

COMMUNICATION SCIENCES
AND
ENGINEERING

XVIII. STATISTICAL COMMUNICATION THEORY*

Prof. Y. W. Lee
Prof. A. G. Bose
Prof. M. Schetzen
Prof. H. L. Van Trees
Prof. V. R. Algazi
R. Alter
R. F. Bauer
C. J. Boardman

J. D. Bruce
A. M. Bush
M. Chessman
G. Crimi
S. Greenblatt
T. G. Kincaid
D. C. Kurtz

D. E. Nelsen
A. V. Oppenheim
R. E. Parente
M. A. Rich
P. M. Rogo
M. J. Schaffer
D. W. Steele
C. I. M. Uchoa

RESEARCH OBJECTIVES

This group is interested in a variety of problems in statistical communication theory. Our current research is concerned primarily with the following problems:

1. Work continues on the analysis and application of the "Two-State Modulation System" that was first described in Quarterly Progress Report No. 66 (pages 187-189). A static analysis of the system has been made, and work now in progress on the dynamic analysis is directed toward both control and power-amplification applications.

2. During the past year algorithms have been developed whereby optimum nonuniform quantizers can be designed when the quantizer input is either a signal or a signal contaminated by noise. This study will continue with emphasis placed upon the evaluation of these optimum quantizers. The effects of linear pre-emphasis and post-emphasis on the optimum quantizer will be investigated. Also, attempts will be made to apply algorithms similar to these quantizer algorithms to other forms of nonlinear filtering.

3. The use of Linear Algebra in the analysis and characterization of nonlinear systems is being investigated. This study has led to a generalization of the principle of superposition and a canonical form for systems satisfying this generalized principle.

4. Theoretical work predicts that the threshold level in multidimensional demodulation schemes can be reduced by use of more sophisticated demodulators. Experimental work is being conducted to verify these predictions.

5. The study of the performance of optimum and nonoptimum filters with emphasis on qualitative aspects of their behavior has continued. An investigation is being made of the limits on the performance of nonlinear filters when some of the message characteristics, such as average power, peak power, power spectrum, and so forth, are known.

6. Many physical processes can be phenomenologically described in terms of a large number of interacting oscillators. A theoretical and experimental investigation is being made.

7. A nonlinear system can be characterized by a set of kernels. The synthesis of a nonlinear system involves the synthesis of these kernels. A study of efficient methods for synthesizing these kernels continues.

8. A method for the construction of function generators was reported in Quarterly Progress Report No. 71 (pages 176-178). Work on the method, both theoretical and experimental, is in progress.

9. The central idea in the Wiener theory of nonlinear systems is to represent the output of a system by a series of orthogonal functionals with the input of the system being a white Gaussian process. An attempt is being made to extend the orthogonal representation to other types of inputs that may have advantages in the practical application of the theory.

* This work was supported in part by the National Science Foundation (Grant G-16526), the National Institutes of Health (Grant MH-04737-03), and in part by the National Aeronautics and Space Administration (Grant NsG-496).

(XVIII. STATISTICAL COMMUNICATION THEORY)

10. A study is being made to relate the random variations in the composition of magnetic recording tape to the noise introduced into a signal recorded on the tape. This will involve an experimental study of the variations in the packing density, orientation, and magnetic properties of the Fe_2O_3 particles making up the magnetic coating, in order to verify a theoretical relation between these properties and the noise.

Y. W. Lee

A. GENERALIZED SUPERPOSITION

1. Introduction

The ease of analysis and characterization of linear systems stems primarily from the fact that they satisfy the principle of superposition. Through the use of this principle, the response of a linear system to inputs that are representable as linear combinations of a set of building blocks can be described by its response to each of the building blocks. When the building blocks are impulses, for example, the system is described through the superposition integral; when the building blocks are complex exponentials and the system is time invariant, it is described through its system function.

