OPERATIONS RESEARCH CENTER

Working Paper

A Fluid Model of Spillback and Bottleneck Phenomena for
Determining Dynamic Travel Times

by

S. Kachani
G.Perakis

OR 356-01   August 2001

MASSACHUSETTS INSTITUTE
OF TECHNOLOGY

A Fluid Model of Spillback and Bottleneck Phenomena for Determining Dynamic Travel Times*

Soulaymane Kachani and Georgia Perakis
Massachusetts Institute of Technology,
77 Massachusetts Avenue, E53-359,
Cambridge, MA 02139.
Phone: (617) 253-8277
Email: georgiap@mit.edu

August, 2001

Abstract

In this paper we introduce travel time models that incorporate spillback and bottleneck phenomena. In particular, we study a model for determining the link travel times for drivers entering a link as well as drivers already in the link but whose travel times are affected by a significant change in traffic conditions (e.g. spillback or bottleneck phenomena). To achieve this goal, we extend the fluid dynamics travel time models proposed by Perakis [11] and subsequently by Kachani and Perakis [6], [7], to also incorporate such phenomena. These models utilize fluid dynamics laws for compressible flow to capture a variety of flow patterns such as the formation and dissipation of queues, drivers' response to upstream congestion or decongestion and drivers' reaction time. We propose variants of these models that explicitly account for spillback and bottleneck phenomena. Our investigation considers both separable and non-separable velocity functions.

1 Introduction

Congestion in transportation networks has been growing tremendously in the recent years and has become an acute problem. In particular, Barnhart *et al.* [2] have estimated that congestion costs around $100 billion each year to Americans alone in the form of lost productivity. Therefore, it is critical to alleviate it by understanding its nature and devising efficient congestion control strategies. Congestion phenomena in urban and highway transportation systems are inherently dynamic. The fast-growing field of Intelligent Vehicle Highway Systems (IVHS) is a case in point. An important component that links with congestion phenomena is travel time that is, the time it takes drivers to reach their destinations in the network taking into consideration what time they started their trip during the day. Our goal in this paper is to continue on the work by Perakis [11] and subsequently by Kachani and Perakis [6], [7], in order to provide insight on the nature of travel times (that is, how delays and traffic patterns are formed and dynamically change) and on the role of information in transportation networks. In particular, in this paper we extend our previous work on modeling travel times by also directly incorporating spillback and bottleneck phenomena in our models.

1

Traffic patterns and phenomena such as flow circulation, queues formation and dissipation, spillback and shock waves, are striking evidence that traffic flows resemble gas and water flows. It is therefore natural to apply physical laws of fluid dynamics for compressible flow to model traffic flow patterns.

Lighthill and Whitham [8], and Richards [12] introduced the first continuum approximation of traffic flows using kinematic wave theory (see Haberman [5] for a detailed analysis). The dynamic nature of these models gave them instant credibility. Indeed, with the increase of urban and highway congestion, the time variations of flow are too important to be neglected. Dynamic traffic flow modeling has captured the focus of many researchers in the transportation area. A variety of dynamic traffic flow models (microscopic and macroscopic) have since been proposed in the literature.

The purpose of this paper is to devise models that will allow us to determine travel times in a transportation network operating under a dynamic setting. In particular, we wish to model link travel times for drivers entering a link as well as for drivers already in the link but whose travel times are affected by a significant change in the traffic conditions (e.g. spillback or bottleneck phenomena). Practitioners in the transportation area have been using several families of travel time functions. Akcelik [1] proposed a polynomial-type travel time function for links at signalized intersections. The BPR function [10], that is used to estimate travel times at priority intersections, is also a polynomial function. Finally, Meneguzzer *et al.* [9] proposed an exponential travel time function for all-way-stop intersections. Our goal is to lay the theoretical foundations for using these polynomial and exponential families of travel time functions in practice. While most analytical models in traffic modeling assume an a priori knowledge of drivers' travel time functions, in this paper, travel time is part of the model and comes as an output. To determine travel times, in this paper we propose an extension of the fluid dynamics travel time models proposed by Perakis [11] and subsequently by Kachani and Perakis [6], [7]. We extend these models in order to: (i) account for spillback and bottleneck phenomena but also (ii) determine travel times of drivers entering a link as well as of drivers already in the link. This allows us to update our estimates of travel times during the trip in each link as they may be affected by changes in traffic conditions.

To achieve our goals:

- We propose two models to estimate travel times as functions of exit flow rates where we account for spillback and bottleneck phenomena: the Spillback Polynomial Travel Time (SPTT) Model and the Spillback Exponential Travel Time (SETT) Model. These models incorporate inflow, outflow and storage capacity constraints.

- We propose an extension of the general framework in Kachani and Perakis [6] for the analysis of the SPTT Model.

- Based on piecewise linear and piecewise quadratic approximations of the flow rates, we propose several classes of travel time functions for the separable SPTT and SETT models. We further establish a connection between these travel time functions.

- We extend the analysis of the SPTT Model to non-separable velocity functions in the case of acyclic networks.

The paper is organized as follows. In Section 2, we provide some useful notations. In Section 3, we propose an analytical dynamic travel time model. We show how this model accounts for spillback and bottleneck phenomena. To make the problem more tractable, we derive two approximations of this model: the Spillback Polynomial Travel Time Model (SPTT Model) and the Spillback Exponential Travel Time Model (SETT Model). In Section 4, we examine the case of separable velocity functions. Based on an approximation of exit flows by piecewise linear and piecewise quadratic functions, we

2

propose several classes of travel time functions for the problem that rely on a variety of assumptions. We analyze the relationship between these travel time functions and show that the assumptions we impose are indeed reasonable. In Section 5, we extend the analysis of the SPTT Model to the case of non-separable velocity functions in acyclic networks. Finally, in Section 6 we discuss future steps for the study of this model.

2 The Hydrodynamic Theory of Traffic Flow and a General Travel Time Model

In Subsection 2.1, we summarize the notation that we use throughout the paper. In Subsection 2.2, we establish a relationship between path and link flows.

2.1 Notation

The physical transportation network is represented by a directed network $G = (N, I)$, where N is the set of nodes and I is the set of directed links. Index w denotes an Origin-Destination (O-D) in the set W of origin destination pairs. Index P denotes the set of paths and index P_w denotes the set of paths between O-D w.

Path variables:

$\|P\|$	:	number of paths in the network;
x_p	:	position on path p;
L_p	:	length of path p;
$T_p(L_p, t)$	:	path travel time function on path p starting the trip at time t;

Link variables:

$\|I\|$	:	number of directed links in the network;
x_i	:	position on link i;
L_i	:	length of link i;
$f_i(x_i, t)$	:	flow rate on link i at position x_i at time t;
$f(0, t)$	:	vector of departure link flow rates;
$C_i^{in}(t)$	:	inflow capacity rate of link i at time t;
$C_i^{out}(t)$	:	outflow capacity rate of link i at time t;
$T_i(x_0, x_i, t)$	:	travel time on link i to reach position x_i of a driver departing at position x_0 at time t;
$T_i(L_i, t)$	:	link traversal time of of a driver departing on link i at time t
	=	$T_i(0, L_i, t)$;
$u_i(x_i, t)$	:	traffic speed on link i at position x_i at time t;
$k_i(x_i, t)$	:	traffic density on link i at position x_i at time t;
u_i^{max}	:	maximum traffic speed on link i;
k_i^{max}	:	maximum traffic density on link i;
f_i^{max}	:	storage capacity rate of link i;

Link-path flow variables:

ip	:	a link-path pair;
$i^- p$	:	predecessor of link i on path p;
δ_{ip}	=	1 if link i belongs to path p, and 0 otherwise;
L_{ip}	:	length from the origin of path p until the beginning of link i;
$T_{ip}(L_{ip}, t)$	:	partial path travel time function from the origin of path p until the beginning of link i starting the trip at time t;

3

Time variables:

$[0, T]$: Time period.

