XXII. PROCESSING AND TRANSMISSION OF INFORMATION*

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RESEARCH OBJECTIVES AND SUMMARY OF RESEARCH

1. Communications

The work of the group is focused on the dual problems of ascertaining the best performance that can be attained with a communication system, and developing efficient techniques for actually achieving performance substantially this good.

a. Coding Techniques

An experimental facility permitting the application of sequential decoding to a wide variety of modems and channels has now been completed and tested.[1] The facility consists of a portable data acquisition system and a special programming language for use with a PDP-6 general-purpose computer. The data acquisition system will accept analog outputs from a wide variety of channel demodulators and, by means of sample-and-hold and analog-to-digital converter circuits, will record the outputs together with appropriate timing information on a digital tape recorder. Circuitry and software permit a ready transfer of data from the recorder to the computer memory (DEC-tape). The special programming language, a modification of FORTRAN, permits experimenters without detailed programming background to write efficient programs for implementing a variety of sequential decoding algorithms, collect performance statistics, and control the program by means of a light pen on the basis of a real-time display of the tree search. This facility has already proved useful in debugging a modified sequential decoding algorithm and in obtaining preliminary data on the performance of a short constraint length convolutional code operating on a simulated deep-space link. This work continues.

On the theoretical side of sequential decoding, it has been shown[2] that the probability distribution of the number of computations, L, required to decode a digit is of the form $KL^{-\alpha}$ when α can be upper- and lower-bounded as a function of the channel and transmission rate. Techniques are now being investigated to avoid this computational problem

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by using several sequential decoders in parallel.

A new coding and decoding technique, called concatenated coding, has been investigated.[3] By using this technique, decoding can be accomplished for any transmission rate less than capacity with a computational complexity that is an algebraic function of the constraint length and with an error probability that is an exponentially decaying function of the constraint length. Some examples indicate that this technique might have a number of practical applications.

The time-bandwidth product of the bubble-tank has been increased and instrumentation for discrete signaling at both high and low energy-to-noise ratios completed. Instrumentation for analog signaling is in the process of development. This equipment will be used in the coming year to investigate experimentally the performance of various coding techniques on randomly time-variant dispersion channels.

R. G. Gallager, I. M. Jacobs

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2. J. E. Savage, "The Computation Problem with Sequential Decoding," Ph.D. Thesis, Department of Electrical Engineering, M.I.T., 1965 (to appear as Technical Report 439 of the Research Laboratory of Electronics and Technical Report 371 of Lincoln Laboratory, M.I.T.).
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b. Optical Communications

Preliminary studies of communication systems employing optical frequencies have indicated three topics to which the concepts and techniques of modern communication theory may most profitably be addressed. They are (i) the import of quantum electrodynamics for the characteristics of efficient communication systems, (ii) the relevant description of device noise as it affects the performance of communication systems, and (iii) the statistical characterization of the atmosphere as a propagation channel at optical frequencies.

A fundamental comprehension of these topics requires familiarity with a mixture of probability theory, physics, and experimental methods. Accordingly, a substantial effort has been invested in the acquisition of the prerequisite background in physics and optical techniques. Concurrent studies of specific modulators, demodulators, and channels have been completed and summarized in Quarterly Progress Report No. 79 (page 205).

R. S. Kennedy

c. Theoretical Bounds on Error Performance

A considerable amount of research has been done on finding the reliability of various channel models. The reliability of a channel as a function of transmission rate is the exponent with which the probability of decoding error can be made to vanish with increasing block length. For channels with additive nonwhite Gaussian noise, upper and lower bounds on reliability have been found which agree both for rates between R_{crit} and capacity and in the limit as the rate approaches zero.[1]

Other classes of channels being actively investigated are randomly time-variant channels and parallel coupled channels without crosstalk. Efforts are also being made to achieve a fundamental understanding of the effect of a lack of time synchronization on error probability.

R. G. Gallager, I. M. Jacobs

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1. P. M. Ebert, "Error Bounds for Parallel Communication Channels," Sc.D. Thesis, Department of Electrical Engineering, M.I.T., 1965 (to appear as Technical Report 448 of the Research Laboratory of Electronics).

d. Vocoded Speech

A voice-excited vocoder has been successfully simulated on the IBM 7094 computer and satisfactory quality of voice has been reconstructed[1] with 220-660 cps frequency used as the voice-excitation band. Experiments are under way to convert the vocoder output to a binary data stream and investigate the effect of channel coding techniques on the reconstructed voice.

