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A. FORCE DENSITY IN POLARIZABLE MATTER OBTAINED FROM THE PRINCIPLE OF VIRTUAL WORK

The force density in a moving, deforming polarizable material with a particular single-valued constitutive law relating the polarization $\overline{P}_0$ and electric field $\overline{E}_0$ in the rest frame has been obtained previously from Hamilton's principle.[1] The principle of virtual work offers an alternate way of obtaining the same result. The advantage of this approach is that it is less dependent on the assumed constitutive law. In fact, the results obtained are valid for any lossless dispersive polarizable fluid (e.g., gyroelectric fluid) in which the constitutive law relating $\overline{P}_0$ and $\overline{E}_0$ is an integro-differential equation in time.

The principle of virtual work in a form that is suitable for the determination of force densities in continua starts from the law of energy conservation. We consider a moving and deforming medium and single out a moving and deforming volume V of the medium enclosed by a surface S that is assumed to move along with the local velocity $\overline{v}(\overline{r}, t)$ of the medium. Denote the tensor stress in the medium by $\overline{\overline{t}}$, the power flow passing the surface by $\overline{s}$, the energy density by w, and the power conversion density by ϕ. The law of energy conservation is then

$$\oint_S d\overline{a} \cdot \overline{\overline{t}} \cdot \overline{v} + \oint_S \overline{s} \cdot d\overline{a} + \frac{d}{dt} \int_V w \, dv = \int_V \phi \, dv. \tag{1}$$

It should be noted that $\overline{s}$ is the power passing through the surface S, such as electromagnetic power or heat flow; the contribution to the time rate of change of the energy in V from the work of the surface stresses is contained in the first integral. As stated, Eq. 1 is of general validity and applies to dissipative or active systems and is not limited to nonrelativistic motion as long as the various terms are all evaluated in the laboratory frame. The law of energy conservation is made into a principle of virtual work by restricting it to nondissipative systems, to small virtual velocities $\overline{v}$, and to systems for which the quantities $\overline{\overline{t}}$, $\overline{s}$, w, and ϕ may be expressed in terms of the velocity $\overline{v}$, and the stress $\overline{\overline{t}}_0$, energy, force, power conversion, and momentum densities w_0, $\overline{f}_0$, ϕ_0, and $\overline{G}_0$ in the rest frame. Once such relations are known (or postulated) it is possible to find $\overline{\overline{t}}_0$, $\overline{f}_0$, and $\overline{G}_0$ in terms of $\overline{s}_0$, w_0, and ϕ_0. Some of these relations are simple. Because $\overline{\overline{t}}$ is a force per unit area, it is, in the nonrelativistic limit (to first order in $\overline{v}$) equal to its rest-frame counterpart

$$\overline{\overline{t}} = \overline{\overline{t}}_0. \tag{2}$$

The power flow density is also invariant

$$\bar{s} = \bar{s}_0. \tag{3}$$

The power conversion density in the laboratory frame ϕ, is related to ϕ_0 in the rest frame by

$$\phi = \phi_0 + \bar{f}_0 \cdot \bar{v}. \tag{4}$$

The second term takes into account the power conversion density contributed by the force density $\bar{f}_0$ working on a fluid moving with the velocity $\bar{v}$.

The relation between w and w_0 is less intuitive if power flow and momentum densities are present in the rest frame. An expression between w and w_0 is obtained if one postulates that the energy-momentum tensor

$$\begin{bmatrix} \bar{\bar{t}} + \bar{v}\bar{G} & i\,\dfrac{\bar{s} + \bar{v}w}{c} \\ \\ ic\bar{G} & -w \end{bmatrix} \tag{5}$$

transforms as a four-tensor. One finds for the 4,4-component

$$w = w_0 + \frac{\bar{v} \cdot \bar{s}_0}{c^2} + \bar{G}_0 \cdot \bar{v}. \tag{6}$$

By introducing (2)-(6) in Eq. 1, using the particle (or mass) density n, one has, to first order in $\bar{v}$,

$$\nabla \cdot \bar{s}_0 + n\frac{d}{dt}\left(\frac{w_0}{n}\right) + \frac{\partial}{\partial t}\left(\frac{\bar{v} \cdot \bar{s}_0}{c^2}\right) - \phi_0 = -\nabla \cdot (\bar{\bar{t}}_0 \cdot \bar{v}) - \frac{\partial}{\partial t}(\bar{G}_0 \cdot \bar{v}) + \bar{f}_0 \cdot \bar{v}. \tag{7}$$