2.2 Relationship between path and link variables

After determining travel time functions on the network's links, we need to determine the travel times to traverse the network's paths. Determining path travel times becomes complicated due to the dynamic nature of traffic. Two approaches have been proposed in the literature to address this problem. The first approach assumes that travelers consider only the current travel time information in the network. That is, travelers compute their path travel time at time t as the sum of all the link travel times along their route, based on the current information available to the travelers at time t. For example, up-to-the-minute radio broadcasts could be a source of such information. This type of travel time function is called instantaneous travel time (see for example Boyce, Ran and Leblanc [3]). The second approach assumes that travelers consider predictions or estimates of travel times. That is, the travel time to traverse a path is the summation of the link travel times that the traveler experiences when he/she reaches each link along the path (see for example Friesz *et al.* [4]). Traveler information systems could provide, for example, such information.

In this paper, we follow the second approach. To illustrate this, let us first consider a network with one path p that consists of two links, 1 and 2. The second approach seems to imply that

$$T_p(L_p, t) = T_1(L_1, t) + T_2(L_2, t + T_1(L_1, t)).$$

Since $L_{2p} = L_1$ and $T_{2p}(L_{2p}, t) = T_1(L_1, t)$, it follows that:

$$T_p(L_p, t) = T_1(L_1, t) + T_2(L_2, t + T_{2p}(L_{2p}, t)).$$

Similarly, if we add a third link 3 to path p, it follows that:

$$
\begin{aligned}
T_p(L_p, t) &= T_1(L_1, t) + T_2(L_2, t + T_1(L_1, t)) + T_3(L_3, t + T_1(L_1, t) + T_2(L_2, t + T_1(L_1, t))) \\
&= T_1(L_1, t) + T_2(L_2, t + T_{2p}(L_{2p}, t)) + T_3(L_3, t + T_{3p}(L_{3p}, t)).
\end{aligned}
$$

The above formulas easily extend to the general case as follows:

$$T_p(L_p, t) = \sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip}, t)) \delta_{ip}.$$

3 A General Travel Time Model and Two Approximation Models

3.1 A General Travel Time Model

In this subsection, we propose a general model for travel time functions. Subsequently, in the next subsection, we show how this model also accounts for spillback and bottleneck phenomena.

As in Perakis' model [11], we make the following three assumptions:
A1 Links in the network have no exits.
A2 The velocity function u_i on link i can be expressed as a function $\widehat{u}_i$ that depends only on the vector

4

of density functions.

A3 For all t, the exit link flow rate $f_i(L_i, t + \tau_i)$ can be approximated by $h_i^t(\tau_i)$, a continuously differentiable function of τ_i.

Below, we introduce a family of velocity functions that verifies Assumption (A2). We consider the general case of non-separable velocity functions. We model link interactions by considering that the velocity of link i, at position x_i and at time t, can be expressed as a function $\hat{u}_i(k, \nabla k) = u_i^{max} - b_i(u_i^{max})^2 k_i + \sum_{j \in B(i)} \alpha_{ij}(x_i) R_{ij}(\overline{x}_j, t - \Delta_{ij})$, where b_i is a constant; $\alpha_{ij}(x_i)$ is the density correlation function between link i and link j and depends on the position x_i on link i; R_{ij} is a function of k_j and ∇k_j; $\overline{x}_j$ is a fixed position of a detector of density on link j; Δ_{ij} is a propagation time between link j and link i; and $B(i)$ denotes a set of links neighboring link i. In Sections 3 and 4, we consider separable velocity functions (e.g. $\alpha_{ij}(.) = 0$). In Section 5, we consider the more general case of non-separable velocity functions for acyclic networks.

Below, we provide the general model:

<div align="center">

Model 1

For all $t \in [0, T]$, we have:

</div>

$$\frac{\partial f_i(x_i, T_i)}{\partial x_i} + \frac{\partial k_i(x_i, T_i)}{\partial T_i} = 0, \qquad \text{for all } i \in I, \tag{1}$$

$$f_i(L_i, t + T_i) = min(h_i^t(T_i), C_i^{out}(t)), \qquad \text{for all } i \in I, \tag{2}$$

$$f_i(0, t) \le C_i^{in}(t), \qquad \text{for all } i \in I, \tag{3}$$

$$f_i(x_i, t) = k_i(x_i, t) u_i(x_i, t), \qquad \text{for all } i \in I, \tag{4}$$

$$u_i = \hat{u}_i(k, \nabla k), \qquad \text{for all } i \in I, \tag{5}$$

$$\frac{dT_i(x_i, t)}{dx_i} = \frac{1}{u_i}, \qquad \text{for all } i \in I, \tag{6}$$

$$T_i(0, t) = 0, \qquad \text{for all } i \in I, \tag{7}$$

$$T_p(L_p, t) = \sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip}, t)) \delta_{ip}, \qquad \text{for all } p \in P. \tag{8}$$

Notice in Model 1 that given exit link flow rate functions $f_i(L_i, t)$, $i \in I$, inflow and outflow link capacity functions $C_i^{in}(t)$ and $C_i^{out}(t)$, $i \in I$, and a density velocity relationship $u_i = \hat{u}_i(k, \nabla k)$, $i \in I$, the Dynamic Travel Time Problem is the problem of determining $f_i(x_i, t)$ $i \in I$, $u_i(x_i, t)$ $i \in I$, $k_i(x_i, t)$ $i \in I$, $T_i(x_i, t)$ $i \in I$, and $T_p(x_p, t)$ $p \in P$ as functions of x_i $i \in I$, x_p $p \in P$ and $t \in [0, T]$.

3.2 Modeling Flexibility

As in the case of Model 1 and as we will illustrate next for the SPTT and the SETT models, our models rely on Assumption (A3), that is, that at departure time t, the exit flow rates $f_i(L_i, t + \tau_i)$ on link i of length L_i can be approximated by piecewise continuoulsy differentiable functions $h_i^t(\tau_i)$. This analysis differs from the one in Kachani and Perakis [6] where the same assumption was made on the entrance (i.e. position 0 on link i) link flow rates $f_i(0, t + \tau_i)$. While the two assumptions seem to be similar, the assumption in this paper enables us to:

- Account directly for spillback and bottleneck phenomena. Spillback of queues is a common phenomenon in transportation networks. Such phenomena might occur in the case of congested traffic (e.g. rush hour) or accidents, where queues form and may backward propagate from a link to its upstream link. This happens whenever the head of a queue reaches the head of a link and the inflow capacity of that link is lower than the flow coming from upstream. The assumption we make on the exit link flow rates $f_i(L_i, t)$ at the tail of link i will enable us to take into account link inflow rate and outflow rate capacities more easily. Our previous version accounts for spillback and bottleneck

phenomena only implicitly via approximations.

