

## XIX. CIRCUIT THEORY

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### RESEARCH OBJECTIVES

The general objective of this research is to extend our knowledge of the properties of electric networks, active or passive, reciprocal or nonreciprocal, linear or nonlinear. The topics that are being investigated are: (a) circuit models of linear nonreciprocal networks; (b) topological methods of analysis for linear nonreciprocal networks; (c) properties of RC networks that contain dynamically redundant elements; (d) applications of higher geometry to the general problem of impedance transformations; (e) circuit models of piecewise-linear traffic networks; (f) valuated algebras and their relationship to impedance networks and switching networks; and (g) general gain-bandwidth considerations.

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### A. SOME LIMITATIONS OF AMPLIFYING DEVICES

The properties of amplifying devices that are invariant to imbedding in various types of networks were investigated. If we assume that a device can be approximated by an arbitrary, linear, resistive network with terminal energy storage (i. e., a series inductance or shunt capacitance is present at every accessible terminal pair of an otherwise purely resistive device) there are  $(3n + 1)n/2$  parameters that must be specified to completely describe the device ( $n$  is the number of independent energy storage elements). It can be shown, however, that there are only  $(n+1)n/2$  properties that are invariant to imbedding the device in a transformer network. The specification of a device by its invariant properties thus reduces the number of parameters by approximately a factor of three; it should be noted, however, that this simplification destroys much of the original identity of the device. Such properties as the available gain without transformers cannot be determined from the invariant properties. If an unlimited supply of ideal transformers is allowed, the invariant properties offer a complete description of the device.

#### 1. Frequency Invariants

The invariant properties of a device can be specified in an infinite variety of ways, but preliminary work indicates that the most useful invariants are those with the dimensions of frequency. The frequency invariants of a multiterminal device are considered in detail in a technical report which is in preparation; the present discussion will be limited to three terminal devices all of which have energy storage elements of the same kind. In particular, let us assume that a device is described by conductance and

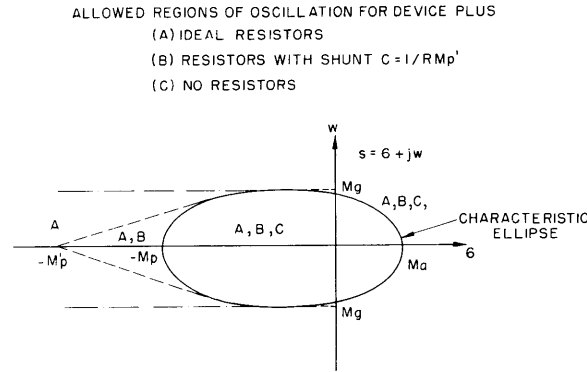


Fig. XIX-1. Characteristic ellipse and allowed regions of oscillation.

capacitance matrices, the capacitance matrix being both symmetric and positive definite and the conductance matrix arbitrary (Eq. 1). The dual case of a resistive network with series inductance is identical in principle and need not be considered separately.

$$[I] = [Y][E] \quad [Y] = [G] + s[C] \quad (1)$$

A set of invariants which is particularly convenient for this type of device is:

$$-Ma = \text{smallest eigenvalue of } [G + G^t][C]^{-1/2} \quad (2)$$

$$Mp = \text{largest eigenvalue of } [G + G^t][C]^{-1/2} \quad (3)$$

$$\pm jMg = \text{eigenvalues of } [G - G^t][C]^{-1/2} \quad (4)$$

These M parameters are invariant to imbedding the device in a transformer network and can be interpreted as follows:

$2Ma$  is the maximum possible rate of energy generation (i. e., negative dissipation) per unit energy stored; thus it is a measure of the active properties of the device.

$2Mp$  is the maximum possible rate of energy dissipation per unit energy stored; thus it is a measure of the passive properties of the device.

$Mg$  is the maximum rate at which energy can be transmitted losslessly through the device per unit energy stored; thus it is a measure of the gyrator properties of the device.

We can also express these limitations in terms of a characteristic ellipse in the complex plane. It is readily shown that if an arbitrary number of a particular device is imbedded in an arbitrary transformer network, all of the natural frequencies of the resultant network must lie within an ellipse bounded by  $s = Ma$ ,  $s = -Mp$ , and  $s = \pm jMg$ ,

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as shown in Fig. XIX-1. If a passive conductance  $G$  with shunt capacity  $C$  is available, there is another invariant,  $Mp' = G/C$ . The allowed region for natural frequencies is then extended to include the region bounded by the ellipse and tangents from  $s = Mp'$  to the ellipse, as shown in Fig. XIX-1.

