IX. STATISTICAL COMMUNICATION THEORY

Prof. Y. W. Lee	D. A. George	T. G. Stockham
Prof. A. G. Bose	I. M. Jacobs	R. E. Wernikoff
M. B. Brilliant	A. H. Nuttall	C. E. Wernlein, Jr.
D. A. Chesler		G. D. Zames

A. INVARIANCE OF CORRELATION FUNCTIONS UNDER NONLINEAR TRANSFORMATIONS

In previous work on the invariance property (1), the output crosscorrelation function was required to be proportional to the input crosscorrelation function for all values of the argument. This statement is adequate if (a particular) one of the input processes has a zero mean. However, the statement must be revamped for arbitrary inputs, as follows. Consider the system of Fig. IX-1, in which $i_1(t)$ and $i_2(t)$ are stationary random processes, and f is a time-invariant nonlinear no-memory network. We define the input and output crosscorrelation functions as

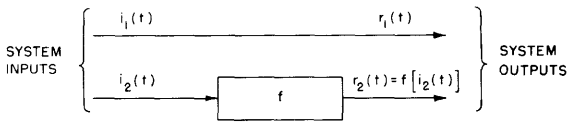

Fig. IX-1. Nonlinear transformation of one input process.

$$\phi(\tau) = \overline{i_1(t)\, i_2(t+\tau)}$$

and

$$\Phi_f(\tau) = \overline{r_1(t)\, r_2(t+\tau)}$$

$$= \overline{i_1(t)\, f[i_2(t+\tau)]}$$

Now, if we require

$$\Phi_f(\tau) = C_f\, \phi(\tau) \quad \text{for all } \tau \tag{1}$$

where C_f is a constant dependent only on f, then for f a constant, say k, we would have

$$\mu_1 k = C_f\, \phi(\tau)$$

if we let $\mu_1 = \overline{i_1(t)}$. This requires $\phi(\tau)$ to be a constant, and is obviously much too restrictive. We therefore define the ac input and output crosscorrelation functions as

$$\phi_{ac}(\tau) = \overline{[i_1(t) - \mu_1][i_2(t+\tau) - \mu_2]} = \phi(\tau) - \mu_1\mu_2$$

and

$$\Phi_{f_{ac}}(\tau) = \overline{[r_1(t) - \mu_1]\left[r_2(t+\tau) - \lambda_{2_f}\right]} = \Phi_f(\tau) - \mu_1\, \lambda_{2_f}$$

where $\mu_2 = \overline{i_2(t)}$, and $\lambda_{2_f} = \overline{r_2(t)}$.

The invariance property is now stated as

$$\Phi_{f_{ac}}(\tau) = C_f \phi_{ac}(\tau) \quad \text{for all } \tau \tag{2}$$

This form is identical with Eq. 1, when $\mu_1 = 0$, but does not suffer when f is a constant. For then we have $0 = C_f \phi_{ac}(\tau)$, and, consequently, $C_f = 0$. This leaves $\phi_{ac}(\tau)$ arbitrary. As a result of requirement 2, some of the previous results (1) are slightly changed, as follows.

Letting

$$g(x_2, \tau) = \int_{-\infty}^{\infty} (x_1 - \mu_1)\, p(x_1, x_2; \tau)\, dx_1 \tag{3}$$

where $p(x_1, x_2; \tau)$ is the joint probability density function of the two input processes, we have

$$\phi_{ac}(\tau) = \int x_2\, g(x_2, \tau)\, dx$$

and

$$\Phi_{f_{ac}}(\tau) = \int f(x_2)\, g(x_2, \tau)\, dx_2$$

There follows then, if we assume that the invariance property holds, the separability of g:

$$g(x_2, \tau) = h(x_2)\, \phi_{ac}(\tau) \quad \text{almost everywhere in } x_2 \tag{4}$$

where $h(x_2)$ is a function only of x_2.

If $i_1(t)$ and $i_2(t)$ are the same process, $i_1(t) = i_2(t)$, assuming that Eq. 2, and therefore Eq. 4, holds, we have

$$g(x_2, \tau) = \frac{x_2 - \mu}{\sigma^2}\, p(x_2)\, \phi_{ac}(\tau)$$

from which it follows that

$$h(x_2) = \frac{x_2 - \mu}{\sigma^2}\, p(x_2)$$

It is seen that the essential change is to operate on the ac component of the input process rather than on the input process directly. The extension to nonstationary inputs and time-varying devices is straightforward.