- Determine travel times not only for drivers who are entering a link but also for drivers who are already in the link. In the latter case, we may need to update our estimates of travel times due to a significant change in traffic conditions. Indeed, once we determine the travel time for a driver entering a link, if spillback occurs while this driver is still in the link, then the travel time might change dramatically. Hence there is a need to re-compute it based on the new traffic conditions. To the best of our knowledge, the current literature of macroscopic analytical dynamic travel time models does not address these issues, and as a result does not allow the computation of travel times for drivers who are already in a link.

Moreover, capacity constraints (3) and (2) enable us to realistically model the inflow and outflow link capacity rates $C_i^{in}(t)$ and $C_i^{out}(t)$. These capacities are functions of time as they may depend on traffic conditions. Furthermore, the hydrodynamic theory of traffic flow implicitly accounts for the link storage capacity rate f_i^{max}. This storage capacity, also called road capacity (see Haberman [5]), is positive and always bounded from above by $u_i^{max} k_i^{max}$.
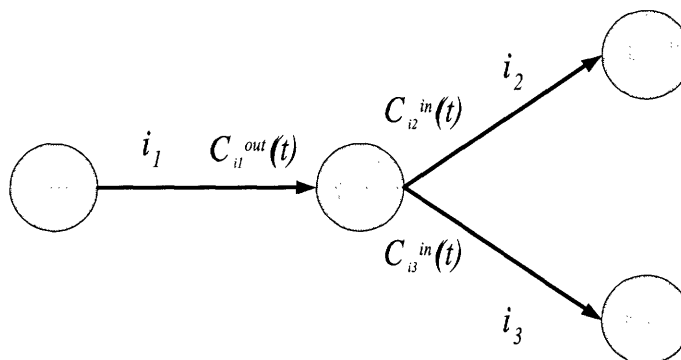

Figure 1: A Three-Link-Network Example

To illustrate our modeling flexibility, let us consider the case of a three-link-network (as shown in Figure 1) with two paths: $p_1 = (i_1, i_2)$ and $p_2 = (i_1, i_3)$. We assume that during a time period $[t, t + \Delta]$, travelers on link i_1 make the approximation that the exit link flow rate for subsequent times $t + T_{i_1}$ is linear in terms of the travel time T_{i_1}. That is,

$$f_{i_1}(L_{i_1}, t + T_{i_1}) = h_{i_1}^t(T_{i_1}) = A_{i_1}(t) + B_{i_1}(t)T_{i_1}.$$

We also assume that during the time period $[t, t + \Delta]$, the inflow and outflow link capacity rates $C_i^{in}(t)$ and $C_i^{out}(t)$ are constant.

The simplest case would be to consider that the exit link flow rate is constant (e.g. $B_{i_1}(t) = 0$). For instance we can have:

$$f_{i_1}(L_{i_1}, t + T_{i_1}) = A_{i_1}(t) = min(C_{i_1}^{out}(t), min(f_{i_2}(0, t), C_{i_2}^{in}(t)) + min(f_{i_3}(0, t), C_{i_3}^{in}(t))).$$

Another situation could correspond to the case where the exit flow rate on link i_1 is below the outflow capacity rate $C_{i_1}^{out}(t)$, the departure flow rate on link i_2 is below the inflow capacity rate $C_{i_2}^{in}(t)$, while

6

the departure flow rate on link i_3 is below the inflow capacity rate $C_{i_3}^{in}(t)$. In this case, drivers in link i_1 might consider that the exit flow rate is linear in the following form:

$$f_{i_1}(L_{i_1}, t + T_{i_1}) = C_{i_3}^{in}(t) + f_{i_2}(0, t) + \frac{df_{i_2}(0, t)}{dt} T_{i_1}.$$

A third situation could correspond to the case where the exit flow rate on link i_1 is below the outflow capacity rate $C_{i_1}^{out}(t)$, and the departure flow rate on links i_2 and i_3 are respectively below the inflow capacity rates $C_{i_2}^{in}(t)$ and $C_{i_3}^{in}(t)$. In this case, drivers in link i_1 might consider that the exit flow rate is linear in the following form:

$$f_{i_1}(L_{i_1}, t + T_{i_1}) = f_{i_1}(L_{i_1}, t) + \frac{df_{i_1}(L_{i_1}, t)}{dt} T_{i_1}.$$

Model 1 is hard to analyze in its current form. For this reason, in the following two subsections, we consider two simplified models in the case of separable velocity functions (where $\alpha_{ij}(.) = 0$). This will give rise to the Separable Spillback Polynomial Travel Time Model (Separable SPTT Model) and the Separable Spillback Exponential Travel Time Model (Separable SETT Model). In Section 4, we provide a detailed analysis of the two models.

In addition to Assumptions (A1)-(A3) introduced in Subsection 3.1, we further assume that:
A4 $\hat{u}_i(.)$ is a separable and linear function of the density k_i. That is, $\hat{u}_i(k_i) = u_i^{max} - b_i(u_i^{max})^2 k_i$, where $b_i = -\frac{1}{(u_i^{max})^2} \frac{d\hat{u}_i(0)}{dk_i}$.
A5 The term $\frac{1}{(u_i^{max})^2} \ll 1$.

Remarks:

- Note that Assumption (A4) is a particular case of the density-velocity relationship introduced in Subsection 3.1 where we consider that $\alpha_{ij}(.) = 0$.

- Our analysis in Section 4 will rely on the separability Assumption (A4). However, in Section 5, we relax this assumption and consider the non-separable case as formulated in Subsection 3.1.

- As an example, consider an arc i with speed limit of 40 miles per hour. Then $(u_{max}^i)^2 = 1,600$. This example demonstrates that Assumption (A5) is reasonable.

Our goal is to propose analytical forms for travel time functions through the solution of Model 1. To achieve this, the first step is to eliminate some of the variables involved in the model. We eliminate the density variables by expressing them as functions of the flow rates.

Lemma 1 *[6] Under Assumption (A4), the link density as a function of the link flow rate function can be expressed as:*

$$k_i = \frac{1}{2b_i u_i^{max}}(1 - (1 - 4b_i f_i)^{\frac{1}{2}}). \tag{9}$$

7

3.3 Separable Spillback Polynomial Travel Time Model

In this subsection, we consider an approximation of the density flow rate relationship (9). This approximation enables us to express conservation law (1) in terms of the link flow rates only. We present a formulation of the Separable Spillback Polynomial Travel Time Model.

3.3.1 A Preliminary Result

Lemma 2 *[6] Under Assumptions (A4)-(A5), the link density as a function of the link flow rate function can be expressed as:*

$$k_i = \frac{f_i}{u_i^{max}} + \frac{b_i f_i^2}{u_i^{max}}.$$

(10)

Using the above result, the following theorem provides a first-order partial differential equation satisfied by the link flow rate functions.

Theorem 1 *[6] Under Assumptions (A4)-(A5) and equation (10), the link flow rate functions f_i are solutions of the first-order partial differential equation:*

$$\frac{\partial f_i}{\partial t} + \frac{u_i^{max}}{1 + 2b_i f_i} \frac{\partial f_i}{\partial x_i} = 0.$$

(11)

Assumption (A3) provides a boundary condition.

