Potential Games and Competitive Scheduling in Wireless Networks

by

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Abstract

This thesis studies a game theoretic model for scheduling transmissions among multiple self-interested users in a wireless network with fading. Our model involves a finite number of mobile users transmitting to a common base station under time-varying channel conditions. A distinguishing feature of our model is the assumption that the channel quality of each user is affected by global and time-varying conditions at the base station, resulting in each user observing a common channel state. Each user chooses a transmission policy that maximizes its utility function, which captures a natural trade-off between throughput and power. The transmission policy specifies how transmissions should be scheduled as a function of the time-varying common channel state observed by each user.

We make three main contributions. First, we establish the existence of a Nash equilibrium of this game and characterize the set of equilibria. We investigate the efficiency properties of these equilibria, and study a related aggregate utility maximization problem, to serve as a benchmark for the performance of the equilibria. We quantify the efficiency loss in the game comparing the optimal solution of the aggregate utility maximization problem, to the best and worst equilibria in terms of the aggregate utility. We show that the performance of the worst equilibrium can be arbitrarily bad (in terms of the aggregate utility), but the efficiency loss of the best equilibrium can be bounded as a function of a technology-related parameter.

Our second contribution is to study various distributed mechanisms to reach an equilibrium of this game. We use the theory of potential games to establish convergence of such mechanisms to an equilibrium. To this end, we study conditions under which the scheduling game is a potential game. This necessitates extending the known necessary conditions for the existence of ordinal potential in games. In this thesis, we show that the scheduling game has a twice continuously differentiable ordinal potential if and only if a rate alignment condition holds.

In our third contribution, we investigate the related question of characterizing the "distance" of an arbitrary game to an exact potential game. We provide a new framework based on combinatorial Hodge theory for projecting an arbitrary game to the set of exact potential games. We prove that the equilibria of a game are ϵ-

equilibria of its projection, where ϵ is bounded by the projection error. Moreover, we show that the projection of a game to the set of exact potential games can be calculated using distributed consensus algorithms.

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Chapter 1

Introduction

In this chapter we briefly discuss the role of game theory in control of wireless networks. We also present an outline and a summary of the contributions of this thesis.

1.1 Context

Traditional network optimization approach assumes a single network administrator which has full access to all information in the network and designs algorithms that optimize a single network-wide objective among obedient users. Modern networks, on the other hand, have emerged from the interconnections of autonomous entities and heterogeneous users with diverse set of application requirements. This naturally has led to a distributed control paradigm in which network control functions are delegated to end users who make their decisions independently according to their own performance objective (e.g., [36, 35, 24, 25, 37]).

In scheduling and resource allocation problems in wireless networks, agents compete for the available resources, such as bandwidth. Noncooperative game theory is a natural theoretical framework for analysis and management of the competition in wireless networks and it provides a robust and distributed control paradigm. Such control paradigms allow having network domains, in which situation-aware users take autonomous decisions with regard to their network usage, based on the current network conditions and their individual preferences. In the recent literature, it is possible to see many applications of game theory to analysis and design of resource allocation mechanisms in wireless networks (see, e.g., [3, 1, 25, 24, 34, 35, 26, 49], and [37] for a survey).

A strategic form game consists of a set of players, a utility function and a set of strategies defined for each player. In applications of game theory to the study of wireless networks, users in the network are treated as players, objectives of players are modeled by utility functions and the interactions between the users are analyzed. Frequently, Nash equilibrium is used to study the outcome of the interactions of players in a game. This solution concept defines an outcome of a game from which no user has incentive to deviate unilaterally.

It is well-known that noncooperative behavior in networks results in inefficiency in terms of the aggregate system utility. Aggregate utility maximization problem is often studied as a benchmark problem to quantify the inefficiency of equilibria. In games efficiency is commonly defined in terms of the ratio of the aggregate utility in equilibria to the optimal solution of the this problem. In particular, two quantities that are used to study the efficiency loss are the *price of stability* and *price of anarchy* in the system, [5, 46, 48, 47]. These quantities represent the ratio of the aggregate utility in the best and the worst equilibria to that of the optimal solution of the benchmark problem. Price of stability and price of anarchy are frequently analyzed in the literature for determining the quality of the equilibrium solution in network problems (see e.g. [14, 22, 34, 35]).

An important challenge in game-theoretic models for networks is the development of dynamics that converge to a Nash equilibrium. Much of the literature on this topic focuses on dynamics that involve simple update rules by the players. However, in general, simple dynamics do not converge to an equilibrium of a game. Potential games are an important class of games in which simple dynamics converge (see e.g. [41, 56, 28]). There are also extensions of potential games, such as ordinal potential games, which have similar properties in terms of dynamics. The common feature of these games is the existence of a potential function which represents the quality of the different strategy profiles jointly for all users.

The general framework which we consider in this thesis is that of users who obtain some information about the network (e.g. channel quality) and accordingly control their transmission parameters. More specifically, we study the scenario in which finitely many users schedule their transmissions to a common base station, while the channel quality between the users and the base station is time varying. In this work, we model the interactions between the users in the network as a game and provide a detailed analysis of this scheduling game. In particular, we study the Nash equilibria, efficiency properties of the described game and provide dynamics that converge to an equilibrium of the game.

1.2 Related Literature

In recent years game theory has found applications in various problems in the communications literature, [38, 23, 26, 49, 55, 57, 33]. In this section we present a brief overview of game-theoretic approaches to resource allocation in networks.

Today's communication systems rely on transmission protocols in order to utilize the scarce resources available, such as bandwidth and energy. Centralized control protocols for these systems are not feasible due to the large size and complicated interconnection structures of communication networks. This leads to distributed control protocols for the control of communication networks. An example is the TCP/IP standard on which Internet is based. Such protocols rely heavily on cooperation of users in the network with the assigned control rule but in many cases, users have incentive to not to obey the control rule. This makes game theory a useful tool in the analysis of networks [38, 23, 26].

Game theory has found applications to the power allocation problem in wireless networks. A frequently studied channel model in these problems is the code division multiple access (CDMA) channel [49, 1, 26, 51]. In [51] the authors consider a power control game for a CDMA system with single base station where utility of each user is a function of its transmit power and signal-to-interference ratio. They show that the achieved equilibrium is inefficient and by supplementary pricing mechanism the quality of the equilibrium can be improved. In [49], the results of [51] are extended to systems with multiple base stations and different pricing schemes are studied.

In [23], the authors consider resource allocation in time varying multiple access channels with users limited by average power constraints. They show that the optimal operating point (in terms of aggregate throughput) coincides with the unique Nash equilibrium of the proposed game. Another work related to multiple access channels is [38]. This paper considers games in multiple access channels where users have quality of service constraints. The authors discuss various utility functions and their implications on the communication systems. The strategy spaces of users consist of a selection of different parameters including choice of transmit powers, transmission rates, modulation scheme, and utilities in the game are defined as a function of these parameters. The authors discuss the properties of the Nash equilibria of the resulting games and quantify the effect of different network parameters on energy efficiency and network capacity.

There is also work related to resource allocation games in collision channels. This channel model differs from the CDMA channels as in collision channels, transmissions at a given time slot are successful only if a single user attempts transmission during this time slot. In [35], the authors consider a model for an uplink collision channel. The channel quality process is assumed to be time varying and independent across the users. Each user aims at minimizing its power investment while satisfying a minimum throughput demand. The authors study the conditions under which equilibrium in the game exists. They show that there are at most two equilibria in the game and if multiple equilibria exist one equilibrium is strictly better than the other for all users in terms of the power investment. The authors also suggest a fully distributed mechanism that converges to the good quality equilibrium. Using a similar model in [36] the authors show that when additional power levels are made available to users in the system, a paradoxical behavior is observed, i.e. the equilibrium quality decreases when more power levels are present in the system.

The wireless network game that is considered in this thesis is related to the games considered in [35, 36], however it significantly differs in the assumptions on the channel quality processes and utilities of users.

1.3 Contributions and Thesis Outline

In this thesis we study a game theoretic model for distributed scheduling in wireless collision channels. We consider a wireless network, where finitely many users interact over a shared collision channel. Channel quality of each user is affected by *global* and time varying conditions. Each user independently adjusts its transmission parameters in order to maximize its payoff which is a function of the trade-off between throughput and power.

Our main results related to the scheduling game can be summarized as follows,

- We study the existence of Nash equilibrium and its properties. We show that equilibrium always exists but it is not unique. In fact there can be uncountably many equilibria in the game.

- We then consider the efficiency loss in the system to determine the quality of the game solution. To this end, we first study the social welfare (aggregate utility) maximization problem in the network to serve as a benchmark to determine the quality of the game solution. We show that the social welfare maximization proplem is a nonconvex optimization problem. We prove that under self-interested user behavior, the equilibrium performance can be arbitrarily bad. Nevertheless, the efficiency loss at the best equilibrium can be bounded as a function of a technology parameter, which accounts both for the mobiles' power limitations and the underlying channel quality.

- We present various dynamics that ensure convergence to a Nash equilibrium in the game. In particular, we show that best-response dynamics converge to an equilibrium in finite time under certain update rules. To do this, we exploit the structure of the strategy spaces of users and utilize the properties of potential games. We also empirically verify the convergence of the dynamics to an equilibrium.

This thesis not only studies the results for the specific wireless network game, but also contributes to the theory of potential games. In the analysis of the scheduling game we use the properties of potential and ordinal potential games to obtain results about the game dynamics. However, in the literature there are no easy to check conditions for studying the existence of ordinal potential in games. This necessitates the study of the conditions on existence of ordinal potential in games.

Although there has been much work in the literature on the necessary and sufficient conditions for the existence of exact potential, the conditions for the existence of ordinal potential are not well understood [41]. In [54, 43], the authors present conditions for the existence of an ordinal potential, however these conditions are not easily checkable. In particular for continuous games, different tools are necessary to study the existence of an ordinal potential.

Exact potential games have many desirable properties, however the class of exact potential games is a "small" subset of the space of games. This motivates the study of the class of games that are "approximately" potential games. To this end, in this thesis we suggest an approach for finding a potential game that is close in some sense to a given game. For this we apply ideas previously used in the context of ranking problems to the theory of potential games.

Combinatorial Hodge theory is a tool that is used in ranking problems to determine the inconsistency in the pairwise rankings [21]. Pairwise comparisons (or rankings) of different alternatives contain inconsistencies if there is no order representing the preferences. For example, if three alternatives a, b, c are considered and pairwise rankings indicating $a > b$, $b > c$, $c > a$ are present (where $>$ represents the preference relation between alternatives) the pairwise rankings are inconsistent.

In a game utility of a user represents rankings of different strategy profiles by this particular user. The game is a potential game if the rankings given by different users are consistent, i.e. if user interests are aligned with a global performance goal. We use the ideas from combinatorial Hodge theory to study the inconsistency in the pairwise rankings of the strategy profiles when the game is not a potential game. This leads to a framework for projecting an arbitrary game to the set of exact potential games.

Our results related to the conditions on existence of ordinal potential in games and projections to the set of exact potential games are summarized below.

- We consider a strategic form game, with finitely many users and study the sets of exact potential games and its extensions. We show that the set of exact potential games is convex, whereas the set of ordinal potential games is nonconvex. We also show that the set of exact potential games is a "small" subspace of the space of games.

- Secondly, we consider continuous games, where the strategy space of each player is a nonempty closed bounded subset of an Euclidean space. Assuming that players have differentiable utility functions we obtain necessary conditions for existence of a continuously differentiable ordinal potential function. Some of the results obtained here are in the same spirit as [16, 32], which provide easy to check conditions for existence of utilities representing preferences of agents in an economy.

- We apply our results on ordinal potential games to the scheduling game studied in this thesis. We show that the game has a continuously differentiable ordinal potential if and only if a symmetry condition holds in the game.

- We study the problem of projection of finite games to the set of exact potential games. The projection framework enables us to find a potential game that is closest to a given game in a well defined norm.

- We show that the projections can be obtained with a distributed procedure requiring some information exchange between the players of a game. Additionally we prove that each equilibrium of the initial game is an ϵ-equilibrium of the projected game and each equilibrium of the projected game is an ϵ-equilibrium of the initial game.

1.3.1 Outline

The rest of this thesis is organized as follows. In Chapter 2 we provide definitions and some known results about games with emphasis on potential games. We also state the known results in the literature about the existence of potential in games. Additionally, we introduce the combinatorial Hodge Theory which is related to the projection framework considered in this thesis. In Chapter 3, we introduce the scheduling game and we provide the results obtained for this game. In Chapter 4 we focus on the existence of potential in games and the projections of games to the set of exact potential games. In Chapter 5 we present a summary of our results as well as future directions for research.

Chapter 2

Background

In this chapter we give an overview of basic notions of game theory with emphasis on the potential games. In Section 2.1 we discuss concepts of equilibrium, efficiency loss and dynamics in games. In this section, we also introduce potential games and discuss various generalization of potential games. Moreover, the results in the literature related to the existence of potential in games are presented.

In Section 2.2, we introduce combinatorial Hodge theory and discuss its relation to the problem of projecting an arbitrary game to the set of exact potential games. We provide an application of Hodge theory to ranking problems, as a similar approach is used in Chapter 4 for projections of games. Additionally, we provide the notations and basic results, related to our projection framework.

2.1 Game Theory and Potential Games

In this section we provide a basic introduction to game theory. We formally define games and discuss different solution concepts in games. We focus on properties of potential games and give a summary of results on efficiency loss and dynamics in games.

2.1.1 Basic Definitions and Notations

Game theory is the study of multi-person decision problems. A mathematical model of a game considers interactions of a number of decision makers, often referred to as agents or players. Agents are assumed to have their individual objectives and act according to their objectives. The aim of game theory is to analyze strategic interactions between different agents in a system.

In this thesis we restrict ourselves to **strategic form games**. A strategic form game consists of:

- **Set of players**, which is usually assumed to be finite. We denote the set of players by $\mathcal{M} = \{1, \ldots M\}$.

- **Strategy space** E^m for each player $m \in \mathcal{M}$, which is the set of actions a player can take. We denote the joint strategy space of all players by $E = \prod_{m \in \mathcal{M}} E^m$.

- **Utility function** $u^m : E \to \mathbb{R}$ for each player $m \in \mathcal{M}$.

We use $x^m \in E^m$ to denote a strategy of player m. A collection $x = (x^1, \ldots, x^M)$ of strategies of all players is referred to as a **strategy profile**. The utility function of a player assigns a payoff to a given strategy profile, and payoff of a player is affected by strategies of other players. Usually the set of all players but m is denoted by $-m$ and these players are referred to as **opponents of player** m. The set of actions for opponents of player m is denoted by E^{-m}. We denote a strategic game with given set of players, strategy spaces and utility functions as $\mathcal{G} = \langle \mathcal{M}, \{E^m\}_{m \in \mathcal{M}}, \{u^m\}_{m \in \mathcal{M}} \rangle$. As a short hand notation we use $u_{all} = \{u^1, \ldots u^M\}$ for the collection of all utilities in the game.

The set of actions available in a game are often referred to as pure strategies. An extension of pure strategies is mixed strategies where mixed strategy of a player can be defined as a probability distribution over the set of its pure strategies. In this thesis we are concerned only with pure strategies and term strategy refers to pure strategies.

In strategic form games the underlying assumption is that preferences of players are captured through the utility functions, i.e. a strategy profile x is preferred over strategy profile y by player m if and only if $u^m(x) > u^m(y)$. Players are assumed to be non-cooperative, each player acts independently to improve its payoff. They are also rational in the sense that they utilize strategies with better payoffs. This assumption leads to an equilibrium concept for games, namely the Nash Equilibrium.

Definition 2.1.1 (Nash Equilibrium). *A Nash equilibrium is a strategy profile from which no player can unilaterally deviate and improve its payoff. Formally, a strategy profile* $x \triangleq (x^1, \dots, x^M)$ *is a Nash equilibrium point if*

$$x^m \in \operatorname*{argmax}_{\tilde{x}^m \in E^m} u^m(\tilde{x}^m, x^{-m}), \quad \text{for every } m \in \mathcal{M} \tag{2.1}$$

Note that Nash equilibrium of a game represents a stable outcome of a strategic form game as when a Nash equilibrium is reached rational players do not deviate from this strategy profile. This makes Nash equilibrium one of the most frequently used solution concepts for games, and in this thesis for the most part we restrict ourselves to this solution concept. By the definition above a Nash equilibrium is a pure strategy profile if it exists. Such an equilibrium is also known as pure Nash equilibrium, we simply refer to it as Nash equilibrium.

The notion of **best response** is closely related to Nash equilibrium. The set of best responses of a player to its opponents strategies x^{-m} is given by,

$$BR^m(\mathbf{x}^{-m}) = \operatorname*{argmax}_{\tilde{\mathbf{x}}^m \in E^m} u^m(\tilde{\mathbf{x}}^m, \mathbf{x}^{-m}) \tag{2.2}$$

and it stands for the set of strategies which maximize the payoff of player m given strategies of other players. This implies that, a Nash equilibrium is a strategy profile in which all players utilize their best responses.

Note that given the definition of Nash equilibrium it is not clear whether it always exists. In Table 2.1 we present the matching pennies game, which has no Nash Equilibrium. In this game players 1 and 2 announce heads (H) or tails (T) simultaneously

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

Table 2.1: Matching Pennies Game

	H	T
H	1, 1	-1, -1
T	-1, -1	1, 1

Table 2.2: Modified Matching Pennies Game

and if their announcements match player 1 and 2 receive payoffs 1 and -1 respectively, and if they do not match payoffs become -1 and 1. In Table 2.1, the left most column stands for actions of first player and top most row stands for actions of second player. Given strategies of both players in the corresponding box first number stands for the payoff of player 1 and the second number stands for the payoff of player 2. It can be seen from this table that none of the strategy profiles do satisfy the definition of Nash equilibrium.

If the strategy space of each player in a strategic form game with finitely many players is finite then the game is referred to as a **finite game**. On the other hand, if the strategy spaces of players are nonempty compact metric spaces and the utility functions are continuous then the game is said to be a **continuous game**.

The matching pennies game suggests that Nash equilibrium may not exist in finite games. An interesting result on existence of Nash equilibrium in continuous games is given in Section 2.1.2.

We note that if Nash equilibrium exists it need not be unique, an example can be obtained by modifying the payoffs in the matching pennies game. Assume that in the new matching pennies game both players receive 1 if they announce the same outcome and both receive -1 otherwise. The payoffs of the modified game are as given in Table 2.2. In this game strategy profiles (H, H) and (T, T) both satisfy the definition of Nash equilibrium.

We conclude this section with a related solution concept, namely the ϵ-equilibrium.

Definition 2.1.2 (ϵ-Equilibrium). *ϵ-equilibrium is a strategy profile from which no* *player can unilaterally deviate and improve its payoff more than ϵ. Formally, a strategy profile $x \triangleq (x^1, \ldots, x^M)$ is an ϵ-equilibrium if*

$$u^m(x^m, x^{-m}) \geq u^m(\tilde{x}^m, x^{-m}) - \epsilon, \quad \textit{for every } \tilde{x}^m \in E^m \textit{ and } m \in \mathcal{M} \tag{2.3}$$

Note that every Nash equilibrium is an ϵ-equilibrium with $\epsilon = 0$. This equilibrium concept refers to strategy profiles that are approximately an equilibrium.

2.1.2 Existence of Nash Equilibrium

In this section we discuss existence of a Nash equilibrium in continuous games. The theorems that show existence of a Nash equilibrium in continuous games are usually derived utilizing Kakutani's fixed point theorem, [20, 44, 15, 19]. Below we state a well known existence result without proving it (see [19]).

Theorem 2.1.1. *Consider a strategic form game with strategy spaces E^m being nonempty compact convex subsets of an Euclidean space. If the payoff function of each player is continuous in joint strategies and quasi-concave in its strategy, there exists a pure-strategy Nash Equilibrium.*

We make use of this theorem in Chapter 3 to conclude existence of a Nash equilibrium. Note that the above theorem also implies existence of equilibria for finite games when mixed strategies are utilized.

2.1.3 Potential Games

Potential games is a class of games in which preferences of all players are aligned with a global objective [41, 31]. This feature is desirable as it makes potential games easier to analyze and it also ensures that simple dynamics such as best response dynamics and fictitious play converge to an equilibrium in potential games [28, 56, 31, 50]. Another reason for potential games to receive attention is its relation to congestion games. Congestion games, which was defined in [45], is an important class of games for economics. As shown in [41] every finite potential game is isomorphic to a congestion game. In this section we focus on the basic properties of potential games and state the conditions under which a game is a potential game, and we defer the results about dynamics in potential games to Section 2.1.5.

We start by giving definitions of **exact and ordinal potential games**.

Definition 2.1.3 (Exact Potential Game). *A game is called an exact potential game if a function* $\Phi : E \to \mathbb{R}$ *such that*

$$\Phi(x^m, x^{-m}) - \Phi(y^m, x^{-m}) = u^m(x^m, x^{-m}) - u^m(y^m, x^{-m}), \tag{2.4}$$

for all $m \in \mathcal{M}$*,* $x^m, y^m \in E^m$*,* $x^{-m} \in E^{-m}$ *exists.*

Definition 2.1.4 (Ordinal Potential Game). *A game is called an ordinal potential game if a function* $\Phi : E \to \mathbb{R}$ *such that*

$$\Phi(x^m, x^{-m}) - \Phi(y^m, x^{-m}) > 0 \Leftrightarrow u^m(x^m, x^{-m}) - u^m(y^m, x^{-m}) > 0, \tag{2.5}$$

for all $m \in \mathcal{M}$*,* $x^m, y^m \in E^m$*,* $x^{-m} \in E^{-m}$ *exists.*

The functions Φ, satisfying the conditions in Definitions 2.1.3 and 2.1.4 are called exact potential function and ordinal potential function respectively. We refer to exact potential functions and ordinal potential functions as potential functions in short. Observe that ordinal potential games is an extension of exact potential games as can be seen from the definitions. Hence, every exact potential game is an ordinal potential game. Another extension of exact potential games is **weighted potential games**. In weighted potential games potential function Φ satisfies,

$$\Phi(x^m, x^{-m}) - \Phi(y^m, x^{-m}) = w^m \left(u^m(x^m, x^{-m}) - u^m(y^m, x^{-m})\right), \tag{2.6}$$

instead of (2.4), where $w^m \in R$ is a positive weight corresponding to player m. Clearly weighted potential games are also ordinal potential games.

Definitions 2.1.3 and 2.1.4 imply that the potential function is an aggregate representation of utility functions of the players. This enables a more tractable analysis of equilibria in exact and ordinal potential games as shown in the following lemma [41].

Lemma 2.1.1. *Let Φ be an (ordinal) potential function for $\mathcal{G} = \langle \mathcal{M}, \{E^m\}_{m\in\mathcal{M}}, \{u^m\}_{m\in\mathcal{M}}\rangle$. The equilibrium set of $\mathcal{G}$ coincides with the equilibrium set of $\tilde{\mathcal{G}} = \langle \mathcal{M}, \{E^m\}_{m\in\mathcal{M}}, \{\Phi\}_{m\in\mathcal{M}}\rangle$.*

This lemma has two implications. First given a potential function Φ, a strategy profile x is a Nash Equilibrium if and only if for all $m \in \mathcal{M}$

$$\Phi(x^m, x^{-m}) \geq \Phi(y^m, x^{-m}) \quad \text{for all } y^m \in E^m \tag{2.7}$$

Second, Lemma 2.1.1 also suggests a way of finding Nash equilibria. It can be seen from (2.7) that maxima of the potential function correspond to equilibria for both continuous and finite games. This implies that for finite potential games a pure Nash equilibrium always exists as the maximum of the potential function always exists.

However, a strategy profile which is not even a local maximum of the potential may be a Nash equilibrium of a potential game as can be seen from the next example.

Example 2.1.1. *Consider a game with utilities $u^1(x,y) = u^2(x,y) = \Phi(x,y) = e^{-(x-1)^2-(y-1)^2} + e^{-(x+1)^2-(y+1)^2} - e^{-(x-1)^2-(y+1)^2} - e^{-(x+1)^2-(y-1)^2}$ where $x \in E^1 = \mathbb{R}$ represents strategies of player 1 and $y \in E^2 = \mathbb{R}$ represents strategies of player 2. Note that for this potential function $\Phi(x,0) = \Phi(0,y) = 0$ for all $x, y \in \mathbb{R}$, hence none of the players have incentive to deviate from $(x,y) = (0,0)$ and this strategy profile is a Nash equilibrium. On the other hand, as Figure 2-1 shows this point is not a local maximum of the potential.*

It should also be noted that not all local maxima of the potential correspond to equilibria of the game, this is illustrated in the next example.

Example 2.1.2. *Consider a game with utilities $u^1(x,y) = u^2(x,y) = (1-x^2)cos(\pi y)e^{-y^2}$ where $x \in [-1,1]$ represents strategies of player 1 and $y \in [-4,4]$ represents actions of player 2. It can be seen that this is a potential game with potential $\Phi(x,y) =$*

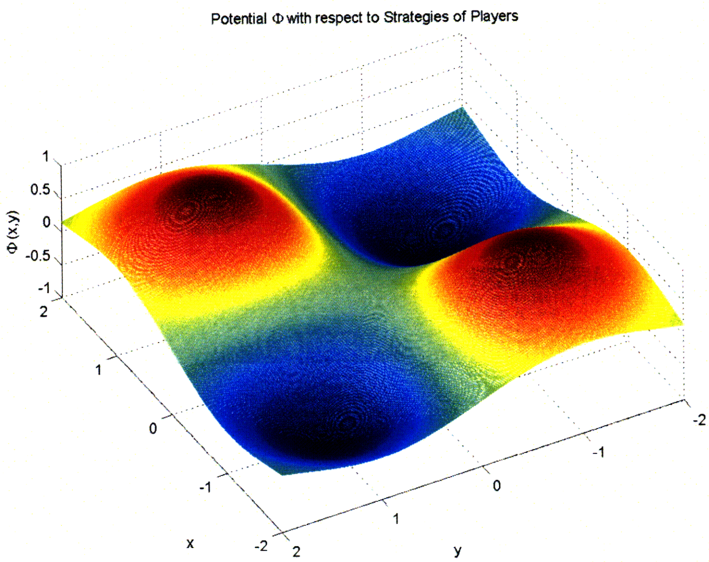


Figure 2-1: Nash equilibrium need not be a local maximum of the potential

$(1-x^2)cos(\pi y)e^{-y^2}$. *On the other hand as Figure 2-2 indicates the potential has multiple local maxima for which* $x=0$ *but the potential is maximized for* $y=0$ *and from all other local maxima player* 2 *has incentive to modify its strategy and move to* $y=0$ *implying that the only equilibrium of the game is* $(x,y)=(0,0)$.

In view of the desirable properties of potential games, an important question is to provide conditions under which a game has a potential function. In the following we provide a brief overview of known conditions from the literature [41, 54, 53].

Definition 2.1.5 (Path- Improvement Path - Closed Path). *A path is a collection of strategy profiles* $\gamma=(x_0,\dots x_N)$ *such that* x_i *and* x_{i+1} *differ in the strategy of exactly one player where* $x_i \in E$ *for* $i \in \{0,1,\dots N\}$. *A path is an improvement path if* $u^{m_i}(x_i) > u^{m_i}(x_{i-1})$ *where* m_i *is the player strategy of which differs between* x_i *and* x_{i-1}. *If for a path* $\gamma=(x_0,\dots x_N)$, *we have* $x_0=x_N$, *then the path is referred as a closed path (or cycle).*

The length of a path $\gamma=(x_0,\dots x_N)$ is N. The transition from strategy profile x_{i-1} to x_i is called as step i of the path. We say a closed path is *simple* if no strategy profile other than the first and the last strategy profiles is repeated along the path.

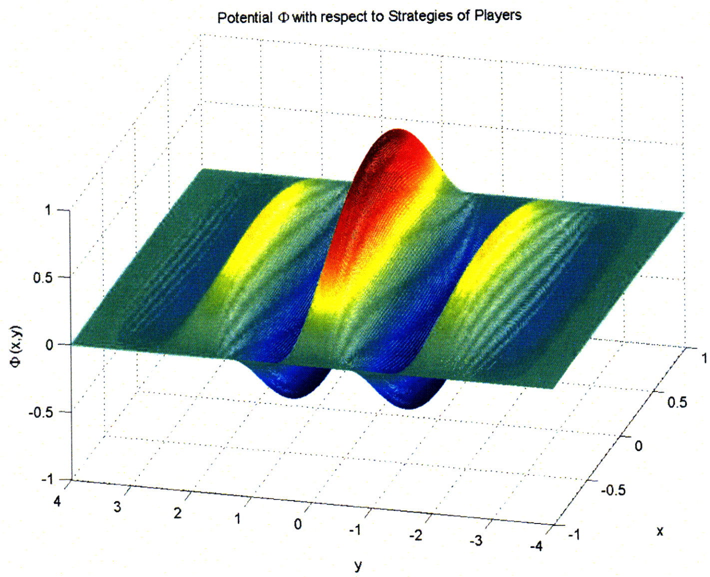


Figure 2-2: Local maximum of the potential need not be an equilibrium

In [41], authors present conditions on the existence of an exact potential in finite games that involve paths. For any path $\gamma = (x_0, \ldots x_N)$ let $I(\gamma, u_{all})$ represent the "cost improvement" along path γ, i.e.,

$$I(\gamma, u_{all}) = \sum_{i=1}^{N} u^{m_i}(x_i) - u^{m_i}(x_{i-1}), \tag{2.8}$$

where m_i denotes the player changing its strategy in the ith step of the path. The following theorem from [41] presents a characterization of exact potential games using conditions on cost improvement along closed paths (or cycles).