A variation of the invariance property may be worth noting. Suppose that

$$\Phi_{f_{ac}}(\tau) = \sum_{k=0}^{N} C_{f_k} [\gamma(\tau)]^k \quad \text{for all } \tau, \quad N \geq 2$$

where $C_{f_o}, \ldots, C_{f_N}$ are constants dependent only on f, and $\gamma(\tau)$ is an arbitrary function. We can then show that g is separable as

$$g(x_2, \tau) = \sum_{k=0}^{N} h_k(x_2) [\gamma(\tau)]^k$$

where the functions $h_o(x_2), \ldots, h_N(x_2)$ depend only on x_2. It also follows that

$$\phi_{ac}(\tau) = \sum_{k=0}^{N} d_k [\gamma(\tau)]^k$$

where $d_o, \ldots, d_N$ are constants.

A. H. Nuttall

References

1. A. H. Nuttall, Invariance of correlation functions under nonlinear transformations, Quarterly Progress Report, Research Laboratory of Electronics, M.I.T., April 15, 1957, p. 69.

B. OPTIMUM LINEAR FILTERS FOR FINITE TIME INTEGRATIONS

In some applications, a short-time estimate of the mean of a random process is desired. One way to accomplish this is to pass the random process through a lowpass filter for T seconds, and call the output divided by T the mean of the input process (except for a known scaling constant). But, because of the limited time T of integration (filtering), the output at time T will be a random variable with fluctuations around its mean value. It is of interest to reduce these fluctuations as much as possible, and still restrict ourselves to finite time integrations. Since we are only integrating for T seconds, the only part of the impulse response of the lowpass filter that accomplishes any smoothing of the input process is that part for $0 \leq t \leq T$, where we define h(t) as the response at time t to a unit impulse applied at time 0. We therefore restrict ourselves to impulse responses that are nonzero only for $0 \leq t \leq T$, and attempt to distribute the area of h(t) over the interval (0, T) so as to accomplish an optimum smoothing of the input process. Here we define smoothing as reduction of the ratio of the average fluctuating power to the squared average component in the output process. That is, we try to maximize the output signal-to-noise power ratio.

The solution for a special type of input process has been obtained by Davenport,

Johnson, and Middleton (1). Their results show that (for their particular random process) the impulse response of the optimum linear filter should be constant over the open interval (0, T), with impulses at the end points 0 and T, and that the "flat" filter, in which the impulse response is absolutely constant over the closed interval (0, T) is not much worse than the optimum linear filter for any value of T. In this report the flat filter will be compared with the optimum linear filter for a broader class of input processes.

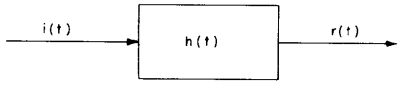

Fig. IX-2. Linear network.

Consider the system of Fig. IX-2, in which i(t) is a stationary random process, h(t) is the impulse response of the linear network, and r(t) is the stationary output process. Then

$$r(t) = \int_0^T h(u)\, i(t-u)\, du = \int h(u)\, i(t-u)\, du \tag{1}$$

since

$$h(t) = \begin{cases} 0 & \text{for } t < 0, \quad t > T \\ \text{arbitrary} & \text{for } 0 \leqslant t \leqslant T \end{cases} \tag{2}$$

[All integrals without limits are over the range $(-\infty, \infty)$.]

We define an error E as

$$E = \frac{\overline{[r(t) - \overline{r(t)}]^2}}{[\overline{r(t)}]^2} \tag{3}$$

and attempt to minimize E subject to the constraint in definition 2. Substitution of Eq. 1 in Eq. 3 gives

$$E = \frac{\iint h(u)\, h(v)\, \phi(v-u)\, du\, dv}{(\bar{i})^2 \iint h(u)\, h(v)\, du\, dv} \tag{4}$$

where $\phi(u)$ is the autocorrelation function of the ac component of the input process. That is,

$$\phi(u) = \phi_{i-\bar{i}}(u) = \phi_i(u) - (\bar{i})^2$$

It is obvious from Eq. 4 that if $h_1(t)$ is a solution to the minimization problem, then

$ch_1(t)$ is also a solution, in which c is an arbitrary constant, since it gives the same error E. Therefore, for convenience, we set

$$\int h(t)\, dt = 1 \tag{5}$$

Applying standard variational techniques to Eq. 4, subject to the constraint in Eq. 5, we obtain the following integral equation for h(t):

$$\int h(u)\, \phi(v-u)\, du = \lambda \quad \text{for } 0 \leq v \leq T \tag{6}$$

where λ is a Lagrangian multiplier (arbitrary constant) to be determined later from the constraint expressed in Eq. 5.