This new conservation law (11) is the basis of the SPTT Model that we present below and analyze in Sections 4 and 5.

3.3.2 Model Formulation

In this subsection, we introduce the SPPT Model. To explicitly account for spillback and bottleneck phenomena in this model, we introduce a boundary condition at the end of the link (see Equation (13)). This boundary condition will allow us to solve partial differential equation (12). Theorem 1 gives rise to the following formulation:

SPTT Model

For all $t \in [0, T]$:

$$\frac{\partial f_i}{\partial t} + \frac{u_i^{max}}{1 + 2b_i f_i} \frac{\partial f_i}{\partial x_i} = 0, \qquad \text{for all } i \in I, \tag{12}$$

$$f_i(L_i, t + T_i) = min(h_i^t(T_i), C_i^{out}(t)), \qquad \text{for all } i \in I, \tag{13}$$

$$f_i(0, t) \leq C_i^{in}(t), \qquad \text{for all } i \in I, \tag{14}$$

$$k_i = \frac{f_i}{u_i^{max}} + \frac{b_i f_i^2}{u_i^{max}}, \qquad \text{for all } i \in I, \tag{15}$$

$$u_i = \frac{f_i}{k_i}, \qquad \text{for all } i \in I, \tag{16}$$

$$\frac{dT_i(x_i, t)}{dx_i} = \frac{1}{u_i}, \qquad \text{for all } i \in I, \tag{17}$$

$$T_i(0, t) = 0, \qquad \text{for all } i \in I, \tag{18}$$

$$T_p(L_p, t) = \sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip}, t))\delta_{ip}, \qquad \text{for all } p \in P. \tag{19}$$

As we discussed in Section 3.2, equation (13) allows us to explicitly account for spillback and bottleneck phenomena. This analysis differs from the one in Kachani and Perakis [6] where a similar assumption, $f_i(0, t + T_i) = h_i^t(T_i)$, was made on the entrance link flow rates $f_i(0, t)$. Equation (12) is a first-order partial differential equation in the link flow rate f_i. Solving this PDE is the bottleneck operation in the solution of this model.

If we assume that equations (12) and (13) possess a continuously differentiable solution f_i, then, equations (15) and (16) determine the density function k_i and the velocity function u_i. The ordinary differential equation (17), under boundary condition (18), determines travel times on the network's links. Finally, path travel times follow from equation (19). Therefore, if we assume that equations (12) and (13) possess a continuously differentiable solution f_i, the SPTT Model, as formulated by equations (12)-(19), also possesses a solution.

Remark: Note that equations (15) and (16) simplify the travel time differential equation (17) into

$$\frac{dT_i(x_i, t)}{dx_i} = \frac{1 + b_i f_i}{u_i^{max}}.\tag{20}$$

3.4 Separable Spillback Exponential Travel Time Model

In this subsection, we take a different approach. We use the exact expression of the density flow rate relationship (9) to derive a conservation law in terms of the link flow rates. We then approximate this equation to obtain a first-order partial differential equation in terms of the link flow rates. We present a formulation of the Separable Spillback Exponential Travel Time Model and provide a necessary and sufficient condition for existence of a solution.

3.4.1 A Preliminary Result

Theorem 2 *[6] Under Assumption (A4), the link flow rate functions f_i are solutions of the partial differential equation:*

$$\frac{\partial f_i}{\partial t} + u_i^{max}(1 - 4b_i f_i)^{\frac{1}{2}}\frac{\partial f_i}{\partial x_i} = 0.\tag{21}$$

Furthermore, under Assumption (A5), the link flow rate functions f_i are solutions of the first-order partial differential equation:

$$\frac{\partial f_i}{\partial t} + u_i^{max}(1 - 2b_i f_i)\frac{\partial f_i}{\partial x_i} = 0.\tag{22}$$

Assumption (A3) provides a boundary condition.

This new conservation law (22) is the basis of the SETT Model that we present below and analyze in Sections 4 and 5.

3.4.2 Model Formulation

Theorem 2 gives rise to the following formulation:

9

SETT Model

For all $t \in [0, T]$:

$$\frac{\partial f_i}{\partial t} + u_i^{max}(1 - 2b_i f_i)\frac{\partial f_i}{\partial x_i} = 0, \qquad \text{for all } i \in I, \tag{23}$$

$$f_i(L_i, t + T_i) = min(h_i^t(T_i), C_i^{out}(t)), \qquad \text{for all } i \in I, \tag{24}$$

$$f_i(0, t) \le C_i^{in}(t), \qquad \text{for all } i \in I, \tag{25}$$

$$k_i = \frac{f_i}{u_i^{max}} + \frac{b_i f_i^2}{u_i^{max}}, \qquad \text{for all } i \in I, \tag{26}$$

$$u_i = \frac{f_i}{k_i}, \qquad \text{for all } i \in I, \tag{27}$$

$$\frac{dT_i(x_i, t)}{dx_i} = \frac{1}{u_i}, \qquad \text{for all } i \in I, \tag{28}$$

$$T_i(0, t) = 0, \qquad \text{for all } i \in I, \tag{29}$$

$$T_p(L_p, t) = \sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip}, t))\delta_{ip}, \qquad \text{for all } p \in P. \tag{30}$$

Equation (23) is a first-order partial differential equation in the link flow rate f_i. Solving this PDE is the bottleneck operation in the solution of this model. Moreover, equation (24) provides the boundary condition for this equation. It explicitly accounts for spillback and bottleneck phenomena by assuming that the exit link flow rates $f_i(L_i, t)$ can be approximated by continuoulsy differentiable functions $h_i(t)$.

Assuming that equations (23) and (24) possess a continuously differentiable solution f_i, equations (25) and (27) determine the density k_i and the velocity u_i. The ordinary differential equation (28) under its boundary condition (29) determines travel times on the network's links. Finally, path travel times follow from equation (30). Therefore, if we assume that equations (23) and (24) possess a continuously differentiable solution f_i, the SETT Model, as formulated by equations (23)-(30), also possesses a solution.

Remark: Replacing equations (25) and (27) in equation (28), leads to the same equation as for the SPTT Model, that is

$$\frac{dT_i(x_i, t)}{dx_i} = \frac{1 + b_i f_i}{u_i^{max}}. \tag{31}$$

4 Analysis of Separable Velocity Functions

In this section, we study the SPTT and the SETT Models in further details.

In particular, in Subsection 4.1, we extensively analyze the SPTT Model for piecewise linear and piecewise quadratic functions $h_i^t(T_i)$ (see Assumption (A3)). We show how Model 1 reduces in this case to the analysis of a single ordinary differential equation. We provide families of travel time functions.

In Subsection 4.2, we analyze the SETT Model by approximating the initial flow rate with piecewise linear functions $h_i^t(T_i)$. Moreover, we show why the analysis of the SETT Model is more complex than the one of the SPTT Model. Finally, we propose a family of travel time functions. In Subsection 4.3, we summarize our results and show how the families of travel time functions we propose in Subsections 4.1 and 4.2 relate.

4.1 Separable SPTT Model

In this subsection, we analyze the SPTT Model for piecewise linear and piecewise quadratic approximations of exit flow rates. We provide families of travel time functions under a variety of assumptions.

4.1.1 A General Framework for the Analysis of the SPTT Model

The purpose of this subsection is to provide a general framework for the analysis of the SPTT Model that reduces the problem to solving a single ordinary differential equation.

