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ANALYSIS OF THE UNCAPACITATED DYNAMIC LOT SIZE PROBLEM

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ABSTRACT

In this paper we provide worst case error bounds for several heuristics for the uncapacitated dynamic lot size problem. We propose two managerially oriented procedures and show that they have a relative worst case error bound equal to two, and develop similar analyses for methods known as the "Silver and Meal" heuristics, the part period balancing heuristics, and economic order quantity heuristics (expressed in terms of a time supply of demand). We also present results on aggregation and partitioning of the planning horizon.

1. Introduction

Due to their importance to production planning and inventory control, lot size problems have been widely studied ([12] and [7]). In particular, these problems play a key role in materials requirement planning ([6] and [8]).

In this paper, we study heuristics for the following uncapacitated version of the lot size problem:

$$(P) \quad \text{Min} \sum_{i=1}^{N} \sum_{t=1}^{T} [s_{it}\delta(X_{it}) + h_{it}I_{it} + v_{it}X_{it}]$$

$$\text{s.t.} \quad X_{it} - I_{it} + I_{i,t-1} = d_{it} \qquad i=1,\dots,N;\ t=1,\dots,T$$

$$X_{it}, I_{it} \geq 0 \qquad i=1,\dots,N;\ t=1,\dots,T$$

$$\delta(X_{it}) = \begin{cases} 1 & \text{if } X_{it} > 0 \\ 0 & \text{otherwise} \end{cases} \qquad i=1,\dots,N \quad t=1,\dots,T$$

where the decision variables are:

X_{it}, the number of units of product i to be produced in time period t,

I_{it}, the inventory of product i carried from period t to period t+1,

and the parameters are:

s_{it}, the set-up cost of producing product i in period t,

h_{it}, the unit cost for holding product i from period t to period t+1,

v_{it}, the unit production costs for product i in period t,

d_{it}, the demand of product i in period t.

Problem (P) is separable in i and reduces to N single product uncapacitated lot size problems.

Sometime ago, Wagner and Whitin [9] proposed an efficient $O(T^2)$ dynamic programming algorithm for solving problem (P). The algorithm has not been extensively used in practice, however, due to the difficulty that managers have in understanding it and because it can be time consuming when applied to problems involving tens of thousands of components. As a consequence, several O(T) heuristics have been developed for the uncapacitated lot size problem. In this paper, we provide worst case error bounds for some of these heuristics and we analyze aggregation procedures. In section two, we show that even though the Silver and Meal heuristic has performed very well in several simulation tests presented in the literature [7], the heuristic's worst case errors can be arbitrarily bad. In the same section, we propose two simple procedures with a worst case relative error bound equal to two, and analyze a part period balancing method and a heuristic based on the economic order quantity expressed as a time supply [7]. In section three, we study the effect of reducing the planning horizon. We show that the worst case error, when the multi-facility single-item problem is partitioned in two sub-problems, does not exceed the sum of the costs associated with a single set-up in each facility. In section four, we address issues of aggregation, proving that when two products present certain proportionality in their parameters, there is an optimal strategy that applies to both. We also study worst case errors for the cases where the proportionality conditions are not applicable. The results of this section apply, in particular, to the diagnostic analysis of inventory systems and complement those of Bitran, Valor and Hax [1]. Finally, in the last section we present conclusions and topics for further investigation.

2. Analysis of Some Heuristics for the Uncapacitated Lot Size Problem

As we have already noted, because practitioners have difficulty understanding optimization based methods such as the Wagner and Whitin algorithm, they frequently use simpler, non-optimal, algorithms. Among these is the Silver and Meal heuristic. In computational studies, this procedure has performed well when compared to others in the literature [7, p. 317]. Nevertheless, as we show, its worst case error bound can be arbitrarily bad. Moreover, we show that other well-known heuristics also perform badly in a worst case sense, and propose two heuristics that have much better worst case performance.

Throughout this section, we will deal with single-item production problems and so we drop the index i from the formulation (P) (e.g. we use d_t instead of d_{it}). The results are derived for the case where $v_t = v$ and therefore the corresponding costs can be ignored.

The Silver and Meal Heuristic

To set notation, we briefly describe the Silver and Meal heuristic. Assume we are in the beginning of the first period (or that we are in any subsequent period where we will produce) and we want to determine the quantity Q to be produced in this period. The total cost associated with a production quantity Q that satisfies demands for n periods is given by

$$TC(n) = \text{set-up cost} + \text{carrying costs to the end of period } n.$$

The procedure selects n as the first period that minimizes (locally) the total cost per unit time, that is, if $AC(t) = TC(t)/t$ denotes the average cost per period, then

$$AC(1) \geq AC(2) \geq \dots \geq AC(n)$$

$$AC(n) < AC(n+1).$$

In the next proposition, we establish a worst case error bound for this heuristic.

Proposition 2.1: The worst case relative error for the Silver and Meal heuristic can be arbitrarily large.

Proof: We prove the proposition by means of a sequence of examples. Let n be a positive integer. Consider a lot size problem with parameters

$$s_t = 1,\ h_t = 1 \text{ for all } t, \text{ and}$$

$$d_1 > 0,\ d_2 = 0, \ldots, d_n = 0,\ d_{n+1} = \frac{1}{n^2} + \varepsilon_n$$

where ε_n is a small positive quantity, and s_t, h_t, and d_t denote respectively the set-up cost, holding cost, and the demand at time period t. Applying the Silver and Meal heuristic, we have

$$AC(1) = 1 > AC(2) = \frac{1}{2} > \ldots > AC(n) = \frac{1}{n}$$

$$AC(n+1) = \frac{1 + \frac{1}{n} + n\varepsilon_n}{n+1} = \frac{1}{n} + \left(\frac{n}{n+1}\right)\varepsilon_n > AC(n).$$

Two set-ups occur, one at period 1 and the second at period n+1. The total cost given by the heuristic is $Z_H = 2$. The optimal solution is to produce everything at period 1. The total cost of the optimal solution is

$$Z_0 = 1 + n(1/n^2 + \varepsilon_n) = 1 + 1/n + n\varepsilon_n.$$

Hence,

$$\frac{Z_H}{Z_0} = \frac{2}{1 + 1/n + n\varepsilon_n}.$$

If $n\varepsilon_n \to 0$, this ratio tends to 2 as $n \to \infty$.

Consider next a lot size problem with parameters

$$s_t = 1,\ h_t = 1, \text{ and}$$

$$d_1 > 0,\ d_2 = \ldots = d_n = 0,\ d_{n+1} = 1/n^2 + \varepsilon_n,$$

$$d_{n+2} = \ldots = d_{2n} = 0,\ d_{2n+1} = 1/n^2 + \varepsilon_n.$$

According to the Silver-Meal heuristic, the production will occur in periods 1, n+1, and 2n+2, with total costs $Z_H = 3$. The optimal strategy is still to produce everything in period 1. The total cost of the optimal solution is

$$Z_0 = 1 + n(1/n^2 + \varepsilon_n) + (2n)(1/n^2 + \varepsilon_n) = 1 + 3/n + 3n\varepsilon_n$$

and

$$\frac{Z_H}{Z_0} = \frac{3}{1 + 3/n + 3n\varepsilon_n} .$$

If $n\varepsilon_n \to 0$, this ratio tends to 3 as $n \to \infty$.

By increasing the size of the problem and duplicating the demand pattern as we have, the relative error can be made arbitrarily large. ▨

Economic Order Quantity and Part-Period Balancing Heuristics

We next consider two other procedures used in practice: an economic order quantity heuristic expressed as a time supply of demand and a part-period balancing heuristic. The economic order quantity expressed as a time supply applies to problem with stationary costs, i.e. for all t, $s_t = s$ and $h_t = h$. The economic time supply is determined from the average demand rate $\bar{D}$ by the formula

$$T_{EOQ} = \frac{EOQ}{\bar{D}} = \sqrt{\frac{2s}{\bar{D}h}}$$

rounded to the nearest integer greater than zero. That is, the item is produced in a quantity large enough to cover exactly the demands of this integer number of periods.

