CAPACITATED TREES, CAPACITATED ROUTING, AND ASSOCIATED POLYHEDRA

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Abstract

We study the polyhedral structure of two related core combinatorial problems: the subtree cardinality-constrained minimal spanning tree problem and the identical customer vehicle routing problem. For each of these problems, and for a forest relaxation of the minimal spanning tree problem, we introduce a number of new valid inequalities and specify conditions for ensuring when these inequalities are facets for the associated integer polyhedra. The inequalities are defined by one of several underlying support graphs: (i) a *multistar*, a "star" with a clique replacing the central vertex; (ii) a *clique cluster*, a collection of cliques intersecting at a single vertex, or more generally at a "central" clique; and (iii) a *ladybug*, consisting of a multistar as a head and a clique as a body. We also consider packing (generalized subtour elimination) constraints, as well as several variants of our basic inequalities, such as partial multistars, whose satellite vertices need not be connected to all of the central vertices. Our development highlights the relationship between the capacitated tree and capacitated forest polytopes and a so-called path-partitioning polytope, and shows how to use monotone polytopes and a set of simple exchange arguments to prove that valid inequalities are facets.

1 Introduction

The minimal spanning tree problem and the traveling salesman problem are two of the most noted and heavily studied of all combinatorial optimization models. The operations research and applied mathematics communities have devoted enormous energies to developing structural properties and efficient algorithms for solving these problems. In large part, the community's interest in these problems stems from two sources: (i) the problems' inherent attractiveness as mathematical objects, and (ii) the fact that they represent core models both for network design and for a wide variety of routing and scheduling applications.

Although the research community knows much about the structure of the integer programming formulations of these problems, it has established surprisingly few results about capacitated versions of these models. Indeed, when we first undertook the research reported in this study, essentially nothing had been reported in the open literature about the polyhedral structure of the capacitated models we are investigating.

Capacitated versions of the minimal spanning tree and the traveling salesman problems have numerous applications as diverse as vehicle routing, facility location, and telecommunication systems planning. For example, a basic problem in telecommunications is the design of a centralized processing network. These networks have one or more central processors that must be linked via a tree topology to several remote terminals which have specified demands that one of the central processors must serve. In practice, the amount of traffic that any link in the network can carry is limited. For the practical situation in which any outgoing link from the central processor can handle only a fixed number of customers, the same for each link, the problem becomes the type of capacitated minimal spanning tree problem that we treat in this paper.

In the past, the research community's ability to solve these capacitated problems by optimization methods (rather than heuristics) has been very limited, in part because the community has not developed a good integer programming representation of the problems. In this paper we address this issue; we undertake a study of the polyhedral structure of two related capacitated versions of these problems: (i) the *subtree cardinality-constrained minimal spanning tree problem* (which we also call the *K-capacitated minimal spanning tree problem*), and (ii) the *capacitated identical customer vehicle routing problem*. Both of these problems are deceptively simple to state, but are rather complex mathematically. For the subtree cardinality-constrained minimal spanning tree problem, for a given graph with edge costs, we wish to find a minimum cost tree with the property that no subtree off a designated root vertex contains more than K (a given nonnegative integer) vertices. (We refer to any tree satisfying this condition as a *K-capacitated tree*.) For the identical customer vehicle routing problem, we wish to find a minimum cost set of routes, all originating and terminating from a given depot, with the properties that no two routes intersect at any vertex other than the depot, and no route contains more than K customers. Note that if we eliminate the last link of each route, then any feasible set of routes becomes a feasible solution to the subtree cardinality-constrained minimal spanning tree problem. This well known observation, which served as the key idea for Held and Karp's [1970] noted approach to the traveling salesman problem, highlights the intimate connection between these two problems. Therefore, it is not too surprising to discover a close connection between the polyhedral structure of these two problems as well. Indeed, this paper merges and extends two studies that we had independently undertaken: two of us had studied the subtree cardinality-constrained minimal spanning tree problem, and one of us the identical customer vehicle routing problem.

From the perspective of computational complexity theory and from a practical viewpoint as well, both the subtree cardinality-constrained minimal spanning tree problem and the identical customer vehicle routing problem are difficult to solve. Papadimitriou [1978] has shown that the tree problem is strongly NP-hard for values of K between 3 and $n/2$ (n is the number of vertices in the underlying network); Bienstock [1987], using a reduction from three-dimensional matching, has proven that even the problem of recognizing whether an (incomplete) graph contains a 3-capacitated spanning tree is NP-complete. Since the identical customer vehicle routing problem contains the traveling salesman problem as a special case (when $K = n$ and the edges incident to the depot are sufficiently expensive), this problem is at least as difficult as its notorious TSP cousin. Certain special versions of the problems might be easily solved, however. For example, when $K = 2$, both problems reduce to an easily solved nonbipartite matching problem. In this instance it is possible to give a complete polyhedral description of the underlying integer program (see Hall and Magnanti [1990] for the tree problem).

From an algorithmic perspective, most of the research concerning both the subtree cardinality-constrained minimal spanning tree problem and the (general) vehicle routing problem has focused on heuristic methods. Since several comprehensive surveys describe the state of the art for the vehicle routing problem (for example, see Bodin et al. [1983], Laporte and Nobert [1987], and Magnanti [1981]), we will not review these developments here. We note that although researchers have developed several very clever algorithmic approaches for this problem class, the exact solution to large-scale problems (for example those with more than fifty nodes) has remained elusive. Indeed, only recently Fisher [1990] reported solving a 100-customer problem to proven optimality.

The subtree cardinality-constrained minimal spanning tree problem has received much less attention in the literature. Gavish and Altinkemer [1985] have studied approximation algorithms (i.e., heuristics with guarantees on their worst-case behavior) for the version of the problem whose costs satisfy the triangle inequality. Modifying a technique used to study the traveling salesman problem, they devise a $(3 - 2/K)$-approximation algorithm, that is, an algorithm guaranteed to deliver a tree of cost at most $(3 - 2/K)$ times the cost of the optimal tree. They also give a $(4 - 4/K)$-approximation algorithm for a general capacitated minimal spanning tree problem in which vertices have arbitrary weights and each subtree off the root vertex has a total vertex weight constraint. Of course by Bienstock's result, no polynomial time approximation algorithm will provide a constant error guarantee for problems with arbitrary costs (i.e., costs that do not satisfy the triangle inequality) unless $P = NP$ (see Garey and Johnson [1979]).

Most of the research on the capacitated spanning tree problem has focused upon developing heuristic procedures that produce feasible solutions; how far these solutions are from optimality is unclear. These heuristic approaches fall into three classes: (i) greedy (merging) algorithms [Esau and Williams 1966, Kershenbaum and Chou 1974, Gavish and Altinkemer 1986], (ii) *second-order* greedy algorithms [Karnaugh 1976, Kershenbaum, Boorstyn and Oppenheim 1980], and (iii) clustering algorithms, for problem defined in the Euclidean plane [McGregor and Shen 1977, Sharma 1983]. Gavish and Altinkemer [1986] present a survey of these three approaches.

Researchers have also attempted to solve the capacitated minimal spanning tree problem by branch and bound [Chandy and Lo 1973, Chandy and Russell 1972, Elias and Ferguson 1974, Kershenbaum and Boorstyn 1983] and by Benders' decomposition [Gavish 1982]. These methods have generally been disappointing: for branch and bound, the solution approach seems limited to problems with at most 20 vertices; Benders' decomposition has been largely unsuccessful, even for problems with from six to twelve vertices.

Gavish [1983,1984] has recently had more success using an augmented Lagrangian based algorithm which generates lower bounds on the value of the optimal capacitated minimum spanning tree. His computational approach is the only work we know that compares heuristic solution values with non-trivial lower bounds. Nevertheless, the gaps between the bounds are still fairly large (on the order of 20 per cent) and point to the need for a better polyhedral representation of the integer programming formulation of the problem.

Bousba and Wolsey [1989] report good computational experience for a related, more general model.

The work reported in this paper is motivated by two considerations. First, although recent research has permitted researchers to solve large scale uncapacitated network design problems and uncapacitated network routing problems, capacitated versions of these problems appear to be computationally elusive. For example, Balakrishnan, Magnanti and Wong [1989] have solved to near optimality uncapacitated network design problems with as many as 500 0-1 variables (design arcs) and 2500 commodities, and Padberg and Rinaldi [1989] have solved traveling salesman problems with as many as 2,392 vertices. The subtree cardinality-constrained minimal spanning tree problem and the identical customer vehicle routing problems are, in a sense, the simplest core capacitated network design and network routing models and so insight concerning these models might prove to be valuable in extending the results for uncapacitated problems to more general capacitated models.

Second, mounting evidence in many application domains has demonstrated the impressive potential of using strong cutting plane methods for solving a variety of pure and mixed integer programs (see Hoffman and Padberg [1985] and Wolsey [1989] for surveys). In particular, research on the simpler traveling salesman problem has shown that the type of polyhedral results that we consider in this paper have been very useful, indeed essential, in solving large-scale problems to optimality (with a guarantee of optimality). This experience suggests that a better understanding of the polyhedral structure of capacitated network design and routing problems might lead to effective algorithms. Nevertheless, to the best of our knowledge until now no one has investigated the polyhedral structure of the capacitated minimal spanning tree problem, and only two studies, conducted in parallel with the research we are reporting, have studied the polyhedral structure of the (identical customer) vehicle routing problem. Cornuéjols and Harche [1989] have focused on extensions of valid inequalities (for example the well known comb inequalities) for the traveling salesman problem that use *routing* type structure of these problems. In work not reported here, one of us (Araque [1989b, 1989c]) has also studied these type of inequalities. Cornuéjols and Harche [1989] and Campos, Corberan and Mota [1989] have also studied a certain set of extended subtour breaking constraints that we also consider in this paper. In contrast, in this paper we focus on a graph-based approach that does not ensue from natural analogs of the traveling salesman problem.

In the next section, we introduce formulations for the two problems we are studying as well as a relaxed forest version of the subtree cardinality-constrained minimal spanning tree problem, and a reformulation of the identical customer vehicle routing problem as a certain path-partitioning problem. We also establish relationships between these problems, and particularly between their valid inequalities and facets. These relationships permit us, for the most part, to treat the two problems simultaneously or as modest variants of each other; on some occasions, however, we need to treat the problems differently because the vehicle routing and path-partitioning problems are more constrained than the minimal spanning tree problem (because of the degree-2 constraints of the vertices in a path). Therefore, we introduce some inequalities that are facets for only the path-partitioning polytope.

In this section we also introduce a general proof technique, based upon certain exchange arguments and comparison of certain generic solutions, that permits us to streamline many of our proofs in later sections.

In Section 3 we provide a general introduction of the various types of inequalities we develop in the rest of the paper and some motivating examples to illustrate why the inequalities are needed. Sections 4 through 6 contain technical details for establishing when the inequalities we introduce are valid and facet-inducing. In Section 4 we discuss tree and packing (or generalized subtour elimination) constraints. In Section 5, we consider *multistar* subgraphs which are star subgraphs, but with a clique replacing the central node of the star. In Section 6, we consider a certain type of *ladybug* graph structure and associated inequalities. A ladybug subgraph has two components: a head which is a multistar and a body which is a clique. As we show, under certain circumstances these subgraphs yield facet-inducing inequalities. In this section, we also consider several *partial multistar* variants of the basic multistar inequalities. In Section 7, we consider a graph structure which we refer to as a *clique cluster*: it is composed of a set of cliques that meet at a common vertex. Again, we show that certain classes of clique clusters are facet-inducing.

In a concluding section, we suggest possible directions for future research and point out some generalizations of the results reported in this paper.

2 Preliminary Results

2.1 Path Partitioning, Forest, and Tree Polytopes

We begin by introducing notation needed for discussing graph structures. Let $G = (V, E)$ be a complete, undirected graph. For $i, j \in V$, we write ij, ji, or e to represent the undirected edge $e = \{i,\ j\} \in E$. For a subset of nodes $S \subseteq V$, we denote the set of all edges between nodes of S by $E(S) = \{ij\ :\ i,\ j \in S\}$; for any two disjoint subsets $S, U \subseteq V$, we denote the set of edges with one vertex in each of these sets as $E(S, U) = \{e = ij\ :\ i \in S,\ j \in U\}$. For $v \in V$, we denote the set of edges incident to v as $\delta(v) = E(\{v\}, V \backslash \{v\})$. For a set of edges $B \subseteq E$ and a vector $\mathbf{x} = (x_e : e \in E)$, we frequently use the notation $X(B) = \sum_{e \in B} x_e$. If $f \in E$, we also let $\mathbf{e}_f$ denote an incident vector with $|E|$ components which has value 1 corresponding to the element f of E and value 0 otherwise.

Throughout the discussion, we label the nodes of V as $1, \ldots, n$. For the minimal spanning tree problem and the vehicle routing problem, we include an additional special node 0 and define the problems with respect to the complete graph on $\{0, 1, \ldots, n\}$.

To formulate the subtree cardinality-constrained minimal spanning tree problem as an integer linear program, we introduce some notation. The central root node is the special node 0. We let K denote the subtree capacity, and c_e denote the cost of edge e, for all $e \in E(V \cup \{0\}) = E \cup \{0j : j = 1, \ldots, n\}$. The decision variables,for all $e \in E(V \cup \{0\})$, are

$$x_e = \begin{cases} 1, & \text{if edge } e \text{ is in the tree;} \\ 0, & \text{otherwise.} \end{cases}$$

The integer programming formulation is

$$\text{Minimize} \quad \sum_{e \in E} c_e x_e + \sum_{j=1}^{n} c_{0j} x_{0j}$$

subject to

$$\sum_{e \in E} x_e + \sum_{j=1}^{n} x_{0j} = n \tag{2.1}$$

$$\sum_{e \in E(S)} x_e \leq |S| - \left\lceil \frac{|S|}{K} \right\rceil, \qquad S \subseteq V,\ |S| \geq 2 \tag{2.2}$$

$$\sum_{e \in E(U)} x_e \leq |U| - 1, \qquad U \subseteq V \cup \{0\},\ |U| \geq 2,\ 0 \in U \tag{2.3}$$

$$x_e \quad \in \{0,1\}, \qquad e \in E.$$

Equality (2.1) is true of all trees. Inequalities (2.3) are some of the rank inequalities for trees: if more than $|U|$ edges connect the nodes of a subset U, then that set of edges must contain a cycle. Inequalities (2.2) are similar to (2.3), except that they reflect the capacity constraint: if the set S does not contain the root node, then the nodes of S must be contained in is at least $\lceil |S|/K \rceil$ different subtrees off of the root.

Let X be the set of solutions to this integer program. Then we define T to be conv(X), the convex hull of K-capacitated trees.

A closely related problem is the capacitated forest problem. If we delete node 0 and all edges incident to node 0 from a capacitated spanning tree, the resulting graph structure is a forest, each of whose components contains at most K nodes. The following integer program models the problem of finding the minimal capacitated forest on nodes $1, \ldots, n$:

$$\text{Minimize} \qquad \sum_{e \in E} c_e x_e$$

subject to

$$\begin{aligned} \sum_{e \in E(S)} x_e &\leq |S| - \left\lceil \frac{|S|}{K} \right\rceil, & S \subseteq V,\ |S| \geq 2 \\ x_e &\in \{0,1\}, & e \in E. \end{aligned}$$

If X is the set of solutions to this integer program, then we define F to be conv(X). Section 2.4 explains the relationship between T and F; in particular, we show that any facet of F is a facet of T as well.

Next we turn to a related problem, the identical customer vehicle routing problem. We let K represent truck capacity, c_{ij} the cost of traveling between nodes i and j, and, by analogy to the tree problem, we let node 0 be the depot and nodes $1, \ldots, n$ the customers. The decision variables for our integer programming formulation are

$$x_{ij} = \begin{cases} 1, & \text{if } i \text{ and } j \text{ are consecutive customers on the same route;} \\ 0, & \text{otherwise,} \end{cases}$$

$$\text{for all } i, j \in V;$$

$$z_i = \begin{cases} 2, & \text{if } i \text{ is alone on a single route;} \\ 1, & \text{if } i \text{ is the first or last customer on a route with at least two customers;} \\ 0, & \text{otherwise,} \end{cases}$$

$$\text{for } i = 1, \ldots, n.$$

We can then formulate the following integer program to determine the minimum-cost routing:

$$\text{Minimize} \quad \sum_{e \in E} c_e x_e + \sum_i c_{0i} z_i$$

subject to

$$\begin{aligned} \sum_{j=1}^{n} x_{ij} + z_i &= 2, && i = 1, \ldots, n \\ \sum_{e \in E(S)} x_e &\leq |S| - \left\lceil \tfrac{|S|}{K} \right\rceil, && S \subseteq V,\ |S| \geq 2 \\ x_e &\in \{0, 1\}, && e \in E \\ z_i &\in \{0, 1, 2\}, && i = 1, \ldots, n. \end{aligned}$$

The variables z_i can be treated as slack variables and eliminated from the formulation to yield a new formulation in the x-variables with edge-costs given by the saving $s_{ij} = c_{0i} + c_{0j} - c_{ij}$ as defined by Clarke and Wright [1964]:

$$\text{Maximize} \quad \sum_{e \in E} s_e x_e$$

subject to

$$\begin{aligned} \sum_{j=1}^{n} x_{ij} &\leq 2, && i = 1, \ldots, n \\ \sum_{e \in E(S)} x_e &\leq |S| - \left\lceil \tfrac{|S|}{K} \right\rceil, && S \subseteq V,\ |S| \geq 2 \\ x_e &\in \{0, 1\}, && e \in E. \end{aligned}$$

This reformulation of the vehicle routing problem leads to a reinterpretation of the problem in terms of a graph model. A feasible solution to the second integer program, on the node set $V = \{1, \ldots, n\}$, consists of a collection of paths, each containing at most K nodes; we refer to this graph structure as a ***K-capacitated path-partition***, and to the corresponding optimization problem as the ***path-partitioning problem.*** If X is the set of solutions to the integer program, then we define PP to be $\text{conv}(X)$, the convex hull of solutions to the path-partitioning problem.

Observe that any K-capacitated path-partition is a K-capacitated forest, as well. Consequently, $PP \subseteq F$, and any facet of PP that is a valid inequality of F must also be a facet of F. We use this fact frequently in the proofs to follow.

2.2 Exchange Arguments

In proving that certain valid inequalities are facets, we will use the following familiar argument (which amounts to a dual or "indirect" proof).

Suppose that P is a full-dimensional polyhedron in $\mathbb{R}^m$ and the inequality $\mathbf{a}\mathbf{x} \leq a_0$ is a valid face of P (i.e, all points in P satisfy this inequality, and at least one point of P lies on the hyperplane defined by the inequality). Then this inequality is a facet of P if it satisfies the property that if $\boldsymbol{\alpha}^T\mathbf{x} \leq \alpha_0$ for all points $\mathbf{x}$ contained in the face $Q = \{\mathbf{x} \in P \mid \mathbf{a}\mathbf{x} = a_0\}$ of P, then $(\boldsymbol{\alpha}, \alpha_0)$ is a multiple of $(\mathbf{a}, a_0)$. One way to establish this result is to identify $k \geq m$ points $\mathbf{x}_0, \ldots, \mathbf{x}_k$ contained in Q with the property that any solution to the system

$$
\begin{aligned}
\boldsymbol{\alpha}^T\mathbf{x}_0 &= \alpha_0 \\
\boldsymbol{\alpha}^T\mathbf{x}_1 &= \alpha_0 \\
&\vdots \\
\boldsymbol{\alpha}^T\mathbf{x}_k &= \alpha_0
\end{aligned}
$$

in the variables $(\boldsymbol{\alpha}, \alpha_0)$ is the vector $(\mathbf{a}, a_0)$ or some multiple of this vector.

We note that the same proof technique applies to situations in which P is not full-dimensional, as is the case for the polyhedron T. If this case, though, we need to show that $\mathbf{a}^T\mathbf{x} \leq a_0$ is a proper face of P—that is, some point of P does not lie on the face defined by this inequality—and that any solution $(\boldsymbol{\alpha}, \alpha_0)$ to the previous system is of the form $(\boldsymbol{\alpha}, \alpha_0) = \lambda_0(\mathbf{a}, a_0) + \sum_1^p \lambda_i(\mathbf{b}_i, b_{0i})$ for some weights $\lambda_0, \lambda_1, \ldots, \lambda_p$ with $\lambda_0 \geq 0$. In this expression, $(\mathbf{b}_i, b_{0i})$ are the variable and right-hand side coefficients of the system of implicit equations satisfied by all solutions of P. As we show in Section 2.3, for the polyhedron T, the only such equation is $X(E(V \cup \{0\})) = n$.

In many cases, such as the polyhedra defined over graphs that we are considering, it is possible to define the vectors $\mathbf{x}_j$ two or three at a time to make inferences about some of the coefficients of the vector $\boldsymbol{\alpha}$. Since the same elementary constructions recur frequently in our development, to avoid duplicating our arguments and to highlight the ideas we are using, we feel it might be useful to collect and formalize some of these arguments at the outset of our discussion in the form of the following straightforward exchange arguments. Throughout this discussion, $P \subseteq \mathrm{IR}^m$ is an arbitrary polyhedron, and F and PP are the forest and path-partitioning polytopes, respectively. We think of P as being defined on K_n, the complete graph on n vertices.

Lemma 2.1 *(Simple-exchange). Suppose $\mathbf{x}, \mathbf{y} \in P$ satisfy $\boldsymbol{\alpha}^T\mathbf{x} = \alpha_0$ and $\boldsymbol{\alpha}^T\mathbf{y} = \alpha_0$, and that for two given edges f and g, $x_f = 1, x_g = 0, y_f = 0, y_g = 1$ and $x_e = y_e$ for all edges $e \neq f, g$. Then the coefficients α_f and α_g are equal.*

Proof: Subtract one equation from the other to obtain

$$\alpha_f x_f - \alpha_g y_g = 0,$$

or $\alpha_f = \alpha_g$ since $x_f = y_g = 1$. ■

The following two easy extensions of Lemma 2.1 will prove to be extremely useful in streamlining the proofs to follow. Often, a candidate facet has many solutions satisfying it at equality that differ among themselves very little. This structure can be exploited.

