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Transactions Papers

New Closed-Form Bounds on the Performance of Coding in Correlated Rayleigh Fading

Dwight K. Hutchenson, *Member, IEEE*, and Daniel L. Noneaker, *Senior Member, IEEE*

Abstract—New, simple bounds are presented for the probability of error in a binary hypothesis test for communications using diversity signaling in correlated Rayleigh fading. The bounds are developed in the context of pairwise error-event probabilities in decoding an error-correction code. A long-standing conjecture regarding the form of worst-case error events in exponentially correlated Rayleigh fading is also proven. The utility of the results is illustrated by their application to transfer-function bounds on the probability of bit error for a system using a convolutional code. The closed-form transfer-function bounds are shown to be tighter than previously developed transfer-function bounds for communications in exponentially correlated Rayleigh fading.

Index Terms—Fading channels, Rayleigh channels, error-correction codes, convolutional codes, diversity reception.

I. INTRODUCTION

MANY communication systems employ code-symbol interleaving to minimize the effect of channel memory on the detection of a message received over a fading channel. For most scenarios of interest, however, the residual effect of fading after interleaving must still be accounted for in the evaluation of system performance. One instance receiving frequent attention concerns a system with binary antipodal modulation, a correlated Rayleigh-fading channel, and soft-decision maximum-likelihood decoding. Various approaches are used to obtain an upper bound on the probability of bit error in this circumstance; among these are transfer-function bounds on the probability of error [1].

For a system using a convolutional code, the union bound on the probability of bit error is expressed as an infinite series in pairwise error-event probabilities [1], each of which can be expressed in closed form in terms of the eigenvalues of an associated covariance matrix for the Rayleigh-fading channel

[2]. (See also the numerous references in [3].) The union-bound series cannot be expressed in a closed form in general, however; thus in practice it can be used only to determine a partial-sum approximation.

A closed-form upper bound on the probability of bit error for a system using a convolutional code is given by any transfer-function bound on the union bound. A transfer-function bound is obtained using any upper bound on the pairwise error-event probability that is a linear combination of geometric functions of the error event's Hamming weight. Transfer-function bounds have been obtained for channels with independent Rayleigh fading [4], [5], and similar results have been obtained for correlated Rayleigh fading [6, equation (23)], [7]. Closed-form expressions for the *exact* union bound on the probability of bit error have been developed in terms of an improper integral [8] and in terms of a proper integral [9]; both expressions are applicable only to channels with independent fading, however.

In this paper we introduce several new bounds on the pairwise error-event probability for communications in correlated Rayleigh fading with an arbitrary time-correlation function. We also show that for exponentially correlated fading [10] and any error event of a given Hamming weight, the pairwise error-event probability and its Chernoff bound are no greater than the corresponding values for an error event of the same Hamming weight formed by consecutive channel symbols at the decoder input. From these results we develop improved transfer-function bounds on the probability of bit error for soft-decision maximum-likelihood sequence detection of a convolutional code with exponentially correlated Rayleigh fading and perfect channel-state information.

Our new results for the pairwise error-event probability and the probability of bit error can be adapted to yield similar bounds for uniform trellis codes. The results are also directly applicable to performance analysis of uncoded diversity signaling. They may prove useful in conjunction with other bounding techniques, such as Gallager bounds [11], as well.

The system and channel are described in Section II. Previous results on the exact pairwise error-event probability are reviewed in Section III. In Section IV we develop improved geometric-form bounds and related integral bounds on the pairwise error-event probability. Results on the pairwise error-event probability that are specific to the exponentially

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D. K. Hutchenson was with the Holcombe Department of Electrical and Computer Engineering, Clemson University, Clemson, SC 29634 USA. He is now with MIT Lincoln Laboratory, Lexington, MA 02420 USA (e-mail: dwight.hutchenson@ll.mit.edu).

D. L. Noneaker is with the Holcombe Department of Electrical and Computer Engineering, Clemson University, Clemson, SC 29634 USA (e-mail: danno@ces.clemson.edu).

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correlated channel are developed in Sections V and VI. The results of Sections IV and V are used in Section VII to obtain several new transfer-function bounds on the probability of bit error for exponentially correlated Rayleigh fading. Examples are considered in Section VIII to illustrate the improvement the new bounds provide over the best previously developed transfer-function bound, and conclusions are summarized in Section IX.

II. SYSTEM AND CHANNEL MODELS

Each binary code word $\underline{c} = (c_1, c_2, \dots, c_L)$ is transmitted using binary antipodal modulation. The channel is piecewise constant with a baseband-equivalent channel gain of α_k during the k th channel-symbol interval that is a complex-valued, zero-mean Gaussian random variable with unit variance. The received signal is subjected to additive white Gaussian noise with a double-sided power spectral density of $N_0/2$. The average energy per received channel symbol is E_c ; thus the channel-symbol signal-to-noise ratio at the receiver is

$$\text{SNR} = \frac{E_c}{N_0}.$$

The complex correlator output for the k th code symbol at the receiver is the code-symbol statistic

$$Z_k = \sqrt{E_c T} \alpha_k (-1)^{c_k} + N_k,$$

where T is the channel-symbol duration and (N_k) are i.i.d. zero-mean, complex-valued Gaussian random variables with variance $N_0 T/2$. The maximum-likelihood sequence detector chooses the code sequence with the largest path metric, which uses perfect channel-state information and is given by

$$M(\underline{c}) = \sum_{k=1}^L \text{Re}\{(-1)^{c_k} \alpha_k^* Z_k\}.$$

In Sections III and IV, an arbitrary autocorrelation function is considered for the discrete-time Gaussian random process $(\alpha_1, \dots, \alpha_L)$. In Sections V through VIII, however, attention is restricted to a channel that has the autocorrelation function given by

$$\text{Cov}(\alpha_k, \alpha_j) = q^{|k-j|} \quad (1)$$

where q is the *covariance parameter* of the channel, with $0 \leq q \leq 1$. The random process thus has a geometric time-correlation function; it is a piecewise-constant approximation to the Rayleigh-fading channel with an exponential time-correlation function. Following common usage (e.g., [7]), the piecewise-constant channel is referred to in this paper as the exponentially correlated channel.