As a first step towards establishing the main result of this subsection, we introduce the classical method of characteristics in fluid dynamics. Haberman [5] provides a detailed analysis of this method. Along the characteristic line that passes through $(x_i, t + T_i)$ with slope $\frac{1+2b_i f_i}{u_i^{max}}$, the solution $f_i(x_i, t + T_i)$ of equation (12) remains constant. If $(L_i, t + s_i(x_i, t + T_i))$ denotes a point on this characteristic line, we have

$$f_i(x_i, t + T_i) = h_i^t(s_i(x_i, t + T_i)). \tag{32}$$

Note that this definition of the characteristic function $s_i(.)$ differs from the one of Haberman [5] and Perakis [11]. Indeed, in this paper, to explicitly account for spillback and bottleneck phenomena, the boundary condition is on the exit link flow rates $f_i(L_i, t)$. Therefore, we consider that $(L_i, t + s_i(x_i, t + T_i))$ is a point of the characteristic line. In Perakis [11], the boundary condition was on the departure link flow rates $f_i(0, t)$ instead and as a result, $(0, t + s_i(x_i, t + T_i))$ was considered as a point of the characteristic line. By using similar arguments as in [11], one can easily establish that

$$s_i(x_i, t + T_i) = \frac{T_i u_i^{max} - (x_i - L_i) - 2b_i(x_i - L_i)h_i^t(s_i(x_i, t + T_i))}{u_i^{max}}. \tag{33}$$

We introduce two new variables $m_i(.)$ and $g_i(.)$ defined by $m_i(s_i) = \frac{b_i}{u_i^{max}}(h_i^t(s_i) - A_i)$ and $g_i(x_i, t) = T_i(x_i, t) - \frac{1+b_i A_i}{u_i^{max}}(x_i - L_i)$.

Theorem 3 *(General framework) The SPTT Model reduces to solving the following ordinary differential equation:*

$$\frac{ds_i}{dx_i} = \frac{-m_i(s_i) - \frac{b_i A_i}{u_i^{max}}}{1 + 2(x_i - L_i)m_i'(s_i)}, \tag{34}$$

with $s_i(x_0, t) = \frac{(L_i - x_0)(1 + 2b_i A_i)}{u_i^{max} - 2b_i B_i(L_i - x_0)}$ as an initial condition, and where x_0 denotes the position of a driver on link i. The link flow rate functions and the link travel time functions follow from:

$$f_i(x_i, t) = h_i^t(s_i) \tag{35}$$

$$T_i(x_i, t) = s_i + \frac{(x_i - L_i) + 2b_i(x_i - L_i)h_i^t(s_i)}{u_i^{max}}. \tag{36}$$

Proof: Introducing g_i, m_i and A_i in equations (33) and (20), we derive the following two relations:

$$s_i = g_i - 2x_i m_i(s_i) - \frac{b_i A_i}{u_i^{max}} x_i, \tag{37}$$

$$\frac{dg_i}{dx_i} = m_i(s_i). \tag{38}$$

11

From equations (37) and (38), it follows that

$$\begin{aligned}
\frac{ds_i}{dx_i} &= \frac{dg_i}{dx_i} - 2m_i(s_i) - 2x_i\frac{ds_i}{dx_i}m_i'(s_i) - \frac{b_iA_i}{u_i^{max}} \\
&= -m_i(s_i) - 2x_i\frac{ds_i}{dx_i}m_i'(s_i) - \frac{b_iA_i}{u_i^{max}}.
\end{aligned}$$

Hence, $\frac{ds_i}{dx_i}(1 + 2x_im_i'(s_i)) = -m_i(s_i) - \frac{b_iA_i}{u_i^{max}}$. Then the results of the theorem follow.

$\square$

4.1.2 Piecewise Linear Exit Link Flow Rate Functions

In this subsection, we apply the general framework to simplify the analysis of the piecewise linear approximation of exit flow rates.

We assume that during a time period $[t, t + \Delta]$, drivers make the approximation that the exit link flow rate for subsequent times $t + T_i$ is linear in terms of the travel time T_i. That is,

$$f_i(L_i, t + T_i) = h_i^t(T_i) = A_i(t) + B_i(t)T_i. \tag{39}$$
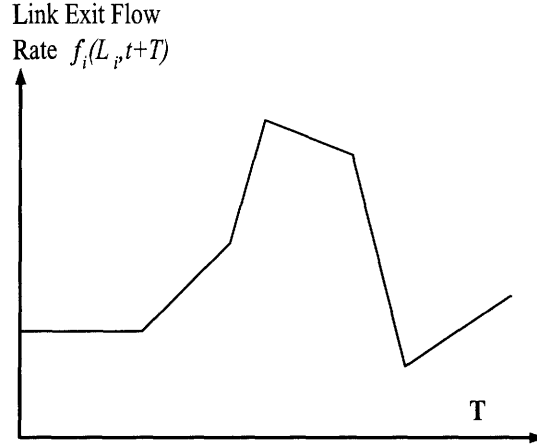
Link Exit Flow
Rate $f_i(L_i, t+T)$

Figure 2: A Possible Profile of Approximated Exit Flow Rates

Over the time period $[0, T]$, this results into a piecewise linear approximation of link exit flow rates as shown in Figure 2.

Remark:
Note that using the method of characteristics, a necessary and sufficient condition for the existence of solution of the SPTT Model in this case is:

$$B_i(t) < \frac{u_i^{max}}{2b_iL_i}. \tag{40}$$

We call the system of equations (12)-(19) and (39) the Linear SPTT Model. Next, we provide a closed form solution for the Linear SPTT Model.

Theorem 4 *If (40) holds, then:*

(i) The Linear SPTT Model possesses a solution,

(ii) The link flow rate functions $f_i(x_i, t + T_i)$ are continuously differentiable,

$$f_i(x_i, t + T_i) = \frac{B_i(t)u_i^{max}T_i - B_i(t)(x_i - L_i) + A_i(t)u_i^{max}}{u_i^{max} + 2b_iB_i(t)(x_i - L_i)}, \tag{41}$$

(iii) The link travel time $T_i(x_0, x_i, t)$ to reach position x_i of a driver initially at position x_0 at time t is given by:

$$T_i(x_0, x_i, t) = \frac{(x_i - x_0)}{u_i^{max}} + \left(\frac{L_i - x_0}{u_i^{max}} + \frac{A_i(t)}{B_i(t)}\right)\left((1 + \frac{x_i - x_0}{\frac{u_i^{max}}{2b_iB_i} - (L_i - x_0)})^{\frac{1}{2}} - 1\right). \tag{42}$$

Proof: Since $h_i^t(s_i) = A_i + B_is_i$, it follows that $m_i(s_i) = \frac{b_iB_i}{u_i^{max}}s_i$. Replacing in equation (34), we obtain

$$\frac{ds_i}{dx_i} = \frac{-\frac{b_iB_i}{u_i^{max}}s_i - \frac{b_iA_i}{u_i^{max}}}{1 + 2(x_i - L_i)\frac{b_iB_i}{u_i^{max}}}.$$

The above equation can be written as the following separable equation:

$$\frac{ds_i}{s_i + \frac{A_i}{B_i}} = -\frac{\frac{b_iB_i}{u_i^{max}}dx_i}{1 + 2(x_i - L_i)\frac{b_iB_i}{u_i^{max}}}. \tag{43}$$