The principle of superposition in its usual form is a statement of the definition of linearity and hence, by definition, cannot be satisfied by a nonlinear system. It can be generalized, however, in such a way that it encompasses a wide class of nonlinear systems. This report is concerned with a generalization of the principle of superposition, and an investigation of the class of nonlinear systems which obeys this generalized principle. The investigation has been carried out within the framework of linear algebra. Rather than bury the discussion under the formalism of linear algebra, however, we shall give only a general discussion of the approach used and the results obtained. In future reports, details in the analysis will be considered.

2. Homomorphic Systems

Linear algebra deals with linear transformations between vector spaces. The operations of vector addition and scalar multiplication, which impose an algebraic structure on the vector spaces, satisfy the algebraic postulates that we normally associate with addition of time functions and multiplication of time functions by scalars. There are, however, many other operations, which satisfy these same postulates, that can be performed on time functions. Multiplication of time functions, for example, is commutative and associative. Similarly, convolution is a commutative and associative binary operation. Transformations between such vector spaces, although linear in an algebraic sense, may be nonlinear in a more conventional interpretation.

Systems that are characterized by such transformations satisfy a generalized principle of superposition. Specifically, if "o" denotes the binary operation on the inputs

and " \square " denotes the binary operation on the outputs, then

$$T[v_1(t) \circ v_2(t)] = w_1(t) \square w_2(t) \quad (1)$$

where

$$T[v_1(t)] = w_1(t),$$

$$T[v_2(t)] = w_2(t),$$

and T is the system transformation. Also, if the combination of an input $v(t)$ with a scalar λ is denoted by $[\lambda \succ v(t)]$ and the combination of an output $w(t)$ with a scalar λ is denoted by $[\lambda/w(t)]$, then

$$T[\lambda \succ v(t)] = \lambda/w(t) \quad (2)$$

where

$$T[v(t)] = w(t).$$

If, for example,

$$T[v(t)] = e^{v(t)}, \quad (3)$$

then

$$T[v_1(t) + v_2(t)] = w_1(t) w_2(t)$$

and

$$T[\lambda v_1(t)] = [w_1(t)]^\lambda$$

for all inputs $v_1(t)$ and $v_2(t)$ and all scalars λ .

Because of the algebraic interconnection between addition and scalar multiplication, scalar multiplication can be interpreted in terms of the addition operation when the scalar is rational. Multiplication by irrational scalars can be taken to be a continuous extension of the definition for rational scalars. Similarly, the operation " \succ " can be

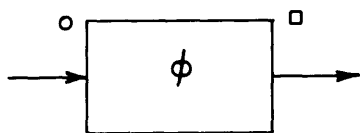


Fig. XVIII-1. Representation of a general homomorphic system.

interpreted in terms of the operation " \circ ". Hence, the operations under which a system satisfies the generalized principle of superposition can usually be summarized by the operations " \circ " and " \square ". A system obeying the generalized principle of superposition

(XVIII. STATISTICAL COMMUNICATION THEORY)

will be referred to as a homomorphic system. The operation "o" will be referred to as the input operation and the operation "□" will be referred to as the output operation. A homomorphic system with system transformation ϕ will be denoted as shown in Fig. XVIII-1. For example, a system with transformation

$$T[x] = y = x^k \tag{4}$$

is homomorphic with multiplication as the input and output operation and, hence, would be denoted in the manner shown in Fig. XVIII-2.

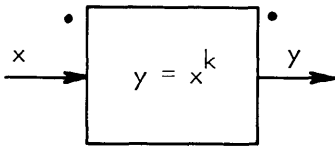


Fig. XVIII-2. Example of a homomorphic system with multiplication as both the input operation and the output operation.

Although the few examples of homomorphic systems which have been given are memoryless, that is, operate only on instantaneous values of the input, this is not a restriction on homomorphic systems in general. The entire class of linear systems, many of which have memory, is homomorphic with addition as both the input and output operation. When the canonical form for homomorphic systems is discussed, it will be clear that many homomorphic systems with memory exist.