The derivatives are given with respect to the laboratory-frame coordinates. If one knows the expressions for $\bar{s}_0$, w_0, and ϕ_0 in terms of the physical variables, if one makes use of the equations of motion of these physical variables, and if one retains only terms up to first order in $\bar{v}$, the left-hand side of Eq. 7 becomes a first-order expression in $\bar{v}$. Comparing the result with the right-hand side, one finds $\bar{f}_0$, $\bar{G}_0$, and $\bar{\bar{t}}_0$ from a term-by-term indentification. This procedure is best illustrated by the following example.

We apply the principle of virtual work to a dielectric fluid, without free charge or free current. This fluid contains electric dipoles, with density (number per unit volume) n and dipole moment $\bar{P}/n$. The fluid energy (per unit volume) in the rest frame of the material may be assigned to each dipole: w_0/n_0. The rate of change $\partial/\partial t_0(w_0/n_0)$ is caused in two ways. Energy is fed into the dipole by the electric field in the rest frame (where the subscript 0 on t denotes it as the rest-frame coordinate) when the dipole moment is changed. Thus, one contribution is

$$\overline{E}_0 \cdot \frac{\partial}{\partial t_0}\left(\frac{\overline{P}_0}{n_0}\right). \tag{8}$$

This would be the only contribution if the microscopic and macroscopic electric fields were the same, and if there were no fluid pressure. Since the two are generally not the same, there is an additional contribution that is proportional to the time rate of change of the density

$$\frac{\pi_0}{n_0}\frac{\partial n_0}{\partial t_0}. \tag{9}$$

The physical meaning of π_0 will emerge later on. The time rate of change of the energy per dipole is

$$\frac{\partial}{\partial t_0}\left(\frac{w_0}{n_0}\right) = \overline{E}_0 \cdot \frac{\partial}{\partial t_0}\left(\frac{\overline{P}_0}{n_0}\right) + \frac{\pi_0}{n_0^2}\frac{\partial n_0}{\partial t_0}. \tag{10}$$

The rate at which power (per unit volume) is fed into the fluid by the electromagnetic field can be obtained on a macroscopic basis from the Poynting theorem

$$\nabla \cdot (\overline{E}\times\overline{H}) + \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\mu_0 H^2\right) = -\overline{E} \cdot \overline{J}_{pol}, \tag{11}$$

where $\overline{J}_{pol}$ is the current density associated with the polarization,

$$\overline{J}_{pol} = \frac{\partial \overline{P}}{\partial t} + \nabla \times (\overline{P}\times\overline{v}). \tag{12}$$

Therefore, by applying this equation in the rest frame, the rest-frame power conversion density is

$$\phi_0 = \overline{E}_0 \cdot \overline{J}_{pol0}. \tag{13}$$

The power-flow vector of the fluid system in the rest frame, $\overline{s}_0$, is zero. Since the partial derivative with respect to the rest-frame time is equal to the substantive derivative in the laboratory frame, the left-hand side of (7) is thus

$$n_0\overline{E}_0 \cdot \frac{d}{dt}\left(\frac{\overline{P}_0}{n_0}\right) + \frac{\pi_0}{n_0}\frac{dn_0}{dt} - \overline{E}_0 \cdot \overline{J}_{pol0} = F. \tag{14}$$

All terms in (14) must be expressed in terms of the physical variables in the laboratory frame because we want to make use of the equation of motion to bring (14) into the form of the right-hand side of (7), where all derivatives are given with respect to laboratory-frame coordinates. To eliminate the zero-order terms in Eq. 14, we use the transformation laws, correct to first order in $\overline{v}$,

$$\overline{P}_0 = \overline{P} \tag{15}$$

$$\overline{E}_0 = \overline{E} + \overline{v} \times \mu_0 \overline{H} \tag{16}$$

$$n_0 = n \tag{17}$$

$$\overline{J}_{pol0} = \overline{J}_{pol} - \rho_{pol}\overline{v} = \frac{\partial \overline{P}}{\partial t} + \nabla \times (\overline{P} \times \overline{v}) + \overline{v}\nabla \cdot \overline{P} = \frac{\partial \overline{P}}{\partial t} + \nabla \cdot (\overline{v}\overline{P}) - \overline{P} \cdot \nabla \overline{v}. \tag{18}$$