Theorem 2.1.2. *A game $\mathcal{G}$ is an exact potential game if and only if for all simple closed paths, γ, $I(\gamma, u_{all}) = 0$. Moreover, it is sufficient to check closed paths of length 4.*

The claim that $I(\gamma, u_{all}) = 0$ for every simple closed path γ can be seen rewriting (2.8) with the exact potential function and observing that the the canonical sum along γ should be equal to 0. Having $I(\gamma, u_{all}) = 0$ for all simple closed paths implies that the game is an exact potential game as in this case a potential function can be constructed by setting the potential equal to zero at an arbitrary strategy profile, say x and setting $I(\gamma, u_{all})$ equal to the potential of strategy profile y if γ is a path connecting x to y. In proving that this is a valid potential functino the property that $I(\gamma, u_{all}) = 0$ for all closed paths is used. This also gives a procedure for constructing the potential function in a potential game. Note that this function is unique up to an additive constant. On the other hand, enumerating and checking all 4 step closed paths may be computationally infeasible for checking the existence of exact potential.

In [54], authors present necessary and sufficient conditions for the existence of an ordinal potential in games. Similar to [41], they present conditions on the existence of a potential by constructing paths of different strategy profiles. Authors define **weak improvement cycle** as a closed path at every step of which player whose strategy is modified has a nonnegative change in its utility and at least at one step the change in payoffs is strictly positive. An obvious necessary condition for the existence of an ordinal potential is that no weak improvement cycle exists in the game. It can be seen that if this condition does not hold, the value of the potential cannot remain constant along a cycle.

The main result of [54] is summarized in the following theorem:

Theorem 2.1.3. *A countable game is an ordinal potential game if and only if the set of strategies does not contain any weak improvement cycles.*

By definition, a game is a potential game if and only if there exists a potential function which represents the preferences of each player among the strategy profiles for which only its strategy is changing. Existence of a weak improvement cycle implies that there is no potential function or ranking of strategy profiles that is consistent with these preferences of players. If a weak improvement cycle exists in a game we say that there are *inconsistent preference relations* in the game.

In [54] the paths over which utility of the players modifying their strategy does not decrease are referred to as nondeteriorating path, and equivalence classes are defined on E by stating two strategy profiles x and y belong to the same equivalence class if there are nondeteriorating paths from x to y and from y to x. An order relation $\succ$ on equivalence classes is defined as follows: two difference equivalence classes $[x]$

and $[y]$ satisfy $[x] \succ [y]$ if there is a nondeteriorating path from y to x. The set of equivalence classes is said to be properly ordered if there exists a function defined on the set of equivalence classes that is order preserving. The set of equivalence classes A is said to be order dense if there exists a countable subset of equivalence classes B such that for any $x, z \in A - B$ there exists $y \in B$ such that $z \succ y$, $y \succ x$.

An extension of Theorem 2.1.3 states that for an uncountable game if set of equivalence classes a contains countable order dense set then the game is an ordinal potential game if and only if no weak improvement cycles exist.

Similar to the result related to exact potential games, it is not clear how one can determine the existence of weak improvement cycles in a systematic and computationally feasible way, possibly by avoiding enumeration of all the cycles. For uncountable games, although finding a cycle that a weak improvement cycle implies that the game is not an ordinal potential game, concluding that the game is an ordinal potential game is difficult if not impossible. Note that unlike the result in [41] for ordinal potential games it is not sufficient to study cycles of length 4, hence the number of cycles one has to check increases significantly for ordinal potential games.

There are also results on the existence of an exact potential in games with differentiable utility functions. One such result due to [41] is stated below.

Theorem 2.1.4. *Let $\mathcal{G}$ be a game in which the strategy spaces are intervals of real numbers, i.e. $x^m \in E^m \subset \mathbb{R}$ for all $m \in \mathcal{M}$. Suppose the utilities are twice continuously differentiable. Then $\mathcal{G}$ is a potential game if and only if,*

$$\frac{\partial^2 u^m}{\partial x^m \partial x^k} = \frac{\partial^2 u^k}{\partial x^m \partial x^k} \quad \text{for all } m, k \in \mathcal{M}. \tag{2.9}$$

This statement can be extended to games where the strategy spaces are compact subsets of $n-$dimensional Euclidean spaces. However, to our knowledge in the literature there are no easy to check conditions for ordinal potential games that is similar to Theorem 2.1.4.

2.1.4 Efficiency Loss in Games

In games we are frequently interested in comparing the quality of the equilibria to a centralized, system-optimal solution. Recently, there has been much work in quantifying the **efficiency loss** incurred by the selfish behavior of players in networked systems (see [46, 47, 12, 22]). The two concepts which are most commonly used in this context are the **price of anarchy** (PoA), and **price of stability** (PoS). These concepts stand for the quality of the best and worst equilibria when compared to a globally optimal solution in terms of some well defined quality measure.

A frequently used quality measure is the **aggregate utility** in the system given by,

$$u(x) = \sum_{m \in \mathcal{M}} u^m(x). \tag{2.10}$$

We define the set of globally optimal solutions by $\operatorname{argmax}_{x \in E} u(x)$. Note that defining such metrics for efficiency loss is usually not very meaningful for specific game instances, as we are interested in obtaining efficiency loss characterization for a class of games. The usual practice is to characterize PoA and PoS for a class of games. We use the notation $\mathcal{I}_{\mathcal{G}}$ for the set of **game instances** of interest and $\mathcal{I} \in \mathcal{I}_{\mathcal{G}}$ for a game instance in this set.

Below we give a formal definition of PoA and PoS.

Definition 2.1.6 (Price of Stability - Price of Anarchy). *For every $\mathcal{I} \in \mathcal{I}_{\mathcal{G}}$, denote by $N_{\mathcal{I}}$ the set of Nash equilibria, and let $x^*_{\mathcal{I}}$ be an optimal strategy profile in terms of the aggregate utility. PoS and PoA are defined as follows:*

$$PoS = \sup_{I \in \mathcal{I}_{\mathcal{G}}} \inf_{x \in N_I} \frac{u(x^*_{\mathcal{I}})}{u(\mathbf{x})}, \tag{2.11}$$

$$PoA = \sup_{I \in \mathcal{I}_{\mathcal{G}}} \sup_{x \in N_I} \frac{u(x^*_{\mathcal{I}})}{u(x)}. \tag{2.12}$$

These quantities are well studied for game classes such as congestion games, [14, 17, 5]. Also properties of potential games are used in the literature to bound the efficiency loss, for details see [5, 42, 31].

2.1.5 Dynamics in Games

We already introduced Nash equilibrium as a solution concept in games. An important question is how a game reaches to an equilibrium. This question is usually answered by theoretical models of dynamics in games. In this section we mention two important classes of dynamics in games: best-response dynamics and fictitious play. Detailed surveys of dynamics in games can be found in [56, 18].

Perhaps the most natural mechanism for (distributed) convergence to an equilibrium relies on a player's **best response**, which in general is a player's strategy that maximizes its own utility, given the strategies of other players. An informal description of a general best-response mechanism is simple: *Each player updates its strategy from time to time through a best response (2.2).*

This model assumes that players are not aware of the utilities of other players and given a strategy profile players independently update their strategies to greedily maximize their payoffs. Variations of these dynamics can be obtained depending on the update schedules of players. Updates may take place simultaneously or sequentially at a prescribed order or randomly decided at each time slot. Another variation of these dynamics is **better response dynamics** in which given a strategy profile x, player m updates its strategy not necessarily to a strategy in $BR^m(x^{-m})$ but to an arbitrary strategy y^m which satisfies $u^m(y^m, x^{-m}) \geq u^m(x^m, x^{-m})$.

Best or better response dynamics do not converge to an equilibrium in general, however for finite potential games, it is possible to show convergence of such dynamics to an equilibrium. This is due to the finite improvement property, which is defined next.

Definition 2.1.7 (Finite Improvement Property (FIP)). *A game is said to have the finite improvement property if every improvement path is finite.*

The following lemma from [41] can be used to show convergence of the best response dynamics to an equilibrium in finite potential games.

Lemma 2.1.2. *Every finite potential game has the finite improvement property.*

This lemma follows since along an improvement path, the potential has to increase at each step but due to the fact that the game is a finite game the improvement path has to terminate in finitely many steps. This implies that best and better response dynamics should terminate in finite games in finitely many steps provided that potential increases at every step. We leave the precise descriptions of the update rules that are used in these dynamics to Chapter 3 where we discuss convergent dynamics for a wireless scheduling game.

Another widely used dynamic is the **fictitious play** [56, 18, 40, 28]. In fictitious play, agents act as if their opponents are utilizing stationary strategies. It is assumed that players update their strategies at times $t \in Z_+$. Another assumption is that at time $t+1$ players have observed the actions of all players up to time t and they have access to the empirical average of the number of times their opponents utilize each strategy profile $x^{-m} \in E^{-m}$. Player m assumes that this empirical average is a realization of a randomized stationary strategy its opponents are utilizing and chooses a strategy to maximize its expected payoff. It is known that fictitious play converges to a Nash equilibrium in potential games [56].

Note that both of the dynamics described here are myopic in the sense that players are trying to maximize the payoff at the time of their updates. There are more complicated dynamics in which players have memory and take strategic actions as a function of their past observations, [56]. For simplicity, in this thesis, we just consider the best response dynamics described in this section.

2.2 Projections of Games to the Set of Exact Potential Games

Despite their desirable properties, the set of exact potential games is a "small" subset of the space of games. This motivates us to study the class of games that are "close" to a potential game. Our approach relies on projecting an arbitrary game to the set of exact potential games and using the projection error to quantify the distance of this game from the set of exact potential games. The projection also shows us how to modify the utility functions in a minimal way to obtain a game with desirable properties of a potential game.

The task of designing games to achieve a specific outcome is studied in game theory under *mechanism design* [19, 42]. The projection approach allows modifying payoffs of players to obtain a potential game with desirable properties. Therefore, this approach is similar to mechanism design in spirit. However it should be noted that in mechanism design problems the focus is mainly on designing a game with small efficiency loss, whereas in the projection approach the goal is to obtain a game with the desirable properties of potential games.

The problem of obtaining a projection of a game to the set of exact potential games may be important from the perspective of cooperative control problems. The control of several autonomous agents working towards a common global objective is usually addressed by cooperative control problems. Recently game-theoretic models have attracted attention in the context of distributed cooperative control problems. The general framework utilized in this approach is to endow agents with utility functions designed to ensure that collective behavior of users drives the system to operate at a Nash equilibrium which is the same as or close to a global optimum [7, 29, 27, 30]. However, if the resulting game is not a potential game, then there exists inconsistent preference relations in the joint strategy space, and simple dynamics may not converge to an equilibrium due to the inconsistencies. On the other hand, the projection approach can be used to eliminate the inconsistencies and obtain a potential game. Moreover, provided that the projection error is small, a Nash equilibrium of the projected game will be close to the globally optimal solution in terms of the performance. Therefore, the projection methods may help in designing game-theoretic models for cooperative control systems in which simple dynamics converge to a good quality equilibrium of the game.

There is an interesting connection between potential games and the ranking problems. In ranking problems, it is assumed that a set of alternatives and the data (or rankings) which corresponds to cardinal scores assigned to these alternatives is given.

The input data can be incomplete or inconsistent, and the main objective is to find a score function defined on the set of alternatives, representing the input data. This score function is frequently referred to as the *global ranking*.

In certain settings the input data represents the amount an alternative is preferred over another. This data may be obtained by pairwise comparisons of the given cardinal scores and often referred to as *pairwise rankings* or *pairwise comparisons*. If three alternatives a, b, c are considered and pairwise rankings indicating $a > b$, $b > c$, $c > a$ are present (where $>$ represents the preference relation between alternatives) the provided pairwise ranking among these three alternatives is *locally inconsistent*. A global ranking which represents the input data cannot contain such inconsistencies, and is *globally consistent*.

In a game utility of each player represents the ranking of strategy profiles by this player. For each player m, consider the pairwise comparison of strategy profiles which differ by the strategy of player m. A game is an exact potential game if and only if for each player the pairwise comparisons obtained from its utility function match with the pairwise comparisons obtained from a potential function. Hence, potential games are games in which rankings given by different players can be represented by a potential function. Therefore, the question of finding a global ranking that represents a set of possibly inconsistent rankings is related to finding a potential function that represents a collection of utilities in the best possible way.

In recent works, tools from combinatorial Hodge Theory has been used in ranking problems. In particular, the recent paper [21] represents a given collection of pairwise rankings as a vector field and uses the Helmholtz decomposition of a vector field to determine a global ranking representing the pairwise rankings in the best possible way.

The Helmholtz Decomposition allows decomposition of a vector field into three vector fields:

- Gradient flow (globally consistent component)

- Harmonic flow (locally consistent but globally inconsistent component)

- Curl flow (locally inconsistent component).

The gradient flow is the consistent part in the given pairwise rankings that actually creates the global ranking. The curl flow represents the local inconsistency in the pairwise rankings. Note that the local inconsistencies by definition involve three alternatives. If there is an inconsistency in the pairwise rankings that can only be observed by checking more than 3 pairwise comparisons, the consistency is not local and is a part of the Harmonic flow. The approach in [21] enables us to construct a global ranking if the given pairwise rankings are globally consistent. For the case, when there is no global ranking representing a given collection of pairwise rankings, this approach also characterizes the nature of inconsistencies in the pairwise rankings.

In the rest of this section we give a brief overview of the Combinatorial Hodge Theory and its application to ranking problems.

2.2.1 Combinatorial Hodge Theory

The objective of this section is to provide the results from combinatorial Hodge Theory that will be used for the projection of games to the set of exact potential games (see Chapter 4). We introduce basic definitions, notation and discuss preliminary results related to combinatorial Hodge Theory.

Let E denote a set of alternatives[1], we define by $C_0 = \{f| \ f : E \to \mathbb{R}\}$ the set of functions defined on E. We study the comparisons of different alternatives but we assume that not all alternatives are *comparable*. We denote the *set of pairs of comparable alternatives* by $A \subset E \times E$, and we say that alternatives $\mathbf{p}, \mathbf{q}$ are comparable if $(\mathbf{p}, \mathbf{q}) \in A$. Furthermore we assume that $(\mathbf{p}, \mathbf{q}) \in A$ if and only if $(\mathbf{q}, \mathbf{p}) \in A$. Similarly we define the set of 3 *cliques of comparable alternatives*, $T = \{(\mathbf{p}, \mathbf{q}, \mathbf{r}) | (\mathbf{p}, \mathbf{q}), (\mathbf{q}, \mathbf{r}), (\mathbf{p}, \mathbf{r}) \in A\}$.

[1]We use the same notation for the set of alternatives and the set of strategy profiles, given the connection between the ranking problems and the problem of finding an exact potential function which represents an arbitrary game

We define an indicator function of comparable alternatives $W : E \times E \to \mathbb{R}$ as

$$W(\mathbf{p}, \mathbf{q}) = \begin{cases} 1 & \text{if } (\mathbf{p}, \mathbf{q}) \in A \\ 0 & \text{otherwise} \end{cases} \tag{2.13}$$

It will be convenient to represent the comparable alternatives using the graph $G = (E, A)$, where E is the set of nodes (or alternatives) and A is the set of edges in the graph. An edge is present between the alternatives that are comparable.

Pairwise comparisons represent how much an alternative is valued over another. A *pairwise comparison* on the set of alternatives is defined as $X : E \times E \to \mathbb{R}$ such that

$$X(\mathbf{p}, \mathbf{q}) = \begin{cases} -X(\mathbf{q}, \mathbf{p}) & \text{if } (\mathbf{p}, \mathbf{q}) \in A \\ 0 & \text{otherwise.} \end{cases} \tag{2.14}$$

We denote the set of pairwise comparisons from $E \times E$ to $\mathbb{R}$ by C_1. By (2.14) it follows that $X(\mathbf{p}, \mathbf{p}) = 0$ for all $X \in C_1$. The pairwise comparisons correspond to edge flows on G.

Similar to the edge flows, we define *triangular flow* of the alternatives $\Psi : E \times E \times E \to \mathbb{R}$ such that

$$\Psi(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \Psi(\mathbf{q}, \mathbf{r}, \mathbf{p}) = \Psi(\mathbf{r}, \mathbf{p}, \mathbf{q}) = -\Psi(\mathbf{q}, \mathbf{p}, \mathbf{r}) = -\Psi(\mathbf{p}, \mathbf{r}, \mathbf{q}) = -\Psi(\mathbf{r}, \mathbf{q}, \mathbf{p}), \tag{2.15}$$

and $\Psi(\mathbf{p}, \mathbf{q}, \mathbf{r}) = 0$ if $(\mathbf{p}, \mathbf{q}, \mathbf{r}) \notin T$. We denote the set of triangular flows from $E \times E \times E$ to $\mathbb{R}$ by C_2.

Next we define operators that will be used in the analysis of the projection problem. In the following, assume that $\phi \in C_0$ is an arbitrary function. We first define the *combinatorial gradient operator* $\delta_0 : C_0 \to C_1$, given by

$$(\delta_0\phi)(\mathbf{p}, \mathbf{q}) = W(\mathbf{p}, \mathbf{q})(\phi(\mathbf{q}) - \phi(\mathbf{p})), \tag{2.16}$$

for all $\mathbf{p}, \mathbf{q} \in E$, $\phi \in C_0$.

An operator which is used in the characterization of the inconsistencies is *the curl* *operator* $\delta_1 : C_1 \to C_2$, which is defined for all $X \in C_1$ and $(\mathbf{p}, \mathbf{q}, \mathbf{r}) \in T$ as,

$$(\delta_1 X)(\mathbf{p}, \mathbf{q}, \mathbf{r}) = X(\mathbf{p}, \mathbf{q}) + X(\mathbf{q}, \mathbf{r}) + X(\mathbf{r}, \mathbf{p}). \tag{2.17}$$

We denote the adjoints of operators δ_0 and δ_1 by δ_0^* and δ_1^* respectively. For $k \in \{0, 1\}$, given an inner product $\langle \cdot, \cdot \rangle_k$ on C_k adjoint of δ_k, $\delta_k^* : C_{k+1} \to C_k$ is the operator which satisfies,

$$\langle \delta_k f_k, g_{k+1} \rangle_{k+1} = \langle f_k, \delta_k^* g_{k+1} \rangle_k, \tag{2.18}$$

for all $f_k \in C_k$, $g_{k+1} \in C_{k+1}$. We drop the subscript in the inner product notation if the space in which it is defined is clear from the context. We next present particular inner products in spaces C_0, C_1 and C_2 that are used in our projection framework.

We assume that for $\phi_1, \phi_2 \in C_0$,

$$\langle \phi_1, \phi_2 \rangle = \sum_{\mathbf{p} \in E} \phi_1(\mathbf{p}) \phi_2(\mathbf{p}). \tag{2.19}$$

For $X, Y \in C_1$, we define the inner product on C_1 as

$$\begin{aligned} \langle X, Y \rangle &= \frac{1}{2} \sum_{(\mathbf{p}, \mathbf{q}) \in E \times E} W(\mathbf{p}, \mathbf{q}) X(\mathbf{p}, \mathbf{q}) Y(\mathbf{p}, \mathbf{q}) \\ &= \frac{1}{2} \sum_{(\mathbf{p}, \mathbf{q}) \in A} X(\mathbf{p}, \mathbf{q}) Y(\mathbf{p}, \mathbf{q}) \end{aligned} \tag{2.20}$$

For $\Psi_1, \Psi_2 \in C_2$ the inner product on C_2 satisfies

$$\langle \Psi_1, \Psi_2 \rangle = \sum_{(\mathbf{p}, \mathbf{q}, \mathbf{r}) \in T} \Psi_1(\mathbf{p}, \mathbf{q}, \mathbf{r}) \Psi_2(\mathbf{p}, \mathbf{q}, \mathbf{r}). \tag{2.21}$$

Using these definitions and the definition of the adjoint, the operators δ_0^* satisfies

$$(\delta_0^* X)(\mathbf{p}) = - \sum_{\mathbf{q} | (\mathbf{p}, \mathbf{q}) \in A} W(\mathbf{p}, \mathbf{q}) X(\mathbf{p}, \mathbf{q}). \tag{2.22}$$

Equivalently, since $W(\mathbf{p},\mathbf{q}) = 0$ for $(\mathbf{p},\mathbf{q}) \notin A$

$$(\delta_0^* X)(\mathbf{p}) = -\sum_{\mathbf{q}\in E} W(\mathbf{p},\mathbf{q}) X(\mathbf{p},\mathbf{q}). \tag{2.23}$$

Note that δ_0^* operates like the divergence operator of calculus, for this reason we sometimes refer to the operator $-\delta_0^*$ as the *divergence operator*.

The domains and codomains of mappings $\delta_0, \delta_1, \delta_0^*, \delta_1^*$ are summarized in (2.24) and (2.25):

$$C_0 \xrightarrow{\delta_0} C_1 \xrightarrow{\delta_1} C_2 \tag{2.24}$$

and

$$C_0 \xleftarrow{\delta_0^*} C_1 \xleftarrow{\delta_1^*} C_2. \tag{2.25}$$

We sometimes use the notations *grad*, *div*, *curl* instead of δ_0, $-\delta_0^*$, δ_1 respectively.

The functions in C_0 can be represented by vectors of length $|E| = h$. This simply requires indexing all alternatives and constructing a vector with the ith entry equal to the value of the function evaluated at the ith alternative. Using this alternative description, we have $C_0 = \mathbb{R}^h$. Similarly C_1 can be expressed as a vector of length $h \times h$ however as elements of C_1 is a subset of the set of functions from $E \times E$ to $\mathbb{R}$, it follows that $C_1 \subset \mathbb{R}^{h\times h}$. Note that the operators defined so far are linear and this makes the alternative descriptions as the operators can be expressed in terms of matrices.

Another operator which is used in the study of the projections is the analogue of the graph Laplacian, $\Delta_0 : C_0 \to C_0$, which is given by,

$$\Delta_0 = \delta_0^* \circ \delta_0, \tag{2.26}$$

where $\circ$ represents the composition of the operators. To simplify the notation we sometimes drop $\circ$ and use $\Delta_0 = \delta_0^* \delta_0$. The reason this operator is named as the graph Laplacian becomes apparent once it is expressed in the matrix representation. Using the matrix representation for the Laplacian and substituting the definitions of δ_0 and

δ_0^*, the Laplacian can be expressed as

$$[\Delta_0]_{\mathbf{p},\mathbf{q}} = \begin{cases} \sum_{\mathbf{r}\in E} W(\mathbf{p},\mathbf{r}) & \text{if } \mathbf{p}=\mathbf{q} \\ -1 & \text{if } \mathbf{p}\neq\mathbf{q} \text{ and } (\mathbf{p},\mathbf{q})\in A \\ 0 & \text{otherwise,} \end{cases} \tag{2.27}$$

where $[\Delta_0]_{\mathbf{p},\mathbf{q}}$ denotes the entry of the matrix with row index equal to the index of $\mathbf{p}$ and column index equal to index of $\mathbf{q}$. This is precisely the definition of Laplacian matrix of a graph with node set E and arc set A.

A related operator is denoted by Δ_1,

$$\Delta_1 = \delta_1^* \circ \delta_1 + \delta_0 \circ \delta_0^*, \tag{2.28}$$

and it is the discrete analogue of the Helmholtz operator [21].

The operators δ_0 and δ_1 are closed, i.e., $\delta_1 \circ \delta_0 = 0$. This implies the well known identities in vector calculus such as $curl \circ grad = 0$, $div \circ curl^* = 0$. Moreover this property is used in the proof of the decomposition theorem discussed below.

Let $X \in C_1$ be a pairwise ranking. If X is derived from a global ranking on E, then X is said to be *globally consistent*. Equivalently X is globally consistent if $X = \delta_0\phi$ for some $\phi \in C_0$. Here ϕ is the potential function or global ranking corresponding to X. This suggests that the set of globally consistent pairwise rankings can be represented by $im(grad)$, the image of the $grad$ operator. By closedness of δ_0 and δ_1, $\delta_1 X = 0$ for a globally consistent pairwise ranking X. We define locally inconsistent pairwise rankings as the pairwise rankings for which $(\delta_1 X)(\mathbf{p},\mathbf{q},\mathbf{r}) = X(\mathbf{p},\mathbf{q}) + X(\mathbf{q},\mathbf{r}) + X(\mathbf{r},\mathbf{p}) \neq 0$ for some $(\mathbf{p},\mathbf{q},\mathbf{r}) \in T$. Note that $ker(\delta_1)$ is the set of pairwise rankings which have no local inconsistencies (as curl is zero), but it turns out that this set is not equal to $im(grad)$, and the pairwise rankings belonging to difference of these sets are the harmonic rankings, which are locally consistent but globally inconsistent. These observations are formalized with the following decomposition theorem from [21].

Theorem 2.2.1 (Helmholtz Decomposition Theorem[2]). *C_1 admits an orthogonal decomposition*

$$C_1 = im(\delta_0) \oplus ker(\Delta_1) \oplus im(\delta_1^*) \tag{2.29}$$

where $ker(\Delta_1) = ker(\delta_1) \cap ker(\delta_0^*)$.

The statement of the above theorem can alternatively be written as

$$C_1 = im(grad) \oplus ker(\Delta_1) \oplus im(curl^*) \tag{2.30}$$

for $ker(\Delta_1) = ker(\delta_1) \cap ker(\delta_0^*) = ker(curl) \cap ker(div)$.

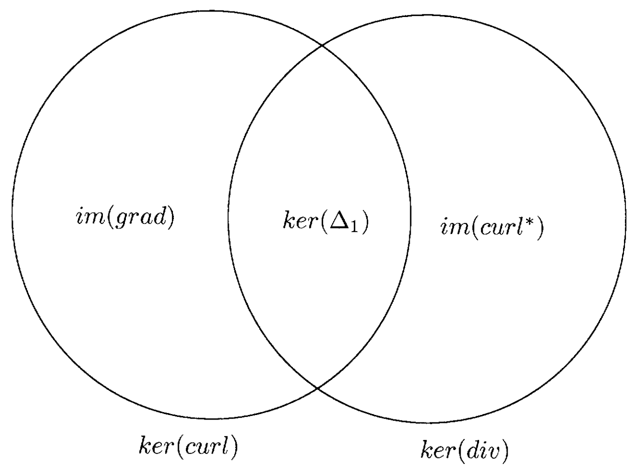


Figure 2-3: Helmholtz Decomposition in C_1

This decomposition theorem implies that the space of flows admits three different orthogonal components. First component is the image of the gradient, which stands for the globally consistent flows. Locally inconsistent component can be found as the image of adjoint of curl operator or alternatively as the orthogonal complement of the kernel of the *curl* operator. Globally inconsistent but locally consistent flows are defined by the kernel of the Δ_1, which is essentially the intersection of the kernels of the curl and the divergence operators. This also implies that kernel of the curl operator consists of locally consistent flows (that may or may not be globally consistent)

[2]Hodge Decomposition theorem gives a generalization of Helmholtz Decomposition theorem for higher dimensions (for details see [21]).

and kernel of the divergence consists of inconsistent components which are locally or globally inconsistent. These relations are summarized in Figure 2-3.

2.2.2 Application of Hodge Theory to Ranking Problems

In this section, we briefly summarize an application of the Helmholtz decomposition to ranking problem as presented in [21]. We use similar ideas in the analysis of projections on the set of exact potential games in Chapter 4.

The decomposition theorem can be used to determine if there is inconsistency in a given set of pairwise comparisons and it allows to determine whether the inconsistency is local or global. In this section we assume that the pairwise comparisons denoted by $\bar{Y}$ are given and a global ranking, s, representing these comparisons is of interest. This problem can be formulated as the following optimization problem:

$$\min_{s \in C_0} \|\delta_0 s - \bar{Y}\|_2^2, \tag{2.31}$$

where $\|X\|_2^2 = \langle X, X \rangle$ for the inner product defined in (2.20).