R. G. Gallager, I. M. Jacobs

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1. R. A. Carpenter, "Study of Excitation in Computer Simulated Voice-Excited Vocoder," S.M. Thesis, Department of Electrical Engineering, M.I.T., 1965.

e. Switching Circuits and Logical Machines

Closely related work is concerned with switching theory and its application to computation problems. During the past year, effort has been concentrated on a study of generalized threshold functions and their application to the problem of detecting malfunctions in computing circuitry. A mathematical formulation of the realizability requirements for multithreshold and compound-weight functions has provided an iterative synthesis procedure for the compound-weight case. This formulation has also clarified the relationships among the various types of threshold functions.

Further investigations will be made of multithreshold and compound-weight realizations. Also, a study will be made of the relationships among (i) the graphical description of a machine, (ii) the physical structure of the machine, and (iii) the problem of detecting malfunctions in the machine.

F. C. Hennie III

A. TRUNCATING A CONVOLUTIONAL CODE

1. Introduction

The sequential nature of a convolutional code requires that the code be periodically truncated when used on a one-way communication system. This enables the system to continue to decode blocks of data even after a block is received which is too noisy to be decoded. The problem of efficiently truncating a convolutional code will be discussed in this report.

A binary convolutional code is generated as the information digits are shifted into a shift register which is initially set with all zeros in it. For every shift of the input data, ν code digits are obtained by tapping ν modulo-2 adders. Each adder is connected to a different set of register stages. Thus, knowledge of the connections to the modulo-2 adders determines a unique mapping of information digits into code digits.

Conventionally, a convolutional code, generated by a K-stage shift register, is truncated by having a tail of $K - 1$ zeros follow the block of L information digits that are to be coded[1] (see also Sec. XXII-B). This implies that the rate per use of the channel over the entire code block is

$$R = \frac{\text{Log}_2 2^L}{(L+K-1)\nu} = \frac{R_N}{1 + \frac{K-1}{L}} \text{ bits,} \tag{1}$$

where R_N, the rate that would result if there was no loss owing to truncation, is $1/\nu$. We see that, for a low rate loss, L must be made considerably larger than the length of the shift register K. (For example, in an experimental simulation (see Sec. XXII-B), an L of twenty times K was used.) It should be noted, however, that a very large block length will deteriorate the code's performance. This results because both the probability of error on decoding a block[1, 2] and the probability of a buffer overflow[3, 4] when using sequential decoding are bounded by functions which increase linearly with L. A scheme for truncation which enables one to use smaller block lengths than these will be considered.

2. Analysis

The error performance of a convolutional code can be studied by using a decoding procedure that makes a decision on each successive digit by selecting the most probable sequence over the constraint length of the digit in question. (Sequential decoding achieves similar performance by means of an algorithm that is much easier to implement.) The constraint length of the i^{th} digit, N_i, is the number of code digits influenced by this information digit. Generally, over this constraint length there will be $K_i - 1$ information digits following the i^{th} digit. Thus, one of 2^{K_i} possible sequences of information digits is selected in order to decode the i^{th} digit. The average probability of error, over the ensemble of all possible coder connections, for decoding the i^{th} digit with the past digits known, is bounded[1] by

$$P(e_i) < 2^{K_i-1} 2^{-N_i R_o} < 2^{-N_i\left[R_o - \frac{K_i}{N_i}\right]}, \tag{2}$$

where R_o is a constant determined by the channel. Furthermore, the probability of an error on decoding the entire block of L digits can be upperbounded[1] by

$$P(e) < \sum_{i=1}^{L} P(e_i). \tag{3}$$

For a convolutional code without any truncation, we have

$$P(e_i) < 2^{-K\nu\left[R_o - \frac{1}{\nu}\right]} \qquad \text{for } 1 \leq i \leq L - K + 1, \tag{4}$$

since these digits pass through the shift register with an $N_i = K\nu$ and $K_i = K$. Note that the exponent in (4) is positive, since R_o, the computational cutoff rate for sequential decoding, is greater than $R_N = 1/\nu$. Now for $i > L - K + 1$, N_i decreases in steps of ν and K_i decreases in steps of 1, so that upon looking at the exponent in (2) we see that

$$N_i(R_o - K_i/N_i) > N_{i+1}\left(R_o - \frac{K_{i+1}}{N_{i+1}}\right) \tag{5}$$

since

$$N_i R_o - K_i > N_i R_o - K_i - (\nu R_o - 1), \tag{6}$$

which implies that $P(e_i)$ is increasing with i. To avoid having a high probability of an error on the digits near the end of the block, the constraint lengths must be increased.