The class of homomorphic systems is a very general class and, in fact, can be shown to include any invertible system. To see this, consider a system with an invertible system transformation ϕ . Let o denote any input operation consistent with the algebraic restrictions stated previously. Let □, the output operation, be defined as

$$w_1(t) \square w_2(t) = \phi \left[\phi^{-1}(w_1) \circ \phi^{-1}(w_2) \right], \tag{5}$$

and let

$$\lambda/w(t) = \phi \left[\lambda \circ \phi^{-1}(w) \right]. \tag{6}$$

It is easily verified that with this choice of output operations the system is homomorphic. Thus any invertible system is homomorphic under any choice of input operation. It can further be shown that for any homomorphic system, the output operation is specified uniquely by the input operation, together with the system transformation. Hence, we are assured that the output operation defined by Eqs. 5 and 6 is the only output operation under which the system is homomorphic, when the input operations have been specified. The construction of the output operation by means of Eqs. 5 and 6 does not necessarily aid in the analysis of homomorphic systems, for it requires a precise characterization of the system transformation. It does, however, allow the construction of examples of

homomorphic systems as an aid to developing the theory, and by virtue of the uniqueness of the output operation, examples constructed in this way will not rely on a trivial choice for the output operation.

In summary, then, the class of homomorphic systems includes a wide variety of non-linear systems. In particular, it includes all invertible systems, as well as many systems that are not invertible.

3. Canonical Representation of Homomorphic Systems

For any choice of input operation \circ , there exists an invertible homomorphic system with addition as the output operation. This system is determined entirely by the operation \circ . This fact leads to a convenient and useful representation of homomorphic systems.

Consider a general homomorphic system as shown in Fig. XVIII-1. By virtue of the existence of an invertible homomorphic system with \circ as the input operation and

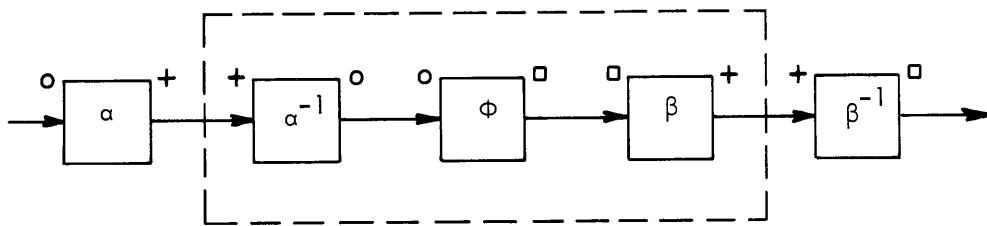


Fig. XVIII-3. Equivalent representation of a general homomorphic system.

addition as the output operation, and an invertible homomorphic system with \square as the input operation and addition as the output operation, the system of Fig. XVIII-1 can be represented in the form of Fig. XVIII-3. The system enclosed in the dotted line, however,

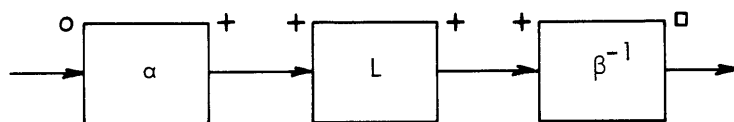


Fig. XVIII-4. Canonical representation of a general homomorphic system.

is a linear system, and thus the system of Fig. XVIII-1 can be represented by the system shown in Fig. XVIII-4. Hence, any homomorphic system can be represented as the cascade of three systems, in which the first and last are dependent only on the input and output operations, respectively, and the second system is a linear system. Furthermore,

it is easily verified that for any choice of the linear system the cascade will be a homomorphic system with input operation \circ and output operation \square . Hence, to generate the entire class of homomorphic systems having specified input and output operations, we determine the systems α and β from knowledge of the input and output operations and then consider all choices for the linear system L .