This problem is essentially projection of a given flow on the space of globally consistent flows. It is proved in [21] that the solution for this problem is given by,

$$s^* = \Delta_0^\dagger \delta_0^* \bar{Y}, \tag{2.32}$$

where $\dagger$ stands for the pseudo inverse. The result is immediately obtained by noting that the projection error is orthogonal to the image space of δ_0 thus optimal s satisfies $\delta_0^* \delta_0 s = \Delta_0 s = \delta_0^* \bar{Y}$.

It is possible to analyze the residual component after projection on the space of consistent flows. The residual component, $R^* = \delta_0 s^* - \bar{Y}$, can further be projected on the local and global inconsistency components giving more insight about the source of inconsistency. The Helmholtz decomposition theorem implies that $R^* = proj_{im(curl^*)} \bar{Y} + proj_{ker(\Delta_1)} \bar{Y}$. It can be shown that local inconsistency component can be found using

$$proj_{im(curl^*)}\bar{Y} = curl^*(curl\ curl^*)^{\dagger} curl\bar{Y} \tag{2.33}$$

which is obtained by projecting $\bar{Y}$ to $im(curl^*)$ and mapping it back to C_1 ensuring that component with zero curl disappears.

2.2.3 Notation Used in the Projection Problem

In this thesis we consider using the Helmholtz decomposition framework for projecting games to the set of exact potential games. This necessitates some additional notation which we introduce next.

In the problem of projection on the set of exact potential games, our objective is to obtain a potential function (global ranking) representing a given game, defined on the set of strategy profiles. Therefore, the set of alternatives is equal to the set of strategy profiles E, for the projection problem.

In the following, C_0 denotes the set of real valued functions defined on $E = E^1 \times \cdots \times E^M$ (such as utilities and potentials) and C_1 denotes the set of pairwise comparisons. We assume the game is finite, and as discussed earlier we equivalently use $C_0 = \mathbb{R}^h$, and $C_1 \subset \mathbb{R}^{h \times h}$ where $|E^m| = h_m$ for all $m \in \mathcal{M}$ and $|E| = \prod_{m \in \mathcal{M}} h_m = h$.

In potential games, we are interested in pairwise comparisons between strategy profiles that differ in the strategy of a single player. Therefore, we say that the strategy profiles that differ in the strategy of a single player are *comparable* and redefine $W : E \times E \to \mathbb{R}$ as

$$W(\mathbf{p}, \mathbf{q}) = \begin{cases} 1 & \text{if } \mathbf{p}, \mathbf{q} \text{ differ in the strategy of a single player} \\ 0 & \text{otherwise} \end{cases} \tag{2.34}$$

We define a similar function, $W^m : E \times E \to \mathbb{R}$ as

$$W^m(\mathbf{p}, \mathbf{q}) = \begin{cases} 1 & \text{if } \mathbf{p}, \mathbf{q} \text{ differ in the strategy of player } m \text{ only} \\ 0 & \text{otherwise.} \end{cases} \quad (2.35)$$

W^m is an indicator function which is equal to 1 if $\mathbf{p}$ and $\mathbf{q}$ differ in the strategy of player m. We say that such strategy profiles are *comparable by player m*. Note that

$$W(\mathbf{p}, \mathbf{q}) = \sum_{m \in \mathcal{M}} W^m(\mathbf{p}, \mathbf{q}), \quad (2.36)$$

and $W^m(\mathbf{p}, \mathbf{q})W^k(\mathbf{p}, \mathbf{q}) = 0$ for all $\mathbf{p}, \mathbf{q} \in E$ and $k, m \in \mathcal{M}$ such that $k \neq m$.

Given $\phi \in C_0$, define $D_m : C_0 \to C_1$ such that

$$(D_m\phi)(\mathbf{p}, \mathbf{q}) = W^m(\mathbf{p}, \mathbf{q}) \left(\phi(\mathbf{q}) - \phi(\mathbf{p})\right). \quad (2.37)$$

Similar to the gradient operator, D_m quantifies the difference between the strategy profiles, on the other hand it can be nonzero only for strategy profiles comparable by player m. Note that this is a linear operator and when we consider C_0 and C_1 as subsets of Euclidean spaces. D_m can be treated as a $h^2 \times h$ matrix. The motivation for introducing this operator, as can be seen in next section, stems from the fact that the pairwise comparisons we deal with have a special structure that can be represented in terms of the operator D_m. We denote the adjoint of the D_m by D_m^* for all $m \in \mathcal{M}$.

We define $D : C_0^M \to C_1$, where $C_0^M = C_0 \times \cdots \times C_0$, such that

$$(Du)(\mathbf{p}, \mathbf{q}) = \sum_{m \in \mathcal{M}} W^m(\mathbf{p}, \mathbf{q}) \left(u^m(\mathbf{q}) - u^m(\mathbf{p})\right), \quad (2.38)$$

for $u = \{u^m\}_{m \in \mathcal{M}}$. It can be seen that $Du = \sum_{m \in \mathcal{M}} D_m u^m$. Given a collection of utilities this operator constructs the pairwise comparisons which are comparable.

The operator D_m and its adjoint D_m^* $m \in \mathcal{M}$, are closely related to δ_0 and δ_0^*. Note that by definition, $W^m(\mathbf{p}, \mathbf{q})(\delta_0\phi)(\mathbf{p}, \mathbf{q}) = (D_m\phi)(\mathbf{p}, \mathbf{q})$. Summing this over m

and using (2.36), for any $\phi \in C_0$, $\mathbf{p}, \mathbf{q} \in E$ it follows that

$$\delta_0 \phi = \sum_{m \in \mathcal{M}} D_m \phi. \tag{2.39}$$

We define operators $\Lambda_m : C_1 \to C_1$ for $m \in \mathcal{M}$ such that

$$(\Lambda_m X)(\mathbf{p}, \mathbf{q}) = W^m(\mathbf{p}, \mathbf{q}) X(\mathbf{p}, \mathbf{q}) \tag{2.40}$$

for all $X \in C_1$, $\mathbf{p}, \mathbf{q} \in E$. By definition Λ_m is a self adjoint scaling operator. It can be seen that

$$D_m = \Lambda_m \delta_0, \tag{2.41}$$

from the definitions of D_m and Λ_m. Hence,

$$D_m^* = \delta_0^* \Lambda_m. \tag{2.42}$$

This enables expressing the operator D_m^* explicitly. Given some $X \in C_1$, we have

$$(D_m^* X)(\mathbf{p}) = -\sum_{\mathbf{q} \in E} W^m(\mathbf{p}, \mathbf{q}) X(\mathbf{p}, \mathbf{q}) \tag{2.43}$$

which follows from (2.23) and the fact that $W^m(\mathbf{p}, \mathbf{q}) W(\mathbf{p}, \mathbf{q}) = W^m(\mathbf{p}, \mathbf{q})$.

From (2.43), (2.23) and (2.36) it follows that

$$\delta_0^* = \sum_{m \in \mathcal{M}} D_m^*. \tag{2.44}$$

Another implication of (2.42) is that $D_k^* D_m = 0$ if $k \neq m$, i.e. the image spaces of D_m $m \in \mathcal{M}$ are orthogonal. To see this note that

$$D_k^* D_m = \delta_0^* \Lambda_k \Lambda_m \delta_0 \tag{2.45}$$

and $(\Lambda_k \Lambda_m X)(\mathbf{p}, \mathbf{q}) = 0$ for all $\mathbf{p}, \mathbf{q} \in E$ and $X \in C_1$.

The orthogonality enables us to exploit the properties of the previously defined

Laplacian operator. Observe that

$$\begin{aligned}\Delta_0 = \delta_0^*\delta_0 = \delta_0^* \sum_{m\in\mathcal{M}} D_m = \sum_{k\in\mathcal{M}} D_k^* \sum_{m\in\mathcal{M}} D_m \\ = \sum_{m\in\mathcal{M}} D_m^* D_m.\end{aligned} \tag{2.46}$$

Here the second equality follows from the fact that $\delta_0 = \sum_{m\in\mathcal{M}} D_m$. Noting that $(\delta_0\phi)(\mathbf{p},\mathbf{q}) = (D_m\phi)(\mathbf{p},\mathbf{q})$ if $W^m(\mathbf{p},\mathbf{q}) = 1$ it also follows that

$$D_m^* D_m = D_m^*\delta_0. \tag{2.47}$$

Observe that D_m^* acts as the divergence operator, when W^m instead of W is utilized for the definition of the gradient operator and the inner product on C_1. Together with (2.47) this suggests that $D_m^* D_m = D_m^*\delta_0$ is a Laplacian with respect to the new weights. This leads us to define a new Laplacian operator $\Delta_{0,m} = D_m^* D_m$ and from (2.46) it follows that

$$\Delta_0 = \sum_{m\in\mathcal{M}} \Delta_{0,m}. \tag{2.48}$$

We illustrate the graphs used for defining Laplacians Δ_0 and $\Delta_{0,m}$ in Figure 2-4. In this figure, each player has three strategies and node (i,j) represents the strategy profile in which players 1 and 2 use strategies i and j respectively. $\Delta_{0,1}$, $\Delta_{0,2}$ correspond to the Laplacians of the graphs of strategy profiles comparable by player 1 and player 2. Hence, according to Figure 2-4, $\Delta_{0,1}$, $\Delta_{0,2}$ are defined on the graphs where edges are shown with dashed and solid lines respectively. On the other hand Δ_0 is defined on the graph for which all edges (dashed and solid) are present.

In our analysis kernel and orthogonal complement of the kernel of D_m plays a key role. It is known that for a linear operator L, $L^\dagger L$ (where $\dagger$ denotes the pseudo inverse) is the operator for projection on the orthogonal complement of the kernel of L. We define a projection operator for projection on the orthogonal complement of kernel of D_m as follows,

$$proj_m = D_m^\dagger D_m. \tag{2.49}$$

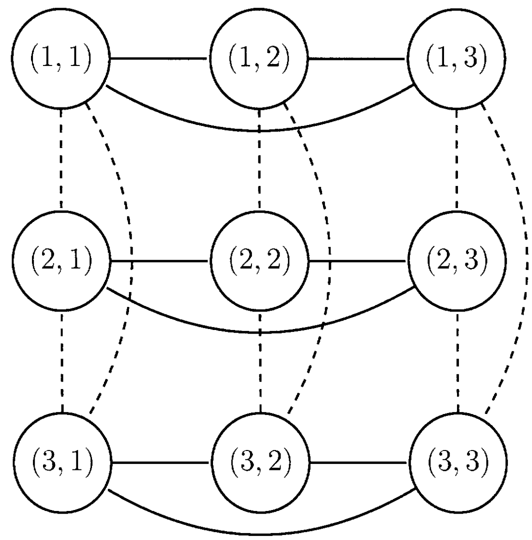


Figure 2-4: Graphs used for defining different Laplacians in a game with 2 players.

Note that this operator is very similar to $\Delta_{0,m}$.

For a given game with collection of utilities $\{u^m\}_{m\in\mathcal{M}} \in C_0^M$, we refer to $proj_m u^m$ and $(I - proj_m)u^m$ are respectively the *strategic* and *nonstrategic* components of the utility of player m. The motivation behind this is that $(I - proj_m)u^m$ denotes the projection of the utility u^m to the kernel of D_m and hence $D_m(I - proj_m)u^m = 0$. On the other hand, entries of $D_m u^m = D_m proj_m u^m$ indicates the pairwise rankings of different strategy profiles by player m. Therefore, $proj_m u^m$ contains all the strategic information of player m whereas $I - proj_m u^m$ stands for the nonstrategic component of u^m.

Due to the special structure of the underlying graph there is a relationship between the projection operator $proj_m = D_m^\dagger D_m$ and Laplacian operator $\Delta_{0,m} = D_m^* D_m$. Next theorem establishes this relationship between $D_m^\dagger D_m$ and $D_m^* D_m$.

Theorem 2.2.2. *Let $|E^m| = h_m$ for all $m \in \mathcal{M}$. Then, for all $m \in \mathcal{M}$, $D_m^* D_m = c^m D_m^\dagger D_m$ where $c_m = \frac{h_m}{h_m - 1} > 1$.*

Proof. The proof relies on the fact that $D_m^* D_m = \Delta_{0,m}$ is a Laplacian operator. As weights $W^m(\mathbf{p}, \mathbf{q}) = 1$, if and only if $\mathbf{p}, \mathbf{q} \in E$ differ in the strategy of player m the underlying graph has edges between strategy profiles which differ only in the strategy of player m. For a fixed m, it can be seen that strategy profile $\mathbf{p} = (\mathbf{p}^m, \mathbf{p}^{-m})$ has edges to strategy profiles $(\mathbf{q}^m, \mathbf{p}^{-m})$ for all $\mathbf{q}^m \in E^m$, $\mathbf{q}^m \neq \mathbf{p}^m$ but to none of the strategy profiles $(\mathbf{q}^m, \mathbf{q}^{-m})$ for $\mathbf{q}^{-m} \neq \mathbf{p}^{-m}$. This implies that the graph over which $\Delta_{0,m}$ is defined has $|E^{-m}| = \prod_{k\neq m} h_k$ components (each $\mathbf{p}^{-m} \in E^{-m}$ creates a different component), each of which has $|E^m| = h_m$ elements. Note that all strategy profiles in a component are connected, thus the underlying graph consists of $|E^{-m}|$ components, each of which is a complete graph with $|E^m|$ nodes.

The Laplacian of an unweighted complete graph with n nodes has eigenvalues 0 and $\frac{n}{n-1}$, where the multiplicity of nonzero eigenvalues is $n-1$ [13]. Each component of $\Delta_{0,m}$ has eigenvalues 0 and $\frac{h_m}{h_m-1}$ with multiplicities 1 and $h_m - 1$ respectively. Therefore, $\Delta_{0,m}$ has eigenvalues 0 and $\frac{h_m}{h_m-1}$ where the multiplicity of nonzero eigenvalues is $(h_m - 1)\prod_{k\neq m} h_k = h - \prod_{k\neq m} h_k$. This suggests that the dimension of the kernel of $\Delta_{0,m}$ is $\prod_{k\neq m} h_k$.

Observe that the kernel of $\Delta_{0,m} = D_m^* D_m$ contains the kernel of D_m. For every $\mathbf{q}^{-m} \in E^{-m}$ define $\nu_{\mathbf{q}^{-m}} \in C_0$ such that

$$\nu_{\mathbf{q}^{-m}}(\mathbf{p}) = \begin{cases} 1 & \text{if } \mathbf{q}^{-m} = \mathbf{p}^{-m} \\ 0 & \text{otherwise} \end{cases} \tag{2.50}$$

It is easy to see that $\nu_{\mathbf{p}^{-m}} \perp \nu_{\mathbf{q}^{-m}}$ for $\mathbf{p}^{-m} \neq \mathbf{q}^{-m}$ and $D_m \nu_{\mathbf{p}^{-m}} = 0$ for all $\mathbf{p}^{-m} \in E^{-m}$. Thus, for all $\mathbf{q}^{-m}$, $\nu_{\mathbf{q}^{-m}}$ belongs to the kernel of D_m and by mutual orthogonality of these functions D_m has dimension at least $|E^{-m}| = \prod_{k\neq m} h_k$. As the dimension of the kernel of $\Delta_{0,m}$ is $\prod_{k\neq m} h_k$ and it contains kernel of D_m, this implies that the kernels of D_m and $\Delta_{0,m}$ coincide.

Thus $\Delta_{0,m}$ maps any $\nu \in C_0$ in the kernel of D_m to zero and scales the ν in the orthogonal complement of the kernel by $\frac{h_m}{h_m-1}$. On the other hand $D_m^\dagger D_m$ is a projection operator and it has eigenvalue 0 for all functions in the kernel of D_m and 1 for the functions in the orthogonal complement of kernel of D_m. This implies that

$$\Delta_{0,m} = \frac{h_m}{h_m - 1} D_m^\dagger D_m, \tag{2.51}$$

as the claim suggests. □

In the proof of the previous theorem we also established that the kernels of D_m and $\Delta_{0,m}$ are equal and have dimension $\prod_{k\neq m} h_k$. We make use of this fact in Chapter 4 and for future reference we state the following lemma.

Lemma 2.2.1. *Kernels of D_m and $\Delta_{0,m}$ are equal and have dimension $\prod_{k\neq m} h_k$.*

In the projection problem the distance of the initial game from the set of exact potential games will be important. To quantify the deviation of a game from a potential game we next introduce some useful norms. For $\phi \in C_0$, let

$$||\phi||_2 = \langle \phi, \phi \rangle^{\frac{1}{2}} = \left(\sum_{\mathbf{p}\in E} \phi^2(\mathbf{p}) \right)^{\frac{1}{2}}. \tag{2.52}$$

For the collection of utilities $u = \{u^m\}_{m\in\mathcal{M}} \in C_0^M$ define the following norm,

$$||u||_2 = \left(\sum_{m\in\mathcal{M}} \langle u^m, u^m \rangle \right)^{\frac{1}{2}} = \left(\sum_{m\in\mathcal{M}} ||u^m||_2^2 \right)^{\frac{1}{2}}. \tag{2.53}$$

For a game $\mathcal{G}$ assume that $\{u^m\}_{m\in\mathcal{M}}$ and $\{v^m\}_{m\in\mathcal{M}}$ are the two different collections of utilities denote by u and v respectively. We use the notation $u-v = \{u^m - v^m\}_{m\in\mathcal{M}}$ to denote the collection of utilities where utility function of player m is $u^m - v^m$. The norm of $u - v$ is expressed as

$$||u - v||_2 = \left(\sum_{m\in\mathcal{M}} ||u^m - v^m||_2^2 \right)^{\frac{1}{2}}. \tag{2.54}$$

Finally we define a weighted norm for $X \in C_1$ as follows

$$||X||_2 = \langle X, X \rangle^{\frac{1}{2}} = \left(\frac{1}{2} \sum_{\mathbf{p},\mathbf{q}\in E} W(\mathbf{p},\mathbf{q}) X^2(\mathbf{p},\mathbf{q}) \right)^{\frac{1}{2}}. \tag{2.55}$$

The notations that are used in this thesis regarding the projection on the set of potential games is summarized in the Table 2.3.

$\mathcal{G}$	The game instance $\langle \mathcal{M}, \{E^m\}_{m\in\mathcal{M}}, \{u^m\}_{m\in\mathcal{M}} \rangle$.
$\mathcal{M}$	The set of players, $\{1, \dots M\}$.
E^m	The set of actions player m has, $E^m = \{1 \dots h_m\}$.
E	The joint action space $\prod_{m\in\mathcal{M}} E^m$.
u^m	The utility function of player m, $u^m : E \to \mathbb{R}$; in the sequel utilities are treated as vectors of length $\lvert E \rvert$.
u	The collection of utilities of all players, $u = \{u^m\}_{m\in\mathcal{M}} \in C_0^M$.
W^m	Function indicating whether strategy profiles are comparable by player m or not, $W^m : E \times E \to \{0, 1\}$.
W	Function indicating whether strategy profiles are comparable or not, $W : E \times E \to \{0, 1\}$.
C_0	Space of utilities, $C_0 = \{u^m \mid u^m : E \to \mathbb{R}\}$, if u^m is represented as a vector then $C_0 = \mathbb{R}^{\lvert E \rvert}$.
C_1	Space of pairwise comparison functions from $E \times E$ to $\mathbb{R}$, C_1 functions can be represented as vectors of length $\lvert E \rvert^2$.
δ_0	The gradient operator. $\delta_0 : C_0 \to C_1$, for $\phi \in C_0$ satisfies $(\delta_0 \phi)(\mathbf{p}, \mathbf{q}) = W(\mathbf{p}, \mathbf{q}) \left(\phi(\mathbf{q}) - \phi(\mathbf{p}) \right)$.
D_m	$D_m : C_0 \to C_1$, for $\phi \in C_0$ satisfies $(D_m \phi)(\mathbf{p}, \mathbf{q}) = W^m(\mathbf{p}, \mathbf{q}) \left(\phi(\mathbf{q}) - \phi(\mathbf{p}) \right)$.
D	$D : C_0^M \to C_1$, for $u \in C_0^M$ satisfies $(Du)(\mathbf{p}, \mathbf{q}) = \sum_{m\in\mathcal{M}} W^m(\mathbf{p}, \mathbf{q}) \left(u^m(\mathbf{q}) - u^m(\mathbf{p}) \right)$.
δ_0^*	$\delta_0^* : C_1 \to C_0$ is the adjoint of the operator δ_0.
Δ_0	Laplacian operator for the graph of all comparable strategy profiles. $\Delta_0 : C_0 \to C_0$ satisfies $\Delta_0 = \delta_0^* \delta_0$.
$\Delta_{0,m}$	Laplacian operator for the graph of comparable strategies by player m. $\Delta_{0,m} : C_0 \to C_0$ satisfies $\Delta_{0,m} = D_m^* D_m = D_m^* \delta_0$.
$proj_m$	Projection operator used for projection on the orthogonal complement of kernel of D_m. $proj_m : C_0 \to C_0$ satisfies $proj_m = D_m^\dagger D_m$.

Table 2.3: Summary of Notations

Note that by the definitions of the operators in Table 2.3 it can be seen that a game is an exact potential game if and only if there exists a $\Phi \in C_0$ such that

$$D_m u^m = D_m \Phi \text{ for all } m \in \mathcal{M}. \tag{2.56}$$

This alternative definition for exact potential games is used in Chapter 4.

Chapter 3

Competitive Scheduling in Wireless Collision Channels

In this chapter, we consider a wireless collision channel, shared by a finite number of users who wish to optimally schedule their transmissions based on a natural trade-off between throughput and power. The channel quality between each user and the base station is randomly time-varying and observed by the user prior to each transmission decision. The bulk of the research in the area has been carried under a simplified assumption that the channel state processes of different users are independent (see e.g.,[4, 35]). In practice, however, there are global system effects, which simultaneously affect the quality of all transmissions (e.g., thermal noise at the base station, or common weather conditions). Consequently, a distinctive feature of our model is that the state processes of different users are *correlated*. As an approximating model, we consider in this chapter the case of *full* correlation, meaning that all users observe the same state prior to transmission. A fully correlated state can have a positive role of a coordinating signal, in the sense that different states can be "divided" between different users. On the other hand, such state correlation increases the potential deterioration in system performance due to noncooperation, as users might transmit simultaneously when good channel conditions are available.

The rest of this chapter is organized as follows. In Section 3.1 we describe the network and game models studied in this chapter. In Section 3.2 we define a related optimization problem to determine a system optimal solution of the scheduling problem. In Section 3.3 we quantify the efficiency loss in the scheduling game. We discuss the best response dynamics and their convergence properties in Section 3.4.

3.1 The Model and Preliminaries

We consider a wireless network, shared by a finite set of mobile users $\mathcal{M} = \{1, \ldots, M\}$ who transmit at a fixed power level to a common base station over a shared collision channel. Time is slotted, so that each transmission attempt takes place within slot boundaries that are common to all. Transmission of a user is successful only if no other user attempts transmission simultaneously. Thus, at each time slot, at most one user can successfully transmit to the base station. To further specify our model, we start with a description of the channel between each user and the base station (Section 3.1.1), ignoring the possibility of collisions. In Section 3.1.2, we formalize the user objective and formulate the noncooperative game which arises in a multi-user shared network.

3.1.1 The Physical Network Model

Our model for the channel between each mobile (or *user*) and the base station is characterized by two basic elements.

a. Channel state process. We assume that the channel state between mobile m and the base station evolves as a stationary process $H^m(t)$, $t \in \mathbb{Z}_+$ (e.g., Markovian) taking values in a set $\mathcal{H}^m = (1, 2, \ldots, \bar{h}_m)$ of $\bar{h}_m$ states. The stationary probability that mobile m observes state $i \in \mathcal{H}^m$ at any time t is given by π_i^m.

b. Expected data rate. We denote by $R_i^m > 0$ the expected data rate (or simply, the rate)[1] that user m can sustain at any given slot as a function of the current state $i \in \mathcal{H}^m$. We further denote by $\mathcal{R}^m = \{R_1^m, R_2^m \ldots, R_{\bar{h}_m}^m\}$ the set of all data rates for user m, and define $\mathcal{R} = \mathcal{R}^1 \times R^2 \cdots \times \mathcal{R}^M$. For convenience, we assume that for every $m \in \mathcal{M}$ the expected data rate R_i^m strictly decreases in the state index i, so that

[1] Say, in bits per second.

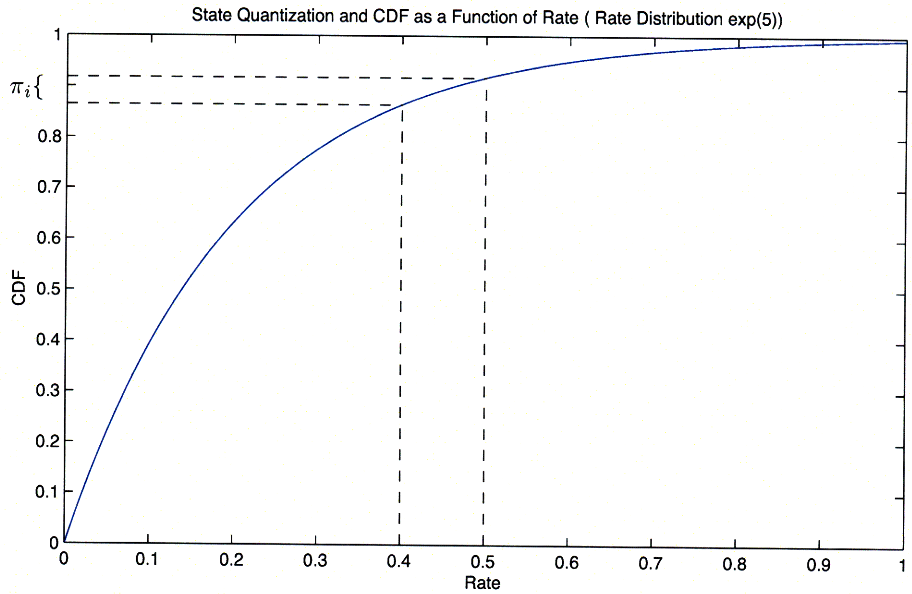


Figure 3-1: State Quantization Example

$R_1^m > R_2^m \cdots > R_{\bar{h}_m}^m$, i.e., state 1 represents the "best state" in which the highest rate can be achieved.

Note that the actual channel quality may still take continuous values, which each user reasonably quantifies into a finite number of information states. Using the cumulative density function of the underlying channel quality, the expected data rates and their associated steady state probabilities can be obtained. The motivation behind considering a discrete state process rather than the actual channel quality is the technical inability of mobiles to sense and process a continuum of channel quality information.

Figure 3-1 exemplifies the quantization in the channel quality. In the figure rates are assumed to be normalized to 1 and rates between 0.4 and 0.5 are represented by a single discrete state i, steady state occurrence probability of which is π_i. π_i and corresponding expected rate can be calculated from the underlying cumulative density function (CDF) of the rate distribution.

A central assumption in this chapter is that the state processes of different users are fully correlated, as we formalize below.

Assumption 3.1.1 (Full Correlation). *All users observe the same channel state $H(t)$ in any given time t. That is, for every mobile $m \in \mathcal{M}$: (i) $\mathcal{H}^m = \mathcal{H} = \{1, 2 \ldots, h\}$, (ii) $\pi_i^m = \pi_i$ for every $i \in \mathcal{H}$ and (iii) $H^m(t) = H(t)$ (where $\mathcal{H}$ is the common state space, and $\pi = (\pi_1, \ldots, \pi_h)$ is its stationary distribution).*

We emphasize that although all mobiles observe the same state, the corresponding rates R_i^m need *not* be equal across mobiles, i.e., in our general model we do *not* assume that $R_i^m = R_i^k$, $m, k \in \mathcal{M}$, $i \in \mathcal{H}$. The case where the latter condition does hold will be considered as a special case in Section 3.4.

The above model can be used to capture the following network scenario. The channel state corresponds to *global* conditions that affect all user transmissions. Examples may include thermal noise at the base station and weather conditions (that play a central role, e.g., in satellite networks), which affect all mobiles in a similar manner. The state information can be communicated from the base-station to all mobiles via a feedback channel. After obtaining the state information at the beginning of each slot, a user may respond by adjusting its coding scheme in order to maximize its data rate on that slot. The rate R_i^m thus takes into account the quality of the current state i, the coding scheme adapted by the user, and "local" characteristics, such as the user's transmission power, the location relative to the base station and (local) fast-fading effects. We emphasize that R_i^m is an *average* quantity, which averages possible fluctuations in local channel conditions, which usually occur at a faster time-scale relative to the change in the global channel state (see, e.g., [39]). This assumption is commensurate with practical considerations, as mobiles usually cannot react to fast local changes.