We now restrict ourselves to a special class of input processes, namely, those for which the input power density spectrum for the ac component has only poles:

$$\Phi(\omega) = \frac{1}{\sum_{i=0}^{N} a_i\, \omega^{2i}} \tag{7}$$

where

$$\Phi(\omega) = \frac{1}{2\pi} \int \phi(u)\, e^{-j\omega u}\, du$$

This power density spectrum approximates a number of practical spectra by proper choice of the constants $\{a_i\}$.

For this power spectrum (Eq. 7) the solution to Eq. 6 is of the form (see ref. 2)

$$h(t) = A + C_1\delta(t) + \ldots + C_N\delta^{(N-1)}(t) + D_1\delta(t-T) + \ldots + D_N\delta^{(N-1)}(t-T)$$

$$0 \leq t \leq T \tag{8}$$

where A, $C_1, \ldots, C_N$, $D_1, \ldots, D_N$ are 2N + 1 unknown constants, and $\delta^{(\nu)}(t)$ is the ν^{th} derivative of the unit impulse function $\delta(t)$. These constants are determined by substituting Eq. 8 in Eq. 6 and requiring equality for $0 \leq v \leq T$. This results in 2N equations. Substitution of Eq. 8 in Eq. 5 yields one more equation. The composite 2N + 1 equations determine h(t). This is a straightforward but tedious procedure. In the present case we are not greatly interested in the explicit form of the optimum impulse response, but rather in the attendant error, defined in Eq. 3, as compared with the error caused by the flat filter, for which

$$h_f(t) = \frac{1}{T}, \quad 0 \leq t \leq T \tag{9}$$

From the forms of the optimum filter, expressed in Eq. 8, and the flat filter, expressed in Eq. 9, we see that the flat filter would be much easier to synthesize. Then, if the errors caused by the two filters are not widely different, the flat filter can be used to perform the desired smoothing rather than the optimum filter. Accordingly, only the errors for the flat and optimum filters will be evaluated, and not their explicit forms. It will be shown that the evaluation of the error caused by the optimum filter is easier to obtain than the exact impulse response of the optimum filter. Thus the smoothing of the filter may be investigated without knowledge of the exact impulse response, which entails a lengthy calculation.

Substituting Eq. 6 in Eq. 4 and using Eq. 5, we obtain the expression for the minimum error:

$$E_{min} = \frac{\lambda}{(\bar{i})^2} \tag{10}$$

We must evaluate, therefore, only λ, which is obtained by substituting Eq. 8 in Eq. 6 and by requiring that the left-hand side be independent of v for $0 \leq v \leq T$:

$$A \int_0^T \phi(v-u)\, du + C_1\phi(v) + \ldots + C_N\phi^{(N-1)}(v)$$

$$+ D_1\phi(T-v) + \ldots + D_N\phi^{(N-1)}(T-v) = \lambda \quad \text{for } 0 \leq v \leq T$$

or

$$\lambda = A \int_0^v \phi(u)\, du + C_1\phi(v) + \ldots + C_N\phi^{(N-1)}(v)$$

$$+ A \int_0^{T-v} \phi(u)\, du + D_1\phi(T-v) + \ldots + D_N\phi^{(N-1)}(T-v) \quad \text{for } 0 \leq v \leq T$$

Now, let us write

$$\lambda = g_1(v) + g_2(T-v) \quad \text{for } 0 \leq v \leq T$$

where

$$g_1(v) = A \int_0^v \phi(u)\, du + C_1\phi(v) + \ldots + C_N\phi^{(N-1)}(v) \tag{11}$$

and

$$g_2(T-v) = A \int_0^{T-v} \phi(u)\, du + D_1\phi(T-v) + \ldots + D_N\phi^{(N-1)}(T-v)$$