Lemma 2.2 *(Triple-exchange for forests). Let S be a subset of vertices. Suppose that, for any three distinct nodes i, j, and k of S, some $0-1$ vector $\mathbf{x}$ in F corresponding to a forest in which i, j, and k are on the same tree satisfies the equality*

$$\boldsymbol{\alpha}^T\mathbf{x} = \alpha_0. \tag{2.4}$$

Suppose further that we can connect the vertices on the tree containing i, j and k in any way and the incidence vectors of the forests so generated still satisfy (2.4). Then all of the coefficients of $E(S)$ in $\boldsymbol{\alpha}$ are equal.

Proof: Fix $\mathbf{x}$. Without loss of generality, assume that the edges ij and jk are part of the forest. By exchanging edge ik for edge ij and applying the simple-exchange argument, we have $\alpha_{ij} = \alpha_{ik}$. Since i, j and k are arbitrary, we immediately conclude that any two edges in $E(S)$ with a common vertex have the same coefficient. Now consider two edges in $E(S)$ with no common vertices, say edges ij and kl. Using the triplets i, j, k and j, k, l, we obtain $\alpha_{ij} = \alpha_{jk} = \alpha_{kl}$. ∎

Lemma 2.3 *(Triple-exchange for paths). Let S be a subset of vertices. Suppose that, for any three distinct nodes i, j, and k of S, some $0-1$ vector $\mathbf{x}$ in PP satisfies the equality*

$$\boldsymbol{\alpha}^T \mathbf{x} = \alpha_0, \tag{2.5}$$

and the path-partition corresponding to $\mathbf{x}$ contains a path of the form $k - j - i - \cdots$, with k as an endpoint. Suppose, further, that if we permute the vertices in this path, we obtain another vector satisfying (2.5). Then all of the coefficients of $E(S)$ in $\boldsymbol{\alpha}$ are equal.

Proof: First, let $\mathbf{x}$ correspond to a partition containing the path $k - j - i - \cdots$, and let $\mathbf{y}$ be obtained from $\mathbf{x}$ by permuting nodes k and j so that the resulting partition contains the path $j - k - i - \cdots$. From the simple-exchange lemma applied to $\mathbf{x}$ and $\mathbf{y}$, we obtain $\alpha_{ik} = \alpha_{ij}$. The result now follows by the same arguments given in the proof of Lemma 2.2. ∎

Lemma 2.4 *(In-and-Out edges). Suppose $\mathbf{x} \in P$ satisfies $\boldsymbol{\alpha}^T \mathbf{x} = \alpha_0$, and that $x_f = 0$ for a given edge f. If it is possible to switch the edge f in and out of the solution and remain on the face $\boldsymbol{\alpha}^T \mathbf{x} = \alpha_0$ (i.e., if we can change x_f to 1 and still obtain a vector in P satisfying the equation $\boldsymbol{\alpha}^T \mathbf{x} = \alpha_0$), then $\alpha_f = 0$.*

Proof: Subtracting the equation $\boldsymbol{\alpha}^T \mathbf{x} = \alpha_0$ from $\boldsymbol{\alpha}^T(\mathbf{x} + \mathbf{e}_f) = \alpha_0$ yields $\alpha_f = 0$. ∎

Of course, Lemmas 2.1, 2.2, and 2.4 apply to the polytope T as well.

2.3 Dimensions of PP, F, and T

The dimension of a polyhedron P is defined as the dimension of the smallest affine space containing P. In general, facet proofs are easier for full-dimensional polyhedra than for lower-dimensional ones. As shown by the following lemma, both F and PP are full-dimensional. Let $m = n(n-1)/2$.

Lemma 2.5 *Dim* F = *dim* $PP = m$.

Proof: Since $PP \subseteq F \subseteq \mathbb{R}^m$, it suffices to show that dim $PP = m$. To do so, we need simply to exhibit a set of $m+1$ affinely independent vectors contained in PP. The $\mathbf{0}$-vector along with the unit vectors $\mathbf{e}_1, \ldots, \mathbf{e}_m$ comprise such a set ($\mathbf{e}_i$ contains a 1 in the i-th component and a 0 elsewhere). ∎

The polytope T does not have full dimension, since every tree satisfies

$$X(E(V \cup \{0\})) = n. \tag{2.6}$$

Lemma 2.6 *Dim* $T = (n+1)n/2 \ - 1$.

Proof: Recall that $T \subseteq \mathbb{R}^{(n+1)n/2}$. To prove the lemma, it is sufficient to show that (2.6) is the only equality (up to a multiplicative factor) containing T. Consider an arbitrary equality

$$\boldsymbol{\alpha}\mathbf{x} = \alpha_0 \tag{2.7}$$

containing T, and consider the star-shaped tree consisting of all of the edges incident to the root, or $\mathbf{y} = (y_{0i} = 1,\ i = 1\ldots, n;\ y_e = 0,\ e \in E)$. Clearly $\mathbf{y}$ satisfies (2.7). Now consider two arbitrary nodes $i, j \neq 0$, and replace edge $0i$ or $0j$ in $\mathbf{y}$ by edge ij. Since these new solutions also satisfy (2.7), by the simple-exchange argument (Lemma 2.1) we have

$$\alpha_{0i} = \alpha_{0j} = \alpha_{ij}. \tag{2.8}$$

Since i and j were arbitrary, all of the edges incident to the root have the same coefficient in (2.7); and thus by (2.8), all of the edges do. Thus (2.7) is indeed a multiple of (2.6). ∎

2.4 Extending facets from F to T

The following lemma emphasizes the close relationship between F and T. It states that any facet for F can be trivially lifted to T.

Lemma 2.7 *Any facet for* F *which is not induced by a nonnegativity constraint is a facet for* T.

Proof: Let $\mathbf{a}^T\mathbf{x} \leq a_0$ be a facet for F such that $a_0 > 0$. (Note that $a_0 \not< 0$, since the $\mathbf{0}$-vector is valid for $\mathbf{a}^T\mathbf{x} \leq a_0$.) Since F is a full-dimensional polytope in $\mathbb{R}^m$, it contains m linearly independent 0–1 vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m \in \mathbb{R}^m$ satisfying $\mathbf{a}^T\mathbf{x} = a_0$. We can arrange these vectors as rows of a matrix $\mathbf{X}$, and their linear independence implies that the matrix is invertible. Each vector $\mathbf{x}_i$ is the incidence vector of a forest with n vertices. By adding edges to connect each component of the forest to the root vertex, we can obtain vectors in T of the form $\tilde{\mathbf{x}}_i = (\mathbf{x}_i, \mathbf{y}_i)$ with $\mathbf{y}_i$ corresponding to the vector of edges incident to the root.

Now, let us assume that

$$\boldsymbol{\gamma}^T\mathbf{x} + \boldsymbol{\beta}^T\mathbf{y} \leq \gamma_0 \tag{2.9}$$

defines a facet of T containing the face defined by $\mathbf{a}^T\mathbf{x} = a_0$; that is, every solution (x, y) of T satisfying $\mathbf{a}^T\mathbf{x} = a_0$ also satisfies 2.9. We need to show that for some scalars $\eta > 0$ and μ, $\boldsymbol{\gamma} = \eta\mathbf{a} + \mu\mathbf{1}$ and $\boldsymbol{\beta} = \mu\mathbf{1}$ (in the first equality $\mathbf{1}$ is the m-dimensional vector of all ones, and in the second equality it is an n-dimensional vector).

First, we establish $\boldsymbol{\beta} = \mu\mathbf{1}$. Since the face $\mathbf{a}^T\mathbf{x} = a_0$ is non-trivial, for each $i, j \leq n$ it contains a feasible vector $\mathbf{x}_l$ for some $1 \leq l \leq m$ with a 1 in the component corresponding to the edge ij. We can extend this vector to a vector in T so that i is connected to the root. By exchanging that edge with the edge connecting j and the root, by the simple-exchange argument (Lemma 2.1) we can show that the coefficients in (2.9) of edges $0i$ and $0j$ are equal; and since i and j were chosen arbitrarily, we see that all of the edges incident to the root have the same coefficients in (2.9). Thus $\boldsymbol{\beta} = \mu\mathbf{1}$, for some scalar μ.

Next we show that $\boldsymbol{\gamma} = \eta\mathbf{a} + \mu\mathbf{1}$ for some scalar η. If the vector $\mathbf{x}_i$ defines a forest with c_i components, then $\mathbf{1}^T\mathbf{x}_i = n - c_i$ and so (here, $\mathbf{1}$ is a vector of m ones)

$$\mathbf{X}\mathbf{1} = \sum_{i=1}^{m}(n - c_i)\mathbf{e}_i, \quad \text{and} \tag{2.10}$$

$$\mathbf{X}\mathbf{a} = a_0\mathbf{1}. \tag{2.11}$$

Since $\tilde{\mathbf{x}}_i = (\mathbf{x}_i, \mathbf{y}_i)$ satisfies (2.9) at equality, we have

$$\boldsymbol{\gamma}^T\mathbf{x}_i + \boldsymbol{\beta}^T\mathbf{y}_i = \boldsymbol{\gamma}^T\mathbf{x}_i + \mu c_i = \gamma_0,$$

which yields the system

$$\mathbf{X}\boldsymbol{\gamma} = \gamma_0\mathbf{1} - \sum_{i=1}^{m}\mu c_i\mathbf{e}_i.$$

Since $\mathbf{X}$ is invertible, we can solve for $\boldsymbol{\gamma}$ and use (2.11) to obtain

$$\boldsymbol{\gamma} = \mathbf{X}^{-1}\gamma_0\mathbf{1} - \mathbf{X}^{-1}\sum_{i=1}^{m}\mu c_i\mathbf{e}_i = \frac{\gamma_0}{a_0}\mathbf{a} - \mathbf{X}^{-1}\sum_{i=1}^{m}\mu c_i\mathbf{e}_i.$$

Substituting from (2.10), rearranging, and applying (2.11) yields

$$\begin{aligned}\boldsymbol{\gamma} &= \frac{\gamma_0}{a_0}\mathbf{a} - \mathbf{X}^{-1}\left(\sum_{i=1}^{m}\mu c_i\mathbf{e}_i - \mu n\mathbf{1} + \mu n\mathbf{1}\right)\\ &= \frac{\gamma_0}{a_0}\mathbf{a} - \mathbf{X}^{-1}(-\mu\mathbf{X}\mathbf{1}) - \mathbf{X}^{-1}\mu n\mathbf{1}\\ &= \frac{\gamma_0}{a_0}\mathbf{a} + \mu\mathbf{1} - \frac{\mu n}{a_0}\mathbf{a}\\ &= \frac{\gamma_0 - \mu n}{a_0}\mathbf{a} + \mu\mathbf{1}.\end{aligned}$$

Taking $\eta = (\gamma_0 - \mu n)/a_0$ establishes the relationship.

Finally, we observe that $\eta > 0$. Since (2.9) defines a facet of T, we have $\gamma_0 - \mu n \neq 0$. It is easy to verify that this quantity is strictly positive by substituting the incidence vector of the tree with every node connected directly to the root in (2.9), yielding $\mu n \leq \gamma_0$. Since $a_0 > 0$, we have $\eta > 0$. ∎

2.5 Trivial Inequalities

Finally, we establish that, in general, the trivial inequalities $x_e \geq 0$ and $x_e \leq 1$ are facets.

Proposition 2.8 *For the polytopes PP, F, and T, if $K \geq 2$ and $n \geq 2$ the inequalities*

$$x_e \geq 0 \tag{2.12}$$

are facets, for all $e \in E$; and if $K \geq 3$ and $n \geq 2$, the inequalities

$$x_e \leq 1 \tag{2.13}$$

are facets, for all $e \in E$.

Proof: Clearly these inequalities are valid for *PP*. Suppose that $\boldsymbol{\alpha}\mathbf{x} \leq \alpha_0$ is a facet for *PP* satisfied at equality by every feasible solution satisfying (2.13) at equality, for some edge $e \in E$. Consider the particular solution containing only the edge e. Since for $K \geq 3$ a solution containing e and exactly one other edge f also satisfies (2.13) at equality and is feasible for *PP*, by Lemma 2.4 (the in-and-out argument) $\alpha_f = 0$ for all $f \neq e$. This conclusion is sufficient to show that $\boldsymbol{\alpha}\mathbf{x} \leq \alpha_0$ is a multiple of (2.13), and thus (2.13) is a facet of PP.

The argument for (2.12) is completely analogous. Since (2.12) and (2.13) are valid for F, they are facets of F, as well; and by Lemma 2.7, (2.13) are also facets of T. ∎

We observe that when $K = 2$, inequalities (2.13) are not facets for any of the polytopes (unless $n = 2$), since they are dominated by the valid inequalities $x_{ij} + x_{jk} + x_{ik} \leq 1$ on any three nodes i, j, and k.

Proposition 2.9 *For $K \geq 2$ and $n \geq 2$, the inequalities*

$$x_{0i} \leq 1 \tag{2.14}$$

are facets for T; and for $K \geq 3$ and $n \geq 3$, the inequalities

$$x_{0i} \geq 0 \tag{2.15}$$

are facets for T.

Proof: First, we observe that (2.14) and (2.15) are valid for T, and that there are solutions satisfying these inequalities as a strict inequality. Let $\boldsymbol{\alpha}\mathbf{x} \leq \alpha_0$ be a facet of T satisfied at equality by every feasible solution satisfying (2.14) at equality for some fixed $i \in V$. We need to show that this facet is a linear combination of (2.6) and (2.14). In particular, the star-solution $\mathbf{x} = (x_{0j} = 1,\ j = 1, \ldots, n;\ x_e = 0,\ e \in E)$ satisfies (2.14) at equality. By substituting edge jk for $0j$ in $\mathbf{x}$, for $j \neq i$, using the simple-exchange lemma we see that $\alpha_{0j} = \alpha_{jk}$ for all $j \neq i$, $k = 1, \ldots, n$; and by the same argument as that in Lemma 2.6, $\alpha_e = \alpha_f$ for all $e, f \neq 0i$. Thus $\boldsymbol{\alpha}\mathbf{x} = \alpha_0$ is a linear combination of (2.14) and (2.6).

Similar arguments establish that the inequalities (2.15) are facets under the given conditions. ∎

When $K = 2$, the inequalities (2.15) are not facets. Certain facets described in Sections 4 and 5 dominate these inequalities. For example, the inequality $X(E(S)) \leq |S| - 1$, for $S = (V \cup \{0\}) \setminus \{i\}$, described in Section 4, dominates (2.15) if $K = 2$.

3 Facets for the Forest, Tree and Path-partitioning Polytopes: an Introduction

In the following sections, we discuss a number of different inequalities and facets for the forest, tree, and path-partitioning polytopes. Since the description of these inequalities, the associated facet proofs, and the conditions for ensuring that the inequalities are facets are fairly delicate, our discussion is necessarily quite detailed. Therefore, before launching into this development, in this section we will first provide a brief preview of these results, introducing some of the various inequalities and providing some examples to demonstrate their value in cutting off fractional solutions from the linear programming relaxation of the problems. We hope that this more introductory and informal discussion will provide some useful motivation for the remainder of this paper as well as some insight concerning the nature of the various polytopes.

Consider any valid inequality

$$\sum_{e \in E(N)} a_e x_e \leq a_0$$

for one of the polytopes that we are considering. We will refer to the subgraph of the original network (V, E) spanned by those edges e with $a_e \neq 0$ as the *support graph* of this inequality. Since essentially all of the inequalities that we will be considering have a natural interpretation in terms of their associated support graphs, as we introduce the various inequalities we will describe their support graphs, illustrate feasible solutions on the support graphs, and identify fractional solutions from one of the linear programming relaxations LP^F, LP^T and LP^{PP} that the inequality cuts off. In most cases, if we make appropriate restrictions on the problem data and the configuration of the support graph, the inequalities will be facets for all of the polytopes F, T, and/or PP; we will draw our examples from the most restrictive of these polytopes, that is, from PP.

We will introduce four types of inequalities:

(1) multistars (large and small),

(2) partial multistars,

(3) ladybugs, and

(4) clique clusters.

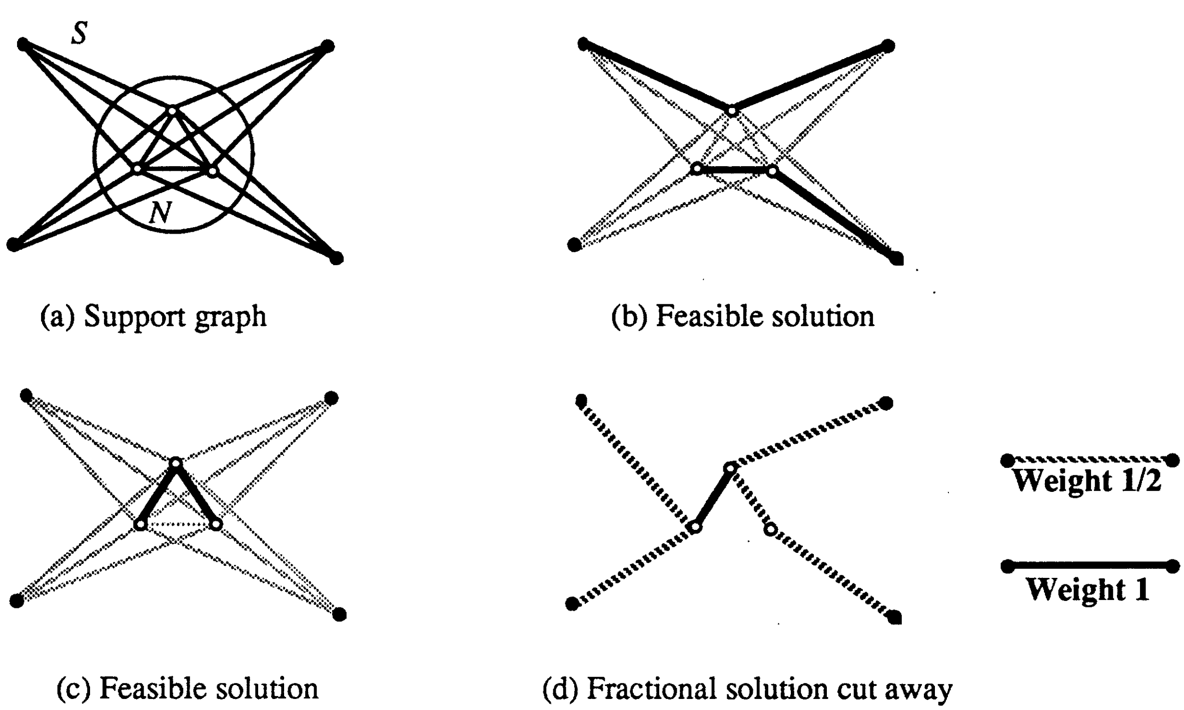


FIGURE 3.1: Multistar example for $K = 3$

Within in each type, we will describe several variants. For example for the multistars, we describe both "large" and "small" multistars.

Multistars

The support graph of a basic multistar looks like a simple star graph except that a clique replaces the central vertex of the star. That is, a multistar consists of the complete subgraph on a set of ***nucleus*** vertices N, together with a set of ***satellite*** vertices S and the edges connecting every satellite vertex to every nucleus vertex. The support graph for the example shown in Figure 3.1 has four satellite vertices and a three-vertex nucleus. As shown in Figures 3.1b and 3.1c, since $K = 3$ for this example, any feasible solution to the path-partitioning problem can contain at most four edges from the support graph of this multistar. Moreover, note that if we assign a weight of $K = 3$ to the edges in the nucleus of the multistar with and a weight of one to those edges joining the nucleus vertices and the satellite vertices, then the total weight of the support graph of the multistar in any feasible solution to the path-partitioning polytope is at most six. Therefore, we can write the valid inequality

$$3X(E(N)) + X(E(N,S)) \leq 6.$$

Note that the fractional solution shown in Figure 3.1d violates this inequality; it has a total weight of 6 1/2. We might note that this particular fractional solution is an extreme point of the packing path-partitioning polytope, that is, the linear programming formulation of the problem with all of the degree-2 constraints, the packing constraints (2.2), and the trivial inequalities (2.12) and (2.13).

In general, the multistar inequality is of the form

$$KX(E(N)) + X(E(N,S)) \leq (K-1)|N|.$$

If we impose certain conditions on the sizes of the nucleus and satellite sets, ***and*** if these two sets contain all the vertices V of the underlying network, then this type of *large multistar* inequality is a facet of each of the polytopes F, T, and PP (the conditions required for the polytope PP are more restrictive than those for the forest and tree polytopes). There is, however, another set of facets, which we refer to as *small multistars*, whose vertex set need not exhaust all of the vertices V of the original network. In this case the coefficient b of the nucleus edges is less than K and the right-hand side of the inequality is less than $(K-1)|N|$. That is, the inequality is of the form

$$bX(E(N)) + X(E(N,S)) \leq \text{RHS}.$$

This small multistar is a facet of each of the three polytopes for certain choices of the constants b and RHS if we impose appropriate restrictions on the relationships between the sizes of the nucleus and satellite sets (see Section 5).