As an example, let us return to the homomorphic system of Fig. XVIII-2. The system defined by Eq. 3 is an invertible homomorphic system with addition as the input operation and multiplication as the output operation. Hence, its inverse, the natural logarithm, is an invertible homomorphic system with multiplication and addition as the input and output operations, respectively. It should be clear that the system of Fig. XVIII-2 can be represented as shown in Fig. XVIII-5. The entire class

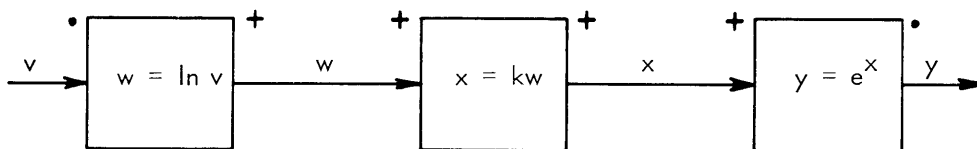


Fig. XVIII-5. Representation of the system of Fig. XVIII-2 in the canonical form of Fig. XVIII-4.

of homomorphic systems with multiplication as both the input and output operation can be generated by replacing the amplifier of gain K by other linear systems.

The canonical representation of Fig. XVIII-4 is effectively a substitution of variables which reduces a homomorphic system to a linear system. The substitution of variables is dependent only on the input and output operations of the homomorphic system. It can be shown that if the system α is memoryless, then the operation \circ is memoryless, that is, it is an operation on the instantaneous values of the inputs. Similarly, if β is memoryless, then the output operation \square must also be memoryless. Furthermore, it can be shown that if \circ and \square are memoryless operations, then all of the memory in the homomorphic system can be concentrated in the linear portion.

4. Systems with Nonadditive Feedback

Consider a feedback system in which the forward and reverse paths contain homomorphic systems and the signal fed back is combined with the input according to some binary operation \circ , as shown in Fig. XVIII-6. The output operation of ϕ_1 and the input operation of ϕ_2 are identical. Also, the input operation of ϕ_1 and the output operation of ϕ_2 are taken to be the same as \circ . If ϕ_1 and ϕ_2 are replaced by their canonical representations, the system of Fig. XVIII-7 results. Because α and α^{-1} are homomorphic, some elementary block diagram manipulations permit the system of Fig. XVIII-7 to be

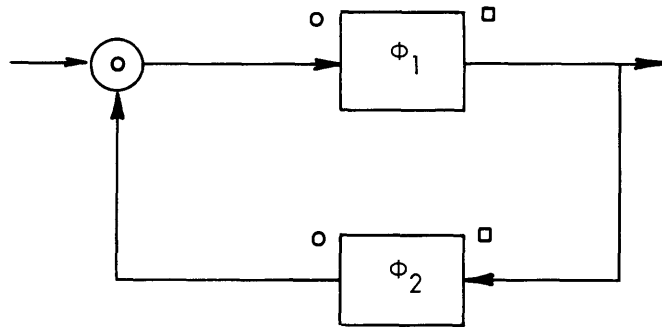


Fig. XVIII-6. Homomorphic feedback system with nonadditive feedback.

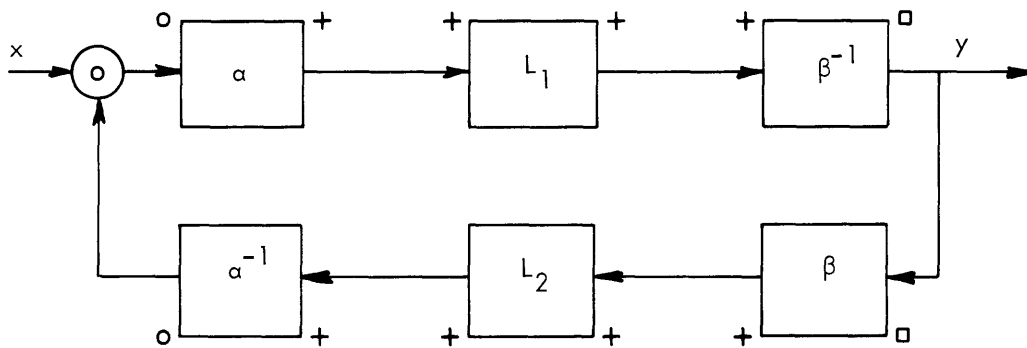


Fig. XVIII-7. Equivalent representation of a feedback system having the form shown in Fig. XVIII-6.