3.1.2 User Objective and Game Formulation

In this subsection we describe the user objective and the noncooperative game which arises as a consequence of the user interaction over the collision channel. In addition, we provide some basic properties and examples for the Nash equilibrium of the underlying game.

Basic Definitions

The basic assumption of our model is that users always have packets to send, yet they are free to determine their own transmission schedule in order to fulfill their objectives. Furthermore, users are unable to coordinate their transmission decisions.

Our focus in this chapter is on *stationary* transmission strategies, in which the decision of whether to transmit or not can depend (only) on the current state. A formal definition is provided below.

Definition 3.1.1 (Stationary Strategies). *A stationary strategy for user m is a mapping $\sigma^m : \mathcal{H} \rightarrow [0,1]^h$. Equivalently, σ^m is represented by an h-dimensional vector $\mathbf{p}^m = (p_1^m, \ldots, p_h^m) \in [0,1]^h$, where the i-th entry corresponds to user m's transmission probability when the observed state is i.*

We denote the the strategy profiles for the wireless scheduling game by $\mathbf{p} = (\mathbf{p}^1, \ldots \mathbf{p}^M)$.

For a given strategy profile $\mathbf{p}$, we define below the Quality of Service (QoS) measures that determine user performance. Let B^m be the (fixed) transmission power of user m per transmission attempt, and denote by $\tilde{P}^m(\mathbf{p}^m)$ its average power investment, as determined by its strategy $\mathbf{p}^m$. Then clearly, $\tilde{P}^m(\mathbf{p}^m) = B^m \sum_{i=1}^h \pi_i p_i^m$ for every user m. We normalize the latter measure by dividing it by B^m, and consider henceforth the normalized power investment, given by

$$P^m(\mathbf{p}^m) = \sum_{i=1}^{h} \pi_i p_i^m. \tag{3.1}$$

For simplicity, we shall refer to $P^m(\mathbf{p}^m)$ as the power investment of user m. We assume that each user m is subject to an individual power constraint $0 < \bar{P}^m \leq 1$, so that any user strategy $\mathbf{p}^m$ should obey

$$P^m(\mathbf{p}^m) \leq \bar{P}^m. \tag{3.2}$$

The vector of power constraints is denoted by $\bar{\mathbf{P}} = (\bar{P}^1, \ldots, \bar{P}^M)$.

The second measure of interest is the mobile's average throughput, denoted by $T^m(\mathbf{p}^m, \mathbf{p}^{-m})$. The average throughput of every user m depends on the transmission success probability at any given state i, $\prod_{k \neq m}(1 - p_i^k)$. Hence,

$$T^m(\mathbf{p}^m, \mathbf{p}^{-m}) = \sum_{i=1}^{h} \pi_i R_i^m p_i^m \prod_{k \neq m} (1 - p_i^k). \tag{3.3}$$

Each user wishes to optimize a natural trade-off between throughput and power, which is captured by maximizing the following utility function

$$u^m(\mathbf{p}^m, \mathbf{p}^{-m}) = T^m(\mathbf{p}^m, \mathbf{p}^{-m}) - \lambda^m P^m(\mathbf{p}^m), \tag{3.4}$$

subject to the power constraint (3.2), where $\lambda^m \geq 0$ is a user dependent trade-off coefficient. We use the notation $\lambda = (\lambda^1, \ldots, \lambda^M)$ for the vector of all users' trade-off coefficients; note that each game *instance* can now be formally described by the tuple $\mathcal{I} = \{\mathcal{M}, \mathcal{R}, \pi, \lambda, \bar{\mathbf{P}}\}$.

The term $\lambda^m P^m(\mathbf{p}^m, \mathbf{p}^{-m})$ in (3.4) can be viewed as the power cost of the mobile. The user utility thus incorporates both a "hard" constraint on power consumption (in the form of (3.2)), but also accounts for mobile devices that do not consume their power abilities to the maximum extent, as energy might be a scarce resource, the usage of which needs to be evaluated against the throughput benefit. We note that the utility (3.4) accommodates the following special cases:

- *Fully "elastic" users.* By setting $P^m(\mathbf{p}^m) = 1$, a user practically does not have a hard constraint on power usage. Accordingly, the optimal operating point of the user is determined solely by the tradeoff between power and throughput, as manifested by the factor λ^m. The fully elastic user case has been considered in the wireless games literature in different contexts (see, e.g., [1]).

- *Power-cost neutral users.* Consider a user with $\lambda^m = 0$. Such a user is interested only in maximizing its throughput subject to a power constraint. This form of utility has been examined, e.g., in [4] and [25].

Nash Equilibrium

The strategy spaces of users are affected by the power constraints and the strategy space of user m can be expressed as:

$$E^m = \{\mathbf{p}^m | P^m(\mathbf{p}^m) \leq \bar{P}^m, 0 \leq \mathbf{p}^m \leq 1\}. \tag{3.5}$$

As previously stated in Chapter 2 the joint strategy space is denoted by $E = \prod_{m \in \mathcal{M}} E^m$.

In the described game the Nash equilibrium always exists as we summarize below.

Theorem 3.1.1. *There always exists a pure Nash equilibrium for the game.*

Proof. E^m is a compact, nonempty, convex subset of an Euclidean space for all $m \in \mathcal{M}$ by (3.5). The payoff function of each user is a continuous function. Moreover, the payoff of user m is linear in its strategy. Using Theorem 2.1.1 a pure Nash equilibrium of this game exists. □

We conclude this section by examples which point to some interesting features of the underlying game. The first example shows that there are possibly infinitely many Nash equilibria.

Example 3.1.1. *Consider a game with two users, m, k, and two states 1, 2. Let $\pi_1 = \pi_2 = \frac{1}{2}$, $R_1^m = R_1^k = 10$, $R_2^m = R_2^k = 5$, $\lambda^m = \lambda^k = 2$,and $\bar{P}^m = 0.8$, $\bar{P}^k = 0.3$. It can be easily shown that the strategy profile $(p_1^m,\ p_2^m,\ p_1^k,\ p_2^k) = (1,\ 0.6,\ 0,\ x)$ is an equilibrium of the game, for every $x \in [0, 0.6]$.*

The next example demonstrates that the behavior of the system in an equilibrium can sometimes be counterintuitive. For example, states which lead to lower expected rates can be utilized (in terms of the total power investment) more than higher quality states.

Example 3.1.2. *Consider a game with two users, m, k, and two states 1, 2. Let $\pi_1 = \pi_2 = \frac{1}{2}$, $R_1^m = R_1^k = 8$, $R_2^m = R_2^k = 3$, $\lambda^m = \lambda^k = 1$ and $\bar{P}^m = 0.8$, $\bar{P}^k = 0.3$. The unique equilibrium of this game instance is given by $(p_1^m,\ p_1^m,\ p_1^k,\ p_2^k) =$*

(1, 0.6, 0, 0.6). *Observe that the total power investment at state* 1 (0.5) *is lower than the total power investment at state* 2 (0.6).

Both examples demonstrate some negative indications as to the *predictability* of the Nash equilibrium. Not only the number of equilibria is unbounded, but also we cannot rely on monotonicity results (such as total power investment increasing with the quality of the state) in order to provide a rough characterization of an equilibrium. At the same time, these observations motivate the study of performance-loss bounds at *any* equilibrium point, and also of network dynamics that can converge to a predictable equilibrium point. Both directions would be examined in the sequel.

3.1.3 Existence of Nash Equilibrium for General Strategy Spaces

In this section we assume that users are not necessarily constrained to stationary strategies and they can utilize nonstationary strategies as well. We show the existence of the Nash equilibrium by showing that equilibria among stationary strategies are actually equilibria among general set of strategies. In other words assuming the system operates at an equilibrium of the stationary strategies then none of the users have incentive to utilize a nonstationary strategy.

The model for nonstationary strategies is slightly different than that of stationary strategies. At each time slot, regardless of the state of the channel and actions of other users, each user has two possible actions. We denote the set of possible actions for user m by $A^m = \{0, 1\}$, where 1 corresponds to transmitting and 0 corresponds to idling. $\mathcal{A} = \{A^m\}_{m=1}^M$ is the joint action space. It is assumed that users may randomize their actions over possible actions at each time slot; we accordingly denote the set of probability distributions over A^m at time slot t by $\Sigma^m(t)$.

We define a general strategy of user m by $\mathbf{s}^m = \{s^m(1), s^m(2) \dots\} \in \Sigma^m$ where $s^m(t) \in \Sigma^m(t)$ is a probability distribution through which the user m chooses its action at time slot t, and $\Sigma^m = \Sigma^m(1) \times \Sigma^m(2) \dots$ is the collection of probability distributions at all time slots. We denote the strategies of all users, or strategy profile, by $\mathbf{s} = \{\mathbf{s}^1, \mathbf{s}^2 \ldots \mathbf{s}^M\}$ and strategies of users other than m by $\mathbf{s}^{-m}$.

Strategies of users may depend on the past history of the system. The history of user m may include, states observed in the past, actions of user m in the past, collision history, and perhaps some additional information. We denote the history of user m at time t by $y^m(t) \in \mathbf{Y}^m(t)$, where $\mathbf{Y}^m(t)$ is the set of all possible realizations of history of user m up to time t. $s^m(t)$ is a mapping from the history up to time t to the set of probability distributions over the action space, or $s^m(t) : \mathbf{Y}^m(t) \to \Sigma^m(t)$.

Metrics presented in equations (3.1), (3.3) are related to expected average power and throughput and determined under the assumption of stationarity in user actions. We next introduce the utility and constraints used in the nonstationary counterpart of the previously defined problem. To that end, we define the expected average power and expected average utility as,

$$P^m_{av}(\mathbf{s}^m) = \limsup_{T \to \infty} \frac{1}{T} E[\sum_{t=1}^{T} I^m(t)] \tag{3.6}$$

and

$$u^m_{av}(\mathbf{s}^m, \mathbf{s}^{-m}) = \limsup_{T \to \infty} \frac{1}{T} E[\sum_{t=1}^{T} R^m(t) I^m(t) (\prod_{k \neq m} (1 - I^k(t)) - \lambda^m)] \tag{3.7}$$

respectively. Here $I^m(t)$ is an indicator variable that is equal to 1 if user m transmits a packet at time slot t and equal to 0 otherwise. $R^m(t)$ is a random variable which is equal to the rate of a successful transmission for user m in time slot t, this quantity is random as it depends on the realization of the channel state. (3.6) is simply the expected average number of transmissions and (3.7) follows by noting that the first term is the expected average rate of successful transmissions and the second term is the expected average cost of transmissions. Note that although in equations (3.6),(3.7), $\mathbf{s}^m$ and $\mathbf{s}^{-m}$ do not appear explicitly, the statistics of indicator variables are determined by these quantities and hence expected average power and utility are functions of $\mathbf{s}^m$ and $\mathbf{s}^{-m}$.

In the new game formulation the strategy space of user m will be denoted as follows,

$$E_{av}^{m} = \{\mathbf{s}^m | P_{av}^{m}(\mathbf{s}^m) \leq \bar{P}^m, \mathbf{s}^m \in \Sigma^m\}. \tag{3.8}$$

Note that each stationary strategy profile, $\mathbf{p}$ corresponds to a strategy profile among general strategies that is denoted by $\mathbf{s}(\mathbf{p})$ and satisfies,

$$\mathbf{Pr}(I^m(t) = 1 \mid y^m(t), \text{ state } i \text{ is observed at tine } t) = p_i^m. \tag{3.9}$$

It is easy to see that (using (3.6) and (3.7))

$$P_{av}^{m}(\mathbf{s}^m(\mathbf{p})) = P^m(\mathbf{p}^m) \tag{3.10}$$

and

$$u_{av}^{m}(\mathbf{s}^m(\mathbf{p}), \mathbf{s}^{-m}(\mathbf{p})) = u^m(\mathbf{p}^m, \mathbf{p}^{-m}), \tag{3.11}$$

if the channel state process is stationary. Hence for a game instance if $\mathbf{p} \in E$ then the corresponding stationary general strategy $\mathbf{s}(\mathbf{p}) \in E_{av} = \prod_{m \in \mathcal{M}} E_{av}^{m}$ and moreover players get same payoffs. Next we state the main theorem of this section. Note that the theorem assumes that the underlying channel state process is Markovian.

Theorem 3.1.2. *For a game instance $\mathcal{I} = \{\mathcal{M}, \mathcal{R}, \pi, \lambda, \bar{\mathbf{P}}\}$, assume that $\mathbf{p}$ is a Nash equilibrium among stationary strategies. If the channel state process is Markovian then $\mathbf{s}(\mathbf{p})$ is a Nash equilibrium among general strategies.*

Proof. We prove the statement by showing that, at $\mathbf{s}(\mathbf{p})$ no user has an incentive to adopt a nonstationary strategy. In order to simplify the notation we denote the general strategy corresponding to the stationary strategy $\mathbf{p}$ by $\mathbf{s}^m = \mathbf{s}^m(\mathbf{p})$ for any user m.

Assume that the claim is wrong and user m has a strictly better payoff by utilizing an optimal strategy $\hat{\mathbf{s}}^m$ which is not necessarily stationary. Since all users other than m are utilizing stationary strategies and since the channel state process is Markovian it follows that finding $\hat{\mathbf{s}}^m$, maximizing (3.7) subject to (3.8) is a constrained Markov decision problem [2, 6]. Moreover the resulting Markov decision problem has finitely many states (state space is simply $\mathcal{H}$) and it is known that there exists an optimal stationary solution for this problem [10]. Let $\mathbf{q}^m$ be the described optimal stationary solution, then it follows that

$$u^m(\mathbf{q}^m, \mathbf{p}^{-m}) = u_{av}^m(\hat{\mathbf{s}}^m, \mathbf{s}^{-m}) > u_{av}^m(\mathbf{s}^m, \mathbf{s}^{-m}) = u^m(\mathbf{p}^m, \mathbf{p}^{-m}). \tag{3.12}$$

and hence $\mathbf{p}$ cannot be an equilibrium as user m has incentive to switch from $\mathbf{p}^m$ to $\mathbf{q}^m$. Thus we obtain a contradiction and $\mathbf{s}(\mathbf{p})$ is a Nash equilibrium among general strategies as claimed.

Therefore, a pure Nash equilibrium among stationary strategies is a Nash equilibrium among general strategies. □

As stationary strategies are also Markovian the previous theorem implies that for the problem formulation studied in this chapter a Nash equilibrium of the game in which all users play stationary strategies remains to be a Nash equilibrium if users are allowed to utilize nonstationary strategies.

In the rest of this chapter we restrict ourselves to the stationary game formulation which was previously described.

3.2 Social Welfare and Threshold Strategies

In this section we characterize the optimal operating point of the network. This characterization allows us to study the efficiency loss due to self-interested behavior (Section 3.3).

An optimal strategy profile in our system is a strategy profile that maximizes the aggregate user utility. Formally, $\mathbf{p}^*$ is an optimal strategy profile if it is a solution to the Social Welfare Problem (SWP), given by

$$(\text{SWP}) \qquad \max_{\mathbf{p} \in E} \quad u(\mathbf{p}), \tag{3.13}$$

where

$$u(\mathbf{p}) = \sum_m T^m(\mathbf{p}) - \lambda^m P^m(\mathbf{p}). \tag{3.14}$$

We note that (SWP) is a non-convex optimization problem. To see this we make use of the Hessian matrix of $u(\mathbf{p})$, denoted by $\nabla^2 u(\mathbf{p})$. The entries of the Hessian for a function $f(\mathbf{x})$ can be given as $\nabla^2 f(\mathbf{x})_{i,j} = \frac{\partial f}{\partial x_j \partial x_i}(\mathbf{x})$. We use the fact that Hessian of a concave function is negative semidefinite at every point in its domain and the objective function in convex maximization problems is concave [8, 11].

Lemma 3.2.1. *(SWP) is a nonconvex optimization problem.*

Proof. The definition of $u(\mathbf{p})$ in (3.14) reveals that the diagonal of the $\nabla^2 u(\mathbf{p})$ is always equal to zero. Hence, trace of the Hessian, or equivalently the sum of eigenvalues of the Hessian is equal to zero for any $\mathbf{p}$. But for $\mathbf{p} \in E$ such that $p_i^m \in (0,1)$ for all m, i, the Hessian is not identically equal to zero, or equivalently all eigenvalues of it can not be equal to zero. Thus, the Hessian is neither negative nor positive semidefinite. Since Hessian is not negative semidefinite, $u(\mathbf{p})$ is not a concave function of its argument and hence the (SWP) is not a convex optimization problem. □

For a further characterization of (SWP), we require the definitions stated below.

Definition 3.2.1 (Partially and Fully Utilized States)**.** *Let $\mathbf{p}^m$ be some strategy of user m. Under that strategy, state i is* partially *utilized by user m if $p_i^m \in (0,1)$; state i is fully utilized by the user if $p_i^m = 1$.*

Definition 3.2.2 (Threshold Strategies)**.** *A strategy $\mathbf{p}^m$ of user m is a threshold strategy, if the following conditions hold: (i) User m partially utilizes* at most *one state, and (ii) If user m partially utilizes exactly one state, then the power constraint (3.2) is active (i.e., met with equality). A strategy profile $\mathbf{p} = (\mathbf{p}^1, \ldots, \mathbf{p}^M)$ is a threshold strategy profile if $\mathbf{p}^m$ is a threshold strategy for every $m \in \mathcal{M}$.*

We next provide a per-state utilization bound for *any* optimal solution of (SWP).

Proposition 3.2.1. *Let $\mathbf{p}$ be an optimal solution of (SWP). Then $\sum_m p_i^m \leq 1$ for every $i \in \mathcal{H}$.*

The significance of the above result is that in case that all mobiles use the same power level B for transmission, then the total energy investment at any optimal solution is bounded above by Bh, where h is the number of states. Note that this bound does not depend on the number of mobiles. The per-state utilization bound will play a key role in Section 3.3, while bounding the overall efficiency loss in the system.

Proof. For the proof, we shall make use of the partial derivatives of the aggregate utility, given by

$$\frac{\partial u(\mathbf{p})}{\partial p_i^m} = \pi_i(R_i^m \prod_{k \neq m}(1-p_i^k) - \lambda_i^m - \sum_{l \neq m} p_i^l R_i^l \prod_{k \neq m,\ l}(1-p_i^k)), \tag{3.15}$$

for any $m \in \mathcal{M}$. Let $\rho_i = \prod_k (1-p_i^k)$. For any $\mathbf{p}$ such that $p_i^k < 1$ for all $k \in \mathcal{M}$, (3.15) can be rewritten as

$$\begin{aligned} \frac{\partial u(\mathbf{p})}{\partial p_i^m} &= \pi_i \frac{\rho_i}{1-p_i^m}\left(R_i^m - \frac{\lambda_i^m}{\rho_i}(1-p_i^m) - \sum_{l \neq m} \frac{p_i^l}{1-p_i^l} R_i^l\right) \\ &= \pi_i \frac{\rho_i}{1-p_i^m}\left(\frac{R_i^m}{1-p_i^m} - \frac{\lambda_i^m}{\rho_i}(1-p_i^m) - \sum_{l} \frac{p_i^l}{1-p_i^l} R_i^l\right). \end{aligned} \tag{3.16}$$

Let $\mathbf{p}$ be an optimal solution of (SWP) and consider some state i. If this state is used by a user with probability 1 then obviously no other user transmits at this state and the claim immediately follows.

The claim obviously holds if no user utilizes the state. Hence, assume that in the optimal solution state i is partially used by some users. Let $K_i \subset \mathcal{M}$ be the subset of users that partially utilize state i. Let

$$m \in \operatorname*{argmin}_{l \in K_i} \frac{R_i^l}{1-p_i^l}. \tag{3.17}$$

Since $\mathbf{p}$ is optimal it follows that

$$\frac{\partial u(\mathbf{p})}{\partial p_i^m} \geq 0, \tag{3.18}$$

as otherwise the aggregate utility can be improved by decreasing p_i^m. Substituting (3.16) in (3.18) and recalling that $1 - p_i^k > 0$ for every $k \in K_i$, we obtain that

$$\frac{R_i^m}{1-p_i^m} - \frac{\lambda_i^m}{\rho_i}(1-p_i^m) - \sum_l \frac{p_i^l}{1-p_i^l} R_i^l \geq 0, \tag{3.19}$$

hence

$$\frac{R_i^m}{1-p_i^m} \geq \sum_l \frac{p_i^l}{1-p_i^l} R_i^l \geq \frac{R_i^m}{1-p_i^m} \sum_l p_i^l, \tag{3.20}$$

where the last inequality follows from (3.17). (3.20) immediately implies that $\sum_l p_i^l \leq 1$. □

We next introduce some ideas from linear programming which are used in the study of threshold strategies. A linear program is an optimization problem with linear objective function and linear equality or inequality constraints. Linear constraints lead to a polyhedral feasible region. For a linear program on n variables, points of the feasible region where n linearly independent constraints are active (i.e. satisfied with equality) are called as extreme points of the feasible region. In linear programs with bounded feasible regions there always exists an optimal solution which is an extreme point of the feasible region [9]. Note that the threshold strategies of a user correspond to the extreme points of its strategy space.

The main result of this section is stated below.

Theorem 3.2.1. *There exists an optimal solution of (SWP) where all users employ threshold strategies.*

Proof. Let $\hat{\mathbf{p}}$ be an optimal solution of (SWP) and, define the function $g_{\hat{\mathbf{p}}}^m : E^m \to \mathbb{R}$ as follows:

$$g_{\hat{\mathbf{p}}}^m(\mathbf{p}^m) \triangleq u(\mathbf{p}^m, \hat{\mathbf{p}}^{-m}) - u(\hat{\mathbf{p}}^m, \hat{\mathbf{p}}^{-m}). \tag{3.21}$$

The function $g_{\hat{\mathbf{p}}}^m(\cdot)$ quantifies the change in the aggregate utility, if user m utilizes a strategy $\mathbf{p}^m$ instead of $\hat{\mathbf{p}}^m$. Consider the following optimization problem,

$$\begin{aligned} \max \quad & g^m_{\hat{\mathbf{p}}}(\mathbf{p}^m) \\ s.t. \quad & \mathbf{p}^m \in E^m. \end{aligned} \tag{3.22}$$

If an optimal solution of this maximization problem is $\mathbf{p}^m$ it follows from the definition of $g^m_{\hat{\mathbf{p}}}$ that $(\mathbf{p}^m, \hat{\mathbf{p}}^{-m})$ is an optimal solution of (SWP).

Observe that $g^m_{\hat{\mathbf{p}}}(\mathbf{p}^m)$ is linear in $\mathbf{p}^m$. Since E^m is by definition a polyhedron (see (3.5)), (3.22) is a linear optimization problem. Therefore, an optimal solution of (3.22) exists at an extreme point of E^m, and it follows that there exists an optimal solution of (SWP) in which user m utilizes a threshold strategy.

Note that in the above argument starting from an arbitrary optimal solution of (SWP), we achieve an optimal solution of (SWP), in which all users but m utilize the same strategy and user m utilizes a threshold strategy. Repeating the same argument for all users it follows that there exists an optimal solution of (SWP) where all users utilize threshold strategies. □

Due to the non-convexity of (SWP), we cannot rely on first order optimality conditions for the characterization of the optimal solution. Nonetheless, Theorem 3.2.1 indicates that there always exist an optimal solution with some well-defined structure, which is used in the next section for comparing the performance of the optimal solution, to performance of equilibria.

3.3 Efficiency Loss

We proceed to examine the extent to which selfish behavior affects system performance. That is, we are interested in comparing the quality of the obtained equilibrium points to the centralized, system-optimal solution (3.13). Recently, there has been much work in quantifying the efficiency loss incurred by the selfish behavior of users in networked systems (see [46] for a comprehensive review). As discussed in Chapter 2 price of anarchy (PoA) and price of stability (PoS) are commonly used concepts to quantify the efficiency loss. The performance measure that we consider here in order to evaluate the quality of a network working point is naturally the aggregate user utility (3.14).

The standard definitions of PoA and PoS consider all possible instances of the associated game. Recall that in our specific framework, a game instance is given by the tuple $\mathcal{I} = \{\mathcal{M}, \mathcal{R}, \pi, \lambda, \bar{\mathbf{P}}\}$. The next example shows that the performance at the *best* Nash equilibrium can be arbitrarily bad compared to the socially optimal working point.

Example 3.3.1. *Consider a network with two users m and k and two channel states. Let $\pi_1 = \pi_2 = \frac{1}{2}$, $\bar{P}^m = \frac{1}{4}$, $\bar{P}^k = \frac{1}{2}$. Assume that $R_2^m = R_2^k = \epsilon$, $R_1^m = 4$, $R_1^k = 4\epsilon$, $\lambda^m = \lambda^k = \frac{\epsilon}{2}$. The socially optimal working point is given by $\hat{\mathbf{p}} = (\hat{p}_1^m, \hat{p}_2^m, \hat{p}_1^k, \hat{p}_2^k) = (\frac{1}{2}, 0, 0, 1)$ and the unique equilibrium is $\bar{\mathbf{p}} = (\bar{p}_1^m, \bar{p}_2^m, \bar{p}_1^k, \bar{p}_2^k) = (0, \frac{1}{2}, 1, 0)$. Note that $u(\hat{\mathbf{p}}) = 1 + \frac{\epsilon}{8}$, while $u(\bar{\mathbf{p}}) = \frac{3\epsilon}{2}$. Hence, $\frac{u(\hat{\mathbf{p}})}{u(\bar{\mathbf{p}})} > \frac{2}{3\epsilon}$, which goes to infinity as $\epsilon \to 0$.*

The above example suggests that if we consider all possible game instances $\{\mathcal{M}, \mathcal{R}, \pi, \lambda, \bar{\mathbf{P}}\}$, then equilibrium performance can be arbitrarily bad. However, we note that for a *given* mobile technology, some elements within any game instance cannot obtain all possible values. Specifically, π is determined by the technological ability of the mobiles to *quantize* the actual channel quality into a finite number of "information states" as described in Section 3.1. Naturally, one may think of several measures for quantifying the quality of a given quantization. We represent the quantization quality by a single parameter $\pi_{max} \triangleq \max_{i \in \mathcal{H}} \pi_i$, under the understanding that smaller π_{max}, the better is the quantization procedure. In addition, a specific wireless technology is obviously characterized by the power constraint $\bar{P}^m$. Again, we represent the power-capability of a given technology by a single parameter $P_{min} = \min_{m \in \mathcal{M}} \bar{P}^m$. Finally, we determine the *technological quality* of a set of mobiles through the scalar $Q = \frac{\pi_{max}}{P_{min}}$.

We consider next the efficiency loss for a given technological quality Q. Denote by $\mathcal{I}_{Q_0}$ the subset of all game instances such that $Q = Q_0$. We provide below modified definitions of price of stability (PoS) and price of anarchy (PoA) which take the quality parameter into account.

Definition 3.3.1 (Price of Stability - Price of Anarchy). *For every game instance $\mathcal{I}$, denote by $N_{\mathcal{I}}$ the set of Nash equilibria, and let $p^*_{\mathcal{I}}$ be an optimal strategy profile. Then for any fixed Q, the PoS and PoA are defined as*

$$PoS(Q) = \sup_{\mathcal{I} \in \mathcal{I}_Q} \inf_{\mathbf{p} \in N_{\mathcal{I}}} \frac{u(p^*_{\mathcal{I}})}{u(\mathbf{p})}, \tag{3.23}$$

$$PoA(Q) = \sup_{\mathcal{I} \in \mathcal{I}_Q} \sup_{\mathbf{p} \in N_{\mathcal{I}}} \frac{u(p^*_{\mathcal{I}})}{u(\mathbf{p})}. \tag{3.24}$$

We next provide upper and lower bounds for $PoS(Q)$ under the assumption that $Q < 1$ (note that an the unbounded price of stability in Example 3.3.1 was obtained for $Q > 1$). The upper bound on the price of stability follows from the next proposition.

Proposition 3.3.1. *Fix $Q < 1$. Let $\hat{\mathbf{p}}$ be some threshold strategy profile, and let $u(\hat{\mathbf{p}})$ be the respective aggregate utility (3.14). Then there exists an equilibrium point $\bar{\mathbf{p}}$ whose aggregate utility is not worse than $u(\hat{\mathbf{p}})(1-Q)^2$. That is, $\frac{u(\hat{\mathbf{p}})}{u(\bar{\mathbf{p}})} \leq (1-Q)^{-2}$.*

Proof. The key idea behind the proof is to start from a threshold strategy profile $\hat{\mathbf{p}}$ and to reach an equilibrium point by some iterative process. In each step of the process we obtain the worst-case performance loss, which leads to the overall loss in the entire procedure.