Partial Multistars

For the path-partitioning polytope, we can also generalize the multistars in another way. Instead of including all the edges connecting the nucleus vertices and the satellite vertices, the support graph contains only those edges that are incident to a subset $\bar{N}$ of the nucleus vertices; we refer to this subset as the *connector vertices*. In Section 6 we will develop facet versions of these inequalities with one, two, and three connector vertices. Figure 3.2 shows the support graph of a partial multistar with one connector vertex v, as well as feasible solutions, and a fractional solution that we wish to cut away. As shown in Figures 3.2b and 3.2c, the support graph for this partial multistar can contain at most three edges in the induced subgraph corresponding to any feasible solution to the path-partitioning problem. Moreover, note that the nucleus set can contain at most two of these edges, and that if a feasible solution uses two of these edges then it cannot use any edge to a satellite vertex. Consequently, if we assign weights of two to the edges joining vertices in

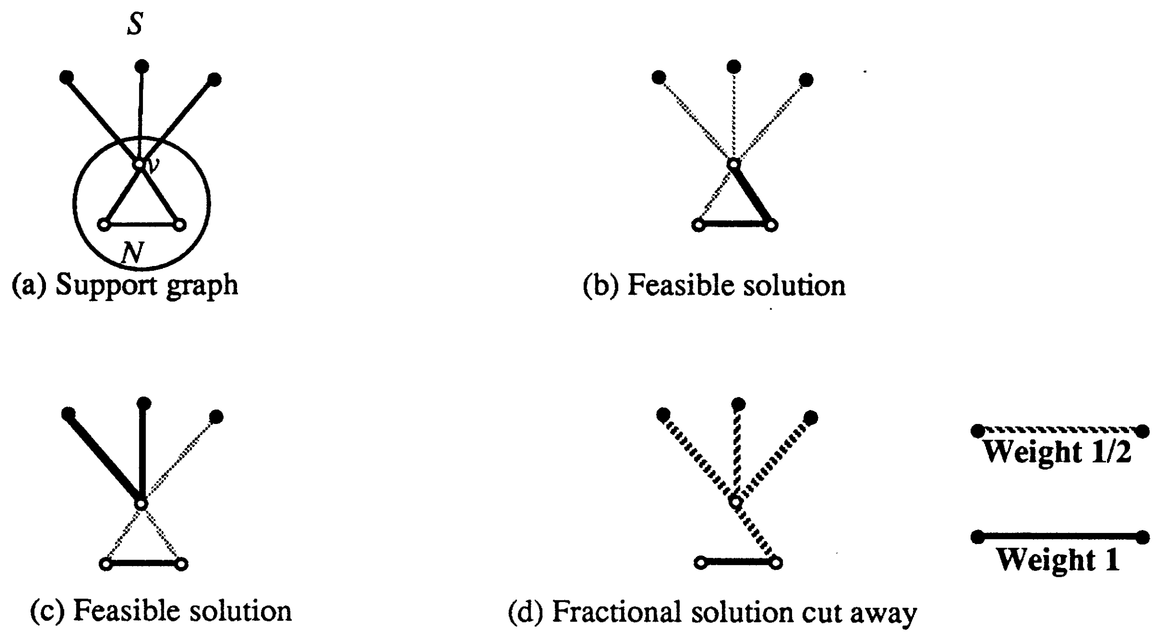


FIGURE 3.2: Partial Multistar example for $K = 3$

the nucleus set and assign a weight of one to the edges joining the connector vertex v to the satellite nodes, then the total weight of the edges in the partial multistar can be no more than four. Therefore, we can write the following valid inequality

$$2X(E(N)) + X(E(v, S)) \leq 4$$

which cuts away the fractional solution in Figure 3.2d. In general, if $\bar{N}$ is the connector set, $|\bar{N}| \leq 3$, we can write a partial multistar inequality of the form

$$aX(E(N)) + X(E(\bar{N}, S)) \leq \text{RHS}$$

and for an appropriate choice of the coefficients a and RHS, and appropriate size restrictions on the size of N and S, this inequality defines a facet of the path-partitioning polytope. Section 6 describes the details and also introduces another related facet defined on the same support graph as the partial multistar.

Ladybugs

A ladybug has a support graph with two components: its ***head*** is a multistar (N, S), and its ***body*** is the complete subgraph on another set of vertices, B, all of which are connected to every vertex in the nucleus set N of the multistar. In this instance, we might interpret the edges connecting the satellite vertices S to

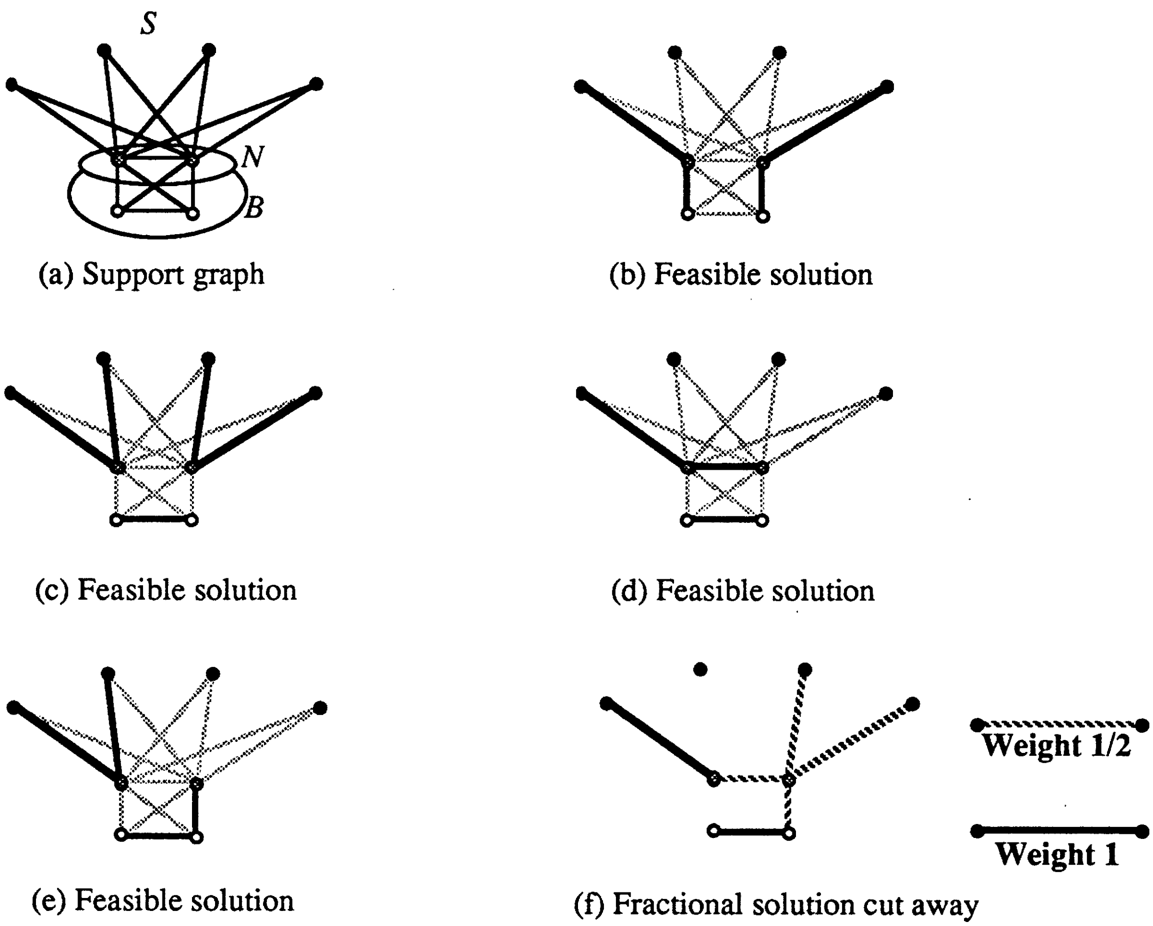
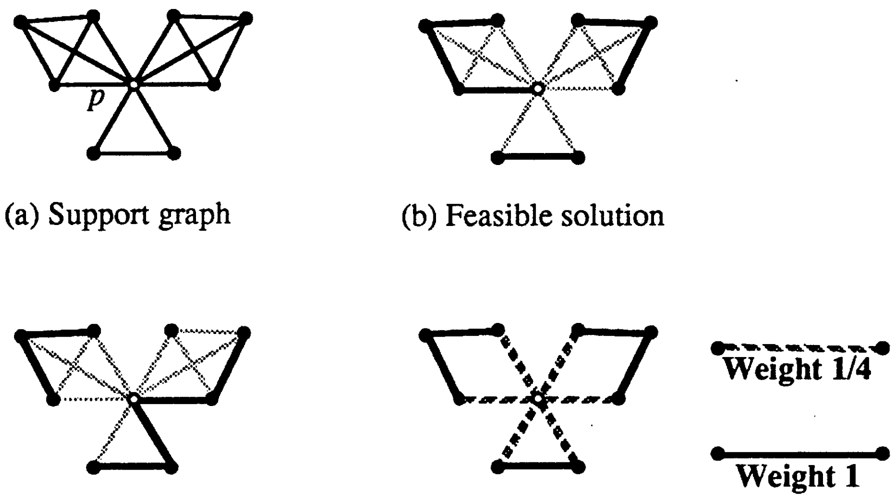


FIGURE 3.3: Ladybug example for $K = 3$

the nucleus set N as the set of antennae of the ladybug. Figure 3.3 gives a 8-vertex example of a ladybug, together with the induced subgraphs corresponding to feasible solutions to the path-partitioning polytope, and a fractional solution to be cut off by the ladybug inequality. In this example, the head has six vertices: four are satellite vertices and two are in the nucleus. The body of this ladybug has two vertices. In this case $K = 3$. Note from Figures 3.3b–3.3e that the support graph induced by any feasible solution to the path-partitioning polytope contains at most five edges. Moreover, note that if we weight the edges joining the nucleus nodes with a weight of three, the edges incident to the body vertices with a weight of two, and the edges joining the nucleus vertices and the satellite vertices with a weight of one, then the overall weight of any solution is at most six. Therefore, we can write the valid inequality

$$2X(E(B)) + 2X(E(B, N)) + 3X(E(N)) + X(E(N, S)) \leq 6.$$



(a) Support graph (b) Feasible solution

(c) Feasible solution (d) Fractional solution cut away

FIGURE 3.4: Clique cluster example for $K = 5$

Note that the fractional solution given in Figure 3.3f does not satisfy this inequality. This fractional solution does satisfy all of the degree-2 constraints, all the subtour elimination (packing) constraints, and all of the large and small multistar constraints. In fact, it is possible to show that it is an extreme point of the linear programming problem defined by these constraints.

In general, the ladybug constraints have the form

$$dX(E(B)) + dX(E(B,N)) + KX(E(N)) + X(E(N,S)) \leq \text{RHS}$$

for some appropriate choice of the coefficient d and of the right-hand side coefficient RHS. In Section 6 we specify the values of these coefficients as a function of the problem data and show when the ladybug inequalities define facets.

Clique clusters

The support graph of a basic clique cluster looks somewhat like a flower and captures some of the properties of cliques and of a cutset around a single vertex. As shown in Figure 3.4, a ***clique cluster*** is a collection of subsets of vertices $C_1, C_2, \ldots, C_t$, each inducing a subgraph that forms one of the cliques or *petals* of the cluster. The clusters intersect at a single central vertex p, which we will refer to as the *nucleus* of the cluster. The support graph for the example shown in Figure 3.4 has three petals, defined by two cliques with four vertices and one clique with three vertices. As shown in Figures 3.4b and 3.4c, since

$K = 5$ for this example, and because the petals of the clique cluster all intersect at the nucleus vertex, any feasible path-partitioning solution can contain at most 6 edges from the support graph of this clique cluster. Therefore, we can write the valid inequality

$$X(E(C_1)) + X(E(C_2)) + X(E(C_3)) \leq 6.$$

Note that the packing (subtour elimination) inequalities would require that this sum be less than only $9 - \lceil 9/5 \rceil = 7$. So the clique cluster inequality is stronger when applied to this subgraph. Figure 3.4d shows an example of a fractional solution that satisfies all of the packing inequalities, but not the clique cluster inequality. In general the clique cluster inequality is of the form

$$\sum_{j=1}^{t} X(E(C_j)) \leq \text{RHS},$$

the right-hand side coefficient RHS being defined by the structure of the clique cluster and the value of K (see Section 7).

One natural generalization of the clique cluster would be to replace the nucleus vertex by a more general graph. Figure 3.5 gives an example with $K = 3$ in which the nucleus contains two vertices p and q, each of the cliques in the clique cluster has four vertices, and the cliques have the nucleus as their common intersection. As shown in Figure 3.5b–3.5d, the number of edges in the support graph of any feasible solution can be at most five. But also note that if we weight the nucleus edge pq with a weight of two and every other edge with a weight of one, then the total weight of the edges from the clique cluster in any feasible solution can be no more than five. The fractional solution shown in Figure 3.5e violates this inequality, even though it satisfies all of the packing inequalities as well as all of the clique-cluster inequalities with a single-vertex nucleus. If we let C_1, C_2, C_3 denote the three 4-vertex petal cliques (including the nucleus nodes p and q) shown in Figure 3.5 let $C = C_1 \cup C_2 \cup C_3 - N$, and let $N = \{p, q\}$ denote the nucleus, then this extended clique cluster inequality becomes

$$X(E(C_1 - N)) + X(E(C_2 - N)) + X(E(C_3 - N)) + X(E(N, C)) + 2X(E(N)) \leq 5.$$

In general, if $C = C_1 \cup C_2 \cup \cdots \cup C_t - N$, the extended clique cluster inequality has the form

$$\sum_{j=1}^{t} X(E(C_j - N)) + X(E(N, C)) + bX(E(N)) \leq \text{RHS}. \tag{3.1}$$

The coefficient $b \leq K$ and the right-hand side RHS are again determined by the structure of the clique cluster and the capacity K (see Section 7).

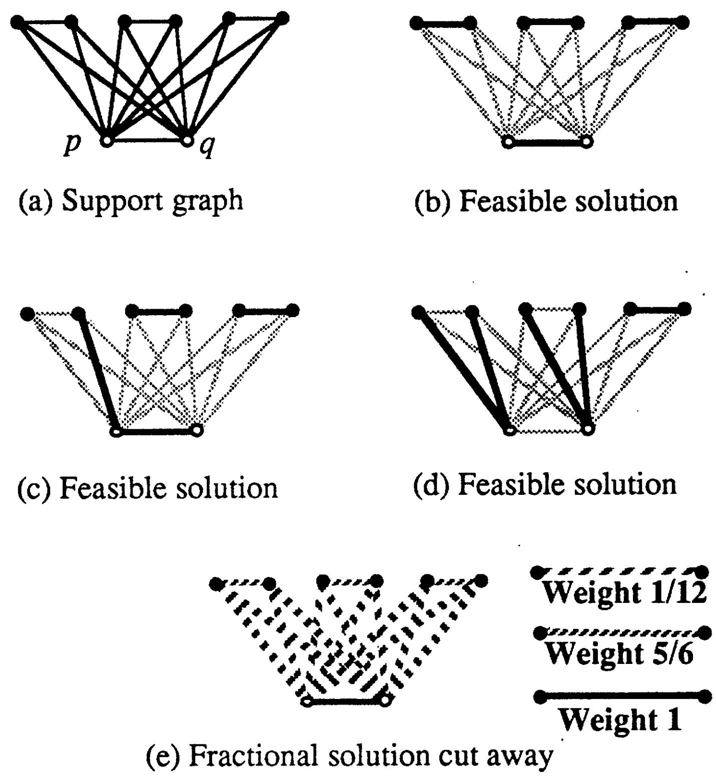


FIGURE 3.5: A generalization of a Clique Cluster for $K = 3$

Notice that for this general form, we can view the ladybugs as special cases of clique clusters for which the clique cluster nucleus is the ladybug head nucleus, and the petals are the nucleus plus a single node and the nucleus plus the body. Viewed as a clique cluster, the ladybug has a more complicated coefficient structure than that given by 3.1. At the end of Section 7, we discuss this very general structure, which can be viewed as subsuming all of our inequalities, and we indicate the combinatorial problems associated with determining what coefficients are needed to construct facets.

4 Packing or Generalized Subtour Elimination Constraints

We begin with a class of valid inequalities that appears in the integer programming formulation of the polytopes PP, F, and T.

Proposition 4.1 *Let $N \subseteq V$ with either*

(i) $|N| \bmod K \neq 0$, $K \geq 2$,

(ii) $N = V$, $K \geq 3$.

Then the packing inequality

$$X(E(N)) \leq |N| - \left\lceil \frac{|N|}{K} \right\rceil \tag{4.1}$$

defines a facet of the polytopes PP, F, and T.

Proof (Validity): A simple packing argument establishes the validity of (4.1). Since the capacity constraints imply that the nodes of N must be partitioned into at least $\lceil |N|/K \rceil$ components, any solution of PP, F, or T contains at most $|N| - \lceil |N|/K \rceil$ edges of $E(N)$.

(Facet): We shall prove that (4.1) is a facet of PP, and by Lemma 2.7 and $PP \subseteq F$, we shall have proved that (4.1) is a facet for all three polytopes. Let $|N| = aK + b$, $1 \leq b \leq K - 1$, or $b = K$ and $N = V$. We refer to the generic solutions A, B, and C in Figure 4.1, each of which contains a paths with K nodes apiece and one path with b nodes. Since any edge ij with $i, j \notin N$ can be in or out of a solution satisfying (4.1) at equality, by Lemma 2.4 $\alpha_{ij} = 0$ for these edges.

Next for problems that satisfy condition (i), we consider edge uv in the solution in Figure 4.1A. Since edge uv is an in-and-out edge, $\alpha_{uv} = 0$; and since u and v are arbitrary nodes of N and $V \backslash N$, respectively, we have $\alpha_{ij} = 0$ for all edges ij in the cutset $E(N, V \backslash N)$. (If $V \backslash N$ is empty, as in condition (ii), then this argument is unnecessary.)

It remains to show that all coefficients of edges in $E(N)$ are equal. For $K \geq 3$, by the triple-exchange argument (Lemma 2.3) this result is immediate. For condition (i) and $K = 2$, the simple-exchange argument (Lemma 2.1) applied to Figure 4.1B implies that $\alpha_{uv} = \alpha_{uw}$, and applied to Figure 4.1C implies that $\alpha_{uw} = \alpha_{wj}$, so that $\alpha_{uv} = \alpha_{wj}$. Since u, v, w, and j were chosen arbitrarily, we have the result. ∎

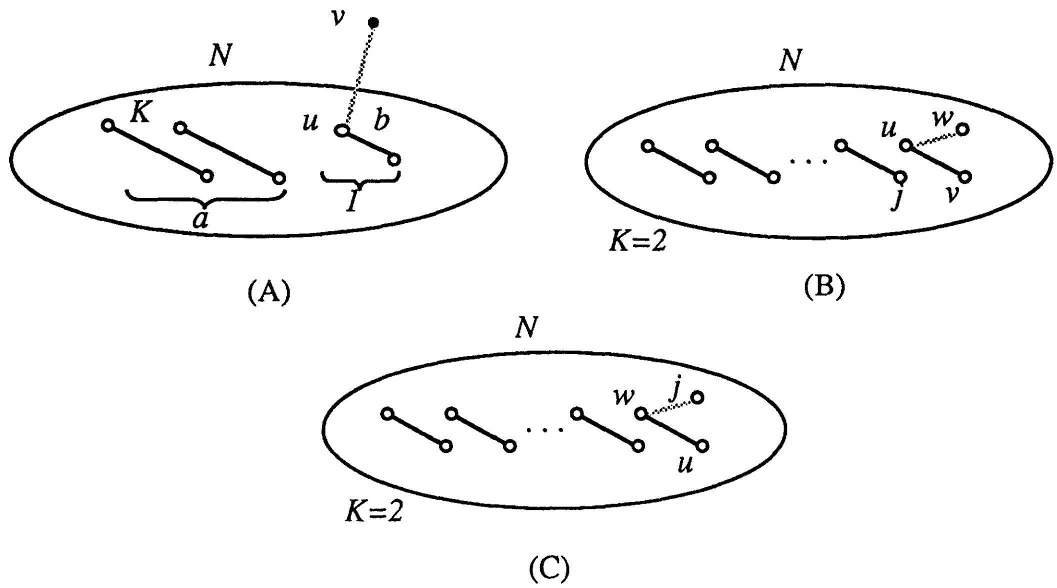


FIGURE 4.1: Generic solutions for the packing facets

A set of inequalities closely related to the packing inequalities are the tree inequalities (2.3) for T. By arguments very similar to those just given, we can prove the following (see Hall [1989]):

Proposition 4.2 *Let $N \subseteq V \cup \{0\}$, with node $0 \in N$ and $N \neq V$. Then the tree inequality*

$$X(E(N)) \leq |N| - 1$$

is a facet of T.

We observe that these inequalities correspond to those for the unconstrained minimum spanning tree. Only those inequalities containing node 0 are facets, however, since otherwise the inequalities are dominated by the inequalities (4.1).

5 Multistar Inequalities

In this section we describe those multistars that define facets of the polytopes F, T and PP. First, we introduce two different types of multistars that define facets of F and T:

(i) "large" multistars with vertex set $N \cup S = V$ (see Proposition 5.1), and

(ii) "small" multistars whose vertex set $N \cup S$ might be smaller than V, but that are facets only when we impose some tight conditions relating $|N|$ and $|S|$ (see Proposition 5.2).

To prove that both types of multistars define facets, we identify some generic solutions and use the exchange arguments introduced in Section 2.

Because the polytope PP is defined by a more tightly constrained version of the integer programming formulation defining the polytope F, since it contains the degree-2 constraints, we might expect that some of the multistar inequalities that are facets for F become lower dimensional faces for PP. Therefore, we might expect the pool of facet-inducing multistars for PP to be smaller than those for F, as reflected by the extra requirements that need to be imposed on the problem data. Propositions 5.3 and 5.4 identify the conditions that we require in the case of the polytope PP for the large and small multistars described in Propositions 5.1 and 5.2. The proofs are somewhat different because the generic solutions we use must satisfy the degree-2 constraints. In addition, the introduction of the degree-2 constraints produces a new set of multistar inequalities which are facets of PP and that we describe in Proposition 5.5. Like the large multistars, they have $N \cup S = V$ but the described resulting inequalities have smaller coefficients. And unlike the multistars described in Propositions 5.3 and 5.4, most of those described in Proposition 5.5 do not yield valid inequalities for the polytope F. The exceptional case is described at the end of the section.

For each of the inequalities we have considered, we have attempted to give the best possible set of conditions under which they define facets. Relaxing the conditions imposed by our hypotheses would either cause the inequalities to become invalid, or reduce the dimension of the faces they define so that they are no longer are facets.

5.1 Forest and Tree Polytopes

Proposition 5.1 *(Large multistars) Let $K \geq 3$ and let M be a multistar with a nucleus set N and a satellite set S satisfying either of the following two sets of conditions:*

(i) $|N| = aK + b, \quad 1 \leq b \leq K-1, \quad a, b$ *nonnegative integers,*
$S = V - N$ *and* $|S| \geq 2K - b$.

(ii) $|N| = aK, \quad a$ *a positive integer,*
$S = V - N$ *and* $|S| \geq K + 1$.

Then the multistar inequality

$$KX(E(N)) + X(E(N,S)) \leq (K-1)|N| \tag{5.1}$$

defines a facet of the polytopes F and T.

