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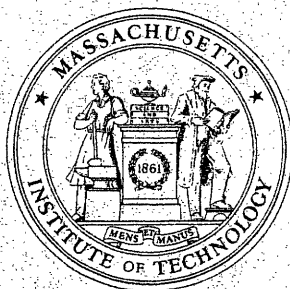
working paper

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THE INFINITE-HORIZON DYNAMIC LOT-SIZE PROBLEM
WITH CYCLIC DEMAND AND COSTS

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0. INTRODUCTION

In this paper we consider an infinite-horizon, dynamic lot-size problem with cyclic demand and costs. This problem is a natural extension of the finite-horizon problem first studied by Wagner and Whitin [10]. After formulating the infinite-horizon problem, we interpret the problem as a minimal cost-to-time ratio circuit problem [2]. With this interpretation we establish directly that an optimal policy is periodic and specify an efficient algorithm for finding the optimal policy. Finally we indicate how these results pertain to simple extensions of the problem, first allowing backorders and then allowing a discounted cost criterion.

1. AVERAGE COST PROBLEM

Define T to be the number of periods per cycle. For each cycle we define the following parameters, all of which are nonnegative integers:

d_i = unit demand in i^{th} period of each cycle, $i=1,2,\ldots,T$;

h_i = unit holding cost for carrying inventory into the $i+1^{st}$ period, $i=1,2,\ldots,T-1$, or into the first period of the next cycle for $i=T$;

f_i = fixed setup cost for the ith period, $i=1,2,\ldots,T$;

v_i = variable unit production cost for the ith period, $i=1,2,\ldots,T$.

In the infinite-horizon dynamic lot-size problem with cyclic demand and costs, we assume each cycle of T periods is identical and repeats itself indefinitely. We denote the ith period of cycle r as period i^r. By convention we understand the notation i^r-1 to denote the prior period, and, in particular, to denote period T^{r-1} when $i=1$; also, $i^r < j^s$ implies either $s > r$ or $s = r$ and $j > i$.

The decision variables for this problem are as follows:

P_i^r = quantity produced in period i^r, $i=1,2,\ldots,T$, $r=1,2,\ldots$;

I_i^r = on-hand inventory at the end of i^r, $i=1,2,\ldots,T$, $r=1,2,\ldots$.

The problem statement is to determine these production and inventory values so that all demand is met without backordering and average long-run production and inventory cost per period is minimized.

This problem statement would seem to be an appropriate representation of settings with a strong seasonal or cyclic demand component. This cyclic property may occur due to a natural product seasonality, or may be induced from the composition of cyclic purchasing patterns of a set of customers, i.e. customer A buys 100 units once every three periods... Furthermore, the study of the infinite-horizon problem should provide insight to and supplement the work on planning horizons for the dynamic lot-size problem (Wagner and Whitin [10], Eppen, Gould, and Pashigian [3], Zabel [11], Lundin and Morton [9], Chand and Morton [1]).

Representation as a Minimum Cost-to-Time Ratio Circuit Problem

We assume that at least one holding cost is positive; else it is optimal to produce an infinite amount on the first occurance of the period with minimum unit production cost. Let $H = h_1 + \ldots + h_T$, and let f_{max} be the largest fixed setup cost. We claim that we may restrict attention to solutions satisfying the following two properties:

P1: For all r, $I_{i-1}^r P_i^r = 0$ for $i=2,\ldots,T$ and $I_T^{r-1} P_1^r = 0$.

P2: There are at most $T \cdot f_{max}/H$ consecutive periods in which a positive amount of inventory is held.

Property P1 is the immediate counterpart to Theorem 1 in [10] and has the same proof, which we omit. We prove the validity of P2 via an interchange argument. Suppose that we produce in period i^r and store at least one unit for $kT+j$ consecutive periods for some integers $k > f_{max}/H$, $j \geq 0$. Now consider a modified policy in which production in period i^r

is decreased by one unit, while production in period i^{r+k} is increased

by one unit with the incremental unit being held for j periods. The net

savings in cost for the modified policy is at least $kH-f_i$, which is positive

by assumption. Therefore, we can "improve" any policy for which P2 does

not hold.