As we might expect, this heuristic performs particularly poorly when there is a significant variability in the demand pattern. The following proposition illustrates this point.

Proposition 2.2: The economic order quantity expressed as a time supply scheme for solving the uncapacitated lot size problem can be arbitrarily bad.

Proof: Let $\varepsilon >$ be given and let n be a given positive integer. Consider a problem having the parameters

$$s = 1,\ h = 1,\ T = n$$

$$d_1 = 2n - \varepsilon,\ d_t = \varepsilon/(n-1) \qquad t=2,3,\ldots,T$$

$$\bar{D} = \frac{\sum_{t=1}^{T} d_t}{T} = 2n/n = 2.$$

According to the heuristic, $T_{EOQ} = 1$, hence, we produce in every period. The cost associated with this plan is $Z_H = n$. The optimal plan is to produce everything in the first period. The optimal cost is

$$Z_0 = 1 + \varepsilon/(n-1) + 2\varepsilon/(n-1) + \ldots + (n-1)\varepsilon/(n-1) = 1 + \varepsilon n/2.$$

Hence, $\frac{Z_H}{Z_0} = \frac{n}{1+n\varepsilon/2}$. If $n\varepsilon \to 0$ and $n \to \infty$, we can make the relative error arbitrarily bad.

In the part-period balancing heuristic, the production in any period when we produce is chosen so that the total carrying costs for the periods covered by the production quantity is made as close as possible to the set-up cost. We make the assumption, when applying this heuristic, that if a set-up occurs in period t-1 and the holding cost is larger than the set up cost in period t, then we also set-up in period t. We will again assume stationary fixed costs $s_t = s$ and holding costs $h_t = h$. Then,

Proposition 2.3: When applied to the uncapacitated single-item lot size problem, the worst case relative error of the part-period balancing heuristic is bounded by 3.

Proof: Assuming that when applying this heuristic, we incur n set-ups. Then,

$$Z_H = ns + H$$

where H corresponds to the total holding costs. But H is bounded by 2ns, since between two consecutive set-ups and after the last set-up, the holding cost does not exceed 2s.

Let Z_0 denote the cost associated with the optimal plan. Then, $Z_0 \geq ns$ since if r and u correspond to two consecutive set-up periods determined by the heuristic, the optimal strategy must incur a cost (holding or a set-up) at least as large as one set-up cost in the interval (R,u]. There are (n-1) of such intervals, plus the initial set-up.

Hence, $\frac{Z_H}{Z_0} \leq \frac{ns + 2ns}{ns} = 3.$ ▨

The worst case bound derived in Proposition 2.3 is tight as is shown in the following example. Consider the uncapacitated problem with parameters:

$h = 1,\ s = 1$

$d_{1+3i} = \varepsilon_1 > 0$ $\qquad i=0,1,2,\ldots,n-1$

$d_{2+3i} = \varepsilon_2 > 0$ $\qquad i=0,1,2,\ldots,n-1$

$d_{3+3i} = s - \varepsilon_3$ $\qquad i=0,1,2,\ldots,n-1$

where $0 < \varepsilon_2 < \varepsilon_3 < s$, $s > 2\varepsilon_3$, $s > \varepsilon_1+\varepsilon_2$, and $s > 2\varepsilon_2+\varepsilon_3$.

By applying the part-period balancing heuristic, we incur set-ups at periods $t = 1+3i$, $i=0,1,2,\ldots,n-1$. The total costs associated with this plan is

$$Z_H = n + (\varepsilon_2 + 2 - 2\varepsilon_3)n = 3n + n\varepsilon_2 - 2n\varepsilon_3.$$

The optimal solution is to produce at periods $t = 3+3i$, $i=0,1,\ldots,n-1$ and period 1. The optimal cost is

$$Z_0 = n + 1 + (n-1)(\varepsilon_1+2\varepsilon_2) + \varepsilon_2 = (n+1) + (2n-1)\varepsilon_2 + (n-1)\varepsilon_1.$$

Therefore,

$$\frac{Z_H}{Z_0} = \frac{3n + n\varepsilon_2 - 2n\varepsilon_3}{(n+1)+(2n-1)\varepsilon_2 + (n-1)\varepsilon_1}$$

If $n\varepsilon_2 \to 0$, $n\varepsilon_3 \to 0$, $n\varepsilon_1 \to 0$, and $n \to \infty$, then

$$\frac{Z_H}{Z_0} \to 3.$$

Two Other Heuristics

We next consider two more heuristic procedures where we assume without loss of generality that $d_1 > 0$. The two heuristics are closely related. They use the same basic rule, produce for an integral number of periods, the number being chosen so that holding cost in total first exceeds a set-up cost. In Heuristic 1, this rule is applied in a forward manner and in Heuristic 2, it is applied backwards. More formally, the heuristics are described as follows.

Heuristic 1 (forward):

Step 0: Set-up at t=1.

Step 1: Let $H = 0$, $t^* = t$.

Step 2: Let $t = t+1$. If $t > T$, go to step 4, otherwise, let

$$H = H + \sum_{q=t^*}^{t-1} h_q d_t.$$

If $H > s_t$, go to Step 3, otherwise go to Step 2.

Step 3: Let

$$X_{t^*} = \sum_{q=t^*}^{t-1} d_q.$$

Go to Step 1.

Step 4: Let

$$X_{t^*} = \sum_{q=t^*}^{t-1} d_q.$$

Stop.

Heuristic 2 (backward):

Step 0: Let t=T.

Step 1: Let H = 0, $t^* = t$.

Step 2: Let t = t-1. If t = 0, go to Step 4, otherwise, let

$$H = H + h_t \sum_{q=t+1}^{t^*} d_q .$$

If $H > s_t$, go to Step 3, otherwise go to Step 2.

Step 3: Let

$$X_{t+1} = \sum_{q=t+1}^{t^*} d_q . \quad \text{Go to Step 1.}$$

Step 4: Let

$$X_1 = \sum_{q=1}^{t^*} d_q . \quad \text{Stop.}$$

We next identify a few properties of solutions obtained by applying these procedures.

Assume constant set-up costs and suppose that the production periods for Heuristic 1 are

$$t_1 = 1 < t_2 < \ldots < t_{n_1} . \tag{2.1}$$

Let the production periods determined by Heuristic 2 be

$$t_1' = 1 < t_2' < \ldots < t_{n_2}' . \tag{2.2}$$

Then,

Lemma 2.1: Consider period t_j' for some j > 1. If for some i,

$$t_i < t_j' \leq t_{i+1}, \quad \text{then } t_{j-1}' \leq t_i .$$

Proof: Assume $t_{j-1}' > t_i$. Then,

$$t_i < t_{j-1}' < t_j' \leq t_{i+1} . \tag{2.3}$$

Also, by Heuristic 1,

$$\sum_{t=t_i+1}^{t_{i+1}-1} \Big(\sum_{q=t_i}^{t-1} h_q \Big) d_t \leq s \tag{2.4}$$

and by Heuristic 2,

$$\sum_{t=t'_{j-1}}^{t'_j-1} \Big(\sum_{q=t'_{j-1}-1}^{t-1} h_q \Big) d_t > s. \tag{2.5}$$

Since h_q and d_t are non-negative, (2.3), (2.4), and (2.5) are not consistent. Hence,

$$t'_{j-1} \leq t_i .$$

▨

Lemma 2.2: Consider period t_k for some $k > 1$. If for some j,

$$t'_j \leq t_k < t'_{j+1}, \quad \text{then } t_{k-1} < t'_j .$$

Proof: Parallels the proof of Lemma 2.1. ▨

Proposition 2.4 (Interleaving Property): Heuristics 1 and 2 generate an equal number of production periods, i.e., $n_1 = n_2$. Moreover $t'_2 \leq t_2 \leq t'_3 \leq t_3 \leq \ldots \leq t'_{n_2} \leq t_{n_1}$.