### 2. The Characteristic Ellipse

The characteristic ellipse described above has some interesting properties; a few of these are summarized below.

a. The foci of this ellipse are the natural frequencies of the device with the terminals open-circuited. The stability of the device is thus determined by

$$\operatorname{Re} \left\{ \frac{Ma - Mp}{2} + \left[ \left( \frac{Ma + Mp}{2} \right)^2 - Mg^2 \right]^{1/2} \right\} < 0 \rightarrow \text{stable} \\ > 0 \rightarrow \text{unstable} \quad (5)$$

b. If the device is open-circuit stable, the maximum frequency of oscillation,  $\omega_m$ , for any number of the device imbedded in an arbitrary network containing transformers, resistors, and capacitors, is given by the intersection of the ellipse with the  $s = j\omega$  axis.

$$\omega_m = Mg \left[ 1 - \left( \frac{Ma - Mp}{Ma + Mp} \right)^2 \right]^{1/2} \quad (6)$$

c. The dc unilateral power gain,  $U$ , is given by

$$\frac{U}{U - 1} = \frac{Mg^2}{MaMp} \quad (7)$$

Since the  $M$  parameters are invariant,  $U$  is invariant, as Mason has shown (1).

d. If a number of device  $D_1$ , described by  $Ma_1$ ,  $Mp_1$ , and  $Mg_1$  are combined to create a "new" three-terminal device  $D_2$ , described by  $Ma_2$ ,  $Mp_2$ , and  $Mg_2$ , the necessary and sufficient condition that  $D_2$  be realizable is that the ellipse describing  $D_2$  lie wholly within the characteristic ellipse of  $D_1$ . Unless  $D_1$  has the same  $M$  parameters as  $D_2$ , it will be necessary to use two and only two of  $D_1$  to synthesize a single  $D_2$ . If a capacitor is available, it may be possible to effect the synthesis with a single  $D_1$ . If a resistor described by  $Mp'$  is available, the characteristic ellipse of  $D_2$  may lie anywhere within the triangular-elliptical region which was described previously and is shown in Fig. XIX-1.

e. If we use properties (c) and (d), we see that it is possible to construct a "new" device with the same  $Ma$  and  $Mp$  but with a lower  $U/(U-1)$ . Thus we come to the

surprising conclusion that it is easier to increase than to decrease  $U$ . In principle, we can realize a  $U$  of infinity merely by combining two devices, or one device and a positive resistor, in a transformer network (assuming that the original device has a  $U$  greater than unity).

### 3. Device Synthesis

As an example of the application of some of these ideas to network synthesis, let us attempt to use a triode to construct a unilateral amplifying device that has equal input and output admittances,  $y_{11} = y_{22} = g + sc$ , with  $g/c = W$ , and with an available unilateral gain  $U$ . From the preceding analysis, we can determine the maximum  $U$  that can be realized for a prescribed "bandwidth,"  $W$ .

If the triode is described by

$$[G] = \begin{bmatrix} 0 & 0 \\ g_m & g_m/\mu \end{bmatrix} \quad [C] = \begin{bmatrix} C_g & -C_m \\ -C_m & C_p \end{bmatrix} \quad C_g C_p > C_m^2 \quad (8)$$

we can readily calculate the  $M$  parameters

$$\begin{aligned} 2Ma &= g_m / [C_1^2 + C_2^2]^{1/2} + C_1 & C_1 &= C_g/\mu + C_m \\ 2Mp &= g_m / [C_1^2 + C_2^2]^{1/2} - C_1 & C_2^2 &= C_g C_p - C_m^2 \\ 2Mg &= g_m / C_2 \end{aligned} \quad (9)$$

The device which we desire to synthesize is described by a circle with center at  $s = -W$  and radius  $U^{1/2}W$ . Noting that the radius of the circle is the voltage gain-bandwidth product, we can readily show that the maximum possible  $U^{1/2}W$  is dependent on  $W$ , as shown in Fig. XIX-2. If  $Mg$  and  $Mp'$  are large, we have the relation

$$[U^{1/2} - 1] W \leq Ma \quad (10)$$

which is similar to the usual gain-bandwidth product that has been previously discussed (2). For the special case of a pentode, we have  $1/\mu = 0$ ,  $C_m = 0$ , so that  $Ma = Mp = Mg$ . If we allow ideal resistors with  $Mp' = -\infty$ , we have the relation

$$U^{1/2}W = Ma \quad (11)$$

which is identical with the familiar gain-bandwidth product.