III. EVALUATION OF THE PAIRWISE ERROR-EVENT PROBABILITY

The pairwise error-event probability at the decoder output can be expressed in a closed form as shown in [12]. This section summarizes results from that earlier paper and defines notation used in our development of new results presented in the subsequent sections. Assume without loss of generality that the transmitted code sequence is $\underline{0}$, the all-zeros sequence. Let $\hat{\underline{c}}$ represent a code sequence of Hamming weight d . The receiver chooses $\hat{\underline{c}}$ over $\underline{0}$ if $M(\hat{\underline{c}}) > M(\underline{0})$.

Let $\underline{A} = \{\alpha_{i_1}, \dots, \alpha_{i_d}\}$ denote the channel gains corresponding to the d positions of the erroneous code symbols in $\hat{\underline{c}}$. Conditioned on \underline{A} , the statistics $\{Z_{i_1}, \dots, Z_{i_d}\}$ are independent, complex-valued, Gaussian random variables. Thus the conditional pairwise error-event probability given \underline{A} is given by

$$\Pr\{M(\hat{\underline{c}}) > M(\underline{0}) \mid \underline{A}\} = Q\left(\sqrt{\frac{2E_c}{N_0}} \underline{A}^H \underline{A}\right).$$

The covariance matrix of \underline{A} , Σ_A , is referred to as the *channel covariance matrix* for the corresponding error event. It is Hermitian; thus it can be represented by its spectral decomposition [13, Theorem 5.2.1]

$$\Sigma_A = U \Lambda U^H.$$

The matrix U is unitary, and $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_d\}$ where (λ_k) are the eigenvalues of Σ_A , and without loss of generality, $0 \leq \lambda_1 \leq \dots \leq \lambda_d$. If $\underline{Y} \triangleq U^H \underline{A}$, it follows that $\underline{Y}^H \underline{Y} = \underline{A}^H \underline{A}$, and the covariance matrix of \underline{Y} is $\Sigma_Y = \Lambda$. Thus $\text{Var}[Y_k] = \lambda_k$ for $1 \leq k \leq d$. Since $\Sigma_Y = \Lambda$ is diagonal, the random variables $\{Y_1, \dots, Y_d\}$ are uncorrelated and thus independent (since they are jointly Gaussian).

The pairwise error-event probability can be expressed as

$$\Pr\{M(\hat{\underline{c}}) > M(\underline{0})\} = \int_{x=0}^{\infty} Q(\sqrt{x}) f_X(x) dx \quad (2)$$

where $X \triangleq \frac{2E_c}{N_0} \sum_{k=1}^d |Y_k|^2$. (Each summand has an exponential distribution.) A closed-form expression for (2) can be obtained, but the form of the expression depends on the number and multiplicity of the distinct eigenvalues of Σ_A . If $\lambda_k = \lambda$ for $1 \leq k \leq d$, for example, the random variable X has a gamma distribution; its probability density function is

$$f_X(x) = \frac{x^{d-1}}{\Gamma(d)(2E_c/N_0)^d} \exp\left(-\frac{x}{2E_c/N_0}\right)$$

for $x \geq 0$. Substitution into (2) and standard techniques of integration yield

$$\Pr\{M(\hat{\underline{c}}) > M(\underline{0})\} = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{E_c/N_0}{1 + E_c/N_0}} \cdot \sum_{i=1}^d \binom{2i}{i} [4(1 + E_c/N_0)]^{-i}. \quad (3)$$

IV. BOUNDS ON THE PAIRWISE ERROR-EVENT PROBABILITY FOR THE GENERAL CORRELATED CHANNEL

In Section VII, new transfer-function bounds are developed for the probability of bit error in a system using a convolutional code. Each transfer-function bound requires a bound on the pairwise error-event probability that is a linear combination of geometric functions of the Hamming weight of the error event. New bounds of the latter type are developed in this section. In this section and subsequent sections, $P(\Sigma)$ is used to denote the pairwise error-event probability for the error event with channel covariance matrix Σ . Similarly, $P_C(\Sigma)$ is used to denote the Chernoff bound on the same probability.

A. Rational-Polynomial Bounds

The Chernoff bound on the pairwise error-event probability is given by

$$P(\Sigma_A) \leq P_C(\Sigma_A) = \frac{1}{2} \prod_{k=1}^d (1 + \lambda_k E_c / N_0)^{-1}, \quad (4)$$

as in [7, equation (7)]. (Development of the bound is described in [14].) A looser, geometric-form upper bound on the pairwise error-event probability is obtained from any non-negative lower bound λ_{lb} on the eigenvalues of Σ_A . Since $\lambda_{lb} \leq \lambda_k$ for all k , it follows from (4) that

$$P(\Sigma_A) \leq \frac{1}{2} (1 + \lambda_{lb} E_c / N_0)^{-d} \quad (5)$$

as in [7, equation (9b)].

A new geometric-form upper bound on the pairwise error-event probability is obtained from any non-negative lower bound λ_{lb} and any upper bound λ_{ub} on the eigenvalues of Σ_A . Its development uses the following lemma, and it is stated as the subsequent theorem.

Lemma 1: Suppose $0 \leq \lambda_{lb} \leq \lambda_{ub}$ and $C \geq 0$. For any x , $0 \leq x \leq 1$,

$$(1 + \lambda^* C)^{-1} \leq (1 + \lambda_{lb} C)^{-x} (1 + \lambda_{ub} C)^{-(1-x)},$$

where

$$\lambda^* = x \lambda_{lb} + (1 - x) \lambda_{ub}.$$

Proof: The lemma follows directly from Jensen's inequality [15] applied to $\log[(1 + zC)^{-1}]$, for $z > 0$. \square

Theorem 1:

$$P(\Sigma_A) \leq \left[(1 + \lambda_{lb} E_c / N_0)^{-y} \times (1 + \lambda_{ub} E_c / N_0)^{-(1-y)} \right]^d, \quad (6)$$

where $0 \leq y \leq 1$ and satisfies

$$1 = y \lambda_{lb} + (1 - y) \lambda_{ub}.$$

Proof: The proof is given in the appendix.