Proof: We have that

$$\sum_{t=t_{n_1}+1}^{T} \Big(\sum_{q=t_{n_1}}^{t-1} h_q \Big) d_t \leq s \quad \text{by Heuristic 1} \tag{2.6}$$

$$\sum_{t=t'_{n_2}}^{T} \Big(\sum_{q=t'_{n_2}-1}^{t-1} h_q \Big) d_t > s \quad \text{by Heuristic 2} \tag{2.7}$$

and

$$\sum_{t=t'_{n_2}+1}^{T} \Big(\sum_{q=t'_{n_2}}^{t-1} h_q \Big) d_t \leq s \quad \text{by Heuristic 2.} \tag{2.8}$$

(2.6), (2.7), and (2.8) imply that $t'_{n_2} \leq t_{n_1}$. (2.9)

By Heuristic 1,

$$\sum_{t=t_{n_1-1}+1}^{t_{n_1}} \left(\sum_{q=t_{n_1-1}}^{t-1} h_q \right) d_t > s. \tag{2.10}$$

(2.1) and (2.8) imply $t_{n_1-1} < t'_{n_2}$. (2.11)

(2.1), (2.2), (2.9), (2.11), and successive applications of Lemmas 2.1 and 2.2 will prove the desired result. ▨

Let Z_{Hi} be the objective value of the feasible solution to the lot size problem when Heuristic i, i=1,2, is applied, and let Z_0 be the corresponding optimal value.

Proposition 2.5: $\frac{Z_{Hi}}{Z_0} \leq 2.$

Proof: We prove the result for Heuristic 1. Similar arguments lead to the same result for Heuristic 2.

Assume that (2.1) holds. Then,

$$Z_{H1} = n_1 s + H$$

where H corresponds to the holding cost and s is the set-up cost.

Since a set-up cost is incurred whenever the holding cost exceeds a set-up cost, it follows that $H \leq n_1 s$. Hence,

$$Z_{H1} \leq 2n_1 s.$$

A lower bound on Z_0 can be obtained as follows. An optimal solution will necessarily have holding cost plus set-up cost at least equal to s in the interval $(t_i, t_{i+1}]$. There are (n_1-1) of such intervals implying a cost of at least $(n_1-1)s$. Since a set-up is incurred in period $t_1=1$ as well, a lower bound on the optimal value is $n_1 s$. Therefore,

$$\frac{Z_{H1}}{Z_0} \leq \frac{2n_1 s}{n_1 s} = 2.$$ ▨

The worst case bound derived in Proposition 2.5 is tight as is shown in the following example. Consider the uncapacitated lot size problem with parameters:

$$d_{2i+1} = \varepsilon_2 \qquad i=1,\dots,m$$

$$d_{2i} = 1-\varepsilon_1 \qquad i=1,\dots,m$$

$$d_1 = 1$$

$$s = 1, \quad h = 1$$

$$T = 2m + 1$$

$$1 > \varepsilon_2 > \varepsilon_1 > 0.$$

By applying Heuristic 1, production will occur in the odd periods. So,

$$Z_{H1} = (m+1) + m(1-\varepsilon_1).$$

An optimal solution will have set-ups at even periods and at t=1. Hence,

$$Z_0 = m + 1 + m\varepsilon_2.$$

Therefore,

$$\frac{Z_{H1}}{Z_0} = \frac{2m + 1 - m\varepsilon_1}{m + 1 + m\varepsilon_2}.$$

If $m \to \infty$, $m\varepsilon_1 \to 0$, and $m\varepsilon_2 \to 0$,

$$\frac{Z_{H1}}{Z_0} \to 2.$$

A similar example shows that the worst case bound of 2 is tight for Heuristic 2 as well.

Based on worst case performance, the suggested algorithms are more attractive than the Silver-Meal heuristic and the other two heuristics discussed earlier.

If demands are bounded from below, we can improve the error bound in

Proposition 2.5. We assume, without loss of generality, that $h_t \geq 1$, $t=1,2,\ldots,T$.

Proposition 2.6: Let s be the set-up cost and suppose that $d_t \geq ps$, $t=1,2,\ldots,T$, for some constant p. Let r and u denote two consecutive production points determined by Heuristic 1 or 2. Then, if $p \leq 1$, the cost of the heuristic solution in the interval (r,u] is at most (2-p) times the cost of any solution in this interval; if $p > 1$, the solution is optimal.

Proof: If $p > 1$, the holding cost always exceeds a set-up cost. Therefore, producing in every period is the optimal strategy.

For $p \leq 1$, let H denote the holding cost of the solution given by Heuristic i in this interval. Then, the heuristic's cost c_H in the interval is s + H. By the rules of the heuristic, $H \leq s$ and $H + (h_r + h_{r+1} + \ldots + h_{u-1})d_u$ is greater than s. Consider any other solution and let c be the corresponding cost in the interval. If this solution does not produce in the interval, it incurs a holding cost of at least max (s, H + ps) (since $d_u \geq ps$).

Consequently, if it does not produce,

$$\frac{c_H}{c} \leq \max \{\frac{H+s}{c} : H \leq s,\ c \geq \max(s, H+ps)\} =$$

$$= \frac{H+s}{\max(ps+H,s)} = \frac{H+ps+(1-p)s}{\max(ps+H,s)} \leq \frac{H+ps}{H+ps} + \frac{(1-p)s}{s} = 2-p.$$

If this solution produces once in the interval (r,u], either

(i) $u = r+1$ in which case $c = c_H = s$ or

(ii) $u > r+1$ in which case the new solution must incur a holding cost of at least ps for some period in the interval (r,u].

Therefore,

$$\frac{c_H}{c} \leq \frac{s+H}{s+ps} \leq \frac{2s}{s+ps} = \frac{2}{1+p} \leq 2-p \quad \text{for} \quad 0 \leq p \leq 1.$$

If the solution produces more than once, then $\frac{c_H}{c} \leq \frac{s+H}{2s} \leq 1$. ▨

Corollary 2.1: Let s be the set-up cost and suppose that $d_t \geq ps$, t=1,...,T for some constant $p \leq 1$. Then, $\sup \frac{Z_{Hi}}{Z_0} \leq 2 - p$.

Proof: We prove the result for Heuristic 1 only. Similar arguments can be used for Heuristic 2. Consider (2.1). The corresponding cost is $s + \sum_{i=1}^{n_1-1} a_i + a_{n_1}$ where a_i is the cost of the interval $(t_i, t_{i+1}]$ and a_{n_1} is the holding cost beyond period t_{n_1}. The cost of the optimal solution is $s + \sum_{i=1}^{n_1-1} b_i + b_{n_1}$ where b_i is the cost in the interval $(t_i, t_{i+1}]$ and b_{n_1} is the cost beyond period t_{n_1}. By the nature of the heuristic $a_{n_1} < s$ and hence $b_{n_1} \geq a_{n_1}$. Since $\frac{a_i}{b_i} \leq 2 - p$, $i=1,2,\ldots,n_1-1$ by Proposition 2.6, $\frac{s + \sum_{i=1}^{n_1} a_i}{s + \sum_{i=1}^{n_1} b_i} \leq 2\text{-}p$.

Therefore, the supremum does not exceed 2-p. ▨

Some of the worst case error bounds that we have presented assume no limitations on the length of the planning horizons. It remains to be studied how such error bounds change if the planning horizon is fixed.

3. Reducing the Planning Horizon

In this section we compute the cost of reducing the planning horizon in an uncapacitated lot size problem. Usually, demand forecasts deteriorate towards the end of the planning horizon. By considering a reduced number of time periods, we operate with a more accurate forecast and handle smaller data bases.

Consider the single product, multi-facility model with concave costs:

$$\text{(F)} \quad \text{Min} \sum_{t=1}^{T} \sum_{j=1}^{M} [C_{jt}(X_{jt}) + H_{jt}(I_{jt})]$$

$$\text{s.t.} \quad I_{jt} = I_{j,t-1} + X_{jt} - X_{j+1,t} \qquad j=1,\ldots,M-1;\ t=1,\ldots,T$$

$$I_{Mt} = I_{M,t-1} + X_{Mt} - D_t \qquad t=1,\ldots,T$$

$$X_{jt} \geq 0, \quad I_{jt} \geq 0 \qquad j=1,\ldots,M;\ t=1,\ldots,T$$

$$I_{j0} = I_{jT} = 0 \qquad j=1,\ldots,M$$

where the index j indicates the facility and the index t, the time period.

