could consider lifting both restrictions: allow more complex strategies, with users perhaps declaring their entire utility function to the market;

and also, allow price discrimination so that each user is charged a personalized per-unit price for the resource. The best known solution employing both these generalizations is the Vickrey-Clarke-Groves (VCG) approach to eliciting utility information (Vickrey 1961, Clarke 1971, Groves 1973). We review VCG mechanisms in Section 5.1.

In this section we are interested in deciding whether the same outcome can be realized while preserving restriction (1) above, but removing restriction (2): that is, can mechanisms with “simple” strategy spaces that employ price discrimination achieve full efficiency? In Section 5.2, we present a class of mechanisms, inspired by the VCG class, in which users only submit scalar strategies; we show that these *scalar strategy VCG (SSVCG) mechanisms* have desirable efficiency properties. In particular, we establish existence of an efficient Nash equilibrium, and under an additional condition, we also establish that all Nash equilibria are efficient. We extend this result to general convex environments in Section 5.3, and discuss an application to multicommodity flow routing.

5.1. VCG Mechanisms

In the Vickrey-Clarke-Groves (VCG) class of mechanisms, the basic approach is to let the strategy space of each user r be the set \mathcal{U} of possible utility functions, as defined in Assumption 1, and structure the payments made by each user so that the payoff of each user r has the same form as the objective function in the problem *SYSTEM*. Thus, in a VCG mechanism, each user is simply asked to *declare* their utility function; of course, if the payments are not structured properly, there is no guarantee that individuals will make truthful declarations. For each r , we use \tilde{U}_r to denote the declared utility function of user r , and use $\tilde{\mathbf{U}} = (\tilde{U}_1, \dots, \tilde{U}_R)$ to denote the vector of declared utilities.

Suppose that user r receives an allocation d_r , but has to make a payment t_r ; we use the notation t_r to distinguish from the bid w_r of Section 3. Then the payoff to user r is:

$$P_r(t_r, d_r) = U_r(d_r) - t_r.$$

On the other hand, the social objective (1) can be written as:

$$U_r(d_r) + \sum_{s \neq r} U_s(d_s).$$

Comparing the preceding two expressions, the most natural means to align user objectives with the social planner's objectives is to define the payment t_r as the negation of the sum of the utilities of all users other than r .

Formally, given a vector of declared utility functions $\tilde{\mathbf{U}}$, a VCG mechanism chooses the allocation $\mathbf{d}(\tilde{\mathbf{U}})$ as an optimal solution to *SYSTEM* for the declared utility functions $\tilde{\mathbf{U}}$. For simplicity, let $\mathcal{X} = \{\mathbf{d} \geq 0 : \sum_r d_r \leq C\}$; this is the feasible region for *SYSTEM*(C, R, \mathbf{U}). Then for a VCG mechanism, we have:

$$\mathbf{d}(\tilde{\mathbf{U}}) \in \arg \max_{\mathbf{d} \in \mathcal{X}} \sum_r \tilde{U}_r(d_r). \quad (22)$$

The payments are structured so that:

$$t_r(\tilde{\mathbf{U}}) = - \sum_{s \neq r} \tilde{U}_s(d_s(\tilde{\mathbf{U}})) + h_r(\tilde{\mathbf{U}}_{-r}). \quad (23)$$

Here h_r is an arbitrary function of the declared utilities of users other than r ; since user r cannot affect this term through the choice of \tilde{U}_r , he chooses \tilde{U}_r to maximize:

$$P_r(d_r(\tilde{\mathbf{U}}), t_r(\tilde{\mathbf{U}})) = U_r(d_r(\tilde{\mathbf{U}})) + \sum_{s \neq r} \tilde{U}_s(d_s(\tilde{\mathbf{U}})).$$

Now note that given $\tilde{\mathbf{U}}_{-r}$, the above expression is bounded above by:

$$\max_{\mathbf{d} \in \mathcal{X}} \left[U_r(d_r) + \sum_{s \neq r} \tilde{U}_s(d_s) \right].$$

But since $\mathbf{d}(\tilde{\mathbf{U}})$ satisfies (22), user r can achieve the preceding maximum by truthfully declaring $\tilde{U}_r = U_r$. Since this optimal strategy does not depend on the utility functions ($\tilde{U}_s, s \neq r$) declared by the other users, we recover the well-known fact that in a VCG mechanism, truthful declaration is a dominant strategy for user r . This discussion is summarized in the following well-known proposition; see, e.g, Green and Laffont (1979).

PROPOSITION 1. *Consider a VCG mechanism defined according to (22) and (23). Then, declaring $\tilde{U}_r = U_r$ is a dominant strategy for each user r . Furthermore, under these strategies, the resulting allocation is efficient.*

For our purposes, the interesting feature of the VCG mechanism is that it elicits the true utility functions from the users, and in turn (because of the definition of $\mathbf{d}(\tilde{\mathbf{U}})$) chooses an efficient allocation. However, it does so with a high degree of required communication: each user submits an entire utility function, essentially an infinite-dimensional object. In the next section, we explore a class of mechanisms inspired by the VCG mechanisms, but with limited communication requirements.

5.2. Scalar Strategy VCG Mechanisms

We now consider a class of mechanisms where each user's strategy is a submitted utility function (as in VCG mechanisms), except that users are only allowed to choose from a given single parameter family of utility functions. One cannot expect such mechanisms to have efficient dominant strategy equilibria, and we will focus instead on the efficiency properties of the resulting Nash equilibria.

Formally, *scalar strategy VCG* (SSVCG) mechanisms allow users to choose from a given family of utility functions $\bar{U}(\cdot; \theta)$, parameterized by $\theta \in (0, \infty)$.⁶ We make the following assumptions about this family.

ASSUMPTION 2. 1. For every $\theta > 0$, the function $\bar{U}(\cdot; \theta) : d \mapsto \bar{U}(d; \theta)$, defined for $d \geq 0$, is strictly concave, strictly increasing, continuous, and continuously differentiable for $d > 0$.⁷

2. For every $\gamma \in (0, \infty)$ and $d \geq 0$, there exists a $\theta > 0$ such that $\bar{U}'(d; \theta) = \gamma$.⁸

Given $\boldsymbol{\theta}$, the mechanism chooses $\mathbf{d}(\boldsymbol{\theta})$ such that:

$$\mathbf{d}(\boldsymbol{\theta}) = \arg \max_{\mathbf{d} \in \mathcal{X}} \sum_r \bar{U}(d_r; \theta_r). \quad (24)$$

Since $\bar{U}(\cdot; \theta_r)$ is strictly concave for each r , the solution $\mathbf{d}(\boldsymbol{\theta})$ is uniquely defined. (Note the similarity between (22) and (24).)

⁶ Note that, by contrast with Section 4, the choice of bid θ by a user indexes a utility function, rather than a demand function. However, this is not crucial: if a user with utility function U maximizes $U(d) - pd$ (i.e., the user acts as a price taker), the solution yields the demand function $D(p) = (U')^{-1}(p)$. Up to an additive constant, the utility function and demand function can be recovered from each other. Thus, we could define SSVCG mechanisms where users submit demand functions from a parameterized class. We define our SSVCG mechanisms in terms of utility functions to maintain consistency with the standard definition of VCG mechanisms in the literature.

⁷ This is almost the same as Assumption 1, except that we require strict concavity, do not require nonnegativity, and allow $U'_r(0)$ to be infinite.

⁸ Since we do not assume differentiability with respect to θ , the only differentiation of \bar{U} is with respect to the first coordinate d , and $\bar{U}'(d; \theta)$ will always stand for the derivative with respect to d .

By analogy with the expression (23), the monetary payment by user r is:

$$t_r(\boldsymbol{\theta}) = - \sum_{s \neq r} \bar{U}(d_s(\boldsymbol{\theta}); \theta_s) + h_r(\boldsymbol{\theta}_{-r}). \quad (25)$$

Here h_r is a function that depends only on the strategies $\boldsymbol{\theta}_{-r} = (\theta_s, s \neq r)$ submitted by the users other than r . While we do not advocate any particular choice of h_r , a natural candidate is to define $h_r(\boldsymbol{\theta}_{-r}) = \sum_{s \neq r} \bar{U}(d_s(\boldsymbol{\theta}_{-r}); \theta_s)$, where $\mathbf{d}(\boldsymbol{\theta}_{-r})$ is the aggregate utility maximizing allocation excluding user r . This leads to a natural scalar strategy analogue of the Clarke pivot mechanism (Clarke 1971). Note that, of course, the per-unit price paid by each user will typically be different.