In the rest frame, the law of conservation of particles is

$$\nabla \cdot \overline{v} = -\frac{1}{n_0} \frac{dn_0}{dt}. \tag{19}$$

If this is substituted in Eq. 18, we find

$$\overline{J}_{pol0} = n_0 \frac{d}{dt} \frac{\overline{P}_0}{n_0} - \overline{P}_0 \cdot \nabla \overline{v} \tag{20}$$

to first order in $\overline{v}$. Now, use of Eqs. 14, 19, and 20 results in

$$F = \overline{P}_0 \cdot (\nabla \overline{v}) \cdot \overline{E}_0 - \pi_0 \nabla \cdot \overline{v} = \overline{v} \cdot (\nabla \pi_0 - \nabla \cdot \overline{P}_0 \overline{E}_0) - \nabla \cdot \left[(\pi_0 \overline{\overline{\delta}} - \overline{P}_0 \overline{E}_0) \cdot \overline{v}\right] \tag{21}$$

and hence from Eq. 7 we recognize that the rest-frame stress tensor is

$$\overline{\overline{t}}_0 = \pi_0 \overline{\overline{\delta}} - \overline{P}_0 \overline{E}_0, \tag{22}$$

the rest-frame momentum density is

$$\overline{G}_0 = 0, \tag{23}$$

and the rest-frame force density is

$$\overline{f}_0 = \nabla \pi_0 - \nabla \cdot \overline{P}_0 \overline{E}_0. \tag{24}$$

When these quantities are transformed to an arbitrary reference frame, we find

$$\overline{\overline{t}} = \pi_0 \overline{\overline{\delta}} - \overline{P}(\overline{E} + \overline{v} \times \mu_0 \overline{H}) \tag{25}$$

$$\overline{s} = \overline{v}(\pi_0 + w) - \overline{P}(\overline{E} \cdot \overline{v}) \tag{26}$$

$$w = w_0 + \overline{v} \cdot \overline{G}, \tag{27}$$

where $\overline{G}$ is the relativistic momentum associated with the energy and stress in the fluid

$$\overline{G} = \overline{v} \frac{\gamma^2 w_0}{c^2} + \overline{v} \cdot \frac{\gamma^2}{c^2} \left[\pi_0 \overline{\overline{\delta}} - \overline{P}(\overline{E} + \overline{v} \times \mu_0 \overline{H})\right]. \tag{28}$$

Now let us add the stress-energy tensor for the polarization to the stress-energy tensor for the fields. The result is

$$\overline{\overline{T}}_{tot} = \overline{\overline{t}}_{tot} + \overline{v}\overline{G}_{tot} = \overline{\overline{\delta}}\left(\pi_0 + \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\mu_0 H^2\right) - (\epsilon_0\overline{E}+\overline{P})\overline{E} - \overline{P}(\overline{v}\times\mu_0\overline{H}) + \overline{v}\overline{G} \tag{29}$$

$$\overline{G}_{tot} = \overline{G} + \frac{\overline{E}\times\overline{H}}{c^2} \tag{30}$$

$$\overline{s}_{tot} = \overline{v}\pi_0 + \overline{v}(w_0 + \overline{v}\cdot\overline{G}) - \overline{P}(\overline{E}\cdot\overline{v}) + \overline{E}\times\overline{H} \tag{31}$$

$$w_{tot} = w_0 + \overline{v}\cdot\overline{G} + \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\mu_0 H^2. \tag{32}$$

The divergence of the stress-energy tensor yields a force density of electromagnetic origin

$$\overline{f}_{em} = (\overline{P}\cdot\nabla)\overline{E} + n\frac{d}{dt}\left(\frac{\overline{P}}{n}\right)\times\mu_0\overline{H} + \overline{v}\times(\overline{P}\cdot\nabla)\mu_0\overline{H} - \nabla\pi_0 - n\frac{d}{dt}\frac{\overline{G}}{n}, \tag{33}$$

which is in agreement with the force density for the same situation predicted by means of Hamilton's principle.[2]

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References

1. P. Penfield, Jr., Force of electromagnetic origin in fluids, Quarterly Progress Report No. 70, Research Laboratory of Electronics, M.I.T., July 15, 1963, pp. 82-87.

2. Ibid., page 86, Eq. 32.