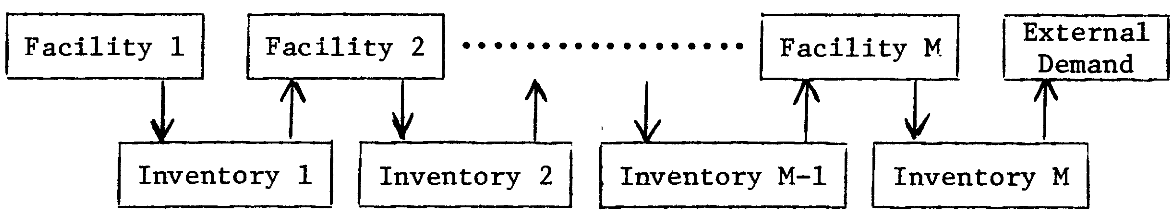

Figure 1: Multi-Facility Model in Series

We assume that the costs are of the form

$$C_{jt}(X_{jt}) = v_j X_{jt} + s_j \delta(X_{jt})$$

$$H_{jt}(I_{jt}) = h_j I_{jt}$$

where

$$\delta(X_{jt}) = \begin{cases} 1 & \text{if } X_{jt} > 0 \\ 0 & \text{if } X_{jt} = 0 \end{cases}$$

Partitioning Problem (F) in two subproblems, the first with T_1 periods and the second with the remaining $T_2 = T-T_1$ periods, we obtain:

$$\text{(F1)} \quad \text{Min} \sum_{t=1}^{T_1} \sum_{j=1}^{M} [v_j X_{jt} + s_j \delta(X_{jt}) + h_j I_{jt}]$$

$$\text{s.t.} \quad I_{jt} = I_{j,t-1} + X_{jt} - X_{j+1,t} \qquad t=1,\ldots,M-1;\ t=1,\ldots,T_1$$

$$I_{Mt} = I_{M,t-1} + X_{Mt} - D_t \qquad t=1,\ldots,T_1$$

$$X_{jt},\ I_{jt} \geq 0 \qquad j=1,\ldots,M;\ t=1,\ldots,T_1$$

$$I_{j0} = 0 \qquad j=1,\ldots,M$$

and

$$\text{(F2)} \quad \text{Min} \sum_{t=T_1+1}^{T} \sum_{j=1}^{M} [v_j X_{jt} + s_j \delta(X_{jt}) + h_j I_{jt}]$$

$$\text{s.t.} \quad I_{jt} = I_{j,t-1} + X_{jt} - X_{j+1,t} \qquad j=1,\ldots,M-1;\ t=T_1+1,\ldots,T$$

$$I_{Mt} = I_{M,t-1} + X_{Mt} - D_t \qquad t=T_1+1,\ldots,T$$

$$X_{jt},\ I_{jt} \geq 0 \qquad j=1,\ldots,M;\ t=T_1+1,\ldots,T$$

$$I_{jT_1} = I_{jT} = 0 \qquad j=1,\ldots,M$$

Let $v(Y)$ denote the optimal value of Problem Y. Then,

Proposition 3.1: $0 \leq v(F1) + v(F2) - v(F) \leq \sum_{j=1}^{M} s_j$.

Proof: Assume that in the optimal solution of (F), the set-ups, at Facility j, occurred at periods

$$1 \leq t_{j1} \leq t_{j2} \leq \cdots \leq t_{jk_j} \leq t_{jk_j+1} \leq \cdots \leq t_{jn_j} \leq T.$$

Hence,

$$v(F) = \sum_{j=1}^{M} n_j s_j + \sum_j H_j + C$$

where H_j corresponds to the holding cost associated with the inventories at facility j, and

$$C \text{ is equal to } \sum_{t=1}^{T} D_t \sum_{j=1}^{M} v_j .$$

Let $t^* = T_1+1$. Then, for some k_j

$$1 \leq t_{j1} < t_{j2} < \ldots < t_{jk_j} < t^* \leq t_{jk_j+1} < \ldots < t_{jn_j} \leq T.$$

Consider the following feasible solution to (F):

a) the set-ups, at Facility j, are incurred at periods

$$t_{j1}, t_{j2}, \ldots, t_{jk_j}, t^*, t_{jk_j+1}, \ldots, t_{jn_j}; \text{ and}$$

b) the production at periods

$$t_{j1}, t_{j2}, \ldots, t_{jk_j-1}, t_{jk_j+1}, \ldots, t_{jn_j}$$

are the same as in the optimal solution to (F);

c) at period t_{jk_j} we produce the necessary amount so that $I_{T_1} = 0$, and at time period t^* we produce the difference between the amount we were supposed to produce at period t_{jk_j} in the optimal solution to (F) and this new quantity.

The objective value, denoted by f_F^v, for this feasible solution satisfies

$$f_F^v \leq \sum_{j=1}^{M} (n_j+1)s_j + \sum_j H_j' + C$$

where H_j' is the new holding cost at facility j.

The inequality is due to the fact that we might not have to produce at t^* (the difference is zero). Note that $H_j' \leq H_j$ since we are carrying at most the same inventory as in the optimal solution to (F).

If we partition this feasible solution, the two components are feasible in (F1) and (F2), respectively. Denoting by f_{F1}^v and f_{F2}^v the corresponding objective values in (F1) and (F2), and noting that $v(F1) \leq f_{F1}^v$ and $v(F2) \leq f_{F2}^v$, we have

$$v(F1) + v(F2) \leq f_F^v = f_{F1}^v + f_{F2}^v$$

$$\leq \sum_{j=1}^{M} (n_j+1)s_j + \sum_{j=1}^{M} H_j' + C \leq \sum_{j=1}^{M} n_j s_j + \sum_{j=1}^{M} H_j + C + \sum_{j=1}^{M} s_j$$

$$= v(F) + \sum_{j=1}^{M} s_j . \qquad (3.1)$$

Since the combination of any feasible solution to (F1) and (F2) forms a feasible solution to (F)

$$v(F) \leq v(F1) + v(F2) . \qquad (3.2)$$

From (3.1) and (3.2)

$$0 \leq v(F1) + v(F2) - v(F) \leq \sum_{j=1}^{M} s_j .$$

Assume that the demands are not known with reasonable accuracy, for example, for more than T/2 periods into the future. Proposition 3.1 states that by partitioning the problem into two problems of size T/2 we incur, in the worst case, an extra cost equal to $\sum_{j=1}^{M} s_j$. This extra cost can be viewed as the maximum value of the information on the demands for the last T/2 periods.

A special instance of Proposition 3.1 is the single product, single facility problem or more generally, the multiproduct uncapacitated lot size problem. The worst case is now a set-up for the single product case and the sum of the set-ups of each product in the multiproduct case.

Note that Proposition 3.1 could be derived for the case of non-constant set-up costs, non-increasing production costs, and no restrictions on the holding costs. In this case, we would have:

$$0 \leq v(F1) + v(F2) - v(F) \leq \sum_{j=1}^{M} s_{jT_1+1} .$$

4. Aggregation

In this section we suggest an aggregation procedure for solving a multi-product uncapacitated lot size problem. We define a basic aggregate product which production strategy dictates the production plans for all products in the aggregation. Worst case error bounds are derived. The aggregation suggested avoids, in part, detailed demand forecasts and limits the amount of data manipulation of inventory systems.