Given h_r , the payoff to user r is:

$$P_r(d_r(\boldsymbol{\theta}), t_r(\boldsymbol{\theta})) = U_r(d_r(\boldsymbol{\theta})) + \sum_{s \neq r} \bar{U}(d_s(\boldsymbol{\theta}); \theta_s) - h_r(\boldsymbol{\theta}_{-r}).$$

A strategy vector $\boldsymbol{\theta}$ is a *Nash equilibrium* if no user can profitably deviate through a unilateral deviation, i.e., if for all users r there holds:

$$P_r(d_r(\boldsymbol{\theta}), t_r(\boldsymbol{\theta})) \geq P_r(d_r(\theta'_r, \boldsymbol{\theta}_{-r}), t_r(\theta'_r, \boldsymbol{\theta}_{-r})), \quad \text{for all } \theta'_r > 0. \quad (26)$$

We start with the following key lemma, proven using an argument analogous to Proposition 1.

LEMMA 1. *The vector $\boldsymbol{\theta}$ is a Nash equilibrium of a SSVCG mechanism if and only if for all r :*

$$\mathbf{d}(\boldsymbol{\theta}) \in \arg \max_{\mathbf{d} \in \mathcal{X}} \left[U_r(d_r) + \sum_{s \neq r} \bar{U}(d_s; \theta_s) \right]. \quad (27)$$

Proof. Fix a user r . Since θ_r does not affect h_r , from (26) user r will choose θ_r to maximize the following effective payoff:

$$U_r(d_r(\boldsymbol{\theta})) + \sum_{s \neq r} \bar{U}(d_s(\boldsymbol{\theta}); \theta_s). \quad (28)$$

The optimal value of the objective function in (27) is certainly an upper bound to user r 's effective payoff (28). Thus, given a vector $\boldsymbol{\theta}$, if (27) is satisfied for all users r , then (26) holds for all users r , and we conclude $\boldsymbol{\theta}$ is a Nash equilibrium.

Conversely, given a vector $\boldsymbol{\theta}$, suppose that (27) is not satisfied for some user r . We will show $\boldsymbol{\theta}$ cannot be a Nash equilibrium. Since \mathcal{X} is compact, an optimal solution exists to the problem in (27)

for user r ; call this optimal solution \mathbf{d}^* . The vector \mathbf{d}^* must satisfy the first order optimality conditions (analogous to (12)-(13)), which only involve the first derivatives $U'_r(d_r^*)$ and $(\bar{U}'(d_s^*; \theta_s), s \neq r)$. Suppose now that user r chooses $\theta'_r > 0$ such that $\bar{U}'(d_r^*; \theta'_r) = U'_r(d_r^*)$. Then, \mathbf{d}^* also satisfies the optimality conditions for the problem (24). Since $\mathbf{d}(\theta'_r, \boldsymbol{\theta}_{-r})$ is the unique optimal solution to (24) when the strategy vector is $(\theta'_r, \boldsymbol{\theta}_{-r})$, we must have $\mathbf{d}(\theta'_r, \boldsymbol{\theta}_{-r}) = \mathbf{d}^*$. Thus we have:

$$\begin{aligned} P_r(d_r(\boldsymbol{\theta}), t_r(\boldsymbol{\theta})) &< U_r(d_r^*) + \sum_{s \neq r} \bar{U}(d_s^*; \theta_s) - h_r(\boldsymbol{\theta}_{-r}) \\ &= U_r(d_r(\theta'_r, \boldsymbol{\theta}_{-r})) + \sum_{s \neq r} \bar{U}(d_s(\theta'_r, \boldsymbol{\theta}_{-r}); \theta_s) - h_r(\boldsymbol{\theta}_{-r}) \\ &= P_r(d_r(\theta'_r, \boldsymbol{\theta}_{-r}), t_r(\theta'_r, \boldsymbol{\theta}_{-r})). \end{aligned}$$

(The first inequality follows from the assumption that (27) is not satisfied for user r .) We conclude that (26) is violated for user r , so $\boldsymbol{\theta}$ is not a Nash equilibrium. \square

The following corollary states that there exists a Nash equilibrium which is efficient. Furthermore, at this efficient Nash equilibrium, all users truthfully reveal their utilities in a local sense: each user r chooses θ_r so that the declared marginal utility $\bar{U}'(d_r(\boldsymbol{\theta}); \theta_r)$ is equal to the true marginal utility $U'_r(d_r(\boldsymbol{\theta}))$.

COROLLARY 1. *For any SSVCG mechanism, there exists an efficient Nash equilibrium $\boldsymbol{\theta}$ defined as follows: Let \mathbf{d}^S be an optimal solution to $\text{SYSTEM}(C, R, \mathbf{U})$. Each user r chooses θ_r so that $\bar{U}'(d_r^S; \theta_r) = U'_r(d_r^S)$. The resulting allocation satisfies $\mathbf{d}(\boldsymbol{\theta}) = \mathbf{d}^S$.*

Proof. By Assumption 2, each user r can choose θ_r so that $\bar{U}'(d_r^S; \theta_r) = U'_r(d_r^S)$. For this vector $\boldsymbol{\theta}$, it is clear that $\mathbf{d}(\boldsymbol{\theta}) = \mathbf{d}^S$, since the optimal solution to (24) is uniquely determined, and the optimality conditions for (24) involve only the first derivatives $\bar{U}'(d_r(\boldsymbol{\theta}); \theta_r)$. By the same argument it also follows that \mathbf{d}^S is an optimal solution in (27). Since $\mathbf{d}(\boldsymbol{\theta}) = \mathbf{d}^S$, we conclude that (27) is satisfied for all r , and thus $\boldsymbol{\theta}$ is a Nash equilibrium. \square

We note that, as in classical VCG mechanisms, there can be additional, possibly inefficient, Nash equilibria, as the following example shows.

EXAMPLE 2. Consider a system with R identical users with strictly concave utility function U . Suppose user 1 chooses θ_1 so that $\bar{U}'(C; \theta_1) > U'(0)$, and every other user r chooses θ_r so that $\bar{U}'(0; \theta_r) < U'(C)$. We then have $\mathbf{d}(\boldsymbol{\theta}) = (1, \mathbf{0})$, so that the entire resource is allocated to user 1. Since $U'(C) \leq U'(0)$, it follows that (27) is satisfied for all users r , and we conclude $\boldsymbol{\theta}$ is a Nash equilibrium. However, this Nash equilibrium is inefficient: the unique optimal solution to $SYSTEM(C, R, \mathbf{U})$ is symmetric, and allocates C/R units of the resource to each of the R users.

The equilibrium in the preceding example involves a “bluff”: user 1 declares such a high marginal utility at C that all other users concede. One way to preclude such equilibria is to enforce an assumption that guarantees participation. The next proposition assumes that at least two users have infinite marginal utility at zero allocation; this guarantees that all Nash equilibria are efficient.

PROPOSITION 2. *Suppose that Assumptions 1 and 2 hold, except that $U'_r(0) = \infty$ for at least two users r . Suppose that $\boldsymbol{\theta}$ is a Nash equilibrium. Then $\mathbf{d}(\boldsymbol{\theta})$ is an optimal solution to $SYSTEM(C, R, \mathbf{U})$.*

Proof. Note that Lemma 1 applies even if we allow $U'_r(0)$ to be infinite. Suppose, without loss of generality, that $U'_1(0) = U'_2(0) = \infty$. Let $\mathbf{d} = \mathbf{d}(\boldsymbol{\theta})$. We observe that users 1 and 2 must have positive allocations at equilibrium, from (27). The optimality conditions for (27) thus imply that for $i = 1, 2$, and for each $s \neq i$, we have:

$$U'_i(d_i) = \bar{U}'(d_s; \theta_s), \quad \text{if } d_s > 0; \quad (29)$$

$$\geq \bar{U}'(d_s; \theta_s), \quad \text{if } d_s = 0. \quad (30)$$

In particular, $U'_1(d_1) = \bar{U}'(d_2; \theta_2)$ and $U'_2(d_2) = \bar{U}'(d_1; \theta_1)$. Furthermore, the optimality conditions for the problem (24) yield $\bar{U}'(d_1; \theta_1) = \bar{U}'(d_2; \theta_2)$. From these relations we first conclude that $U'_1(d_1) = \bar{U}'(d_1; \theta_1) = \mu$ for some scalar μ . Further, if we consider the optimality conditions for (27) for an arbitrary user r , we obtain $U'_r(d_r) = \bar{U}'(d_1; \theta_1) = U'_1(d_1) = \mu$, if $d_r > 0$, and $U'_r(d_r) \leq \bar{U}'(d_1; \theta_1) = U'_1(d_1) = \mu$, if $d_r = 0$. But these are exactly the sufficient conditions for

optimality for \mathbf{d} for the problem $SYSTEM(C, R, \mathbf{U})$, with Lagrange multiplier μ . \square

Intuitively, for efficiency to hold, we need to have at least two actively “competing” users. In the previous result, this is guaranteed because the two users with infinite marginal utility at zero allocation will want strictly positive rate in any equilibrium.