Proof (Validity): We give a simple packing argument to establish the validity of (5.1). Let $\mathbf{x}$ be any solution maximizing the left-hand side of (5.1) over F, and which has 0 coefficients corresponding to the edges not in the support graph of the inequality. Note that this solution corresponds in a natural way to a bin packing of all the vertices in N and a few of the vertices in S. Suppose that the bin packing has c bins each with a capacity of at most K vertices, that every bin contains at least one vertex in N, and that vertices in different bins are not connected directly by an edge in the solution $\mathbf{x}$. Because the edges in $E(N)$ have a larger coefficient in (5.1) than do the edges in $E(N,S)$, all the nucleus vertices in the same bin constitute a connected subgraph. Since there are c of those components (i.e. bins) with a total capacity of cK vertices, the bins contain at most $cK - |N|$ spaces for the vertices in S. We tie these vertices to the nucleus using edges in $E(N,S)$, each with a coefficient of 1 in (5.1). The left-hand side of (5.1) becomes

$$K(|N| - c) + (cK - |N|) = (K-1)|N|.$$

(Facet): We will refer to the generic solutions A to E in Figure 5.1.

The vertices inside each of the ovals are part of the nucleus. The number of free satellite vertices in the solutions A, B and C corresponds to the lower bound on S imposed by the conditions in (i) and (ii). As

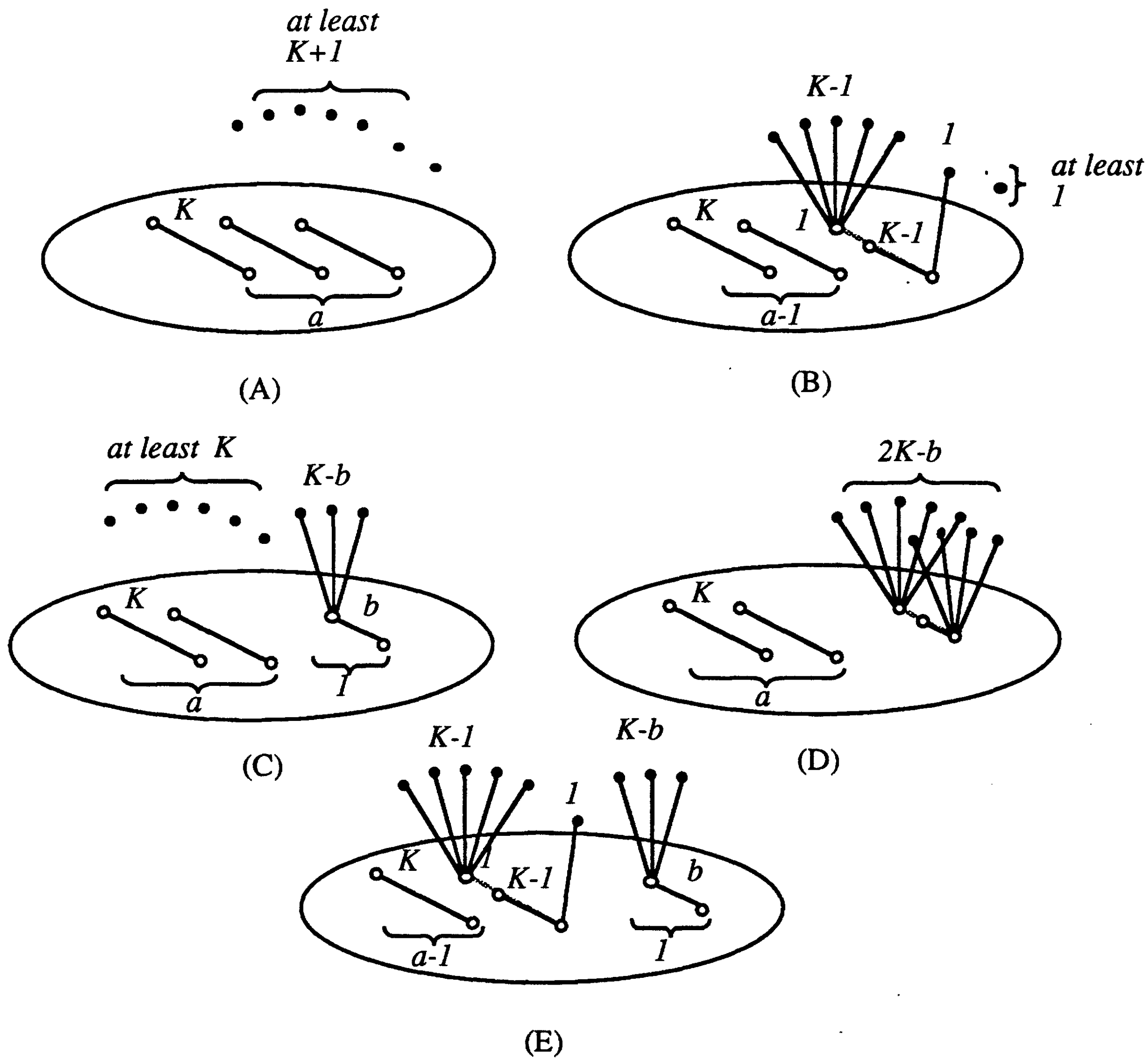


FIGURE 5.1: Generic solutions for the multistar facet (5.1)

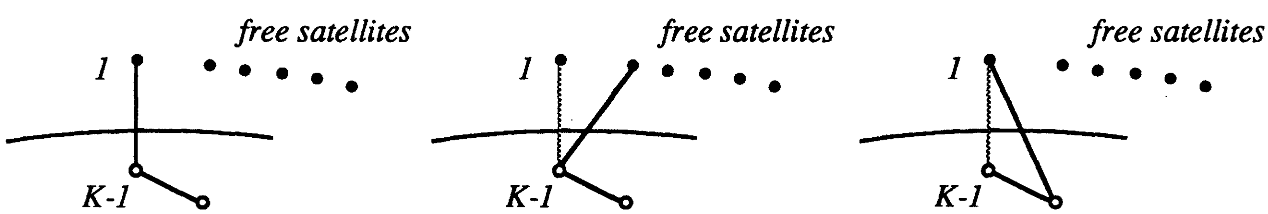


FIGURE 5.2: Exchanges

mentioned in Section 2, we assume that $\boldsymbol{\alpha}^T\mathbf{x} \leq \alpha_0$ defines a facet of F containing all the points that satisfy (5.1) as an equality. We consider three different values for the cardinality of N, namely $|N| = aK, |N| = aK + 1$ and $|N| = aK + b$ for $2 \leq b \leq K - 1$.

For $|N| = aK$ we consider only the generic solutions A and B. Since we can regard any edge with both endpoints in S as an in-and-out edge in the generic solution A, $\alpha_{ij} = 0$ for $i, j \in S$. Now consider the component of the generic solution B containing $K - 1$ vertices from N and one vertex from S. Exchange the edge in solution B connecting the single satellite vertex to the nucleus vertex with any other edge connecting a free satellite to any nucleus vertex in the same component, as in Figure 5.2 because we are free to choose the satellite vertex and the vertex in the nucleus arbitrarily, the simple-exchange argument (Lemma 2.1) implies that $\alpha_{ij} = \lambda$ for $i \in N, j \in S$ for some constant λ. By subtracting the equations corresponding to the generic solutions A and B, we immediately have $\alpha_{ij} = K\lambda$ for $i, j \in N$ and so

$$\boldsymbol{\alpha}^T\mathbf{x} = \lambda(KX(E(N)) + X(E(N, S))).$$

This result shows that the multistar inequality (5.1) is a facet when $|N| = aK$.

For the remaining two cases we use the other three generic solutions. We can consider any edge joining two of the (at least K) free satellite vertices in the generic solution C as an in-and-out edge, and so $\alpha_{ij} = 0$ for both $i, j \in S$. Since $K - b \geq 1$ the solution C contains at least one edge $e^\star$ from $E(N, S)$. We can exchange this edge for any edge connecting the same nucleus vertex with any one of the free satellite vertices. The simple-exchange argument proves that $\alpha_{ij} = \lambda_i$ for $i \in N$ and $j \in S$, i.e., the coefficient is independent of j.

Now, we distinguish between the two cases. If $|N| = aK + b$ for $2 \leq b \leq K - 1$, then we can exchange the edge $e^\star$ for an edge joining a different nucleus vertex in the same component (which exists since $b \geq 2$) and the same satellite vertex. The simple-exchange argument shows that $\lambda_i = \lambda$ for $i \in N$. By subtracting the equations corresponding to the generic solutions C and D (note that the generic solution D does not exist if $b = 1$), we see that $\alpha_{ij} = K\lambda$ for $i, j \in N$ so that $\boldsymbol{\alpha}^T\mathbf{x} = \lambda(KX(E(N)) + X(E(N, S)))$ and the multistar inequality (5.1) is a facet when $|N| = aK + b$ for $2 \leq b \leq K - 1$.

If $|N| = aK + 1$ and $a = 0$, our prior results already show that the inequality $\boldsymbol{\alpha}^T\mathbf{x} \leq \alpha_0$ is a scalar multiple of (5.1). If $|N| = aK + 1$ and $a \geq 1$, we must use the generic solution E to show that $\lambda_i = \lambda$ for $i \in N$. We apply a simple-exchange argument using the component containing $K - 1$ nucleus vertices and one satellite vertex, connecting this satellite vertex to a different nucleus vertex in the same component, as in Figure 5.2. This exchange shows that $\lambda_i = \lambda$ for $i \in N$. By subtracting the generic solutions C and E, we see that $\alpha_{ij} = K\lambda$ for $i, j \in N$ and consequently that the inequality $\boldsymbol{\alpha}^T \mathbf{x} \leq \alpha_0$ is again a scalar multiple of (5.1). The result for the polytope T follows from Lemma ??. ■

Proposition 5.2 *(Small multistars) Let $K \geq 3$ and let M be a multistar with a nucleus set N and a satellite set S satisfying both of the following conditions:*

(i) $|N| + |S| = aK + b, \quad 1 \leq a, \qquad 2 \leq b \leq K-1, \quad a, b$ *nonnegative integers,*

(ii) $b < |S| < (K-1)|N|$.

Then the multistar inequality

$$bX(E(N)) + X(E(N,S)) \leq b(|N| - a - 1) + |S| \tag{5.2}$$

defines a facet of the polytopes F and T.

Proof (Validity): Let $\mathbf{x}$ be any solution maximizing the left-hand side of (5.2) over F, and which has zero coefficients corresponding to the edges not in the support graph of the inequality. Then, the support graph of $\mathbf{x}$ corresponds to a forest. Because of the special structure of that support graph, every tree in this forest has either vertices in N and S, or vertices in N only, or is a single vertex in S. Since the inequality coefficients for the edges in $E(N)$ in (5.2) are larger than those for the edges in $E(N,S)$, all the nucleus vertices in the same component must be joined together, i.e., if we remove the satellite vertices, the nucleus vertices in the same component of the forest still induce a connected subgraph. Let c_N be the number of trees containing a nucleus vertex, and let c_S be the number of trees that are isolated satellites. Then, the left-hand side of (5.2) becomes

$$b(|N| - c_N) + |S| - c_S.$$

Maximizing this expression is equivalent to minimizing $bc_N + c_S$. Now,

$$bc_N + c_S \geq b\left\lceil \frac{|N| + |S| - c_S}{K} \right\rceil + c_S = b\left\lceil \frac{aK + b - c_S}{K} \right\rceil + c_S = ab + b\left\lceil \frac{b - c_S}{K} \right\rceil + c_S.$$

Consider the last term as a function of c_S. Note that it has value $b(a+1)$ for $c_S = 0$, then it increases as c_S increases, until $c_S = b$ when it jumps downward and becomes $b(a+1)$ again; the next jump downward

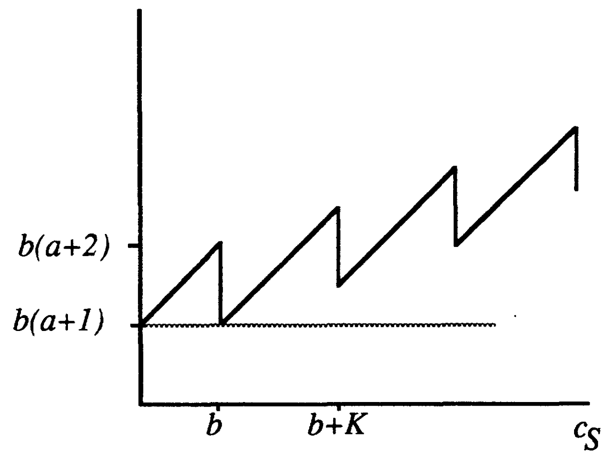


FIGURE 5.3: Function $ab + b\left\lceil \frac{b-c_S}{K} \right\rceil + c_S$

of b units occurs after c_S has increased by K additional units, and the value of this function continues to increase by K units and then fall by b units (see Figure 5.3). Consequently the function has only two minima for nonnegative values of c_S. Therefore, the left-hand side in (5.2) is at most $b|N| + |S| - b(a+1)$, proving the validity of the inequality.

(Facet): We break the proof into three different subcases. We obtain these cases by looking at how $|S|$ decomposes modulus $(K-1)$, and then considering the possible values for $|N|$ that are consistent with the restrictions imposed by (i) and (ii).

(CASE 1) $\quad |S| = t, \quad b < t \leq K-1,$

$\qquad |N| = (K-t) + b + uK;$

(CASE 2) $\quad |S| = s(K-1) + t, \quad 1 \leq s, \quad 1 \leq t \leq K-2,$

$\qquad |N| = s + (b-t), \quad 1 \leq b-t \leq K-2;$

(CASE 3) $\quad |S| = s(K-1) + t, \quad 1 \leq s, \quad 1 \leq t \leq K-1,$

$\qquad |N| = s + (K-t) + b + uK.$

In the first case $a = u+1$, in the second $a = s$, and in the third $a = s+u+1$. In the remainder of the proof, we refer to the generic solutions A1, B1, A2, B2, A3, and B3 shown in Figure 5.4. The numerical reference corresponds to one of the three cases we have identified above. We eliminate the numerical reference (i.e., refer to A instead of A1, A2 or A3) when the argument applies to all three cases.

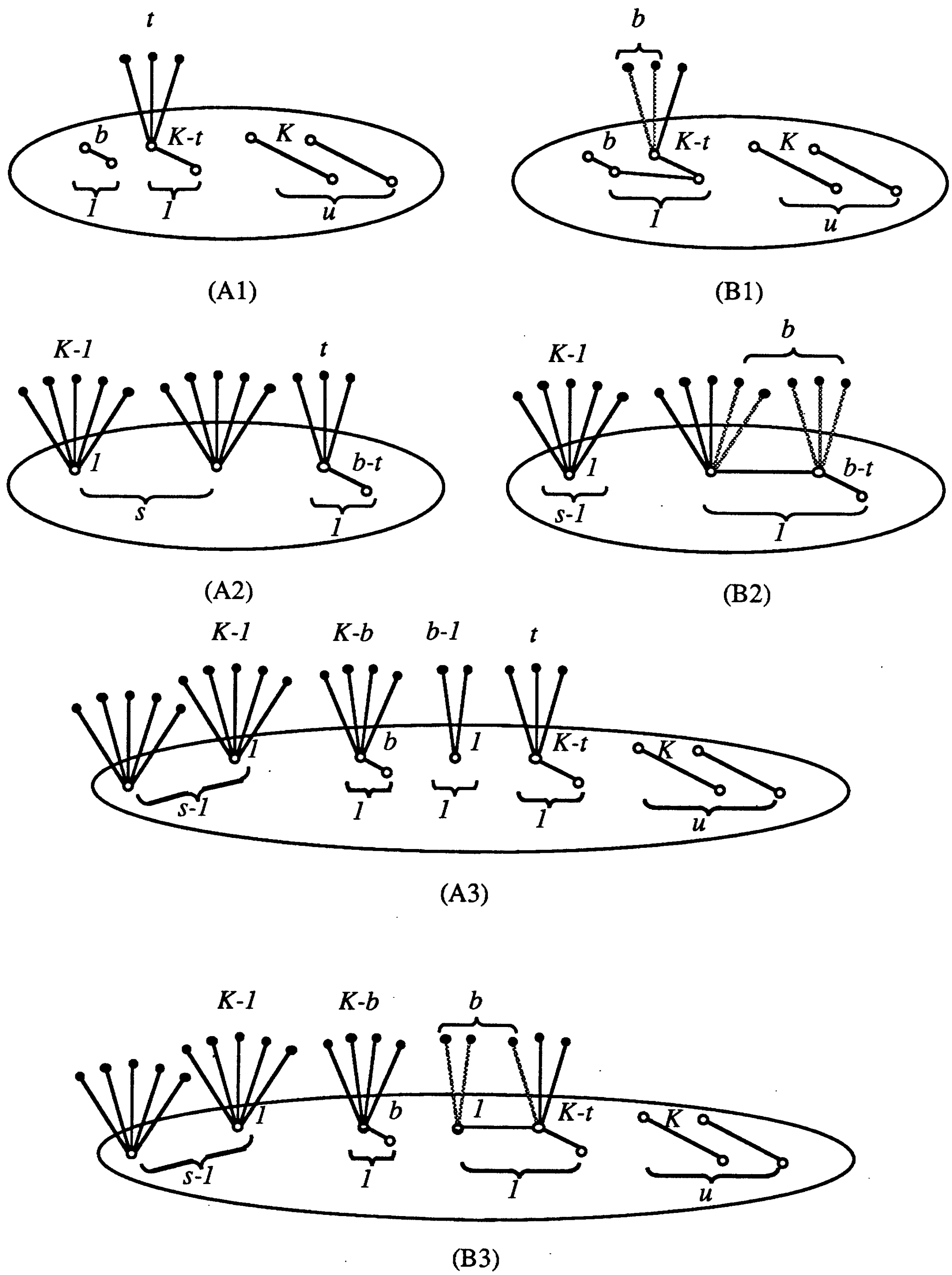


FIGURE 5.4: Generic solutions for the multistar facet (5.2)

Let $\boldsymbol{\alpha}^T\mathbf{x} \leq \alpha_0$ be a facet of F containing all the points satisfying the multistar inequality (5.2) as an equality. Since the generic solution B has $b \geq 2$ free satellite vertices, we can treat an edge between two of those satellites, or an edge between one of them and a vertex not in $N \cup S$, as an in-and-out edge. Therefore $\alpha_{ij} = 0$ for $i, j \in S$ and $i \in S, j \notin N \cup S$. Also, note that the generic solution B contains one component with at least one satellite vertex and at least two nucleus vertices; in B1, that component has $K - t + b \geq 2$ vertices in N and $t - b > 0$ vertices in S; in B2, because of the case 2 conditions, it has $b - t + 1 \geq 2$ vertices in N and $K - 1 - (b - t) \geq 1$ in S; in B3, it has $b \geq 2$ vertices in N and $K - b \geq 1$ vertices in S. Therefore, we can replace an edge between $i \in N$ and $j \in S$ in that component by an edge between i and any free satellite vertex, or by an edge between j and any other nucleus vertex on the same component. By the simple-exchange argument, we conclude that $\alpha_{ij} = \lambda$ for $i \in N, j \in S$ for some constant λ. Next, note that the generic solution A contains one component with b vertices, and at least one of them is in the nucleus; the edge between that nucleus vertex an any vertex not in $N \cup S$ is an in-and-out edge because $b \leq K - 1$, and so we can connect another vertex to that component. Since the nucleus vertex is arbitrary, we have that $\alpha_{ij} = 0$ for $i \in N, j \notin N \cup S$. And finally, subtracting the equations corresponding to the generic solutions A and B, we obtain $\alpha_{ij} = b\lambda$ for $i, j \in N$, which allows us to conclude

$$\boldsymbol{\alpha}^T\mathbf{x} = \lambda(bX(E(N)) + X(E(N,S))),$$

which implies that the multistar inequality (5.2) defines a facet of F, and also of T. ■

5.2 Path-Partitioning Polytope

We consider the polytope PP. Since $PP \subseteq F$, we expect to be able to again use the multistar inequalities described in the last subsection. In translating our previous results to this problem setting, however, we would like to stress three different points. First, although our results for the path-partitioning polytope are similar to those for the tree and forest polytopes, we do need to impose stronger conditions to guarantee that the multistar inequalities are still facet-inducing. Second, we obtain the validity of the multistar inequalities for the path-partitioning polytope for free, i.e., they inherit their validity from their validity for the larger polytope F. Third, we need to rewrite the proofs for establishing that the inequalities are facets because we must base those proofs on generic solutions that satisfy the degree-2 constraints, which is not the case in the generic solutions in Figures 5.1 and 5.4. However, the same exchange arguments apply.

In addition, because of the addition of the degree-2 constraints, we are able to define a new multistar inequality for the path-partitioning polytope (see Proposition 5.5). This inequality defines a facet for the polytope PP, but is not a valid inequality for the polytopes F and T.

Proposition 5.3 (*Large multistars*) *Let $K \geq 3$ and let M be a multistar with a nucleus set N and a satellite set S satisfying either of the following two sets of conditions:*

(i) $|N| = aK + b, \quad 1 \leq b \leq K-1, \quad a, b$ *nonnegative integers,*

$S = V - N$ *and* $|S| \geq 2K - b$,

$a \geq (2K - b)/2 - 2$.

(ii) $|N| = aK, \quad a$ *a positive integer,*

$S = V - N$ *and* $|S| \geq K + 1$.

$a \geq K/2 - 1$, *except for* $K = 4$. *In this case* $a \geq 2$.

Then the multistar inequality

$$KX(E(N)) + X(E(N,S)) \leq (K-1)|N| \tag{5.3}$$

defines a facet of the polytope PP.

Proof (Validity): The inequality (5.3) is valid for the polytope F, hence it is also valid for $PP \subseteq F$.

(Facet): As in Proposition 5.1, the proof relies on some manipulations of certain generic solutions. In this case we must build paths and not trees, so the constructions become somewhat trickier. The basic generic solution has s paths with two satellite vertices and $K-2$ nucleus vertices, t paths with one satellite vertex and $K-1$ nucleus vertices, u paths with K nucleus vertices, and some free satellite vertices which are part of no path (see Figure 5.5). We will choose the values of s, t, u depending on the case being considerated (see Table I). In all cases, we need to check that the chosen values are well defined in the sense that they actually are nonnegative numbers, and sometimes are even strictly positive. Table II indicates some properties of the generic solutions defined in Table I. (Note that by definition t equals either 0 or 1, except for the case $K = 4$ for the solution D.)