1. Let $\acute{\mathcal{H}}$ be the set of states such that each state $i \in \acute{\mathcal{H}}$ satisfies $\hat{p}^m_i > 0$ for some $m \in \mathcal{M}$. For each $i \in \acute{\mathcal{H}}$, define

$$m_i \in \operatorname*{argmax}_{\{k \in \mathcal{M} | 0 < p^k_i \leq 1\}} R^k_i - \lambda^k. \tag{3.25}$$

If the set $\operatorname{argmax}_{\{k \in \mathcal{M} | 0 < p^k_i \leq 1\}} R^k_i - \lambda^k$ is not a singleton, m_i is chosen arbitrarily from the elements of the set. Consider a modified strategy profile $\mathbf{q}$ of the original strategy profile $\hat{\mathbf{p}}$, given by

$$q^k_i = \begin{cases} 1 & \text{if } i \in \acute{\mathcal{H}},\ k = m_i \\ 0 & \text{otherwise.} \end{cases} \tag{3.26}$$

Let $N = \{m_i | i \in \acute{\mathcal{H}}\}$. Note that from the definition of a threshold strategy and

(3.25) it follows that the transmission probability of any user in $\mathbf{q}$ is strictly larger than the transmission probability in $\hat{\mathbf{p}}$ at most for a single state (namely the partially used state, if such exists).

2. The strategy profile $\mathbf{q}$ can be infeasible, as the power constraint of every $m \in N$ can be violated by investing extra power in partially used states. Note that if $k \notin N$, strategy $\mathbf{q}^k \leq \hat{\mathbf{p}}^k$ is feasible. Also $q_i^k \in \{0, 1\}$ for all $k \in \mathcal{M}$, $i \in \mathcal{H}$, and no two users utilize the same state.

   Let $\Delta\bar{P}^m$, $m \in N$, denote the maximum additional power investment required for user m to utilize strategy $\mathbf{q}^m$ instead of $\hat{\mathbf{p}}^m$ (recall that each user partially uses at most a single state). This quantity is obviously bounded by π_{max}, since fully utilizing any state requires at most π_{max} amount of additional power. Set $\Delta\bar{P}^m = 0$ if the strategy of m is already feasible. We next obtain a feasible strategy by modifying $\mathbf{q}$.

3. Consider the following optimization problem,

$$\begin{aligned} \overline{BR}^k(\tilde{\mathbf{p}}^k, \mathbf{p}^{-k}) = \underset{\mathbf{p}^k}{\operatorname{argmax}} \quad & u^k(\mathbf{p}^k, \mathbf{p}^{-k}) \\ s.t. \quad & \sum_{i \in \mathcal{H}} \pi_i p_i^k \leq \sum_{i \in \mathcal{H}} \pi_i \tilde{p}_i^k \\ & 0 \leq p_i^k \leq 1. \end{aligned} \tag{3.27}$$

   $\overline{BR}^k(\tilde{\mathbf{p}}^k, \mathbf{p}^{-k})$ denotes the threshold best response of user k to $\mathbf{p}^{-k}$ assuming that the power investment in the optimization problem is less than or equal to the power investment under $\tilde{\mathbf{p}}^k$. Due to the linearity of $u^k(\mathbf{p}^k, \mathbf{p}^{-k})$ in $\mathbf{p}^k$, the problem becomes a linear optimization problem and a threshold strategy solving (3.27) always exists. Define

$$\gamma_k = \max\{j \in \mathcal{H} \cup \{h+1\} | \sum_{i=j}^{h} \pi_i q_i^k \geq \Delta\bar{P}^k\}.$$

   By convention, assume that $\sum_{i=h+1}^{h} \alpha_i = 0$ for any α_i, thus $\gamma_k = h+1$ for any feasible strategy $\mathbf{q}^k$.

Consider the following iterative algorithm

(a) Set $\bar{M} = \emptyset$, $\mathbf{p} = \mathbf{q}$.

(b) Choose $k \in \operatorname{argmin}_{l \in \mathcal{M} - \bar{M}} \gamma_l$ (if the set is not a singleton choose an arbitrary k in the set).

(c) If $\gamma_k < h + 1$ modify $\mathbf{q}^k$ to

$$\bar{q}_i^k = \begin{cases} q_i^k & \text{if } i < \gamma_k \\ \frac{\sum_{l=\gamma_k}^{h} \pi_l q_l^k - \Delta \bar{P}^k}{\pi_{\gamma_k}} & \text{if } i = \gamma_k \\ 0 & \text{if } i > \gamma_k \end{cases} \tag{3.28}$$

else set $\bar{\mathbf{q}}^k = \mathbf{q}^k$. Let $\mathbf{w}^k = \overline{BR}^k(\bar{\mathbf{q}}^k, \mathbf{p}^{-k})$.

(d) For any $i \in \mathcal{H}$ if $0 < w_i^k < 1$ set $w_i^k = 0$.

(e) Set $\mathbf{p} = (\mathbf{w}^k, \mathbf{p}^{-k})$, $\bar{M} = \bar{M} \cup \{k\}$. If $\bar{M} = \mathcal{M}$ terminate, else go to step 3b.

Let $\mathbf{w}$ denote the strategy profile that is achieved upon termination of the above algorithm. It can be readily seen that $\mathbf{w}$ is feasible and all states up to some threshold $\bar{i}$ are used with probability 1 (each state $i \leq \bar{i}$, $i \in \mathcal{H}$ is used by a single user with probability 1), while the remaining states are not used at all. The $0 - 1$ property follows from step 3d. The threshold state phenomenon can be easily proved, as otherwise one obtains a contradiction with the optimality of $\mathbf{w}^k$ in step 3c for some k.

4. Let $\bar{\mathbf{p}}$ be a Nash equilibrium such that $\mathbf{w}$ has the same transmission probability assignment as $\bar{\mathbf{p}}$ for states $i \leq \bar{i}$. Such $\bar{\mathbf{p}}$ is guaranteed to exist by considering a reduced game where only states $i > \bar{i}$ are considered and the power budgets are given by $\bar{P}^m - \sum_{i=\bar{i}}^{h} \pi_i w_i^m$, for all $m \in \mathcal{M}$.

We next show that the efficiency loss between $\hat{\mathbf{p}}$ and any $\bar{\mathbf{p}}$ is bounded by some fraction of $u(\hat{\mathbf{p}})$, where $\hat{\mathbf{p}}$ is the initial optimal threshold solution. To that end, we consider the efficiency loss incurred in the transition from $\hat{\mathbf{p}}$ to $\bar{\mathbf{p}}$ through the path $\hat{\mathbf{p}} \to \mathbf{q} \to \mathbf{w} \to \bar{\mathbf{p}}$.

$\hat{\mathbf{p}} \to \mathbf{q}$: Note that $u(\mathbf{q}) \geq u(\hat{\mathbf{p}})$, since

$$\begin{aligned}
u(\hat{\mathbf{p}}) &= \sum_{i\in\mathcal{H}} \pi_i \sum_l \hat{p}_i^l (R_i^l \prod_{k\neq l}(1-\hat{p}_i^k) - \lambda^l) \\
&\leq \sum_{i\in\acute{\mathcal{H}}} \pi_i \sum_l \acute{p}_i^l (R_i^l \prod_{k\neq l}(1-\acute{p}_i^k) - \lambda^l) \\
&\leq \sum_{i\in\acute{\mathcal{H}}} \pi_i \sum_l \acute{p}_i^l (R_i^l - \lambda^l) \\
&\leq \sum_{i\in\acute{\mathcal{H}}} \pi_i (R_i^{m_i} - \lambda^{m_i}) \sum_l \acute{p}_i^l \\
&\leq \sum_{i\in\acute{\mathcal{H}}} \pi_i (R_i^{m_i} - \lambda^{m_i}) = u(\mathbf{q}),
\end{aligned} \tag{3.29}$$

where $\sum_{l\in\mathcal{M}} \acute{p}_i^l \leq 1$ and $\acute{p}_i^m \leq \hat{p}_i^m$ for all $m \in \mathcal{M}$, $i \in \mathcal{H}$. The existence of $\acute{p}_i^m$ for all $m \in \mathcal{M}$, $i \in \mathcal{H}$ satisfying the first inequality follows by considering the aggregate utility maximization problem for each state $i \in \mathcal{H}$ separately and using the fact that at any optimal strategy $\mathbf{p}$, $\sum_l p_i^l \leq 1$, as Proposition 3.2.1 suggests.

$\mathbf{q} \to \mathbf{w}$: For any user $m \in \mathcal{M}$ if $p_i^k = 0$ for $k \neq m$ whenever $p_i^m > 0$, then $u^m(\mathbf{p}^m, \mathbf{p}^{-m}) = \sum_{i\in\mathcal{H}} \pi_i p_i^m (R_i^m - \lambda^m)$. Hence the payoff is a weighted linear combination of the power invested in different states. Now due to linearity, by assuming: (i) $\sum_{i\in\mathcal{H}} \pi_i p_i^m \geq \beta$ and (ii)$\sum_{i=j+1}^h \pi_i p_i^m \leq \alpha$ it follows that if user m modifies its strategy $\mathbf{p}^m$ to a strategy $\tilde{\mathbf{p}}^m$ such that transmission probabilities in states $i > j$ for a fixed j are set to zero, then $u^m(\mathbf{p}^m, \mathbf{p}^{-m}) - u^m(\tilde{\mathbf{p}}^m, \mathbf{p}^{-m}) \leq \frac{\alpha}{\beta} u^m(\mathbf{p}^m, \mathbf{p}^{-m})$. This follows since α amount of power which is utilized in lower weights contribute at most $\frac{\alpha}{\beta} u^m(\mathbf{p}^m, \mathbf{p}^{-m})$ to the user's payoff.

In step 3 of the algorithm, observe that modifying actions of a user does not affect the payoffs of other users. Let u^m be the initial payoff of user m in this step, then $\sum_{m\in\mathcal{M}} u^m = u(\mathbf{q})$. Denote the payoff of user m after step 3c by

$\hat{u}^m$, it follows that $\hat{u}^m \geq u^m(1 - \frac{\pi_{max}}{P_{min}})$ since for users satisfying $\Delta \bar{P}^m = 0$ the payoff actually increases when playing the best response. Also, for users with $\Delta \bar{P}^m > 0$, at least P_{min} is invested in the system and these users stop investing π_{max} amount of power in their worst states in step 3c (as in (3.28)). Then playing best response, the aggregate utility can only increase and it is larger than $u^m(1 - \frac{\pi_{max}}{P_{min}})$.

Similarly, in step 3d of algorithm, every user m using a state partially invests at least P_{min} amount of power and it stops investing at most π_{max} amount of power in its worst states. Denote its final payoff by $\bar{u}^m$. Then, $\bar{u}^m \geq \hat{u}^m(1 - \frac{\pi_{max}}{P_{min}}) \geq u^m(1 - \frac{\pi_{max}}{P_{min}})^2$. Since users who do not utilize any state partially do not modify their strategies, it follows that $u(\mathbf{q})\left(1 - \frac{\pi_{max}}{P_{min}}\right)^2 = \sum_{m \in \mathcal{M}} \left(1 - \frac{\pi_{max}}{P_{min}}\right)^2 u^m \leq \sum_{m \in \mathcal{M}} \bar{u}^m = u(\mathbf{w})$.

$\mathbf{w} \to \bar{\mathbf{p}}$: Finally, it can be seen that $u(\mathbf{w}) \leq u(\bar{\mathbf{p}})$. Since $\bar{p}_i^k = w_i^k$ for $i \leq \bar{i}$, $k \in \mathcal{M}$ and the contribution of remaining states to the aggregate utility can not be negative as in this case, at least one user can improve his payoff by setting the transmission probabilities in states $i > \bar{i}$ equal to zero and this contradicts with the fact that $\bar{\mathbf{p}}$ is a Nash equilibrium. To summarize,

$$u(\hat{\mathbf{p}})(1 - \frac{\pi_{max}}{P_{min}})^2 \leq u(\mathbf{q})(1 - \frac{\pi_{max}}{P_{min}})^2 \leq u(\bar{\mathbf{p}}) \tag{3.30}$$

Hence, $\frac{u(\hat{\mathbf{p}})}{u(\bar{\mathbf{p}})} \leq \frac{1}{(1 - \frac{\pi_{max}}{P_{min}})^2}$ as the claim suggests.

□

Recalling that there always exists an optimal threshold strategy profile (Theorem 3.2.1), immediately establishes the following.

Corollary 3.3.1. *Let $Q < 1$. Then $PoS(Q) \leq (1 - Q)^{-2}$.*

The above result implies that for P_{min} fixed, a finer quantization of the channel quality results in a better upper bound for the PoS, which approaches 1 as $\pi_{max} \to 0$.

It is also possible to obtain a lower bound on the PoS for any given Q as the next proposition suggests.

Proposition 3.3.2. *Let $Q < 1$. Then $PoS(Q) \geq \left(1 - \frac{1}{\lfloor \frac{1}{Q}+1 \rfloor}\right)^{-1}$.*

Proof. We present a parameterized example achieving the PoS lower bound for a given Q. Consider a game with two players m and k. Let Q be fixed and define $j = \lfloor \frac{1}{Q} + 1 \rfloor$. Choose P_{min} such that $P_{min} < \frac{1}{1+j}$ and $\pi_{max} = P_{min}Q$. Let $\mathcal{H} = \{1, 2, \ldots h\}$ and $h > j$. Consider the system with $\pi_i = \pi = \frac{P_{min}}{j} + \epsilon < \pi_{max}$ for sufficiently small ϵ at states $i \in \{1, 2, \ldots j\}$ (i.e., the best j states). Also assume that $\pi_h = \pi_{max}$ and the remaining π_i are chosen so that $\sum_{i \in \mathcal{H}} \pi_i = 1$. Let $\bar{P}^m = \bar{P}^k = P_{min}$, $\lambda^m = \lambda^k = \lambda$, where λ will be specified along the sequel. Choose rates as $(R_1^m, R_2^m \ldots R_h^m) = (10 + r_1, 10 + r_2 \ldots 10 + r_j, r_{j+1} \ldots r_h)$, $(R_1^k, R_2^k \ldots R_h^k) = (\bar{r}_1, \bar{r}_2 \ldots \bar{r}_j, \bar{r}_{j+1}, \ldots \bar{r}_h)$, for $r_h < \cdots < r_{j+1} < \lambda < r_j < \cdots < r_1 < \delta$ and $\bar{r}_h < \cdots < \bar{r}_{j+1} < \lambda < \bar{r}_j < \cdots < \bar{r}_1 < \delta$ for some δ. In this setting, the optimal solution is $\mathbf{p^k} = 0$, $\mathbf{p^m} = (1, \ldots, 1, 1-\bar{\epsilon}, 0, \ldots 0)$ where state j is the partially used state, and $\bar{\epsilon}$ is a function of ϵ that satisfies $\bar{\epsilon} \to 0$ as $\epsilon \to 0$. On the other hand, the best Nash equilibrium satisfies $p_i^m = 1$ for $i < j$ and $p_i^m = 0$ for $j \geq i$ whereas $p_j^k = 1$ and $p_i^k = 0$ for $i \neq j$ (where we choose λ such that $\lambda < \bar{\epsilon}\bar{r}_j$). Now choosing ϵ and δ sufficiently small (so that the contribution of the terms such as r_i and $\bar{r}_i$ to the aggregate utility is negligible) the aggregate utility is approximately $10\pi j$ in the central optimum, whereas it is $10\pi(j-1)$ in the best Nash equilibrium, hence

$$PoS(Q) \geq \frac{j}{j-1} = \left(1 - \frac{1}{\lfloor \frac{1}{Q} + 1 \rfloor}\right)^{-1}.$$

□

Observe that, for $Q \ll 1$ or for $Q = \frac{1}{n} + \epsilon$ for some integer n and $0 < \epsilon \ll 1$, $\lfloor \frac{1}{Q} + 1 \rfloor \approx \frac{1}{Q}$ and hence $PoS \geq \frac{1}{1-Q}$ for such Q. Note that $PoS(Q) \leq (1-Q)^{-2}$ by Corollary 3.3.1, the gap between the upper and lower bound remains a subject for on-going work.

We conclude this section by showing that the PoA is unbounded for any Q.

Proposition 3.3.3. *For any given Q, $PoA(Q) = \infty$*

Proof. The proof is constructive and follows from an example. Fix Q, $\mathcal{M}$ and consider a game instance with $\mathcal{H} = \{1, 2 \dots h\}$ for $h > Q^{-1}$. Let $R_i^m = R_i$, $\pi_i = \pi_{max} = \frac{1}{h}$, $\lambda^m = \lambda$, $\bar{P}^m = \frac{\pi_{max}}{Q}$ for every $m \in \mathcal{M}$ and $i \in \mathcal{H}$. Assume that $\sum_{i \in \mathcal{H}} \sqrt[M-1]{\frac{\lambda}{R_i}} = h - Q^{-1}$, and $R_i > \lambda$ for every $i \in \mathcal{H}$ (it is always possible to construct such a problem instance for a given Q by choosing h and $\{R_i\}_{i \in \mathcal{H}}$) properly. It can be seen that there exists an equilibrium $\mathbf{p}$ for every such game instance which satisfies

$$p_i^m = 1 - \sqrt[M-1]{\frac{\lambda}{R_i}} \quad \text{for every } m \in \mathcal{M},\, i \in \mathcal{H}, \tag{3.31}$$

which yields $u(\mathbf{p}) = \sum_{m \in \mathcal{M}} u^m(\mathbf{p}) = 0$ at this equilibrium. Note that the given strategy profile is feasible since for any $m \in \mathcal{M}$,

$$\begin{aligned} \sum_{i \in \mathcal{H}} \pi_i p_i^m &= 1 - \frac{1}{h} \sum_{i \in \mathcal{H}} \sqrt[M-1]{\frac{\lambda}{R_i}} \\ &= 1 - \frac{h - Q^{-1}}{h} = \frac{\pi_{max}}{Q} = \bar{P}^m \end{aligned} \tag{3.32}$$

The aggregate utility at an optimal solution is obviously greater than 0, as $R_i > \lambda$ for every $i \in \mathcal{H}$, leading to an unbounded PoA.

□

The above result indicates that despite technological enhancements (which result in a low Q), the network can still arrive at bad-quality equilibria with unbounded performance loss. This negative result emphasizes the significance of mechanisms or distributed algorithms, which preclude such equilibria. We address this important design issues in the next section.

3.4 Best-Response Dynamics

A Nash equilibrium point for our game represents a strategically stable working point, from which no user has incentive to deviate unilaterally. In this section we address the question of if and how the system arrives at an equilibrium, which is of great importance from the system point of view. As discussed in Section 3.3, the set of equilibria can vary with respect to performance. Hence, we conclude this section by briefly discussing how to lead the system to good quality equilibria.

3.4.1 Convergence Properties

In Chapter 2 best response dynamics was defined. In this chapter we discuss the use of best response dynamics to ensure convergence to an equilibrium of the scheduling game.

The best-response mechanism, is not guaranteed to converge to an equilibrium in our game without imposing additional assumptions. We specify below the required assumptions. Our convergence analysis relies on establishing the existence of a *potential* function under a certain condition, which we refer to as the *rate alignment* condition. The rate alignment condition is defined as follows.

Assumption 3.4.1 (Rate Alignment Condition). *The set of user rates* $\{R_i^m\}_{i\in\mathcal{H},m\in\mathcal{M}}$ *is said to be aligned if there exist per-user positive coefficients* $\{c^m\}_{m\in\mathcal{M}}$ *and per-state positive constants* $\{R_i\}_{i\in\mathcal{H}}$ *such that*

$$R_i^m = c^m R_i \tag{3.33}$$

for every $m \in \mathcal{M}$ *and* $i \in \mathcal{H}$. *The rate alignment condition is satisfied if user rates are aligned.*

The coefficient c^m above reflects user m's relative quality of transmissions, which is affected mainly by its transmission power and location relative to the base station. While the rate alignment condition might not hold for general and heterogeneous mobile systems, a special case of interest which satisfies (3.33) is the *symmetric-rate* case, i.e., $c^m = c$ for every $m \in \mathcal{M}$. Rate-symmetry is expected in systems where mobiles use the same technology (transmission power and coding scheme), and where "local" conditions, such as distance from the base station, are similar.

Theorem 3.4.1. *Under Assumption 3.4.1, our game is an ordinal potential game* *with a potential function given by*

$$\phi(\mathbf{p}) = -\sum_{i=1}^{h} \pi_i R_i \prod_{k\in\mathcal{M}} (1-p_i^k) - \sum_{i=1}^{h} \sum_{k\in\mathcal{M}} \pi_i \frac{\lambda^k}{c^k} p_i^k \tag{3.34}$$

Proof. Consider two different strategy profiles $\mathbf{p}$, $\mathbf{q}$ such that

$$\begin{aligned} \mathbf{p} &= (\mathbf{p}^m, \mathbf{p}^{-m}) \\ \mathbf{q} &= (\mathbf{q}^m, \mathbf{p}^{-m}) \end{aligned} \tag{3.35}$$

Observe that

$$\begin{aligned} \phi(\mathbf{p}) - \phi(\mathbf{q}) &= -\sum_i \frac{R_i^m}{c^m} \pi_i ((1-p_i^m) - (1-q_i^m)) \prod_{k\neq m} (1-p_i^k) - \sum_i \frac{\lambda^m}{c^m} \pi_i (p_i^m - q_i^m) \\ &= \frac{1}{c^m} (\sum_i \pi_i p_i^m (R_i^m \prod_{k\neq m} (1-p_i^k) - \lambda^m) - \sum_i \pi_i q_i^m (R_i^m \prod_{k\neq m}(1-p_i^k) - \lambda^m)) \\ &= \frac{1}{c^m} (u^m(\mathbf{p}) - u^m(\mathbf{q})) \end{aligned} \tag{3.36}$$

Since $c^m > 0$, the above equality implies that the game is an ordinal potential game. □

Theorem 3.4.1 also indicates that the game is also a weighted potential game where weight of each player is equal to $\frac{1}{c^m}$ hence if $c^m = 1$ for every $m \in \mathcal{M}$, then the game is an exact potential game.

In the following, we assume that users restrict themselves to threshold strategies (see Definition 3.2.2). Since our focus is on best response dynamics this assumption is natural as whenever a user updates its strategy there always exists a threshold strategy that maximizes the performance of that user. Moreover, it turns out that despite the fact that the game we are interested in is a continuous game, convergence takes place in finitely many update periods if users only utilize threshold strategies.

Throughout this section, we assume that users may update their strategy at a slower time-scale compared to their transmission rates. For simplicity, we assume that user updates may take place only every T_E time slots and refer to T_E as the *update period.*

For our convergence result, we require the following set of assumptions.

Assumption 3.4.2.

(i) The user population is fixed.

(ii) Rates are aligned (see Assumption 3.4.1).

(iii) The transmission-success probabilities $\prod_{k \neq m}(1-p_i^k), i \in \mathcal{H}$ *are perfectly estimated by each user before each update.*

Consider the following mechanism.

Definition 3.4.1 (Round-Robin BR Dynamics)**.** *Strategy updates take place in a round-robin manner and at each update period only a single user may modify its strategy. The user who is chosen for update modifies its strategy to a threshold strategy from the set* $BR^m(\mathbf{p}^{-m})$*, if the modification strictly improves its utility.*

As the utility of each user is linear in its actions and the strategy space is a polyhedron best responses of users can be found by solving a linear program. Hence, $BR^m(\mathbf{p}^{-m})$ always contains an extreme point of E^m. As extreme points correspond to threshold strategies in the system there always exists a threshold strategy in $BR^m(\mathbf{p}^{-m})$.

The next lemma suggests that our game is a finite game if users are restricted to playing threshold strategies, and further provides a bound on the number of threshold strategy profiles for any given game instance.

Lemma 3.4.1. *For a given game instance with* M *users and* h *states the number of threshold strategy profile is bounded by* $(2e)^{M(h+1)}$.

Proof. Observe that for any user $m \in \mathcal{M}$, its threshold strategies are the extreme points of the feasible region E^m. Similarly each threshold strategy profile $\mathbf{p}$ corresponds to an extreme point of the joint feasible region E. The idea behind the proof is to upper bound the number of extreme points of the joint feasible region or equivalently the number of threshold strategies in the system.

In general, a polyhedral region that is a subset of $\mathbb{R}^n$ and is defined by k constraints is represented by the polyhedron $\{x| \, A\mathbf{x} \leq b\}$, where A is a $k \times n$ matrix and $b \in \mathbb{R}^k$ is a constant vector. Now, at any extreme point of this polyhedron, at least n linearly independent constraints are active, and such constraints define extreme points, hence there are at most $\binom{k}{n}$ threshold strategies.

In our problem, each user has h decision variables and a total of $2h+1$ constraints. Hence, in total we have $M(2h+1)$ constraints and Mh variables. Thus, the number of threshold strategies is bounded by

$$\begin{aligned}\binom{M(2h+1)}{Mh} &= \binom{M(2h+1)}{M(h+1)} \\ &\leq \left(\frac{eM(2h+1)}{M(h+1)}\right)^{M(h+1)} \leq (2e)^{M(h+1)},\end{aligned} \tag{3.37}$$

where the first inequality follows from the inequality $\binom{m}{n} \leq (\frac{em}{n})^n$. □

Relying on the above lemma, we have the following convergence result.

Theorem 3.4.2. *Let Assumption 3.4.2 hold. Then Round-Robin best response dynamics converge in finitely many update periods to an equilibrium point. In addition, the number of update periods required for convergence is upper bounded by* $M(2e)^{M(h+1)}$.

Proof. Utilizing Round-Robin best response dynamics players are restricted to playing threshold strategies after first M updates. By restricting users to threshold strategies, the underlying game becomes a finite game (i.e., the game has a finite action space as Lemma 3.4.1 suggests), with a potential function given by (3.34). As such, the finite improvement property (FIP) in potential games (see Chapter 2), holds: Any sequence of updates, which results in strict improvement in the utility of the user who is modifying its strategy, terminates after finitely many updates. Moreover, each finite improvement path terminates at a Nash equilibrium.

By Lemma 3.4.1 the number of threshold strategies is bounded by $(2e)^{M(h+1)}$. Observing that no strategy profile can occur more than once during the updates (as the potential strictly increases with each update), this implies that number of updates required for convergence is bounded by $(2e)^{M(h+1)}$. By Definition 3.4.1, a user who can strictly improve its utility can be found in every M update periods. Hence, the number of update periods required for convergence is bounded by $M(2e)^{M(h+1)}$. $\square$

We emphasize that the restriction to threshold strategies is commensurate with the users' best interest. Not only there always exists such best-response strategy, but also it is reasonably easier to implement.

We discuss next some important considerations regarding the presented mechanism and the assumptions required for its convergence. The best response dynamics as described in Definition 3.4.1, requires synchronization between the mobiles, which can be done centrally by the base station or by a supplementary distributed procedure. We emphasize that the schedule of updates is the only item that needs to be centrally determined. Users are free to choose their strategies according to their own preferences, which are usually private information. Assumption 3.4.2(iii) entails the notion of a quasi-static system, in which each user responds to the steady state reached after preceding user update. This approximates a natural scenario where users update their transmission probabilities at much slower time-scales than their respective transmission rates. An implicit assumption here is that the update-period T_E is chosen large enough to allow for accurate estimation of the transmission-success probabilities. We leave the exact determination of T_E for future work. We emphasize that users need not be aware of the specific transmission probabilities p_i^m of other users. Indeed, in view of (3.4), only the transmission-success probabilities $\prod_{k\neq m}(1-p_i^k)$, $i \in \mathcal{H}$ are required. These can be estimated by sensing the channel and keeping track of idle slots.

A last comment relates to the rate-alignment condition. The convergence results in this section rely on establishing a potential function for the underlying game, which is shown to exist when rates are aligned. In next chapter, we show that in a system of three states or more, the alignment condition is not only sufficient, but also necessary for the existence of a continously differentiable potential function. This suggests that novel methods would have to be employed for establishing convergence of dynamics under more general assumptions.

Next we relax the deterministic update schedule (round-robin updates) of the previous theorem. Consider the following set of dynamics,

Definition 3.4.2 (Randomized Best Response Dynamics). *Let* $f_{\mathcal{M}} : \mathcal{M} \to [0,1]$ *be a probability mass function defined on set* $\mathcal{M}$ *such that* $f_{\mathcal{M}}(k) > 0$ *for all* $k \in \mathcal{M}$. *Start from a strategy profile* $\mathbf{p}$. *At each update period,*

1. *Randomly choose one user in* $\mathcal{M}$ *using distribution* $f_{\mathcal{M}}$.
2. *Let* m *be the user chosen in the previous step, if* m *has an estimation of* $\mathbf{p}^{-m}$ *set* $\hat{\mathbf{p}}^m$ *to a threshold best response of user* m *in* $BR^m(\mathbf{p}^{-m})$, *else set* $\hat{\mathbf{p}}^m = \mathbf{p}^m$.
3. *If user* m *has better payoff utilizing* $\hat{\mathbf{p}}^m$ *then let* $\mathbf{p} = (\hat{\mathbf{p}}^m, \mathbf{p}^{-m})$, *otherwise do not modify* $\mathbf{p}$.