The results of this section demonstrate that by relaxing the assumption that the resource allocation mechanism must set a single price, we can in fact significantly improve upon the efficiency guarantee of Theorem 1. We note that Maheswaran and Basar (2004) (as well as the earlier work, Maheswaran 2003) and Yang and Hajek (2006a) have (independently of each other) presented a class of mechanisms for this resource allocation problem that operate as follows. Each user r chooses a “bid” $\theta_r \geq 0$. Define $T = \sum_s \theta_s$, and $T_{-r} = \sum_{s \neq r} \theta_s$. Their mechanism allocates in proportion to the bids, so $d_r^{MY}(\boldsymbol{\theta}) = \theta_r C / T$. Each user r makes a payment defined by:

$$t_r^{MY}(\boldsymbol{\theta}) = \theta_r (\varphi(T) - \varphi(T_{-r})),$$

where φ is a strictly increasing, continuously differentiable function, such that in addition the function $u \mapsto u^2 \varphi'(u) / C$ is a strictly increasing, onto function from $[0, \infty)$ to $[0, \infty)$. Under these assumptions, define $\bar{U}(x; \theta) = -\theta \varphi(\theta/x)$. Then it can be verified that: (1) \bar{U} satisfies Assumption 2; and (2) the corresponding SSVCG mechanism is the Maheswaran-Basar/Yang-Hajek mechanism, so that $\mathbf{d}(\boldsymbol{\theta}) = \mathbf{d}^{MY}(\boldsymbol{\theta})$, and $\mathbf{t}(\boldsymbol{\theta}) = \mathbf{t}^{MY}(\boldsymbol{\theta})$.⁹ Yang and Hajek have shown that as long as at least two users r have $U_r'(0) = \infty$, then all Nash equilibria are efficient; this matches the result of Proposition 2. A similar result is derived by Maheswaran and Basar for allocation of a single infinitely divisible resource, by assuming a single non-strategic “virtual” user who always bids ε for the resource; Nash equilibria of the resulting mechanism become efficient as ε approaches zero. In conclusion, our class of SSVCG mechanisms includes the mechanism of Maheswaran (2003), Maheswaran and Basar (2004) and Yang and Hajek (2006a) as a special case.

⁹ A small technical issue arises here because $\bar{U}'(0, \theta) = \infty$ for all $\theta > 0$, which violates the last part of Assumption 2; however, the development in this section can be adapted to cover this case.

5.3. General Convex Environments

Despite the fact that we have formulated SSVCG mechanisms for the specific choice of \mathcal{X} corresponding to allocation of a single resource, our results hold for far more general convex environments. Formally, we continue to assume that each user r is to receive an allocated quantity d_r . However, we now assume that the feasible resource allocations are determined by constraints involving the vector \mathbf{d} , as well as some auxiliary variables $\mathbf{y} = (y_{R+1}, \dots, y_{R+M})$. As an example, \mathbf{y} may be the path flows in a multicommodity flow model, in which case d_r is the total rate allocated to user r . We make the following assumption about \mathcal{X} .

ASSUMPTION 3. *The set \mathcal{X} of feasible allocations \mathbf{d} is of the form*

$$\mathcal{X} = \{\mathbf{d} \geq \mathbf{0} : \text{there exists } \mathbf{y} \in \mathbb{R}^M, \mathbf{y} \geq \mathbf{0},$$

$$\text{such that } g_j(\mathbf{d}, \mathbf{y}) \leq 0, \quad j = 1, \dots, J\},$$

for some given functions g_j .

Further:

1. *The set \mathcal{X} is compact;*
2. *For each j , the function g_j is convex and differentiable on an open set containing $(\mathbb{R}^+)^{R+M}$;*
3. *There exists a pair $(\bar{\mathbf{d}}, \bar{\mathbf{y}}) \in \mathbb{R}^{R+M}$ such that $g_j(\bar{\mathbf{d}}, \bar{\mathbf{y}}) < 0$ for all j .*

In the preceding assumption, M denotes the dimension of the space of auxiliary variables, and J the number of convex inequality constraints. Although we assume differentiability in condition (2), this is not crucial; we assume differentiability for technical simplicity.

REMARK 11. Condition (3) in the preceding assumption ensures that the Slater constraint qualification holds for all the optimization problems we consider; see, e.g., Bertsekas et al. (2003). We note that all our results apply to the more general setting where in addition to the convex inequality constraints, the set \mathcal{X} is constrained by *affine equality constraints* of the form $h_k(\mathbf{d}, \mathbf{y}) = 0$, $k = 1, \dots, K$. In this case condition (3) in Assumption 3 is modified to require that there exists a pair $(\bar{\mathbf{d}}, \bar{\mathbf{y}})$ in the *relative interior* of the set $\{(\mathbf{d}, \mathbf{y}) : h_k(\mathbf{d}, \mathbf{y}) = 0\}$, such that $g_j(\bar{\mathbf{d}}, \bar{\mathbf{y}}) < 0$ for

all j ; with this modification the Slater constraint qualification once again applies. All our results continue to hold in this setting, if we simply represent the k th affine equality constraint by the two equivalent inequality constraints $h_k(\mathbf{d}, \mathbf{y}) \leq 0$ and $h_k(\mathbf{d}, \mathbf{y}) \geq 0$.

We define an efficient allocation as the solution to the following generalized definition of *SYSTEM*:

SYSTEM:

$$\text{maximize } \sum_{r=1}^R U_r(d_r) \tag{31}$$

$$\text{subject to } \mathbf{d} \in \mathcal{X}. \tag{32}$$

It is straightforward to check that under Assumption 3, nearly all the basic results established above continue to hold; in particular, Proposition 1, Lemma 1, and Corollary 1 continue to hold, with proofs nearly identical to their existing versions. Thus (27) continues to provide a necessary and sufficient condition for characterization of Nash equilibria, and from Corollary 1 we know that *there always exists at least one efficient Nash equilibrium*.

It is natural to search for an analog of Proposition 2. In particular, we would like a more general sufficient condition that guarantees that *all* Nash equilibria are efficient. We now provide such a guarantee, under additional conditions on the structure of the constraints.

We first introduce some notation. Given a vector $\boldsymbol{\theta}$ of user strategies, we denote by $\mathbf{z} = (\mathbf{d}, \mathbf{y})$ the optimal solution $\mathbf{d} = \mathbf{d}(\boldsymbol{\theta})$ to the problem (24), together with optimal values of the auxiliary variables \mathbf{y} . (Recall that \mathbf{d} , the vector of resulting allocations, is uniquely determined, by strict concavity of the objective function in (24).) Given such a solution, we let

$$\mathcal{P} = \{k \in \{1, \dots, R + M\} : z_k > 0\} \tag{33}$$

be the indices associated with variables whose value is nonzero, and we let

$$\mathcal{J} = \{j : g_j(\mathbf{z}) = 0\} \tag{34}$$

be the set of constraint indices that are binding. For each k , we define the vector $\mathbf{D}(k)$ as:

$$\mathbf{D}(k) = \left(\frac{\partial g_j}{\partial z_k}(\mathbf{z}), j \in \mathcal{J} \right). \quad (35)$$

For example, if the constraints are of the form $\mathbf{A}\mathbf{z} - \mathbf{b} \leq \mathbf{0}$, for some matrix \mathbf{A} and some vector \mathbf{b} , and if all constraints are binding, then $\mathbf{D}(k)$ is the k 'th column of \mathbf{A} .