Since the last digit has the highest probability of error, an N_L will be found so that $P(e_L)$ has the same exponent as was the case for $i \leq L - K + 1$. Equating exponents, we have

$$N_L\left[R_o - \frac{1}{N_L}\right] = K\nu\left[R_o - \frac{1}{\nu}\right]. \tag{8}$$

which yields

$$N_L = (K-1)\left(\nu - \frac{1}{R_o}\right) + \nu. \tag{9}$$

Since before truncation $N_L = \nu$, the amount of code length added is

$$N_o = (K-1)\left(\nu - \frac{1}{R_o}\right). \tag{10}$$

This added code length can be obtained by having just

$$K_o = (K-1)\left(1 - \frac{R_N}{R_o}\right) \tag{11}$$

zeros follow the block of L information digits. Since only K_o zeros are shifted into the K-stage shift register, the information digits still left in the register would be dumped before the new block of information digits enters the register.

We must now verify the fact that no further code length increase is required for the range $L - K + 1 < i < L$. For $L - K + 1 < i \leq L - K + K_o + 1$, N_i remains constant at $K\nu$ and K_i decreases in steps of 1, so that for this range $P(e_i)$ is decreasing. For $L - K + K_o + 1 < i < L$, N_i decreases in steps of ν and K_i decreases in steps of 1, so as in (5) and (6), $P(e_i)$ increases with i. But $P(e_i)$ is bounded by $P(e_L)$, so the code length increase given by (10) is all that is needed.

For this form of truncation, $P(e_i)$ is constant for $i \leq L - K + 1$, then starts to decrease and finally increases to its original value as i approaches L. It is possible, by shortening the constraint lengths in the range $L - K + 1 < i < L$, to have a uniform $P(e_i)$ for all i. Since the same total code length increase of N_o is still required, however, no improvement in rate loss is achieved.

3. Discussion

The essential result obtained from this analysis is that the increase in length required to truncate a convolutional code without deteriorating its error performance is

$$N_o = (K-1)\left(\nu - \frac{1}{R_o}\right) = N_{conv}\left[1 - \frac{R_N}{R_o}\right], \tag{12}$$

where the conventional length is

$$N_{conv} = (K-1)\nu \tag{13}$$

The rate over the whole block, including the added length for truncation, is

$$R = \frac{L}{L\nu + N_o} = \frac{R_N}{1 + \frac{K-1}{L}\left(1 - \frac{R_N}{R_o}\right)} \text{ bits} \tag{14}$$

which is to be compared with (1) for the conventional case.

For inefficient operation, when R_N is much less than R_o, only a small gain is achieved over the ordinary method of truncation. For efficient operation when R_N

approaches R_o, the amount of length added decreases toward zero. This enables one to improve the code's performance by having an L that is only slightly greater than K and yet still have R sufficiently close to R_N.

D. Chase

References

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2. J. M. Wozencraft and B. Reiffen, Sequential Decoding (John Wiley and Sons, Inc., New York, 1961).
3. J. E. Savage, "The Computation Problem with Sequential Decoding," Ph. D. Thesis, Department of Electrical Engineering, M. I. T., February 1965.
4. I. M. Jacobs and E. Berlekamp, "A Lower Bound to the Distribution of Computation for Sequential Decoding" (unpublished).

B. SIMULATION OF SEQUENTIAL DECODING FOR A TELEMETRY CHANNEL

A coherent binary antipodal (PSK) signaling scheme, utilizing convolutional encoding and sequential decoding, has been proposed for use in a deep space telemetry system.[1,2] The purpose of this research is to study the performance of the sequential decoder by computer simulation.

1. Description of the System and its Simulation

In the signaling scheme, successive binary information and parity digits emerging from a convolution encoder are transmitted as binary antipodal waveforms with energy E. The transmitted waveform corresponding to a particular information or parity digit is assumed to be received together with independent additive white Gaussian noise with power spectral density $N_o/2$. The resultant received waveform is crosscorrelated with the positive binary waveform by sampling the output of a matched filter. This sample is quantized into one of eight possible levels (3-bit quantization). The cascade of modulator, additive white Gaussian noise channel, matched filter, sampler, and quantizer is equivalent to a discrete memoryless channel with two input letters and 8 output letters. The original binary information sequence is extracted from the output of this channel with a sequential decoder.