A direct consequence of P1 and P2 is that an optimal policy exists

such that a positive P_i^r is just sufficient to cover all demand requirements

from period i^r up to but not including period j^{r+k} where $i^r < j^{r+k}$ and

$k < f_{max}/H$. Hence, we have either $P_i^r = 0$ or $P_i^r = D_{ij}+kD$ for some j,k

integer with $i^r < j^{r+k}$, $k < f_{max}/H$ and

$$D = d_1 + d_2 + \ldots + d_T ,$$

$$D_{ij} = \begin{cases} d_i + \ldots + d_{j-1} & \text{if } i < j \\ 0 & \text{if } i = j \\ - D_{ji} & \text{if } i > j \end{cases} .$$

If we consider only solutions satisfying P1 and P2, the resulting

problem is a minimum cost-to-time ratio circuit problem. This interpreta-

tion is an infinite-horizon version of Zangwill's [13] interpretation of

the finite-horizon dynamic lot-size problem as a shortest path problem.

We construct the graph $G = (V,E)$ for the cyclic lot-size problem as

follows: The vertex set is $V = \{1,\ldots,T\}$ with one vertex for each period

of a cycle. For each pair i,j of vertices and for each $k < f_{max}/H$ such

that $i^0 < j^k$, we have an edge (i,j) with <u>transit time</u> k. A unit flow on

this edge corresponds to setting $P_i^r = D_{ij}+kD$ for some cycle r. The cost

on this edge, c_{ij}^k, equals the production and holding costs associated

with the specified P_i^r from period i^r up to but not including period j+k.

The dynamic lot-size problem is to circulate one unit of flow through

the above graph so as to minimize the ratio of the flow cost to the transit

time for the flow. This is exactly the "tramp steamer problem", (also called the minimum cost-to-time ratio circuit problem) as proposed and solved by Dantzig et al. [2]. A directed circuit in the graph whose total transit time is t corresponds to a production schedule that has initial and terminal inventories of 0 and repeats after exactly t cycles. If this schedule is repeated infinitely often, then the average cost per cycle is the net cost of the circuit divided by t.

An immediate consequence of the correspondence to the "tramp steamer problem" is that for the infinite-horizon dynamic lot-size problem an optimal policy exists that is periodic. That is, there is some optimal policy given by P_i^r such that $P_i^r = P_i^{r+t}$ where t is integer and denotes the periodicity of the policy. In particular, t equals the transit time of the optimal circuit in the minimum cost-to-time ratio circuit problem.

The difficulty with the above representation is the number of multiple edges. Lawler [8] shows that the minimum cost-to-time circuit problem may be solved in $0 \ (|V| \ |E| \ \log \ (t^* + c^* + T))$ steps, where t^* is the maximum transit time and c^* is the maximum edge cost. In our case, such an algorithm is $0[T^3 \cdot f_{max}/H \cdot \log \ M^*]$ steps where $M^* = \max(f_{max}, v_{max}, d_{max}, h_{max}, T)$, and this is not necessarily polynomial in the data. Of course, one can improve the results slightly by showing a priori that certain edges cannot appear in an optimal cycle; however, we can improve dramatically on this result. In the next section we present an $0(T^3 \log \ M^*)$ algorithm, which is in many cases better than $0(|E|) = 0(T^2 \cdot f_{max}/H)$.

An Efficient Implementation

A standard technique for determining a minimum cost-to-time ratio circuit is an iterative procedure based on the following observation:
Remark. Let G be a directed graph, and for each edge e let c_e and t_e denote its cost and transit time. Let λ be a real number and let

$\hat{c}_e = c_e - \lambda t_e$ be the <u>reduced cost</u> of edge e. Then any circuit C in G has a cost-to-time ratio of at most λ if and only if the reduced cost of C is nonnegative [7].

The technique based on this remark is to use binary search to find the minimum value of λ for which there is no circuit with negative reduced cost. At each iteration for each pair i,j we select that edge (i,j) with minimum reduced cost and ignore all other edges from i to j. In the next section we show that we can do this efficiently. The net time for computing whether there is a circuit of negative length is $O(T^3)$ via the Bellman-Ford algorithm [7]. The number of iterations is at most $2 \log T + \log c_{max} + \log t_{max}$ where c_{max} and t_{max} are the maximum edge cost and transit time. To see this, note that the cost of a circuit is bounded above by $T \cdot c_{max}$ and bounded below by 1 while the transit time is bounded above by $T \cdot t_{max}$ and below by 1. Thus $T^2 \cdot c_{max} \cdot t_{max}$ bounds the number of values that the ratio of cost-to-time may take; the maximum number of iterations for a binary search is the logarithm, which is $2 \log T + \log c_{max} + \log t_{max}$. To obtain an explicit upper bound on the computation time, we calculate c_{max} and t_{max} in the next section.