An example of lower and upper bounds λ_{lb} and λ_{ub} applicable to any covariance matrix Σ_A are those obtained from Geršgorin's Theorem [13, Theorem 10.6.1], though the resulting lower bound λ_{lb} is useful (positive) only if the matrix is strictly diagonally dominant [13].

B. Integral Bounds

In this subsection, the generalization to correlated Rayleigh fading of an equality developed in [9] is used in developing further bounds. The function $Q(x)$ can be expressed as the proper integral [9]

$$Q(x) = \frac{1}{\pi} \int_{\theta=0}^{\pi/2} \exp\left(\frac{-x^2}{\sin^2 \theta}\right) d\theta.$$

Use of this representation in (2) leads to an exact proper-integral expression for the pairwise error-event probability

$$P(\Sigma_A) = \frac{1}{\pi} \int_{\theta=0}^{\pi/2} \prod_{k=1}^d \left[\frac{\sin^2 \theta}{\sin^2 \theta + \lambda_k E_c / N_0} \right] d\theta \quad (7)$$

as in [16, equation (7)].

The two approaches considered in the previous section can be mimicked here to obtain integral bounds that are appropriate for use with the code's transfer function. The first upper bound is obtained by noting that $(\sin^2 \theta + \lambda_{lb} E_c / N_0)^{-1} \geq (\sin^2 \theta + \lambda_k E_c / N_0)^{-1}$ for all k so that

$$P(\Sigma_A) \leq \frac{1}{\pi} \int_{\theta=0}^{\pi/2} \left[\frac{\sin^2 \theta}{\sin^2 \theta + \lambda_{lb} E_c / N_0} \right]^d d\theta \quad (8)$$

following [17, equation (22)].

Application of Lemma 1 with $C = E_c / (N_0 \sin^2 \theta)$ leads immediately to the second, tighter upper bound

$$P(\Sigma_A) \leq \frac{1}{\pi} \int_{\theta=0}^{\pi/2} \left[(\sin^2 \theta) (\sin^2 \theta + \lambda_{lb} E_c / N_0)^{-y} \cdot (\sin^2 \theta + \lambda_{ub} E_c / N_0)^{-(1-y)} \right]^d d\theta, \quad (9)$$

where $0 \leq y \leq 1$ and satisfies

$$1 = y \lambda_{lb} + (1 - y) \lambda_{ub}.$$

If λ_{lb} and λ_{ub} are chosen such that yd is an integer, partial-fraction expansion of (9) results in an alternative expression as the difference of two terms of the form of (3). The same approach, followed by application of [18, equation (5.A.3)], results in the difference of two expressions in the form of the Gauss hypergeometric function if yd is not an integer.

V. BOUNDS ON THE PAIRWISE ERROR-EVENT PROBABILITY FOR THE EXPONENTIALLY CORRELATED CHANNEL

Results developed in this section concerning the pairwise error-event probability are employed in the new transfer-function bounds presented in Section VII. Each of the three key results in this section addresses the comparison of the pairwise error-event probability of an arbitrary error event with bounds on the pairwise error-event probability of a corresponding error event of a special form. Theorems 2 and 3 of this section are used in Section VII to obtain a new polynomial-form transfer-function bound and a new integral-form transfer-function bound, respectively. Theorem 7 of this section is used in Section VII to obtain an improvement of the two new transfer-function bounds; it also proves a long-standing conjecture concerning the "worst case" error event of a given Hamming weight on the exponentially correlated Rayleigh-fading channel.

A. Minimum-Spacing Error Events

Consider the notional error event of Hamming weight d corresponding to d consecutive code symbols in the code sequence detected by the decoder, which we will refer to as the *minimum-spacing error event* of weight d . (For a particular code and a given value of d , it may be that no such error event is actually possible.) Recall that the autocorrelation function of the channel gains in a Rayleigh-fading channel with exponential time correlation is given in (1).

Let $\Sigma_{ms}(d)$ denote the channel covariance matrix for the minimum-spacing error event of weight d . It is shown in [2] that the eigenvalues of $\Sigma_{ms}(d)$ are bounded by

$$\left(\frac{1-q}{1+q}\right) \leq \lambda_k \leq \left(\frac{1+q}{1-q}\right) \quad (10)$$

for all k . (The upper bound also follows from Geršgorin's Theorem.) Moreover, from the implicit solution given in [2] for the eigenvalues of $\Sigma_{ms}(d)$, it follows that the bounds given by (10) are asymptotically tight as $d \rightarrow \infty$. Thus they are the tightest fixed bounds which are applicable to the minimum-spacing error events for all values of d .

B. Other Error Events

Suppose B is the channel covariance matrix for an arbitrary error event of weight d . Let \hat{B} denote the channel covariance matrix for a new notional error event that results from the insertion of an additional zero at some location in the code sequence for the original error event. Therefore, the two matrices can be expressed as

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \quad \text{and} \quad \hat{B} = \begin{pmatrix} B_{11} & qB_{12} \\ qB_{21} & B_{22} \end{pmatrix} \quad (11)$$

where the dimensions of B and \hat{B} are d -by- d , and the dimensions of B_{11} are i -by- i for some i , $1 \leq i \leq d-1$. The length- d eigenvectors of B are denoted $\underline{b}_1, \dots, \underline{b}_d$ with corresponding eigenvalues $\gamma_1 \leq \dots \leq \gamma_d$. The length- d eigenvectors of \hat{B} are denoted $\hat{\underline{b}}_1, \dots, \hat{\underline{b}}_d$ with corresponding eigenvalues $\hat{\gamma}_1 \leq \dots \leq \hat{\gamma}_d$. Because B and \hat{B} are covariance matrices, both are nonnegative definite.

Lemma 2: All eigenvalues of \hat{B} are within the interval $[\gamma_1, \gamma_d]$.

Proof: The proof of the lemma is given in [14].