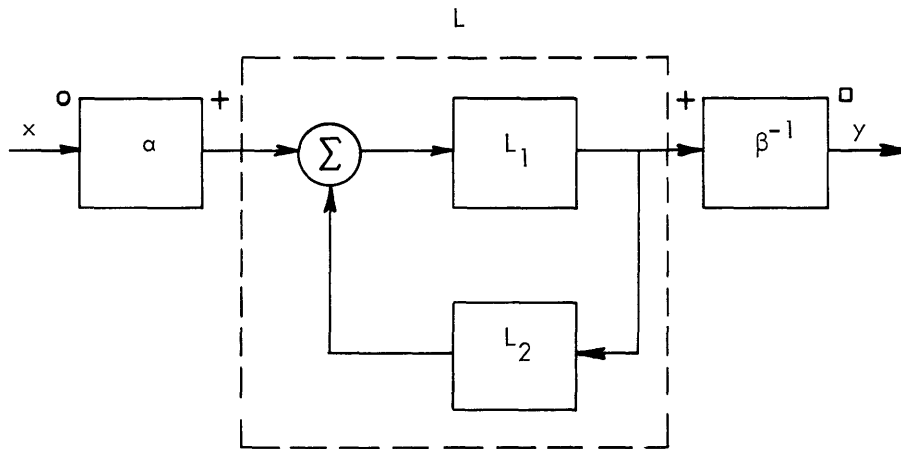


Fig. XVIII-8. Canonical representation of a homomorphic feedback system with nonadditive feedback.

(XVIII. STATISTICAL COMMUNICATION THEORY)

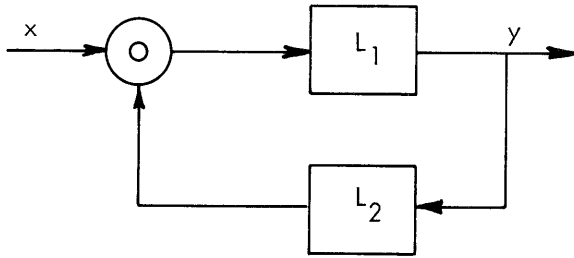


Fig. XVIII-9. General feedback system with linear elements and nonadditive feedback.

redrawn as shown in Fig. XVIII-8. Hence, the system of Fig. XVIII-6 is homomorphic with the same input and output operations as ϕ_1 . Furthermore, the linear system L in its canonical representation is an additive feedback system with the linear portion of ϕ_1 in the forward path and the linear portion of ϕ_2 in the feedback path.

If the systems in the forward and feedback paths are linear systems but the feedback operation is not addition, then the nonlinearity in the system cannot be removed from the feedback loop as it could in the system of Fig. XVIII-6. The notion of homomorphic feedback systems, however, does permit representation of this type of feedback system as an additive feedback system with nonlinearities in the forward path. Specifically, consider the feedback system of Fig. XVIII-9. It was stated previously that there will always exist an invertible homomorphic system with "o" as the input operation and addition as the output operation. If we denote this system by γ , then the system of

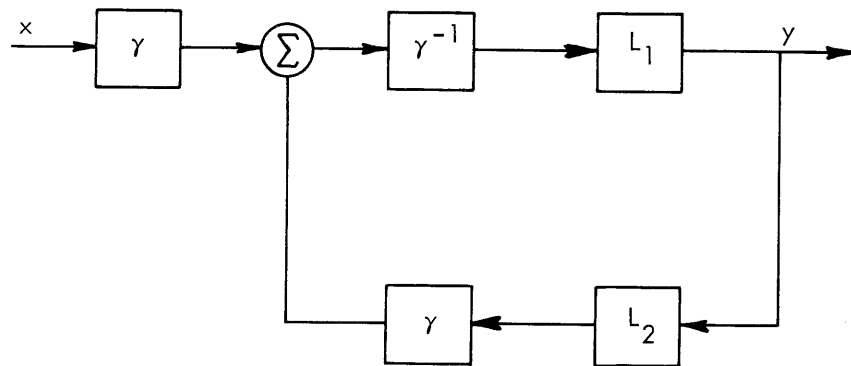


Fig. XVIII-10. Equivalent representation of the feedback system of Fig. XVIII-9.