<u>Calculating Costs and Transit Times</u>

In the following calculations of the costs c_{ij}^k, let a_{ij}^k denote the holding cost component in c_{ij}^k. We therefore have the following relation:

$$c_{ij}^k = f_i + v_i(D_{ij} + kD) + a_{ij}^k \tag{1}$$

The cost a_{ij}^k is computed as[*]

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[*] We find it useful to define a_{ij}^0 for $i > j$ even though the corresponding c_{ij}^0 is not defined; a_{ij}^0 for $i > j$ is the holding cost savings from meeting demand requirements from period j^0 to i^0 and is needed for (5).

$$a^0_{ij} = \sum_{\ell=i}^{j-1} h_\ell D_{\ell+1,j} \qquad \text{for } i < j,$$

$$a^0_{ij} = 0 \qquad \text{for } i = j, \qquad (2)$$

$$a^0_{ij} = \sum_{\ell=j}^{i-1} h_\ell D_{j,\ell+1} = \sum_{\ell=j}^{i-1} h_\ell D_{ji} - a^0_{ji} \qquad \text{for } i > j,$$

with

$$a^1_{ii} = \left[\sum_{\ell=i}^{T-1} h_\ell (D - D_{i,\ell+1}) \right] + h_T D_{1i} + \left(\sum_{\ell=1}^{i-1} h_\ell D_{\ell+1,i} \right) \qquad (3)$$

and

$$a^k_{ii} = k\, a^1_{ii} + \frac{k(k-1)}{2} DH \qquad \text{for } k > 1. \qquad (4)$$

We can compute the remaining values of a^k_{ij} from

$$a^k_{ij} = a^k_{ii} + a^0_{ij} + k D_{ij} H \qquad \text{for } k \geq 1. \qquad (5)$$

To explain (5) we consider two cases: $i < j$ and $i > j$. For $i < j$, the cost difference between holding inventory from i^0 to $j^k (a^k_{ij})$ and holding inventory from i^0 to i^k (a^k_{ii}) is the cost of holding D_{ij} units from i^0 to i^k $(k D_{ij} H)$ plus the holding cost from i^k to j^k (a^0_{ij}). For $i > j$, the cost for holding inventory from i^0 to j^k (a^k_{ij}) equals the holding cost from i^0 to i^k (a^k_{ii}) minus the incremental cost incurred to satisfy requirements (D_{ji}) from j^k to i^k. This incremental cost is the cost of holding D_{ji} units from i^0 to i^k $(k D_{ji} H = -k D_{ij} H)$ minus the cost savings from j^k to i^k (a^0_{ij}).

Note that t_{max} is bounded by $k^* = \lfloor f_{max}/H \rfloor$ as before, and c_{max} is bounded above by $f_{max} + v_{max} \cdot k^* \cdot D + k^* \cdot T \cdot h_{max} \cdot D + k^*(k^* - 1)DH/2$. Thus the number of iterations $(2 \log T + \log c_{max} + \log t_{max})$ is $0 (\log M^*)$,

where $M^* = \max(f_{max}, v_{max}, d_{max}, h_{max} T)$. We now show that the algorithm is $O(T^3 \log M^*)$ by showing that at each iteration the reduced costs may be calculated in $O(T^2)$ steps, and thus do not add to the complexity of the algorithm.

At each iteration the minimum reduced cost for an edge (i,j) given parameter λ is found by setting the transit time k to be max $\{o, k^*\}$ for i < j and to be max $\{1, k^*\}$ for $i \geq j$ where

$$k^* = \lceil (\lambda - a_{ii}^1 - v_i D - HD_{ij})/HD \rceil . \qquad (6)$$

From (1)-(5) this choice of k minimizes the reduced cost for edge (i,j). Hence for each pair (i,j) at each iteration of the algorithm the minimum cost edge is found in a constant number of elementary operations.