Lemma 3: If the channel is exponentially correlated, the eigenvalues of the channel covariance matrix for each weight- d error event are within the interval $[\lambda_{ms,1}, \lambda_{ms,d}]$, where $\lambda_{ms,1}$ and $\lambda_{ms,d}$ are the minimum and maximum eigenvalues, respectively, of $\Sigma_{ms}(d)$.

Proof: The code sequence of any weight- d error event is obtained from the code sequence of the minimum-spacing error event of weight d by inserting a finite number of zeros into the latter sequence. Through repeated application of Lemma 2, it follows that eigenvalues from all weight- d error events fall within the range $[\lambda_{ms,1}, \lambda_{ms,d}]$. \square

Lemma 4: If the channel is exponentially correlated, the eigenvalues of the channel covariance matrix for any error event (of any weight) are bounded by (10).

Proof: This follows immediately from Lemma 3, because (10) applies to $\lambda_{ms,1}$ and $\lambda_{ms,d}$. \square

The following two theorems follow directly from Lemma 4 and are used in the development of the transfer-function bounds in Subsections VII-A and VII-B.

Theorem 2: If the channel is exponentially correlated, the bounds on the pairwise error-event probability in (5) and (6) hold for all weight- d error events if the eigenvalue bounds in (10) are used for λ_{lb} and λ_{ub} , respectively.

Theorem 3: If the channel is exponentially correlated, the bounds on the pairwise error-event probability in (8) and (9) hold for all weight- d error events if the eigenvalue bounds in (10) are used for λ_{lb} and λ_{ub} , respectively.

Stronger results are obtained from further consideration of the relationship between the matrices B and \hat{B} . Define a d -by- d matrix function of u by

$$C(u) = (aI + B_{11}) - u^2 B_{12}(aI + B_{22})^c B_{21}$$

where $a \geq 0$ is a constant and $(aI + B_{22})^c$ is the c -inverse of $aI + B_{22}$ defined by

$$(aI + B_{22})(aI + B_{22})^c(aI + B_{22}) = (aI + B_{22}).$$

If $(aI + B_{22})^{-1}$ exists, the matrix $C(u)$ is referred to as the *Schur complement* [13] of $aI + B_{22}$ in the matrix

$$aI + \begin{pmatrix} B_{11} & uB_{12} \\ uB_{21} & B_{22} \end{pmatrix}.$$

Following [19, Theorem 8.2.1], $|aI + B| = |aI + B_{22}| \cdot |C(1)|$ and $|aI + \hat{B}| = |aI + B_{22}| \cdot |C(q)|$.

Lemma 5: $\underline{x}^T C(1) \underline{x} \leq \underline{x}^T C(q) \underline{x}$, \forall length- d vector \underline{x} .

Proof: The proof is given in the appendix.

Lemma 6: $|aI + \hat{B}| \geq |aI + B|$ for any constant $a \geq 0$.

Proof: The proof is given in the appendix.

Theorem 4: If the channel is exponentially correlated and Σ_A is the channel covariance matrix for an error event of weight d , $|aI + \Sigma_A| \geq |aI + \Sigma_{ms}(d)|$.

Proof: The code sequence of any weight- d error event is obtained from the code sequence of the minimum-spacing error event of weight d by inserting a finite number of zeros into the latter sequence. The result thus follows from repeated application of Lemma 6. \square

Theorem 5: If the channel is exponentially correlated and Σ_A is the channel covariance matrix for an error event of weight d , then $|\Sigma_A| \geq |\Sigma_{ms}(d)|$.

Proof: The result follows immediately from Theorem 4 with $a = 0$. \square

Note that Theorem 5 also follows immediately from [7, equation (11)].

Recall that for the error event with channel covariance matrix Σ , $P(\Sigma)$ denotes its pairwise error-event probability, and $P_C(\Sigma)$ denotes the Chernoff bound on the probability.

Theorem 6: If the channel is exponentially correlated and Σ_A is the channel covariance matrix for an error event of weight d , then $P_C(\Sigma_A) \leq P_C(\Sigma_{ms}(d))$.

Proof: The result follows immediately from (4) and application of Theorem 4 with $a = (E_c/N_0)^{-1}$. \square

The result of Theorem 6 is stated as part of a theorem in [7, Proposition 1] for the more general exponentially correlated Rician-fading channel, but no details of a proof are given therein.

The result below also follows from Theorem 4; it is used in the development of the transfer-function bounds in Subsection VII-C.

Theorem 7: If the channel is exponentially correlated and Σ_A is the channel covariance matrix for an error event of weight d , $P(\Sigma_A) \leq P(\Sigma_{ms}(d))$.

Proof: Application of Theorem 4 with $a = \frac{\sin^2 \theta}{E_c/N_0}$ yields the inequality

$$\prod_{k=1}^d (\sin^2 \theta + \lambda_k E_c/N_0) \geq \prod_{k=1}^d (\sin^2 \theta + \lambda_{m,s,k} E_c/N_0)$$

for any θ . It follows that

$$\begin{aligned} \frac{1}{\pi} \prod_{k=1}^d \left[\frac{\sin^2 \theta}{\sin^2 \theta + \lambda_k E_c/N_0} \right] \\ \leq \frac{1}{\pi} \prod_{k=1}^d \left[\frac{\sin^2 \theta}{\sin^2 \theta + \lambda_{m,s,k} E_c/N_0} \right]. \end{aligned} \quad (12)$$

Integration of each side of (12) with respect to θ over the range $[0, \pi/2]$ and comparison of the resulting expressions with (7) yields the desired result. \square

The result of Theorem 7 has been utilized as an (unproven) “folk theorem” in some previous work (such as [6] and [20]). Note that Theorem 2 and Theorem 3 follow from Theorem 7 and the results in Section IV. Theorem 7 applies only to the exponentially correlated channel in general; yet if Σ_1 and Σ_2 are the channel covariance matrices for two error events in any correlated Rayleigh-fading channel and $|cI + \Sigma_1| \geq |cI + \Sigma_2|$ for all $c \geq 0$, then $P(\Sigma_1) \leq P(\Sigma_2)$.