PROPOSITION 3. *Suppose that Assumptions 1-3 hold, except that $U'_r(0)$ is allowed to be infinite. Suppose that $\boldsymbol{\theta}$ is a Nash equilibrium, $\mathbf{d} = \mathbf{d}(\boldsymbol{\theta})$ is the optimal solution to the problem (24), and \mathbf{y} is an associated optimal vector of auxiliary variables. Define \mathcal{P} , \mathcal{J} , and $\mathbf{D}(k)$ according to (33), (34), and (35), respectively. Suppose furthermore that for every user $r \in \{1, \dots, R\}$, the vector $\mathbf{D}(r)$ is in the span of the vectors $\mathbf{D}(s)$, $s \neq r$, $s \in \mathcal{P}$. Then \mathbf{d} is an efficient allocation.*

Proof. Let $\boldsymbol{\theta}$ be a Nash equilibrium, and let $\mathbf{z} = (\mathbf{d}, \mathbf{y})$ correspond to an optimal solution to the problem (24). Since $\boldsymbol{\theta}$ is a Nash equilibrium, \mathbf{z} is also an optimal solution to the problem (27), for each $r \in \{1, \dots, R\}$.

To simplify the argument, we first assume that $\mathcal{J} = \{1, \dots, J\}$, so that all constraints are binding. Let $\boldsymbol{\lambda} \geq 0$ and $\boldsymbol{\mu}(r) \geq 0$ be vectors of Lagrange multipliers associated with the optimal solution \mathbf{z} for the problem (24), and the problem (27) for user r , respectively. Such Lagrange multipliers are guaranteed to exist by Assumption 3, because the Slater constraint qualification is satisfied; see, e.g., Bertsekas et al. (2003), Section 5.4. To simplify the presentation, we define $U_k(d) = 0$, for $k > R$ and all $d \geq 0$. Then the optimality conditions for the problem (24) are:

$$\bar{U}'(z_k; \theta_k) = \boldsymbol{\lambda}^\top \mathbf{D}(k), \quad \text{if } z_k > 0, k \in \{1, \dots, R\}; \quad (36)$$

$$\bar{U}'(z_k; \theta_k) \leq \boldsymbol{\lambda}^\top \mathbf{D}(k), \quad \text{if } z_k = 0, k \in \{1, \dots, R\}; \quad (37)$$

$$U'_k(z_k) = \boldsymbol{\lambda}^\top \mathbf{D}(k), \quad \text{if } z_k > 0, k \in \{R+1, \dots, R+M\}; \quad (38)$$

$$U'_k(z_k) \leq \boldsymbol{\lambda}^\top \mathbf{D}(k), \quad \text{if } z_k = 0, k \in \{R+1, \dots, R+M\}. \quad (39)$$

Similarly, the optimality conditions for the problem (27) for user r , $r \in \{1, \dots, R\}$, are:

$$\bar{U}'(z_k; \theta_k) = \boldsymbol{\mu}(r)^\top \mathbf{D}(k), \quad \text{if } z_k > 0, k \in \{1, \dots, R\}, k \neq r; \quad (40)$$

$$\bar{U}'(z_k; \theta_k) \leq \boldsymbol{\mu}(r)^\top \mathbf{D}(k), \quad \text{if } z_k = 0, \quad k \in \{1, \dots, R\}, \quad k \neq r; \quad (41)$$

$$U'_k(z_k) = \boldsymbol{\mu}(r)^\top \mathbf{D}(k), \quad \text{if } z_k > 0, \quad k \in \{r, R+1, \dots, R+M\}; \quad (42)$$

$$U'_k(z_k) \leq \boldsymbol{\mu}(r)^\top \mathbf{D}(k), \quad \text{if } z_k = 0, \quad k \in \{r, R+1, \dots, R+M\}. \quad (43)$$

From (36), (38), (40), and (42), we deduce that for all $r \in \{1, \dots, R\}$ and $k \in \mathcal{P}$ such that $k \neq r$, we have:

$$\boldsymbol{\mu}(r)^\top \mathbf{D}(k) = \boldsymbol{\lambda}^\top \mathbf{D}(k). \quad (44)$$

The preceding equality holds because z_k is positive for all $k \in \mathcal{P}$.

Fix a user $r \in \{1, \dots, R\}$. By assumption, there exists a vector of coefficients $(\alpha_{rk}, k \in \mathcal{P}, k \neq r)$ such that:

$$\mathbf{D}(r) = \sum_{\{k \in \mathcal{P}: k \neq r\}} \alpha_{rk} \mathbf{D}(k). \quad (45)$$

Suppose first that $z_r = 0$. We have:

$$U'_r(z_r) \leq \boldsymbol{\mu}(r)^\top \mathbf{D}(r) = \sum_{\{k \in \mathcal{P}: k \neq r\}} \alpha_{rk} \boldsymbol{\mu}(r)^\top \mathbf{D}(k) = \sum_{\{k \in \mathcal{P}: k \neq r\}} \alpha_{rk} \boldsymbol{\lambda}^\top \mathbf{D}(k) = \boldsymbol{\lambda}^\top \mathbf{D}(r).$$

The inequality follows from the optimality condition (43). The first equality follows from (45).

The second equality follows from (44). The third equality uses (45) again. If instead $z_r > 0$, then by using (42), we see that the inequality above holds with equality. Combining this result with (38)-(39), we conclude that:

$$U'_k(z_k) = \boldsymbol{\lambda}^\top \mathbf{D}(k), \quad \text{if } z_k > 0, \quad k \in \{1, \dots, R+M\};$$

$$U'_k(z_k) \leq \boldsymbol{\lambda}^\top \mathbf{D}(k), \quad \text{if } z_k = 0, \quad k \in \{1, \dots, R+M\}.$$

These are precisely the optimality conditions for the problem *SYSTEM* (with Lagrange multiplier vector $\boldsymbol{\lambda}$), and therefore \mathbf{d} is an efficient allocation.

If $\mathcal{J} \neq \{1, \dots, J\}$, so that some constraints are not binding, the preceding argument continues to hold: we need only consider the constraints in \mathcal{J} , since the Lagrange multipliers associated with nonbinding constraints are zero. □

REMARK 12. For the case where the constraints defining \mathcal{X} are linear, of the form $\mathbf{Az} - \mathbf{b} \leq \mathbf{0}$, the linear dependence condition of Proposition 3 is equivalent to the following: Given a Nash equilibrium and $r \in \{1, \dots, N\}$, if we perturb the r th component of \mathbf{z} , then it is always possible to feasibly modify the components z_k , $k \neq r$, $k \in \mathcal{P}$, so that those constraints that are binding (i.e., the constraints in \mathcal{J}) remain binding.

To gain some intuition for Proposition 3, suppose that we have no auxiliary variables, that the set \mathcal{X} is a polyhedron, and that \mathcal{X} has an extreme point with positive components. At that extreme point, and in the absence of degeneracy, there are exactly R constraints that are binding. In this case, the vectors $\mathbf{D}(k)$ are linearly independent, and therefore the linear dependence condition of Proposition 3 fails to hold. Intuitively, for the condition to hold, we need to have a number of actively “competing” users (i.e., users with positive d_r) which is greater than the effective number of resources (i.e., the number of binding constraints). (Notice that the violation of this condition is behind the failure observed in Example 2.)

The linear dependence condition is satisfied in a broad class of problems. The following presents an important example: multicommodity flow routing.

EXAMPLE 3 (MULTICOMMODITY FLOW). In this model, R users send traffic through a network of L links. Each link ℓ has a given finite positive capacity, denoted $C_\ell > 0$. Each user r has available a collection $P(r)$ of paths on which she can send traffic; each path $p \in P(r)$ is identified with the set of links it goes through, so that $p \subset \{1, \dots, L\}$. Let $P = \bigcup_r P(r)$, and let $M = |P|$. By duplicating paths if necessary, we assume without loss of generality that each path $p \in P$ is associated with exactly one user. The set \mathcal{X} is then defined as follows:

$$\mathcal{X} = \left\{ \mathbf{d} \geq \mathbf{0} : \text{there exists } \mathbf{y} \geq \mathbf{0} \text{ with} \right.$$

$$\left. \sum_{p \in P(r)} y_p = d_r, \quad r = 1, \dots, N; \text{ and} \right. \quad (46)$$

$$\left. \sum_{p \in P: \ell \in p} y_p \leq C_\ell, \quad \ell = 1, \dots, L \right\} \quad (47)$$

As discussed in Remark 11, \mathcal{X} satisfies a variation of Assumption 3, and therefore Proposition 3 still applies.