Consider the single product uncapacitated lot size problem.

$$\text{(P1)}\quad \text{Min} \sum_{t=1}^{T} [s_t\delta(X_t) + h_tI_t + v_tX_t]$$

$$\text{s.t.}\quad X_t - I_t + I_{t-1} = d_t \qquad t=1,\ldots,T$$

$$X_t,\ I_t \geq 0 \qquad t=1,\ldots,T$$

$$\delta(X_t) = \begin{cases} 1 & \text{if } X_t > 0 \\ 0 & \text{otherwise} \end{cases} \qquad t=1,\ldots,T$$

An optimal solution for this problem follows a "Wagner-Whitin" strategy, i.e.,

$$X_tI_{t-1} = 0 \qquad t=1,2,\ldots,T \text{ (assuming } I_0=0).$$

Using this result we can formulate an equivalent facility location problem. In this formulation the binary variable y_t denotes whether or not we produce in period t and θ_{tj} denotes the fraction of the demand in period j that is produced in period t.

$$\text{(Q)}\quad \text{Min} \sum_{t=1}^{T}\sum_{j=t}^{T} c_{tj}\theta_{tj} + \sum_{t=1}^{T} s_ty_t$$

$$\text{s.t.}\quad \sum_{t=1}^{j} \theta_{tj} = 1 \qquad j=1,\ldots,T$$

$$0 \leq \theta_{tj} \leq y_t \leq 1 \qquad t=1,\ldots,T;\ j \geq t$$

$$y_t \text{ integer} \qquad t=1,\ldots,T$$

where $c_{tj} = (h_t + h_{t+1} + \dots + h_{j-1} + v_t)d_j \qquad t=1,\dots,T;\ j \geq t.$

Note that since $\sum_{t=1}^{j} \theta_{tj} = 1$ and $0 \leq \theta_{tj} \leq y_t \leq 1$, it follows that at least one of the y_t's will be non-zero. If the y_t's are integers, the remaining constraints form a unimodular matrix implying that the θ_{tj}'s will be integers in the optimal solution.

We study this formulation to compute bounds for the errors due to the product aggregation.

Proposition 4.1: If in Problem (Q) the cost coefficients are replaced by $c'_{tj} = kc_{tj} + \alpha_j$ and $s'_t = ks_t$ for some $k > 0$ and any α_j, the optimal strategy will not change (i.e., the location of the facilities in the optimal solution are the same for both cost structures).

Proof: Let U denote the feasible set of Problem (Q). Then,

$$\operatorname*{Min}_{(\theta,y)\varepsilon U} \left(\sum_{t=1}^{T} \sum_{j=t}^{T} c'_{tj}\theta_{tj} + \sum_{t=1}^{T} s'_t y_t \right) =$$

$$= \operatorname*{Min}_{(\theta,y)\varepsilon U} \left\{ \sum_{t=1}^{T} \sum_{j\geq t} [(kc_{tj} + \alpha_j)\theta_{tj}] + \sum_{t=1}^{T} (ks_t y_t) \right\} =$$

$$= \operatorname*{Min}_{(\theta,y)\varepsilon U} \left\{ k\left[\sum_{t=1}^{T} \sum_{j\geq t}^{T} c_{tj}\theta_{tj} + \sum_{t=1}^{T} s_t y_t \right] + \sum_{t=1}^{T} \sum_{j\geq t} \alpha_j \theta_{tj} \right\} =$$

$$= \operatorname*{Min}_{(\theta,y)\varepsilon U} \left\{ k\left[\sum_{t=1}^{T} \sum_{j\geq t} c_{tj}\theta_{tj} + \sum_{t=1}^{T} s_t y_t \right] + \sum_{j=1}^{T} \sum_{t=1}^{j} \alpha_j \theta_{tj} \right\} =$$

$$= \operatorname*{Min}_{(\theta,y)\varepsilon U} \ k\left[\sum_{t=1}^{T} \sum_{j=t}^{T} c_{tj}\theta_{tj} + \sum_{t=1}^{T} s_t y_t \right] + \sum_{j=1}^{T} \alpha_j \sum_{t=1}^{j} \theta_{tj} \} .$$

But,

$$\sum_{t=1}^{j} \theta_{tj} = 1 \qquad \text{for any } (\theta,y)\varepsilon U$$

Hence,

$$\underset{(\theta,y)\varepsilon U}{\text{Min}} \left(\sum_{t=1}^{T} \sum_{j=t}^{T} c'_{tj}\theta_{tj} + \sum_{t=1}^{T} s'_t y_t \right) =$$

$$= \sum_{j=1}^{T} \alpha_j + k \underset{(\theta,y)\varepsilon U}{\text{Min}} \left(\sum_{t=1}^{T} \sum_{j=t}^{T} c_{tj}\theta_{tj} + \sum_{t=1}^{T} s_t y_t \right)$$

Proposition 4.1 suggests a way to group products. If product 1 has cost coefficients c_{tj} and s_t and product 2 has cost coefficients c'_{tj} and s'_t such that $c'_{tj} = kc_{tj} + \alpha_j$, $s'_t = ks_t$ for some $k > 0$ and $\alpha_j \varepsilon R$, we can solve just one of the problems because both have the same optimal strategy.

From now on we assume that for each product the set-up costs, holding costs, and production costs are constant over time. In this case, since the total demand is fixed, the production costs are fixed as well and can be eliminated.

As an example of this result, suppose that we have two products, 1 and 2 having, respectively, set-up costs s_1 and s_2, holding costs h_1 and h_2, demands d_{1t} and d_{2t} at time t. If the parameters of the two products satisfy the conditions:

$$c^2_{tj} = kc^1_{tj} + \alpha_j \text{ and } s^2_t = ks^1_t \text{ for some } k > 0 \text{ and } \alpha_j \varepsilon R \text{ then,}$$

$$s_2 = ks_1 \text{ or } k = \frac{s_2}{s_1}.$$

$$c^2_{tj} = (j-t)h_2 d_{2j}$$

$$c^1_{tj} = (j-t)h_1 d_{1j}$$

$$c^2_{tj} = kc^1_{tj} + \alpha_j = k(j-t)h_1 d_{1j} + \alpha_j$$

$$(j-t)h_2 d_{2j} = k(j-t)h_1 d_{1j} + \alpha_j \qquad (4.1)$$

Expression (4.1) holds for all $j \geq t$. In particular setting, j=t it implies that $\alpha_j = 0$. Therefore,

$$(j-t)h_2 d_{2j} = k(j-t)h_1 d_{1j}.$$

For $j > t$,

$$h_2 d_{2j} = k h_1 d_{1j} \quad \text{or}$$

$$\frac{d_{2j}}{d_{1j}} = \frac{k h_1}{h_2} = \text{constant} = \alpha^{-1} .$$

Formally,

Corollary 4.1: If two uncapacitated lot size problems have parameters satisfying

$$\frac{d_{1t}}{d_{2t}} = \alpha \qquad t=1,\ldots,T \qquad (4.2)$$

$$k = \frac{s_2}{s_1} \text{ and } \frac{h_2}{k h_1} = \alpha \qquad (4.3)$$

then, the optimal production strategies for both problems coincide.

Conditions (4.2) and (4.3) are equivalent to those used by Manne [5].

Unfortunately, relation (4.2) and/or (4.3) do not always hold in practice. Therefore, we assume that the parameters are related in the following general way:

$$\frac{d_{1t}}{d_{2t}} = \alpha + \phi_t \quad \text{for some } \phi_t \varepsilon R$$

where

$$\alpha \triangleq \frac{\sum_{t=1}^{T} d_{1t}}{\sum_{t=1}^{T} d_{2t}}$$

and

$$\left(\frac{s_1}{h_1}\right) / \left(\frac{s_2}{h_2}\right) = \alpha\eta \quad \text{for some } \eta \varepsilon R.$$

In the remainder of this section we establish worst case error bounds for the cases where we use the optimal strategy for product 1 as solution for product 2, when relation (4.2) and/or (4.3) are not satisfied exactly.

4.1 Condition (4.2) is not satisfied and (4.3) holds

Assume that

$$\frac{d_{1t}}{d_{2t}} = \alpha + \phi_t \text{ or } d_{1t} = \alpha d_{2t} + \phi_t d_{2t} \qquad t=1,2,\dots,T,$$

$$\phi_t \varepsilon R, \ \alpha \varepsilon R^+ \ \text{and} \ \eta = 1.$$

If we solve Problem (Q) for product 1, we have:

$$\min_{(\theta,y)\varepsilon U} \{ \sum_{t=1}^{T} \sum_{j\geq t} (j-t)h_1 d_{1j}\theta_{tj} + \sum_{t=1}^{T} s_1 y_t\} =$$

$$= \min_{(\theta,y)\varepsilon U} \{ \sum_{t=1}^{T} \sum_{j\geq t} [(j-t)h_1(d_{2j}\alpha + \phi_j d_{2j})\theta_{tj}] + \sum_{t=1}^{T} s_1 y_t\} =$$

$$= \min_{(\theta,y)\varepsilon U} \{ \sum_{t=1}^{T} \sum_{j\geq t} (j-t)h_1\alpha d_{2j}\theta_{tj} + \sum_{t=1}^{T} s_1 y_t + \sum_{t=1}^{T} \sum_{j\geq t} (j-t)h_1\phi_j d_{2j}\theta_{tj}\} .$$

From (4.3), $\alpha h_1 = \frac{h_2}{k} = \frac{h_2 s_1}{s_2}$.