From the form of $\Phi(\omega)$ given in Eq. 7 (for $\Phi(\omega)$ that contains only single-order poles), it follows that

$$\phi(u) = \sum_{i=1}^{N} a_i e^{-\beta_i u} \quad \text{for } u \geq 0 \tag{12}$$

where a_i and β_i are complex, and $\mathrm{Re}(\beta_i) > 0$ for all i. Therefore $g_1(v)$ has terms only of the form $e^{-\beta_i v}$ for $v \geq 0$, and $g_2(T-v)$ has terms only of the form $e^{+\beta_i v}$ for $v \leq T$. Therefore, not only must $g_1(v) + g_2(T-v)$ be independent of v for $0 \leq v \leq T$, but also $g_1(v)$ and $g_2(T-v)$ must each individually be independent of v for $0 \leq v \leq T$, because $e^{-\beta_i v}$ dependence cannot be cancelled by $e^{\beta_i v}$ dependence. (For $\Phi(\omega)$ that contains higher-order poles, the same conclusions hold for $g_1(v)$ and $g_2(T-v)$.) Thus, $g_1(v)$ is independent of v for $0 \leq v \leq T$.

But, by looking at the form of $g_1(v)$ as given in Eq. 11, we see that, since $\phi(u)$ is a well-behaved function (see Eq. 12), $g_1(v)$ must be independent of v for $v \geq 0$. Similarly, $g_2(T-v)$ must be independent of v for $v \leq T$. Therefore

$$g_1(v) = k_1 \quad \text{for } v \geq 0 \tag{13}$$

and

$$g_2(T-v) = k_2 \quad \text{for } v \leq T \tag{14}$$

Therefore

$$A \int_0^{v} \phi(u)\, du + C_1\phi(v) + \ldots + C_N\phi^{(N-1)}(v) = k_1, \quad v \geq 0 \tag{15}$$

and

$$A \int_0^{T-v} \phi(u)\, du + D_1\phi(T-v) + \ldots + D_N\phi^{(N-1)}(v) = k_2, \quad v \leq T \tag{16}$$

If we let $v = \infty$ in Eq. 15, and $v = -\infty$ in Eq. 16, we obtain

$$A \int_0^{\infty} \phi(u)\, du = k_1 = k_2 \tag{17}$$

Now, if we substitute Eq. 17 in Eq. 15, N equations result, from which

$C_1, C_2, \ldots, C_N$ can be expressed in terms of A. Substitution of Eq. 17 in Eq. 16 results in N equations for $D_1, D_2, \ldots, D_N$ in terms of A. Since only one set of constants, $C_1, \ldots, C_N$, will satisfy Eq. 15 for given A, it is obvious that $D_1 = C_1, \ldots, D_N = C_N$, since Eq. 16 can be obtained from Eq. 15, as far as v-dependence goes, by substituting T - v for v. Thus, the optimum impulse response is symmetrical (even) about the point T/2. Thus we have

$$g_1(v) = g_2(v) = k_1 \quad \text{for } 0 \leq v \leq T$$

and

$$g_1(v) + g_2(T-v) = \lambda \quad \text{for } 0 \leq v \leq T$$

Therefore

$$g_1(v) = g_2(v) = \frac{\lambda}{2} \quad \text{for } 0 \leq v \leq T$$

Also, by Eq. 13 and Eq. 14, we have

$$g_1(v) = \frac{\lambda}{2} = k_1 \quad \text{for } v \geq 0 \tag{18}$$

and

$$g_2(T-v) = \frac{\lambda}{2} \quad \text{for } v \leq T$$

Therefore, from Eqs. 10, 17, and 18, we obtain

$$E_{min} = \frac{A \int \phi(u)\, du}{(\bar{i})^2} = \frac{2\pi A \Phi(0)}{(\bar{i})^2} \tag{19}$$

Thus, to evaluate the minimum error, we must evaluate only one constant, A, and not the N + 1 constants, $A, C_1, \ldots, C_N$ ($D_1 = C_1, \ldots, D_N = C_N$). This will simplify our work considerably, for higher values of N.

For the flat filter, the error becomes

$$E_f = \frac{1}{(\bar{i})^2 T^2} \int_0^T \int_0^T \phi(u-v)\, du\, dv$$

$$E_f = \frac{2}{(\bar{i})^2 T^2} \int_0^T (T-u)\, \phi(u)\, du$$

$$E_f = \frac{2}{(\bar{i})^2 T^2} \int \Phi(\omega) \frac{e^{j\omega T} - j\omega T - 1}{(j\omega)^2}\, d\omega \tag{20}$$

which can be compared with Eq. 19 for specific cases. We remark that the results for E_{min} hold only for the case in which the input power density spectrum has only poles; that is, $\Phi(\omega)$ is given by Eq. 7. Several interesting and practical examples follow.