We now make the following assumptions. First, we assume that for all users r , we have $U'_r(0) = \infty$. Second, we assume that for each link ℓ , there are two users $r_1(\ell)$ and $r_2(\ell)$ for which the path $\{\ell\}$, consisting of the single link ℓ , is the only candidate path. The latter assumption guarantees sufficient competition at every link; informally, it requires that competition for resources from flows travelling a short distance is nonnegligible. (A similar assumption was used by Yang and Hajek in Yang and Hajek (2007).) Under these assumptions, we now show that Proposition 3 applies, and thus establish that any resulting Nash equilibrium is efficient.

We only need to verify that the linear dependence condition in Proposition 3 is satisfied at a Nash equilibrium. To check this, let θ be a Nash equilibrium, and let $\mathbf{z} = (\mathbf{d}, \mathbf{y})$ be a corresponding optimal solution to the problem (24), so $\mathbf{d} = \mathbf{d}(\theta)$. We will use the equivalent statement of the linear dependence condition provided in Remark 12.

First note that because $U'_r(0) = \infty$, at a Nash equilibrium every variable d_r is nonzero. Furthermore, for those users that have a single path p , the corresponding path variable y_p is also nonzero.

Let us consider perturbing a variable d_r , say by adding δ to it. Let $p \in P(r)$ be such that $y_p > 0$. Then, we can add δ to y_p , so that the binding constraint (46) for that particular user r remains binding. In addition, for every link ℓ on the path p , there exists a user $r_i(\ell) \neq r$, that has a single path that only uses this link. By decreasing $d_{r_i(\ell)}$ and the associated component of \mathbf{y} by δ , the binding constraint (46) for $r_i(\ell)$ remains binding. Furthermore, if constraint (47) was binding, it remains binding. We conclude that the condition of Remark 12 holds, as required; and thus the result of Proposition 3 applies.

We conclude this section by noting several extensions. First, it is possible to construct a similar class of mechanisms when each user r has a utility function $U_r(\mathbf{x}_r)$ that depends on a *vector* of resources \mathbf{x}_r . In this case, user r needs a strategy space of the same dimension as the vector \mathbf{x}_r . The structure of the mechanism is then analogous to the one described in Section 5.2.

As for standard VCG mechanisms, the mechanisms we have proposed are certainly not *budget balanced*; that is, $\sum_r t_r(\boldsymbol{\theta}) \neq 0$ in general. However, one can consider a variety of forms for the functions h_r to mitigate this effect; a choice similar to the “pivot” mechanisms of Clarke (1971) results if $h_r(\boldsymbol{\theta}_{-r}) = -\sum_{s \neq r} \bar{U}(x_s(\boldsymbol{\theta}_{-r}); \theta_s)$, where $x_s(\boldsymbol{\theta}_{-r})$ is the optimal resource allocation if user r is removed from the system. In this mechanism all payments t_r are negative, and a user r *makes* a payment if and only if his presence affects the optimal resource allocation.

6. Discussion and Conclusions

This paper studies the efficiency achievable by scalar-parameterized mechanisms, particularly in allocation of a single infinitely divisible resource. In Section 4, we show our first main result: when a single price is used for all users, then within a class of reasonable market-clearing mechanisms the Kelly mechanism uniquely minimizes the worst case efficiency loss when users are price anticipating. An intermediate result also shows that allocating the resource in proportion to users’ bids (as in the Kelly mechanism) is a natural consequence of simple assumptions on the mechanism. In Section 5, we show that relaxing the single price assumption actually allows to significantly improve the performance of the mechanism: in our second main result, we demonstrate a class of mechanisms that exhibit efficient Nash equilibria. We also argue that these SSVCG mechanisms can be extended to much more general resource allocation environments, and provide conditions under which *all* Nash equilibria are efficient. We then successfully apply SSVCG mechanisms to a standard multicommodity flow setting.

Compared to the two major results described above, there is another possibility: we might relax the assumption of one-dimensional bids, while preserving the single price restriction on the market mechanism. Consider, for example, mechanisms where users submit *arbitrary* demand functions (giving demand as a function of price), and a single price is chosen to clear the market. Using an analog of the argument in Klemperer and Meyer (1989), it is straightforward to show that for the model we are considering, there exist games with arbitrarily inefficient Nash equilibria. The argument involves showing that essentially any feasible outcome (consisting of a price and allocation vector) can be sustained at a Nash equilibrium.

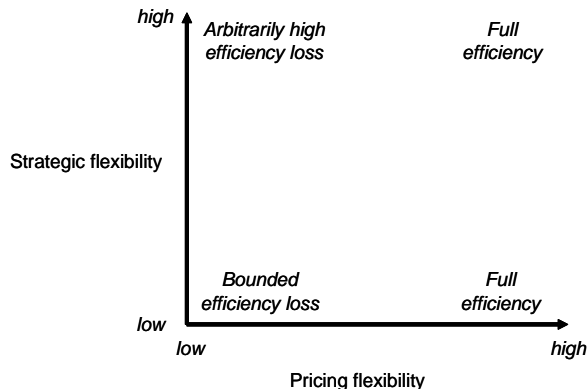


Figure 2 *Strategic flexibility vs. pricing flexibility.* The horizontal (resp., vertical) axis increases with pricing flexibility (resp., strategic flexibility), and the annotations reflect a summary of current understanding. This paper discusses efficiency when mechanisms have restricted strategic flexibility (i.e., one-dimensional bids).

Our results, as well as those of earlier work such as Maheswaran (2003), Maheswaran and Basar (2004), Yang and Hajek (2006a) and Yang and Hajek (2007), suggest that there is an interesting interplay between the *strategic flexibility* granted to users, and the *pricing flexibility* granted to the mechanism. When users are allowed great flexibility in submitting their bids, then full efficiency is possible with price differentiation (the VCG approach, where users’ bids consist of entire utility functions), while arbitrarily high inefficiency results if only a single price is allowed (as in the preceding paragraph). On the other hand, when users are allowed to use only one-dimensional signals, then with a single price the most efficient mechanism (i.e., the Kelly mechanism) suffers a nonzero efficiency loss, while with price differentiation full efficiency can again be achieved (as established for SSVCG mechanisms, and their precursors in Maheswaran and Basar (2004) and Yang and Hajek (2007)). Note, in particular, the critical insight from this stream of work that *when pricing flexibility is limited, restricting strategic flexibility may actually increase efficiency.* These results are summarized in Figure 2.

We briefly note several extensions and open directions. First, Section 4 considers only a single resource. For general network topologies, we can consider a game where users submit individual bids to each link in the network, similar to the approach taken by Johari and Tsitsiklis (2004).

In this case, if each link is endowed with a smooth market-clearing mechanism satisfying the assumptions of Section 4, then the worst case efficiency loss will be minimized if each link uses the Kelly mechanism.

A second extension of the work in Section 4 applies to markets with supply function bidding. Motivated by current problems in market design for electric power systems, we consider a model where multiple producers compete to satisfy an inelastic demand. Demand for electricity, particularly in the short run, is characterized by low elasticity with respect to price, i.e., changes in price do not lead to significant changes in the level of demand; see, e.g., Stoft (2002), Section 1-7.3. A basic model for electricity market operation involves *supply function bidding* (Klemperer and Meyer 1989): each generator submits a supply function expressing their willingness to produce electricity as a function of the market clearing price. A single price is then chosen to ensure that supply matches the inelastic demand. In Johari (2004), we consider *restrictions* on the supply functions which can be chosen by the generators, and aim to design these restrictions so that nearly efficient allocations are achieved even if firms are price anticipating. We prove a characterization theorem, similar to Theorem 1, describing the best possible efficiency guarantees, within a class of mechanisms in which the generators are restricted to submitting a supply function chosen from within a restricted, one-parameter family. Details of this work can be found in Johari (2004).