VI. THE EFFECT OF THE COVARIANCE PARAMETER ON THE PAIRWISE ERROR-EVENT PROBABILITY FOR THE EXPONENTIALLY CORRELATED CHANNEL

Consider an error event with Hamming weight d . Let j_i represent the position of the i th “one” in the code sequence of the error event. The spacing between consecutive “ones” is represented by the vector $\underline{\Delta} = (\Delta_1, \Delta_2, \dots, \Delta_{d-1})$, where $\Delta_i = j_{i+1} - j_i$. Let $\Sigma_{\underline{\Delta}}(q)$ represent the covariance matrix corresponding to this error event if the covariance parameter is q , $0 \leq q \leq 1$.

Theorem 8: If the channel is exponentially correlated and $\underline{\Delta}$ is the spacing vector for an error event of Hamming weight d , then $P_C(\Sigma_{\underline{\Delta}}(q_1)) \leq P_C(\Sigma_{\underline{\Delta}}(q_2))$ for $q_1 \leq q_2$.

Proof: Let $B(1) = \Sigma_{\underline{\Delta}}(q_2)$. For $1 \leq i \leq d-1$, let $B_{11}(i)$ denote the upper left i -by- i submatrix of $B(i)$, and define $\hat{B}(i)$ based on $B(i)$ according to (11) with $q = (q_1/q_2)^{\Delta_1}$. From Lemma 6 with $a = (E_c/N_0)^{-1}$, $P_C(B(i)) \geq P_C(\hat{B}(i))$. But $\hat{B}(d-1) = \Sigma_{\underline{\Delta}}(q_1)$; thus $P_C(\Sigma_{\underline{\Delta}}(q_1)) \leq P_C(\Sigma_{\underline{\Delta}}(q_2))$. \square

The result of Theorem 8 is stated as the second part of a theorem in [7, Proposition 1] for the more general exponentially correlated Rician-fading channel; yet a new, stronger result for the Rayleigh-fading channel also follows from Theorem 8.

Theorem 9: If the channel is exponentially correlated and $\underline{\Delta}$ is the spacing vector for an error event of Hamming weight d , then $P(\Sigma_{\underline{\Delta}}(q_1)) \leq P(\Sigma_{\underline{\Delta}}(q_2))$ for $q_1 \leq q_2$.

Proof: The same iterative argument is followed as in the proof of Theorem 8, except that Lemma 6 is applied with $a = \frac{\sin^2 \theta}{E_c/N_0}$ at each step. Integration of each side of the resulting inequality with respect to θ over the range $[0, \pi/2]$, as in the proof of Theorem 7, yields the desired result. \square

A decrease in the covariance parameter of the exponentially correlated Rayleigh-fading channel thus results in a decrease

in both the pairwise error-event probability and its Chernoff bound.

VII. BOUNDS ON THE PROBABILITY OF BIT ERROR FOR THE EXPONENTIALLY CORRELATED CHANNEL

The standard union bound on the probability of bit error for a system with convolutional coding is given by

$$P_b \leq \sum_{\underline{c} \in \mathcal{C} \setminus \underline{0}} i(\underline{c}) \Pr\{M(\underline{c}) > M(\underline{0})\} \quad (13)$$

where \mathcal{C} is the infinite set of finite-weight code sequences of the code and $i(\underline{c})$ is the weight of the information sequence that maps to \underline{c} (i.e., the code sequence’s *information weight*). The infinite series cannot be expressed in a closed form, but a closed-form upper bound on it can be obtained using the code’s transfer function.

The transfer function of the convolutional code, denoted $T(D, I)$, is a power series in the indeterminate variables D and I in which the summand $a_{j,k} D^j I^k$ indicates that the code has $a_{j,k}$ distinct error events of Hamming weight j and information weight k [21]. Any bound on the pairwise error-event probability that is a linear combination of geometric functions of the Hamming weight of the error event can be used in conjunction with the transfer function to obtain a closed-form upper bound on the union bound on the probability of bit error (which is thus an upper bound on the actual probability of bit error).

A. Rational-Polynomial Bounds

If the bound on the pairwise error-event probability is a geometric function of the Hamming weight, $(g(E_c/N_0))^d$, the resulting transfer-function bound for a code of rate b/n is given by

$$P_b \leq \frac{1}{2b} \left. \frac{dT(D, I)}{dI} \right|_{D=g(E_c/N_0), I=1}. \quad (14)$$

The bound in (5) and the result of Theorem 2 together yield the transfer-function bound of (14) with

$$g(E_c/N_0) = \left[1 + \left(\frac{1-q}{1+q} \right) \frac{E_c}{N_0} \right]^{-1} \quad (15)$$

for the exponentially correlated Rayleigh-fading channel. A tighter transfer-function bound for the same channel is obtained by using (6) instead of (5). From (10) it follows that (6) is true for $y = (1+q)/2$. This results in the transfer-function bound of (14) with

$$\begin{aligned} g(E_c/N_0) = \left[1 + \left(\frac{1-q}{1+q} \right) \frac{E_c}{N_0} \right]^{-(1+q)/2} \\ \times \left[1 + \left(\frac{1+q}{1-q} \right) \frac{E_c}{N_0} \right]^{-(1-q)/2}. \end{aligned} \quad (16)$$