TABLE I Parameter choices for the generic solutions

Generic Solution	s	t	u
A	$\lfloor \frac{K-1-b}{2} \rfloor$	$1 + \lceil \frac{K-1-b}{2} \rceil - \lfloor \frac{K-1-b}{2} \rfloor$	$a - \lceil \frac{K-1-b}{2} \rceil$
B	$\lfloor \frac{2K-b}{2} \rfloor$	$\lceil \frac{2K-b}{2} \rceil - \lfloor \frac{2K-b}{2} \rfloor$	$a + 2 - \lceil \frac{2K-b}{2} \rceil$
C	0	0	a
D ($K \geq 5$)	$\lfloor \frac{K}{2} \rfloor$	$\lceil \frac{K}{2} \rceil - \lfloor \frac{K}{2} \rfloor$	$a + 1 - \lceil \frac{K}{2} \rceil$
D ($K = 4$)	1	2	$a - 2$
D ($K = 3$)	1	1	$a - 1$

TABLE II Characteristics of the generic solutions

Generic Solution	Number of paths	Vertices in N	Vertices in S	Free satellites
A	$a+1$	$aK+b$	$K-b$	at least K
B	$a+2$	$aK+b$	$2K-b$	at least 0
C	a	aK	0	at least K+1
D ($K \geq 5$)	$a+1$	aK	K	at least 1
D ($K = 4$)	$a+1$	$4a$	4	at least 1
D ($K = 3$)	$a+1$	$3a$	3	at least 1

Assume that $\boldsymbol{\alpha}^T\mathbf{x} \leq \alpha_0$ defines a facet of PP containing all the points that satisfy (5.3) as an equality. First, we consider the situation in (i) and the generic solutions A and B. Note that A is well defined. Since

$$a \geq \frac{2K-b}{2} - 2 = \frac{K-3+K-1-b}{2} \geq \frac{K-1-b}{2} \geq 0,$$

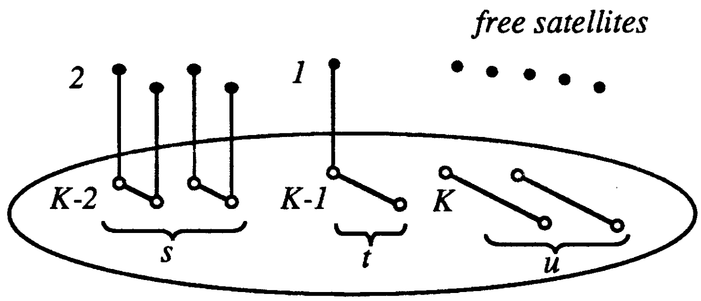


FIGURE 5.5: Generic solution for the multistar facet (5.3)

a is nonnegative; by definition $t \geq 1$; and since

$$2s + t \leq 1 = (K - 1 - b) = K - b,$$

the solution contains at least $K \geq 2$ free satellite vertices. Similarly, we can see that B is well defined. In A, we can consider any edge joining two of the free satellite vertices as an in-and-out edge, and use it to prove that $\alpha_{ij} = 0$ for $i, j \in S$. Since the solution A contains at least one path $\mathcal{P}$ with $K - 1 \geq 2$ vertices from N and 1 from S, we can make simple-exchanges like those in Figure 5.2. The first two of these exchanges permit us to conclude that α_{ij}, $(i \in N,\ j \in S)$ is independent of j, and the third exchange permits us to conclude that it is also independent of i; thus for some constant λ, $\alpha_{ij} = \lambda$ for $i \in N$ and $j \in S$.

If in addition $K \geq 4$, then the path $\mathcal{P}$ contains at least 3 nucleus vertices and we can apply the triple-exchange argument to show that for some constant μ, $\alpha_{ij} = \mu$ for all $i, j \in N$. By subtracting the equations corresponding to the generic solutions A and B, we obtain $\mu = K\lambda$ and so the inequality $\boldsymbol{\alpha}^T\mathbf{x} \leq \alpha_0$ is a scalar multiple of the multistar inequality (5.3).

If $K = 3$, we cannot apply the triple-exchange argument because the path $\mathcal{P}$ contains only 2 nucleus vertices. However, we can break the path as shown in Figure 5.6. Subtracting the equations corresponding to the generic solution A and the solution obtained after breaking up the path shows that $\alpha_{ij} = 3\lambda$ for $i, j \in N$. Therefore, the inequality $\boldsymbol{\alpha}^T\mathbf{x} \leq \alpha_0$ is again a scalar multiple of the multistar inequality (5.3).

In case (ii), we use the generic solutions C and D. Note that the generic solution D is well defined for $K \geq 5$ or $K = 3$ because $a \geq \frac{K}{2} - 1$, and for $K = 4$ because $a \geq 2$. Applying the triple-exchange argument to the generic solution C shows that $\alpha_{ij} = \mu$ for all $i, j \in N$ and some constant μ. Moreover, in the same solution C we can choose any edge connecting two free satellite vertices as an in-and-out edge, implying

Proposition 5.4 (*Small multistars*) *Let $K \geq 3$ and let M be a multistar with a nucleus set N and a satellite set S satisfying both of the following conditions:*

(i) $|N| + |S| = aK + b, \quad 1 \leq a, \qquad 2 \leq b \leq K-1, \quad a, b$ *nonnegative integers,*

(ii) $b < |S|, \qquad \left\lfloor \frac{|S|}{2} \right\rfloor \leq a.$

Then the multistar inequality

$$bX(E(N)) + X(E(N,S)) \leq b(|N| - a - 1) + |S| \tag{5.4}$$

defines a facet of the polytope PP.

Proof (Validity): The validity of (5.4) for *PP* follows from its validity for the polytope F.

(Facet): We construct two generic solutions in the following way. For the generic solution A, we partition the set of vertices into three groups:

- $\left\lfloor \frac{|S|-1-b}{2} \right\rfloor$ disjoint paths containing $K-2$ vertices from N and two from S.
- $1 + \left\lceil \frac{|S|-1-b}{2} \right\rceil - \left\lfloor \frac{|S|-1-b}{2} \right\rfloor \geq 1$ disjoint paths containing $K-1$ vertices from N and 1 from S.
- $a - \left\lceil \frac{|S|-1-b}{2} \right\rceil - 1$ disjoint paths containing K vertices from N and none from S.

The resulting partition has a paths and uses all the vertices in N and $|S| - b$ of the vertices in S. The generic solution is well defined since

$$|S| > b \Rightarrow \left\lfloor \frac{|S| - 1 - b}{2} \right\rfloor \geq 0$$

and

$$2\left\lfloor \frac{|S|}{2} \right\rfloor + 1 \geq |S|, \quad b \geq 2 \Rightarrow 2\left\lfloor \frac{|S|}{2} \right\rfloor - 2 \geq |S| - b - 1 \Rightarrow a - 1 \geq \left\lfloor \frac{|S|}{2} \right\rfloor - 1 \geq \left\lceil \frac{|S| - 1 - b}{2} \right\rceil,$$

so the number of paths in each group of the partition is nonnegative. Note that at least one path contains $K-1$ vertices from N and one from S.

For the generic solution B, we construct another partition with three types of components:

- $\left\lfloor \frac{|S|-1-b}{2} \right\rfloor$ disjoint paths containing $K-2$ vertices from N and two from S.

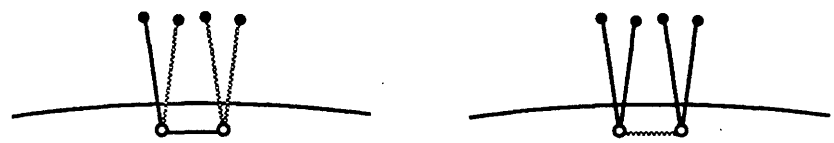


FIGURE 5.6: Breaking the path for the case $K = 3$

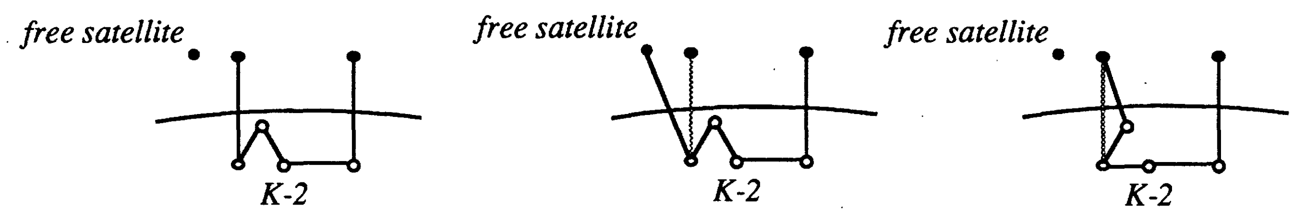


FIGURE 5.7: Exchanges

that$\alpha_{ij} = 0$ for all $i, j \in S$.

If $K \geq 5$, the solution D contains at least one path with $K - 2$ vertices from N and two vertices from S on which we can perform exchanges like those in Figure 5.7. Making the first two of these exchanges shows obtain that α_{ij} $(i \in N, j \in S)$ is independent of j; making the first and third exchange shows that α_{ij} is independent of i, so that necessarily $\alpha_{ij} = \lambda$ for all $i \in N, j \in S$. Finally, subtracting the equations for the generic solutions C and D, we obtain $\mu = K\lambda$ and conclude the proof for this case.

If $K = 4$ or $K = 3$, we note that the solution D contains at least one path with 1 satellite vertex and $K - 1$ nucleus vertices, and at least one free satellite. On this path, we can perform simple-exchanges like those in Figure 5.2 to prove that $\alpha_{ij} = \lambda$ for $i \in N, j \in S$. Again, subtracting the generic solutions C and D yields $\mu = K\lambda$, completing the proof. ∎

In the same way that we tightened the conditions stated in Proposition 5.1 to obtain facets of the polytope *PP*, in Proposition 5.3 we can also strengthen the conditions to ensure that the small multistars are facets for the path-partitioning polytope.

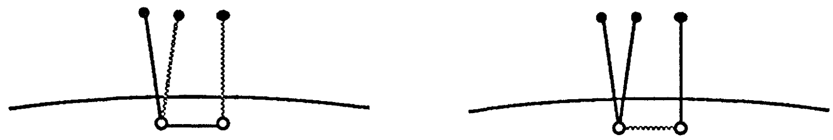


FIGURE 5.8: Breaking up the path when $K = 3$ and the solution contains two free satellite vertices

- $a - \left\lfloor \frac{|S|-1-b}{2} \right\rfloor$ disjoint paths containing K vertices from N and none from S.
- One path containing the remaining b vertices, i.e, it contains $b-1$ vertices from N and one from S if $|S|$ is odd, or b vertices from N if $|S|$ is even.

These $a+1$ paths use all the vertices in N and in S. We can use the path with $K-1$ nucleus vertices and 1 satellite vertex in the generic solution A and the exchanges shown in Figure 5.2 to prove that $\alpha_{ij} = \lambda$ for all $i \in N, j \in S$. Also, since the solution A contains $b \geq 2$ free satellite vertices, we can use the in-and-out edge argument on any edge connecting them to show that $\alpha_{ij} = 0$ for all $i, j \in S$, and on any edge connecting one of those free satellite vertices and any vertex not in the multistar to show that $\alpha_{ij} = 0$ for all $i \in S, j \notin N \cup S$. We consider the path with b elements in the generic solution B. Since that path has at most one satellite vertex, at least one of its endpoints must be a nucleus vertex, and we can connect that vertex to a vertex outside the support graph of the multistar without violating any of the degree-2 constraints. The in-and-out edge argument shows that $\alpha_{ij} = 0$ for all $i \in N, j \notin N \cup S$. To continue with the proof, we consider two cases, one with $K \geq 4$ and another with $K = 3$.

If $K \geq 4$, we can use the path with $K-1$ nucleus vertices in the generic solution A, together with the triple-exchange argument on the vertices in the nucleus, to prove that $\alpha_{ij} = \mu$ for all $i, j \in N$. Subtracting the equations for the generic solutions A and B yields $\mu = b\lambda$.

If $K = 3$, then $b = 2$ and the generic solution A contains only two free satellite vertices. Nevertheless, we can break up the path with two nucleus vertices and one satellite vertex as shown in Figure 5.8, to obtain the condition $\alpha_{ij} = b\lambda$ for all $i, j \in N$. ∎

Propositions 5.3 and 5.4 prescribe conditions under which the large and small multistars are facets for the path-partitioning polytope. For this polytope we also can describe another set of "composite" facets. Like the large multistars, the support graph for these inequalities contains all the nodes, but like the small multistars, the coefficients of the arcs connecting the nucleus nodes are smaller than the capacity K.

Proposition 5.5 (*Intermediate multistars*) *Let $K \geq 4$ and let M be a multistar with a nucleus set N and a satellite set S satisfying the following three conditions:*

(i) $|N| = s(K-2) + (b-2), \quad 3 \leq b \leq K-1, \quad s, b$ *nonnegative integers,*

(ii) $|S| = V - N$,

(iii) $|S| \geq 2s + 2 \geq b$, *but* $|S| \neq b$.

Then the multistar inequality

$$bX(E(N)) + X(E(N,S)) \leq b\,(|N| - (s+1)) + 2s + 2 \tag{5.5}$$

defines a facet of the polytope PP.

Proof (Validity): We give an recursive argument to show that

$$iX(E(N)) + X(E(N,S)) \leq i\,(|N| - (s+1)) + 2s + 2$$

for $i = 2, \ldots, b$. For $i = 2$

$$\begin{aligned} 2X(E(N)) + X(E(N,S)) &= \sum_{v \in N} X(star(v)) \\ &\leq 2|N| \\ &= 2\,(|N| - (s+1)) + 2s + 2. \end{aligned}$$

Now, if $i + 1 \leq b < K$, then

$$\begin{aligned} (i+1)X(E(N)) + X(E(N,S)) &= \frac{K-i-1}{K-i}(iX(E(N)) + X(E(N,S))) \\ &\quad + \frac{1}{K-i}(KX(E(N)) + X(E(N,S))) \\ &\leq \frac{K-i-1}{K-i}(i\,(|N| - (s+1)) + 2s + 2) \\ &\quad + \frac{1}{K-i}(K-1)|N| \\ &= (i+1)\,(|N| - (s+1)) + 2s + 2 + \frac{K-b}{K-i}. \end{aligned}$$

The inequality in these expressions follows from the recursion and the inequality

$$KX(E(N)) + X(E(N,S)) \leq (K-1)|N|$$

that is established in the Proposition 5.3. Since $(K-b)/(K-i) < 1$, after rounding down the right-hand side we obtain the desired result.

(Facet): We assume that $\boldsymbol{\alpha}^T\mathbf{x} \leq \alpha_0$ induces a facet of *PP* containing all the points that satisfy (5.5) as an equality. Although we consider two cases, we define a generic solution A that can be used in both cases. We construct this solution by partitioning the set of vertices into $s+1$ paths:

- s disjoint paths with $K-2$ vertices from N and two from S.
- 1 path with $b-2$ vertices from N and two from S.

The $s+1$ components of this partition use all the vertices in N and $2s+2$ vertices in S—S contains this many vertices because of condition (iii).

If $|S| > 2s+2 = b$, then necessarily $K \geq 5$ since b is even and $3 \leq b \leq K-1$. Also, we can construct another generic solution B by partitioning all the $s(K-2)+b-2 = sK$ vertices in N into a collection of s disjoint paths, each with K vertices. Using the triple-exchange argument on this solution shows that $\alpha_{ij} = \mu$ for all $i,j \in N$. Moreover, we can choose any edge between two satellite vertices as an in-and-out edge, implying that $\alpha_{ij} = 0$ for all $i,j \in S$. Since $|S| > 2s+2$, the generic solution A contains at least one free satellite vertex. By attaching this satellite vertex with any of the paths having $K-2$ nucleus vertices and using exchanges like those shown in Figure 5.7, we find that for some λ, $\alpha_{ij} = \lambda$ for all $i \in N, j \in S$. Subtracting the equations corresponding to the generic solutions A and B gives $\mu = b\lambda$, and concludes the proof for this case.

If $|S| \geq 2s+2 > b$ we are able to conclude only that $K \geq 4$. We produce a generic solution C by partitioning the set of vertices into three different types of paths:

- $s - 1 - \lceil \frac{b-3}{2} \rceil$ disjoint paths containing $K-2$ vertices from N and two from S.
- $1 + \lceil \frac{b-3}{2} \rceil - \lfloor \frac{b-3}{2} \rfloor \geq 1$ disjoint paths containing $K-1$ vertices from N and 1 from S.
- $\lfloor \frac{b-3}{2} \rfloor$ disjoint paths containing K vertices from N and none from S.

The resulting partition has s paths, uses all the vertices in N and $2s + 2 - b$ vertices in S, and so the solution has at least two free satellite vertices. Also,

$$2s + 2 > b \Leftrightarrow 2s + 1 \geq b \Leftrightarrow s - 1 \geq (b - 3)/2 \geq 0,$$

so the number of paths in each class is nonnegative and the generic solution is well defined. We can use the in-and-out edge argument on edges joining the free satellite vertices to show that $\alpha_{ij} = 0$ for all $i, j \in S$. Since at least one path contains $K - 1$ nucleus vertices and 1 satellite vertex, we can make the exchanges shown in Figure 5.2 and use the triple-exchange argument on the (at least 3) nucleus vertices on this path to establish the conditions $\alpha_{ij} = \lambda$ for all $i \in N, j \in S$, and $\alpha_{ij} = \mu$ for all $i, j \in N$. Subtracting the equations corresponding to the generic solutions A and C now shows that $\mu = b\lambda$. ∎

To conclude this section, we note that the intermediate multistars are not always valid for the forest and tree polytopes. Indeed, if $|S| > 2s + 2$ the generic solution A in the proof of Proposition 5.5 has at least one free satellite vertex. This vertex can be attached to any of the nucleus vertices on the path with $b \leq K - 1$ vertices. The resulting forest does not satisfy the degree-2 constraints but is feasible for the polytope F, and violates the right-hand side of (5.5) by 1 unit. If $|S| = 2s + 2$, the inequality defines a facet for the forest and tree polytopes. It satisfies all the conditions in Proposition 5.2 and reduces to one of the small multistars.

6 Ladybug and Partial Multistar inequalities

This section considers two additional sets of inequalities: the ladybugs and the partial multistars. We begin (in Proposition 6.1) by studying ladybug inequalities for the forest and tree polytopes F and T. We then introduce some additional conditions (in Proposition 6.2) to guarantee that the ladybug inequalities also become facets of the path-partitioning polytope PP. Next we introduce the partial multistar inequalities which define facets only for PP. Due to their structure, these inequalities look like a transition between a ladybug and a multistar inequality, and that is the reason for our choice of their name. We can regard them as a ladybug whose edges in the nucleus have the same coefficient as the edges in the body, or as a multistar whose nucleus vertices are not all connected to the satellite vertices. We thus refer to those vertices in the nucleus that are actually connected to the satellite set as "special vertices".

In Propositions 6.3 and 6.4 we introduce some partial multistars with one and two special vertices. Propositions 6.5 and 6.6 describe two different kinds of partial multistars with three special vertices. We close the section by describing (in Proposition 6.7) an inequality whose support graph looks like a ladybug with a head set having two vertices in N. Only one of these two vertices is connected to the satellite set S.

Although the partial multistars might appear to have a very specific, special structure, some empirical evidence of Araque (1989a,b,c) suggests that they can play an active role in the solution of vehicle routing problems by cutting-plane methods. In contrast, at this point we have no such empirical evidence for the ladybug constraints.

6.1 Ladybugs

Proposition 6.1 (*Ladybugs*) *Let $K \geq 3$ and let L be a ladybug with a body B and a head (N, S) satisfying both of the following conditions:*

(i) *The head set is a multistar with a nucleus set N and a satellite set S satisfying any one of the two conditions in Proposition 5.1, except that $S = V - N \cup B$ instead of $S = V - N$,*

(ii) $|B| = cK + d, \quad 2 \leq d \leq K - 1, \quad c, d$ *nonnegative integers.*

Then the ladybug inequality

$$dX(E(B)) + dX(E(B,N) + KX(E(N)) + X(E(N,S)) \leq d(|B| - c - 1) + (K-1)|N| \tag{6.1}$$

defines a facet of the polytopes F and T.

Proof (Validity): We give a packing argument to verify the validity of this inequality. Let $\mathbf{x}$ be the incidence vector of a forest maximizing the left-hand side of the ladybug inequality (6.1) over F; we can assume that this vector has 0 components corresponding to the edges not in the support graph of the inequality. The vector $\mathbf{x}$ induces in a natural way a packing of the vertices into bins of capacity K if we pack the vertices in the same tree in the same bin. If we disregard any tree that is just an isolated satellite vertex, then $\mathbf{x}$ corresponds to a bin packing of all the vertices in B, N and a few vertices in S (those connected to vertices in N), and every bin contains at least one vertex in $B \cup N$. Because the left-hand side coefficient (K) of the edges in $E(N)$ in (6.1) is larger than the other coefficients (d or 1), a solution that maximizes the value of the left-hand side has the property that all the nucleus vertices in the same bin are members of a connected subgraph. The bins can be classified into three different groups, namely:

1. bins containing only vertices from B and no vertex from S,
2. bins with vertices both from B and N, and possibly some vertices from S,
3. bins with vertices from N but none from B, and possibly some vertices in S.