All the work in this paper is concerned with *quasilinear environments*, i.e., users' utilities are measured in monetary units. Further, our models assume users have unlimited budgets. An interesting set of parallel issues arises if one relaxes these assumptions. Indeed, we note that the Kelly mechanism is quite similar to the celebrated Shapley-Shubik "trading post" mechanism (Shapley and Shubik 1977). In the Shapley-Shubik mechanism, users maximize utility (which may not be quasilinear), and users are subject to a budget constraint. These games thus exhibit very different equilibria from the Kelly mechanism. An interesting open direction concerns evaluating the efficiency properties of the Shapley-Shubik mechanism, and determining the extent to which the Shapley-Shubik mechanism is an "optimal" design. Zhang et al. have considered notions of optimality such as envy-freeness in the setting where users have budget constraints; see Zhang (2005)

and Feldman et al. (2005) for details.

We note that all our work considers efficiency from a *static* standpoint. This is natural as a starting point, and our positive results give proof-of-concept of potentially viable mechanism designs. However, a complete approach also requires attention to the dynamic setting. Several works have recently considered dynamics for convergence to efficient equilibrium. Yang and Hajek (2007) have shown that a form of myopic best response dynamics can converge to an efficient Nash equilibrium of the SSVCG mechanism in the multicommodity flow setting, while the ascending auction mechanism of Ausubel (2006) provides a mechanism with limited communication requirements and at least one efficient subgame perfect equilibrium. The former analyzes a stylized model of dynamics assuming a certain model of user behavior; the latter proposes a dynamic mechanism design assuming fully rational user behavior. Clearly, a significant open direction is to develop a unified treatment of the efficiency of resource allocation mechanisms with restricted communication requirements, in a *dynamic* setting.

Ultimately, however, the most intriguing questions raised by this work are captured by Figure 2. The results of this paper and earlier work in the area suggest that in designing efficient markets, strategic flexibility can be increased only if pricing flexibility is available. However, we have only studied this issue in the extremes depicted in Figure 2. In particular, we have not quantitatively characterized the *tradeoffs* between strategic flexibility and pricing flexibility. For example, we have studied mechanisms with a single price per resource, and with perfect price discrimination. A natural question arises: what are the gains in efficiency if a two-tiered pricing system is used? How many prices are needed for a bounded efficiency loss guarantee if users have strategy spaces parameterized by a fixed number of dimensions? The answers to such questions remain interesting directions for future work.

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Omitted Proofs

Step 2, Proof of Theorem 1: A user's payoff is concave if he is price taking. The condition that a uniform market-clearing price must exist implies that for any fixed $\theta > 0$, the range of $D(\mu, \theta)$ must contain $(0, \infty)$ as μ varies in $(0, \infty)$. Now suppose that for fixed $\theta > 0$, there exist $\mu_1, \mu_2 > 0$ with $\mu_1 \neq \mu_2$ such that $D(\mu_1, \theta) = D(\mu_2, \theta) = d$, where $d > 0$. Let $C = 2d$ and let $R = 2$. Then for $\boldsymbol{\theta} = (\theta, \theta)$, there cannot exist a unique market-clearing price $p_D(\boldsymbol{\theta})$; so we conclude that $D(\cdot, \theta)$ is monotonic, and strictly monotonic in the region where it is nonzero.

Let $I \subset (0, \infty)$ be the set of $\theta > 0$ such that $D(\mu, \theta)$ is monotonically nondecreasing in μ . From the preceding paragraph, we conclude that if $\theta \in (0, \infty) \setminus I$, then $D(\mu, \theta)$ is necessarily monotonically nonincreasing in μ . Further, if $\theta \in I$, then $D(\mu, \theta) \rightarrow \infty$ as $\mu \rightarrow \infty$, and $D(\mu, \theta) \rightarrow 0$ as $\mu \rightarrow 0$; on the other hand, if $\theta \in (0, \infty) \setminus I$, then $D(\mu, \theta) \rightarrow 0$ as $\mu \rightarrow \infty$, and $D(\mu, \theta) \rightarrow \infty$ as $\mu \rightarrow 0$.

Suppose $I \neq (0, \infty)$ and $I \neq \emptyset$; then choose $\theta \in \partial I$, the boundary of I . Choose a sequence $\theta_n \in I$ such that $\theta_n \rightarrow \theta$; and choose another sequence $\hat{\theta}_n \in (0, \infty) \setminus I$ such that $\hat{\theta}_n \rightarrow \theta$. Fix μ_1, μ_2 with $0 < \mu_1 < \mu_2$, such that $D(\mu_1, \theta) > 0$ and $D(\mu_2, \theta) > 0$. Then we have $D(\mu_1, \theta_n) \leq D(\mu_2, \theta_n)$, and $D(\mu_1, \hat{\theta}_n) \geq D(\mu_2, \hat{\theta}_n)$. Taking limits as $n \rightarrow \infty$, we get $D(\mu_1, \theta) \leq D(\mu_2, \theta)$, and $D(\mu_1, \theta) \geq D(\mu_2, \theta)$, so that $D(\mu_1, \theta) = D(\mu_2, \theta)$. But this is not possible, since $D(\cdot, \theta)$ must be strictly monotonic in the region where it is nonzero. Thus $I = (0, \infty)$ or $I = \emptyset$.

We will use Step 1 to show $D(\mu, \theta)$ is concave in $\theta \geq 0$ for fixed $\mu > 0$. Since $D(\mu, \theta)$ is continuous, it suffices to show that $D(\mu, \theta)$ is concave for $\theta > 0$. Suppose not; fix $\theta > 0$, $\bar{\theta} > 0$, and $\delta \in (0, 1)$ such that:

$$D(\mu, \delta\theta + (1 - \delta)\bar{\theta}) < \delta D(\mu, \theta) + (1 - \delta)D(\mu, \bar{\theta}). \quad (\text{EC.1})$$

Note this implies in particular that either $D(\mu, \theta) > 0$ or $D(\mu, \bar{\theta}) > 0$. We assume without loss of generality that $D(\mu, \theta) > 0$. Let $C^R = RD(\mu, \theta)$, and let $\boldsymbol{\theta}^R = (\theta, \dots, \theta) \in (\mathbb{R}^+)^R$. To emphasize the dependence of the market-clearing price on the capacity, we will let $p_D(\bar{\boldsymbol{\theta}}; C)$ denote the market-clearing price when the composite strategy vector is $\bar{\boldsymbol{\theta}}$ and the capacity is C . We will show that

for any $\theta' > 0$, if $\mu^R = p_D(\boldsymbol{\theta}^{R-1}, \theta'; C^R)$, then $\mu^R \rightarrow \mu$ as $R \rightarrow \infty$. First note that by definition, we have $D(\mu^R, \theta') + (R-1)D(\mu^R, \theta) = RD(\mu, \theta)$; or, rewriting, we have:

$$\frac{1}{R}D(\mu^R, \theta') + \left(1 - \frac{1}{R}\right)D(\mu^R, \theta) = D(\mu, \theta). \quad (\text{EC.2})$$

Now note that as $R \rightarrow \infty$, the right hand side remains constant. Suppose that $\mu^R \rightarrow \infty$. Since $I = (0, \infty)$ or $I = \emptyset$, either $D(\mu^R, \theta'), D(\mu^R, \theta) \rightarrow 0$, or $D(\mu^R, \theta'), D(\mu^R, \theta) \rightarrow \infty$; in either case, the equality (EC.2) is violated for large R . A similar conclusion holds if $\mu^R \rightarrow 0$ as $R \rightarrow \infty$. Thus we do not have $\mu^R \rightarrow 0$ or $\mu^R \rightarrow \infty$ as $R \rightarrow \infty$. Choose a convergent subsequence, such that $\mu^{R_k} \rightarrow \hat{\mu}$, where $\hat{\mu} \in (0, \infty)$. From (EC.2), we must have $D(\hat{\mu}, \theta) = D(\mu, \theta)$. But as established above, since $D(\cdot, \theta)$ is strictly monotonic in the region where it is nonzero, this is only possible if $\hat{\mu} = \mu$. We conclude that the following three limits hold:

$$\begin{aligned} \lim_{R \rightarrow \infty} p_D(\boldsymbol{\theta}^R; C^R) &= \mu; \\ \lim_{R \rightarrow \infty} p_D(\boldsymbol{\theta}^{R-1}, \bar{\theta}; C^R) &= \mu; \\ \lim_{R \rightarrow \infty} p_D(\boldsymbol{\theta}^{R-1}, \delta\theta + (1-\delta)\bar{\theta}; C^R) &= \mu. \end{aligned}$$

The remainder of the proof is straightforward. From (EC.1), for R sufficiently large, we must have:

$$\begin{aligned} D(p_D(\boldsymbol{\theta}^{R-1}, \delta\theta + (1-\delta)\bar{\theta}; C^R), \delta\theta + (1-\delta)\bar{\theta}) \\ < \delta D(p_D(\boldsymbol{\theta}^R; C^R), \theta) + (1-\delta)D(p_D(\boldsymbol{\theta}^{R-1}, \bar{\theta}; C^R), \bar{\theta}). \end{aligned}$$

This violates the conclusion of Step 1, so we conclude $D(\mu, \theta)$ is concave in $\theta \geq 0$ given $\mu > 0$. A similar argument shows that $\mu D(\mu, \theta)$ is convex in θ , by using the fact that $p_D(\boldsymbol{\theta})D(p_D(\boldsymbol{\theta}), \theta_r)$ must be convex in θ_r for nonzero $\boldsymbol{\theta}$. Combining these results yields the desired conclusion.