EXAMPLE 1: N = 0.

Then, $h(t) = A = 1/T$, and $h_f(t) = 1/T$.

Therefore, for white noise, the optimum filter is the flat filter.

EXAMPLE 2: N = 1.

Then, $h(t) = A + C[\delta(t) + \delta(t-T)]$.

If we use Eq. 5, $AT + 2C = 1$.

From Eqs. 11, 18, and 17,

$$g_1(0) = C\,\phi(0) = A\,\pi\,\Phi(0)$$

Therefore

$$A = \frac{1}{T + \dfrac{2\pi\,\Phi(0)}{\phi(0)}}$$

Then, from Eq. 19, we have

$$E_{min} = \frac{\phi(0)}{(\bar{i})^2}\,\frac{1}{1 + T\,\dfrac{\phi(0)}{2\pi\,\Phi(0)}}$$

Defining $E_o = \dfrac{\phi(0)}{(\bar{i})^2}$ (the no-smoothing error), and an effective correlation time T_c as

$$\int_0^\infty \phi(u)\,du = \phi(0)\,T_c = \pi\,\Phi(0) \tag{21}$$

we have

$$E_{min} = \frac{E_o}{1 + \dfrac{T}{2T_c}}$$

Thus, in this case, only the ratio of integration time T to effective correlation time T_c is of importance in determining the effective smoothing. We note that $T_c = \dfrac{1}{\omega_o}$ where ω_o is the value of ω for which $\Phi(\omega)$ is one-half of its maximum value $\Phi(0)$.

For the flat filter, from Eq. 20, we have

$$E_f = E_o \, 2 \, \frac{e^{-\frac{T}{T_c}} + \frac{T}{T_c} - 1}{(T/T_c)^2}$$

In order to compare the two filters, the ratios E_{min}/E_o and E_f/E_o are plotted in Fig. IX-3a. We see that the flat filter is almost as good as the optimum filter for all values of integration time T. This process was studied by Davenport, Johnson, and Middleton (1).