If c_i denotes the number of bins in class i, and B_N denotes the total number of vertices in B distributed into bins of the second class, then the value obtained when we substitute $\mathbf{x}$ is into the left-hand side of (6.1) is at most

$$d(|B| - B_N - c_1) + dB_N + K(|N| - c_2 - c_3) + (K(c_2 + c_3) - |N| - B_N) = d(|B| - c_1) - B_N + (K-1)|N|.$$

The first term on the left accounts for the packing of the vertices in class 1, the second accounts for the packing of the B_N vertices from B into bins of class 2, the third accounts for the packing of the nucleus vertices in classes 2 and 3, and the last term accounts for the filling up of any empty space left in the bins of the last two classes with vertices from S. Maximizing this expression is equivalent to minimizing $dc_1 + B_N$. But

$$dc_1 + B_N \geq d\left\lceil \frac{|B| - B_N}{K} \right\rceil + B_N = cd + d\left\lceil \frac{d - B_N}{K} \right\rceil + B_N,$$

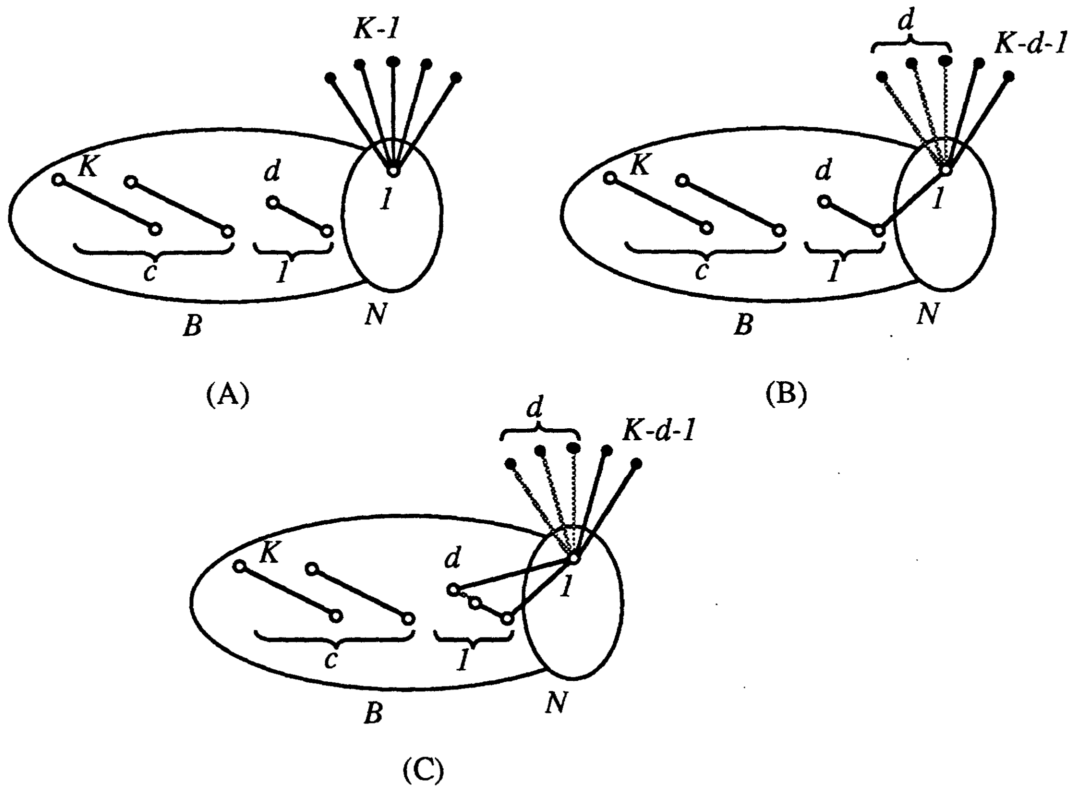


FIGURE 6.1: Generic solutions for the ladybug facet (6.1)

and we have already proved in Proposition 5.2 that the minimum value of the rightmost expression is $d(c+1)$, attained when $B_N = 0$ or $B_N = d$. Consequently, the left-hand side of the ladybug inequality (6.1) is at most $d(|B| - c - 1) + (K-1)|N|$, and the inequality is valid for the forest and tree polytopes F and T.

(Facet): Let $\boldsymbol{\alpha}^T\mathbf{x} \leq \alpha_0$ be a facet of F containing all the points in the face defined by the ladybug inequality (6.1). We can extend the generic solutions A to E shown in Figure 5.1 so that they become generic solutions of (6.1) by just adding some new components corresponding to the body B of the ladybug. These new components are a forest containing c paths each with K vertices, and one additional path with d vertices (see Figure 6.1). Since the head set defines a multistar which is a facet of the forest polytope on the vertex set $V - B$, we can conclude as in the proof of Proposition 5.1 that $\alpha_{ij} = 0$ for all $i, j \in S$, $\alpha_{ij} = \lambda$ for all $i \in N, j \in S$, and $\alpha_{ij} = K\lambda$ for all $i, j \in N$. Also, note that if we arrange the head set using a generic solution like 5.1A or 5.1C, then we can treat any edge between a free satellite vertex and the body component with d vertices as an in-and-out edge. Therefore, $\alpha_{ij} = 0$ for all $i \in B, j \in S$. Note also that if

$|B| \geq 3$, then we can apply the triple-exchange argument to show that for some constant μ, $\alpha_{ij} = \mu$ for all $i, j \in B$. If $|B| = 2$, this fact is trivial since $E(B)$has only one edge.

To continue, note that it is always possible to obtain a generic solution for the multistar (N, S) with one nucleus vertex connected to $K-1$ satellite vertices, as we can see from Figures 5.1B, D, and E. Since $2 \leq d \leq K-1$, we can perform exchanges like those in the Figure 6.1. The exchanges corresponding to configurations A and B prove that $\alpha_{ij} = d\lambda$ for all $i \in N, j \in B$, and those corresponding to configurations A and C prove that $\mu = d\lambda$. ∎

In the next proposition we show how to specialize the facet conditions for ladybugs for the smaller polytope *PP*. The validity of the inequalities for *PP* is a direct consequence of their validity for F. We need to revise the proof that the inequalities define facets of *PP*, though, to accomodate both the generic solutions satisfying the degree-2 constraints and the stronger set of hypotheses.

Proposition 6.2 (*Ladybugs*) *Let $K \geq 3$ and let L be a ladybug with a body B and a head (N, S) satisfying both of the following conditions:*

(i) *The head set is a multistar with nucleus N and satellites S satisfying any one of the two conditions in Proposition 5.3, except that $S = V - N \cup B$ instead of $S = V - N$,*

(ii) $|B| = cK + d, \quad 2 \leq d \leq K-1, \quad c, d$ *nonnegative integers.*

Then the ladybug inequality

$$dX(E(B)) + dX(E(B, N) + KX(E(N)) + X(E(N, S)) \leq d(|B| - c - 1) + (K-1)|N| \qquad (6.2)$$

defines a facet of the polytope PP.

Proof (Validity): Because the inequality is valid for F, it is valid for $PP \subseteq F$.

(Facet): The proof follows the same general structure of the proof of Proposition 6.1, with some minor changes to deal with the need to construct paths instead of trees to obtain valid generic solutions. First, as in the proof of Proposition 6.1 we extend the generic solutions A to D given in Proposition 5.3 (see Figure 5.5) by connecting the vertices in B with c paths each with K vertices, and one path with d vertices. We use the fact that the head set corresponds to a multistar, which is a facet on a polytope on the vertex set

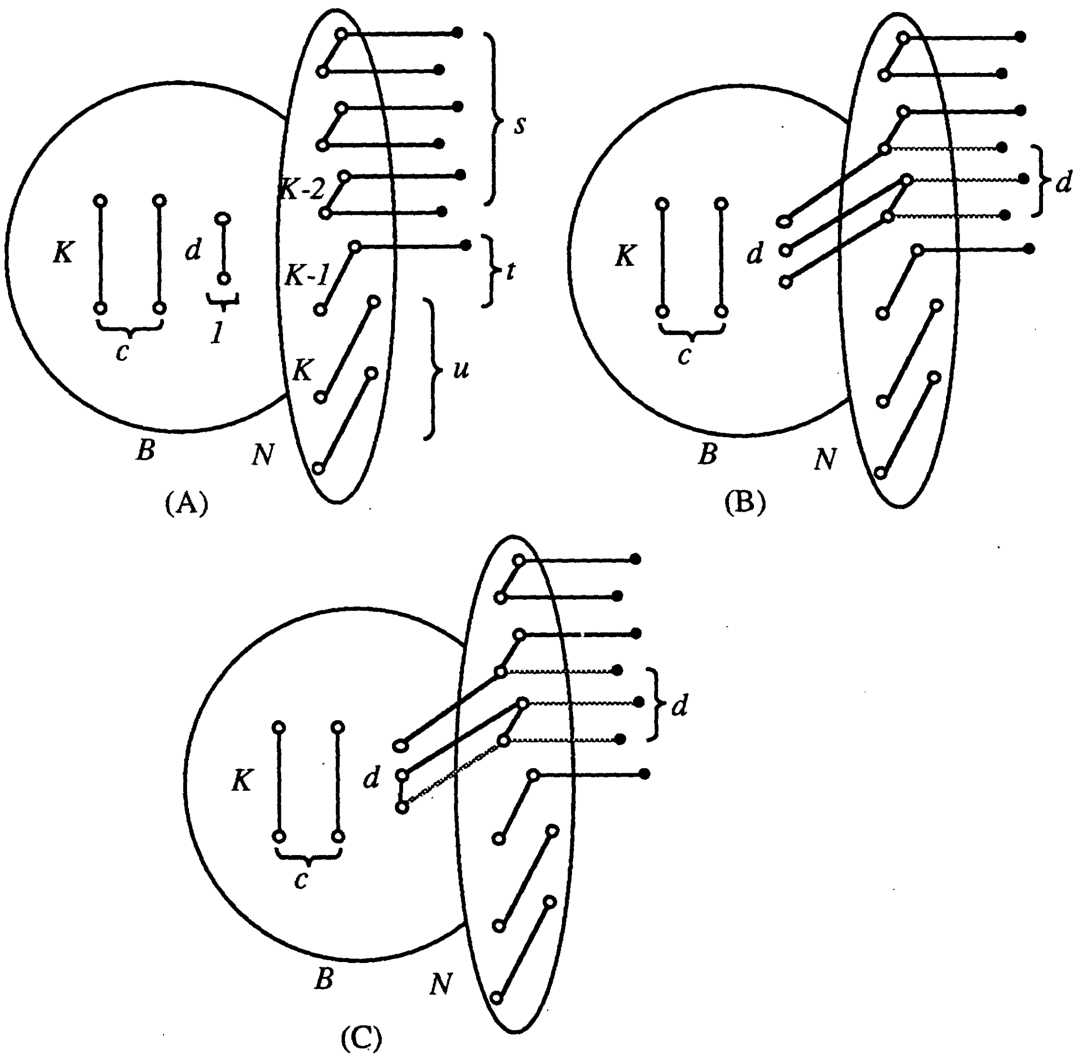


FIGURE 6.2: Generic solutions for the ladybug facet (6.2)

$V - B$, to conclude as in the proof of Proposition 5.3 that $\alpha_{ij} = 0$ for all $i, j \in S$, $\alpha_{ij} = \lambda$ for all $i \in N, j \in S$, $\alpha_{ij} = K\lambda$ for all $i, j \in N$. Also, because the generic solutions A and C have a free satellite vertex, we can treat an edge joining it and the endpoint of the path with d B-vertices as an in-and-out edge. Thus $\alpha_{ij} = 0$ for all $i \in B, j \in S$. If $|B| \geq 3$ we apply the triple-exchange argument to show that for some constant μ, $\alpha_{ij} = \mu$ for all $i, j \in B$.

To complete the proof we must make some exchanges similar to those we made in Proposition 6.1 for the generic solutions shown in Figure 6.1, but using paths instead of trees. The configuration A in Figure 6.2 corresponds to a generic solution with the head set arranged according to the generic solutions B or D in Proposition 5.3, so that at least K satellite vertices are connected to N. In the generic solution B of Figure 6.2, we have substituted d body vertices for d of those satellite vertices; this construction is possible since the solutions contain enough satellite vertices connected to N to perform the exchange. The exchange in 6.2C is realizable if the generic solution B in Figure 6.2 contains a path in the head set with 2 satellite vertices; both the generic solutions B and D have at least one such path since $s \geq 1$ (see Figure 5.5 and Table I). With the simple-exchange between the configurations B and C, we obtain the result $\alpha_{ij} = \mu$ for all $i \in N, j \in B$, and with the exchange between the configurations A and B we see that $\mu = d\lambda$. ∎

6.2 Partial Multistars

We next consider a set of inequalities that depend very heavily on the degree-2 constraints. They define facets of PP, but are not even valid inequalities for the polytopes F and T. We call these inequalities partial multistars, since their support graphs have edges between the satellite vertices and some, but not all, the nucleus vertices.

Proposition 6.3 (*One-connector partial multistars*) *Let $K \geq 3$ and let P be a partial multistar with a nucleus set N, a connector set $\bar{N} \subseteq N$, and a satellite set S. Suppose that*

(i) *$|N| = aK$ for some positive integer a,*

(ii) *$S = V - N$ and $|S| \geq 3$,*

(iii) *$\bar{N} = \{v\}$.*

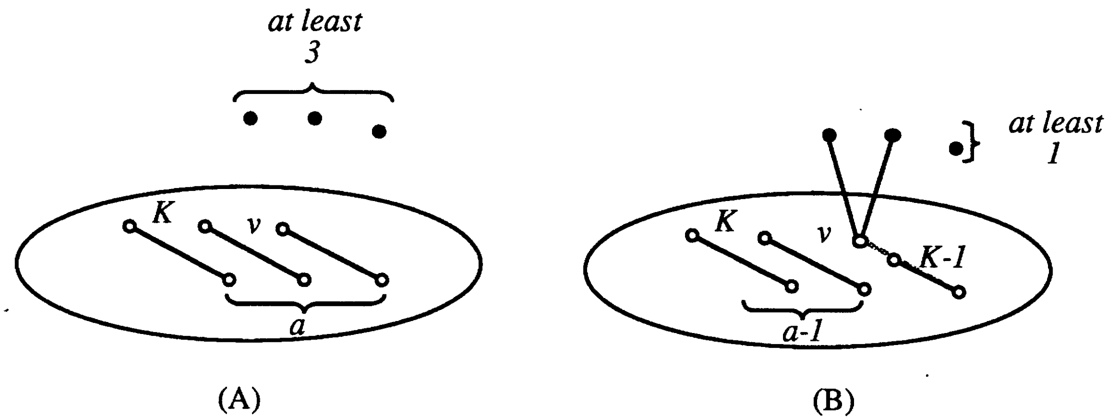


FIGURE 6.3: Generic solutions for the one-connector partial multistar facet (6.3)

Then the partial multistar inequality

$$2X(E(N)) + X(E(\{v\}, S)) \leq 2(|N| - a) \tag{6.3}$$

defines a facet of the polytope PP.

Proof (Validity): Let $\mathbf{x}$ be a vector maximizing the left-hand side of the partial multistar inequality (6.3) over PP. If s denotes the number of vertices in S connected directly to the vertex v, note that s can assume only the values 0,1 and 2 (since the solution must be a set of paths). Consequently,

$$2X(E(N)) + X(E(\{v\}, S)) \leq 2(|N| - \left\lceil \frac{|N| + s}{K} \right\rceil) + s = 2(|N| - a) - 2\left\lceil \frac{s}{K} \right\rceil + s \leq 2(|N| - a).$$

(Facet): We will show that if the face defined by (6.3) is contained in the facet defined by $\boldsymbol{\alpha}^T\mathbf{x} \leq \alpha_0$, then this latter inequality is a scalar multiple of (6.3). Toward that purpose, we use the generic solutions A and B shown in Figure 6.3. Note that in the generic solution A any edge joining two satellite vertices is an in-and-out edge. Similarly, in the generic solution B any edge between a free satellite vertex and an endpoint of the nucleus path of length $K - 1$ is also an in-and-out edge. Therefore, $\alpha_{ij} = 0$ for all $i, j \in S$ and for all $i \in N - \{v\}, j \in S$. By applying the triple-exchange argument to the generic solution A, we obtain $\alpha_{ij} = \mu$ for all $i, j \in N$ and some constant μ. In addition, we can use the generic solution B and some simple-exchanges among the edges used to connect the satellite vertices to v, to show that for some constant λ, $\alpha_{vj} = \lambda$ for all $j \in S$. Finally, subtracting the equations corresponding to both the generic solutions yields $\mu = 2\lambda$. ∎

Proposition 6.4 (*Two-connector partial multistars*) *Let P be a partial multistar with a nucleus set N, a connector set $\bar{N} \subseteq N$, and a satellite set S. Suppose that*

(i) $|N| = aK + 1$ *for some positive integer a,*

(ii) $S = V - N$ *and* $|S| \geq 4$,

(iii) $\bar{N} = \{v_1, v_2\}$,

(iv) $K \geq 4$.

Then the partial multistar inequality

$$2X(E(N)) + X(E(\bar{N}, S)) \leq 2(|N| - a) \tag{6.4}$$

defines a facet of the polytope PP.

Proof (Validity): Given a vector $\mathbf{x} \in PP$, we consider two cases concerning the number of satellite vertices connected to the connector vertices v_1 and v_2. If at least one of the connector vertices, say v_1, is connected to two satellite vertices, we can substitute $\mathbf{x}$ into (6.4) and use the inequality (6.3) to see that (for that particular $\mathbf{x}$)

$$\begin{aligned} 2X(E(N)) + \sum_{i=1}^{2} X(E(\{v_i\}, S)) &\leq 2X(E(N - \{v_1\})) + X(E(\{v_2\}, S)) + 2 \\ &\leq 2(|N| - 1 - a) + 2. \end{aligned}$$

On the other hand, if at most one satellite vertex is connected to each connector vertex, then

$$\begin{aligned} 2X(E(N)) + \sum_{i=1}^{2} X(E(\{v_i\}, S)) &\leq 2X(E(N)) + 2 \\ &\leq 2(|N| - a - 1) + 2. \end{aligned}$$

(Facet): In this case, we will refer to the the generic solutions A,B and C shown in Figure 6.4. Observe that in the generic solution A an edge joining two free satellite vertices is an in-and-out edge. Similarly, in the generic solution C an edge joining a free satellite vertex and an endpoint of the nucleus path of length $K - 1$ is an in-and-out edge. Consequently, $\alpha_{ij} = 0$ for all $i, j \in S$ and for all $i \in N - \{v_1, v_2\}, j \in S$. We can use the generic solution A and the triple-exchange argument to show that for some constant μ, $\alpha_{ij} = \mu$

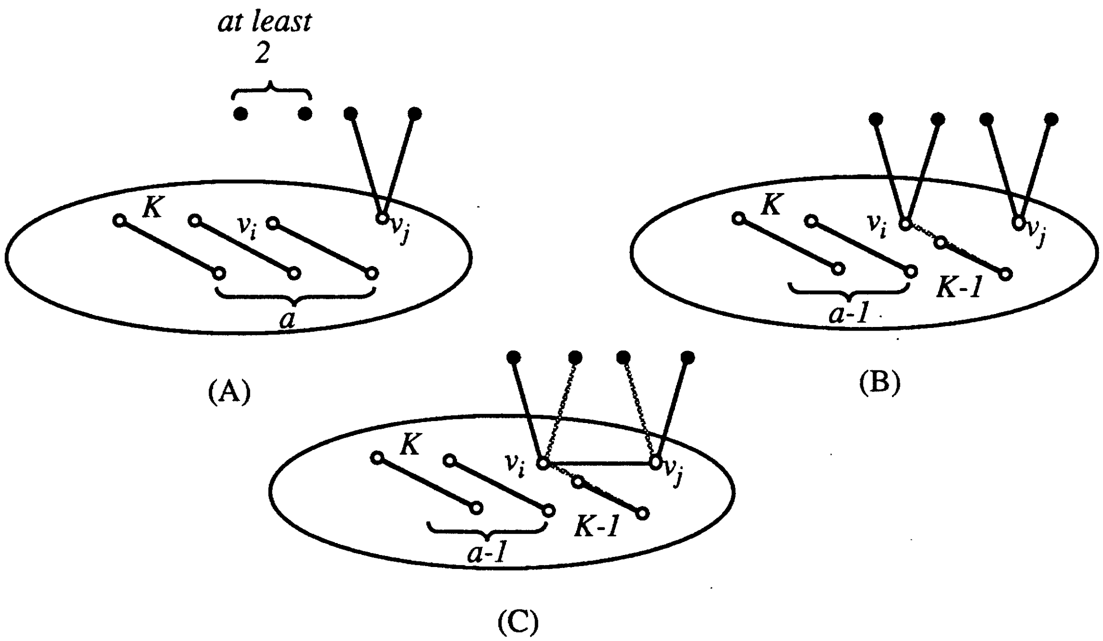


FIGURE 6.4: Generic solutions for the two-connector partial multistar facet (6.4)

for all $i, j \in N$, except for $i = v_1, j = v_2$. In addition, we can use the same generic solution and some simple-exchanges among the edges used to connect satellite vertices to v_j, to see that $\alpha_{v_j i} = \lambda_j$ for all $i \in S$, $j = 1, 2$ and some constants λ_j. Subtracting the equations corresponding to the generic solutions A and B yields $\mu = 2\lambda_1 = 2\lambda_2 = 2\lambda$; subtracting the equations for B and C shows that $\alpha_{v_1 v_2} = 2\lambda$. ∎

Proposition 6.5 (*Three-connector partial multistars*) *Let P be a partial multistar with a nucleus set N, a connector set $\bar{N} \subseteq N$, and a satellite set S. Suppose that*

(i) $|N| = aK$ *for some positive integer a,*

(ii) $S = V - N$ *and* $|S| \geq 4$,

(iii) $\bar{N} = \{v_1, v_2, v_3\}$,

(iv) $K \geq 4$.

Then the partial multistar inequality

$$3X(E(N)) + X(E(\bar{N}, S)) \leq 3(|N| - a) \tag{6.5}$$

defines a facet of the polytope PP.

