

## XXII. MATHEMATICS

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### A. ON EFFECTIVE COMPUTABILITY

The theory of general recursive computability (1) concerns computations that can be carried out by a Turing machine ("T-machine," a finite digital computer with indefinitely large memory). A function (integers into integers) computable on a T-machine is called a "recursive" function. In general, all true assertions of the form "T-machine M computes a function" (as M varies) cannot be effectively listed (and any formal logic is incomplete with respect to them). There has been interest in various "natural" sub-families of the recursive functions, for which T-machines can be effectively listed. The primitive recursive functions are such a subfamily. We suggest the following larger family: the recursive functions with (each of) which we can associate a T-machine M with the property that the assertion "T-machine M computes a function" is provable as a combinatorial statement within ordinary number theory; we require, of course, that M compute the function with which it is associated. We call these functions "p-functions" (provable functions); "p-1-1-functions" and "p-permutations" are similarly defined. The last form a group (the improvement over known properties of the primitive recursive functions) and give a stronger isomorphism between sets of integers than that given by the group of all recursive permutations. Many of the isomorphism results of recursive function theory can be strengthened to isomorphism under p-permutations. Other results include that the recursive sets (a set is recursive if membership in it can be decided by a T-machine) fall into an infinity of isomorphism types of which one is minimum (i. e., most effective) relative to an appropriate ordering by reducibility. We also observe (by simple extension of known results) that the sets of provable theorems of any two consistent mathematical systems containing elementary number theory are isomorphic in this strong sense.

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### References

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### B. A NEW INEQUALITY IN COMMUNICATION THEORY

Consider a continuous signal  $x$  which is transmitted by making it a parameter on which a second random variate  $y$  depends. After receiving  $y$ ,  $x$  is estimated in such a way as to minimize the rms error on both the a priori distribution of  $x$  and the

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conditional distribution of  $y$ , for fixed  $x$ . Inequalities on the rms error are well known in mathematical statistics, but they deal exclusively with the special case in which the exact a priori distribution of  $x$  is unknown, that is to say, when this distribution is most unfavorable.

By extending the argument, with the help of a fundamental identity in communication theory, a lower bound to the root mean square error was found, which reduces to the known bound in the special case mentioned above. In the general case, the approximation is far more sharp. If  $y$  is obtained by adding a random noise to  $x$ , it can be proved that the bound is attained if and only if both the noise and the signal  $x$  are normally distributed.

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### C. VALUATED ALGEBRAS

An abstraction of some relationships common to lattice theory and electric networks has given rise to a new subfield of abstract algebras, called, at present, "valuated algebras." The following is an abstract of some of the basic definitions and the direction of research. They will be presented later in more detail.

Let  $S$  be a set that is closed under binary operations  $\cup$  and  $\cap$ , and let  $G$  be a commutative group under  $+$ . A map  $v$  of  $S$  into  $G$  satisfying  $v(a \cup (b \cup a)) = v((a \cup b) \cup a)$ ,  $v(a \cap (b \cap c)) = v((a \cap b) \cap c)$ , and  $v(a \cup b) + v(a \cap b) = v(a) + v(b)$  is called a "valuation." This valuation is a generalization of lattice valuations (1) and network impedance, but is not to be confused with certain other mathematical valuations (2).

This definition leads directly to an implicit function theorem (series-parallel reciprocity) and consequently to some theorems on valuation congruences. In particular, valuation congruences on  $\cup$  or  $\cap$  are valuation congruences on both simultaneously. Proceeding further along the line of lattice valuations, we are led to several problems in lattices and network transformations. One of the specialized results gives a basis for a theory of information (3).

Self-valuations, valuations in which the  $\cup$ ,  $\cap$  and  $+$  operations are defined on the same set, are sometimes extendable beyond series-parallel graphs (polynomials) into general graphs or generalized networks (4). If  $\cup$  (or  $\cap$ ) distributes over  $+$ , we are led to Boolean rings and switching algebra. If  $+$  distributes over  $\cup$  (or  $\cap$ ), we are led to division semirings,  $\ell$ -groups (1), and resistor networks. A simple extendable self-valuation not satisfying either distributive law is derivable from the solution of C. E. Shannon's switching game.

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#### References

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