B. Integral Bounds

The approach in the previous section can also be applied to the integrand in the integral-form bounds derived in Section IV for the pairwise error-event probability. In each instance, the resulting bound on the probability of bit error has the form of a single-dimensional proper integral with a rational-polynomial integrand. Specifically, for a code of rate b/n it is given by

$$P_b \leq \frac{1}{b\pi} \int_{\theta=0}^{\pi/2} \left. \frac{dT(D, I)}{dI} \right|_{D=g(E_c/N_0, \theta), I=1} d\theta, \quad (17)$$

where $g(E_c/N_0, \theta)$ is determined by the particular geometric-form bound used for the pairwise error-event probability. The use of (8) and the result of Theorem 2 result in the transfer-function bound (17) with

$$g(E_c/N_0, \theta) = (\sin^2 \theta) \times \left[\sin^2 \theta + \left(\frac{1-q}{1+q} \right) \frac{E_c}{N_0} \right]^{-1} \quad (18)$$

for the exponentially correlated Rayleigh-fading channel. A tighter bound on the probability of bit error for the same channel is obtained by using (9) instead of (8). This results in the transfer-function bound of (17) with

$$g(E_c/N_0, \theta) = (\sin^2 \theta) \times \left[\sin^2 \theta + \left(\frac{1-q}{1+q} \right) \frac{E_c}{N_0} \right]^{-(1+q)/2} \times \left[\sin^2 \theta + \left(\frac{1+q}{1-q} \right) \frac{E_c}{N_0} \right]^{-(1-q)/2}. \quad (19)$$

C. Term-by-Term Corrections

The transfer-function bounds in (14) and (17) are looser than the union bound (13), but the former are amenable to exact evaluation whereas the latter is not. A closed-form expression with accuracy closer to that of the union bound can be obtained by replacing the summands in (14) or (17) for a finite subset of the error events with the exact closed-form pairwise error-event probability expressions for those error events. This “term-by-term correction” [22] of the transfer-function bound has been applied previously to transfer-function bounds using standard Chernoff bounds on the pairwise error-event probability for exponentially correlated Rician fading and non-coherent communications [23]. Term-by-term correction requires knowledge of the details of the code sequence associated with each error event (in order to determine the eigenvalues of the corresponding channel covariance matrix) in addition to its information weight.

Term-by-term corrections can be computationally intensive. Correction for error events of the several lowest Hamming weights can require consideration of a large number of error events. The exact structure of each corresponding code sequence (in particular, the placement of non-zero bits within the code sequence) cannot be obtained analytically even from the three-variable complete path-weight enumerator [21] of the code. Instead a search of the code trellis is required. Moreover, separate calculations are required to determine the eigenvalues for the channel covariance matrix and the resulting pairwise error-event probability for each low-weight error event.

This computational burden can be largely eliminated for exponentially correlated Rayleigh fading by using a somewhat weaker correction to the transfer-function bound. By Theorem 7, a looser term-by-term correction is obtained by replacing each summand in (14) or (17) for a finite subset of the error events with the pairwise error-event probability for the minimum-spacing error event of the same Hamming weight. Applying this approach to (14) for term-by-term correction for the error events of weights d_{free} through N results in

$$P_b \leq \frac{1}{2b} \left. \frac{dT(D, I)}{dI} \right|_{D=g(E_c/N_0), I=1} + \frac{1}{b} \sum_{k=d_{free}}^N B_k \left(\Pr\{M(\underline{c}_{ms}(k)) > M(\underline{0})\} - [g(E_c/N_0)]^k / 2 \right), \quad (20)$$

where $\underline{c}_{ms}(k)$ is a sequence of k consecutive ones and B_k is the sum of the information weights of all Hamming-weight- k error events. The term B_k is often tabulated for good codes for low-weight error events; alternatively, it can be obtained from polynomial long division of $\left. \frac{dT(D, I)}{dI} \right|_{I=1}$. Thus, a computer search to determine the details of each error event in the subset is not required. Furthermore, the pairwise error-event probability need be calculated only once for each Hamming weight for which the correction is applied. Applying the same approach to (17) for term-by-term correction for the error events of weights d_{free} through N results in

$$P_b \leq \frac{1}{b\pi} \int_{\theta=0}^{\pi/2} \left(\left. \frac{dT(D, I)}{dI} \right|_{D=g(E_c/N_0, \theta), I=1} - \sum_{k=d_{free}}^N B_k [g(E_c/N_0, \theta)]^k \right) d\theta + \frac{1}{b} \sum_{k=d_{free}}^N B_k \Pr\{M(\underline{c}_{ms}(k)) > M(\underline{0})\}. \quad (21)$$

VIII. ACCURACY OF THE BOUNDS FOR A CONVOLUTIONAL CODE

The accuracy of the three (new) closed-form bounds developed in this paper is compared with the accuracy of the closed-form bound of (14) and (15) (i.e., using the result from [7]). The latter bound is referred to as the *rational-polynomial bound*. The new bound using (14) and (16) is referred to as the *tighter rational-polynomial bound*, the new bound using (17) and (18) is referred to as the *integral bound*, and the new bound using (17) and (19) is referred to as the *tighter integral bound*. Simulation results and bounds are compared by considering the performance of a system using the “NASA standard” constraint-length-seven, rate-1/2 convolutional code [24] over the exponentially correlated Rayleigh-fading channel.

Figure 1 shows the bounds and simulation results for the probability of bit error as a function of E_c/N_0 for a channel with a covariance parameter of $q = 2.83 \cdot 10^{-7}$. For this fading rate, the rational-polynomial bound from (15) and the integral bound from (18) are nearly indistinguishable from their tighter

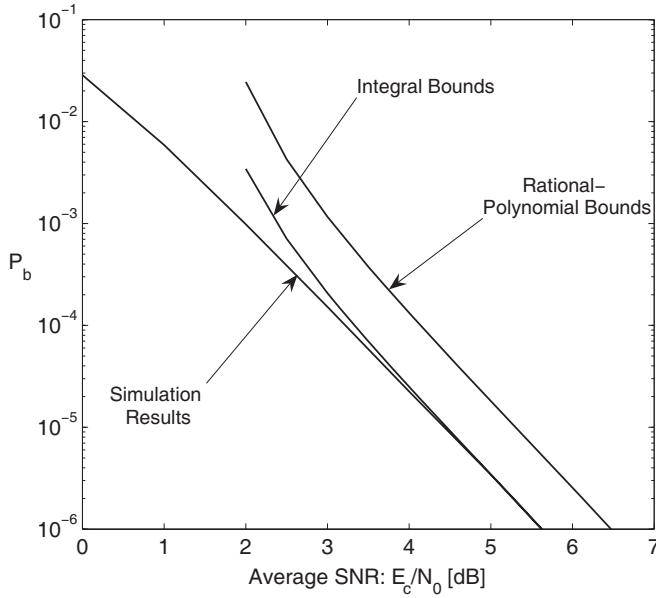


Fig. 1. Bounds and simulation results for the NASA standard, $K = 7$, rate-1/2 convolutional code for $q = 2.83 \cdot 10^{-7}$.

counterparts from (16) and (19), respectively. The integral bounds are much tighter than the rational-polynomial bounds, however. The two integral bounds differ by 0.1 dB from the actual performance if the probability of bit error is 10^{-4} , for example, whereas the two rational-polynomial bounds differ from the actual performance by 0.9 dB.