Integrating both parts using $s_i(x_0, t) = \frac{(L_i - x_0)(1 + 2b_iA_i)}{u_i^{max} - 2b_iB_i(L_i - x_0)}$ (see Bender and Orszag (1978)) gives rise to

$$\frac{s_i + \frac{A_i}{B_i}}{\frac{(L_i - x_0) + \frac{A_iu_i^{max}}{B_i}}{u_i^{max} - 2b_iB_i(L_i - x_0)}} = \frac{1 - 2(L_i - x_i)\frac{b_iB_i}{u_i^{max}}}{(1 - 2(L_i - x_i)\frac{b_iB_i}{u_i^{max}})^{\frac{1}{2}}}.$$

Therefore it follows that

$$s_i(x_i, t) = \frac{(L_i - x_0) + \frac{A_iu_i^{max}}{B_i}}{u_i^{max} - 2b_iB_i(L_i - x_0)}\frac{1}{(1 + \frac{x_i - x_0}{\frac{u_i^{max}}{2b_iB_i} - (L_i - x_0)})^{\frac{1}{2}}} - \frac{A_i}{B_i}. \tag{44}$$

Using equations (35) and (36), the results of the theorem follow.

$\square$

Corollary 1 *Assume that*

$$|B_i(t)| << \frac{u_i^{max}}{2b_iL_i}. \tag{45}$$

Then:

(i) The Linear SPTT Model possesses a solution,

(ii) The link travel time functions $T_i(x_0, x_i, t)$ simplify as follows:

$$T_i(x_0, x_i, t) = \frac{1 + A_i(t)b_i - \frac{(L_i - x_0)b_iB_i(t)}{u_i^{max}}}{u_i^{max} - 2(L_i - x_0)b_iB_i(t)}(x_i - x_0) \tag{46}$$

$$- \frac{1}{2(u_i^{max})^2}(A_i(t)B_i(t)(b_i)^2 + \frac{(B_i(t)b_i)^2(L_i - x_0)}{u_i^{max}})(x_i - x_0)^2.$$

13

Proof:
(i) Note that equation (45) implies that equation (40) holds. Hence, part (i) follows.
(ii) Relationship (45) allows us to use a second order Taylor expansion of equation (42). This leads to equation (46).

$\square$

4.1.3 Piecewise Quadratic Exit Link Flow Rate Functions

In this subsection, we assume that during a time period $[t, t + \Delta]$, drivers make the approximation that the exit link flow rate for subsequent times $t + T_i$ is quadratic in terms of the travel time T_i. That is,

$$f_i(L_i, t + T_i) = h_i^t(T_i) = A_i(t) + B_i(t)T_i + C_i(t)(T_i)^2. \tag{47}$$

Over the time period $[0, T]$, this results into a piecewise quadratic approximation of link exit flow rates as shown in Figure 3.
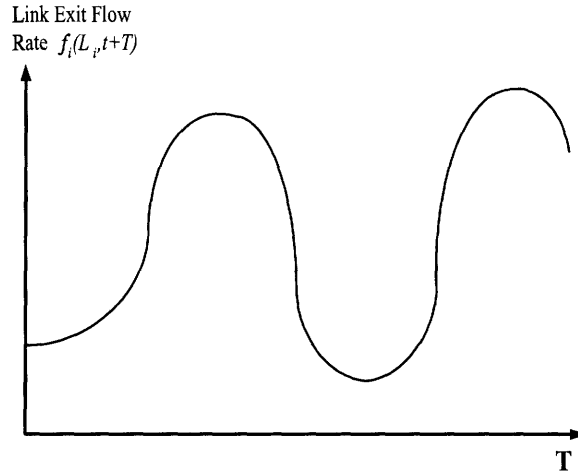

Figure 3: A Possible Profile of Approximated Exit Flow Rates

Note that a necessary and sufficient condition for existence of a solution, is in this case:

$$B_i(t) + 2C_i(t)(t + \Delta) < \frac{u_i^{max}}{2b_i L_i}. \tag{48}$$

We call the system of equations (12)-(19) and (47) the Quadratic SPTT Model. Next, we provide a closed form solution to the Quadratic SPTT Model. Note that when the quadratic term is neglected (i.e. $C_i = 0$), we capture the previously studied case of piecewise linear exit link flow rate functions.

Theorem 5 *Assume that*

$$|B_i(t) + 2C_i(t)(t + \Delta)| << \frac{u_i^{max}}{2b_i L_i}. \tag{49}$$

14

Then, the following holds

(i) The Quadratic SPTT Model possesses a solution.

(ii) The third degree Taylor expansion of the link travel time functions $T_i(x_i, t)$ becomes

$$T_i(x_0, x_i, t) = \frac{1 + A_i(t)b_i - \frac{(L_i - x_0)b_i B_i(t)}{u_i^{max}}}{u_i^{max} - 2(L_i - x_0)b_i B_i(t)}(x_i - x_0)$$

$$- \frac{1}{2(u_i^{max})^2}(A_i(t)B_i(t)(b_i)^2 + \frac{(B_i(t)b_i)^2(L_i - x_0)}{u_i^{max}})(x_i - x_0)^2$$

$$+ (\frac{11A_i(t)(B_i(t))^2(b_i)^3}{6(u_i^{max})^3} + \frac{11(B_i(t))^3(b_i)^3(L_i - x_0)}{6(u_i^{max})^4}$$

$$- \frac{4(A_i(t))^2 C_i(t)(b_i)^3}{3(u_i^{max})^3} - \frac{2A_i(t)B_i(t)C_i(t)(b_i)^3(L_i - x_0)}{(u_i^{max})^4})(x_i - x_0)^3. \tag{50}$$

Proof: The analysis involved in this proof is very tedious and similar to the one in Kachani and Perakis [6]. We do not include it for the sake of brevity.

4.2 Separable SETT Model

In this subsection, we study the SETT Model. We show that the analysis of the SETT Model is more complex than the SPTT Model, and propose a different class of travel time functions for piecewise linear approximations of exit flow rates.

Piecewise Linear Exit Link Flow Rate Functions

We assume that during a time period $[t, t + \Delta]$, drivers make the approximation that the exit link flow rate for subsequent times $t + T_i$ is linear in terms of the travel time T_i (see Figure 2). That is,

$$f_i(L_i, t + T_i) = h_i^t(T_i) = A_i(t) + B_i(t)T_i. \tag{51}$$

A necessary and sufficient condition for the existence of a solution, is in this case:

$$B_i(t) < \frac{u_i^{max}}{2b_i L_i}(1 - 2b_i A_i(t) - 2b_i B_i(t)(t + \Delta))^2. \tag{52}$$

We call the system of equations (23)-(30) and (51) the Linear SETT Model. Next, we provide a closed form solution of the Linear SETT Model.

To make our notation more tractable, we introduce variables $\theta_1 = \frac{b_i B_i(t)}{1 - 2b_i A_i(t)}$, $\theta_2 = \frac{b_i B_i(t)}{u_i^{max}}$ and $\theta_3 = \frac{1 + b_i A_i(t) + \frac{\theta_1(L_i - x_0)}{u_i^{max}}}{u_i^{max}} - \frac{1}{u_i^{max}(1 - 2b_i A_i(t))}$.