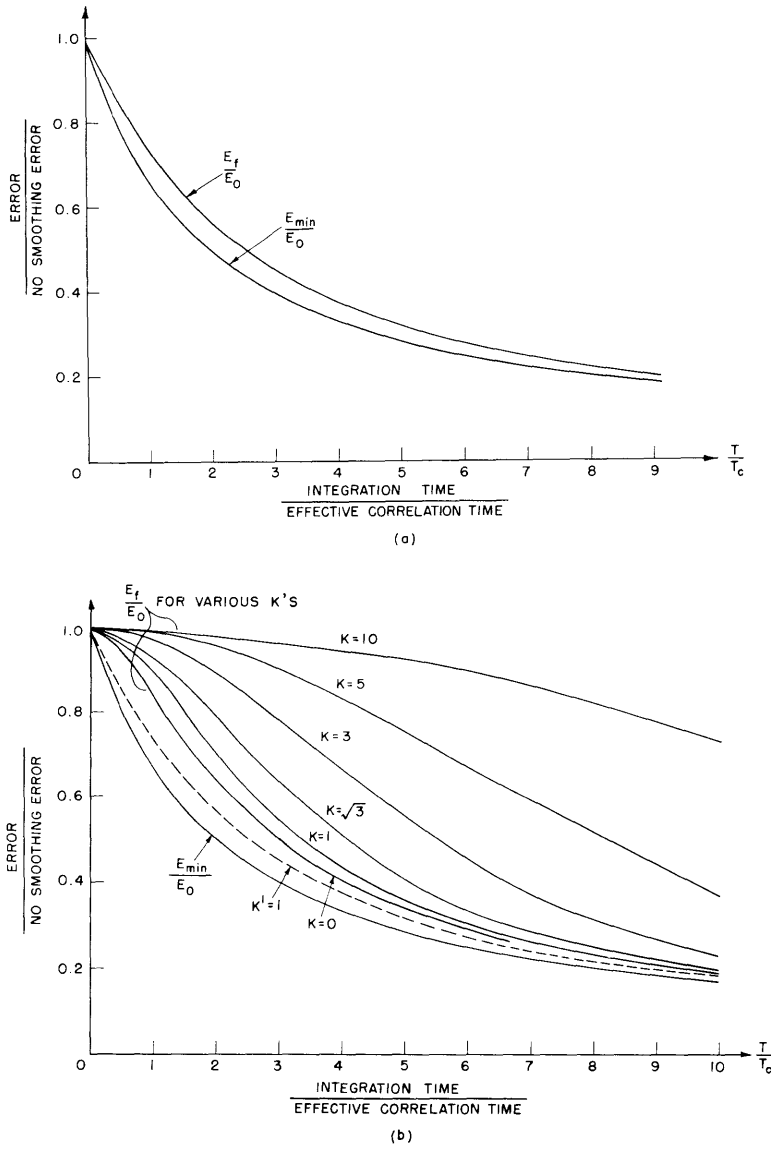

Fig. IX-3. Errors for flat and optimum filters: (a) for N = 1; (b) for N = 2.

EXAMPLE 3: N = 2.

Then $h(t) = A + C_1[\delta(t) + \delta(t-T)] + C_2[\delta^{(1)}(t) + \delta^{(1)}(t-T)]$.

The use of Eq. 5 yields $AT + 2C_1 = 1$.

Also, from Eqs. 11, 18, and 17, we have

$$g_1(0) = C_1\,\phi(0) + C_2\,\phi^{(1)}(0) = A\,\pi\,\Phi(0)$$

But

$$\phi^{(1)}(0) = \int j\omega\,\Phi(\omega)\,d\omega = 0$$

since

$$\Phi(\omega) = \frac{1}{\sum_{i=0}^{2} a_i \omega^{2i}}$$

Therefore

$$A = \frac{1}{T + \dfrac{2\pi\,\Phi(0)}{\phi(0)}}$$

and so

$$E_{min} = \frac{E_o}{1 + T/2T_c}$$

with T_c as defined in Eq. 21. Now,

$$\Phi(\omega) = \frac{1/a_2}{(\omega + \alpha + j\beta)(\omega + \alpha - j\beta)(\omega - \alpha + j\beta)(\omega - \alpha - j\beta)}$$

Let $\alpha = K\beta$, where K measures the degree of damping in the correlation function. Then

$$\Phi(\omega) = \frac{1/a_2}{\omega^4 + 2\beta^2(1 - k^2)\,\omega^2 + \beta^4(1 + k^2)^2} \tag{22}$$

From Eq. 20, we have

$$E_f = E_o \frac{1}{2K x^2}\left[K(3 - K^2)\left(e^{-\frac{2x}{1+K^2}} \cos \frac{2Kx}{1 + K^2} + \frac{2x}{1 + K^2} - 1\right)\right.$$

$$\left. + (3K^2 - 1)\left(\frac{2Kx}{1 + K^2} - e^{-\frac{2x}{1+K^2}} \sin \frac{2Kx}{1 + K^2}\right)\right], \quad 0 \leq K < \infty$$

where $x = T/T_c$. We now plot in Fig. IX-3b the ratios E_{min}/E_o, and E_f/E_o for different values of K.

There is also a case (for N = 2) in which all the poles of $\Phi(\omega)$ lie on the j-axis in the ω-plane. Thus

$$\Phi(\omega) = \frac{1/a_2}{\left(\omega^2 + r_1^2\right)\left(\omega^2 + r_2^2\right)}, \quad r_2^2 \geq r_1^2 > 0$$

is a special case of Eq. 22 if we let

$$K = j \frac{r_2 - r_1}{r_2 + r_1} = jK' \quad (r_2 \geq r_1)$$

and

$$\beta = \frac{r_2 + r_1}{2}$$

Then we obtain

$$\frac{E_f}{E_o} = \frac{1}{2K' x^2}\left[K'(3 + K'^2)\left(\exp\left(-\frac{2x}{1 - K'^2}\right)\cosh \frac{2K' x}{1 - K'^2} + \frac{2x}{1 - K'^2} - 1\right)\right.$$

$$\left. - (3K'^2 + 1)\left(\frac{2K' x}{1 - K'^2} - \exp\left(-\frac{2x}{1 - K'^2}\right)\sinh \frac{2K' x}{1 - K'^2}\right)\right]$$

where $0 \leq K' \leq 1$, and $x = T/T_c$.