Figure 2 shows analogous results for a more slowly time-varying fading channel for which $q = 0.221$. Greater differences between the bounds occur for this channel than for the channel with the smaller covariance parameter. If the probability of bit error is 10^{-4} , the rational-polynomial bound from (15) differs from the actual performance by 2.8 dB, while the tighter rational-polynomial bound from (16) is within 1.2 dB of the actual performance. For the same probability of bit error, the integral bound from (18) and the tighter integral bound from (19) differ from the actual performance by 2.0 dB and 0.4 dB, respectively.

As described in Subsection VII-C, the accuracy of the bounds can be improved using term-by-term corrections. Figure 3 illustrates the value of the corrections for a covariance parameter of $q = 0.860$. Results are shown for the integral bound and tighter integral bound both without term-by-term corrections and with the looser corrections of (21) for error events of Hamming weights 10, 12, and 14 (the lowest three weights for the code). These are compared with the result using the rational-polynomial bound. If the probability of bit error is 10^{-4} , an improvement of 0.4 dB is achieved by applying the term-by-term corrections to the integral bound. For the same probability of bit error, there is an improvement of 0.5 dB if the term-by-term corrections are applied to the tighter integral bound. For this value of the system's covariance parameter, moreover, the looser term-by-term corrections in (21) yield results so close to the exact term-by-term corrections that the two are nearly indistinguishable when plotted using the scale of Figure 3.

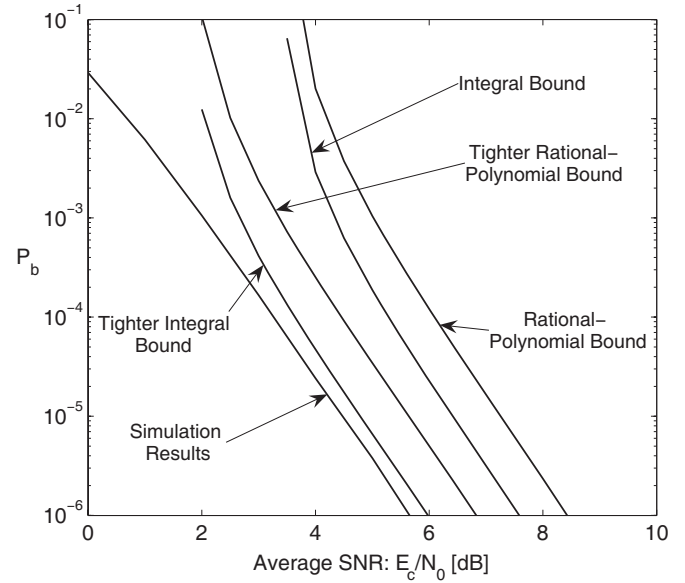


Fig. 2. Bounds and simulation results for the NASA standard, $K = 7$, rate-1/2 convolutional code for $q = 0.221$.

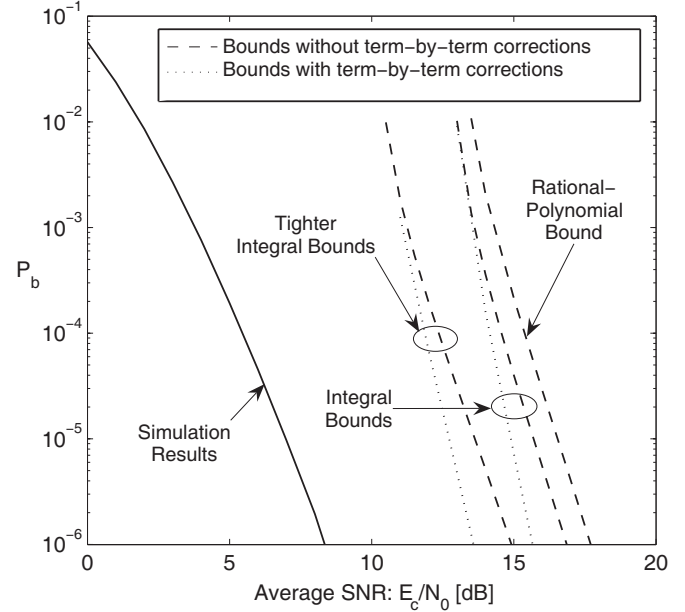


Fig. 3. Term-by-term corrections of integral bounds for the NASA standard, $K = 7$, rate-1/2 convolutional code for $q = 0.860$.

IX. CONCLUSION

New bounds on the pairwise error-event probability are developed for coded communications over a correlated Rayleigh-fading channel in terms of upper and lower bounds on the eigenvalues of the channel covariance matrix. The results are applied to the exponentially correlated Rayleigh-fading channel, and the relationship between the spacing of the erroneous code symbols in an error event and its pairwise error-event probability is examined. It is shown that the minimum-spacing error event of a given Hamming weight results in the largest pairwise error-event probability among all error events of that weight for a binary code.

The bounds on the pairwise error-event probability for the

exponentially correlated channel are used to develop three new closed-form transfer-function bounds on the probability of bit error for convolutional coding and maximum-likelihood decoding. It is shown that the new bounds can be as much as several decibels tighter than previously developed transfer-function bounds. A form of term-by-term corrections is presented that eliminates much of the computational burden of term-by-term corrections with exact pairwise error-event probabilities; it yields a bound that is nearly as tight as that obtained with exact term-by-term corrections.

APPENDIX PROOFS

Proofs are given for Theorem 1 (introduced in Section IV) and Lemmas 5 and 6 (introduced in Section V).