A Good Starting Point

In the special case in which we restrict the edges to transit times of at most $\hat{k}$ (i.e., we would not produce so as to satisfy demand $\hat{k}$ cycles in the future), the minimum cost-to-time ratio circuit can be calculated in $O(\hat{k}^3 T^3)$ steps via the method of Karp and Orlin [5]. One approach to solving the lot-size problem is to solve first a minimum cost-to-time ratio circuit problem in which we consider only edges with transit time 0 or 1; then we may use the resulting optimal ratio λ^* as a starting point to the original problem with no restrictions on transit times.

3. BACKORDER CASE

The preceding analysis of the infinite-horizon lot-size problem can be directly extended to a problem definition that allows backordering of demand. This analysis is analogous to the backorder extension by Zangwill [12] to the finite-horizon lot size problem. We define g_i to

be the unit cost of demand backordered from period i to period i+1 for

i=1,...,T-1 or from period T to the first period of the next cycle for

i=T. Let $G = g_1 + \ldots + g_T$. As before, we let P_i^r and I_i^r denote the

production and inventory in period i^r except that here we interpret a

negative value for I_i^r as backordered demand.

We define period i^r+1 to be a <u>regeneration point</u> if $I_i^r = 0$ and demand

in period i^r+1 is non-zero. Property P1 in the case of backorders may

be restated as follows:

P1': a) If periods i^r and j^s are successive regeneration points, then

        there is exactly one period k^t with $i^r \leq k^t < j^s$ such that

        $P_k^t > 0$; i.e., between successive regeneration points, there is

        exactly one period of positive production.

    b) If P_i^r, $P_j^s > 0$ and $i^r < j^s$, then there exists a period k^t with

        $i^r \leq k^t < j^s$ such that $I_k^t = 0$.

This property is directly analogous to that given by Zangwill [12]

for the finite-horizon problem with backorders. Property P2 is now

supplemented by the following property when backorders are allowed:

P2': There are at most $T \cdot f_{max}/G$ consecutive periods in which demand is

      backordered, and hence at most $R^* = \lfloor T \cdot f_{max}[(1/G)+(1/H)] \rfloor$ periods

      between successive regeneration points.

Now, we may interpret the backorder problem as a minimum cost-to-time

ratio circuit problem. Again we construct the graph $G = (V,E)$ with

vertex set $V = \{1,\ldots,T\}$. Here we interpret the edges as follows: For

each pair i,j of vertices and $k \leq R^*$ there is an associated edge (i,j)

with transit time k; the edge cost c_{ij}^k is the minimum cost of producing,

storing, and backordering so as to satisfy all demand between successive

regeneration points i^r and j^{r+k}, for any cycle r. By property P1', there

are integers ℓ and s such that $i^r \leq \ell^{r+s} < j^{r+k}$ and ℓ^{r+s} is the unique period of production in any optimal solution using the edge (i,j) with transit time k.

The dynamic lot-size problem with backordering is again to circulate one unit of flow through G to minimize the ratio of flow cost to transit time. We show below that the binary search algorithm of the previous section can again be efficiently implemented; for the backorder case the complexity of the algorithm is $O(T^3 \log M^{**})$ where $M^{**} = \max(M^*, g_{max})$.

Our analysis proceeds as before. The maximum number of iterations of the binary search is bounded above by $\log(T^2 c_{max} t_{max}) = \log(T^2 c_{max} R^*)$ $= O(\log M^{**})$. To derive the $O(T^3 \log M^{**})$ bound we show that each iteration may be completed in $O(T^3)$ steps.

An Efficient Implementation

The cost of edge (i,j) with transit time k is given by

$$c_{ij}^k = \min_{\ell,s} \{f_\ell + v_\ell(D_{ij}+kD) + b_{i,\ell}^s + a_{\ell j}^{k-s}\} \qquad (7)$$

where ℓ,s are nonnegative integers such that $i^0 \leq \ell^s < j^k$, and where $b_{i,\ell}^s$ is the total backorder cost from period i^0 up to but not including period ℓ^s assuming that period i^0 is a regeneration point and that period ℓ^s is the next production point after this regeneration point. The values of b_{ij}^k can be computed analogously to the values a_{ij}^k [i.e., equations (2)-(5)], and we omit the recursive formulae. The additional computational time is at most equal to the time to compute the a_{ij}^k's, and does not increase the order of computation for the algorithm.