The curves for E_f/E_o for $0 \leq K' \leq 1$ all lie between the curve E_f/E_o for $K = K' = 0$ and the curve E_{min}/E_o. This follows from the fact that the average output fluctuating power is given by the infinite integral of the squared modulus of the transfer function multiplied by the power density function of the alternating component of the input process, i.e., the numerator of Eq. 4. As K' varies from 0 to 1, the input power density function approaches more closely the power density spectrum for N = 1. The poles at

$\pm jr_2$ have less effect on the shape of $\Phi(\omega)$ near $\omega = 0$. Therefore we should expect that as K' approaches 1, the error will approach the error for N = 1. This (dotted) curve is shown in Fig. IX-3b as K' = 1.

Notice that for $K \leq 1$, the error for the optimum filter (E_{min}/E_o) and the error for the flat filter (E_f/E_o) are not widely different for any T. However, for larger values of K (say, $K \geq 5$), a marked improvement can be achieved, for certain ranges of T, by using the optimum filter rather than the flat filter. For example, for $T/T_c = 10$, the ratio of errors is 4.38 for K = 10. This behavior for large K is to be expected, since $\Phi(\omega)$ has a peak at $\omega = \beta(K^2 - 1)^{1/2}$ (for $K \geq 1$) of value $1/4\, a_2\beta^4K^2$, which may be significantly different from $\Phi(\omega)$ at $\omega = 0$, namely $1/\beta^4(1 + K^2)^2$. Consequently, since the (squared modulus of the) flat filter pays no attention to the location of the peaks of the input power density function, we can expect the optimum network, which does take into account the relative location of the peaks of the input power density function, to reduce the fluctuation power more effectively. Although the improvement with the optimum filter is expected, it is not obvious how great it will be; the exact amount of improvement is given in Fig. IX-3b (for N = 2).

For $N \geq 3$, the solution for A will not be as simple as it is for N = 1 or N = 2, because, now, $\phi^{(2)}(0)$ will not be zero. However, the same remarks hold as before, namely, we need solve only for the constant A and not for the N + 1 constants A, $C_1, \ldots, C_N$, $(D_1 = C_1, \ldots, D_N = C_N)$. We can solve an N^{th}-order determinant that is obtained by substituting Eq. 8 in Eq. 6. In this manner, we never need bother with $C_1, \ldots, C_N$.

We notice that these results, in particular Eq. 19, hold only for the case in which the alternating-component input power density spectrum has no zeros, but only poles. Inclusion of zeros will force the optimum linear filter to be non-constant over the open interval (0, T). This complicates error evaluation.

The form of the density spectrum in Eq. 7 can approximate a number of practical cases by proper choice of the $\{a_i\}$. If N = 2 can be made to approximate a given density spectrum, reference to Fig. IX-3b shows the relative advantage that will be gained with an optimum filter compared with the flat filter. If it is necessary to resort to the optimum filter, then, of course, the constants C_1 and C_2 will have to be evaluated. C_1 is easily evaluated from the results shown in Example 3; but C_2 must be evaluated from Eq. 6.

A. H. Nuttall

References

1. W. B. Davenport, Jr., R. A. Johnson, and D. Middleton, Statistical errors in measurements on random time functions, J. Appl. Phys. 23, 377-388 (1952).

2. L. A. Zadeh and J. R. Ragazzini, An extension of Wiener's theory of prediction, J. Appl. Phys. 21, 645-655 (1950).

C. A VIEWPOINT FOR NONLINEAR SYSTEM ANALYSIS

A function, broadly but rigorously defined, is a relationship between two sets of objects, called the domain and the range of the function, which assigns to each element of the domain a corresponding element of the range, with each element of the range assigned to at least one element of the domain. A function, according to this definition, is itself an object.

A nonlinear system, from a restricted but useful point of view, is a device that accepts a function of time (e.g., a voltage waveform) as an input, and, for each such function, generates a function of time as the output. The correspondence that is thus established meets the conditions of the definition of a function, and the behavior of a nonlinear system can therefore be thought of as being represented by a function.

Two different classes of functions figure in this description. On the one hand, the input and output are relations between sets of real numbers; their domain is an interval of the real numbers, representing values of time, and their range is another interval of the real numbers (representing, e.g., values of voltage). Such functions are conveniently called real functions; they are the kind of functions we are most used to dealing with. On the other hand, the system is represented as a relation between sets of real functions: a function whose domain is the set of all possible inputs and whose range is the set of all possible outputs. Such a function will be called a "hyperfunction." The problem of the representation of nonlinear systems is the problem of the representation of hyperfunctions.