Theorem 6 *Assume that*

$$|B_i(t)| << \frac{u_i^{max}}{2b_i L_i}(1 - 2b_i A_i(t) - 2b_i B_i(t)(t + \Delta))^2. \tag{53}$$

The following holds,

(i) The Linear SETT Model possesses a solution,

(ii) The link characteristic line functions $s_i(.)$ are continuously differentiable and can be expressed as a function of the link travel time functions, that is,

$$s_i(x_i, t) = \frac{T_i u_i^{max}(1 - 2b_i A_i(t)) - (x_i - L_i)}{u_i^{max}(1 - 2b_i A_i(t) + 2b_i B_i(t)T_i)}, \tag{54}$$

(iii) The link travel time $T_i(x_0, x_i, t)$ to reach position x_i of a driver initially at position x_0 at time t is given by:

$$T_i(x_i, t) = \theta_3 \left(\frac{e^{\theta_2(x_i - x_0)} - 1}{\theta_2} \right) + \frac{\theta_1(x_i - x_0)}{\theta_2(u_i^{max})^2}, \tag{55}$$

(iv) If condition (45) holds, the link travel time function $T_i(x_0, x_i, t)$ is

$$T_i(x_0, x_i, t) = \frac{1 + A_i(t)b_i - \frac{(L_i - x_0)b_i B_i(t)}{u_i^{max}}}{u_i^{max} - 2(L_i - x_0)b_i B_i(t)}(x_i - x_0) \tag{56}$$

$$- \frac{1}{2(u_i^{max})^2}\left(A_i(t)B_i(t)(b_i)^2 + \frac{(B_i(t)b_i)^2(L_i - x_0)}{u_i^{max}}\right)(x_i - x_0)^2.$$

Proof: The analysis involved in this proof is quite tedious and similar to [6]. For the sake of brevity, we do not include it.

4.3 Summary and Models Comparison

In summary, we have so far derived two families of travel time functions. The Linear SPTT Model which leads to the polynomial family of travel time functions

$$T_i(x_0, x_i, t) = \frac{(x_i - x_0)}{u_i^{max}} + \left(\frac{L_i - x_0}{u_i^{max}} + \frac{A_i(t)}{B_i(t)}\right)\left((1 + \frac{x_i - x_0}{\frac{u_i^{max}}{2b_i B_i} - (L_i - x_0)})^{\frac{1}{2}} - 1\right), \tag{57}$$

and the Linear SETT Model which leads to the exponential family of travel time functions

$$T_i(x_i, t) = \theta_3 \left(\frac{e^{\theta_2(x_i - x_0)} - 1}{\theta_2} \right) + \frac{\theta_1(x_i - x_0)}{\theta_2(u_i^{max})^2}, \tag{58}$$

where, θ_i, $i \in \{1, 2, 3\}$ defined in Subsection 4.2.

It is very important to note that equations (57) and (58) coincide when $|B_i(t)| << \frac{u_i^{max}}{2b_i L_i}$ holds. That is, they possess the same second order Taylor expansion

$$T_i(x_0, x_i, t) = \frac{1 + A_i(t)b_i - \frac{(L_i - x_0)b_i B_i(t)}{u_i^{max}}}{u_i^{max} - 2(L_i - x_0)b_i B_i(t)}(x_i - x_0) \tag{59}$$

$$- \frac{1}{2(u_i^{max})^2}\left(A_i(t)B_i(t)(b_i)^2 + \frac{(B_i(t)b_i)^2(L_i - x_0)}{u_i^{max}}\right)(x_i - x_0)^2.$$

This relationship seems to indicate that the assumptions made for both the Linear SPTT Model and the Linear SETT Model are indeed reasonable.

Furthermore, the Quadratic SPTT Model gives rise to a more complicated expression of link travel time functions. The third degree Taylor expansion leads to

$$T_i(x_0, x_i, t) = \frac{1 + A_i(t)b_i - \frac{(L_i - x_0)b_i B_i(t)}{u_i^{max}}}{u_i^{max} - 2(L_i - x_0)b_i B_i(t)}(x_i - x_0) \tag{60}$$

$$- \frac{1}{2(u_i^{max})^2}\left(A_i(t)B_i(t)(b_i)^2 + \frac{(B_i(t)b_i)^2(L_i - x_0)}{u_i^{max}}\right)(x_i - x_0)^2$$

$$+ \left(\frac{11A_i(t)(B_i(t))^2(b_i)^3}{6(u_i^{max})^3} + \frac{11(B_i(t))^3(b_i)^3(L_i - x_0)}{6(u_i^{max})^4}\right.$$

$$\left. - \frac{4(A_i(t))^2 C_i(t)(b_i)^3}{3(u_i^{max})^3} - \frac{2A_i(t)B_i(t)C_i(t)(b_i)^3(L_i - x_0)}{(u_i^{max})^4}\right)(x_i - x_0)^3.$$

We observe that

- If the quadratic term is neglected (i.e. $C_i = 0$), then a second order approximation of equation (60) leads to equation (59) and, as one would expect, we fall in the case of the Linear SPTT Model. Hence, it appears that the assumptions made for the Quadratic SPTT Model are also reasonable.

- If the constant term is neglected (i.e. $A_i = 0$), equation (60) provides us with a non-zero third order degree term.

This concludes our analysis of the separable case. In the following section, we study the non-separable case of this problem.

5 A Non-Separable Model

In this section, we generalize the Spillback Polynomial Travel Time Model (SPTT Model) we presented in Subsection 3.3 to the case of non-separable velocity functions. We show how the results obtained for the separable case extend to the non-separable case.

In this section, we consider an acyclic network. The acyclicity assumption will enable us to extend the results of Subsection 4.1 to the case of non-separable velocity functions. We model link interactions by considering that the velocity of link i, at position x_i and at time t, can be expressed as in Subsection 3.1 by

$$\widehat{u}_i(k, \nabla k) = u_i^{max} - b_i(u_i^{max})^2 k_i + \sum_{j \in B(i)} \alpha_{ij}(x_i)R_{ij}(\overline{x}_j, t - \Delta_{ij}), \tag{61}$$

where $\alpha_{ij}(x_j)$ is the density correlation function between link j and link i and depends on the position x_j on link j; R_{ij} is a function of k_j and ∇k_j; $\overline{x}_j$ is a fixed position of a detector of density on link j; Δ_{ij} is a propagation time between link j and link i; and $B(i)$ is the set of predecessors of link i (See Figure 4 for an illustration of the notation in the case of a two-link-network).
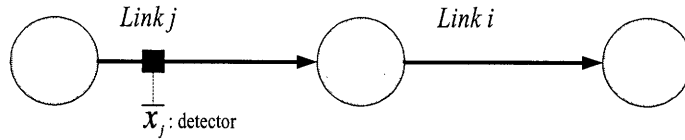

Figure 4: A Network with Two Links

A predecessor of a link i is any link that comes before i on a path. It does not restrict to only the immediate parent of a link. Note that since we consider the case of acyclic networks, we can talk about predecessors of a link. Note as well that the results we will establish for the case, where we consider the set of predecessors, also apply to the case where we consider the set of successors instead.