In the simulation, binary convolution codes with rate 1/7 bits per channel waveform were used. Three signal-to-noise ratios, defined as $10 \log_{10} \frac{E}{N_o}$ db, were considered, -6 db, -6.5 db, and -7 db. The quantization scheme that was chosen for the matched-filter output is shown in Fig. XXII-1. Figure XXII-2 shows the transition probabilities calculated for this quantization scheme for a signal-to-noise ratio of -6 db.

For a channel having a P-letter input alphabet with probability distribution

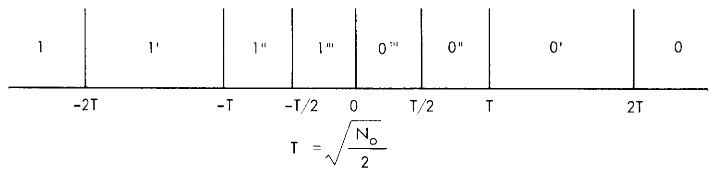

Fig. XXII-1. Quantization scheme. Horizontal axis represents time-sampled matched-filter output.

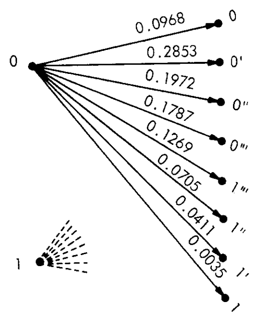

Fig. XXII-2. Transition probabilities for the discrete memoryless channel $\left(10 \text{ Log}_{10} \frac{E}{N_o} = -6 \text{ db}\right)$.

$\{p_i,\ i=1,2,\ldots,P\}$ and Q-letter output alphabet with transition probabilities $\{q_{ij},\ i=1,2,\ldots,P,\ j=1,2,\ldots,Q\}$ Savage[3] and Jacobs and Berlekamp[4] have established bounds which indicate that the distribution of the number of computations per decoded digit, C, is Pareto. That is,

$$\text{pr}(C \geq X) \sim X^{-\alpha}, \qquad (X \gg 1),$$

where α, the Pareto exponent, satisfies the relation

$$R = \frac{E(\alpha)}{\alpha}$$

in which R is the information rate in bits per channel waveform, and

$$E(\alpha) = -\log_2 \sum_{j=1}^{Q} \left(\sum_{i=1}^{P} p_i q_{ij}^{\frac{1}{1+\alpha}} \right)^{1+\alpha}$$

Thus α may be calculated from the transition probabilities and the input probability distribution. For $R = 1/7$, $\alpha = 1.2$ for the channel of Fig. XXII-2. If α is less than 1, the expectation of C is theoretically infinite. The rate corresponding to $\alpha = 1$ is known as R_{comp}. For rates above R_{comp}, the bound on the mean computation diverges. $R_{comp} = E(1)$ is also calculated from the transition probabilities and input distribution. For the transition probabilities of Fig. XXII-2 with $p_1 = p_2 = \frac{1}{2}$, $R_{comp} = .160$.

The signal-to-noise ratio of -6 db and the rate of 1/7 were chosen so that with 21 bits of quantization per information bit at the decoder input, the energy required per bit, E/R, was only 0.4 db greater than the required energy per bit for an infinite-bandwidth, unquantized Gaussian channel with the same values of $N_o/2$ and α^1. In other words, with these parameters, the quantization scheme contributed a degradation of only 0.4 db of required signal energy.

The sequential decoder design was based on the Fano algorithm. No attempt was made to optimize the threshold spacing, since previous experiments[5] have suggested that the mean computation per decoded digit has a very broad minimum with respect to this parameter. The tilted distance function used for the path metric was the usual Fano distance function,[11] with the bias equal to the rate, 1/7 bits per channel waveform.

The simulation of the sequential decoder was accomplished on a PDP-6 computer by means of a sequential decoding simulation facility developed by Niessen.[6] The computer program allows a display of the branches that are being searched, as well as a facility for collecting histograms of quantities of interest in sequential decoding. The simulated sequential decoder could perform approximately 2,500,000 computations per hour. Presumably, a special-purpose machine could be built to increase this speed several orders of magnitude.