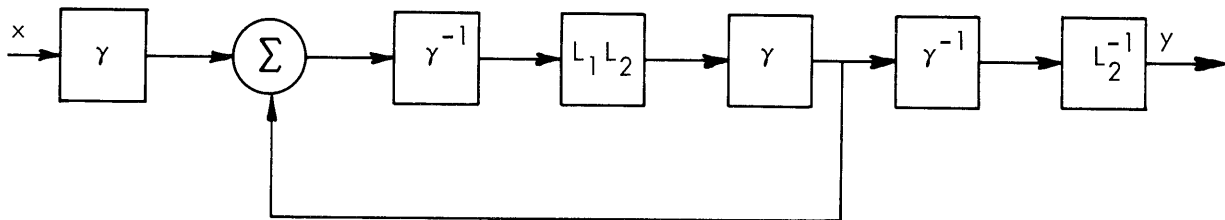


Fig. XVIII-11. Representation of a feedback system having the form shown in Fig. XVIII-9.

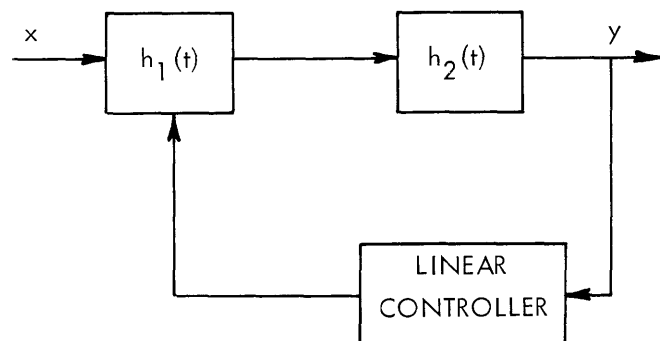


Fig. XVIII-12.
Example of a feedback system of
the form shown in Fig. XVIII-9.

Fig. XVIII-9 can be represented in the form of Fig. XVIII-10. If the linear system L_2 is invertible, then the block diagram of Fig. XVIII-10 can be manipulated into the form of Fig. XVIII-11. The essential feature in the representation of Fig. XVIII-11 is that a system with nonadditive feedback has been represented in a more conventional form.

As an example, we might consider the system shown in Fig. XVIII-12. The systems $h_1(t)$ and $h_2(t)$ are linear systems. The impulse response $h_1(t)$ is linearly dependent on the output $y(t)$. This represents a feedback system in which convolution is the binary feedback operation. Hence, this system can be represented in the form of Fig. XVIII-9, with \circ taken to be convolution.

5. Conclusions

The results obtained to date concerning homomorphic systems seem to indicate that this is a useful means of classifying many nonlinear systems. The canonical representation of these systems permits their investigation in terms of linear systems, for which many analytical tools are available. It is difficult to predict the areas in which homomorphic systems will assume practical significance. It is hoped that as the theory progresses its engineering applications will become clear.

A. V. Oppenheim

B. ENERGY DISTRIBUTION IN TRANSIENT FUNCTIONS

In this report, we shall present some results that have been obtained concerning the distribution of energy in transient functions, $f(t)$, which are zero for $t < 0$. The Laplace transform, $F_1(s)$, of the transient function $f_1(t)$ is

$$F_1(s) = \int_0^{\infty} f_1(t) e^{-st} dt, \quad (1)$$

in which $s = \sigma + j\omega$. Also, the partial energy, which is the energy in the first τ seconds of $f_1(t)$, is

$$E_1(\tau) = \int_0^\tau |f_1(t)|^2 dt. \quad (2)$$

We assume that $F_1(s)$ contains a zero at $s = p_1$ so that it can be written as

$$F_1(s) = [s-p_1] G(s). \quad (3)$$

If the zero in the s -plane is moved to a new position at $s = p_2$, the transient function $f_2(t)$ results for which