Step 5, Proof of Theorem 1: B is an invertible, differentiable, strictly increasing, and concave function on $(0, \infty)$. Note from (10) that:

$$B(p_D(\boldsymbol{\theta})) = \frac{\sum_{r=1}^R \theta_r}{C}. \quad (\text{EC.3})$$

We immediately see that B must be invertible on $(0, \infty)$; it is clearly onto, as the right hand side of (EC.3) can take any value in $(0, \infty)$. Furthermore, if $B(p_1) = B(p_2) = \gamma$ for some prices $p_1, p_2 > 0$, then choosing $\boldsymbol{\theta}$ such that $\sum_{r=1}^R \theta_r / C = \gamma$, we find that $p_D(\boldsymbol{\theta})$ is not uniquely defined. Thus B is one-to-one as well, and hence invertible. Finally, note that since D is differentiable, B must be differentiable as well.

We let Φ denote the differentiable inverse of B . We will show that Φ is strictly increasing and convex. We first note that for nonzero $\boldsymbol{\theta}$ we have:

$$p_D(\boldsymbol{\theta}) = \Phi \left(\frac{\sum_{r=1}^R \theta_r}{C} \right).$$

Let

$$w_r(\boldsymbol{\theta}) = p_D(\boldsymbol{\theta}) D(p_D(\boldsymbol{\theta}), \theta_r) = \Phi \left(\frac{\sum_{s=1}^R \theta_s}{C} \right) \left(\frac{\theta_r}{\sum_{s=1}^R \theta_s} C \right). \quad (\text{EC.4})$$

By Step 1, $w_r(\boldsymbol{\theta})$ is convex in $\theta_r > 0$. By considering strategy vectors $\boldsymbol{\theta}$ for which $\boldsymbol{\theta}_{-r} = \mathbf{0}$, it follows that Φ is convex.

It remains to be shown that Φ is strictly increasing. Since Φ is invertible, it must be monotonic; and thus Φ is either strictly increasing or strictly decreasing. To simplify the argument, we assume that Φ is twice differentiable.¹⁰ We twice differentiate $w_r(\boldsymbol{\theta})$, given in (EC.4). Letting $\mu = \sum_{s=1}^R \theta_s / C$, we have for nonzero $\boldsymbol{\theta}$:

$$\frac{\partial^2 w_r}{\partial \theta_r^2}(\boldsymbol{\theta}) = \Phi''(\mu) \frac{\theta_r}{C^2 \mu} + \frac{2 \sum_{s \neq r} \theta_s}{C^2 \mu^3} (\mu \Phi'(\mu) - \Phi(\mu)). \quad (\text{EC.5})$$

Consider some nonzero $\boldsymbol{\theta}_{-r}$, and take the limit as $\theta_r \rightarrow 0$. The limit of the left-hand side in (EC.5) is nonnegative, by the convexity of $w_r(\boldsymbol{\theta})$ in $\theta_r > 0$. The limit of the first term in the right-hand side of (EC.5) is zero. Since $\Phi(\mu) > 0$, it follows that $\Phi'(\mu) > 0$, so that Φ is strictly increasing. This establishes the desired facts regarding B .

¹⁰ While the most direct argument uses twice differentiability of Φ , it is possible to make a similar argument even if Φ is only once differentiable, by arguing only in terms of increments of Φ .

Step 6, Proof of Theorem 1: Let (C, R, \mathbf{U}) be a utility system. A vector $\boldsymbol{\theta} \geq 0$ is a Nash equilibrium if and only if at least two components of $\boldsymbol{\theta}$ are nonzero, and there exists a nonzero vector $\mathbf{d} \geq 0$ and a scalar $\mu > 0$ such that $\theta_r = \mu d_r$ for all r , $\sum_{r=1}^R d_r = C$, and the following conditions hold:

$$U'_r(d_r) \left(1 - \frac{d_r}{C}\right) = \Phi(\mu) \left(1 - \frac{d_r}{C}\right) + \mu \Phi'(\mu) \left(\frac{d_r}{C}\right), \quad \text{if } d_r > 0;$$

$$U'_r(0) \leq \Phi(\mu), \quad \text{if } d_r = 0.$$

In this case $d_r = D(p_D(\boldsymbol{\theta}), \theta_r)$, $\mu = \sum_{r=1}^R \theta_r / C$, and $\Phi(\mu) = p_D(\boldsymbol{\theta})$. Suppose that $\boldsymbol{\theta}$ is a Nash equilibrium. Since $Q_r(\theta_r; \boldsymbol{\theta}_{-r}) = -\infty$ if $\boldsymbol{\theta} = 0$, (from (7)), we must have $\boldsymbol{\theta} \neq 0$. Suppose then that only one component of $\boldsymbol{\theta}$ is nonzero; say $\theta_r > 0$, and $\boldsymbol{\theta}_{-r} = 0$. Then the payoff to user r is:

$$U_r(C) - \Phi\left(\frac{\theta_r}{C}\right) C.$$

But now observe that by infinitesimally reducing θ_r , user r can strictly improve his payoff (since Φ is strictly increasing). Thus $\boldsymbol{\theta}$ could not have been a Nash equilibrium; we conclude that at least two components of $\boldsymbol{\theta}$ are nonzero. In this case, from (7), and the expressions in (11) and (EC.4), the payoff $Q_r(\bar{\theta}_r; \boldsymbol{\theta}_{-r})$ to user r is differentiable. When two components of $\boldsymbol{\theta}$ are nonzero, we may write the payoff Q_r to user r as follows, using (11) and (EC.4):

$$Q_r(\theta_r; \boldsymbol{\theta}_{-r}) = U_r\left(\frac{\theta_r}{\sum_{s=1}^R \theta_s} C\right) - \Phi\left(\frac{\sum_{s=1}^R \theta_s}{C}\right) \left(\frac{\theta_r}{\sum_{s=1}^R \theta_s} C\right).$$

Differentiating the previous expression with respect to θ_r , we conclude that if $\boldsymbol{\theta}$ is a Nash equilibrium then the following optimality conditions hold for each r :

$$F_r(\boldsymbol{\theta}) = 0 \quad \text{if } \theta_r > 0; \tag{EC.6}$$

$$F_r(\boldsymbol{\theta}) \leq 0 \quad \text{if } \theta_r = 0, \tag{EC.7}$$

where

$$F_r(\boldsymbol{\theta}) = U'_r\left(\frac{\theta_r}{\sum_{s=1}^R \theta_s} C\right) \left(\frac{C}{\sum_{s=1}^R \theta_s} - \frac{\theta_r C}{\left(\sum_{s=1}^R \theta_s\right)^2}\right) - \Phi'\left(\frac{\sum_{s=1}^R \theta_s}{C}\right) \left(\frac{\theta_r}{\sum_{s=1}^R \theta_s}\right)$$

$$-\Phi \left(\frac{\sum_{s=1}^R \theta_s}{C} \right) \left(\frac{C}{\sum_{s=1}^R \theta_s} - \frac{\theta_r C}{\left(\sum_{s=1}^R \theta_s \right)^2} \right).$$

These conditions are equivalent to (14)-(15), if we make the substitutions $\mu = \sum_{s=1}^R \theta_s / C$, and $d_r = D(p_D(\boldsymbol{\theta}), \theta_r)$. Furthermore, in this case we have $\mathbf{d} \geq 0$, $\mu > 0$, $\theta_r = \mu d_r$, $\sum_{r=1}^R d_r = C$, and $p_D(\boldsymbol{\theta}) = \Phi(\mu)$.