Theorem 3.4.3. *Let Assumption 3.4.2 hold. Then the randomized best response dynamics converge to a Nash equilibrium of the game in finitely many update periods with probability* 1.

Proof. As in Theorem 3.4.2 game is a finite ordinal potential game, and has the finite improvement property.

Let K be the length of the longest improvement path, since the game is a finite game there are finitely many improvement paths and K is well defined. Using the randomized best response dynamics at each step assuming that a Nash equilibrium is not reached, with probability at least $\min_{k\in\mathcal{M}} f_{\mathcal{M}}(k) > 0$ a user who has incentive to modify his strategy is chosen for update. The expected number of updates to reach to a Nash equilibrium (N_{NE}) is smaller than the expected time to observe K successes in a Bernoulli process (T_K) with success probability $\min_{k\in\mathcal{M}} f_{\mathcal{M}}(k)$. The latter is simply $\frac{K}{\min_{k\in\mathcal{M}} f_{\mathcal{M}}(k)}$, hence

$$E[N_{NE}] \leq E[T_K] = \frac{K}{\min_{k\in\mathcal{M}} f_{\mathcal{M}}(k)}. \tag{3.38}$$

Thus, with probability 1 a Nash equilibrium is achieved in finitely many updates. By assumption 3.4.2 it follows that convergence to a Nash equilibrium happens in finitely many time slots with probability 1. Now the result follows since when a Nash equilibrium is achieved, none of the users have any incentive to deviate from the Nash equilibrium. □

Theorems 3.4.2 and 3.4.3 imply that using best response update rules and threshold strategies convergence to an equilibrium takes place. Also observe that the equilibrium reached as a result of this update rule is a threshold strategy profile. This leads us to the following corollary.

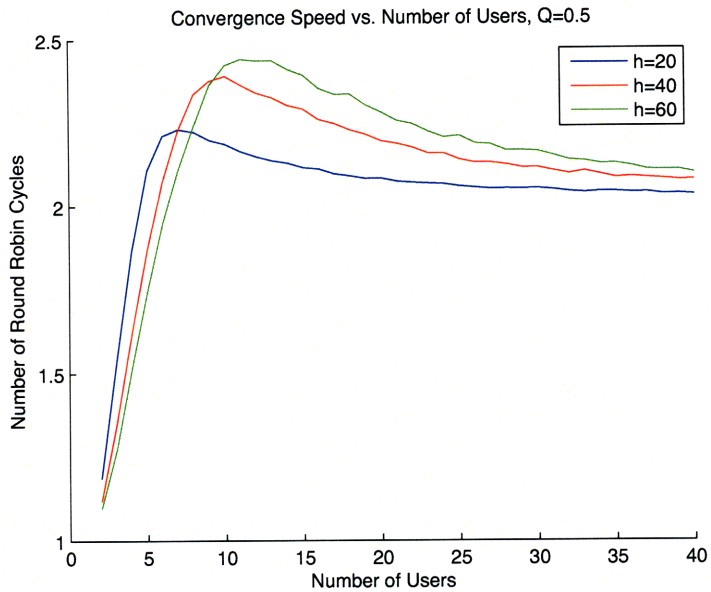
Corollary 3.4.1. *Let rate alignment condition hold. Then, there exists a threshold strategy profile that is also a Nash equilibrium of the game.*

3.4.2 Simulations

The objective in this section is to study through simulations the convergence properties of sequential best-response dynamics. More specifically, we wish to examine the dependence of convergence time on several factors, such as the number of users in the system, the number of states, and the technology factor Q. In all our experiments, we consider a relaxed version of Assumption 3.4.2, where the rate-alignment condition (Assumption 3.4.2(ii)) is not enforced.

The specific setup for our simulations is as follows. We assume that $\pi_i = \frac{1}{h}$ for every $i \in \mathcal{H}$. For given Q, M and h, we construct a significant number of game instances (10000) by randomly choosing in each instance the power constraints $\bar{P}^m$, the tradeoff coefficient λ^m and the associated rates R_i^m for every $m \in \mathcal{M}$, $i \in \mathcal{H}$. We simulate each game instance, and examine the average convergence speed, measured in the number of round-robin iterations (recall that in a round-robin iteration, each user updates its strategy at most once). Figure 3-2 presents the convergence speed results for $Q = 0.5$ and $Q = 0.95$, as a function of the number of users in the system. For the given value of Q, we consider three cases for which number of states, h, is different.

As seen in Figure 3-2, the average number of Round-Robin cycles required for convergence is less than three on average. We emphasize that all game instances



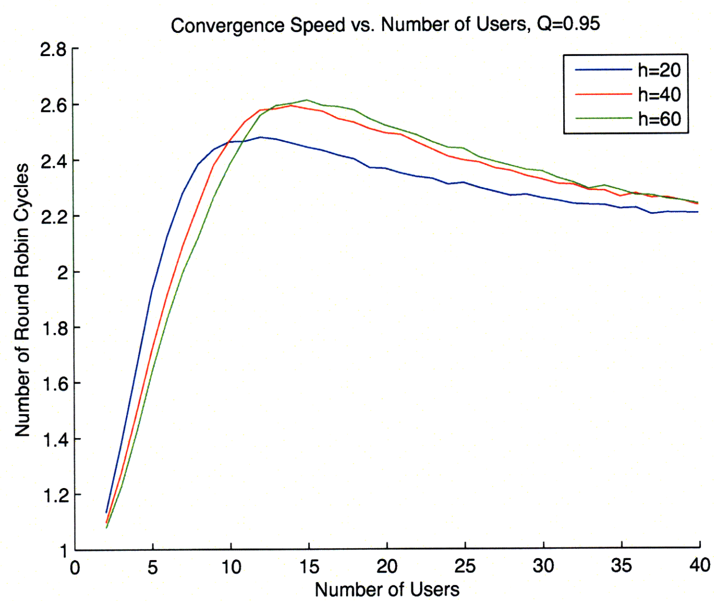


Figure 3-2: Convergence speed as a function of the number of users.

converge without requiring the rate-alignment condition, indicating the possibility to exclude this condition in future analysis of best-response convergence. It can be seen that increasing Q slows down the convergence speed slightly. We observe that all graph curves initially increase as a function of the number of users, and after some point gradually decrease until reaching a fixed number of iterations. This interesting phenomenon can be intuitively justified as follows: When the number of users is relatively small, there is less competition on each state, and convergence is fast. At the other extreme, when the number of users is larger than some threshold, then there are more users who can fully utilize states at the first iteration (see Definition 3.2.1), thereby decreasing the competition at subsequent iterations and leading to faster convergence.

3.4.3 Obtaining Desirable Equilibria

We conclude this section by briefly discussing possible means for obtaining high-quality equilibria in terms of the aggregate utility (3.14). Theorem 3.4.2 introduces a scheme (or mechanism) which assures converge to an equilibrium point in a finite number of steps. However, the resulting equilibrium can be of low quality. Proposition 3.3.1 suggests that if the system is initiated at some threshold strategy profile, then there exists an equilibrium, performance of which cannot deviate by much, compared to the performance at the initial working point. Consequently, one may consider an iterative *hybrid* algorithm, in which a network-management entity forces some initial working-point (a good quality threshold strategy profile), waits enough time until convergence, and if the equilibrium performance is unsatisfactory, enforces a different working point, until reaching a satisfactory equilibrium. The algorithm would rely on the fast convergence to an equilibrium, which is demonstrated in all our simulations, and allows to consider numerous initial working points in plausible time-intervals. The precise requirements and properties of such an algorithm, as well as the means for choosing and enforcing initial working-points, remain as a challenging future direction.

Chapter 4

Potential Games and Projections to the Set of Potential Games

In this chapter we focus on the properties of the set of exact potential games. The main objective of this chapter is to characterize the properties of this set and quantify the "distance" of an arbitrary game to the set of exact potential games. We also provide a condition for checking existence of an ordinal potential function in continuous games, and relate it to the scheduling game described in the previous chapter.

The rest of this chapter is organized as follows. In Section 4.1 we consider the sets of exact, weighted and ordinal potential games and present properties of these sets. In particular, we study some topological properties of these sets. In Section 4.2 we present a necessary condition for the existence of an ordinal potential function in continuous games. Using this condition we prove that the scheduling game presented in Chapter 3 does not have a twice continuously differentiable ordinal potential function unless the rate alignment condition (Assumption 3.4.1) holds. In Section 4.3 we discuss different approaches for projecting a game to the set of exact potential games. We also discuss the distributed implementation of projections and present simulation results.

4.1 Sets of Potential Games

In this section we restrict ourselves to the study of finite games, with set of players $\mathcal{M} = \{1, \dots, M\}$ and strategy spaces $E^m = \{1, \dots, h_m\}$ for all $m \in \mathcal{M}$. In our discussion of topological properties of sets of potential games we assume that a fixed joint strategy space $E = \prod_{m\in\mathcal{M}} E^m$ is given, and the set of games defined on this joint strategy space is of interest.

We denote the set of games with player set $\mathcal{M}$ and joint strategy space E as

$$\mathcal{G}_{\mathcal{M},E} = \{\langle \mathcal{M}, \{E^m\}_{m\in\mathcal{M}}, \{u^m\}_{m\in\mathcal{M}}\rangle | u^m \in C_0 \text{ for all } m \in \mathcal{M}\}. \tag{4.1}$$

It is clear that there is a bijective correspondence between $\mathcal{G}_{\mathcal{M},E}$ and the set C_0^M as $u_{all} = \{u^m\}_{m\in\mathcal{M}} \in C_0^M$ and each $u_{all} \in C_0^M$ uniquely defines a different game instance in $\mathcal{G}_{\mathcal{M},E}$. In the following we use the product space of utilities, C_0^M, to study the space of games. We define the dimension and convexity of the set of games using the properties of C_0^M. We use the terms space of games and the product space of utilities interchangeably.

In Chapter 2 it was discussed that each function in C_0 has a vector representation, hence an alternative representation for C_0 is $\mathbb{R}^{|E|}$. Using this, it can be seen that the dimension of C_0 is equal to $|E| = \prod_{m\in\mathcal{M}} h_m$.

We define the dimension of space of games with joint strategy space E, and set of players $\mathcal{M}$ as the dimension of the product space of utility functions of all players, C_0^M. Dimension of this product space is the sum of the dimension of all spaces in the product. The following lemma characterizes the dimension of the space of games as a function of E and $\mathcal{M}$.

Lemma 4.1.1. *The dimension of the space of games with set of players $\mathcal{M}$, and joint strategy space E is $M \prod_{m\in\mathcal{M}} h_m$.*

Proof. The dimension of C_0 is $|E| = \prod_{m\in\mathcal{M}} h_m$. Therefore, the dimension of C_0^M, or the dimension of space of games with joint strategy space E, can be given by $|\mathcal{M}||E| = M \prod_{m\in\mathcal{M}} h_m$. □

Using the vector representations of utilities we can define the dimension of a a subspace in C_0 or C_0^M as the dimension of the corresponding vector space. Let payoff function of player m be represented by the column vector u^m. Then the column vector

$$u_{all} = \begin{bmatrix} u^1 \\ u^2 \\ \vdots \\ u^M \end{bmatrix} \tag{4.2}$$

belongs to C_0^M. This suggests that each subspace of C_0^M can be studied as a subspace of $\mathbb{R}^{M|E|}$, and dimension of a subspace of games can be calculated from the dimension of the corresponding subspace of $\mathbb{R}^{M|E|}$. In this section, we use this approach to find the dimension of the set of exact potential games.

We next define the notion of convexity that is relevant to our projection framework. We define the convexity of the set of games by making use of the underlying set of utility functions.

Definition 4.1.1. *Let $B \subset \mathcal{G}_{\mathcal{M},E}$. The set B is said to be convex if and only if for any two game instances $\mathcal{G}_1$, $\mathcal{G}_2 \in B$ with collections of utilities $u = \{u^m\}_{m\in\mathcal{M}}$, $v = \{v^m\}_{m\in\mathcal{M}}$ respectively*

$$\langle \mathcal{M}, \{E^m\}_{m\in\mathcal{M}}, \{\alpha u^m + (1-\alpha)v^m\}_{m\in\mathcal{M}}\rangle \in B, \tag{4.3}$$

for all $\alpha \in [0,1]$.

Note that with this definition the convexity of $\mathcal{G}_{\mathcal{M},\{E^m\}_{m\in\mathcal{M}}}$ follows trivially.

We next obtain results on the dimension of the sets of potential games and the convexity properties of these sets.

In Theorem 2.1.2, a condition for a game to be an exact potential game is stated. This theorem implies that a game is an exact potential game if and only if for any simple closed cycle, γ, $I(\gamma, u_{all}) = 0$, where $I(\gamma, u_{all})$ denotes the aggregate change in the payoff over all steps of γ (see Chapter 2).

Enumerating all the simple closed cycles of the game, a necessary and sufficient condition for existence of an exact potential function can be written as a linear equation

$$Lu_{all} = 0 \tag{4.4}$$

for some matrix L. Here Lu_{all} is a vector, ith row of which gives the condition $I(\gamma_i, u_{all}) = 0$ for the ith simple closed cycle γ_i. Note that this is possible since $I(\gamma_i, u_{all}) = 0$ is a linear function of the payoffs in the game for any γ_i. It follows from (4.4) that the set of exact potential games is a subspace in C_0^M.

The dimension of the set of exact potential games is given by the dimension of the null space of L. For $|\mathcal{M}| = 3$ and $h_m = h_0$ for all $m \in \mathcal{M}$, Figure 4.1 shows the dimension of exact potential games and dimension of all games for different h_0.

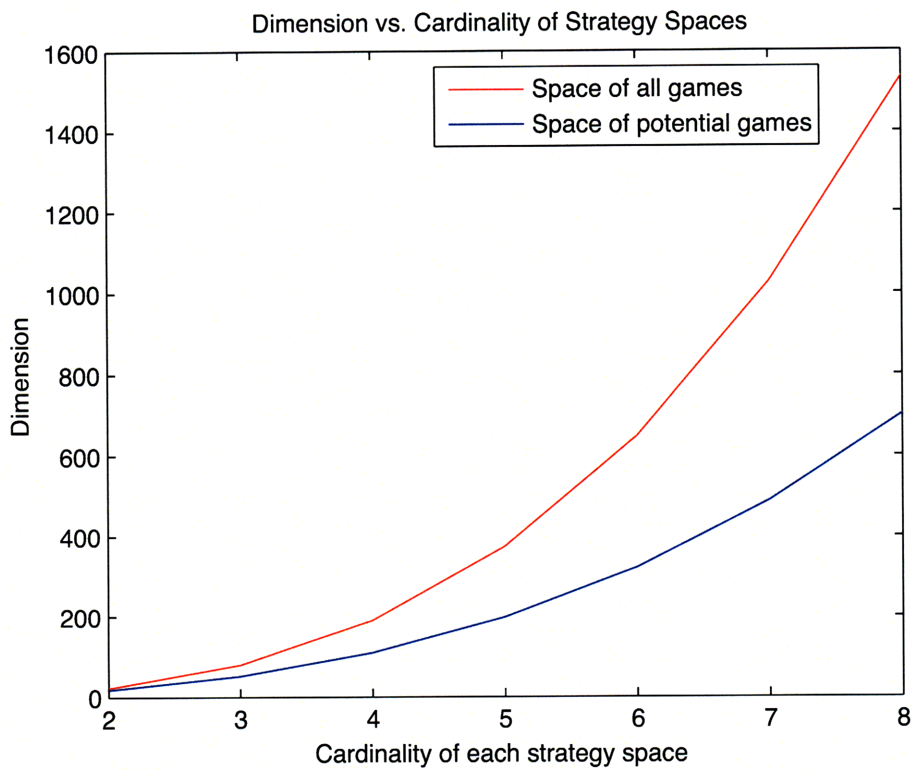


Figure 4-1: Dimensions of set of all games and exact potential games for 3 players with same number of strategies

Using data fitting tools in Figure 4.1, it can be seen that the dimension of the exact potential games for the presented example is given by $h_0^m + mh_0^{m-1} - 1$. The following theorem formalizes this result by providing an exact expression for the dimension of the set of exact potential games.

Theorem 4.1.1. *The dimension of the set of exact potential games is given by,*

$$\prod_{m\in\mathcal{M}} h_m + \sum_{m\in\mathcal{M}} \prod_{k\in\mathcal{M}, k\neq m} h_k - 1. \tag{4.5}$$

Proof. Note that potential function for a game is unique up to a constant. Fix $\mathbf{p}_0 \in E$ and consider the set $B = \{\phi | \phi(\mathbf{p}_0) = 0, \phi \in C_0\}$. Each potential game has a unique potential function in B. For all $\mathbf{p} \neq \mathbf{p}_0$ define $\phi_{\mathbf{p}} : E \to \mathbb{R}$ such that

$$\phi_{\mathbf{p}}(\mathbf{q}) = \begin{cases} 1 & \text{if } \mathbf{q} = \mathbf{p} \\ 0 & \text{otherwise} \end{cases} \tag{4.6}$$

Clearly $\{\phi_{\mathbf{p}} | \mathbf{p} \in E, \mathbf{p} \neq \mathbf{p}_0\}$ is a set of orthogonal basis vectors of B.

The dimension of the set of exact potential games is equivalent to the dimension of,

$$U = \{u | u = \{u^m\}_{m\in\mathcal{M}}, \text{ there exists } \phi \in B \text{ such that } D_m u^m = D_m \phi, \text{for all } m \in \mathcal{M} \}.$$

Consider the system of equations for a fixed $\phi_{\mathbf{p}}$.

$$D_m u^m = D_m \phi_{\mathbf{p}} \quad \text{for all } m \in \mathcal{M}. \tag{4.7}$$

The kernel of D_m has dimension $\prod_{k\neq m} h_k$ as it can be seen from Lemma 2.2.1. Therefore, the kernel of the linear system in (4.7) has a dimension $\sum_{m\in\mathcal{M}} \prod_{k\neq m} h_k = K$. Consider collection of utilities $v_{\mathbf{p}} = \{v_{\mathbf{p}}^m\}_{m\in\mathcal{M}} = \{\phi_{\mathbf{p}}\}_{m\in\mathcal{M}}$. Note that $v_{\mathbf{p}}$ is a solution of (4.7), hence the set of solutions of the system in (4.7) is nonempty and has dimension equal to K.

Let $\{b_i\}_{i=1}^K$ be a basis for the kernel of the linear system in (4.7), where $b_i = \{b_i^m\}_{m\in\mathcal{M}} \in C_0^M$. Note that $D_m b_i^m = 0$ for all $m \in \mathcal{M}$ as b_i belongs to the kernel of the linear system. We claim that $\{v_{\mathbf{p}}\}_{\mathbf{p}\neq\mathbf{p}_0} \cup \{b_i\}_{i=1}^{K}$ is a basis for U. First we prove linear independence of this collection. Assume the linear independence fails, then for some α_i, $\beta_{\mathbf{p}}$ not identically equal to 0,

$$\sum_{i=1}^{K} \alpha_i b_i^m + \sum_{\mathbf{p}\neq\mathbf{p}_0} \beta_{\mathbf{p}} v_{\mathbf{p}}^m = 0 \tag{4.8}$$

for all $m \in \mathcal{M}$. Note that as $\{b_i\}_{i=1}^{K}$ constitute a basis for some subspace it follows that there exists a $\beta_{\mathbf{p}} \neq 0$ for (4.8) to hold.

Multiplying (4.8) by D_m and substituting $v_{\mathbf{p}}^m = \phi_{\mathbf{p}}$,

$$D_m \sum_{\mathbf{p}\neq\mathbf{p}_0} \beta_{\mathbf{p}} \phi_{\mathbf{p}} = 0, \tag{4.9}$$

for all $m \in \mathcal{M}$ as b_i^m is in the kernel of D_m by definition. Since this is true for all m it follows that

$$\sum_{\mathbf{p}\neq\mathbf{p}_0} \beta_{\mathbf{p}} \phi_{\mathbf{p}} = c \tag{4.10}$$

for some constant c. On the other hand

$$\sum_{\mathbf{p}\neq\mathbf{p}_0} \beta_{\mathbf{p}} \phi_{\mathbf{p}}(\mathbf{p}_0) = 0 \tag{4.11}$$

by definition of $\phi_{\mathbf{p}}$ and hence $c = 0$. Thus, (4.10) implies that

$$\sum_{\mathbf{p}\neq\mathbf{p}_0} \beta_{\mathbf{p}} \phi_{\mathbf{p}} = 0 \tag{4.12}$$

but this contradicts with the fact that $\{\phi_{\mathbf{p}}\}_{\mathbf{p}\neq\mathbf{p}_0}$ is a basis for B. Thus $\{v_{\mathbf{p}}\}_{\mathbf{p}\neq\mathbf{p}_0} \cup \{b_i\}_{i=1}^{K}$ is a linearly independent collection.

Next we show that any element of U can be obtained as a linear combination of elements in the collection $\{v_{\mathbf{p}}\}_{\mathbf{p}\neq\mathbf{p}_0} \cup \{b_i\}_{i=1}^{K}$. Let $u \in U$ be a collection of utilities with the corresponding potential $\Phi \in B$. Consider the vector $u - \sum_{\mathbf{p}\in E} \Phi(\mathbf{p}) v_{\mathbf{p}}$ in

C_0^M. Note that for all $m \in \mathcal{M}$,

$$\begin{aligned} D_m(u^m - \sum_{\mathbf{p} \in E} \Phi(\mathbf{p}) v_{\mathbf{p}}^m) &= D_m(u^m - \sum_{\mathbf{p} \in E} \Phi(\mathbf{p}) \phi_{\mathbf{p}}) \\ &= D_m(u^m - \Phi) = 0, \end{aligned} \tag{4.13}$$

where we used the fact that $\sum_{\mathbf{p} \in E} \Phi(\mathbf{p}) \phi_{\mathbf{p}} = \Phi$. Thus $u - \sum_{\mathbf{p} \in E} \Phi(\mathbf{p}) v_{\mathbf{p}}$ lies in the kernel of the system given in (4.7). Therefore, u can be obtained as a linear combination of elements in the collection $\{v_{\mathbf{p}}\}_{\mathbf{p} \neq \mathbf{p}_0} \cup \{b_i\}_{i=1}^K$.

It follows that the dimension of the set of exact potential games is the cardinality of the collection $\{v_{\mathbf{p}}\}_{\mathbf{p} \neq \mathbf{p}_0} \cup \{b_i\}_{i=1}^K$. Noting that $|\{v_{\mathbf{p}}\}_{\mathbf{p} \neq \mathbf{p}_0}| = \prod_{m \in \mathcal{M}} h_m - 1$ and $K = \sum_{m \in \mathcal{M}} \prod_{k \neq m} h_k$ the dimension of the set of exact potential games is obtained as

$$\prod_{m \in \mathcal{M}} h_m + \sum_{m \in \mathcal{M}} \prod_{k \neq m, k \in \mathcal{M}} h_k - 1. \tag{4.14}$$

□

We proceed by studying the convexity properties of the sets of potential games.

Theorem 4.1.2. *The set of weighted potential games and the set of ordinal potential games are not convex.*

Proof. We prove the claim by showing the convex combination of two weighted potential games is not an ordinal potential game. This implies that the sets of both weighted and ordinal potential games are nonconvex since every weighted potential game is an ordinal potential game.

In Table 4.1 we present the payoffs and the potential in a two player game, $\mathcal{G}_1$, where each player has two strategies. Given strategies of both players the first table shows payoffs of players (the first number denotes the payoff of the first player), the second table shows the corresponding potential function. In both tables the first column stands for actions of first player and top row stands for actions of second player. Note that this game is a weighted potential game with weights $w^1 = 1$, $w^2 = 3$.

	A	B
A	0,0	0,4
B	2,0	8,6

	A	B
A	0	12
B	2	20

Table 4.1: Payoffs and potential in $\mathcal{G}_1$

Now with the same notations we define another game $\mathcal{G}_2$ as in Table 4.2. Note that this game is also a weighted potential game with weights $w^1 = 3$, $w^2 = 1$.

	A	B
A	4,2	6,0
B	0,8	0,0

	A	B
A	20	18
B	8	0

Table 4.2: Payoffs and potential in $\mathcal{G}_2$

We consider a game $\mathcal{G}_3$ in which the payoffs are averages (hence convex combinations) of payoffs of $\mathcal{G}_1$ and $\mathcal{G}_2$.

	A	B
A	2,1	3,2
B	1,4	4,3

Table 4.3: Payoffs in $\mathcal{G}_3$

Note that in this game strategy profiles satisfy the preference relations

$$(A, A) > (B, A) > (B, B) > (A, B) > (A, A), \tag{4.15}$$

and the preference relations are strict. Thus this game has a weak improvement cycle and hence it is not an ordinal potential game.

The above example shows that the sets of weighted and ordinal potential games with two players each of which has two strategies is nonconvex. For games in which the joint strategy space is larger the result immediately follows by noting that any game derived from the games in Tables 4.1 and 4.2 by setting the potential functions on the newly introduced strategy profiles equal to 0 and deriving utilities accordingly, is a weighted potential game. However, the convex combinations of these games are not ordinal potential games as the weak improvement cycle in Table 4.3 is preserved in the convex combination. □

The next theorem shows that the set of exact potential games is convex.

Theorem 4.1.3. *The set of exact potential games is convex.*

Proof. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be exact potential games with set of players $\mathcal{M}$, and joint strategy spaces E. Denote the collection of utilities in $\mathcal{G}_1$ and $\mathcal{G}_2$ by $u = \{u^m\}_{m\in\mathcal{M}}$ and $v = \{v^m\}_{m\in\mathcal{M}}$ respectively. Since these games are exact potential games it follows that for all $m \in \mathcal{M}$,

$$D_m u^m = D_m \phi_1 \tag{4.16}$$

and

$$D_m v^m = D_m \phi_2 \tag{4.17}$$

for some $\phi_1, \phi_2 \in C_0$. Consider the convex combination of the utilities $\nu = \{\nu^m\}_{m\in\mathcal{M}} = \{\alpha u^m + (1-\alpha)v^m\}_{m\in\mathcal{M}}$ for $\alpha \in [0,1]$. It follows that for all $m \in \mathcal{M}$,

$$D_m \nu^m = D_m(\alpha\phi_1 + (1-\alpha)\phi_2) \tag{4.18}$$

by the linearity of the operator D_m. Thus, the game with strategy space E, set of players $\mathcal{M}$, collection of utilities ν is an exact potential game, and as α is arbitrary the set of exact potential games is convex. □

4.2 Conditions on the Existence of Differentiable Ordinal Potential in Continuous Games

In this section we obtain a necessary condition for the existence of a continuously differentiable potential function for continuous games. We also use this result to show that the scheduling game introduced in Chapter 3 does not have a twice continuously differentiable ordinal potential.

In this section we assume that for all $m \in \mathcal{M}$, $E^m \subset \mathbb{R}^h$ for some $h \in \mathbb{Z}_+$ is a compact and nonempty set and $u^m(\cdot)$ is twice continuously differentiable in $\mathbf{p}$. We denote the space of twice continuously differentiable functions by $\mathcal{C}^2$.

In the following proposition, we study the relationship between the partial derivatives of ordinal potential function and the utilities of players. We are interested in the set $A^m = \{\mathbf{p} \mid \frac{\partial u^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m} \neq 0, \text{ for some } i \in \mathcal{H} \text{ and } \mathbf{p} \in int(E)\}$ ($int(\cdot)$ denotes interior of a set) for all $m \in \mathcal{M}$. The proposition states that in ordinal potential games for any user $m \in \mathcal{M}$, at any strategy profile $\mathbf{p} \in A^m$, the vector of partial derivatives of the ordinal potential function and that of the utility of user m with respect to the actions of user m are aligned with some alignment function $d^m(\mathbf{p}^m, \mathbf{p}^{-m}) : E \to \mathbb{R}$.

Proposition 4.2.1. *Consider the game* $\mathcal{G} = \langle \mathcal{M}, \{E^m\}, \{u^m(\cdot)\} \rangle$,

(i) If there exists a continuously differentiable ordinal potential function $\phi(\cdot) : E \to \mathbb{R}$ *then for every* $m \in \mathcal{M},\ i \in \mathcal{H},\ \mathbf{p} \in A^m$ *it satisfies,*

$$\frac{\partial \phi(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m} = d^m(\mathbf{p}^m, \mathbf{p}^{-m}) \frac{\partial u^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m}, \tag{4.19}$$

and $d^m(\mathbf{p}) \geq 0$.