At each iteration of the algorithm, for a given value of λ we calculate the reduced costs for each pair of vertices (i,j):

$$\hat{c}_{ij} = \min_k (c_{ij}^k - k\lambda)$$

$$= \min_k \{\min_{\ell,s} [f_\ell + v_\ell(D_{ij} + kD) + b_{i,\ell}^s + a_{\ell j}^{k-s} - k\lambda]\}$$

$$(8)$$

If we substitute t = k-s and rearrange the minimization operations, we
obtain

$$\hat{c}_{ij} = \min_{\ell} \{f_\ell + v_\ell D_{ij} + \min_{s} (sv_\ell D + b_{i,\ell}^s - s\lambda)$$

$$+ \min_{t} (tv_\ell D + a_{\ell j}^t - t\lambda)\} \tag{9}$$

Note that for a set value of ℓ, the determination of t is identical to
the determination of the best edge from vertex ℓ to j for the no-backorder
case. But this can be done by (6) in a constant number of operations.
Similarly, given ℓ the determination of s is immediate by means comparable
to (6). Consequently, since ℓ may take on values 1,2,...,T, the complexity
of the determination of $\hat{c}_{ij}$ is 0(T); since the number of edges is
T(T-1)/2, the determination of $\hat{c}_{ij}$ for all edges is $0(T^3)$ in complexity.

3. DISCOUNTING

In this section we consider the dynamic lot-size problem in which the
objective is to determine the minimum discounted cost for satisfying demands
over an infinite horizon. We let the discount rate be ρ per cycle, and
we let c_{ij}^k denote the minimum cost of production, storage, and backordering
for satisfying demand between regeneration points i^0 and j^k, with costs
being discounted to the present (period 1^0).

Let z_i denote the minimum discounted cost of satisfying all demands
over the infinite horizon starting in period i^0. Then the minimum discounted
cost of satisfying all demands starting in period i^r is $\rho^r z_i$, where ρ^r is
the rth power of ρ. Furthermore, the following recursion uniquely deter-
mines the values for z:

$$z_i = \inf_{j,k} \{c_{ij}^k + \rho^k z_j\}. \tag{10}$$

Once the values of c_{ij}^k are known, the values for z_i may be determined

by a standard technique such as policy improvement [4] or linear programming.

Below we show that it is not necessary to calculate all of the values for

c_{ij}^k; instead, we can save much of the computation time by a preprocessing

of the problem data. For the policy improvement procedure, this preprocess-

ing results in each iteration of the procedure having $O(T^2)$ computational

requirements. For a linear programming approach to (1) this preprocessing

permits solution via Khachian's algorithm [6] in polynomial time.

We note that the straight-forward approach of evaluating c_{ij}^k for all

i, j, k has no immediate upper bound because there is no bound on how far

ahead we might backorder demand; indeed, in some cases the optimal solution

may be to never produce, but rather to backorder all demand for all time.

Furthermore, the $O(T^2)$ result is surprising in that for the average-cost

problem with backordering the amount of computation per iteration just to

compute the reduced costs was found to be $O(T^3)$.

Implementation of Policy Improvement

As before, we restrict ourselves to policies such that between two

successive regeneration points there is exactly one period of production,

and between two production periods there is exactly one regeneration point.

If we let b_{ij}^k and a_{ij}^k denote the _discounted_ costs of backordering and storing,

then equation (10) may be rewritten as

$$z_i = \inf_{j,\ell,k,r} \{b_{i,\ell}^r + \rho^r [f_\ell + v_\ell (D_{ij} + kD) + a_{\ell j}^{k-r}] + \rho^k z_j\},$$

$$\tag{11}$$

where ℓ^r is the production period between regeneration points i^0 and j^k. By

letting t = k-r, and by substituting formulae (A13) and (A17) derived in

the Appendix into formula (11), we obtain

$$z_i = \inf_{\ell,r} \{\beta_{i,\ell} + \rho^r [r\hat{\beta}_{i,\ell} + \bar{\beta}_{i,\ell} + \hat{z}_\ell]\} \tag{12}$$