Since the simulated channel (including the modulator, matched filter, and quantizer) is symmetric from the input, the error probabilities are independent of the input information sequence. Consequently, for convenience, an all-zero information sequence was assumed to be transmitted.

The discrete memoryless channel was simulated on the computer with a random-number generator whose output was quantized in accordance with the calculated transition probabilities. A description of the random-number generator and its statistical properties has been given by Green, Smith, and Klem.[7]

As a consequence of the sequential nature of the decoding process, an unusually large number of computations to decode some digit may unavoidably delay the decoding of all succeeding digits, unless the decoder is "resynchronized" at periodic intervals. For this reason, in the absence of a feedback link, the information-bit stream should be organized into blocks, and a constraint length of zeros inserted between successive blocks before the resulting bit stream is allowed to enter the encoder. Each encoded sequence of information digits and its associated "tail" of zeros is now referred to as a block. Thus the decoder may start to decode from the beginning of any block without the knowledge of any previous decoding decisions. From the last information digit of a block, the decoder is constrained to follow a path for which all information digits till the end of the block are zeros. In the simulation, the number of information bits per block was chosen to be twenty times the constraint length[12]; thus the decrease in effective rate caused by the resynchronization between blocks was 5 per cent, and the resulting increase in the energy required per information bit was just under 1/4 db.

In the application of sequential decoding considered here, the problem of undecoded data overflowing a finite storage buffer does not arise, since it is assumed that the data received from the channel may be recorded and then decoded off-line. In practice, there will be some upper limit on the number of computations that one is willing to do to decode any block. The occasional block for which this limit is exceeded may be termed an "undecodable" block and constitutes a type of system failure. Another type of system failure is the undetected block error, in which errors remain in a block after it has been decoded. Clearly, constraint lengths that are long enough to make the undetected error probability much less than the undecodable block probability would complicate the encoder unnecessarily. Consequently, an important aspect of the simulation was to determine approximately the minimum constraint length necessary to yield an undetected error probability comparable to the undecodable block probability, with a reasonable limit on the maximum number of computations per block. A further practical constraint on the simulated conditions was that the undetected error and undecodable block probabilities be large enough that at least a few of these events would be observed within a reasonable interval of computer running time. With, say, 20 hours of computer time available (and block lengths greater than 300) this would preclude the measurement of undetected block error probabilities less than 10^{-4}

The choice of the convolution code affects the error probability and the mean computation, especially if the constraint length is relatively short, as in our case. Binary convolution codes with optimum "front ends" for use on a binary symmetric channel have been found by computer search techniques.[8,9] It was decided more or less arbitrarily, to use combinations of these codes to yield codes with rate 1/7. The generators used are listed in Table XXII-1. Each generator (corresponding to one of the 6 parity checks per information bit) is listed as a sequence of octal numbers, each representing 3 successive binary digits. Note that two different codes with a constraint length of 18 were tested. The generators of Code I were the 5 generators from an optimized rate 1/6 binary code, together with the generator of an optimized rate 1/2 code.[8] Code II was obtained by replacing the first 7 binary digits of each generator of Code I with optimum "front ends" compiled by Lin and Lyne.[9] Code III was similar to Code II but to each generator was appended 6 binary digits, also obtained from Blustein.[8] Code II, when used with the simulated channel and sequential decoder yielded a rather surprising three-fold increase in error probability over Code I, and consequently Code II was discarded. Optimization of codes for this channel is not considered rewarding, since it is easier to lower the undetected error probability by increasing the constraint length slightly.

During each run, the number of blocks decoded, the total number of computations performed, and the number of undetected block errors were counted. One computation was said to occur each time the decoder compared a path metric to a threshold. An

Table XXII-1. Convolution code generators.

Constraint Length	Generators	
18	I 736036	II 442036
	543122	537122
	652436	552436
	632705	632705
	536106	652106
	607363	723363
24	III 44203601	
	53712256	
	55243601	
	63270547	
	65210662	
	72336305	
36	Same as III, but each was given a randomly chosen 12-bit tail.	

undetected error occurred when the decoder accepted a block, including the tail, but decoded one or more of the information digits in the block as ones.

Programs were written to generate three kinds of histograms for data generated during each run as follows:

i. The distribution of the number of computations per decoded block.

ii. The distribution of the search depth. A search, starting at some node J, constitutes all computations done in the time interval between first reaching J and first reaching the next succeeding node. The search depth is the maximum depth which the decoder backs up during a search.

iii. The distribution of the number of computations per search. This is closely related to the distribution of computation studied by Savage and by Jacobs and Berlekamp.