(ii) If for all $m \in \mathcal{M}$, *utility functions* $u^m(\cdot)$ *are linear in* $\mathbf{p}^m$, *and if there exists a continuously differentiable function* $\phi(\cdot) : E \to \mathbb{R}$ *such that for every* $m \in \mathcal{M},\ i \in \mathcal{H},\ \mathbf{p} \in E$,

$$\frac{\partial \phi(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m} = d^m(\mathbf{p}^m, \mathbf{p}^{-m}) \frac{\partial u^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m}, \tag{4.20}$$

and $d^m(\mathbf{p}) > 0$ *then* $\phi(\cdot)$ *is an ordinal potential function for* $\mathcal{G}$.

Proof. (i) (4.19) implies that for user m the vector of partial derivatives of its utility and the ordinal potential function with respect to $\mathbf{p}^m$ are aligned. Assume that a potential function $\phi(\cdot) : E \to \mathbb{R}$ exists, and assume by contradiction that (4.19) does not hold for some m. Then there exists $\mathbf{p} = (\mathbf{p}^m, \mathbf{p}^{-m})$ and $\mathbf{q} = (\mathbf{q}^m, \mathbf{p}^{-m})$, $\mathbf{p} \in A^m$, $\mathbf{q} \in E$ such that

$$\nabla u^m(\mathbf{p}^m, \mathbf{p}^{-m})^T (\mathbf{q} - \mathbf{p}) > 0 \quad \text{and} \quad \nabla \phi(\mathbf{p}^m, \mathbf{p}^{-m})^T (\mathbf{q} - \mathbf{p}) < 0. \tag{4.21}$$

This implies that the directional derivatives of $u^m(\cdot)$ and $\phi(\cdot)$ in $(\mathbf{q} - \mathbf{p})$ direction have opposite signs at $\mathbf{p}$, and hence there exists some $\epsilon > 0$ small such that

$$u^m(\mathbf{p}^m + \epsilon(\mathbf{q}^m - \mathbf{p}^m), \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m}) > 0, \tag{4.22}$$

whereas

$$\phi(\mathbf{p}^m + \epsilon(\mathbf{q}^m - \mathbf{p}^m), \mathbf{p}^{-m}) - \phi(\mathbf{p}^m, \mathbf{p}^{-m}) < 0, \tag{4.23}$$

which is a contradiction to the assumption that $\phi(\cdot)$ is an ordinal potential function.

(ii) Assume that (4.20) holds for some function $\phi(\cdot)$. Then, for every $\mathbf{p} = (\mathbf{p}^m, \mathbf{p}^{-m}) \in E$ and $\mathbf{q} = (\mathbf{q}^m, \mathbf{p}^{-m}) \in E$

$$\nabla u^m(\mathbf{p}^m, \mathbf{p}^{-m})^T(\mathbf{q} - \mathbf{p}) > 0 \Leftrightarrow \nabla\phi(\mathbf{p}^m, \mathbf{p}^{-m})^T(\mathbf{q} - \mathbf{p}) > 0. \tag{4.24}$$

Observe that since utility of user m is linear in its actions,

$$u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m}) = \nabla u^m(\mathbf{p}^m, \mathbf{p}^{-m})^T(\mathbf{q} - \mathbf{p}). \tag{4.25}$$

Moreover, for all $m \in \mathcal{M}$ linearity implies that

$$\frac{\partial u^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m} = \frac{\partial u^m(\gamma^m, \mathbf{p}^{-m})}{\partial p_i^m} \text{ for all } \gamma^m \in E^m,\ i \in \mathcal{H}. \tag{4.26}$$

Hence, substituting (4.26) in (4.25) yields,

$$u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m}) = \nabla u^m(\gamma^m, \mathbf{p}^{-m})^T(\mathbf{q} - \mathbf{p}), \tag{4.27}$$

for any $\gamma^m \in E^m$.

First we show that $u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m}) > 0 \Rightarrow \phi(\mathbf{q}^m, \mathbf{p}^{-m}) - \phi(\mathbf{p}^m, \mathbf{p}^{-m}) > 0$. If $u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m}) > 0$ then by (4.27) $\nabla u^m(\gamma^m, \mathbf{p}^{-m})^T(\mathbf{q} - \mathbf{p}) > 0$. For $\gamma^m = \alpha\mathbf{q}^m + (1-\alpha)\mathbf{p}^m$, $\alpha \in (0,1)$ this is equivalent to $\nabla u^m(\gamma^m, \mathbf{p}^{-m})^T(\mathbf{q} - (\gamma^m, \mathbf{p}^{-m})) > 0$, and using (4.24), the last inequality implies that $\nabla\phi(\gamma^m, \mathbf{p}^{-m})^T(\mathbf{q} -$

$(\gamma^m, \mathbf{p}^{-m})) > 0$. Now using the fundamental theorem of calculus,

$$\phi(\mathbf{q}^m, \mathbf{p}^{-m}) - \phi(\mathbf{p}^m, \mathbf{p}^{-m}) = \int_{\Gamma^m} \nabla\phi(\mathbf{s})^T d\mathbf{s} > 0, \tag{4.28}$$

where $\Gamma^m = \{(\alpha\mathbf{q}^m + (1-\alpha)\mathbf{p}^m, \mathbf{p}^{-m}) \mid \alpha \in [0,1]\}$. In (4.28) we made use of the fact that in Γ^m, $\mathbf{s}$ is in the form of $(\gamma^m, \mathbf{p}^{-m})$, hence vectors $\mathbf{q} - (\gamma^m, \mathbf{p}^{-m})$ and $d\mathbf{s}$ are always aligned and $\nabla\phi(\cdot)$ is a continuous function.

Next we show that $\phi(\mathbf{q}^m, \mathbf{p}^{-m}) - \phi(\mathbf{p}^m, \mathbf{p}^{-m}) > 0 \Rightarrow u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m}) > 0$. If $\phi(\mathbf{q}^m, \mathbf{p}^{-m}) - \phi(\mathbf{p}^m, \mathbf{p}^{-m}) > 0$ then there exists a $\gamma^m = \alpha\mathbf{q}^m + (1-\alpha)\mathbf{p}^m$ for some $\alpha \in [0,1]$ so that $\nabla\phi(\gamma^m, \mathbf{p}^{-m})^T(\mathbf{q} - \mathbf{p}) > 0$ since otherwise we obtain a contradiction with $\phi(\mathbf{q}^m, \mathbf{p}^{-m}) - \phi(\mathbf{p}^m, \mathbf{p}^{-m}) > 0$ using the integral in (4.28). Combining this with (4.24) it can be obtained that $\nabla u^m(\gamma^m, \mathbf{p}^{-m})^T(\mathbf{q} - \mathbf{p}) > 0$. Hence, (4.25) and (4.26) imply that $u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m}) > 0$.

Therefore, if (4.20) is satisfied with some continuously differentiable $\phi(\cdot)$, the game $\mathcal{G}$ is an ordinal potential game with potential function $\phi(\cdot)$. □

Using Proposition 4.2.1 we can obtain results on the existence of differentiable ordinal potential in the scheduling game. To this end we first state a preliminary result.

Lemma 4.2.1. *In the scheduling game, the set* $B_0 = \{\mathbf{p} \mid \frac{\partial u^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m} \neq 0,$ *for* $m \in \mathcal{M}, i \in \mathcal{H}, \mathbf{p} \in int(E)\}$ *contains a nonempty open subset of the joint feasible action space,* E.

Proof. For every nonempty open subset U of E, there exists an open set V, contained in U such that for every strategy profile $\mathbf{q}$ in V, $\frac{\partial u^k}{\partial p_j^k}(\mathbf{q}) \neq 0$ for a user $k \in \mathcal{M}$ and a state $j \in \mathcal{H}$, since $\frac{\partial u^k}{\partial p_j^k}(\mathbf{q})$ is a continuous function of its argument and the set $\mathbb{R} - \{0\}$ is open. The fact that V is not empty immediately follows from the definition of the utility function $u^k(\cdot)$. Since the above statement is true for an arbitrary open set U and since there are finitely many users and states in the system, there exists a nonempty subset of E which is contained in B_0. □

In the following we denote a nonempty open set of E contained in B_0 by V_0.

The next lemma characterizes the partial derivatives of the utilities for the scheduling game assuming that a $\mathcal{C}^2$ ordinal potential function exists.

Lemma 4.2.2. *Consider the scheduling game with $|\mathcal{M}| > 1$, $|\mathcal{H}| > 2$ and $\mathcal{C}^2$ ordinal potential function. Let ϕ, and alignment functions, $d^m(\cdot)$, $d^k(\cdot)$ be as in (i) of Proposition 4.2.1. For any $m, k \in \mathcal{M}$, there exists $\alpha^{mk} : E \to \mathbb{R}$ such that for every $\mathbf{p} \in V_0$ and for any $i \in \mathcal{H}$,*

$$\frac{\partial d^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^k} = \alpha^{mk}(\mathbf{p}) \frac{\partial u^k(\mathbf{p}^k, \mathbf{p}^{-k})}{\partial p_i^k} \quad \text{for all } i \in \mathcal{H}, \tag{4.29}$$

and

$$\frac{\partial d^k(\mathbf{p}^k, \mathbf{p}^{-k})}{\partial p_i^m} = \alpha^{mk}(\mathbf{p}) \frac{\partial u^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m} \quad \text{for all } i \in \mathcal{H}. \tag{4.30}$$

Proof. Consider two different users $m, k \in \mathcal{M}$ and two different states $i, j \in \mathcal{H}$. Then by Proposition 4.2.1 and by the symmetry of the second derivatives of the potential function it follows that

$$\begin{aligned}\frac{\partial^2 \phi}{\partial p_j^k \partial p_i^m}(\mathbf{p}) =& \frac{\partial}{\partial p_j^k}\left(d^m(\mathbf{p}^m, \mathbf{p}^{-m}) \frac{\partial u^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m}\right) \\ =& \frac{\partial}{\partial p_i^m}\left(d^k(\mathbf{p}^k, \mathbf{p}^{-k}) \frac{\partial u^k(\mathbf{p}^k, \mathbf{p}^{-k})}{\partial p_j^k}\right) = \frac{\partial^2 \phi}{\partial p_i^m \partial p_j^k}(\mathbf{p}),\end{aligned} \tag{4.31}$$

for $\mathbf{p} \in V_0$.

The previous equation is equivalent to,

$$\frac{\partial d^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_j^k} \frac{\partial u^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m} = \frac{\partial d^k(\mathbf{p}^k, \mathbf{p}^{-k})}{\partial p_i^m} \frac{\partial u^k(\mathbf{p}^k, \mathbf{p}^{-k})}{\partial p_j^k}, \tag{4.32}$$

using chain rule and observing that partial derivative of a utility of a user with respect to actions in some state j, is a function of actions of users in state j. (4.32) implies that

$$\frac{\frac{\partial d^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_j^k}}{\frac{\partial u^k(\mathbf{p}^k, \mathbf{p}^{-k})}{\partial p_j^k}} = \frac{\frac{\partial d^k(\mathbf{p}^k, \mathbf{p}^{-k})}{\partial p_i^m}}{\frac{\partial u^m(\mathbf{p}^m, \mathbf{p}^{-m})}{\partial p_i^m}}. \tag{4.33}$$

As i and j are arbitrary and $|\mathcal{H}| > 2$, (4.33) implies that there exists a function

$\alpha^{mk} : E \to \mathbb{R}$ such that

$$\alpha^{mk}(\mathbf{p}) = \frac{\frac{\partial d^m(\mathbf{p}^m,\mathbf{p}^{-m})}{\partial p_i^k}}{\frac{\partial u^k(\mathbf{p}^k,\mathbf{p}^{-k})}{\partial p_i^k}} = \frac{\frac{\partial d^k(\mathbf{p}^k,\mathbf{p}^{-k})}{\partial p_i^m}}{\frac{\partial u^m(\mathbf{p}^m,\mathbf{p}^{-m})}{\partial p_i^m}}. \tag{4.34}$$

for all $i \in \mathcal{H}$. □

The reason for (4.29) and (4.30) to hold is that for fixed $\mathbf{p}$ the system of equations in (4.32) with unknowns equal to partial derivatives of d^m and d^k is a linear system of equations with null space of rank one, and null space vector satisfies (4.29) and (4.30). However, if there are two or less states in the system, this system of equations has a null space with a higher dimension and hence (4.29) and (4.30) does not follow.

The next theorem shows that a $\mathcal{C}^2$ ordinal potential function, does not exist in the game unless the rate alignment condition holds.

Theorem 4.2.1. *Consider a scheduling game with more than a single player and three or more states. The game has a $\mathcal{C}^2$ ordinal potential function if and only if the rate alignment condition (assumption 3.4.1) holds.*

Proof. If assumption 3.4.1 holds, the result follows directly from Theorem 3.4.1.

For the other part of the claim, assume that there exists a $\mathcal{C}^2$ potential function ϕ for the scheduling game.

Observe that there exists $\mathbf{p} \in V_0$ such that $d^k(\mathbf{p}^k, \mathbf{p}^{-k}) \neq 0$ or $d^m(\mathbf{p}^m, \mathbf{p}^{-m}) \neq 0$ since otherwise there exists a neighborhood in which although utility of a user is changing by modifying the policy the potential of the game remains constant. Fix a $\mathbf{p} \in V_0$ such that $d^k(\mathbf{p}^k, \mathbf{p}^{-k}) \neq 0$.

Now using symmetry of partial derivatives of the potential function with respect to p_i^m and p_i^k it is obtained that

$$\begin{aligned}\frac{\partial d^m(\mathbf{p}^m,\mathbf{p}^{-m})}{\partial p_i^k}\frac{\partial u^m(\mathbf{p}^m,\mathbf{p}^{-m})}{\partial p_i^m} + d^m(\mathbf{p}^m,\mathbf{p}^{-m})\frac{\partial^2 u^m(\mathbf{p}^m,\mathbf{p}^{-m})}{\partial p_i^k \partial p_i^m} &= \\ \frac{\partial d^k(\mathbf{p}^k,\mathbf{p}^{-k})}{\partial p_i^m}\frac{\partial u^k(\mathbf{p}^k,\mathbf{p}^{-k})}{\partial p_i^k} + d^k(\mathbf{p}^k,\mathbf{p}^{-k})\frac{\partial^2 u^k(\mathbf{p}^k,\mathbf{p}^{-k})}{\partial p_i^m \partial p_i^k}&\end{aligned} \tag{4.35}$$

Using (4.29) and (4.30) one can see that terms including partial derivatives of

d^m and d^k cancel, and substituting the second partial derivatives of utilities one can achieve,

$$R_i^m d^m(\mathbf{p}^m, \mathbf{p}^{-m}) = d^k(\mathbf{p}^k, \mathbf{p}^{-k}) R_i^k \tag{4.36}$$

Note that (4.36) holds for any i and since $d^k(\mathbf{p}^k, \mathbf{p}^{-k}) \neq 0$ it follows that $d^m(\mathbf{p}^m, \mathbf{p}^{-m}) \neq 0$. Therefore, (4.36) implies that rate alignment condition holds, hence the scheduling game has a $\mathcal{C}^2$ potential function if and only if assumption 3.4.1 holds. □

4.3 Projections to the Set of Exact Potential Games

Given an arbitrary game our goal is to project it to the set of exact potential games. This enables us to quantify how "close" a game is to a potential game and provides insights on how to modify the game (or equivalently the utilities of players) to inherit the desirable properties of potential games. Note that generalizations of exact potential games such as weighted potential games and ordinal potential games have similar desirable properties to those of exact potential games. However, we focus on projections to the set of exact potential games as the sets of weighted and ordinal potential games are nonconvex.

In the next subsections we discuss different approaches for projection to the set of exact potential games. The approach in Section 4.3.1 utilizes the idea of projection of the utility differences of strategy profiles in a game. Similar to the ranking problems, a global function (potential function) that represents the pairwise comparisons (utility differences) in the best possible way is found and then utilities of the projected game are obtained by constructing utilities that agree with the potential and are closest to the initial utilities in 2-norm sense. In Section 4.3.2 for an arbitrary game, we find the potential game with the smallest change in the utilities. In this approach, we do not construct the pairwise comparisons and operate in C_0 space. In Section 4.3.3 we repeat these projections utilizing infinity norm instead of 2 norm. In Section 4.3.4 we relate the equilibria of a game and ϵ-equilibria of its projection. In Section 4.3.5 we discuss a distributed framework for implementing the projections and we present simulation results in section 4.3.6.

4.3.1 Projection in C_1

A potential game by definition satisfies $D_m\phi = D_m u^m$ for all $m \in \mathcal{M}$ where ϕ is some potential function (cf. (2.56) from Chapter 2). Our goal is to find a potential game that is "closest" to an arbitrary given game. In this subsection we discuss a particular projection method in which we first obtain pairwise comparisons in a game, $Du \in C_1$, and then project the pairwise comparisons to the set of consistent pairwise comparisons in C_1, i.e. $\{X | X \in C_1, \delta_0\phi = X \text{ for some } \phi \in C_0\}$. For such a projection in C_1, one can construct a potential function representing the projected pairwise comparison. We then construct the new utility functions utilizing the obtained potential function and the initial utility functions.

More precisely, we are interested in the following projection problem,

$$\begin{aligned} err_1^2(\mathcal{G}) = \min_{X,\ \phi} \quad & ||X - Du||_2^2 \\ s.t. \quad & \delta_0\phi = X, \\ & X \in C_1,\ \phi \in C_0 \end{aligned} \tag{4.37}$$

where X represents a globally consistent pairwise comparison that corresponds to a potential function ϕ and the optimal solution of this problem is the projection of Du to the space of globally consistent pairwise rankings. An equivalent formulation of (4.37) can be obtained as

$$err_1^2(\mathcal{G}) = \min_{\phi \in C_0} \quad ||\delta_0\phi - Du||_2^2. \tag{4.38}$$

Solution of this problem can be found by making use of the Hodge theory as discussed in Section 2.2.2 and the solution is:

$$\phi = \Delta_0^{\dagger}\delta_0^{*}Du \tag{4.39}$$

where $\Delta_0^{\dagger}$ is the pseudo-inverse of the previously defined Laplacian operator. In projection of $\mathcal{G}$, $err_1(\mathcal{G})$ denotes the norm of the projection error in C_1. The obtained

ϕ is the potential function for the projected game.

The utilities that represent the potential and that are close to initial utilities can be constructed by solving an additional optimization problem (for a fixed ϕ, and for all $m \in \mathcal{M}$):

$$\begin{aligned} \hat{u}^m = \arg\min_{\bar{u}^m} \quad & ||u^m - \bar{u}^m||_2^2 \\ s.t. \quad & D_m \bar{u}^m = D_m \phi \\ & \bar{u}^m \in C_0. \end{aligned} \tag{4.40}$$

We refer to solutions of (4.38) and (4.40) as C_1 projection of the game since in C_1 the pairwise rankings are projected to the set of globally consistent pairwise rankings and then utilities and potential are constructed from this projection.

The solution of this projection problem is given by the following theorem.

Theorem 4.3.1. *Solutions of* (4.38) *and* (4.40) *are given by:*

$$\phi = \left(\sum_{m \in \mathcal{M}} \Delta_{0,m} \right)^{\dagger} \sum_{m \in \mathcal{M}} \Delta_{0,m} u^m, \tag{4.41}$$

and

$$\hat{u}^m = (I - proj_m) u^m + proj_m \left(\sum_{k \in \mathcal{M}} \Delta_{0,k} \right)^{\dagger} \sum_{k \in \mathcal{M}} \Delta_{0,k} u_k. \tag{4.42}$$

Proof. The solution of (4.38) is $\phi = \Delta_0^{\dagger} \delta_0^* D u$ as mentioned before in (4.39). Using (2.46) and (2.44) and $Du = \sum_{m \in \mathcal{M}} D_m u^m$ it follows that

$$\phi = \left(\sum_{m \in \mathcal{M}} D_m^* D_m \right)^{\dagger} \sum_{k \in \mathcal{M}} D_k^* \sum_{m \in \mathcal{M}} D_m u^m. \tag{4.43}$$

Due to the orthogonality of image spaces of D_m and D_k for any $k \neq m$ the previous equation becomes,

$$\phi = \left(\sum_{m \in \mathcal{M}} D_m^* D_m \right)^{\dagger} \sum_{m \in \mathcal{M}} D_m^* D_m u^m. \tag{4.44}$$

Given a potential ϕ, we next focus on the solution for the utilities. Note that

(4.40) can be reformulated as

$$\begin{aligned} \hat{u}^m = \phi + \arg\min_{\gamma_m} \quad & ||(u^m - \phi) - \gamma_m||_2^2 \\ s.t. \quad & D_m \gamma_m = 0 \\ & \gamma_m \in C_0, \end{aligned} \tag{4.45}$$

by setting $\gamma_m = \bar{u}^m - \phi$. Observe that (4.45) is the projection problem to the kernel of D_m. $(I - proj_m)(u^m - \phi)$ gives a projection of $(u^m - \phi)$ to the kernel of D_m, therefore the optimal solution of the optimization problem in (4.45) can be obtained as

$$\hat{\gamma}_m = (I - proj_m)(u^m - \phi), \tag{4.46}$$

and thus $\hat{u}^m$ is obtained as a function of the potential,

$$\begin{aligned} \hat{u}^m &= \hat{\gamma}_m + \phi = (I - proj_m)(u^m - \phi) + \phi \\ &= (I - proj_m)u^m + proj_m \phi. \end{aligned} \tag{4.47}$$

Now substituting the potential from (4.44), $\hat{u}^m$ becomes

$$\hat{u}^m = (I - proj_m)u^m + proj_m \left(\sum_{k \in \mathcal{M}} D_k^* D_k \right)^\dagger \sum_{k \in \mathcal{M}} D_k^* D_k u_k. \tag{4.48}$$

The result follows from (4.44) and (4.48) noting that $D_k^* D_k = \Delta_{0,k}$ by definition. □

As discussed earlier (see (2.48) and the discussion following it) $\Delta_{0,m}$ is a Laplacian for the graph constructed on strategy profiles where edges exists between any two strategy profile that are comparable by player m. (4.41) suggests that potential function is a solution of,

$$\left(\sum_{m \in \mathcal{M}} \Delta_{0,m} \right) \phi = \sum_{m \in \mathcal{M}} \Delta_{0,m} u^m. \tag{4.49}$$

Intuitively, the graph Laplacian gives a measure of how much a node is valued over its neighbors. Then for each strategy profile, $\mathbf{p}$, $\Delta_{0,m} u^m$ indicates the value of $\mathbf{p}$ among all strategy profiles that are comparable with $\mathbf{p}$ by m. Hence, (4.49) implies that the potential function represents the aggregate value of each strategy profile according to different players.

In Chapter 2, it was explained that for player m the strategic component of a function $f \in C_0$ is given by $proj_m f$. It can be seen from (4.47) that for player m, the projected utility $\hat{u}^m$, is the sum of the nonstrategic part of the initial utility, u^m, and the strategic part of the potential, ϕ.

Next we relate the projection error $err_1^2(\mathcal{G})$ to $||u - \hat{u}||_2^2$. Observe that given an optimal potential function ϕ,

$$err_1^2(\mathcal{G}) = ||\delta_0 \phi - Du||_2^2 = \sum_{m \in \mathcal{M}} ||D_m(u^m - \phi)||_2^2 = \sum_{m \in \mathcal{M}} \langle (u^m - \phi), D_m^* D_m(u^m - \phi) \rangle, \tag{4.50}$$

by the orthogonality of the image spaces of D_m, $m \in \mathcal{M}$.

On the other hand,

$$\begin{aligned} ||u - \hat{u}||_2^2 &= \sum_{m \in \mathcal{M}} ||u^m - \hat{u}^m||_2^2 = \sum_{m \in \mathcal{M}} ||u^m - \hat{u}^m||_2^2 = \sum_{m \in \mathcal{M}} ||proj_m(u^m - \phi)||_2^2 \\ &= \sum_{m \in \mathcal{M}} \langle proj_m(u^m - \phi), proj_m(u^m - \phi) \rangle \\ &= \sum_{m \in \mathcal{M}} \langle (u^m - \phi), proj_m(u^m - \phi) \rangle \end{aligned} \tag{4.51}$$

where the first line follows from (4.47) and the last line follows from the fact that image of $proj_m$ is orthogonal to image of $I - proj_m$. From (4.50) and (4.51) it follows that $err_1^2(\mathcal{G})$ and $||u - \hat{u}||_2^2$ are not necessarily equal. The next theorem provides an inequality between $err_1^2(\mathcal{G})$ and $||u - \hat{u}||_2^2$.

Theorem 4.3.2. *Let a game $\mathcal{G}$ and its projection $\hat{\mathcal{G}}$ have utilities $u = \{u^m\}_{m \in \mathcal{M}}$ and $\hat{u} = \{\hat{u}^m\}_{m \in \mathcal{M}}$ respectively. Then,*

$$||u - \hat{u}||_2 \leq err_1(\mathcal{G}). \tag{4.52}$$

Proof. Using (4.50), (4.51) and Theorem 2.2.2,

$$\begin{aligned} err_1^2(\mathcal{G}) &= \sum_{m\in\mathcal{M}} \langle (u^m - \phi), D_m^* D_m (u^m - \phi)\rangle \\ &= \sum_{m\in\mathcal{M}} \frac{h_m}{h_m - 1} \langle (u^m - \phi), D_m^\dagger D_m (u^m - \phi)\rangle \\ &\geq \sum_{m\in\mathcal{M}} \langle (u^m - \phi), D_m^\dagger D_m (u^m - \phi) \\ &= ||u - \hat{u}||_2^2. \end{aligned} \tag{4.53}$$

Thus, $||u - \hat{u}||_2 \leq err_1(\mathcal{G})$. □

If $h_m = h_0$ for all $m \in \mathcal{M}$. The above proof also implies that $err_1(\mathcal{G}) = \sqrt{\frac{h_0}{h_0-1}}||u-\hat{u}||_2$.

4.3.2 Projection in C_0

A related optimization problem for finding a projection of a game to the set of exact potential games is studied in this section. Consider,

$$\begin{aligned} err_0^2(\mathcal{G}) = \min_{\phi,\{\bar{u}^m\}_{m\in\mathcal{M}}} \quad & \sum_{m\in\mathcal{M}} ||u^m - \bar{u}^m||_2^2 \\ s.t. \quad & D_m \bar{u}^m = D_m \phi, \\ & \phi,\ \bar{u}^m \in C_0 \quad \text{for all } m \in \mathcal{M}. \end{aligned} \tag{4.54}$$

Observe that in this optimization formulation the norm of change in the utilities is minimized. As the utilities are in C_0, and the pairwise comparisons of utilities are not utilized for projection, the projection approach in this section is different from the approach taken in the previous section. We refer to the projection problem in (4.54) as C_0 projection. The next theorem states the potential function and the utilities obtained from the above optimization formulation.

Theorem 4.3.3. *Optimal ϕ and $\{\hat{u}^m\}_{m\in\mathcal{M}}$ solving* (4.54) *are given by:*

$$\phi = \left(\sum_{m\in\mathcal{M}} proj_m\right)^{\dagger} \sum_{m\in\mathcal{M}} proj_m u^m. \tag{4.55}$$

and

$$\hat{u}^m = (I - proj_m)u^m + proj_m \left(\sum_{k\in\mathcal{M}} proj_k\right)^{\dagger} \sum_{k\in\mathcal{M}} proj_k u_k. \tag{4.56}$$

Proof. The optimization problem in (4.54) can be reformulated as

$$\begin{aligned} \min_{\phi\in C_0} \min_{\{\bar{u}^m\}_{m\in\mathcal{M}}} \quad & \sum_{m\in\mathcal{M}} ||u^m - \bar{u}^m||_2^2 \\ s.t. \quad & D_m\bar{u}^m = D_m\phi, \\ & \bar{u}^m \in C_0 \quad \text{for all } m \in \mathcal{M}, \end{aligned} \tag{4.57}$$

or equivalently

$$\begin{aligned} \min_{\phi\in C_0} \sum_{m\in\mathcal{M}} \min_{\bar{u}^m} \quad & ||u^m - \bar{u}^m||_2^2 \\ s.t. \quad & D_m\bar{u}^m = D_m\phi, \\ & \bar{u}^m \in C_0, \end{aligned} \tag{4.58}$$

since the objective function and the constraints are decoupled for different $\bar{u}^m$.