where

$$\hat{z}_\ell = \inf_{j,t} [\alpha_{\ell j} + t\hat{\alpha}_{\ell j} + \rho^t(\bar{\alpha}_{\ell j} + z_j)] \tag{13}$$

where the constants $(\alpha_{ij}, \hat{\alpha}_{ij}, \bar{\alpha}_{ij})$ and $(\beta_{ij}, \hat{\beta}_{ij}, \bar{\beta}_{ij})$ are derived in the Appendix. The significance of rewriting (11) as (12)-(13) is that we can separate the evaluation of z_i into two components, the first of which is the determination of the number of periods to backorder while the second is the determination of the number of periods to carry positive inventory. We show next that this separation permits us to perform each iteration in a policy improvement algorithm in $O(T^2)$ steps.

At each iteration of a policy improvement algorithm we have a current estimate to the vector $\{z_i\}$. Based on this current estimate, we evaluate (13) to obtain a revised estimate for $\{\hat{z}_\ell\}$, which is used in (12) to obtain an improved estimate to $\{z_i\}$. To evaluate $\hat{z}_\ell$ in (13), suppose we specify a value for j. If $(\bar{\alpha}_{\ell j} + z_j)$ is nonpositive, the best choice for t is 0 if $\ell < j$ and 1 if $\ell \geq j$, since $\hat{\alpha}_{\ell j} \geq 0$ and $0 < \rho < 1$. If $(\hat{\alpha}_{\ell j} + z_j)$ is positive, $(t\hat{\alpha}_{\ell j} + \rho^t (\bar{\alpha}_{\ell j} + z_j))$ is convex in t and takes its minimum value over the set of integers at

$$t^* = \lceil -\log((\bar{\alpha}_{\ell j} + z_j)(1-\rho)/\hat{\alpha}_{\ell j})/\log \rho \rceil; \tag{14}$$

then the best choice for t in (13) is $\max(0,t^*)$ if $\ell < j$ and $\max(1,t^*)$ if $\ell \geq j$. In either case, for a given ℓ and j in (13), we can obtain the optimal choice for t in a constant number of elementary operations. Since j can take on at most T values, the determination of $\hat{z}_\ell$ for any value of

ℓ requires $O(T)$ steps, and thus the computation of $\hat{z}_\ell$ for $\ell = 1, 2, \ldots, T$ is $O(T^2)$ in total.

For a specified value of ℓ in (12), the optimal choice for r is also immediate due to the following remarks:

Remark: For $0 < \rho < 1$ and $C_3 < 0$, we define $f(r) = C_1 + C_2\rho^r + C_3 r\rho^r$ to be evaluated on the set of integers. Then $-f(r)$ is unimodal.

Proof: Consider $f(r+1) - f(r)$ given by $\rho^r[C_3\rho - C_2(1-\rho) - C_3 r(1-\rho)]$. If $C_3 < 0$, we have

$$f(r+1) - f(r) < 0 \qquad \text{for } r < \rho(1-\rho)^{-1} - C_2/C_3,$$

and $\qquad f(r+1) - f(r) > 0 \qquad \text{for } r > \rho(1-\rho)^{-1} - C_2/C_3.$

Hence, for $C_3 < 0$, $-f(r)$ is unimodal with mode at $r = \lceil \rho(1-\rho)^{-1} - C_2/C_3 \rceil$.

With the above remark, having determined $\hat{z}_\ell$ for all ℓ, we can evaluate a particular z_i via (12) in $O(T)$ steps since ℓ may take on T values in the minimization, and since the determination of the optimal choice for r is immediate for a specified value of ℓ in the minimization. To see this, note that if $\hat{\beta}_{i,\ell} < 0$ the above remark applies. If $\hat{\beta}_{i,\ell} \geq 0$, the optimal choice for r for a specified value for ℓ is either at its upper or lower bound, since $\rho^r(r\hat{\beta}_{i\ell} + \hat{\beta}_{i\ell})$ is either unimodal or monotonic in r. Hence, the determination of z_i, $i = 1, 2, \ldots, T$, given $\{\hat{z}_\ell\}$, also requires $O(T^2)$ steps. Consequently, each iteration of a policy improvement procedure takes $O(T^2)$ time; unfortunately we cannot bound the total computational requirements for the policy improvement procedure since we have not found a bound on the number of iterations.