On the other hand, suppose that we have found $\boldsymbol{\theta}$, \mathbf{d} , and μ such that the conditions of Step 6 are satisfied. In this case we simply reverse the argument above; since $Q_r(\bar{\theta}_r; \boldsymbol{\theta}_{-r})$ is concave in $\bar{\theta}_r$ (Condition 2 in Definition 4), if at least two components of $\boldsymbol{\theta}$ are nonzero then the conditions (EC.6)-(EC.7) are necessary and sufficient for $\boldsymbol{\theta}$ to be a Nash equilibrium. Furthermore, if $\mathbf{d} \geq 0$, $\mu > 0$, $\theta_r = \mu d_r$, and $\sum_{r=1}^R d_r = C$, then it follows that $\mu = \sum_{s=1}^R \theta_s / C$, $\Phi(\mu) = p_D(\boldsymbol{\theta})$, and $d_r = D(p_D(\boldsymbol{\theta}), \theta_r)$. Thus the conditions (EC.6)-(EC.7) become equivalent to (14)-(15), as required.

Step 7, Proof of Theorem 1: Let (C, R, \mathbf{U}) be a utility system. Then there exists a unique Nash equilibrium. Our approach will be to demonstrate existence of a Nash equilibrium by finding a solution $\mu > 0$ and $\mathbf{d} \geq 0$ to (14)-(15), such that $\sum_{r=1}^R d_r = C$. If we find such a solution, then at least two components of \mathbf{d} must be nonzero; otherwise, (14) cannot hold for the user r with $d_r = C$. If we define $\boldsymbol{\theta} = \mu \mathbf{d}$, then $\mu = \sum_{s=1}^R \theta_s / C$, so $p_D(\boldsymbol{\theta}) = \Phi(\mu)$; and from (11), we have $d_r = D(p_D(\boldsymbol{\theta}), \theta_r)$. Thus if $\mu > 0$ and $\mathbf{d} \geq 0$ satisfy (14)-(15), then $\boldsymbol{\theta} = \mu \mathbf{d}$ is a Nash equilibrium by Step 6. Consequently, it suffices to find a solution $\mu > 0$ and $\mathbf{d} \geq 0$ to (14)-(15).

We first show that for a fixed value of $\mu > 0$, the equality in (14) has at most one solution d_r . To see this, rewrite (14) as:

$$U'_r(d_r) \left(1 - \frac{d_r}{C} \right) - (\mu \Phi'(\mu) - \Phi(\mu)) \left(\frac{d_r}{C} \right) = \Phi(\mu).$$

Since Φ is convex and strictly increasing with $\Phi(\mu) \rightarrow 0$ as $\mu \rightarrow 0$, we have $\mu \Phi'(\mu) - \Phi(\mu) \geq 0$. Thus the left hand side is strictly decreasing in d_r (since U_r is strictly increasing and concave), from $U'_r(0)$ at $d_r = 0$ to $\Phi(\mu) - \mu \Phi'(\mu) \leq 0$ when $d_r = C$. This implies a unique solution $d_r \in [0, C]$ exists

for the equality in (14) as long as $U_r'(0) \geq \Phi(\mu)$; we denote this solution $d_r(\mu)$. If $\Phi(\mu) > U_r'(0)$, then we let $d_r(\mu) = 0$. Observe that as $\mu \rightarrow 0$, we must have $d_r(\mu) \rightarrow C$, since otherwise we can show that (14) fails to hold for sufficiently small μ .

Next we show that $d_r(\mu)$ is continuous. Since we defined $d_r(\mu) = 0$ if $\Phi(\mu) > U_r'(0)$, and $d_r(\mu) = 0$ if $\Phi(\mu) = U_r'(0)$ from (14), it suffices to show that $d_r(\mu)$ is continuous for μ such that $\Phi(\mu) \leq U_r'(0)$. But in this case continuity of d_r can be shown using (14), together with the fact that U_r' , Φ , and Φ' are all continuous (the latter because Φ is concave and differentiable, and hence continuously differentiable). Indeed, suppose that $\mu_n \rightarrow \mu$ where $\Phi(\mu) \leq U_r'(0)$, and assume without loss of generality that $d_r(\mu_n) \rightarrow d_r$ (since $d_r(\mu_n)$ takes values in the compact set $[0, C]$). Then since μ_n and $d_r(\mu_n)$ satisfy the equality in (14) for sufficiently large n , by taking limits we see that μ and d_r satisfy the equality in (14) as well. Thus we must have $d_r = d_r(\mu)$, so we conclude $d_r(\mu)$ is continuous.

We now show that $d_r(\mu)$ is nonincreasing in μ . To see this, choose $\mu_1, \mu_2 > 0$ such that $\mu_1 < \mu_2$. Suppose that $d_r(\mu_1) < d_r(\mu_2)$. Then, in particular, $d_r(\mu_2) > 0$, so (14) holds with equality for $d_r(\mu_2)$ and μ_2 . Now note that as we move from $d_r(\mu_2)$ to $d_r(\mu_1)$, the left hand side of (14) strictly increases (since U_r is concave). On the other hand, since Φ is convex and strictly increasing with $\Phi(\mu) \rightarrow 0$ as $\mu \rightarrow 0$, we have the inequalities $\mu_2 \Phi'(\mu_2) - \Phi(\mu_2) \geq \mu_1 \Phi'(\mu_1) - \Phi(\mu_1) \geq 0$. From this it follows that the right hand side of (14) strictly decreases as we move from $d_r(\mu_2)$ to $d_r(\mu_1)$ and from μ_2 to μ_1 . Thus neither (14) nor (15) can hold at $d_r(\mu_1)$ and μ_1 ; so we conclude that for all r , we must have $d_r(\mu_1) \geq d_r(\mu_2)$.

Thus for each r , $d_r(\mu)$ is a nonincreasing continuous function such that $d_r(\mu) \rightarrow C$ as $\mu \rightarrow 0$, and $d_r(\mu) \rightarrow 0$ as $\mu \rightarrow \infty$. We conclude there exists at least one $\mu > 0$ such that $\sum_{r=1}^R d_r(\mu) = C$; and in this case $\mathbf{d}(\mu)$ satisfies (14)-(15), so by the discussion at the beginning of this step, we know that $\boldsymbol{\theta} = \mu \mathbf{d}(\mu)$ is a Nash equilibrium.

Finally, we show that the Nash equilibrium is unique. Suppose that there exist two solutions $\mathbf{d}^1 \geq 0, \mu_1 > 0$, and $\mathbf{d}^2 \geq 0, \mu_2 > 0$ to (14)-(15), such that $\sum_{r=1}^R d_r^i = C$ for $i = 1, 2$. Of course, we must have $\mathbf{d}^i = \mathbf{d}(\mu_i)$, $i = 1, 2$. We assume without loss of generality that $\mu_1 \leq \mu_2$; our goal is to show

that $\mu_1 = \mu_2$. Since $d_r(\cdot)$ is nonincreasing, we know $d_r(\mu_1) \geq d_r(\mu_2)$ for all r . Since $\sum_{r=1}^R d_r^i = C$ for $i = 1, 2$, we conclude that $d_r(\mu_1) = d_r(\mu_2)$ for every r . Let r be such that $d_r(\mu_1) = d_r(\mu_2) > 0$. Observe that $\Phi(\mu)$ and $\mu\Phi'(\mu)$ are both strictly increasing in $\mu > 0$, since Φ is strictly increasing and convex. Thus for fixed $d_r > 0$, the equality in (14) has a unique solution μ , so $d_r(\mu_1) = d_r(\mu_2) > 0$ implies $\mu_1 = \mu_2$. Thus (14)-(15) have a unique solution $\mathbf{d} \geq 0$, $\mu > 0$, such that $\sum_{r=1}^R d_r = C$. From Step 6, this ensures the Nash equilibrium $\boldsymbol{\theta} = \mu\mathbf{d}$ is unique as well.