2. Results

a. Undetected Errors

The relative frequencies of undetected errors are tabulated in Table XXII-2, together with 99 per cent confidence bounds on the true error probabilities. Note that the sample sizes were too small to yield accurate estimates of the true error probabilities. The 99 percent confidence bounds were evaluated by using a Chernoff bound for the tail of the binomial probability distribution.[10] If r is the observed number of erroneous blocks out of a total of N,

$$\mathrm{pr}(r \leq Np_1) < \exp\left\{-N\left[H(p_o) - H(p_1) + (p_1 - p_o) \ln \frac{1-p_o}{p_o}\right]\right\}, \qquad \text{for } p_1 < p_o \tag{1}$$

where p_o is the true error probability, and H(p) is the binary entropy function.

To get the 99 per cent confidence bounds, we set the right-hand side of (1) equal to .01 with

$$p_1 = \frac{\text{observed number of undetected errors}}{\text{number of blocks decoded}},$$

and then solve for p_o. Then, on the basis of the observed relative frequencies, the true error probability is less than this value of p_o with 99 per cent probability.

Table XXII-2. Occurrences of undetected block errors.

$10 \log_{10} E/N_o$	Constraint Length	Number of Blocks Decoded	Number of Undetected Errors	99% Confidence bounds on probability of undetected error per block, p_o
-6 db	18	5436	4	.002
-6 db	24	7822	0	.0006
-6.5 db	18	1027	1	.008
-6.5 db	24	1331	0	.003
-7 db	24	167	0	.03
-7 db	36	202 (360 Bit Blocks)	0	.02

It will be noticed that all errors occurred when the constraint length was 18, which suggests a strong dependence on the constraint length. The error probability for sequential decoding has been shown to be bounded by an exponential function of constraint length.[11]

The number of incorrectly decoded binary digits constituting an undetected block error was typically approximately 5. The largest number of erroneous bits per block that was observed was 16. An interesting topic for future research would be a closer study of statistics of the number and positions of erroneous binary digits within a block, and how they are influenced by the code parameters.

b. Distribution of Computations

The calculated Pareto exponents for the three signal-to-noise ratios simulated are given in Table XXII-3.

Table XXII-3. Calculated Pareto exponents.

$10 \log_{10} E/N_o$	Pareto Exponent, a
-6	1.20
-6.5	1.04
-7	0.80

It is seen that for a signal-to-noise ratio of -7 db, a is less than one, which implies an infinite mean for the Pareto distribution. This corresponds to operation at a rate above $R_{comp.}$

The observed average number of computations per decoded digit, including the tails, are tabulated in Table XXII-4. At -7 db, the measured mean varied rather erratically as expected, and the measured "averages" for three different runs are shown for a constraint length of 24. (The law of large numbers does not hold when the mean is infinite.)

Table XXII-4. Average computation per decoded digit.

Constraint Length	$10 \log_{10} E/N_o$		
	-6 db	-6.5 db	-7 db
18	4.52	7.81	–
24	3.22	6.29	12.7, 13.7, 23.2
36	–	–	25.6

Figures XXII-3 and XXII-4 show the distribution of the number of computations per block for constraint lengths of 18 and 24. The tail of the distribution of computation per block is expected to be identical to the tail of the distribution of computation per decoded digit, except for a multiplicative factor that depends on block length. Consequently, the magnitude of the slope of each curve should equal the corresponding Pareto exponent. Figures XXII-3 and XXII-4 show that the magnitude of the slope increases with the signal-to-noise ratio as expected, but the sample sizes are too small to display the true tail behavior. These graphs do allow estimates of the undecodable block criterion that is necessary to achieve an undecodable block probability of 10^{-3} or 10^{-4}. For instance, it appears that a maximum of 200,000 computations, or approximately 10 minutes of computer time per block, must be tolerable to achieve an undecodable block error probability of 10^{-3} when the number of bits per block is 360 and the signal-to-noise ratio is -6 db.