Implementation of Linear Programming

We may solve the discounted problem in polynomial time via the ellipsoidal algorithm [6]. To see this, we rewrite (10) as its equivalent linear program:

$$\text{Min} \quad z_1 + \ldots + z_T \tag{15}$$

subject to

$$z_i \geq c_{ij}^k + \rho^k z_j$$

$$\text{for } i,j=1,2,\ldots,T, \ k=0,1,2,\ldots, \text{ and } i^0 < j^k. \tag{16}$$

Given any vector $\{z_i\}$, we can discover in $O(T^2)$ steps, as shown above, whether it is feasible (and hence optimal) to (15)-(16), and if not, we find a violated constraint. Therefore, the ellipsoidal algorithm runs in polynomial time despite the infinite number of constraints implied by the linear program (15)-(16).

Appendix: Calculating the Parameters for the Discounted Problem

We redefine the parameters f_i, v_i, g_i, and h_i to be costs for period i^0 discounted to 1^0. Similarly, a_{ij}^k (b_{ij}^k) is the discounted cost of storage (backorders) from period i^0 to period j^k as defined earlier. We define $H_i = \sum_{\ell=1}^{T} h_\ell + \rho \sum_{\ell=1}^{i-1} h_\ell$ to be the discounted cost for holding one unit from period i^0 to i^1.

We may calculate the values for a_{ij}^k via recursive formulae analogous to (2) - (5). For $k = 0$ we define

$$a_{ij}^0 = \sum_{\ell=i}^{j-1} h_\ell D_{\ell+1,j} \qquad \text{for } i < j,$$

$$a_{ij}^0 = 0 \qquad \text{for } i = j, \qquad (A1)$$

$$a_{ij}^0 = \sum_{\ell=j}^{i-1} h_\ell D_{j,\ell+1} = \sum_{\ell=j}^{i-1} h_\ell D_{ji} - a_{ji}^0 \qquad \text{for } i > j.$$

To compute the remaining values of a_{ij}^k, we need to use

$$a_{ii}^1 = \left[\sum_{\ell=i}^{T-1} h_\ell (D - D_{i,\ell+1})\right] + h_T D_{1i} + \rho\left(\sum_{\ell=1}^{i-1} h_\ell D_{\ell+1,i}\right) \qquad (A2)$$

Noting that the unit cost of storage from perioc i^0 to period i^k is $(1-\rho^k)(1-\rho)^{-1} H_i$, we may form the following recursive formula for a_{ii}^k:

$$a_{ii}^k = a_{ii}^{k-1} + \rho^{k-1} a_{ii}^1 + (1-\rho^{k-1})(1-\rho)^{-1} H_i D. \qquad (A3)$$

It is now easy to prove inductively from (A3) that

$$a_{ii}^k = (1-\rho^k)(1-\rho)^{-1} a_{ii}^1 + [(k-1) - \rho k + \rho^k](1-\rho)^{-2} H_i D. \qquad (A4)$$

Finally, we can calculate a_{ij}^k for $i \neq j$, $k \geq 1$ from

$$a_{ij}^k = a_{ii}^k + \rho^k a_{ij}^0 + (1-\rho^k)(1-\rho)^{-1} H_i D_{ij} . \qquad (A5)$$

The explanation of (A5) is identical to that given for (5).

Analogous to the inventory cost determination, we can determine the backorder cost b_{ij}^k from i^0 to j^k as follows:

$$b_{ij}^0 = \sum_{\ell=i}^{j-1} g_\ell D_{i,\ell+1} \qquad \text{for } i < j,$$

$$b_{ij}^0 = 0 \qquad \text{for } i = j, \qquad (A6)$$

$$b_{ij}^0 = \sum_{\ell=j}^{i-1} g_\ell D_{\ell+1,i} = \sum_{\ell=j}^{i-1} g_\ell D_{ji} - b_{ji}^0 \qquad \text{for } i > j.$$