$$F_2(s) = [s-p_2] G(s). \quad (4)$$

The results that we shall present in this report concern the difference between the partial energies of $f_1(t)$ and $f_2(t)$ for the case in which the zero in the s -plane is moved parallel to the σ axis. For our derivations, we shall consider only those transient functions for which

$$f_1(0+) = \lim_{s \rightarrow \infty} sF_1(s) < \infty \quad (5a)$$

and

$$f_1(\infty) = \lim_{s \rightarrow 0} sF_1(s) = 0. \quad (5b)$$

To obtain an expression for the difference of the partial energies, we let $g(t)$ be the inverse transform of $G(s)$. Equation 3 then can be expressed in the time domain as

$$f_1(t) = \frac{d}{dt} g(t) - p_1 g(t). \quad (6)$$

Thus the square of the magnitude of $f_1(t)$ is

$$|f_1(t)|^2 = f_1(t) \overline{f_1(t)} = |g'(t)|^2 + |p_1 g(t)|^2 - p_1 g(t) \overline{g'(t)} - \overline{p_1 g(t)} g'(t), \quad (7)$$

in which the prime indicates the derivative and the bar indicates the complex conjugate of the function. In a similar manner, the square of the magnitude of $f_2(t)$ from Eq. 4 is

$$|f_2(t)|^2 = |g'(t)|^2 + |p_2 g(t)|^2 - p_2 g(t) \overline{g'(t)} - \overline{p_2 g(t)} g'(t). \quad (8)$$

Consequently,

$$|f_1(t)|^2 - |f_2(t)|^2 = \left[|p_1|^2 - |p_2|^2 \right] |g(t)|^2 + 2 \operatorname{Re} \{ [p_2 - p_1] g(t) \overline{g'(t)} \}, \quad (9)$$

in which Re means the real part of the quantity within the braces. Since the zero is moved parallel to the σ axis, we have $p_2 - p_1 = \sigma_2 - \sigma_1$ and $|p_1|^2 - |p_2|^2 = \sigma_1^2 - \sigma_2^2$, so that Eq. 9 can be written

$$\begin{aligned}
|f_1(t)|^2 - |f_2(t)|^2 &= \left[\sigma_1^2 - \sigma_2^2 \right] |g(t)|^2 + 2(\sigma_2 - \sigma_1) \operatorname{Re} \{g(t) \overline{g'(t)}\} \\
&= \left[\sigma_1^2 - \sigma_2^2 \right] |g(t)|^2 + (\sigma_2 - \sigma_1) \frac{d}{dt} |g(t)|^2.
\end{aligned} \tag{10}$$

The difference between the partial energies of $f_1(t)$ and $f_2(t)$ is

$$\begin{aligned}
E_1(\tau) - E_2(\tau) &= (\sigma_1^2 - \sigma_2^2) \int_0^\tau |g(t)|^2 dt + (\sigma_2 - \sigma_1) \int_0^\tau \frac{d}{dt} |g(t)|^2 dt \\
&= (\sigma_1^2 - \sigma_2^2) A + (\sigma_2 - \sigma_1) B,
\end{aligned} \tag{11}$$

in which the partial energy of $g(t)$, A , is

$$A = \int_0^\tau |g(t)|^2 dt > 0 \tag{12a}$$

and

$$B = |g(t)|^2 \geq 0. \tag{12b}$$

As a function of σ_2 , Eq. 11 is the equation of a parabola that crosses the σ_2 axis at the points $\sigma_2 = \sigma_1$ and $\sigma_2 = \frac{B}{A} - \sigma_1$. The parabola has a maximum at $\sigma_2 = \frac{B}{2A}$, at which point $E_1(\tau) - E_2(\tau) = A \left[\frac{B}{2A} - \sigma_1 \right]^2$. Figure XVIII-13 is a plot of $E_1(\tau) - E_2(\tau)$ vs σ_2 for