First consider for a fixed ϕ the following optimization problem

$$\begin{aligned} \min_{\bar{u}^m} \quad & ||u^m - \bar{u}^m||_2^2 \\ s.t. \quad & D_m\bar{u}^m = D_m\phi, \\ & \bar{u}^m \in C_0. \end{aligned} \tag{4.59}$$

Defining $\gamma_m = \bar{u}^m - \phi$ an equivalent optimization problem is:

$$\begin{aligned} \min_{\gamma_m} \quad & ||(u^m - \phi) - \gamma_m||_2^2 \\ s.t. \quad & D_m\gamma_m = 0, \\ & \gamma_m \in C_0, \end{aligned} \tag{4.60}$$

where the optimal solutions of (4.59) and (4.60) are related by $\hat{\gamma}_m = \hat{u}^m - \phi$. The optimal solution of (4.60) is the projection of $(u^m - \phi)$ to the kernel of D_m, hence optimal solution can be obtained as $\hat{\gamma}_m = (I - proj_m)(u^m - \phi)$. Therefore,

$$\hat{u}^m = (I - proj_m)u^m + proj_m \phi. \tag{4.61}$$

Using this (4.58) can be reformulated as,

$$\min_{\phi \in C_0} \sum_{m \in \mathcal{M}} ||proj_m(u^m - \phi)||_2^2. \tag{4.62}$$

Let,

$$f(\phi) = \sum_{m \in \mathcal{M}} ||proj_m(u^m - \phi)||_2^2 = \sum_{m \in \mathcal{M}} \langle (u^m - \phi), proj_m(u^m - \phi) \rangle. \tag{4.63}$$

Where the second equality follows from the fact that the images of $proj_m$ and $I - proj_m$ are orthogonal.

Note that the optimal solution of (4.62) satisfies $\nabla f(\phi) = 0$. Thus, it follows that

$$\nabla f(\phi) = \sum_{m \in \mathcal{M}} 2 proj_m(u^m - \phi) = 0, \tag{4.64}$$

or equivalently

$$\sum_{m \in \mathcal{M}} proj_m u^m = \left(\sum_{m \in \mathcal{M}} proj_m \right) \phi, \tag{4.65}$$

which gives an optimal solution of

$$\phi = \left(\sum_{m \in \mathcal{M}} proj_m \right)^{\dagger} \sum_{m \in \mathcal{M}} proj_m u^m. \tag{4.66}$$

Hence optimal solution $\hat{u}^m$ for user m can be rewritten as

$$\hat{u}^m = (I - proj_m)u^m + proj_m \left(\sum_{k \in \mathcal{M}} proj_k \right)^{\dagger} \sum_{k \in \mathcal{M}} proj_k u_k. \tag{4.67}$$

□

Observe that similar to (4.42), (4.56) obtains $\hat{u}^m$ as the sum of the nonstrategic component of u^m and the strategic component of the potential ϕ. On the other hand, since ϕ is a solution of (4.65), we conclude that in C_0 projection the potential is a function which represents the sum of strategic components of utilities of different users.

The next theorem presents conditions under which C_0 and C_1 projections coincide.

Theorem 4.3.4. *Optimal solutions of C_0 projection and C_1 projection coincide when all players have same number of strategies, i.e. $h_m = h_0$ for all $m \in \mathcal{M}$.*

Proof. If for all $m \in \mathcal{M}$, $h_m = h_0$, Theorem 2.2.2 suggests that

$$D_m^* D_m = \frac{h_0}{h_0 - 1} D_m^\dagger D_m. \tag{4.68}$$

Thus, (4.55) can be rewritten as

$$\begin{aligned} \phi &= \left(\sum_{m \in \mathcal{M}} \frac{h_0 - 1}{h_0} D_m^* D_m \right)^\dagger \sum_{m \in \mathcal{M}} \frac{h_0 - 1}{h_0} D_m^* D_m u^m. \\ &= \left(\sum_{m \in \mathcal{M}} D_m^* D_m \right)^\dagger \sum_{m \in \mathcal{M}} D_m^* D_m u^m. \end{aligned} \tag{4.69}$$

which is equivalent to (4.41), hence potential functions in C_0 and C_1 projections coincide. On the other hand as can be seen from Theorems 4.3.1 and 4.3.3 both projections satisfy

$$\hat{u}^m = (I - D_m^\dagger D_m) u^m + D_m^\dagger D_m \phi. \tag{4.70}$$

Hence, the projected utilities are the same and the solutions of C_0 and C_1 projections coincide. □

4.3.3 Projections Using Infinity Norm

In the previous section projections using 2 norm are studied. The 2 norm has the benefit of giving closed form solutions for the studied projection problems. However, the projection problem can still be generalized to other norms. In this section we explore projections using infinity norm.

First we define the infinity norm for functions in C_0 and C_1. Let $\phi \in C_0$, $X \in C_1$ and $u = \{u^m\}_{m\in\mathcal{M}} \in C_0^M$ then

$$||\phi||_\infty = \max_{\mathbf{p}\in E} |\phi(\mathbf{p})|, \tag{4.71}$$

$$||u||_\infty = \max_{m\in\mathcal{M}} ||u^m||_\infty, \tag{4.72}$$

$$||X||_\infty = \max_{\mathbf{p},\mathbf{q}\in E} |W(\mathbf{p},\mathbf{q})X(\mathbf{p},\mathbf{q})|. \tag{4.73}$$

Let $u = \{u^m\}_{m\in\mathcal{M}}$ and $v = \{v_m\}_{m\in\mathcal{M}}$ be two different collections of utility functions. We define the norm of difference of these collections as

$$||u - v||_\infty = \max_{m\in\mathcal{M}} ||u^m - v^m||_\infty. \tag{4.74}$$

In a similar fashion to (4.38) the projection problem can be formulated as,

$$\begin{aligned} \overline{err}_1(\mathcal{G}) = \min \quad & ||\delta_0\phi - Du||_\infty \\ s.t. \quad & \phi \in C_0 \end{aligned} \tag{4.75}$$

and given ϕ, the optimal solution of (4.75), utilities can be constructed as

$$\begin{aligned} \hat{u}^m \in \arg\min \quad & ||u^m - \bar{u}^m||_\infty \\ s.t. \quad & D_m\bar{u}^m = D_m\phi \\ & \bar{u}^m \in C_0. \end{aligned} \tag{4.76}$$

This projection is similar to the C_1 projection discussed in the previous section and we refer to this projection as C_1 projection using infinity norm.

As before we study the projection error and the norm of the difference between the utilities of the initial game and utilities of its projection.

Theorem 4.3.5. *Let a game $\mathcal{G}$ and its C_1 projection using infinity norm have utilities*

$u = \{u^m\}_{m\in\mathcal{M}}$ *and* $\hat{u} = \{\hat{u}^m\}_{m\in\mathcal{M}}$ *respectively. Then,*

$$||u - \hat{u}||_\infty \leq \overline{err}_1(\mathcal{G}). \tag{4.77}$$

Proof. Let ϕ denote the potential of the projection, for all $m \in \mathcal{M}$ fix a strategy $\mathbf{p}_0^m \in E^m$. For all $m \in \mathcal{M}$, define $\bar{u}^m : C_0 \to \mathbb{R}$ such that

$$\bar{u}^m(\mathbf{p}^m, \mathbf{p}^{-m}) = u^m(\mathbf{p}_0^m, \mathbf{p}^{-m}) + \phi(\mathbf{p}^m, \mathbf{p}^{-m}) - \phi(\mathbf{p}_0^m, \mathbf{p}^{-m}), \tag{4.78}$$

for all $\mathbf{p} \in E$. Observe that these utilities satisfy $D_m\bar{u}^m = D_m\phi$. Considering equation (4.76) it follows that

$$||u^m - \hat{u}^m||_\infty \leq ||u^m - \bar{u}^m||_\infty. \tag{4.79}$$

Also observe that

$$\bar{u}^m(\mathbf{p}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m}) = \left(\phi(\mathbf{p}^m, \mathbf{p}^{-m}) - \phi(\mathbf{p}_0^m, \mathbf{p}^{-m})\right) - \left(u^m(\mathbf{p}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}_0^m, \mathbf{p}^{-m})\right).$$

Hence for all $\mathbf{p} = (\mathbf{p}^m, \mathbf{p}^{-m}) \in E$,

$$\begin{aligned} |\bar{u}^m(\mathbf{p}) - u^m(\mathbf{p})| &= \left| \left(\phi(\mathbf{p}^m, \mathbf{p}^{-m}) - \phi(\mathbf{p}_0^m, \mathbf{p}^{-m})\right) - \left(u^m(\mathbf{p}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}_0^m, \mathbf{p}^{-m})\right) \right| \\ &\leq \max_{\mathbf{p},\mathbf{q}\in E} |W(\mathbf{p},\mathbf{q})(\delta_0\phi - Du)(\mathbf{p},\mathbf{q})| = \overline{err}_1(\mathcal{G}), \end{aligned} \tag{4.80}$$

as $W((\mathbf{p}_0^m, \mathbf{p}^{-m}), (\mathbf{p}^m, \mathbf{p}^{-m})) = 1$ (i.e., $(\mathbf{p}_0^m, \mathbf{p}^{-m})$ and $(\mathbf{p}^m, \mathbf{p}^{-m})$ differ in the strategy of a single player).

Taking the maximum over $\mathbf{p} \in E$, $m \in \mathcal{M}$ and utilizing (4.79), it follows that

$$||u - \hat{u}||_\infty \leq \overline{err}_1(\mathcal{G}). \tag{4.81}$$

□

Similar to C_0 projection that is discussed before one can introduce the following projection using infinity norm,

$$\begin{aligned} \overline{err}_0(\mathcal{G}) = \min_{\phi,\{\bar{u}^m\}_{m\in\mathcal{M}}} \quad & \max_{m\in\mathcal{M}} ||u^m - \bar{u}^m||_\infty \\ s.t. \quad & D_m\bar{u}^m = D_m\phi, \\ & \phi,\ \bar{u}^m \in C_0 \quad \text{for all } m \in \mathcal{M}. \end{aligned} \tag{4.82}$$

We refer this projection as C_0 projection using infinity norm.

Note that the C_0 and C_1 projections using infinity norm do not admit closed form solutions. For this reason in the rest of this chapter our main focus will be on projections using 2 norm.

4.3.4 ϵ-equilibria of a Game and its Projection

In the previous subsections, we studied the closest potential game to an arbitrary game. In this section we relate the ϵ-equilibria of these games.

Lemma 4.3.1. *Let $\mathcal{G}$ and $\hat{\mathcal{G}}$ be games with set of players $\mathcal{M}$, strategy space E and with collections of utilities u and $\hat{u}$ respectively.*

1. *Assume, $||u - \hat{u}||_2 \leq \alpha$. Then each equilibrium of $\mathcal{G}$ is an ϵ-equilibrium of $\hat{\mathcal{G}}$ and similarly each equilibrium of $\hat{\mathcal{G}}$ is an ϵ-equilibrium of $\mathcal{G}$ where $\epsilon \leq \sqrt{2}\alpha$.*
2. *Assume, $||u - \hat{u}||_\infty \leq \alpha$. Then each equilibrium of $\mathcal{G}$ is an ϵ-equilibrium of $\hat{\mathcal{G}}$ and similarly each equilibrium of $\hat{\mathcal{G}}$ is an ϵ-equilibrium of $\mathcal{G}$ where $\epsilon \leq 2\alpha$.*

Proof. Note that it is sufficient to prove that each equilibrium of $\mathcal{G}$ is an ϵ-equilibrium of $\hat{\mathcal{G}}$, by symmetry it also follows that each equilibrium of $\hat{\mathcal{G}}$ is an ϵ-equilibrium of $\mathcal{G}$.

Let $\mathbf{p}$ be a Nash equilibrium of $\mathcal{G}$, and $\mathbf{q}$ be a strategy profile that is different than $\mathbf{p}$ in exactly a single strategy (i.e. $W(\mathbf{p},\mathbf{q}) = 1$). Assume that $\mathbf{p}$ and $\mathbf{q}$ differ in the strategy of player m only.

1. As $||u - \hat{u}||_2 \leq \alpha$ it follows that

$$\alpha^2 \geq ||u - \hat{u}||_2^2 \geq (\hat{u}^m(\mathbf{q}) - u^m(\mathbf{q}))^2 + (\hat{u}^m(\mathbf{p}) - u^m(\mathbf{p}))^2 . \tag{4.83}$$

Note that for any $a, b \in \mathbb{R}$; $a^2 + b^2 \leq \alpha^2$ implies that $a - b \leq \sqrt{2}\alpha$. Thus (4.83) implies that

$$\sqrt{2}\alpha \geq (\hat{u}^m(\mathbf{q}) - u^m(\mathbf{q})) - (\hat{u}^m(\mathbf{p}) - u^m(\mathbf{p})) \geq \hat{u}^m(\mathbf{q}) - \hat{u}^m(\mathbf{p}). \tag{4.84}$$

where the last inequality follows as $\mathbf{p}$ is a Nash equilibrium of $\mathcal{G}$ and $\mathbf{p}$, $\mathbf{q}$ differ in the strategy of player m only. Since this is true for an arbitrary $\mathbf{q}$ which is different than $\mathbf{p}$ in exactly a single strategy, $\mathbf{p}$ is an ϵ-equilibrium of the projected game where $\epsilon \leq \sqrt{2}\alpha$.

2. $||u - \hat{u}||_\infty \leq \alpha$ implies that

$$\begin{aligned} 2\alpha &\geq |\,(\hat{u}^m(\mathbf{q}) - u^m(\mathbf{q}))\,| + |\,(\hat{u}^m(\mathbf{p}) - u^m(\mathbf{p}))\,| \\ &\geq (\hat{u}^m(\mathbf{q}) - u^m(\mathbf{q})) - (\hat{u}^m(\mathbf{p}) - u^m(\mathbf{p}))\,. \end{aligned} \tag{4.85}$$

As $\mathbf{p}$ is a Nash equilibrium of $\mathcal{G}$ and $\mathbf{p}$, $\mathbf{q}$ differ in the strategy of player m only (4.85) implies that

$$2\alpha \geq \hat{u}^m(\mathbf{q}) - \hat{u}^m(\mathbf{p}). \tag{4.86}$$

Since this is true for an arbitrary $\mathbf{q}$ which is different than $\mathbf{p}$ in exactly a single strategy, $\mathbf{p}$ is an ϵ-equilibrium of the projected game where $\epsilon \leq 2\alpha$.

□

Corollary 4.3.1. *Let $\mathcal{G}$ be a game and $\hat{\mathcal{G}}$ be its projection.*

1. *If 2 norm is used in the projection (C_0 or C_1) and err denotes the projection error then any equilibrium of $\hat{\mathcal{G}}$ is an ϵ-equilibrium of $\mathcal{G}$ and any equilibrium of $\mathcal{G}$ is an ϵ-equilibrium of $\hat{\mathcal{G}}$ for $\epsilon \leq \sqrt{2}err$.*

2. *If infinity norm is used in the projection (C_0 or C_1) and $\overline{err}$ denotes the projection error then any equilibrium of $\hat{\mathcal{G}}$ is an ϵ-equilibrium of $\mathcal{G}$ and any equilibrium of $\mathcal{G}$ is an ϵ-equilibrium of $\hat{\mathcal{G}}$ for $\epsilon \leq 2\overline{err}$.*

Proof. Claim immediately follows from definitions of the projections, Theorem 4.3.2, Theorem 4.3.5 and Lemma 4.3.1. □

4.3.5 Distributed Implementation

In this section we discuss the distributed implementation of the projections using 2-norm. As Theorems 4.3.1 and 4.3.3 suggest the projected utilities contain a non-strategic component, $(I - proj_m)u^m$, and a strategic component obtained from the potential function. Note that each player can calculate the nonstrategic part of its projected utility on its own. On the other hand, the strategic components for C_1 and C_0 projections are

$$proj_m \hat{u}^m = proj_m \left(\sum_{k \in \mathcal{M}} \Delta_{0,k} \right)^{\dagger} \sum_{k \in \mathcal{M}} \Delta_{0,k} u_k \tag{4.87}$$

and

$$proj_m \hat{u}^m = proj_m \left(\sum_{k \in \mathcal{M}} proj_k \right)^{\dagger} \sum_{k \in \mathcal{M}} proj_k u_k. \tag{4.88}$$

respectively. However, this implies that in order to calculate the projections, all users require the knowledge of $\sum_{k \in \mathcal{M}} \Delta_{0,k} u_k$ or $\sum_{k \in \mathcal{M}} proj_k u_k$ depending on the projection being utilized. On the other hand these quantities can be calculated using distributed averaging or consensus algorithms.

Averaging algorithms are a special case of consensus algorithms with the goal of computing average of the initial values of nodes (or agents) on a graph. The objective of averaging algorithms is to design simple distributed update schemes, which do not require the knowledge of the underlying graph, in order to calculate the average of the initial values of the agents. A widely studied averaging algorithm due to [52] necessitates having agents which update their values by taking a weighted average of their own values and the information received from their neighbors. Given a communication graph, this algorithm can be ensured to converge to the average of the initial values if the weights are properly chosen. For example, it can be shown that convergence to the average of the initial values of agents ($\{x^m(0)\}_{m \in \mathcal{M}}$), takes place in a n node network if at each step each node with d neighbor updates its value to

$$x^m(t+1) = \frac{n-d}{n} x^m(t) + \frac{1}{n} \sum_{k \in N(m)} x^k(t) \qquad (4.89)$$

where $N(m)$ is the set of neighbors of user m, $x^m(t)$ is the value of node m at time t. The convergence of this particular algorithm to the average of the initial values of agents follows as the update matrix with the given weights is doubly stochastic (i.e., if the updates in (4.89) are written as $x(t+1) = Ux(t)$ where x is the vector of x^m matrix U is doubly stochastic) .

Note that as averaging can be done in a distributed manner utilizing the consensus algorithms, the distributed computation of the strategic components in (4.87) and (4.88) is possible. In the simple scheme we suggest for C_1 projection, each player $k \in \mathcal{M}$ calculates $\Delta_{0,k} u^k$ and then, using the update rule described in equation (4.89) players obtain $\frac{1}{M} \sum_{k \in \mathcal{M}} \Delta_{0,k} u^k$ in the limit. Similarly for C_0 projection players first calculate $proj_k u^k$ and the update rule converges to $\frac{1}{M} \sum_{k \in \mathcal{M}} proj_k u^k$. Given $\frac{1}{M} \sum_{k \in \mathcal{M}} \Delta_{0,k} u^k$ or $\frac{1}{M} \sum_{k \in \mathcal{M}} proj_k u^k$ each player can calculate its new utility in the projected game. Hence, the described approach allows distributed implementation of the projections introduced in Section 4.3. Distributed projection approach is illustrated via simulations in the next section.

4.3.6 Simulations

In this section we present a simulation for projection to the set of exact potential games utilizing a distributed algorithm. We assume that the updates follow the update equation (4.89) and projection is found in a distributed manner described in the previous section.

The game we simulate is related to average opinion game of [41]. In average opinion games each player picks a number from a finite set (we assume that $E^m = \{1, 2, 3\}$ for all $m \in \mathcal{M}$) and the payoff of each user is assumed decrease with the deviation of its number from the median of the numbers all players pick. A candidate utility for user m at $\mathbf{p} \in E$ satisfies

$$u^m(\mathbf{p}) = 2\hat{M} - (\hat{M} - \mathbf{p}^m)^2, \tag{4.90}$$

where $\hat{M}$ is the median of $\mathbf{p}^k$, $k \in \mathcal{M}$. It is known that average opinion games are not weighted potential games. By simulations it can be verified that such games are ordinal potential games.

We consider a related graphical game in which each node corresponds to a player and the payoff function of player m is given by

$$u^m(\mathbf{p}) = 2\hat{M} - (\hat{M}^m - \mathbf{p}^m)^2, \tag{4.91}$$

where $\hat{M}^m$ is the median of $\mathbf{p}^k$, $k \in N(m)$ ($N(m)$ is the set of neighbors of player m). Hence the game is an average opinion game on a graph where each player is trying to reach the median opinion of its immediate neighbors.

We run our simulations for the communication graph given in Figure 4.3.6. For

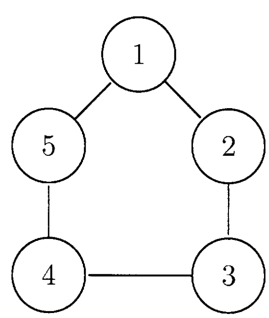


Figure 4-2: The communication graph of players.

the given graph the game is not an ordinal potential game, hence weak improvement cycles exist in the set of strategy profiles. Since all players have same number of strategies, C_0 and C_1 projections coincide for this game.

We assume that players can only communicate with their neighbors in the graph given in Figure 4.3.6. Thus, update equation also relies on this graph.

In Figure 4-3 we plot the utility functions of all players in the initial and the projected games. It can be seen that the projected utilities are very similar to the initial utilities in terms of the payoffs.

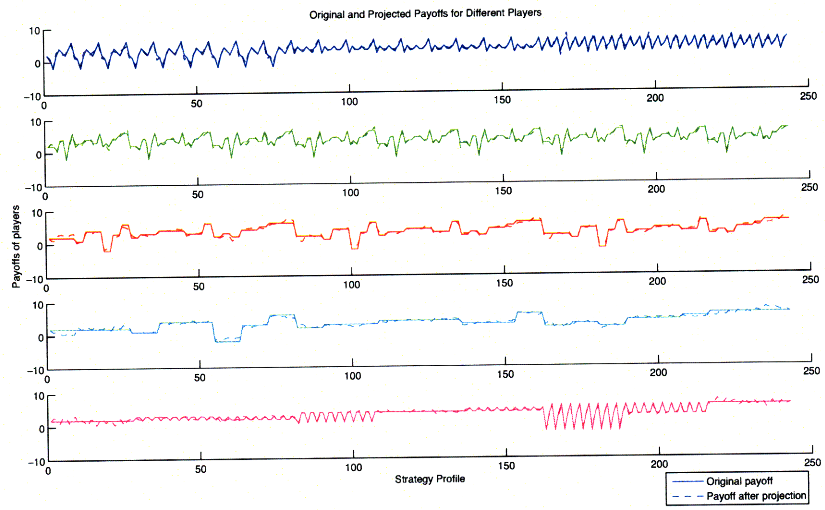


Figure 4-3: Original and projected payoff functions

In Figure 4-4 we plot the projection error at each step of the consensus algorithm. It can be seen from this figure that convergence to the projected game takes place in a small number of steps.

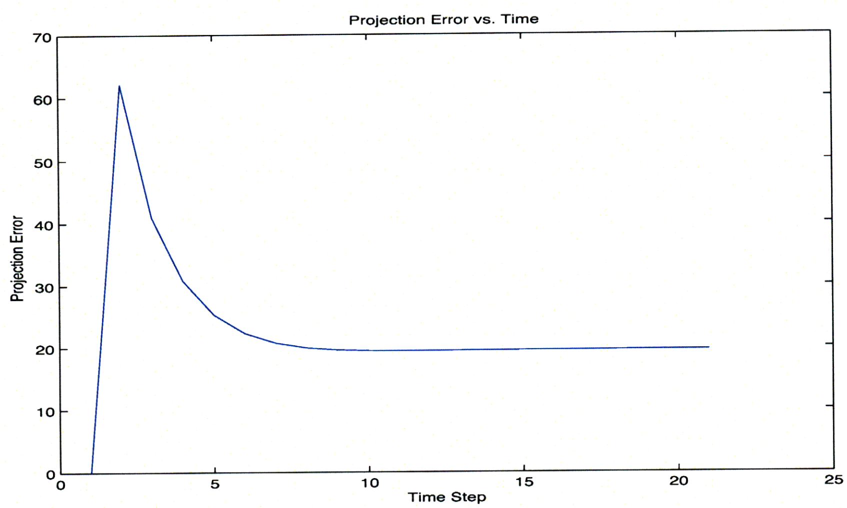


Figure 4-4: Projection Error vs. Time

Next we assume that players utilize their best responses and do projection simultaneously. We also assume that the game is initialized at a randomly chosen strategy profile and at each update each player independently plays its best response with probability 0.5. We assume that the best responses are played according to players payoff function at the time of the update. In Figure 4-5 we plot the aggregate payoffs of players at each step of the consensus algorithm. It can be seen that aggregate payoffs also converge in small number of steps, hence an equilibrium is reached, after a small number of steps. On the other hand, in the initial game it may not be possible to reach an equilibrium with best responses due to weak improvement cycles in the joint strategy space.

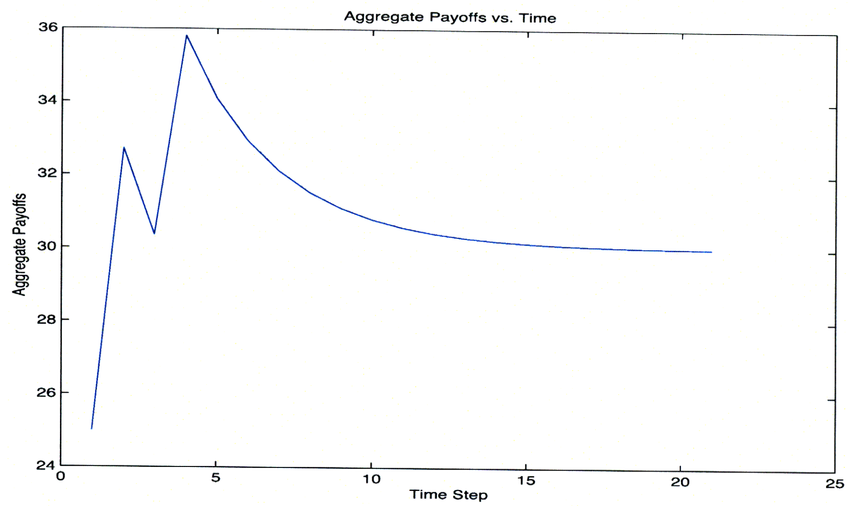


Figure 4-5: Aggregate Payoff vs. Time

Chapter 5

Conclusions

5.1 Summary

In this thesis we have considered a wireless network game, where mobiles interact over a shared collision channel. The novelty in our model is the state correlation assumption, which incorporates the effect of global time-varying conditions on performance. In general, the correlated state can be exploited by the users for time-division of their transmission, which would obviously increase the system capacity. However, we have shown that under self-interested user behavior, the equilibrium performance can be arbitrarily bad. Nevertheless, the efficiency loss at the best equilibrium can be bounded by a function of a technology parameter, which accounts both for the mobiles power limitations and the level of discretization of the underlying channel quality. Importantly, we have shown that under certain assumptions best-response dynamics converge to an equilibrium in finite time, and empirically verified that such dynamics converge fairly fast.

In the study of dynamics of the scheduling game we used the properties of potential games. In order to have a better understanding of potential games we studied the convexity properties and dimensions of the spaces of potential games. We also extended the known results in the literature on the existence of ordinal potential in games and used these new results to show that the scheduling game introduced in this thesis does not have a twice continuously differentiable potential function unless a rate alignment condition holds.

In this thesis, we have also studied the problem of finding a potential game that is close in some sense to a given game. To this end, we have defined different methods for projecting a game to the set of exact potential games. We have obtained closed form solutions for projections using 2-norm, and showed that if 2-norm is used the projection can be obtained with a distributed scheme by making use of the consensus algorithms. Our simulations indicate that a distributed algorithm converges to a potential game in a small number of steps.

Additionally, we obtained a relationship between the equilibria of a game and its projection. Particularly, we showed that the equilibria of a game remain to be ϵ-equilibria of its projection, where ϵ is bounded by the projection error.

5.2 Future Work

We briefly note several extensions and open directions of the presented work.

For the scheduling game the convergence analysis of best-response dynamics under more general conditions is important. It is demonstrated with simulations that convergence to an equilibrium with best responses takes place even when the game is not a potential game. This suggests that new tools rather than the theory of potential games are necessary in order to prove convergence of dynamics when the rate alignment condition does not hold. Another challenging direction is to obtain a tight bound on the price of stability, and examine how the price of anarchy can be bounded while fixing other game parameters besides the technological quality. The fading model we used in this thesis assumed that all users in the network receive the same channel state at all time instants. An extension of the current model is to consider the *partial* correlation case, in which a user reacts to a channel state that incorporates both global and local temporal conditions.

In this thesis we provide a condition for existence of twice continuously differentiable ordinal potential in continuous games. However, it is not true that a potential game with differentiable utility functions always has a differentiable potential function. Therefore, it is necessary to relax this differentiability assumption in order to be able to test existence of ordinal potential in games. Tools from differential calculus and vector calculus may be used in order to relax the differentiability condition on the potential. We leave this as a challenging future problem.

It is interesting to identify other convex sets of games with desirable properties. Given a convex set of games it is possible to extend the current projection framework to obtain projections of games to this set. This approach provides insights on how to modify the game (or equivalently the utilities of players) to obtain a game with desirable properties. One particular example is projecting a game to the set of exact potential games, with convex potential functions. This set is convex and projection of an arbitrary game to this set has a unique equilibrium. Therefore, projection of an arbitrary game on this set may be important, and is a topic for future study.

The distributed projection method presented in this thesis requires each player to exchange its entire payoff matrix with its neighbors. However, it is not clear if a distributed scheme, which obtains projection of a game to the set of exact potential games, exists under communication constraints. Study of distributed projection schemes which work under communication constrains may be important for practical applications of the projections and is left as a future problem.

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