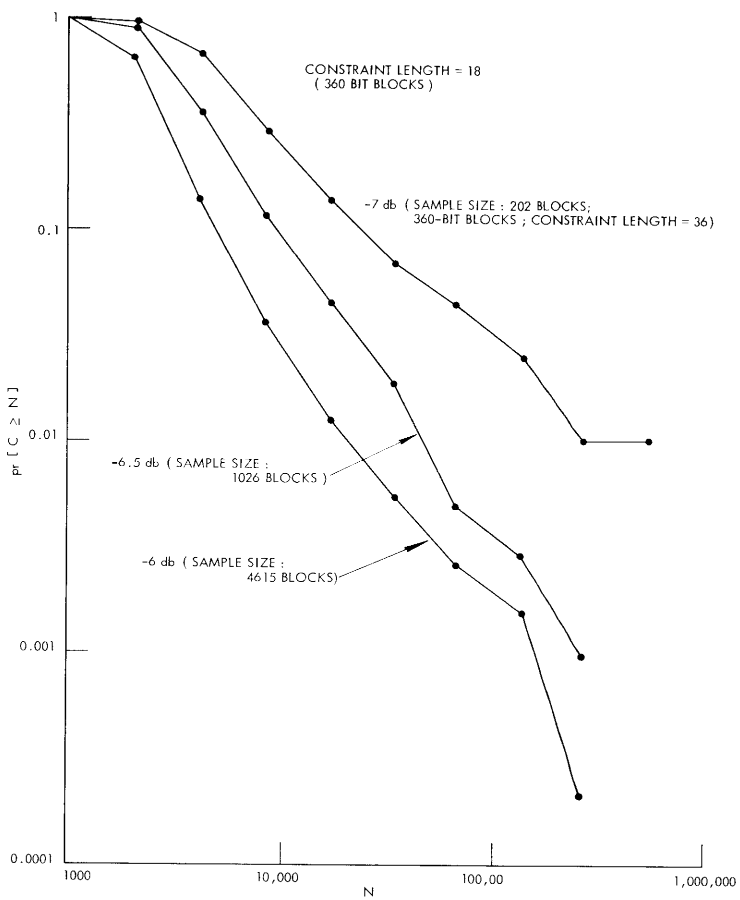

Fig. XXII-3. Cumulative distribution of the number of computations per block, C.

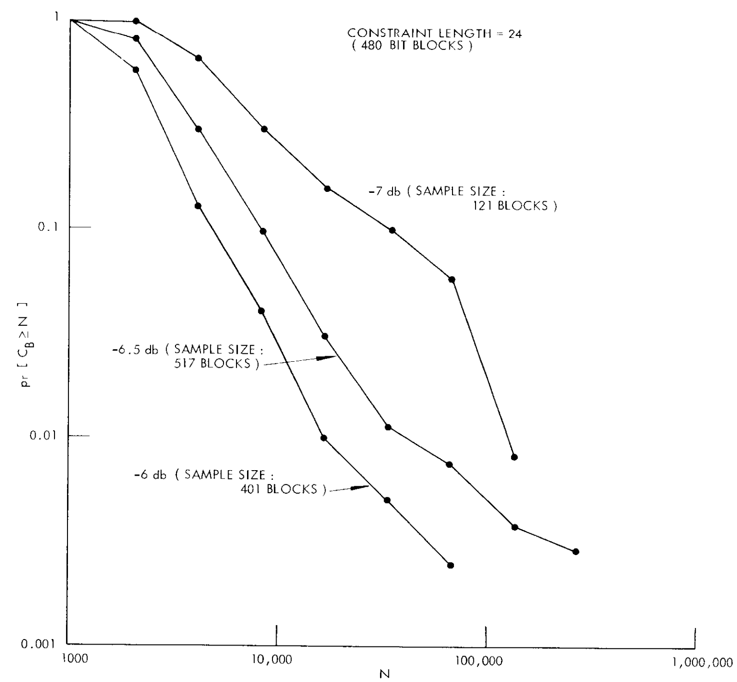

Fig. XXII-4. Cumulative distribution of the number of computations per block, C_B.