Substituting, we obtain

$$\min_{(\theta,y)\varepsilon U} \{ \sum_{t=1}^{T} \sum_{j\geq t} (j-t)h_2 \frac{s_1}{s_2} d_{2j}\theta_{tj} + \sum_{t=1}^{T} \frac{s_1}{s2} s_2 y_t + \sum_{t=1}^{T} \sum_{j\geq t} (j-t)h_1\phi_j d_{2j}\theta_{tj}\} =$$

$$= \min_{(\theta,y)\varepsilon U} \{\frac{s_1}{s_2} [\sum_{t=1}^{T} \sum_{j\geq t} (j-t)h_2 d_{2j}\theta_{tj} + \sum_{t=1}^{T} s_2 y_t] + \sum_{t=1}^{T} \sum_{j\geq t} (j-t)h_1\phi_j d_{2j}\theta_{tj}\}.$$

Let θ^1_{tj}, y^1_t correspond to the optimal solution for product 1, and θ^2_{tj}, y^2_t correspond to the optimal solution for product 2. The optimal values will be denoted by f^1 and f^2 respectively. Denote by f^2_v the objective value for product 2 when the feasible solution θ^1_{tj}, y^1_t is used. Then,

$$f^1 = \underset{(\theta,y)\varepsilon U}{\text{Min}} \left\{ \sum_{t=1}^{T} \sum_{j=t}^{T} c^1_{tj}\theta_{tj} + \sum_{t=1}^{T} s_1 y_t \right\} =$$

$$= \frac{s_1}{s_2} \left\{ \sum_{t=1}^{T} \sum_{j=t}^{T} (j-t)h_2 d_{2j}\theta^1_{tj} + \sum_{t=1}^{T} s_2 y^1_t \right\} + \sum_{t=1}^{T} \sum_{j=t}^{T} (j-t)h_1\phi_j d_{2j}\theta^1_{tj} =$$

$$= \frac{s_1}{s_2} f^2_v + \sum_{t=1}^{T} \sum_{j=t}^{T} (j-t)h_1\phi_j d_{2j}\theta^1_{tj} \geq \frac{s_1}{s_2} f^2 + \sum_{t=1}^{T} \sum_{j=t}^{T} (j-t)h_1\phi_j d_{2j}\theta^1_{tj} .$$

$$f^1 = \underset{(\theta,y)\varepsilon U}{\text{Min}} \left\{ \sum_{t=1}^{T} \sum_{j\geq t} c^1_{tj}\theta_{tj} + \sum_{t=1}^{T} s_1 y_t \right\} \leq \sum_{t=1}^{T} \sum_{j\geq t} c^1_{tj}\theta^2_{tj} + \sum_{t=1}^{T} s_1 y^2_t =$$

$$= \frac{s_1}{s_2} f^2 + \sum_{t=1}^{T} \sum_{j=t}^{T} (j-t)h_1\phi_j d_{2j}\theta^2_{tj}.$$

So,

$$\frac{s_1}{s_2} f^2 + \sum_{t=1}^{T} \sum_{j=t}^{T} (j-t)h_1\phi_j d_{2j}\theta^1_{tj} \leq$$

$$\leq \frac{s_1}{s_2} f^2_v + \sum_{t=1}^{T} \sum_{j=t}^{T} (j-t)h_1\phi_j d_{2j}\theta^1_{tj} \leq$$

$$\leq \frac{s_1}{s_2} f^2 + \sum_{t=1}^{T} \sum_{j=t}^{T} (j-t)h_1\phi_j d_{2j}\theta^2_{tj} .$$

Assuming $\frac{s_1}{s_2} > 0$ and since $\frac{s_2 h_1}{s_1} = \frac{h_2}{\alpha}$, we can write

$$f^2 + \sum_{t=1}^{T} \sum_{j=t}^{T} (j-t) \frac{h_2}{\alpha} \phi_j d_{2j}\theta^1_{tj} \leq$$

$$\leq f^2_v + \sum_{t=1}^{T} \sum_{j=t}^{T} (j-t) \frac{h_2}{\alpha} \phi_j d_{2j}\theta^1_{tj} \leq$$

$$\leq f^2 + \sum_{t=1}^{T} \sum_{j=t}^{T} (j-t) \frac{h_2}{\alpha} \phi_j d_{2j}\theta^2_{tj}.$$

or

$$0 \leq f^2_v - f^2 \leq \sum_{t=1}^{T} \sum_{j=t}^{T} (j-t) \frac{h_2}{\alpha} \phi_j d_{2j}(\theta^2_{tj} - \theta^1_{tj}). \qquad (4.4)$$

We next compute an upper bound on the right hand side of this inequality. Since $0 \leq \theta^1_{tj} \leq 1$, $0 \leq \theta^2_{tj} \leq 1$, $\phi_j(\theta^2_{tj} - \theta^1_{tj}) \leq |\phi_j|$, and thus

$$0 \leq f^2_v - f^2 \leq$$

$$\leq \sum_{t=1}^{T} \sum_{j=t}^{T} (j-t) \frac{h_2}{\alpha} \phi_j d_{2j} (\theta^2_{tj} - \theta^1_{tj}) \leq$$

$$\leq \sum_{j=1}^{T} (j-1) \frac{h_2}{\alpha} |\phi_j| \, d_{2j} \, . \tag{4.5}$$

Let $\phi = \max_{j=1,\ldots,T} |\phi_j|$, it follows that

$$0 \leq f^2_v - f^2 \leq \frac{\phi}{\alpha} \sum_{j=1}^{T} (j-1) h_2 d_{2j} \tag{4.6}$$

Defining $f^2_{max} \triangleq Ts_2 + \sum_{j=1}^{T} (j-1) h_2 d_{2j}$,

$$\frac{f^2_v - f^2}{f^2_{max} - f^2} \leq \frac{\phi}{\alpha} \tag{4.7}$$

Relation (4.7) suggests that we should choose as the basic product the one that gives the smallest value for $\frac{\phi}{\alpha}$.

Another bound can be obtained by defining f^2_{max} as

$$f^2_{max} = \max\{Ts_2, \; s_2 + \sum_{j=1}^{T} (j-1) h_2 d_{2j}\}.$$

In this case,

$$\frac{f^2_v - f^2}{f^2_{max} - f^2} \leq \frac{\frac{\phi}{\alpha} \sum_{j=1}^{T} h_2 (j-1) d_{2j}}{\left| Ts_2 - s_2 - \sum_{j=1}^{T} (j-1) h_2 d_{2j} \right|} = \frac{\frac{\phi}{\alpha} \sum_{j=1}^{T} h_2 (j-1) d_{2j}}{\left| (T-1) s_2 - \sum_{j=1}^{T} (j-1) h_2 d_{2j} \right|}$$

Letting

$$\varepsilon_r = \frac{(T-1) s_2}{\sum_{j=1}^{T} (j-1) h_2 d_{2j}}$$

we have

$$\frac{f_v^2 - f^2}{f_{max}^2 - f^2} \leq \begin{cases} \dfrac{\phi}{\alpha(1-\varepsilon_r)} & \text{if } \varepsilon_r < 1 \\ \dfrac{\phi}{\alpha(\varepsilon_r - 1)} & \text{if } \varepsilon_r > 1 \end{cases}$$

The bound given by (4.5) is attainable, as can be seen by the following example:

Assume T = 4

$$s_1 = 10,\ h_1 = 1,\ d_{11} = 10,\ d_{12} = 0,\ d_{13} = 5,\ d_{14} = 0$$

$$s_2 = 10,\ h_2 = 1,\ d_{21} = 9.5,\ d_{22} = 0,\ d_{23} = 5.5,\ d_{24} = 0$$

Then, $\alpha = 1$ and

$$\phi_1 d_{21} = -0.5$$

$$\phi_2 d_{22} = 0$$

$$\phi_3 d_{23} = 0.5$$

$$\phi_4 d_{24} = 0$$

An optimal strategy for product 1 is to set-up at t = 1 only. This will give $f^1 = 20$.