If a nonlinear system is time-invariant, there is a different, and in some respects simpler, way to represent it. In this case the value of the output at any time t is determined if, without specifying t, we specify, for every x, the value of the input at time $t - x$. Thus, a real number is assigned to each real function, and a function is defined whose domain is a set of real functions and whose range is a set of real numbers. Such a function is called a "functional." The hyperfunction representing the system can be determined from this functional, and the problem of representing the system is reduced to the problem of the representation of functionals.

Because these functionals are functions, methods for their representation can be derived from known methods for the representation of real functions. Except for functions that can be computed by algebraic operations, or can be defined geometrically or by other special means, these methods are methods of approximation. (We include series expansions as approximation methods, because it is impossible to evaluate and sum an infinity of terms; the best that we can do is to compute a partial sum.) Three important methods of approximate representation are: tables of values, power series, and series expansions in orthogonal functions. These three methods all have fairly good generalizations to the approximate representation of functionals.

A rough analog of a table of values is found in the finite-state transducers discussed by Singleton (1). The easiest way to use a table of values is to assign to each number that is not in the table the value corresponding to the nearest number to it that is in the table. This amounts to dividing the domain of the function into a finite number of subsets, and assigning an element of the range to each subset. If we carry this idea over from real functions to functionals, we obtain the finite-state transducer.

The analog of the power series is the analytic functional, which is associated with systems that the author has called "analytic systems." Analytic functionals have been described by Volterra (2), used in a specific problem by Wiener (3) and by Ikehara (4), and further discussed by the author (5).

The method of orthogonal expansions requires the ability to integrate over the domain of the function that is being expanded. Therefore, to expand a functional we must integrate over a set of real functions. How do we define an integral over a set of real functions? The answer is a convenient one for communication theory: If the domain of the functional is a statistical ensemble of functions, then a probability measure is defined on it, and an ensemble average is equivalent to an integral over the set of functions. Methods based on this idea, including a method due to Wiener, have been discussed by Booton (6) and by Bose (7).

Combinations of these methods are possible. Bose used expansion in orthogonal functionals to obtain a finite-state representation. It should also be possible to expand in orthogonal functionals that are themselves defined as analytic functionals.

In applying any method of approximate representation there arises the question of whether or not the function that is to be represented can be approximated arbitrarily closely by the method that is chosen. If it can, and only if it can, we may say that the method is capable of representing the function. But the question itself depends on how we define closeness of approximation, or, in mathematical terms, what topology we define on the hyperfunctions; this, in turn, can be made to depend on topology and measure in the domain and range of the hyperfunction. We can, for example, require the error in the output to be always less than some given number for every input in a certain set; the appropriate mathematical concept, in this case, seems to be the uniform convergence of continuous functions on a compact set. Alternatively, we can require the mean square error in the output, for a certain ensemble of inputs, to be less than a certain amount; the appropriate idea here is convergence in the mean.

The theory of such approximations requires more study. An indication of its importance is that it appears to be connected in some way with the phenomenon of hysteresis.

M. B. Brilliant

References

1. H. E. Singleton, Theory of nonlinear transducers, Technical Report 160, Research Laboratory of Electronics, M.I.T., Aug. 12, 1950.
2. V. Volterra, Leçons sur les Fonctions de Lignes (Gauthier-Villars, Paris, 1913).
3. N. Wiener, Response of a nonlinear device to noise, Report 129, Radiation Laboratory, M.I.T., April 6, 1942.
4. S. Ikehara, A method of Wiener in a nonlinear circuit, Technical Report 217, Research Laboratory of Electronics, M.I.T., Dec. 10, 1951.
5. M. B. Brilliant, Analytic nonlinear systems, Quarterly Progress Reports, Research Laboratory of Electronics, M.I.T., Jan. 15, 1957, pp. 54-61; April 15, 1957, pp. 60-69.
6. R. C. Booton, Jr., The measurement and representation of nonlinear systems, Trans. IRE, vol. CT-1, no. 4, pp. 32-34 (Dec. 1954).
7. A. G. Bose, A theory of nonlinear systems, Technical Report 309, Research Laboratory of Electronics, M.I.T., May 15, 1956.