Proof (Validity): Let $\mathbf{x}$ be a 0–1 vector maximizing the left-hand side of (6.5) over PP. We consider several cases concerning the number of components (paths or single vertices) that $\mathbf{x}$ induces on N. If $\mathbf{x}$ induces at least $a+2$ components, then for that particular $\mathbf{x}$,

$$\begin{aligned} 3X(E(N)) + \sum_{i=1}^{3} X(E(\{v_i\}, S)) &\leq 3(|N| - a - 2) + 2 + 2 + 2 \\ &= 3(|N| - a). \end{aligned}$$

If the induced graph has $a+1$ components and $X(E(\{v_i\}, S)) \leq 1$ for $i = 1, 2, 3$, then

$$\begin{aligned} 3X(E(N)) + \sum_{i=1}^{3} X(E(\{v_i\}, S)) &\leq 3(|N| - a - 1) + 1 + 1 + 1 \\ &= 3(|N| - a). \end{aligned}$$

If the induced graph has $a+1$ components and for at least one connector vertex, say v_1, $X(E(\{v_1\}, S)) = 2$, then one of the $a+1$ components is just the vertex v_1, so that the remaining $aK - 1$ vertices in N belongs to the remaining a components. In any such graph the vertices of N other than v_1 must belong to $a - 1$ paths with K vertices, and one path with $K - 1$ vertices. We can connect only one vertex in S to the latter path, so

$$X(E(\{v_2\}, S)) + X(E(\{v_3\}, S)) \leq 1.$$

Therefore,

$$\begin{aligned} 3X(E(N)) + \sum_{i=1}^{3} X(E(\{v_i\}, S)) &\leq 3(|N| - a - 1) + 2 + 1 \\ &= 3(|N| - a). \end{aligned}$$

If the induced graph has a components, then all the vertices in N belong to a paths each with K vertices, so no vertex in S can be connected to a vertex in N, and therefore

$$\begin{aligned} 3X(E(N)) + \sum_{i=1}^{3} X(E(\{v_i\}, S)) &\leq 3(|N| - a) + 0 \\ &= 3(|N| - a). \end{aligned}$$

(Facet): We assume that the face defined by the partial multistar inequality (6.5) is contained in the facet defined by $\boldsymbol{\alpha}^T \mathbf{x} \leq \alpha_0$, and show that the latter inequality reduces to a scalar multiple of (6.5). We

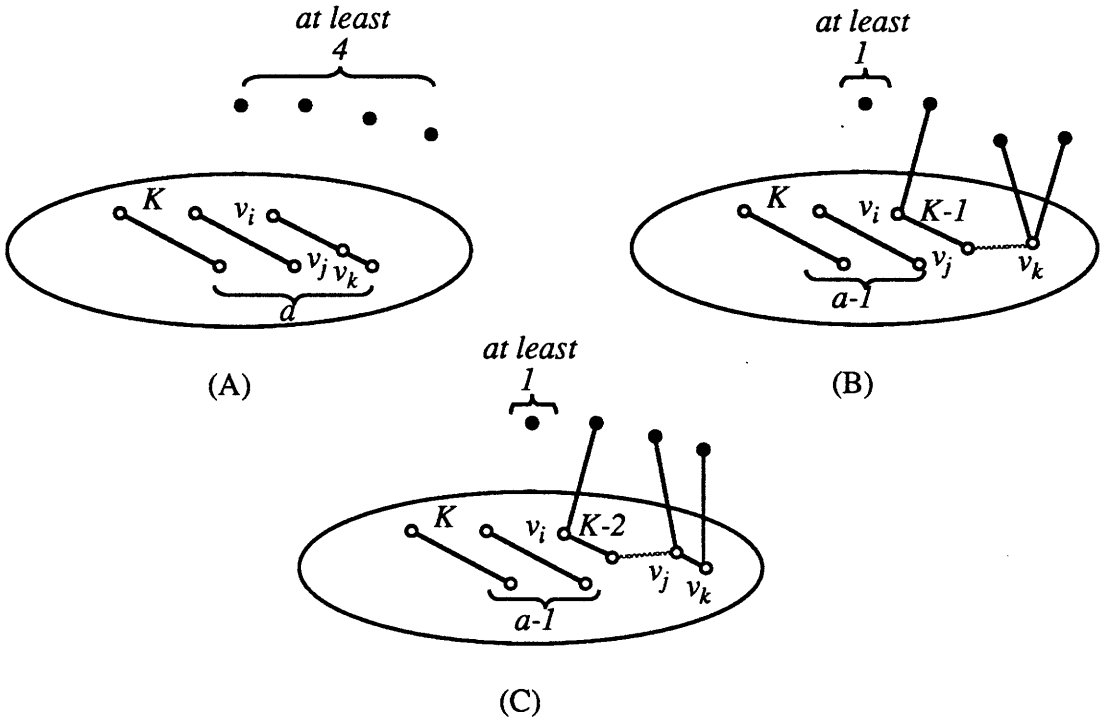


FIGURE 6.5: Generic solutions for the three-connector partial multistar facet (6.5)

refer to the generic solutions A,B and C shown in Figure 6.5. In the generic solution A an edge joining two satellite vertices is an in-and-out edge. Similarly, in the generic solution C an edge joining a free satellite vertex and the vertex in $N - \{v_1, v_2, v_3\}$ which is the endpoint of the segment with $K - 2$ nucleus vertices is an in-and-out edge. Thus, $\alpha_{ij} = 0$ for all $i, j \in S$ and for all $i \in N - \{v_1, v_2, v_3\}, j \in S$. Using the generic solution A and the triple-exchange argument shows that for some constant μ, $\alpha_{ij} = \mu$ for all $i, j \in N$. In addition, we can use the generic solution B and simple exchanges like those shown in Figure 5.2 to see that for some constant λ, $\alpha_{v_i j} = \lambda$ for all $j \in S$, $i = 1, 2, 3$. Subtracting the equations corresponding to the generic solutions A and B yields $\mu = 3\lambda$. ∎

As shown by the next proposition, it is possible to obtain another type of three-connector partial multistar. It is similar to the one described in the previous proposition, but has different coefficients.

Proposition 6.6 (*Three-connector partial multistars*) *Let P be a partial multistar with a nucleus set N, a connector set $\bar{N} \subseteq N$, and a satellite set S. Suppose that*

(i) $|N| = aK$ *for some positive integer* a,

(ii) $S = V - N$ *and* $|S| \geq 5$,

(iii) $\bar{N} = \{v_1, v_2, v_3\}$,

(iv) $K \geq 4$.

Then the partial multistar inequality

$$2X(E(N)) + X(E(\bar{N}, S)) \leq 2(|N| - a) + 1 \tag{6.6}$$

defines a facet of the polytope PP.

Proof (Validity): Using (6.5) we can obtain the inequality

$$\begin{aligned} 2\Big(2X(E(N)) + \sum_{i=1}^{3} X(E(\{v_i\}, S))\Big) &\leq 3X(E(N)) + \sum_{i=1}^{3} X(E(\{v_i\}, S)) \\ &\quad + X(E(N - \{v_1, v_2, v_3\})) + \sum_{i=1}^{3} X(star(v_i)) \\ &\leq 3(|N| - a) + (|N| - 3 - a) + 6. \end{aligned}$$

After dividing each side by 2, we have

$$2X(E(N)) + \sum_{i=1}^{3} X(E(\{v_i\}, S)) \leq 2(|N| - a) + 1.5\,.$$

Rounding down the right-hand side of the inequality, we obtain the partial multistar inequality (6.6).

(Facet): Again, we assume that the face defined by (6.6) is contained in the facet defined by $\boldsymbol{\alpha}^T \mathbf{x} \leq \alpha_0$, and show that this inequality reduces to a scalar multiple of (6.6), using the generic solutions B,C and D shown in Figure 6.6. Note that the only difference between the generic solutions B and C in Figures 6.5 and 6.6 is the minimum number of satellite vertices that are free vertices. Since we can use an edge between two free satellite vertices in B as an in-and-out edge, $\alpha_{ij} = 0$ for all $i, j \in S$. We can use B and C in the same way we did in Proposition 6.5 to prove that $\alpha_{ij} = 0$ for all $i \in N - \{v_1, v_2, v_3\}, j \in S$, and that $\alpha_{v_i j} = \lambda$ for all $j \in S$, $i = 1, 2, 3$ and some constant λ. Subtracting the equations for C and D yields $\alpha_{v_j v_k} = 2\lambda$ for $j, k = 1, 2, 3$. Subtracting the equations for B and D yields $\alpha_{v_j i} = 2\lambda$ for $i \in N - \{v_1, v_2, v_3\}, j = 1, 2, 3$.

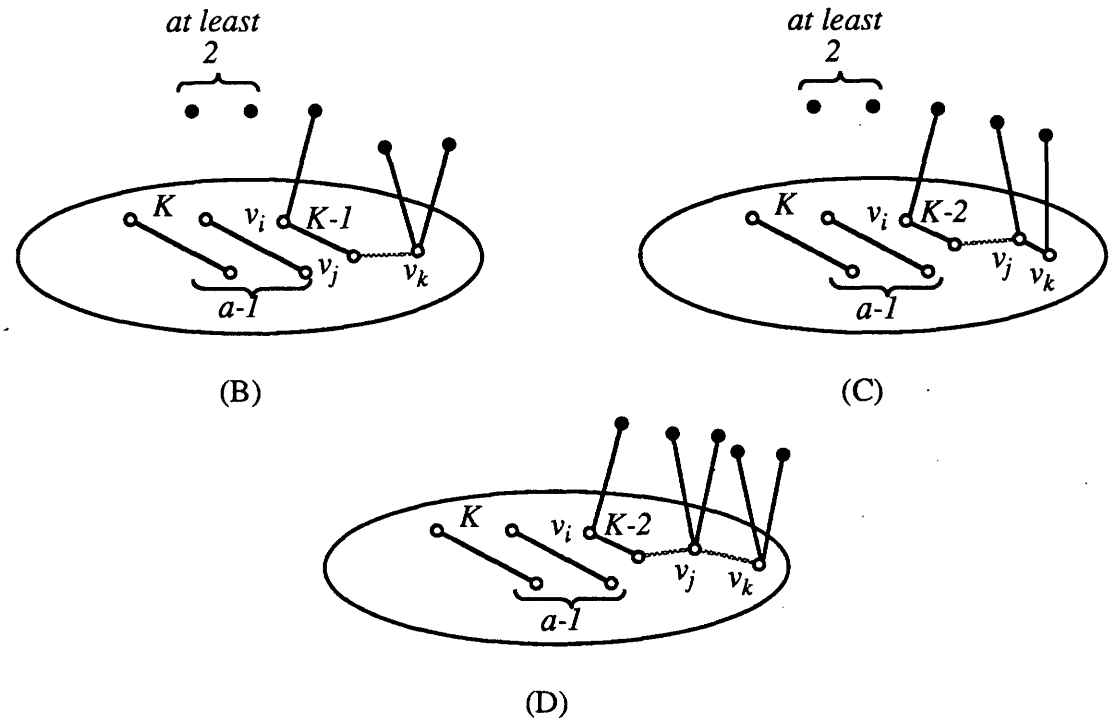


FIGURE 6.6: Generic solutions for the three-connector partial multistar facet (6.6)

It remains to consider just the coefficients of the form α_{ij} for $i, j \in N - \{v_1, v_2, v_3\}$. We deal with these coefficients using different cases. If $K \geq 5$ we can see in the generic solution B, the path containing vertex v_j contains v_i and at least two other nucleus vertices, say l, m. We can arrange the vertices v_j, l, m in any order on the path. Therefore, by the triple-exchange argument, $\alpha_{v_j l} = \alpha_{lm} = 2\lambda$. If $K = 4$ and $a \geq 2$, we may alter the generic solution B by interchanging v_j with a nucleus vertex on a path which initially did not contain any connector vertex. Hence, we obtain a path with v_j and $K - 1$ non-connector vertices and we can use the triple-exchange argument on this path to obtain the conditions $\alpha_{v_j l} = \alpha_{lm} = 2\lambda$ for all $l, m \in N - \{v_1, v_2, v_3\}$. If $K = 4$ and $a = 1$, there is nothing to prove since no edges join two non-connector vertices. ∎

The last inequality that we consider in this section has a structure slightly different from that of a partial multistar.

Proposition 6.7 *Let $K \geq 3$, $N, S \subseteq V$ and $u, v \in N$. Suppose that*

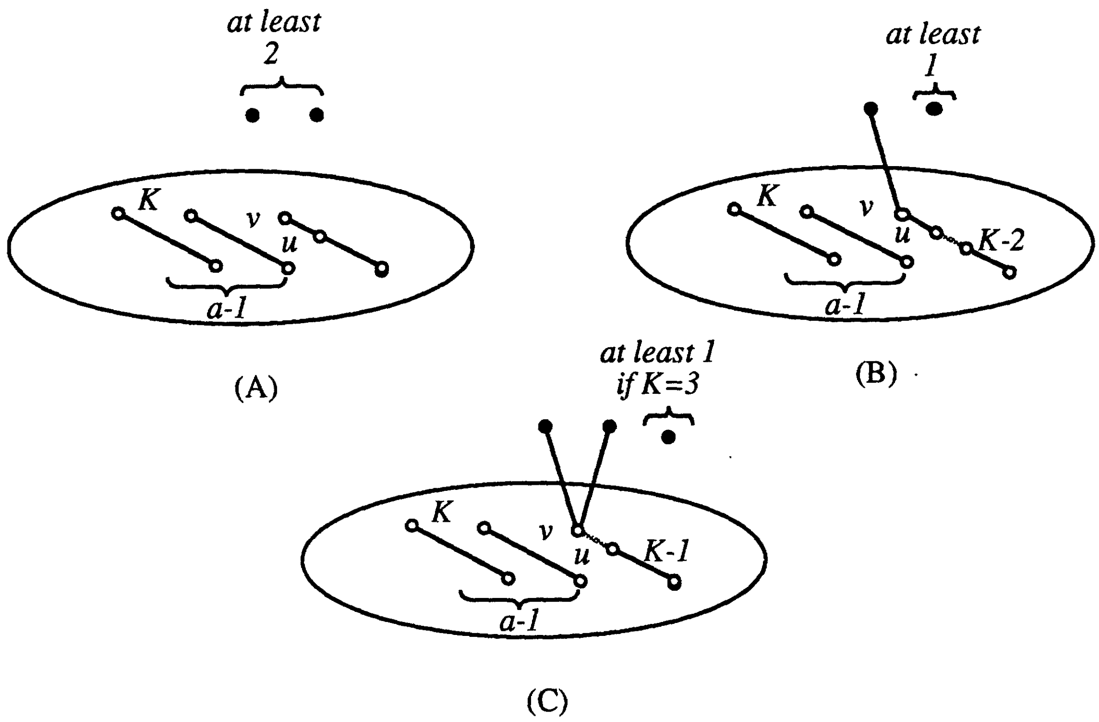


FIGURE 6.7: Generic solutions for the facet (6.7)

(i) $|N| = aK$ *for some positive integer* a,

(ii) $S = V - N$,

(iii) $|S| \geq 2$ *when* $K \geq 4$, *and* $|S| \geq 3$ *when* $K = 3$.

Then the inequality

$$X(E(N)) + X(E(\{v\}, S)) + x_{uv} \leq |N| - a + 1 \tag{6.7}$$

defines a facet of the polytope PP.

Proof (Validity): Using the partial multistar inequality (6.3), we obtain

$$\begin{aligned} 2\Big(X(E(N)) + X(E(\{v\}, S)) + x_{uv}\Big) &= \Big(2X(E(N)) + X(E(\{v\}, S))\Big) + X(E(\{v\}, S)) + x_{uv} \\ &\leq 2(|N| - a) + 2 + 1. \end{aligned}$$

After dividing each side by two and rounding down on the right-hand side, we obtain inequality (6.7).

(Facet): Assume that the face defined by (6.7) is contained in the facet defined by $\boldsymbol{\alpha}^T\mathbf{x} \leq \alpha_0$, and consider the generic solutions A, B and C shown in Figure 6.7. In the generic solution A, an edge joining two satellite vertices is an in-and-out edge. in the generic solution B, any edge joining a free satellite vertex and an endpoint of the path with $K-2$ vertices of $N - \{u, v\}$ is also an in-and-out edge. Therefore, $\alpha_{ij} = 0$ for all $i, j \in S$ and for all $i \in N - \{u, v\}, j \in S$.

If $K \geq 4$, an edge joining a free satellite vertex and vertex u is an in-and-out edge in the generic solution B, so $\alpha_{uj} = 0$ for all $j \in S$. On the other hand, if $K = 3$ we use the generic solution C to obtain the same result. Note that in this case we need the hypothesis that $|S| \geq 3$ to guarantee that some free satellite vertex is connected to vertex u.

If we interchange the places of the vertices u and v in the generic solution A, we obtain another feasible solution. We can use the simple-exchange argument to show that for some constant μ, $\alpha_{uj} = \alpha_{vj} = \mu$ for all $j \in N - \{u, v\}$. If we interchange the satellite vertex connected to vertex v in the generic solution B with any other free satellite vertex, we can use the simple-exchange argument to prove that for some constant λ, $\alpha_{vj} = \lambda$ for all $j \in S$. We can then subtract the equations corresponding to the generic solutions A and B to prove that $\mu = \lambda$, and subtract the equations corresponding to A and C to prove that $\alpha_{uv} = 2\lambda$.

If $K \geq 4$, the path containing vertex u in the generic solution C also contains at least two other vertices in $N - \{u, v\}$. Since we can apply the triple-exchange argument to these vertices, we have that $\alpha_{ij} = \alpha_{uj} = \mu = \lambda$ for all $i, j \in N - \{u, v\}$, concluding the proof for $K \geq 4$.

If $K = 3$ and $a \geq 2$, the path containing vertex u in the generic solution C contains only two vertices, u and another vertex in $N - \{u, v\}$. Nevertheless, we still can apply the triple-exchange argument since we can add to this path a vertex from any of the paths with three vertices. Therefore, $\alpha_{ij} = \alpha_{uj} = \lambda$ for any $i, j \in N - \{u, v\}$. If $K = 3$ but $a = 1$, no edge has both endpoints in $N - \{u, v\}$ and we need to consider no other edges. ∎

7 Clique Clusters and Extensions

This section describes facets based on *clique clusters*, defined in Section 3. First we define and consider a *simple clique cluster*, whose nucleus contains a single node. As was the case for the packing constraints (4.1), the conditions under which the inequalities defined by this graph structure are facets for the path-partitioning polytope and the forest polytope are identical. Next we will consider a straightforward generalization for the tree polytope, which we call *augmented clique-cluster inequalities.* Finally, we hint at further generalizations, show how these encompass the classes in Sections 5 and 6, and explain why it seems difficult to establish the facet conditions for these generalizations.

7.1 Simple Clique Clusters

We describe a simple clique cluster by t sets of nodes $C_1, \ldots, C_t$, whose pairwise intersections consist of the nucleus p of the cluster. We let

$$|C_i| = a_i K + b_i, \quad i = 1, \ldots, t, \quad 2 \leq b_1 \leq b_2 \leq \cdots \leq b_t \leq K.$$

Proposition 7.1 *Consider a simple clique cluster on the node set $S = \bigcup_{i=1}^{t} C_i$ with nucleus p, $t \geq 3$, satisfying the following conditions:*

(i) $b_1 + b_2 \geq K + 2$;

(ii) $b_1 + b_{t-1} + b_t \leq 2K + 2$.

Then the clique-cluster inequality

$$\sum_{i=1}^{t} X(E(C_i)) \leq \sum_{i=1}^{t} (|C_i| - \left\lceil \frac{|C_i|}{K} \right\rceil - 1) \ + 1 \tag{7.1}$$

is a facet for PP, F, and T.

Proof (Validity): Consider any feasible solution for F. We partition the sets $C_1, \ldots, C_t$ into two groups: the first group consists of those C_i for which $E(C_i \backslash \{p\}, p)$ is not empty, and the second is the set of remaining C_i. For each set C_i in the second group, the packing constraint (4.1) on $C_i \backslash \{p\}$ implies that the solution satisfies

$$X(E(C_i)) \leq a_i(K - 1) + b_i - 2. \tag{7.2}$$

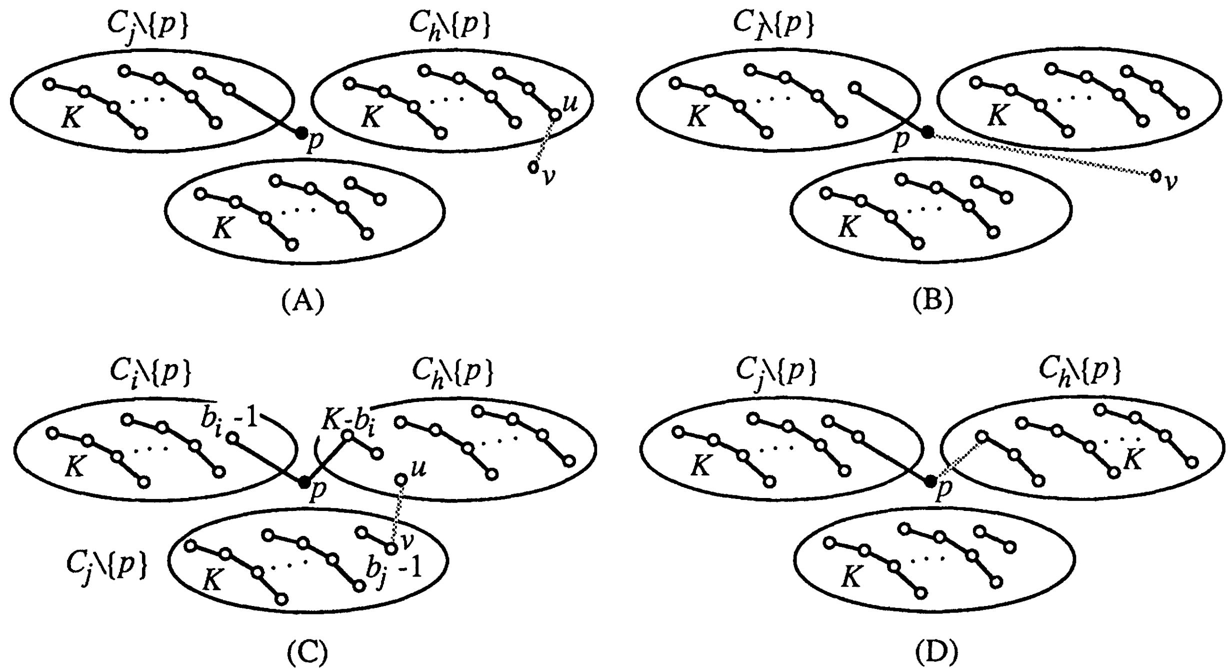


FIGURE 7.1: Generic solutions for the simple clique clusters

Next we consider the first group, and for each set C_i we consider the number of nodes of C_i, other than p, that are in the same component as p: call this number γ_i. Condition (i) implies $b_i + b_j \geq K + 2$ for any two sets C_i and C_j, and thus in particular that $\gamma_j \geq b_j - 1$ for at most one set C_j. For this component, the packing constraint (4.1) on C_j implies

$$X(E(C_j)) \leq a_j(K-1) + b_j - 1. \tag{7.3}$$

For any other set C_i, since $\gamma_i < b_i - 1$, the number of nodes of C_i not attached to p is strictly greater than a_iK; hence they are partition into at least $a_i + 1$ components. Thus all of the nodes of C_i are partitioned into at least $a_i + 2$ components, yielding

$$\begin{aligned} X(E(C_i)) &\leq |C_i| - (a_i + 2) \\ &= a_i(K-1) + b_i - 2. \end{aligned} \tag{7.4}$$

Combining (7.2), (7.3), and (7.4) proves the validity of (7.1).