To carry this example further, let us assume that we wish to use a 6AF4A high-frequency triode to synthesize a device of the kind described above, with  $W = 10^9$  radians

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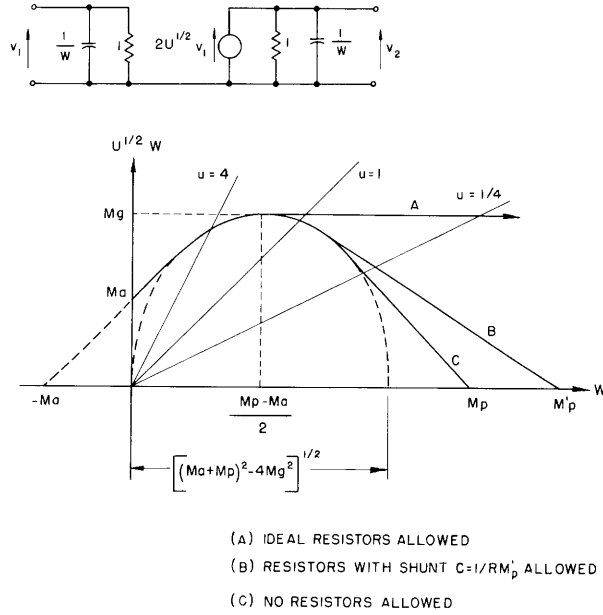


Fig. XIX-2. Maximum gain-bandwidth product versus bandwidth.

per second and maximum unilateral power gain. A 6AF4A triode, when it is biased to its maximum ratings,

$$e_b = 80 \text{ volts}, \quad e_c = -1.1 \text{ volts}, \quad i_b = 28 \text{ ma} \quad (12)$$

is described by

$$g_m = 11 \times 10^{-3} \text{ mhos}, \quad \mu = 20 \quad (13)$$

$$C_g = 4.1 \text{ } \mu\mu\text{f}, \quad C_p = 2.35 \text{ } \mu\mu\text{f}, \quad C_m = 1.9 \text{ } \mu\mu\text{f}$$

If we allow, in addition, some wiring capacities:

$$C_{g'} = 0.4 \text{ } \mu\mu\text{f}, \quad C_{p'} = 0.4 \text{ } \mu\mu\text{f}, \quad C_{m'} = 0.2 \text{ } \mu\mu\text{f} \quad (14)$$

we eventually determine the M parameters:

$$Ma = 0.887 \times 10^9 \text{ sec}^{-1} \quad Mp = 3.62 \times 10^9 \text{ sec}^{-1} \quad (15)$$

$$Mg = 1.79 \times 10^9 \text{ sec}^{-1}$$

Thus, for  $W = 10^9$  radians per second, the maximum possible U is 2.0. A typical circuit for realizing this maximum gain is shown in Fig. XIX-3. Although this circuit requires transformers and may not be practical for most applications, the preceding analysis enables us to establish upper bounds on the gain that can be realized.

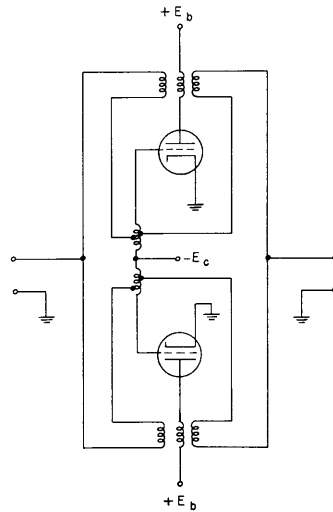


Fig. XIX-3. Typical amplifier synthesis technique.

A technical report describing this work is in preparation. This report discusses the frequency invariants for the general multiterminal device that contains both L and C. The report also describes an amplifier that utilizes only tubes, transformers, and inductors, and is capable of realizing arbitrarily large gains and bandwidths with a Butterworth frequency response. The analysis is extended to show how the frequency invariants are related to the properties of a space-charge-limited device. It is shown that the M parameters are proportional to  $v/d$ , where  $v$  is the maximum electron velocity and  $d$  is a characteristic dimension of the device.

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#### References

1. S. J. Mason, Power gain in feedback amplifiers, Technical Report 257, Research Laboratory of Electronics, M.I.T., Aug. 25, 1953; Trans. IRE, Vol. PGCT-1, No. 2, 20-25 (June 1954).
2. Quarterly Progress Report, Research Laboratory of Electronics, M.I.T., July 15, 1956, pp. 96-101.

#### B. PIECEWISE-LINEAR NETWORK THEORY

This work, which was submitted as a thesis in partial fulfillment of the requirements for the degree of Doctor of Science, Department of Electrical Engineering, M.I.T., May 1956, is in process of publication as Technical Report 315.

T. E. Stern