We let $G_i = \sum_{\ell=i}^{T} g_\ell + \rho \sum_{\ell=1}^{i-1} g_\ell$ be the unit cost of backordering from i^0 to i^1. Then we may calculate b_{ii}^1 as

$$b_{ii}^1 = \sum_{\ell=i}^{T-1} g_\ell D_{i,\ell+1} + g_T (D-D_{1i}) + \rho \sum_{\ell=1}^{i-1} g_\ell (D-D_{\ell+1,i}), \qquad (A7)$$

and we may find b_{ii}^k recursively from

$$b_{ii}^k = b_{ii}^{k-1} + \rho^{k-1} b_{ii}^1 + \rho^{k-1} G_i (k-1) D . \qquad (A8)$$

The interpretation of (A8) is that the increased cost of backordering from i^0 to i^k over backordering from i^0 to i^{k-1} is due to the cost of back-ordering to satisfy demand from i^{k-1} to i^k plus the cost of backordering an additional $(k-1)D$ units from period i^{k-1} to i^k to satisfy demand requirements from i^0 to i^{k-1}.

The value b_{ii}^k may be shown by induction to be:

$$b_{ii}^k = (1-\rho^k)(1-\rho)^{-1} b_{ii}^1 + (1-k\rho^{k-1} + (k-1)\rho^k)(1-\rho)^{-2} \rho G_i D . \qquad (A9)$$

Finally, by reasoning analogous to that used for (5) we obtain for $i \neq j$, $k \geq 1$

$$b^k_{ij} = b^0_{ij} + b^k_{jj} + (1-\rho^k)(1-\rho)^{-1} G_j D_{ij} . \qquad (A10)$$

Now consider the cost expression

$$b^r_{i\ell} + \rho^r [f_\ell + v_\ell (D_{ij} + kD) + a^{k-r}_{\ell j}] \qquad (A11)$$

as used in (11) to equal the total discounted costs between successive regeneration periods i^0 and j^k, where period ℓ^r is the production period between the periods i^0 and j^k. By letting $t=k-r$, we have (A11) equal to

$$\{b^r_{i\ell} + \rho^r [f_\ell + v_\ell (D_{i\ell} + rD)]\} + \rho^r \{v_\ell (D_{\ell j} + tD) + a^t_{\ell j}\} . \qquad (A12)$$

By use of (A4)-(A5) and (A9)-(A10), we may simplify the components of (A12) as follows:

$$\{b^r_{i\ell} + \rho^r [f_\ell + v_\ell (D_{i\ell} + rD)]\} = \beta_{i\ell} + \rho^r (r\hat{\beta}_{i\ell} + \overline{\beta}_{i\ell}) \qquad (A13)$$

where

$$\beta_{i\ell} = b^0_{i\ell} + (1-\rho)^{-1} (b^1_{\ell\ell} + G_\ell D_{i\ell}) + (1-\rho)^{-2} \rho G_\ell D, \qquad (A14)$$

$$\hat{\beta}_{i\ell} = (v_\ell - (1-\rho)^{-1} G_\ell)D , \qquad (A15)$$

$$\overline{\beta}_{i\ell} = f_\ell + v_\ell D_{i\ell} - (1-\rho)^{-1} (b^1_{\ell\ell} + G_\ell D_{i\ell}) + (1-\rho)^{-2} \rho G_\ell D, \qquad (A16)$$

and

$$\{v_\ell (D_{\ell j} + tD) + a^t_{\ell j}\} = \alpha_{\ell j} + t \hat{\alpha}_{\ell j} + \rho^t \overline{\alpha}_{\ell j} \qquad (A17)$$

where

$$\alpha_{\ell j} = v_\ell D_{\ell j} + (1-\rho)^{-1} (a^1_{\ell\ell} + H_\ell D_{\ell j}) - (1-\rho)^{-2} H_\ell D , \qquad (A18)$$

$$\hat{\alpha}_{\ell j} = v_\ell D + (1-\rho)^{-1} H_\ell D , \qquad (A19)$$

$$\bar{\alpha}_{\ell j} = a^0_{\ell j} - (1-\rho)^{-1} (a^1_{\ell \ell} + H_\ell \, D_{\ell j}) + (1-\rho)^{-2} H_\ell D . \tag{A20}$$

Hence, by substitution of (A13) and (A17) into (A12), we obtain formula (12).

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