For the sake of simplicity, let us consider $R_{ij}(.) = k_j(.)$. Moreover, for the sake of simplifying notation, we introduce $J_i = 1 + \sum_{j \in B(i)} \frac{\alpha_{ij}(x_i)}{u_i^{max}} k_j(\overline{x}_j, t - \Delta_{ij})$.

The Non-Separable SPTT Model becomes:

For all $t \in [0, T]$,

$$\frac{\partial f_i}{\partial x_i} + \frac{1}{u_i^{max}}\left(\frac{1}{J_i} + \frac{2b_i f_i}{J_i^3}\right)\frac{\partial f_i}{\partial T_i} = 0, \qquad \text{for all } i \in I, \tag{62}$$

$$f_i(L_i, t + T_i) = min(h_i^t(T_i), C_i^{out}(t)), \qquad \text{for all } i \in I, \tag{63}$$

$$f_i(0, t) \leq C_i^{in}(t), \qquad \text{for all } i \in I, \tag{64}$$

$$k_i = \frac{f_i}{u_i^{max} J_i} + \frac{b_i f_i^2}{u_i^{max} J_i^3}, \qquad \text{for all } i \in I, \tag{65}$$

$$u_i = \frac{f_i}{k_i}, \qquad \text{for all } i \in I, \tag{66}$$

$$\frac{dT_i(x_i, t)}{dx_i} = \frac{1}{u_i}, \qquad \text{for all } i \in I, \tag{67}$$

$$T_i(0, t) = 0, \qquad \text{for all } i \in I, \tag{68}$$

$$T_p(L_p, t) = \sum_{i \in I} T_i(L_i, t + T_{ip}(L_{ip}, t))\delta_{ip}, \qquad \text{for all } p \in P. \tag{69}$$

Note that boundary condition (63) allows us to explicitly account for spillback and bottleneck phenomena.

We assume that the density correlation function $\alpha_{ij}(x_i)$ between link i and link j is a constant function of x_i. In this case, J_i is also a constant function of x_i.

We now consider that during a time period $[t, t + \Delta]$, drivers make the approximation that the link flow rate for subsequent times $t + T_i$ is linear in terms of the travel time T_i. That is, $f_i(0, t + T_i) = h_i^t(T_i) = A_i(t) + B_i(t)T_i$.

The following theorem is an extension of Theorem 4 to the non-separable case with constant density correlation functions.

Theorem 7 If $B_i < \frac{u_i^{max} J_i^3}{2b_i L_i}$ holds, then:
(i) The Non-Separable Linearized SPTT Model possesses a solution,
(ii) The link flow rate functions $f_i(x_i, t + T_i)$ are continuously differentiable, and we have:

$$f_i(x_i, t + T_i) = \frac{B_i(t)u_i^{max}T_i - \frac{B_i(t)(x_i - L_i)}{J_i} + A_i(t)u_i^{max}}{u_i^{max} + \frac{2b_i B_i(t)(x_i - L_i)}{J_i^3}},$$

(iii) The link travel time $T_i(x_0, x_i, t)$ to reach position x_i of a driver initially at position x_0 at time t is given by:

$$T_i(x_0, x_i, t) = \frac{(x_i - x_0)}{u_i^{max} J_i} + \left(\frac{L_i - x_0}{u_i^{max}} + \frac{A_i(t)}{B_i(t)}\right)\left(\left(1 + \frac{x_i - x_0}{\left(\frac{u_i^{max}}{2b_i B_i} - (L_i - x_0)\right)J_i^3}\right)^{\frac{1}{2}} - 1\right). \tag{70}$$

Proof: The proof is similar to the one of Theorem 4.

Note that when the density correlation functions are set to zero, it follows that $J_i = 1$. The results of Theorem 7 then reduce to the results of Theorem 4 from Subsection 4.1.2.

6 Conclusions

In this paper, we took a fluid dynamics approach to determine travel times in a transportation network operating under a dynamic setting. We introduced a model that incorporated spillback and bottleneck

phenomena, that is, drivers whose travel times are affected by a change in traffic conditions during their trip (e.g. spillback or bottleneck phenomena) and therefore need to be updated. We proposed a general travel time model and derived two approximation models: the Spillback Polynomial (SPTT) and the Spillback Exponential (SETT) Travel Time models. We proposed an ordinary differential equation for analyzing the SPTT Model. In the case of separable velocity functions, we proposed families of travel time functions that rely on piecewise linear and piecewise quadratic approximations of exit flow rates. We further established a connection between these travel time functions. Our analysis of the SETT Model applied to the case of separable velocity functions while the analysis of the SPTT Model applied to both separable and non-separable velocity functions. The latter applied to the case of acyclic networks.

Continuing this work, we intend to extend our models to also incorporate the case of non-separable velocity functions as they apply to non-acyclic networks. We intend to extend our models in order to incorporate second-order effects such as reaction of drivers to upstream and downstream congestion as well as second-order link interaction effects. Moreover, we intend to investigate alternate approaches such as queuing models but also connect our models with the dynamic user-equilibrium problem. Finally, we intend to perform a numerical study for realistic networks using our current models in order to investigate how a numerical solution approach compares to an analytical one.

References

[1] R. Akcelik. Capacity of a Shared Lane. *Australian Road Research Board Proceedings*, 14(2):228–241, 1988.

[2] C. Barnhart, P. Ioannou, G. Nemhauser, and H. Richardson. A Report of the National Science Foundation on the Workshop on Planning, Design, Management and Control of Transportation Systems. 1998.

[3] D. Boyce, B. Ran, and L. Leblanc. Solving an instantaneous dynamic user-optimal route choice model. *Transportation Science*, 29(2):128–142, 1995.

[4] T.L. Friesz, D. Bernstein, T.E. Smith, R.L. Tobin, and B-W. Wie. A variational inequality formulation of the dynamic network user-equilibrium problem. *Operations Research*, 41(1):179–191, 1993.

[5] R. Haberman. Mathematical models, mechanical vibrations, population dynamics and traffic flow. *Prentice-Hall, Engelwood Cliffs, New Jersey*, 1977.

[6] S. Kachani and G. Perakis. Modeling Travel Times in Dynamic Transportation Networks; A Fluid Dynamics Approach. *Submitted to Operations Research*, 2001.

[7] S. Kachani and G. Perakis. Second-Order Fluid Dynamics Models for Travel Times in Dynamic Transportation Networks. *Proceedings of the 4th International IEEE Conference on Intelligent Transportation Systems, Oakland, California*, 2001.

[8] M.J. Lighthill and G.B. Whitham. On kinematic waves ii. A theory of traffic flow on long crowded roads. *Proceedings of the royal society of London, Series A*, (229):317–345, 1955.

[9] C. Meneguzzer, D.E. Boyce, N. Rouphail, and A. Sen. Implementation and Evaluation of an Asymmetric Equilibrium Route Choice Model Incorporating Intersection-Related Travel Times. *Report to Illinois Department of Transportation, Urban Transportation Center, University of Illinois, Chicago*, 1990.

[10] "Bureau of Public Roads". Traffic Assignment Manual. *US Department of Commerce, Washington, DC*, 1964.

[11] G. Perakis. The Dynamic User-Equilibrium Problem through Hydrodynamic Theory. *Working paper, Operations Research Center, MIT*, 1997.

[12] P.I. Richards. Shock waves on the highway. *Operations Research*, (4):42–51, 1956.