Using this strategy for product 2, we have

$$f_v^2 = 10 + 11 = 21$$

The optimal strategy for product 2 is to set-up at t = 1 and t = 3. The corresponding optimal value is

$$f^2 = 20.$$

So, $f_v^2 - f^2 = 1 = \sum_{j=1}^{T} (j-1) \frac{h_2}{\alpha} |\phi_j| d_{2j} = 1$

which in this case is also equal to

$$f_v^2 - f^2 = \frac{\phi}{\alpha} \sum_{j=1}^{T} (j-1) h_2 d_{2j} = 1.$$

4.2 Condition (4.3) is not satisfied and (4.2) holds

We next derive bounds for the instances where $\phi_t = 0$, $t=1,\ldots,T$, and $\eta \neq 1$. Then,

$$\eta = \left(\frac{s_1}{h_1 \alpha}\right) / \left(\frac{s_2}{h_2}\right)$$

$$f^1 = \min_{(\theta,y)\varepsilon U} \left\{ \sum_{t=1}^{T} \sum_{j=t}^{T} c_{tj}^1 \theta_{tj} + \sum_{t=1}^{T} s_1 y_t \right\} .$$

Substituting as in subsection 4.1, we obtain

$$f^1 = \frac{\alpha h_1}{h_2} \min_{(\theta,y)\varepsilon U} \{\eta[\sum_{t=1}^{T} \sum_{j=t}^{T} h_2 (j-t) d_{2j} \theta_{tj} + \sum_{t=1}^{T} s_2 y_t] + + (1-\eta) \sum_{t=1}^{T} \sum_{j=t}^{T} h_2 (j-t) d_{2j} \theta_{tj}\} \tag{4.8}$$

or

$$f^1 = \frac{\alpha h_1}{h_2} \min_{(\theta,y)\varepsilon U} \{\eta[\sum_{t=1}^{T} \sum_{j=t}^{T} h_2 (j-t) d_{2j} \theta_{tj} + \sum_{t=1}^{T} s_2 y_t + (\eta-1) \sum_{t=1}^{T} s_2 y_t\} \tag{4.9}$$

Similarly,

$$0 \leq f_v^2 - f^2 \leq \left|\frac{\eta-1}{\eta}\right| \sum_{j=1}^{T} h_2 (j-1) d_{2j} \tag{4.10}$$

or

$$0 \leq f_v^2 - f^2 \leq |\eta-1| \, (T-1) s_2 \tag{4.11}$$

from (4.8) and (4.9), respectively.

To derive the relative error bound we define

$$f^2_{max} \triangleq Ts_2 + \sum_{j=1}^{T} h_2(j-1)d_{2j}.$$

The result is:

$$\frac{f_v^2 - f^2}{f^2_{max} - f^2} \leq \begin{cases} 1 - \frac{1}{\eta} & \text{if } \eta > 1 \\ 1 - \eta & \text{if } \eta < 1 \end{cases}$$

4.3 Both conditions (4.2) and (4.3) do not hold

We now compute bounds for the general case, where $\phi_t \neq 0$ for some t and $\eta \neq 1$. Following similar procedures as in the previous subsections:

$$f^1 = \underset{(\theta,y)\varepsilon U}{\text{Min}} \{\sum_{t=1}^{T} \sum_{j=t}^{T} c_{tj}\theta_{tj} + \sum_{t=1}^{T} s_1 y_t\} =$$

$$= \underset{(\theta,y)\varepsilon U}{\text{Min}} \{(\frac{\alpha h_1}{h_2})\eta[\sum_{t=1}^{T} \sum_{j=t}^{T} h_2(j-t)d_{2j}\theta_{tj} + \sum_{t=1}^{T} s_2 y_t] +$$

$$+ \sum_{t=1}^{T} \sum_{j=t}^{T} (j-t)h_1\phi_j d_{2j}\theta_{tj} + (1-\eta) \sum_{t=1}^{T} \sum_{j=t}^{T} (j-t)h_1 \alpha d_{2j}\theta_{tj}\}$$

and,

$$0 \leq f_v^2 - f^2 \leq \sum_{j=1}^{T} (j-1)h_2 \left|\frac{\phi_j + (1-\eta)\alpha}{\alpha\eta}\right| d_{2j} \leq$$

$$\leq (\frac{\phi}{\alpha\eta} + |\frac{1-\eta}{\eta}|) \sum_{j=1}^{T} (j-1)h_2 d_{2j} \qquad (4.12)$$

The bounds given by (4.10) and (4.12) are also attainable. The following example attains bound (4.12). (To obtain the bound in (4.10), just take the demands for product 2 equal to those of product 1). Let

$s_1 = 10$, $h_1 = 5$, $d_{11} = 4$, $d_{12} = 2$

$s_2 = 6$, $h_2 = 5$, $d_{21} = 3.9$, $d_{22} = 2.1$

$T = 2, \alpha = 1, \eta = \frac{5}{3}$

$\phi_1 d_{21} = -.1$

$\phi_2 d_{22} = .1$

An optimal strategy for product 1 is to produce only at t = 1. Using this strategy for product 2, we would obtain

$$f_v^2 = 6 + 10.5 = 16.5.$$

The optimal solution for product 2 is to produce at t = 1 and t = 2. The optimal value is

$$f^2 = 12.$$

$$f_v^2 - f^2 = 4.5 \text{ and,}$$

$$f_v^2 - f^2 \leq \left(\frac{\phi}{\alpha\eta} + \frac{|1-\eta|}{\eta}\right) \sum_{j=1}^{T} (j-1) h_2 d_{2j} =$$

$$= \frac{5}{5/3}(.1) + \frac{2/3}{5/3} 5(2.1) = .3 + 4.2 = 4.5.$$

So, the bound is tight.

4.4 Product Aggregation

We have been considering product aggregation, not only to avoid excessive computations, but also to avoid excessive detailed demand forecasts. One might argue that it is often easier to assess the demand of a whole family of products than of each single product separately. Also, demands for longer intervals might be easier to assess than for shorter ones. We explore these ideas in what follows. Assume that there are N products and let the aggregate product have demands

$$D_t \triangleq \sum_{i=1}^{N} d_{it} \qquad t=1,2,\ldots,T.$$

We want to compute the parameters S_g and H_g (set-up cost and holding cost, respectively) of this aggregate product in order to minimize the worst case errors bound. Let

$$D \triangleq \sum_{t=1}^{T} D_t, \quad \alpha_i \triangleq \frac{D}{\sum_{t=1}^{T} d_{it}}, \quad \phi_{it} \triangleq \frac{D_t}{d_{it}} - \alpha_i, \quad \text{and}$$

$$\eta_i = (\frac{S_g}{H_g}) / (\frac{\alpha_i s_i}{h_i}) \tag{4.13}$$

As we showed previously,

$$0 \leq f_v^i - f^i \leq (\frac{\phi_i}{\alpha_i \eta_i} + \frac{|1-\eta_i|}{\eta_i}) \sum_{j=1}^{T} (j-1) h_i d_{ij} \tag{4.14}$$

where $\phi_i \triangleq \max_{t=1,\ldots,T} |\phi_{it}|$, and f_v^i is the objective function value for product i corresponding to the solution of the aggregate product. Substituting (4.13) in (4.14) we have

$$0 \leq f_v^i - f^i \leq \frac{s_i}{S_g} H_g(\phi_i + \alpha_i \; |1 - \frac{S_g h_i}{H_g s_i \alpha_i}|) \sum_{j=1}^{T} (j-1) d_{ij} \tag{4.15}$$