Theorem 1:

$$P(\Sigma_A) \leq \left[(1 + \lambda_{lb} E_c/N_0)^{-y} (1 + \lambda_{ub} E_c/N_0)^{-(1-y)} \right]^d,$$

where $0 \leq y \leq 1$ and

$$1 = y\lambda_{lb} + (1 - y)\lambda_{ub}.$$

Proof: Let x_k satisfy $\lambda_k = x_k\lambda_{lb} + (1 - x_k)\lambda_{ub}$. Applying Lemma 1 with $C = E_c/N_0$ to (4) results in

$$\begin{aligned} P(\Sigma_A) &\leq \prod_{k=1}^d (1 + \lambda_{lb} E_c/N_0)^{-x_k} (1 + \lambda_{ub} E_c/N_0)^{-(1-x_k)} \\ &= \left[(1 + \lambda_{lb} E_c/N_0)^{-y} (1 + \lambda_{ub} E_c/N_0)^{-(1-y)} \right]^d, \end{aligned}$$

where $y = \frac{1}{d} \sum_{k=1}^d x_k$. Because the trace of a matrix is equal to the sum of its eigenvalues and $\text{tr}\{\Sigma_A\} = d$, it follows that

$$d = \sum_{k=1}^d \lambda_k = \sum_{k=1}^d (x_k\lambda_{lb} + (1 - x_k)\lambda_{ub}).$$

Dividing by d , it follows that y must satisfy $1 = y\lambda_{lb} + (1 - y)\lambda_{ub}$ and that $\lambda_{lb} \leq 1 \leq \lambda_{ub}$. Thus $0 \leq y \leq 1$. \square

Lemma 5: $\underline{x}^T C(1) \underline{x} \leq \underline{x}^T C(q) \underline{x}$, \forall length- d vector \underline{x} .

Proof: The matrix B is nonnegative definite because it is a covariance matrix. It follows that $aI + B$ is nonnegative definite; and $C(1)$ is nonnegative definite [19, Theorem 12.2.21]. The matrices $aI + B_{11}$ and $aI + B_{22}$ are nonnegative definite because they are principal submatrices of $aI + B$. Therefore, $(aI + B_{22})^c$ is nonnegative definite and

$$\begin{aligned} &\underline{x}^T B_{12} (aI + B_{22})^c B_{21} \underline{x} \\ &= \underline{x}^T B_{21}^T (aI + B_{22})^c B_{21} \underline{x} \\ &= (B_{21} \underline{x})^T (aI + B_{22})^c (B_{21} \underline{x}) \\ &\geq 0, \end{aligned}$$

i.e. $B_{12}(aI + B_{22})^c B_{21}$ is nonnegative definite. But $0 \leq q < 1$; thus,

$$\begin{aligned} &\underline{x}^T C(1) \underline{x} \\ &= \underline{x}^T (aI + B_{11}) \underline{x} - \underline{x}^T B_{12} (aI + B_{22})^c B_{21} \underline{x} \\ &\leq \underline{x}^T (aI + B_{11}) \underline{x} - q^2 \underline{x}^T B_{12} (aI + B_{22})^c B_{21} \underline{x} \\ &= \underline{x}^T C(q) \underline{x}. \end{aligned}$$

Lemma 6: $|aI + \hat{B}| \geq |aI + B|$ for any constant $a \geq 0$. \square

Proof: Let $\eta_1 \leq \eta_2 \leq \dots \leq \eta_n$ represent the eigenvalues of $C(1)$, with corresponding eigenvectors $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n$, and let $\hat{\eta}_1 \leq \hat{\eta}_2 \leq \dots \leq \hat{\eta}_n$ represent the eigenvalues of $C(q)$, with corresponding eigenvectors $\hat{\underline{x}}_1, \hat{\underline{x}}_2, \dots, \hat{\underline{x}}_n$. From Lemma 5 it follows that $\hat{\eta}_k \geq \eta_k$, for $1 \leq k \leq n$ [14]; thus

$$\begin{aligned} |aI + \hat{B}| &= |aI + B_{22}| \cdot |C(q)| \\ &= |aI + B_{22}| \prod_{k=1}^n \hat{\eta}_k \\ &\geq |aI + B_{22}| \prod_{k=1}^n \eta_k \\ &= |aI + B_{22}| \cdot |C(1)| \\ &= |aI + B|. \end{aligned} \quad \square$$

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Dwight K. Hutchenson received the B.S. and M.S. degrees in electrical engineering from Clemson University in Clemson, SC, in 2005 and 2008, respectively, and was a research assistant there at the time of this article's submission. He is currently an associate staff member at the Massachusetts Institute of Technology's Lincoln Laboratory in Lexington, MA. His research interests are in digital communication theory, with an emphasis on error-control coding, modulation techniques, and signal acquisition.



Dr. Daniel L. Noneaker received the B.S. degree (with high honors) from Auburn University in 1977 and the M.S. degree from Emory University in 1979, both in mathematics. He received the M.S. degree in electrical engineering from the Georgia Institute of Technology in 1984, and he was awarded the Ph.D. degree in electrical engineering from the University of Illinois at Urbana-Champaign in 1993.

Dr. Noneaker has industrial experience in both hardware and software design for communication systems. From 1979 to 1982 he was with Sperry-

Univac, Salt Lake City, Utah, and from 1984 to 1988 he was with the Motorola Government Electronics Group, Scottsdale, Arizona. He was a Research Assistant in the Coordinated Science Laboratory, University of Illinois, Urbana, Illinois from 1988 to 1993. Since August 1993 he has been with the Holcombe Department of Electrical and Computer Engineering at Clemson University, Clemson, SC, where he currently holds the positions of Professor and Associate Department Chair.

Dr. Noneaker is engaged in research on wireless communications for both military and commercial applications, including cellular networks and ad hoc mobile radio networks. He has published numerous papers on design and performance analysis for spread-spectrum multiple-access communications, error-control coding for fading channels, signal acquisition, and protocols for packet radio networks.