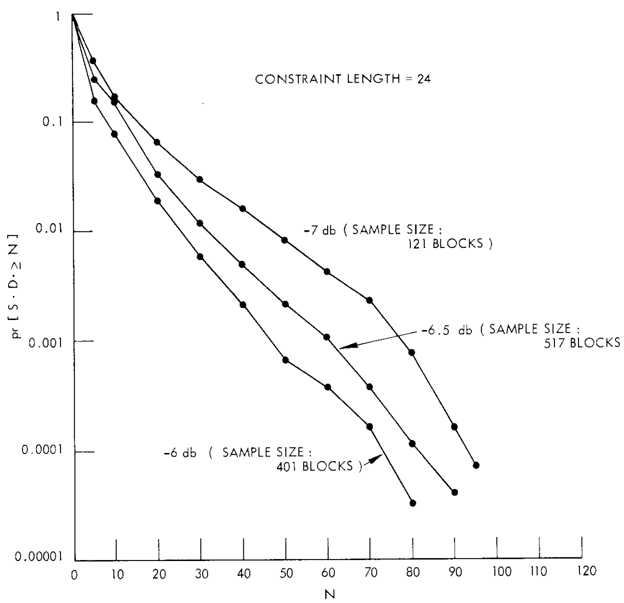

Fig. XXII-5. Cumulative distribution of the search depth, S.D.

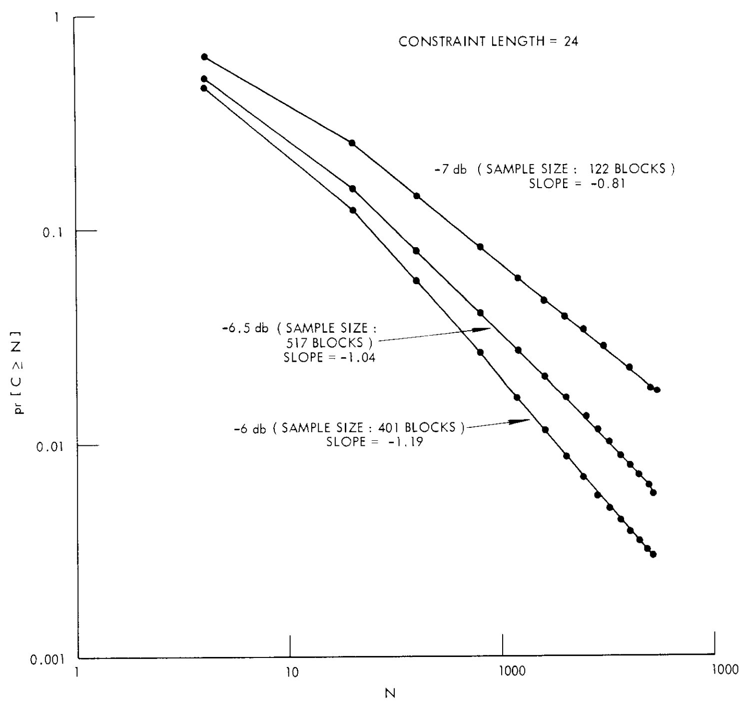

Fig. XXII-6. Cumulative distribution of the number of computations per search, C.

Figure XXII-5 shows the cumulative distribution of the search depth for a constraint length of 24. (The corresponding curves for a constraint length of 18 show no essential differences and are not reproduced here.) As expected, the distribution appears to be an exponential function of the search depth. Note that searches of up to 80 or 100 nodes which caused no errors were observed, even though the constraint length was 24 nodes.

The observed distribution of the number of computations per search for a constraint length of 24 is shown in Fig. XXII-6. The corresponding curves for the other constraint lengths show no noticeable difference from those of Fig. XXII-6. The tails of these curves plot as straight lines whose slopes are equal in magnitude to the corresponding Pareto exponents. The slopes are listed on the graph and agree very closely with the calculated Pareto exponents of Table XXII-4.

c. Operation at a Rate Exceeding R_{comp}

Perhaps the most striking aspect of the simulation was the sustained operation of the system above R_{comp}. With constraint lengths of 24 and 36, 167 720-bit blocks and 202 360-bit blocks, respectively, were decoded without error. The mean computation per decoded digit was high and varied erratically, but in spite of more and longer searches, the dynamics of the sequential search procedure were similar to those for operation below R_{comp}. Most digits were decoded with only a few computations, as Fig. XXII-6 shows.

The implications of the results at -7 db with 720-bit blocks are that continuous operation of this system above R_{comp} may require unreasonable decoding times, but that relatively short excursions above R_{comp}, because of a temporary decrease in received signal power, are tolerable. Figure XXII-6 shows clearly that the Pareto distribution of computation still holds for values of α less than one, that is, for rates greater than R_{comp}. The concatenation of block codes with convolution codes is now being investigated as a means of increasing the Pareto exponent at any rate below channel capacity and thereby reducing the computational variability associated with sequential decoding.

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11. Ibid., Chapter 6.

12. The use of tails of less than a constraint length is considered by D. Chase in Section XXII-A.