Let

$$\beta_i \triangleq \frac{s_i}{S_g} H_g(\phi_i + \alpha_i \; |1 - \frac{S_g h_i}{H_g s_i \alpha_i}|)$$

and

$$\beta \triangleq \max_{i=1,\ldots,N} \{\beta_i\}$$

From (4.15) it follows that

$$0 \leq \sum_{i=1}^{T} (f_v^i - f^i) \leq \sum_{i=1}^{N} \beta_i \sum_{j=1}^{T} (j-1) d_{ij} \leq \beta \sum_{i=1}^{N} \sum_{j=1}^{T} (j-1) d_{ij} =$$

$$= \beta \sum_{j=1}^{T} (j-1) D_j. \tag{4.16}$$

We wish to determine the values of S_g and H_g that minimize

$$\sum_{i=1}^{N} \beta_i \sum_{j=1}^{T} (j-1)d_{ij}.$$

In order to avoid detailed forecasts, we suggest that a problem of the form

$$\text{(R)} \qquad \beta^* = \min_{\substack{0<S_g \\ 0<H_g}} \; \{\max_{i=1,\ldots,N} \beta_i\}$$

be solved. This problem can be written as

$$\text{(R)} \quad \beta^* = \text{Min } \delta$$

$$\text{s.t.} \quad \delta \geq \frac{s_i}{S_g} H_g[\phi_i + \alpha_i(1- \frac{S_g h_i}{s_i H_g \alpha_i})] \qquad i=1,\ldots,N$$

$$\delta \geq \frac{s_i}{S_g} H_g[\phi_i + \alpha_i(\frac{S_g h_i}{s_i H_g \alpha_i} -1)] \qquad i=1,\ldots,N$$

$$H_g > 0, \quad S_g > 0.$$

As we can see, the optimal solution to this problem depends only on the ratio $\frac{H_g}{S_g}$. This implies that we have one degree of freedom to choose one of the two parameters at will.

Define $\quad y \triangleq \frac{H_g}{S_g}$.

Then,

$$\text{(R)} \quad \beta^* = \text{Min } \delta$$

$$\text{s.t.} \quad \delta \geq (s_i\phi_i + s_i\alpha_i)y - h_i \qquad i=1,\ldots,N$$

$$\delta \geq (s_i\phi_i - s_i\alpha_i)y + h_i \qquad i=1,\ldots,N$$

$$y > 0.$$

If we assume

$$f^i_{max} = Ts_i + \sum_{j=1}^{T} h_i(j-1)d_{ij},$$

it follows that

$$\frac{\sum_{i=1}^{N} (f^i_v - f^i)}{\sum_{i=1}^{N} (f^i_{max} - f^i)} \leq \frac{\beta \sum_{j=1}^{T} (j-1)D_j}{\sum_{i=1}^{N} \sum_{j=1}^{T} h_i(j-1)d_{ij}}$$

If

$$H \triangleq \frac{\sum_{i=1}^{N} \sum_{j=1}^{T} h_i(j-1)d_{ij}}{\sum_j (j-1)D_j},$$

then

$$\frac{\sum_{i=1}^{N} (f^i_v - f^i)}{\sum_{i=1}^{N} (f^i_{max} - f^i)} \leq \frac{\beta}{H} \quad . \tag{4.17}$$

Minimizing β will assure that this relative error is made as small as possible.

Let us take a closer look at problem (R). Assuming ideal cases, we should expect that $\beta^* = 0$. We show that this is the case.

Assume that $\phi_{it} = 0$, $i=1,\ldots,N$, $t=1,\ldots,T$; $\frac{D_t}{d_{it}} = \alpha_i$; and $\frac{s_i\alpha_i}{h_i}$ = constant independent of i. Then, problem (R) becomes

$$\begin{aligned} &\text{Min} \quad \delta \\ &\text{s.t.} \quad \delta \geq s_i\alpha_i y - h_i && i=1,\ldots,N \\ &\qquad \delta \geq -s_i\alpha_i y + h_i && i=1,\ldots,N \\ &\qquad y > 0 \end{aligned}$$

If $y = \frac{h_i}{s_i\alpha_i}$, then $\delta = 0$, implying $\beta^* = 0$.

Therefore, for ideal cases, expressions (4.16) and (4.17) for the error bounds perform well.

A potential drawback in this approach is solving problem (R). For the special case presented below, an explicit solution is given.

Let $\frac{s_j \alpha_j}{h_j}$ be constant for j=1,2,...,N. Assuming

$$\alpha_i \geq \phi_i, \qquad j=1,\ldots,N \qquad (4.18)$$

the optimal solution to (R) is

$$y = \frac{h_i}{s_i \alpha_i} \quad \text{and} \quad \beta^* = \max_{i=1,\ldots,N} \{\frac{h_i \phi_i}{\alpha_i}\} .$$

Note that the assumption (4.18) is not that restrictive since we expect α_i to be large.

All the bounds calculated here are "a priori" bounds. Better bounds can be obtained once a solution to the aggregate problem is determined ("a posteriori" bounds).

We have used aggregation of products in a way that differs from other approaches in the literature. Our aggregate model suggests a production strategy to be used as a solution to the initial production problem. In work on aggregation proposed in the literature [2], [3], [4], the aggregate model, once solved, is itself the approximate solution to the initial problem.

Our aggregate optimization model is useful as a tool for the diagnostic analysis of inventory systems. Firms want diagnostic studies to be done cheaply and without much effort. Detailed demand forecasts and large data manipulation are prohibitive. The aggregation suggested takes into consideration these factors.

There are some manipulations that one can do, prior to the aggregation, that will result in a simplified problem (R) for which an explicit solution can be obtained.

Recalling Corollary 4.1, we can construct equivalent problems where new demands are defined for product i, at time period t. The demands will be given by $h_i d_{it}$ and the new holding costs for each product will be equal to one.

Following the same steps as previously proposed, problem (R) becomes:

$$\text{Min } \delta$$

$$\text{s.t. } \delta \geq (s_i\alpha_i' + s_i\phi_i')y - 1 \qquad i=1,\ldots,N$$

$$\delta \geq (-s_i\alpha_i' + s_i\phi_i')\, y + i \qquad i=1,\ldots,N$$

$$y > 0$$

where $\alpha_i' = \frac{\alpha_i}{h_i}$ and $\phi_i' = \frac{\phi_i}{h_i}$.

The solution of the problem above is given by

$$y = \frac{2}{s_k(\alpha_k + \phi_k) + s_j(\alpha_j - \phi_j)}$$

where $s_k(\alpha_k + \phi_k) = \max\limits_{i=1,\ldots,N} \{s_i(\alpha_i + \phi_i)\}$

$$s_j(-\alpha_j + \phi_j) = \max_{i=1,\ldots,N} \{s_i(-\alpha_i + \phi_i)\}$$

and (4.18) is assumed. Therefore,

$$\frac{S_g}{H_g} = \frac{s_k(\alpha_k + \phi_k) + s_j(\alpha_j - \phi_j)}{2} \; .$$

Clearly, the original problem (R) and this new problem are not equivalent. The manipulations proposed led to a simplified problem (R) which would be otherwise obtained if we had used expression (4.14) in our prior developments and defined β_i to be

$$\left(\frac{\phi_i}{\alpha_i\eta_i} + \frac{|1-\eta_i|}{\eta_i}\right) \; .$$

5. Conclusions and Topics for Further Research

In practical settings, the use of simpler and intuitive procedures to solve lot size problems are preferred to the more complex Wagner-Whitin algoirthm. Often, such approximations rely mostly on common sense rules without strong theoretical support. In this paper, we have tried to fill such a gap by providing worst case relative errors for three heuristic procedures used in practice, and two related heuristics that can be seen as variants of part period balancing. Further research remains to be done for some cases with finite horizon and when the demand follows special patterns. Also, probabilistic error bounds would be of much interest for practical purposes.

The approximations suggested in section four provide additional options for managers to solve the uncapaciated problem. The aggregation suggested in this paper assumes constant set-up and holding costs. The general case remains a topic for future research.

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