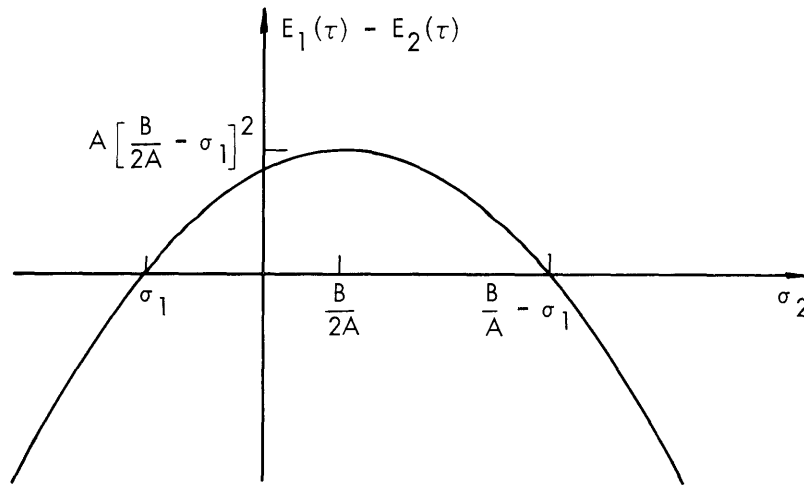


Fig. XVIII-13. Difference of the partial energies versus σ_2 .

$\sigma_1 < 0$. We observe from Fig. XVIII-13 that $E_2(\tau) < E_1(\tau)$ for σ_2 in the range $\sigma_1 < \sigma_2 < \frac{B}{A} - \sigma_1$, and that $E_2(\tau)$ is a minimum for $\sigma_2 = \frac{B}{2A} \geq 0$. Thus for $p_1 \neq 0$ the total energy of $f_2(t)$ which is $E_2(\infty)$ is a minimum for $\sigma_2 = 0$ because $B = 0$ at $\tau = \infty$. We show

that it is a minimum from (3), (5b), and (12b) and the fact that $p_1 \neq 0$, for which we have

$$\begin{aligned} \lim_{\tau \rightarrow \infty} g(\tau) &= \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} \frac{s}{s - p_1} F_1(s) \\ &= -\frac{1}{p_1} \lim_{s \rightarrow 0} sF_1(s) = 0. \end{aligned}$$

This result implies that if the s -plane zeros of a transform, $F(s)$, are moved parallel to the σ axis, then, of all the corresponding transient functions, the transform of the one with the minimum total energy has every one of its zeros on the $j\omega$ axis.

Let us now consider the special case for which $\sigma_2 = -\sigma_1 > 0$. For this case, $|F_1(\omega)| = |F_2(\omega)|$ and, consequently, the energy-density spectrum of $f_1(t)$ is identical with that of $f_2(t)$. Then, from Eq. 11, the difference of their partial energies is

$$\begin{aligned} E_1(\tau) - E_2(\tau) &= -2\sigma_1 B \\ &= -2\sigma_1 |g(\tau)|^2 \geq 0, \end{aligned} \tag{13}$$

since we have assumed that $\sigma_1 < 0$. We thus note that of all transient functions with the same energy-density spectrum, the transform of the one with the greatest partial energy has all its zeros in the left half of the s -plane and the transform of the one with the smallest partial energy has all its zeros in the right half of the s -plane.

As another application of this last result, let $F_2(s) = F(s) \frac{s - p_1}{s + p_1}$ and $F_1(s) = F(s) \frac{s + p_1}{s + p_1} = F(s)$. Then $E_2(\tau) \leq E_1(\tau)$. Thus, in general, the partial energy of the transient input of an all-pass system is greater than that of the output. For example, the partial energy of the Laguerre functions, $l_n(t)$, is a monotonically decreasing function of n , since the Laplace transform of $l_n(t)$ is

$$L_n(s) = \sqrt{2p} \frac{(p-s)^n}{(p+s)^{n+1}} = \frac{p-s}{p+s} L_{n-1}(s).$$

Since the total energy of each Laguerre function is one, this result means that the energy of successive Laguerre functions is delayed, and this delay is a monotonically increasing function of n .

M. Schetzen