(Facet): We will prove that (7.1) is a facet for *PP*, and the result for F and T then follows. We refer to the generic solutions in Figure 7.1. First, we note that for $u, v \notin S$, we may regard edge uv as an in-and-out edge, and thus $\alpha_{uv} = 0$.

Edges in $E(S\backslash\{p\}, V\backslash S)$

Consider the generic solution A in Figure 7.1: each vertex set $C_i\backslash\{p\}$ is partitioned into a_i paths with K nodes and one path with $b_i - 1$ nodes, and exactly one of the latter type of path is attached to the nucleus p. For node u in C_h and $v \notin S$, since $b_h - 1 < K$ we may regard edge uv as an in-and-out edge; since u and v were chosen arbitarily we have $\alpha_{uv} = 0$ for all $u \in S\backslash\{p\}$, $v \notin S$.

Edges in $E(\{p\}, V\backslash S)$

Next consider solution B, which is identical to A except that p is attached specifically to the path of length $b_1 - 1$ in C_1. By condition (ii), $b_1 + b_{t-1} + b_t \leq 2K + 2$, and since $b_1 \leq b_{t-1} \leq b_t$, for $K \geq 3$ we have $b_1 \leq K - 1$. Thus edge pv is an in-and-out edge, and $\alpha_{pv} = 0$ for arbitrary $v \notin S$.

Edges between $C_h\backslash\{p\}$ *and* $C_j\backslash\{p\}$, $h \neq j$

Next consider solution C, in which the path containing p has K nodes in it: $b_i - 1$ nodes from $C_i\backslash\{p\}$, and $K - b_i$ nodes from $C_h\backslash\{p\}$.

The rest of C_h is partitioned into a_h paths of length K and one path of length

$$b_h + b_i - K - 1 = b_h - (K - b_i + 1)$$

(we recall that $b_h + b_i - K - 1 \geq 1$ by condition (i)). Finally, all remaining cliques C_j are packed the usual way, with a_j paths of length K, and one path of length $b_j - 1$.

Suppose that C_i is the smallest set other than C_h and C_j. In that case, we have

$$(b_j - 1) + (b_h + b_i - K - 1) = b_j + b_h + b_i - K - 2 \leq K$$

by condition (ii). Hence the two paths containing nodes u and v are short enough to be joined feasibly, and thus uv is an in-and-out edge. Thus $\alpha_{uv} = 0$. But since u, v, h, and j were chosen arbitrarily, $\alpha_{uv} = 0$ for any two nodes $u, v \neq p$ contained in different sets C_h and C_j.

Edges in $\bigcup_{i=1}^{t} E(C_i)$

It remains to show that all edges in $\bigcup_{i=1}^{t} E(C_i)$ have the same coefficient. Consider solution A. By the triple-exchange lemma (Lemma 2.2), all of the edges in $E(C_j)$ have the same coefficient. But by simple-exchange (see solution D), all of the edges between p and any other node of S have the same coefficient. Thus all of the edges in $\bigcup_{i=1}^{t} E(C_i)$ have the same coefficient. This conclusion establishes that (7.1) is a facet, under conditions (i) and (ii). ∎

7.2 Augmented Clique Cluster Inequalities for T

Next, we focus our attention on a related class of inequalities for the polytope T. Consider a clique cluster given by $C_1, \ldots, C_t$, and let p be defined as before.

Proposition 7.2 *Let $K \geq 3$, and consider a clique cluster with $t \geq 2$ satisfying the following properties:*

(i) $0 \notin C_i$, $i = 1, \ldots, t$;

(ii) $|C_1| \bmod K \neq 1, 2$;

(iii) $|C_i| \bmod K = 0$, $i = 2, \ldots, t$.

Then the augmented clique-cluster inequality

$$X(E(\{0\}, C_1 \backslash \{p\}) + \sum_{i=1}^{t} X(E(C_i)) \leq \sum_{i=2}^{t} (|C_i| - \frac{|C_i|}{K} - 1) + |C_1| \tag{7.5}$$

is a facet for T.

Proof (Validity): Consider any feasible solution for T. We observe that for at most one i, $2 \leq i \leq t$,

$$X(E(C_i)) = |C_i| - |C_i|/K, \tag{7.6}$$

since for such a set node p must be in a connected component of C_i of size at least b_i. For all other $i \geq 2$, $X(E(C_i)) \leq |C_i| - |C_i|/K - 1$. Now, first suppose that for some i, $2 \leq i \leq t$, (7.6) holds. Then $X(E(C_1 \backslash \{p\}, \{p\})) = 0$, and so $X(E(C_1 \cup \{0\})) \leq |C_1| - 2$ by the tree inequality (2.3) on $(C_1 \backslash \{p\}) \cup \{0\}$. On the other hand, if $X(E(C_i)) \leq |C_i| - |C_i|/K - 1$ for all $i \geq 2$, we still have $X(E(C_1 \cup \{0\})) \leq |C_1| - 1$. In both cases, combining these inequalities shows that (7.5) is valid.

(Facet): Let $\mathrm{S} = \bigcup_{i=1}^{t} C_i$, and let $|C_i| = a_i K + b_i$, $i = 1, \ldots, t$, $3 \leq b_i \leq K$. In order to show that (7.5) is a facet of T, we will show that any inequality

$$\boldsymbol{\alpha}\mathbf{x} \leq \alpha_0 \tag{7.7}$$

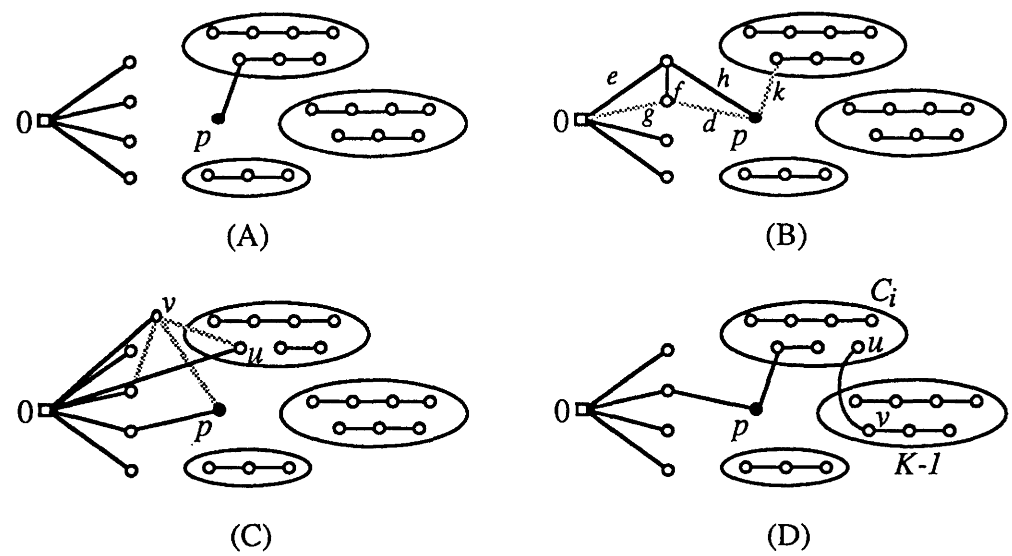


FIGURE 7.2: Generic solutions for the augmented clique cluster inequalities

that is satisfied at equality by every solution satisfying (7.5) at equality must be a convex combination of (7.5) and the equality (2.1). We also need to show that some solution satisfies (7.5) as a strict inequality; this is obvious. We refer to the generic solutions in Figure 7.2. First, we note that for $u, v \notin S$, it is easy to establish that the coefficients α_{uv} are all equal in (7.7); let us call this coefficient β.

Edges in $\bigcup_i E(C_i)$ *and* $E(\{0\}, C_1)$

Consider solution A in Figure 7.2: each of the sets C_l, $l = 2, \ldots, t$, contains $a_l(K-1) + K - 2$ edges except for one set C_i, which is attached to node p and contains $a_i(K-1) + K - 1$ edges. The nodes of $C_1 \backslash \{p\}$ are all attached directly to the root 0. A simple-exchange argument shows that all edges α_{pv}, $v \in \bigcup_{l=2}^{t} E(C_l)$, have the same coefficient. Now consider the set C_i with node p attached. By the triple-exchange argument on $E(C_i)$, we see that all edges in $E(C_i)$ have the same coefficient. Noting that i was chosen arbitrarily and that $\alpha_{pu} = \alpha_{pv}$ for all $u, v \in S$, we have $\alpha_e = \alpha_f$ for all $e, f \in \bigcup_{i=2}^{t} E(C_i)$. Finally, consider solution B. By a simple exchange of edges h and k, we have $\alpha_{hp} = \alpha_{kp}$ for $h \in C_1$, $k \in C_i$, $i \neq 1$. By simple exchanges on d and f, f and g, and e and g, we see that all edges in $E(C_1)$ and $E(C_1 \backslash \{p\}, \{0\})$ have the same coefficient as edges in $\bigcup_{i=2}^{t} E(C_i)$. Thus all edges in the supporting subgraph of (7.5) have the same coefficient in (7.7).

Edges in $E(C_i, \{0\})$, $i \neq 1$, *and* $E(C_i, V \backslash (S \cup \{0\}))$

From solution C, a simple exchange argument shows that $\alpha_{0v} = \alpha_{uv} = \alpha_{0u}$, for $v \notin S$, $v \neq 0$, and $u \in C_i \backslash \{p\}$,

$i = 2, \ldots, t$. Since we know that $\alpha_{0v} = \beta$, we have $\alpha_{uv} = \alpha_{0u} = \beta$. Since u was chosen arbitrarily from $\bigcup_{i=1}^{t} C_i \backslash \{p\}$, all of the edges in $E(C_i \backslash \{p\}, \{0\})$ have coefficient β. It is easy to see from solution C that for $v \in V \backslash S$, $v \neq 0$, and $u \in C_1$ (including possibly $u = p$), $\alpha_{uv} = \beta$, as well.

Edges in $E(C_i, C_j)$, $i \neq j$

Finally, we consider solution D of Figure 7.2: each node of $C_1 \backslash \{p\}$ is connected directly to node 0, and node p is connected to some other node of C_1 and to $K - 2$ nodes of C_i, for some i. The rest of the nodes of $\bigcup_{l=2}^{t} C_l$ are packed into components of size K or $K - 1$, leaving an isolated node u of C_i. Node u is connected directly to node 0. Now consider $v \in C_j$, $j \neq i$ (but possibly $j = 1$), so that v is in a component of size at most $K - 1$. Then a simple exchange on edges uv and $0u$ yields $\alpha_{uv} = \alpha_{0u} = \beta$ in (7.7). Since u and v were arbitrary nodes from arbitrary distinct components, we have that all edges in $\bigcup_{i \neq j} E(C_i, C_j)$ have coefficient β in (7.7).

We have now shown that all edges outside of the supporting graph of (7.5) have the same coefficient β in (7.7). Thus (7.7) must be a convex combination of (7.5) and (2.1), and (7.5) is a facet of T. ∎

7.3 Extensions of Clique Cluster Inequalities

The simple clique cluster inequalities can be generalized to have a nucleus containing more than a single node. In this case, the conditions under which such inequalities are valid inequalities, let alone facets, are difficult to establish. In general these inequalities are not $0-1$ facets. In this section we illustrate these generalizations, show how they encompass the multistar and ladybug inequalities, and indicate the combinatorial difficulties inherent in determining when these inequalities are facets.

For the purposes of this subsection, we denote a general clique cluster by a set of *disjoint* node sets, $C_1, \ldots, C_t$, and an additional disjoint set N. The clique cluster consists of the subgraph $\bigcup_i (E(C_i) \cup E(C_i, N)) \cup E(N)$. The easiest way to indicate the combinatorial subtleties involved is by example.

Consider the clique cluster in Figure 7.3, with $K = 7$, $|N| = 3$, $|C_1| = 4$, and $|C_2| = |C_3| = 3$. In comparing solutions A and B, we see that if N is connected, then at most one set C_i can be attached to N, regardless of which set C_i we choose. This indicates that the coefficients on $E(C_1)$ should be equal to those on $E(C_2)$ or $E(C_3)$. If N is split (see solution C), then all three components (not just two) can be attached

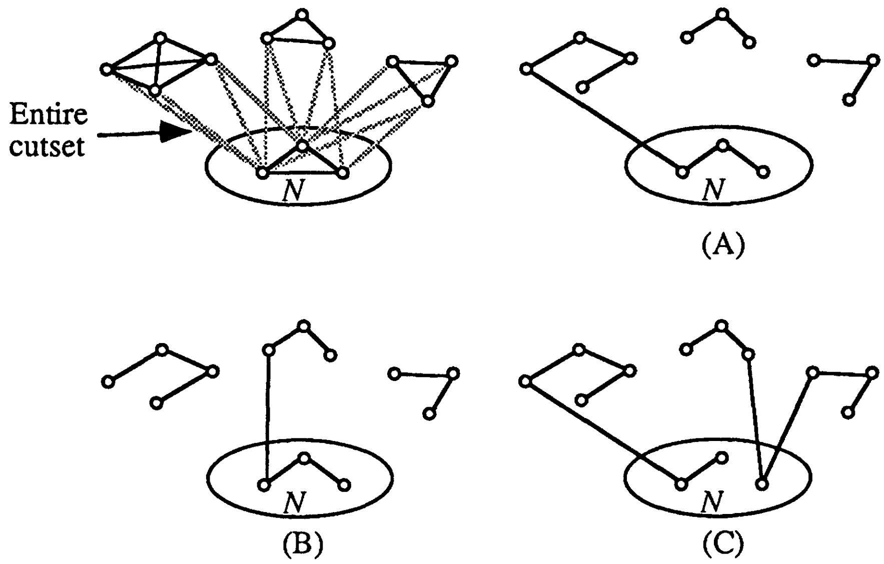


FIGURE 7.3: Solutions on a clique cluster with a multiple-node nucleus, $K = 7$

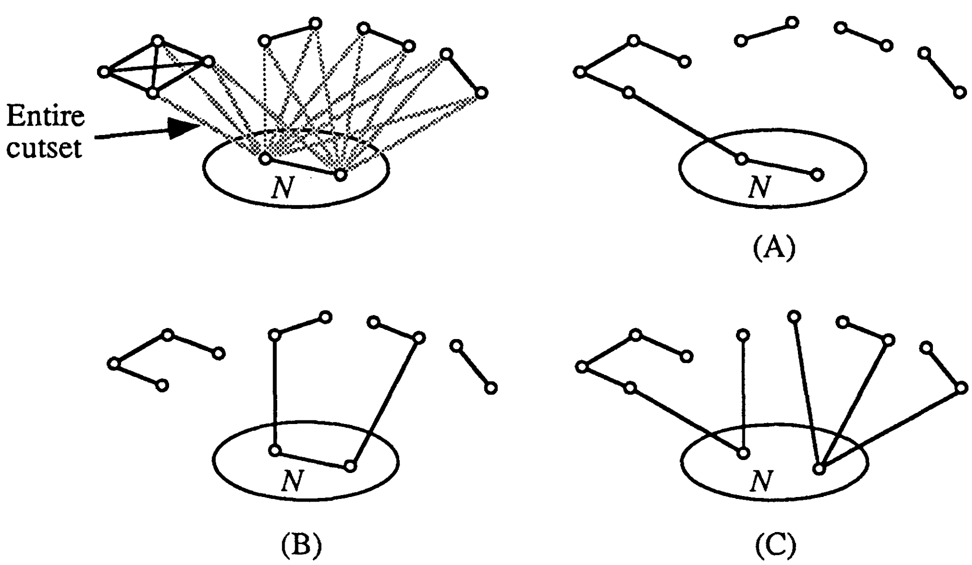


FIGURE 7.4: Solutions on a clique cluster with a multiple-node nucleus, $K = 6$

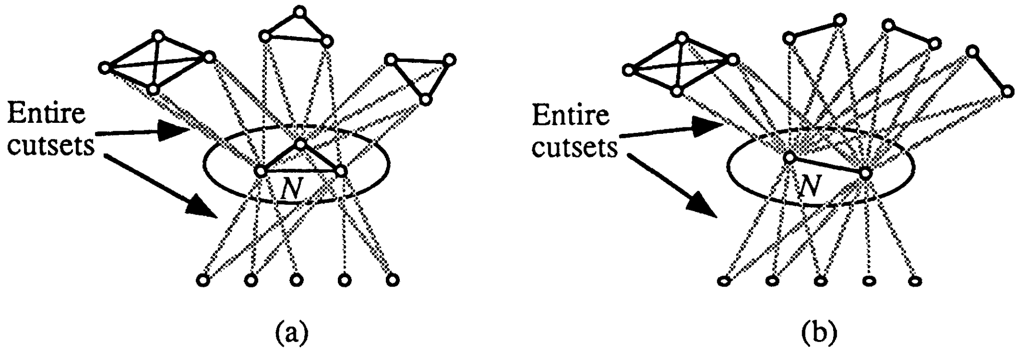


FIGURE 7.5: Modified clique clusters with a multiple-node nucleus, which include the cutset $E(N, V\backslash S)$

to N. The correct valid inequality is thus

$$2X(E(N)) + \sum_{i=1}^{3} X(E(C_i) \cup E(C_i, N)) \leq 12,$$

and actually, this inequality is a facet. On the other hand, in Figure 7.4 we see that the coefficients on the sets C_i are not equal. Here, $K = 6$, $|C_1| = 4$, $|C_2| = |C_3| = |C_4| = |N| = 2$. In this case, the correct valid inequality is

$$6X(E(N)) + 4X(E(C_1) \cup E(C_1, N)) + 2\sum_{i=2}^{4} X(E(C_i) \cup E(C_i, N)) \leq 28.$$

However, this inequality is not a facet; in fact, it can be trivially strengthened to

$$6X(E(N)) + 4X(E(C_1) \cup E(C_1, N)) + 2\sum_{i=2}^{4} X(E(C_i) \cup E(C_i, N)) + X(E(N, V\backslash S)) \leq 28,$$

for $S = N \cup \bigcup_{i=1}^{4} C_i$ (see Figure 7.5B). When we add these additional edges to the example of Figure 7.3 (see Figure 7.5A), we obtain a different valid inequality, with different coefficients:

$$7X(E(N)) + 4X(E(C_1) \cup E(C_1, N)) + 3\sum_{i=2}^{3} X(E(C_i) \cup E(C_i, N)) + X(E(N, V\backslash S)) \leq 42,$$

for $S = N \cup \bigcup_{i=1}^{3} C_i$.

These latter two examples generalize the multistar and ladybug inequalities: rather than having zero or one non-trivial clique attached to the nucleus, respectively, we have an arbitrary number. For this cutset structure, the coefficient on the set $E(C_i) \cup E(C_i, N)$ is proportional to $|C_i| (\bmod K)$. However, we anticipate that it will be difficult to establish when such inequalities are valid. Validity seems to be a function of how well the sets can be packed with respect to the parameter K.

8 Concluding remarks

We close this paper with a few remarks concerning the potential computational implication of our results, and by indicating directions for future research. As we mentioned in the introduction, recently enumerative methods based on strong cutting planes, such as branch-and-cut algorithms, have emerged as an effective computational tool for generating optimal solutions with a guarantee of optimality for large-scale integer programming problems. The impressive computational results obtained by Padberg and Rinaldi [1989] and Grötschel and Holland [1988] in solving traveling salesman problems to guaranteed optimality have sparked very active research concerning other combinatorial problems. The availability of increasing computer power has largely contributed to making these polyhedral methods viable in solving practical, large-scale problems.

A successful branch-and-cut code for a specific combinatorial problem relies on two foundations. First, we need a "good" linear programming representation of the problem—that is, a theoretical understanding of its polyhedral structure, which entails establishing an array of nicely structured facet-inducing inequalities. We have addressed this issue in this paper by presenting a large collection of facet-inducing inequalities for two related problems, highlighting their similarities and differences. The list of inequalities is of course by no means complete. For example, in another paper [Araque 1989] one of us describes a collection of valid "comb inequalities" for the vehicle routing problem.

The second, more practical requirement for developing successful branch-and-cut algorithms is solving the *separation problem* for these inequalities efficiently: namely, given a solution x that satisfies a subset of valid inequalities, determine whether x is feasible for the original problem and if not, exhibit a valid inequality that x violates. We currently know of no polynomial-time separation algorithm for any (non-trivial) class of facets discussed in this paper. In particular, the question of whether the packing or generalized subtour elimination constraints (4.1) can be separated efficiently is a compelling open problem. For computational purposes, however, good separation heuristics may suffice. Hall [1989] reports preliminary computational work using a separation heuristic for the packing or generalized subtour elimination constraints that is based on a heuristic of Padberg and Wolsey [1983] for the traveling salesman subtour elimination constraints. Araque [1989b,c], by visually identifying violated inequalities, has been able to solve some vehicle routing problems to proven optimality. In his work, he solved problems with 20, 34, and 48 customers. As shown in Table III, the packing (or generalized subtour elimination) constraints proved to be quite effective as did the multistar and the partial multistar constraints. We have not used any ladybug or clique cluster constraints since we do not have heuristics to identify them; thus we cannot comment on their practical usefulness.

TABLE III: Computational Experience with some Identical Customer Vehicle Routing Problems

Problem Number	Number of Nodes	Avg.* Number of Added Constraints	Constraint Breakdown
1	20	63	30 % degree-2 60 % packing (gen. subtour elim.) 3 % comb 7 % multistar
2	34	176	19 % degree-2 66 % packing (gen. subtour elim.) 2 % comb 11 % multistar 2 % partial multistar
3	48	318	15 % degree-2 42 % packing (gen. subtour elim.) 12 % comb 13 % multistar 18 % partial multistar

*Averaged over 4 instances of problems 1 and 2 and 1 instance of problem 3.

One extension of this research would, of course, be to further generalize the classes of facets that we have thus far established. Perhaps a more compelling extension would be to generalize our results to the problem with nonhomogeneous node demands (i.e., the nodes or customers have different demands and the routes or the subtrees off of the root must satisfy a capacity constraint). For this problem, we are aware only of some polyhedral results on the packing constraints [Cornuéjols and Harche 1989] and the trivial inequalities. Ultimately, it would be important to tackle more general models that reflect realistic complications: for example problems with many depots, scheduling or time-window considerations, nonhomogeneous truck fleets or nonhomogeneous demands, and/or precedence constraints.

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